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DISTRIBUTIONS ARISING FROM BIRTH AND DEATH PROCESSES AT EQUILIBRIUM AND THEIR EXTENSIONS

Research Report in Mathematics, Number 18, 2018

Symon Mareyan Lesaris

August 2018

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Mathematics Statistics

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Master of Science Project

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Mathematics Statistics

Prepared for The Director
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Abstract

The goal of this project, is to demonstrate the use of special functions; in this case - hypergeometric function in statistics. We start by deriving the basic difference differential equations for birth and death processes at equilibrium and solving it iteratively using different values of λ_n and μ_n. The solution of the basic difference - differential equations are applied and hence obtain distributions to the: (i) Growth models; (ii) Waiting time problems and (iii) Queuing processes as special cases of Birth and Death processes at equilibrium.

The basic difference differential equations are also expressed as ratio of polynomials and the equations are solved to obtain probability distributions in terms probability generating function technique and hypergeometric functions.

Birth and death processes at equilibrium and their extensions based on recursive models of P_{n+1} as a function of P_n and P_{n-1}; Katz, Crow-Bardwell, Panjer's and Kemp's families of recursive models as a ratio of polynomials $\frac{P_{n+1}}{P_n} = \frac{Q(n)}{R(n)}$; Kapur's recursive model as a ratio of polynomials $\frac{P_\iota}{P_{\iota-1}}$.

Note that, Kapur (1978a) generalized birth and death processes are expressed in terms of generalized hypergeometric functions at equilibrium as ratio of polynomials given;

$$\frac{P_\iota}{P_{\iota-1}} = \frac{\lambda_{\iota-1}}{\mu_\iota}, \quad \mu_\iota \neq 0$$

where various cases of λ_ι and μ_ι are solved.

A confluent hypergeometric series distribution is constructed using Kummer's series is used as a tool to construct hypergeometric function from a ratio of polynomials. Some special cases and properties of the distributions arising from these processes are discussed.

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Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

SYMON MAREYAN LESARIS
Reg No. I56/88813/2016

In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

Signature Date

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Dedication

This project is dedicated to the Lord God who has been my shelter in the time of storms, to my parents the Late Wesley Mareyan and Diana Nongera, and to my colleagues Kelvin and Gilbert for their support.

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Symon Mareyan Lesaris

Nairobi, 2018.

1 GENERAL INTRODUCTION

1.1 Background Information

The aim of this thesis is to demonstrate the use of hypergeometric function in statistics. The study of distributions based on difference differential equations arising from stochastic processes. In this case, we studied birth and death processes, where a transition takes place from one state only to a neighboring state. With an arrival (birth) λ_k, there is a transition from state $k(\geq 0)$ to the state $k+1$, and with a service completion there (death) μ_j there is a transition from the state j to the state $(j-1)(j>0)$, the state denoting the number in the system.
The basic difference differential equation in general is note easy to solve in most cases. But can be solved easily at steady state; solving differential equation for birth and death processes as $t \rightarrow \infty$.

1.1.1 A Stochastic Process

The theory of stochastic processes mainly originated from the need of physicists. It began with the study of physical phenomena as random phenomena changing with time. Let t be a parameter assuming values in a set τ, and let $X(t)$ represent a random or stochastic variable for every $t \in \tau$. The family or collection of random variables $\{X(t), t \in \tau\}$ is called a stochastic process.
The parameter or index t is generally interpreted as time and the random variable $X(t)$ as the state of the process at time t. The elements of τ are time points, or epochs, and τ is a linear set, denumerable or non-denumerable.
If τ is countable (or denumerable), then the stochastic process $\{X(t), t \in \tau\}$ is said to a discrete-parameter or discrete-time process, while if τ is an interval of the real line, then the stochastic process is said to be continuous-parameter or continuous-time processes. For instance $\{X_n, n = 0, 1, 2, ...\}$ is a discrete - time and $\{X(t), t \geq 0\}$ is a continuous - time process. The set of all possible values that the random variable $X(t)$ can assume is called the state space of the process; which may be countable or non-countable.

1.1.2 Types of Stochastic Processes

The stochastic processes may be classified into these four types:
(i) Discrete state and Discrete time,

(ii) Continuous state and Discrete time,
(iii) Discrete state and Continuous time, and
(iv) Continuous state and Continuous time.
Therefore, birth-and-death processes is a continuous time and discrete state processes. Birth and Death processes were initiated by David G. Kendall (1948).

1.2 Definitions, Notations and Terminologies

At equilibrium / At steady state: Given the function $P_n(t)$, at steady state / equilibrium it means that $t \to \infty$ implying that $P_n(t)$ is independent of t.
Birth and Death Processes: Is a process where a transition takes place from one state only to a neighboring state. With an arrival (birth) λ_k, there is a transition from state $k(\geq 0)$ to the state $k+1$, and with a service completion there (death) μ_j there is a transition from the state j to the state $(j-1)(j>0)$, the state denoting the number in the system.
λ_n: Birth rate.
μ_n: Death rate.
Δt: It is the time interval between t and $t+\Delta t$.
$0(\Delta t)$: Order Δt means that a function of Δt goes to zero faster than Δt as $\Delta t \to 0$. It is the probability of having two or more births.
pgf.: probability generating function.
ECHS: Extended Confluent Hypergeometric Series.
iid: independent and identically distributed.

1.3 Problem Statement

Solutions of Basic Difference - Differential equations are not easy to obtain; they can be cumbersome and complicated.
Most of the literature concerning birth and death processes involve the coefficients λ and μ are permitted to depend on time. In the analysis, distributions based on difference differential equations arising from birth and death processes at time t which sometimes tends to infinity. Most Birth and Death processes never tends to infinity and for such processes time dependent do not make sense hence the need for birth and death processes at an equilibrium.
Kapur (1978a) obtained generalized birth and death processes at steady state. Though he has identified some special cases, but did not provide their explicit solutions. Mine is to solve the special cases of λ_i and μ_i as a ratio of polynomials and identify distributions arising from the special cases in terms of hypergeometric functions.

1.4 Objectives

1.4.1 Main Objective

The main objective of the project is to obtain distributions arising from birth and death processes at equilibrium and their extensions.

1.4.2 Specific Objective

1. To derive and solve the basic difference - differential equations of birth and death processes at equilibrium.

2. To apply the solution of the basic difference - differential equations and hence obtain distributions to the following:
   (a) Growth model,
   (b) Waiting time problem,
   (c) Queuing processes as special cases of birth and death processes.

3. To express the basic difference - differential equations as ratio of polynomials and solve the equations to obtain probability distributions.

1.5 Methodology

Distributios based on difference differential equations arising from birth and death processes at steady state can be solved using:
1. Iteration technique;
2. Probability generating function technique;
3. Laplace technique; and
4. Others.
In this case, the methods used are:
1. Iteration technique; and
2. Probability generating function technique;
Where, confluent hypergeometric and generalised hypergeometric functions are constructed from birth and death processes at steady state and some special cases and their properties are obtained.

1.6 Literature Review

A general Birth and Death process at equilibrium where the coefficients λ and μ are independent of time was discussed in details by Kendall, D.G. (1948). He also worked on the simple birth - death - and immigration process with zero initial population (Kendall, D.G. 1949).

In waiting line and trunking problems for telephone exchanges were studied long before the theory of stochastic processes was available and had a simulating influence on the development the theory. In particular, Palm's C. (1943) impressive work on waiting lines and trunking problems were useful. Also Palm, C. works on the distribution of repairmen in servicing automatic machines.

Servicing of machines problem was derived by Erlang, A.K. (1878 - 1929). The limiting distribution is an Erlang distribution when only one serviceman is servicing the machine and a binomial distribution when several repairmen are servicing the machines.

Naor, P. (1969) has used the queuing system model M/M/1/K to study the regulations of queue size by levying tolls.

Rue, R.C. and Rosenshine, M. (1981) have extended Naor's arguments to obtain a policy for individual optimum in case of M classes of customers.

In the displaced Poisson destribution introduced by Staff, P.J. (1964), we have $\lambda = r$ is a positive integer. For any $\lambda > 0$ we have a hyperpoisson distribution. Bardwell, G.E. and Crow, E.L. (1964) termed the distribution Sub-Poisson for $\lambda > 1$.

Barton, D.E. (1966) also pointed out that a hyper-poisson distribution can be obtained by considering a truncated Pearson type III mixture of a Poisson distribution.

Kemp, A.W. and Kemp, C.D. (1966) found that if mixing is treated purely as a formal process with the poisson parameter θ taking negative values, then a Hermite distribution can be derived as a Poisson - Normal mixture.

In constructing a generalized Hermite distribution, Gupta, R.P. and Jain, G.C. (1974) considered the variable X given by $X = X_1 + MX_2$, where X_1 and X_2 are independent Poisson random variables with parameters η_1 and η_2 respectively.

The probability generating function of a Hermite distribution is a compound Poisson distribution obtained by considering $S_N = X_1 + X_2 + ... + X_N$, where $X_i's$ are independent and identically distributed binomial random variables with parameters 2 and p, where N is Poisson with parameter λ.

Kemp (1978a) studied the most general case in its steady state and found expressions for the probability distribution, probability generating function, and moments in terms of generalized hypergeometric functions.

1.7 Significance of the Study

Special functions were originally used in Theoretical Physics and Applied Mathematics. Here is a case where they are being used in Statistics.

2 BIRTH AND DEATH PROCESSES AT EQUILIBRIUM WITH APPLICATION TO GROWTH MODELS

2.1 Introduction

Birth-and-Death process is a stochastic process in which jumps from a particular state (number of individuals, cells, lineages etc) are only allowed to neighbouring states. A jump to the right, i.e. increase by one of the number of individuals or similar quantities represents birth, whereas a jump to the left represents death.
Assume that the probability is approximately $\lambda\Delta t$ or $\mu\Delta t$ that in an interval of length Δt, a member will either create a new member or loss a member. More specifically, if $X(t)$ is the size of the population at time t, then $\{X(t), t \geq 0\}$ is birth (death) process with $\lambda_n = n\lambda \quad (\mu_n = n\mu)$ for n=0, 1, 2, . . .

2.2 Derivation of the basic difference differential equations for Birth and Death Processes

Let $X(t)$ be the population size at time t. Further, Let

$$P_n(t) = \text{Prob}[X(t) = n] \tag{2.1}$$

This implies that;

$$P_n(t+\Delta t) = \text{Prob}[X(t) + \Delta(t) = n] \tag{2.2}$$

where Δt is the time interval between t and Δt.
Assume that the following events occur within time interval $(t, t+\Delta t)$:
(1) Only one event occurs; or
(2) No event occurs.
For instance, for births only if $X(t+\Delta t)$; then

$$X(t) = n-1 \quad \text{or} \quad X(t) = n \tag{2.3}$$

For deaths only, if $X(t+\Delta t)=n$; then

$$X(t)=n+1 \quad \text{or} \quad X(t)=n \tag{2.4}$$

Therefore, the following basic assumptions are underlying the birth and death processes with parameters λ_n and μ_n:

2.2.1 Assumptions: $X(t)=n$

1. The probability of a birth occurring within Δt is $\lambda_n \Delta t + 0(\Delta t)$.
2. The probability of a death occurring within Δt is $\mu_n \Delta t + 0(\Delta t)$.
3. The probability of a death and a birth occurring within Δt is $0(\Delta t)$.
4. The probability of no birth and no death occurring within Δt is $1-\lambda_n\Delta t-\mu_n\Delta t-0(\Delta t)$.

Note that: $0(\Delta t)$ is order Δt which means that the function of Δt tends to zero faster than Δt.
i.e. ,

$$\lim_{\Delta t\to 0}\frac{0(\Delta t)}{\Delta t}=0 \tag{2.5}$$

Thus,

$$\begin{aligned}P_n(t+\Delta t) &= \text{Prob}[X(t+\Delta t)=n|X(t)=n-1]\text{Prob}[X(t)=n-1]\\ &+\text{Prob}[X(t+\Delta t)=n|X(t)=n+1]\text{Prob}[X(t)=n+1]\\ &+\text{Prob}[X(t+\Delta t)=n|\text{Prob}[X(t)=n]\end{aligned}$$

Therefore,

$$P_n(t+\Delta t) = \Big(\lambda_{n-1}\Delta t + 0(\Delta t)\Big)P_{n-1}(t) + \Big(\mu_{n+1}\Delta t + 0(\Delta t)\Big)P_{n+1}(t)$$
$$+\Big(1-\lambda_n\Delta t - \mu_n\Delta t - 0(\Delta t)\Big)P_n(t) \quad \textbf{(2.6)}$$

Using the first principle;

$$P_n'(t) = \lim_{\Delta t\to 0}\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} \quad \textbf{(2.7)}$$

where $P_n'(t)$ is something to be derived.
Given the initial condition, we have;

$$P'(t) = \frac{\Big(\lambda_{n-1}\Delta t + 0(\Delta t)\Big)P_{n-1}(t) + \Big(\mu_{n+1}\Delta t + 0(\Delta t)\Big)P_{n+1}(t) + \Big(1-\lambda_n\Delta t - \mu_n\Delta t - 0(\Delta t)\Big)P_n(t) - P_n(t)}{\Delta t}$$
$$= \lim_{\Delta t\to 0}\Big\{\Big(\lambda_{n-1} + \frac{0(\Delta t)}{\Delta t}\Big)P_{n-1}(t) + \Big(\mu_{n+1} + \frac{0(\Delta t)}{\Delta t}\Big)P_{n+1}(t)$$
$$+\Big(-\lambda_n - \mu_n - \frac{0(\Delta t)}{\Delta t}\Big)P_n(t)\Big\}$$

But,

$$\lim_{\Delta t\to 0}\frac{0(\Delta t)}{\Delta t} = 0$$

Therefore, $P_n'(t) = \mu_{n+1}P_{n+1}(t) + \lambda_{n-1}P_{n-1}(t) - (\lambda_n+\mu_n)P_n(t)$

$$\therefore P_n'(t) = \mu_{n+1}P_{n+1}(t) + \lambda_{n-1}P_{n-1}(t) - (\lambda_n+\mu_n)P_n(t); \quad n \geq 1 \quad \textbf{(2.8)}$$

For n=0;

$P_0(t+\Delta t) =$ Prob (death) or Prob (no birth, no death)

$P_0(t+\Delta t) =$ Prob (death) $+$ Prob (no birth, no death)

Since

$\text{Prob}[X(t+\Delta t) = 0|X(t) = -1]\text{Prob}[X(t) = -1] = 0$

Then,

$$\begin{aligned}P_0(t+\Delta t) &= \text{Prob}[X(t+\Delta t)=0|X(t)=1]\text{Prob}[X(t)=1] \\ &\quad + \text{Prob}[X(t+\Delta t)=0|X(t)=0]\text{Prob}[X(t)=0] \\ P_0(t+\Delta t) &= [\mu_1\Delta t + 0(\Delta t)]P_1(t) + [1-\mu_0\Delta t - \lambda_0\Delta t - 0(\Delta t)]P_0(t)\end{aligned}$$

From the first Principle;

$$P_0'(t) = \lim_{\Delta t\to 0}\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t}$$

we have;

$$\begin{aligned}P_0'(t) &= \lim_{\Delta t\to 0}\frac{\Big\{[\mu_1\Delta t + 0(\Delta t)]P_1(t) + [1-\mu_0\Delta t - \lambda_0\Delta t - 0(\Delta t)]P_0(t) - P_0(t)\Big\}}{\Delta t} \\ &= \lim_{\Delta t\to 0}\Big\{[\mu_1 + \frac{0(\Delta t)}{\Delta t}]P_1(t) + [-\mu_0 - \lambda_0 - \frac{0(\Delta t)}{\Delta t}]P_0(t)\Big\}\end{aligned}$$

But;

$$\lim_{\Delta t\to 0}\frac{0(\Delta t)}{\Delta t} = 0$$

Therefore;

$$P_0'(t) = \mu_1 P_1(t) - \Big(\lambda_0 + \mu_0\Big)P_0(t)$$

Since $\mu_n = 0$ for $n = 0$ (i.e., there is no death at μ_0)
Therefore,

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \tag{2.9}$$

Thus, for the general Birth-and-Death processes the basic difference differential equations are:

$$
\begin{aligned}
&P_n'(t) = \mu_{n+1}P_{n+1}(t) + \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t), \quad n \geq 1 \\
&\text{ and} \\
&P_0'(t) = \mu_1 P_1(t) - \lambda_0 P_0(t), \quad \text{for} \quad n = 0.
\end{aligned} \tag{2.10}
$$

At stedy state $t \to \infty$ implying that $P_n(t)$ is independent of t.
Thus,

$$
\begin{gathered}
P_n(t) = P_n \\
\text{and} \\
\lim_{t \to \infty} P_n'(t) = 0
\end{gathered}
$$

Therefore, the basic difference differential equations for the steady state for the general Birth-and-Death processes are:

$$
\begin{aligned}
&0 = \mu_1 P_1 - \lambda_0 P_0, \\
&\text{ and} \\
&0 = \mu_{n+1}P_{n+1} + \lambda_{n-1}P_{n-1} - (\lambda_n + \mu_n)P_n, \quad n \geq 1
\end{aligned} \tag{2.11}
$$

2.3 Recuesive Relation based on three consecutive terms

Let, the probability of birth and death in the time - interval $(t, t + \Delta t)$ be;

$$
P_0(t + \Delta t) = \mu_1 P_1(t)\Delta t + [1 - \lambda_0 \Delta t]P_0(t) + 0(\Delta t) \tag{2.12}
$$

and

$$
P_n(t + \Delta t) = \mu_{n+1}P_{n=1}(t)\Delta t + \lambda_{n-1}P_{n-1}P_{n-1}(t)\Delta t + [1 - \lambda_n \Delta t - \mu_n \Delta t]P_n(t) + 0(\Delta t); \quad n = 1, 2, \ldots \tag{2.13}
$$

Which leads to the system of the basic difference differential equations given by;

$$P_0'(t) = \mu_1 P_1(t) - \lambda_0 P_0(t) \tag{2.14}$$

and

$$P_n'(t) = \mu_{n+1} P_{n+1}(t) + \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t); n \geq 0 \tag{2.15}$$

For the steady state, we have;

$$\lim_{t \to \infty} P_n'(t) = 0, \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Thus, equations (2.14) and (2.15) becomes;

$$0 = \mu_1 P_1 - \lambda_0 P_0 \tag{2.16}$$

$$0 = \mu_{n+1} P_{n+1} + \lambda_{n-1} P_{n-1} - (\lambda_n + \mu_n) P_n; n \geq 0 \tag{2.17}$$

Hence;

Proposition 2.1

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} P_0; \quad n = 1, 2, 3, \ldots, \quad \mu_n \neq 0 \tag{2.18}$$

$$\text{where,} \quad P_0 = \frac{1}{\left\{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \lambda_2 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}\right\}}. \tag{2.19}$$

Proof

Equation (2.16) can be expressed as;

$$\mu_1 P_1 = \lambda_0 P_0$$

$$\Rightarrow P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$\therefore P_1 = \frac{\lambda_0}{\mu_1} P_0. \tag{2.20}$$

When n=1;
Equation (2.17) becomes;

$$0 = \mu_2 P_2 - (\lambda_1 + \mu_1)P_1 + \lambda_0 P_0 \tag{2.21}$$

Substituting (2.20) in equation (2.21), we have;

$$\begin{aligned} 0 &= \mu_2 P_2 - (\lambda_1 + \mu_1)\frac{\lambda_0}{\mu_1}P_0 + \lambda_0 P_0 \\ 0 &= \mu_2 P_2 - \frac{\lambda_0\lambda_1}{\mu_1}P_0 - \lambda_0 P_0 + \lambda_0 P_0 \\ 0 &= \mu_2 P_2 - \frac{\lambda_0\lambda_1}{\mu_1}P_0 \\ \iff \mu_2 P_2 &= \frac{\lambda_0\lambda_1}{\mu_1}P_0 \\ \Rightarrow P_2 &= \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0 \\ \therefore P_2 &= \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0. \end{aligned} \tag{2.22}$$

When n=2;
Equation (2.17) becomes;

$$0 = \mu_3 P_3 - (\lambda_2 + \mu_2)P_2 + \lambda_1 P_1 \tag{2.23}$$

Substituting (2.20) and (2.22) in equation (2.23), we have;

$$\begin{aligned} 0 &= \mu_3 P_3 - (\lambda_2 + \mu_2)\frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0 + \lambda_1\frac{\lambda_0}{\mu_1}P_0 \\ 0 &= \mu_3 P_3 - \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2}P_0 - \frac{\lambda_0\lambda_1}{\lambda_1\lambda_2}P_0 + \frac{\lambda_0\lambda_1}{\lambda_1\lambda_2}P_0 \\ 0 &= \mu_3 P_3 - \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2}P_0 \\ \iff \mu_3 P_3 &= \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2}P_0 \end{aligned}$$

$$\Rightarrow P_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3}P_0$$
$$\therefore P_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3}P_0. \tag{2.24}$$

When n=3;
Equation (2.17) becomes;

$$0 = \mu_4P_4 - (\lambda_3+\mu_3)P_3 + \lambda_2P_2 \tag{2.25}$$

Substituting (2.22) and (2.24) in equation (2.25), we have;

$$\begin{aligned}
0 &= \mu_4P_4 - (\lambda_3+\mu_3)\frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3}P_0 + \lambda_2\frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0\\
0 &= \mu_4P_4 - \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3}P_0 - \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2}P_0 + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2}P_0\\
0 &= \mu_4P_4 - \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3}P_0\\
\Longleftrightarrow \mu_4P_4 &= \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3}P_0\\
\Rightarrow P_4 &= \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3\mu_4}P_0\\
\therefore P_4 &= \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3\mu_4}P_0.
\end{aligned}$$

By Mathematical induction, we have;

$$\begin{aligned}
P_5 &= \frac{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4}{\mu_1\mu_2\mu_3\mu_4\mu_5}P_0,\\
P_6 &= \frac{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5}{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_5}P_0,\\
&.\\
&.\\
&.\\
P_n &= \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n}P_0. \quad n = 1,2,3,\ldots \qquad (2.26)\\
\text{where,} \quad P_0 &= \frac{1}{\left\{1+\sum_{n=1}^{\infty}\frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}\right\}}.
\end{aligned}$$

But;

$$\sum_{n=0}^{\infty} P_n = 1$$
$$\Rightarrow P_0 + P_1 + P_2 + P_3 + \ldots + P_n + \ldots = 1$$
$$P_0 + \frac{\lambda_0}{\mu_1}P_0 + \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0 + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3}P_0 + \ldots + \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n}P_0 + \ldots = 1$$
$$P_0\left\{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} + \ldots + \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n} + \ldots\right\} = 1$$
$$P_0\left\{1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n}\right\} = 1$$

Thus, the probability of ultimate extinction is given by;

$$P_0 = \frac{1}{\left\{1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n}\right\}}.$$

Therefore;

$$P_n = \frac{\frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n}}{\left\{1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n}\right\}}; \quad n = 1, 2, 3, \ldots \tag{2.27}$$

2.4 Recursive Relation based on two consecutive terms

Proposition 2.2

$$\frac{P_n}{P_{n-1}} = \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0 \tag{2.28}$$

Proof
From (2.26), we have;

$$P_n = \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n}P_0. \quad n = 1,2,3,\ldots$$

$$\Rightarrow P_{n-1} = \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-2}}{\mu_1\mu_2\mu_3\ldots\mu_{n-1}}P_0.$$

Therefore,

$$\frac{P_n}{P_{n-1}} = \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-2}\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_{n-1}\mu_n}.\frac{\mu_1\mu_2\mu_3\ldots\mu_{n-1}}{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-2}}$$

$$\therefore \frac{P_n}{P_{n-1}} = \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0 \tag{2.29}$$

2.5 Problem Statement

The problem is to determine P_n using propositions (2.1) and (2.2). For special cases, differential equations in probability generating function will be derived and solved to obtain the means and the variance.

2.6 Population Growth Model

Note that, from (2.11) the basic difference differential equations for the general Birth-and-Death processes at equilibrium are:

$$0 = \mu_1P_1 - \lambda_0P_0,$$
$$0 = \mu_{n+1}P_{n+1} + \lambda_{n-1}P_{n-1} - (\lambda_n + \mu_n)P_n; \quad n \geq 1$$

2.7 Birth, Death and Immigration Process

One interesting variant of simple Birth-and-Death Process is obtained if we add the condition that in an infinititesimal time interval Δt, there is a chance $v\Delta t + 0(\Delta t)$ that a single member will be added to the population by immigration from the outside world. The characteristic feature of immigration effect is that it acts at an expected rate which is independent of the population size.
Here,

$$\lambda_n = n\lambda + v,$$
$$\text{and} \tag{2.30}$$
$$\mu_n = n\mu.$$

For a steady state, the basic difference differential equations are:

$$0 = -vP_0 + \mu P_1 \tag{2.31}$$
$$0 = \mu(n+1)P_{n+1} + \Big(\lambda(n-1) + v\Big)P_{n-1} - (n\lambda + v + n\mu)P_n; \quad n \geq 1 \tag{2.32}$$

Hence,
Proposition 2.3

$$P_n = \begin{pmatrix} \frac{v}{\lambda} + n - 1 \\ n \end{pmatrix} \Big(\frac{\lambda}{\mu}\Big)^n \Big(1 - \frac{\lambda}{\mu}\Big)^{\frac{v}{\lambda}}. \quad \text{for} \quad n = 0, 1, 2, ... \tag{2.33}$$

Proof
Equation (2.31) can be expressed as;

$$\mu P_1 = vP_0$$
$$\Rightarrow P_1 = \frac{v}{\mu}P_0$$
$$\therefore P_1 = \frac{v}{\mu}P_0. \tag{2.34}$$

Solving equation (2.32) iteratively, we have;
When n=1;
Equation (2.32) becomes;

$$0 = -(\lambda + v + \mu)P_1 + vP_0 + 2\mu P_2$$
$$\Rightarrow 2\mu P_2 = (\lambda + \mu + v)P_1 - vP_0 \tag{2.35}$$

Substituting (2.34) in equation (2.35), we get;

$$
\begin{aligned}
2\mu P_2 &= (\lambda+\mu+v)\frac{v}{\mu}P_0 - vP_0\\
&= \Big(\frac{v\lambda+v^2}{\mu}\Big)P_0\\
&= \frac{v}{\mu}(\lambda+v)P_0\\
\Rightarrow P_2 &= \frac{v}{2\mu^2}(\lambda+v)P_0\\
\therefore P_2 &= \frac{v}{2\mu^2}(\lambda+v)P_0. \qquad (2.36)
\end{aligned}
$$

When n=3;
Equation (2.32) becomes;

$$
0 = -(2\lambda+v+2\mu)P_2 + (\lambda+v)P_1 + 3\mu P_3 \qquad (2.37)
$$

Substituting (2.34) and (2.36) in equation (2.37), we get;

$$
\begin{aligned}
0 &= -(2\lambda+v+2\mu)\frac{v}{2\mu^2}(\lambda+v)P_0 + (\lambda+v)\frac{v}{\mu}P_0 + 3\mu P_3\\
0 &= -\frac{v\lambda}{\mu^2}(\lambda+v)P_0 - \frac{v^2}{2\mu^2}(\lambda+v)P_0 - \frac{v}{\mu}(\lambda+v)P_0 + \frac{v}{\mu}(\lambda+v)P_0 + 3\mu P_3\\
0 &= -\frac{v\lambda}{\mu^2}(\lambda+v)P_0 - \frac{v^2}{2\mu^2}(\lambda+v)P_0 + 3\mu P_3\\
\Longleftrightarrow 3\mu P_3 &= \frac{v(\lambda+v)(2\lambda+v)}{\mu.2\mu^2.\mu^2}P_0\\
\Rightarrow P_3 &= \frac{v(\lambda+v)(2\lambda+v)}{\mu.2\mu^2.3\mu^3}P_0\\
\therefore P_3 &= \frac{v(\lambda+v)(2\lambda+v)}{\mu.2\mu^2.3\mu^3}P_0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P_4 &= \frac{v(\lambda+v)(2\lambda+v)(3\lambda+v)}{\mu.2\mu^2.3\mu^3.4\mu^4}P_0,\\
P_5 &= \frac{v(\lambda+v)(2\lambda+v)(3\lambda+v)(4\lambda+v)}{\mu.2\mu^2.3\mu^3.4\mu^4.5\mu^5}P_0,\\
&.\\
&.\\
&.\\
P_n &= \prod_{j=0}^{n-1}\frac{j\lambda+v}{n!\mu^n}P_0. \qquad (2.38)\\
&= \frac{P_0}{n!\mu^n}\prod_{j=0}^{n-1}(j\lambda+v)\\
&= \frac{P_0}{n!\mu^n}\prod_{j=0}^{n-1}\lambda(j+\frac{v}{\lambda})\\
&= \frac{P_0\lambda^n}{n!\mu^n}\prod_{j=0}^{n-1}(j+\frac{v}{\lambda})\\
&= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{\frac{v}{\lambda}(\frac{v}{\lambda}+1)(\frac{v}{\lambda}+2)...(\frac{v}{\lambda}+n-1)\Big\}\\
&= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{(\frac{v}{\lambda}+n-1)(\frac{v}{\lambda}+n-2)...(\frac{v}{\lambda}+1)\frac{v}{\lambda}\Big\}\\
&= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{(\frac{v}{\lambda}+n-1)(\frac{v}{\lambda}+n-2)...(\frac{v}{\lambda}+1)\frac{v}{\lambda}\frac{\Gamma(\frac{v}{\lambda})}{\Gamma(\frac{v}{\lambda})}\Big\}\\
&= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{(\frac{v}{\lambda}+n-1)(\frac{v}{\lambda}+n-2)...(\frac{v}{\lambda}+1)\frac{\Gamma(\frac{v}{\lambda}+1)}{\Gamma(\frac{v}{\lambda})}\Big\}\\
&= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\frac{1}{\Gamma(\frac{v}{\lambda})}\Big\{(\frac{v}{\lambda}+n-1)(\frac{v}{\lambda}+n-2)...(\frac{v}{\lambda}+2)\Gamma(\frac{v}{\lambda}+2)\Big\}\\
&.\\
&.\\
&.
\end{aligned}
$$

$$
\begin{aligned}
\therefore P_n &= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\frac{1}{\Gamma(\frac{v}{\lambda})}\Big\{(\frac{v}{\lambda}+n-1)\Gamma(\frac{v}{\lambda}+n-1)\Big\}\\
P_n &= P_0\Big(\frac{\lambda}{\mu}\Big)^n\frac{1}{n!\Gamma(\frac{v}{\lambda})}\Gamma(\frac{v}{\lambda}+n)
\end{aligned}
$$

Therefore,

$$P_n = P_0\Big(\frac{\lambda}{\mu}\Big)^n \frac{\Gamma(\frac{v}{\lambda}+n)}{n!\Gamma(\frac{v}{\lambda})}$$
$$= P_0\Big(\frac{\lambda}{\mu}\Big)^n \left(\begin{array}{c} \frac{v}{\lambda}+n-1 \\ n \end{array} \right)$$
$$\therefore P_n = \left(\begin{array}{c} \frac{v}{\lambda}+n-1 \\ n \end{array} \right) \Big(\frac{\lambda}{\mu}\Big)^n P_0; \quad n = 1,2,3,... \tag{2.39}$$

But;

$$\sum_{n=0}^{\infty} P_n = 1$$
$$\Rightarrow P_0 + P_1 + P_2 + ... = 1$$
$$P_0 + \frac{v}{\mu}P_0 + \frac{v}{2\mu^2}(\lambda+v)P_0 + ... = 1$$
$$P0\{1 + \frac{v}{\mu} + \frac{v}{2\mu^2}(\lambda+v) + ...\} = 1$$
$$\therefore P_0 = (1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}. \tag{2.40}$$
$$\therefore P_n = \left(\begin{array}{c} \frac{v}{\lambda}+n-1 \\ n \end{array} \right) \Big(\frac{\lambda}{\mu}\Big)^n (1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}; \quad n = 0,1,2,3,... \tag{2.41}$$

This is a Negative Binomial Distribution with parameters $\frac{v}{\lambda}$ and $1-\frac{\lambda}{\mu}$.

Remark (2.1)

(1) In the birth, death and immigration processes, the population either remain constant, increase or decrease.
It may eventually reach zero; however, since there is always a positive immigration rate v, the population will never become extinct. But the population will become extinct as v goes to zero.
i.e., $\lim_{v\to\infty} P_n$.
Using Proposition 2.1

We have;

$$P_n = \frac{\lambda_0\lambda_1\lambda_2\lambda_3...\lambda_{n-1}}{\mu_1\mu_2\mu_3...\mu_n}P_0$$

But, $\lambda_0 = v,\quad \lambda_1 = \lambda + v,\quad \lambda_2 = (2\lambda + v),\quad \lambda_3 = (3\lambda + v),\quad ...,\quad \lambda_{n-1} = (n-1)\lambda + v.$

and

$$\mu_1 = \mu,\quad \mu_2 = 2\mu,\quad \mu_3 = 3\mu,\quad ...,\quad \mu_n = n\mu.$$

$$\therefore P_n = \frac{v(\lambda+v)(2\lambda+v)...[(n-1)\lambda+v]}{\mu 2\mu 3\mu ... n\mu}P_0$$

$$= \frac{v(\lambda+v)(2\lambda+v)...[(n-1)\lambda+v]}{n!\mu^n}P_0$$

$$= \prod_{j=0}^{n-1}\frac{j\lambda+v}{n!\mu^n}P_0$$

$$= \frac{P_0}{n!\mu^n}\prod_{j=0}^{n-1}(j\lambda+v)$$

$$= \frac{P_0}{n!\mu^n}\prod_{j=0}^{n-1}\lambda(j+\frac{v}{\lambda})$$

$$= \frac{P_0\lambda^n}{n!\mu^n}\prod_{j=0}^{n-1}(j+\frac{v}{\lambda})$$

$$= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{\frac{v}{\lambda}(\frac{v}{\lambda}+1)(\frac{v}{\lambda}+2)...(\frac{v}{\lambda}+n-1)\Big\}$$

$$= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{(\frac{v}{\lambda}+n-1)(\frac{v}{\lambda}+n-2)...(\frac{v}{\lambda}+1)\frac{v}{\lambda}\Big\}$$

$$= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{(\frac{v}{\lambda}+n-1)(\frac{v}{\lambda}+n-2)...(\frac{v}{\lambda}+1)\frac{v}{\lambda}\frac{\Gamma(\frac{v}{\lambda})}{\Gamma(\frac{v}{\lambda})}\Big\}$$

$$= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{(\frac{v}{\lambda}+n-1)(\frac{v}{\lambda}+n-2)...(\frac{v}{\lambda}+1)\frac{\Gamma(\frac{v}{\lambda}+1)}{\Gamma(\frac{v}{\lambda})}\Big\}$$

$$\therefore P_n = \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\frac{1}{\Gamma(\frac{v}{\lambda})}\Big\{(\frac{v}{\lambda}+n-1)(\frac{v}{\lambda}+n-2)...(\frac{v}{\lambda}+2)\Gamma(\frac{v}{\lambda}+2)\Big\}$$

.

.

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$$P_n = \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\frac{1}{\Gamma(\frac{v}{\lambda})}\Big\{(\frac{v}{\lambda}+n-1)(\frac{v}{\lambda}+n-2)\Gamma(\frac{v}{\lambda}+n-2)\Big\}$$

Therefore,

$$\begin{aligned}
P_n &= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n \frac{1}{\Gamma(\frac{v}{\lambda})}\Big\{(\frac{v}{\lambda}+n-1)\Gamma(\frac{v}{\lambda}+n-1)\Big\} \\
&= P_0\Big(\frac{\lambda}{\mu}\Big)^n \frac{1}{n!\Gamma(\frac{v}{\lambda})}\Gamma(\frac{v}{\lambda}+n) \\
&= P_0\Big(\frac{\lambda}{\mu}\Big)^n \frac{\Gamma(\frac{v}{\lambda}+n)}{n!\Gamma(\frac{v}{\lambda})} \\
&= P_0\Big(\frac{\lambda}{\mu}\Big)^n \left(\begin{array}{c} \frac{v}{\lambda}+n-1 \\ n \end{array}\right) \\
\therefore P_n &= \left(\begin{array}{c} \frac{v}{\lambda}+n-1 \\ n \end{array}\right)\Big(\frac{\lambda}{\mu}\Big)^n P_0; \quad n = 1, 2, 3, \ldots
\end{aligned}$$

But,

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n &= 1 \\
\text{i.e.,} \quad P_0 + \sum_{n=0}^{\infty} P_n &= 1 \\
P_0[1 + \sum_{n=1}^{\infty} \left(\begin{array}{c} \frac{v}{\lambda}+n-1 \\ n \end{array}\right)\Big(\frac{\lambda}{\mu}\Big)^n] &= 1 \\
P_0 \sum_{n=0}^{\infty} \left(\begin{array}{c} \frac{v}{\lambda}+n-1 \\ n \end{array}\right)\Big(\frac{\lambda}{\mu}\Big)^n &= 1 \\
P_0 \sum_{n=0}^{\infty} (-1)^n \left(\begin{array}{c} -\frac{v}{\lambda} \\ n \end{array}\right)\Big(\frac{\lambda}{\mu}\Big)^n &= 1
\end{aligned}$$

Therefore,

$$P_0 \sum_{n=0}^{\infty} \begin{pmatrix} -\frac{v}{\lambda} \\ n \end{pmatrix} \left(-\frac{\lambda}{\mu}\right)^n = 1$$

$$\therefore P_0(1-\frac{\lambda}{\mu})^{-\frac{v}{\lambda}} = 1$$

$$\therefore P_0 = (1-\frac{\lambda}{\mu})^{-\frac{v}{\lambda}}.$$

$$\therefore P_n = \begin{pmatrix} \frac{v}{\lambda}+n-1 \\ n \end{pmatrix} \left(\frac{\lambda}{\mu}\right)^n (1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}; \quad n = 0,1,2,3,\ldots$$

Which is a Negative Binomial Distribution with parameters $\frac{v}{\lambda}$ and $1-\frac{\lambda}{\mu}$.
where, $\lambda < \mu, \quad \lambda > 0, \quad \mu > 0$.

Using Proposition 2.2

$$\frac{P_n}{P_{n-1}} = \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0$$

But,

$$\lambda_{n-1} = (n-1)\lambda + v,$$

and

$$\mu_n = n\mu.$$

Therefore,

$$\frac{P_n}{P_{n-1}} = \frac{(n-1)\lambda + v}{n\mu}$$

$$= \frac{(n-1+\frac{v}{\lambda})}{\frac{n\mu}{\lambda}}$$

$$\therefore \frac{P_n}{P_{n-1}} = (\frac{v}{\lambda}+n-1)\frac{\frac{\lambda}{\mu}}{n}$$

$$\therefore \frac{P_n}{P_{n-1}} = (\frac{v}{\lambda}+n-1)\frac{\frac{\lambda}{\mu}}{n}.$$

$$\Rightarrow nP_n = \frac{\lambda}{\mu}(\frac{v}{\lambda}+n-1)P_{n-1}; \quad n = 1,2,3,\ldots \tag{2.42}$$

Using the probability generating function technique, we multiply (2.42) by s^n and sum the results over n to obtain;

$$\begin{aligned}
\sum_{n=1}^{\infty} nP_n s^n &= \frac{\lambda}{\mu}\sum_{n=1}^{\infty}(\frac{v}{\lambda}+n-1)P_{n-1}s^n \\
\sum_{n=1}^{\infty} nP_n s^n &= \frac{\lambda}{\mu}.\frac{v}{\lambda}\sum_{n=1}^{\infty}P_{n-1}s^n + \frac{\lambda}{\mu}\sum_{n=1}^{\infty}(n-1)P_{n-1}s^n \\
s\sum_{n=1}^{\infty} nP_n s^{n-1} &= \frac{\lambda}{\mu}.\frac{v}{\lambda}s\sum_{n=1}^{\infty}P_{n-1}s^{n-1} + \frac{\lambda}{\mu}s^2\sum_{n=1}^{\infty}(n-1)P_{n-1}s^{n-1} \\
s\frac{\psi(s)}{\mathrm{d}s} &= \frac{\lambda}{\mu}.\frac{v}{\lambda}s\psi(s) + \frac{\lambda}{\mu}s^2\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \\
\Longleftrightarrow (1-\frac{\lambda}{\mu}s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \frac{\lambda}{\mu}.\frac{v}{\lambda}s\psi(s) \\
\Rightarrow \frac{\mathrm{d}\psi(s)}{\psi(s)} &= \frac{\frac{\lambda}{\mu}.\frac{v}{\lambda}}{(1-\frac{\lambda}{\mu}s)}\mathrm{d}s
\end{aligned}$$

Integrating both sides, we have;

$$\begin{aligned}
\int\frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int\frac{\frac{\lambda}{\mu}.\frac{v}{\lambda}}{(1-\frac{\lambda}{\mu}s)}\mathrm{d}s \\
\ln\psi(s) &= -\frac{v}{\lambda}\ln(1-\frac{\lambda}{\mu}s)+\ln c \\
\therefore \psi(s) &= c_1(1-\frac{\lambda}{\mu}s)^{-\frac{v}{\lambda}}.
\end{aligned}$$

Putting s=1;

$$\begin{aligned}
\psi(1) = 1 &= c_1(1-\frac{\lambda}{\mu})^{-\frac{v}{\lambda}} \\
\Rightarrow c_1 &= (1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}. \\
\therefore \psi(s) &= (1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}(1-\frac{\lambda}{\mu}s)^{-\frac{v}{\lambda}}
\end{aligned}$$

$$\therefore \psi(s) = \Big(\frac{(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu}s)}\Big)^{\frac{v}{\lambda}}. \qquad (2.43)$$

Which is the pgf. for a Negative Binomial Distribution with parameters $\frac{v}{\lambda}$ and $1-\frac{\lambda}{\mu}$.

$$\psi^{'}(s) = \frac{\frac{\lambda}{\mu}\frac{v}{\lambda}(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}}{(1-\frac{\lambda}{\mu}s)^{\frac{v}{\lambda}+1}}$$

$$\psi^{''}(s) = \frac{(\frac{\lambda}{\mu})^2 \cdot \frac{v}{\lambda}(\frac{v}{\lambda}+1)(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}}{(1-\frac{\lambda}{\mu}s)^{\frac{v}{\lambda}+2}}$$

$$= \frac{(\frac{\lambda}{\mu})^2(\frac{v}{\lambda})^2(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}} + (\frac{\lambda}{\mu})^2\frac{v}{\lambda}(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}}{(1-\frac{\lambda}{\mu}s)^{\frac{v}{\lambda}+2}}$$

$$E(X) = \psi(1) = \frac{\frac{\lambda}{\mu}\frac{v}{\lambda}(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}}{(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}+1}}$$

$$\therefore E(X) = \frac{v}{\mu-\lambda}.$$

The variance is given by;

$$Var(X) = \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2$$

$$= \frac{(\frac{\lambda}{\mu})^2(\frac{v}{\lambda})^2(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}} + (\frac{\lambda}{\mu})^2\frac{v}{\lambda}(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}}{(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}+2}} + \frac{\frac{\lambda}{\mu}\frac{v}{\lambda}(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}}{(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}+1}} - [\frac{\frac{\lambda}{\mu}\frac{v}{\lambda}(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}}{(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}+1}}]^2$$

$$= \frac{(\frac{\lambda}{\mu})^2(\frac{v}{\lambda})^2(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}} + (\frac{\lambda}{\mu})^2\frac{v}{\lambda}(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}}{(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}+2}} + \frac{\frac{\lambda}{\mu}\frac{v}{\lambda}(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}}}{(1-\frac{\lambda}{\mu})^{\frac{v}{\lambda}+1}} - \frac{(\frac{\lambda}{\mu})^2(\frac{v}{\lambda})^2}{(1-\frac{\lambda}{\mu})^2}$$

$$= \frac{(\frac{\lambda}{\mu})^2(\frac{v}{\lambda})^2 + (\frac{\lambda}{\mu})^2\frac{v}{\lambda}}{(1-\frac{\lambda}{\mu})^2} + \frac{\frac{\lambda}{\mu}\frac{v}{\lambda}}{(1-\frac{\lambda}{\mu})} - \frac{(\frac{\lambda}{\mu})^2(\frac{v}{\lambda})^2}{(1-\frac{\lambda}{\mu})^2}$$

$$= \frac{v\lambda + v\mu - v\lambda}{(\mu-\lambda)^2}$$

$$\therefore Var(X) = \frac{v\mu}{(\mu-\lambda)^2}.$$

From Gauss hypergeometric series, we have;

$$
\begin{aligned}
{}_2F_1(\frac{v}{\lambda},1;1;\frac{\lambda}{\mu}) &= 1+\frac{\frac{v}{\lambda}.1}{1}\frac{\frac{\lambda}{\mu}}{1!}+\frac{\frac{v}{\lambda}(\frac{v}{\lambda}+1)1(1+1)}{1(1+1)}\frac{(\frac{\lambda}{\mu})^2}{2!}+...+\frac{(\frac{v}{\lambda}+n-1)(1+n-1)}{(1+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\frac{v}{\lambda}(\frac{v}{\lambda}+1)...(\frac{v}{\lambda}+n-1)1(1+1)(1+2)...(1+n-1)}{1(1+1)(1+2)...(1+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(\frac{v}{\lambda}+n-1)...(\frac{v}{\lambda}+1)\frac{v}{\lambda}(1+n-1)...(1+2)(1+1)1}{(1+n-1)...(1+2)(1+1)1}\frac{(\frac{\lambda}{\mu})^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\frac{v}{\lambda}+n)\Gamma(1+n)}{\Gamma(1+n)}.\frac{\Gamma(1)}{\Gamma(\frac{v}{\lambda})\Gamma(1)}\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}
$$

Normalizing, we get;

$$
1 = \sum_{n=0}^{\infty}\frac{\Gamma(\frac{v}{\lambda}+n)\Gamma(1+n)}{\Gamma(1+n)}.\frac{\Gamma(1)}{\Gamma(\frac{v}{\lambda})\Gamma(1)}.\frac{1}{{}_2F_1(\frac{v}{\lambda},1;1;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}
$$

Hence,

$$
\begin{aligned}
P_n &= \text{Prob.}(N=n)\\
&= \frac{\Gamma(\frac{v}{\lambda}+n)\Gamma(1+n)}{\Gamma(1+n)}.\frac{\Gamma(1)}{\Gamma(\frac{v}{\lambda})\Gamma(1)}.\frac{1}{{}_2F_1(\frac{v}{\lambda},1;1;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}
$$

The pgf. in hypergeometric terms is given by;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty}P_n z^n\\
\phi(z) &= \sum_{n=0}^{\infty}\frac{\Gamma(\frac{v}{\lambda}+n)\Gamma(1+n)}{\Gamma(1+n)}.\frac{\Gamma(1)}{\Gamma(\frac{v}{\lambda})\Gamma(1)}\frac{(\frac{\lambda}{\mu}z)^n}{n!}.\frac{1}{{}_2F_1(\frac{v}{\lambda},1;1;\frac{\lambda}{\mu})}\\
\therefore \phi(z) &= \frac{{}_2F_1(\frac{v}{\lambda},1;1;\frac{\lambda}{\mu}z)}{{}_2F_1(\frac{v}{\lambda},1;1;\frac{\lambda}{\mu})}.\\
\phi^{'}(z) &= \frac{v}{\lambda}\frac{\lambda}{\mu}\frac{{}_2F_1(\frac{v}{\lambda}+1,1+1;1+1;\frac{\lambda}{\mu}z)}{{}_2F_1(\frac{v}{\lambda},1;1;\frac{\lambda}{\mu})}\\
\phi^{''}(z) &= \frac{v}{\lambda}(\frac{v}{\lambda}+1)(\frac{\lambda}{\mu})^2\frac{{}_2F_1(\frac{v}{\lambda}+2,3;3;\frac{\lambda}{\mu}z)}{{}_2F_1(\frac{v}{\lambda},1;1;\frac{\lambda}{\mu})}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{Let,} \quad \Lambda_\kappa &= \frac{{}_2F_1(\frac{v}{\lambda}+\kappa, 1+\kappa; 1+\kappa; \frac{\lambda}{\mu})}{{}_2F_1(\frac{v}{\lambda}, 1; 1; \frac{\lambda}{\mu})}; \quad \kappa = 1, 2. \\
E(X) &= \phi^{'}(1) \\
\therefore E(X) &= \frac{v}{\lambda}\frac{\lambda}{\mu}\Lambda_1. \\
Var(X) &= \phi^{''}(1) + \phi^{'}(1) - [\phi^{'}(1)]^2. \\
&= \frac{v}{\lambda}(\frac{v}{\lambda}+1)(\frac{\lambda}{\mu})^2\Lambda_2 + \frac{v}{\lambda}\frac{\lambda}{\mu}\Lambda_1 - [\frac{v}{\lambda}\frac{\lambda}{\mu}\Lambda_1]^2 \\
\therefore Var(X) &= \frac{v}{\lambda}\frac{\lambda}{\mu}\Lambda_1 + \frac{v}{\lambda}(\frac{\lambda}{\mu})^2\{(\frac{v}{\lambda}+1)\Lambda_2 - \frac{v}{\lambda}\Lambda_1^2\}.
\end{aligned}
$$

3 BIRTH AND DEATH PROCESSES AT EQUILIBRIUM WITH APPLICATION TO WAITING LINE PROBLEMS

3.1 Introduction

In this chapter, birth and death processes and some special cases of waiting line and servicing problems are explored. Their basic difference differential equations at equilibrium are stated given different birth λ_n and death μ_n rate values or functions.
The steady states equations are solved iteratively to generate distributions arising from the birth and death process. Some properties and special cases of these distributions have also been discussed.

3.2 The simple trunking problem

Suppose that infinitely many trunks or channels are available, and that the probability of a conversation ending between t and $t+\Delta t$ is $\mu\Delta t + 0(\Delta t)$, (exponential holding time). The incoming calls constitute a traffic of the Poisson type with parameter λ.
Assume that the duration of conversations are mutually independent. If n lines are busy, the probability that one of them will be free within time Δt is then $n\mu\Delta t+)(\Delta t)$. The probability of a new call arriving is $\lambda\Delta t + 0(\Delta t)$. The probability of a combination of several calls, or of a call arriving and a conversation ending is $0(\Delta t)$.
Thus, this implies that;

$$\lambda_n = \lambda,$$

and

$$\mu_n = n\mu.$$

At a stedy state, $t \to \infty$ implying that $P_n(t)$ is independent of t.
Thus,

$$P_n(t) = P_n,$$
and
$$\lim_{t\to\infty} P_n'(t) = 0.$$

Therefore, the basic difference differential equations for the steady state for the simple trunking problem are:

$$0 = \mu P_1 - \lambda P_0 \quad (3.1)$$
$$0 = (n+1)\mu P_{n+1} + (n-1)\lambda P_{n-1} - (\lambda + n\mu)P_n, \quad n \geq 1 \quad (3.2)$$

Hence,
Proposition 3.1

$$P_n = \frac{\lambda^n}{n!\mu^n} e^{-\frac{\lambda}{\mu}} \quad (3.3)$$

Proof
Equation (3.1) can be expressed as;

$$\lambda P_0 = \mu P_1$$
$$\Rightarrow \frac{\lambda}{\mu} P_0 = P_1$$
$$\therefore P_1 = \frac{\lambda}{\mu} P_0. \quad (3.4)$$

Solving steady state equation iteratively, we have;
When n=1;
Equation (3.2) becomes;

$$0 = 2\mu P_2 + \lambda P_0 - (\lambda + \mu)P_1 \quad (3.5)$$

Substituting (3.4) in equation (3.5), we get;

$$
\begin{aligned}
0 &= 2\mu P_2 + \lambda P_0 - (\lambda+\mu)\frac{\lambda}{\mu}P_0 \\
0 &= 2\mu P_2 + \lambda P_0 - \lambda P_0 - \frac{\lambda^2}{\mu}P_0 \\
0 &= 2\mu P_2 - \frac{\lambda^2}{\mu}P_0 \\
\Longleftrightarrow \frac{\lambda^2}{\mu}P_0 &= 2\mu P_2 \\
\Rightarrow \frac{\lambda^2}{2!\mu^2}P_0 &= P_2 \\
\therefore P_2 &= \frac{\lambda^2}{2!\mu^2}P_0. \qquad \textbf{(3.6)}
\end{aligned}
$$

When n=2;
Equation (3.2) becomes;

$$0 = 3\mu P_3 + \lambda P_1 - (\lambda + 2\mu)P_2 \qquad \textbf{(3.7)}$$

Substituting (3.4) and (3.6) in equation (3.7), we get;

$$
\begin{aligned}
0 &= 3\mu P_3 + \lambda\frac{\lambda}{\mu}P_0 - (\lambda+2\mu)\frac{\lambda^2}{2!\mu^2}P_0 \\
0 &= 3\mu P_3 + \frac{\lambda^2}{\mu}P_0 - \frac{\lambda^2}{\mu}P_0 - \frac{\lambda^3}{2\mu^2}P_0 \\
0 &= 3\mu P_3 - \frac{\lambda^3}{2\mu^2}P_0 \\
\Longleftrightarrow \frac{\lambda^3}{2\mu^2}P_0 &= 3\mu P_3 \\
\Rightarrow \frac{\lambda^3}{3!\mu^3}P_0 &= P_3 \\
\therefore P_3 &= \frac{\lambda^3}{3!\mu^3}P_0. \qquad \textbf{(3.8)}
\end{aligned}
$$

When n=3;
Equation (3.2) becomes;

$$0 = 4\mu P_4 + \lambda P_2 - (\lambda + 3\mu)P_3 \tag{3.9}$$

Substituting (3.6) and (3.8) in equation (3.9), we get;

$$\begin{aligned}
0 &= 4\mu P_4 + \lambda \frac{\lambda^2}{2!\mu^2}P_0 - (\lambda + 3\mu)\frac{\lambda^3}{3!\mu^3}P_0 \\
0 &= 4\mu P_4 + \frac{\lambda^3}{2\mu^2}P_0 - \frac{\lambda^3}{2\mu^2}P_0 - \frac{\lambda^4}{3!\mu^3}P_0 \\
0 &= 4\mu P_4 - \frac{\lambda^4}{3!\mu^3}P_0 \\
\Longleftrightarrow \frac{\lambda^4}{3!\mu^3}P_0 &= 4\mu P_4 \\
\Rightarrow \frac{\lambda^4}{4!\mu^4}P_0 &= P_4 \\
\therefore P_4 &= \frac{\lambda^4}{4!\mu^4}P_0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
P_5 &= \frac{\lambda^5}{5!\mu^5}P_0, \\
P_6 &= \frac{\lambda^6}{6!\mu^6}P_0, \\
&\vdots \\
P_n &= \frac{\lambda^n}{n!\mu^n}P_0. \qquad (3.10) \\
&\vdots
\end{aligned}$$

But,

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow P_0\left\{1+\frac{\lambda}{\mu}+\frac{\lambda^2}{2!\mu^2}+\frac{\lambda^3}{3!\mu^3}+\frac{\lambda^4}{4!\mu^4}+...\right\}=1$$

$$P_0\left\{e^{\frac{\lambda}{\mu}}\right\}=1$$

$$\therefore P_0=e^{-\frac{\lambda}{\mu}}. \tag{3.11}$$

Hence,

$$P_1=\frac{\lambda}{1!\mu}e^{-\frac{\lambda}{\mu}},$$

$$P_2=\frac{\lambda^2}{2!\mu^2}e^{-\frac{\lambda}{\mu}},$$

$$P_3=\frac{\lambda^3}{3!\mu^3}e^{-\frac{\lambda}{\mu}},$$

$$P_4=\frac{\lambda^4}{4!\mu^4}e^{-\frac{\lambda}{\mu}},$$

.

.

.

We find by Mathematical induction that;

$$P_n=\frac{\lambda^n}{n!\mu^n}e^{-\frac{\lambda}{\mu}}. \tag{3.12}$$

Thus, the limiting distribution is a Poisson with parameter $\frac{\lambda}{\mu}$. It is independent of the initial state.

Using Proposition (2.1)

We have;

$$P_n=\frac{\lambda_0\lambda_1\lambda_2\lambda_3...\lambda_{n-1}}{\mu_1\mu_2\mu_3...\mu_n}P_0$$

But, $\lambda_0=\lambda_1=\lambda_2=\lambda_3=...=\lambda_{n-1}=\lambda,$

and

$$\mu_1=\mu,\quad \mu_2=2\mu,\quad \mu_3=3\mu,\quad ...,\quad \mu_n=n\mu.$$

$$\therefore P_n=\frac{\lambda\lambda\lambda\lambda...\lambda}{\mu 2\mu 3\mu...n\mu}P_0$$

Therefore,

$$P_n = \frac{\lambda^n}{n!\mu^n} P_0$$

But, from (3.11) above;

$$P_0 = e^{-\frac{\lambda}{\mu}}$$

$$\therefore P_n = \frac{\lambda^n}{n!\mu^n} e^{-\frac{\lambda}{\mu}}.$$

Which is a Poisson distribution with parameter $\frac{\lambda}{\mu}$.

Using Proposition (2.2)

$$\frac{P_n}{P_{n-1}} = \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0$$

But, $\lambda_{n-1} = \lambda$,

and

$$\mu_n = n\mu.$$

$$\therefore \frac{P_n}{P_{n-1}} = \frac{\lambda}{n\mu}.$$

Proposition 3.2

$$\frac{P_n}{P_{n-1}} = \frac{\lambda}{n\mu}$$

$$\Rightarrow n\mu P_n = \lambda P_{n-1} \tag{3.13}$$

Multiplying (3.13) by s^n and sum the results over n we obtain;

$$\sum_{n=0}^{\infty} n\mu P_n s^n = \sum_{n=0}^{\infty} \lambda P_{n-1} s^n$$

$$\mu s \sum_{n=0}^{\infty} n P_n s^{n-1} = \lambda s \sum_{n=0}^{\infty} P_{n-1} s^{n-1}$$

$$s\mu \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = s\lambda \psi(s)$$

$$\frac{\mathrm{d}\psi(s)}{\psi(s)} = \frac{\lambda}{\mu} \mathrm{d}s$$

Integrating both sides, we have;

$$\int \frac{\mathrm{d}\psi(s)}{\psi(s)} = \int \frac{\lambda}{\mu}\mathrm{d}s$$
$$\ln\psi(s) = \frac{\lambda}{\mu}s + c$$
$$\psi(s) = c_1 e^{\frac{\lambda}{\mu}s}$$

Putting s=1;

$$\psi(1) = 1 = c_1 e^{\frac{\lambda}{\mu}}$$
$$\Rightarrow c_1 = e^{-\frac{\lambda}{\mu}}.$$
$$\therefore \psi(s) = e^{-\frac{\lambda}{\mu}} e^{\frac{\lambda}{\mu}s}$$
$$\therefore \psi(s) = e^{-\frac{\lambda}{\mu}(1-s)}.$$

Which is the pgf. for a Poisson distribution with parameter $\frac{\lambda}{\mu}$.

$$\psi^{'}(s) = \frac{\lambda}{\mu} e^{-\frac{\lambda}{\mu}(1-s)}$$
$$\psi^{''}(s) = (\frac{\lambda}{\mu})^2 e^{-\frac{\lambda}{\mu}(1-s)}$$
$$E(X) = \psi^{'}(1)$$
$$\therefore E(X) = \frac{\lambda}{\mu}.$$
$$Var(x) = \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2$$
$$= (\frac{\lambda}{\mu})^2 + \frac{\lambda}{\mu} - [\frac{\lambda}{\mu}]^2$$
$$\therefore Var(X) = \frac{\lambda}{\mu}.$$

From Kummer's confluent hypergeometric series, we have;

$$
\begin{aligned}
{}_1F_1(\flat;\flat;\frac{\lambda}{\mu}) &= 1+\frac{\flat}{\flat}\frac{\frac{\lambda}{\mu}}{1!}+\frac{\flat(\flat+1)}{\flat(\flat+1)}\frac{(\frac{\lambda}{\mu})^2}{2!}+...+\frac{(\flat+n-1)}{(\flat+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\flat(\flat+1)(\flat+2)...(\flat+n-1)}{\flat(\flat+1)(\flat+2)...(\flat+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(\flat+n-1)(\flat+n-2)...(\flat+2)(\flat+1)\flat}{(\flat+n-1)(\flat+n-2)...(\flat+2)(\flat+1)\flat}\frac{\Gamma(\flat)}{\Gamma(\flat)}.\frac{\Gamma(\flat)}{\Gamma(\flat)}.\frac{(\frac{\lambda}{\mu})^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\flat+n)}{\Gamma(\flat+n)}\frac{\Gamma(\flat)}{\Gamma(\flat)}.\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}
$$

Normalizing, we have;

$$
1=\sum_{n=0}^{\infty}\frac{\Gamma(\flat+n)}{\Gamma(\flat+n)}\frac{\Gamma(\flat)}{\Gamma(\flat)}.\frac{1}{{}_1F_1(\flat;\flat;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}
$$

Therefore,

$$
\begin{aligned}
P_n &= \text{Prob.}(N=n)\\
&= \frac{\Gamma(\flat+n)}{\Gamma(\flat+n)}\frac{\Gamma(\flat)}{\Gamma(\flat)}.\frac{1}{{}_1F_1(\flat;\flat;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}
$$

In hypergeometric terms, its pgf. is given as;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty}P_nz^n\\
&= \frac{\Gamma(\flat+n)}{\Gamma(\flat+n)}\frac{\Gamma(\flat)}{\Gamma(\flat)}\frac{(\frac{\lambda}{\mu}z)^n}{n!}.\frac{1}{{}_1F_1(\flat;\flat;\frac{\lambda}{\mu})}\\
\therefore \phi(z) &= \frac{{}_1F_1(\flat;\flat;\frac{\lambda}{\mu}z)}{{}_1F_1(\flat;\flat;\frac{\lambda}{\mu})}.\\
\phi^{'}(z) &= \frac{\lambda}{\mu}\frac{{}_1F_1(\flat+1;\flat+1;\frac{\lambda}{\mu}z)}{{}_1F_1(\flat;\flat;\frac{\lambda}{\mu})}\\
\phi^{''}(z) &= (\frac{\lambda}{\mu})^2\frac{{}_1F_1(\flat+2;\flat+2;\frac{\lambda}{\mu}z)}{{}_1F_1(\flat;\flat;\frac{\lambda}{\mu})}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_1F_1(\flat+\kappa;\flat+\kappa;\frac{\lambda}{\mu})}{{}_1F_1(\flat;\flat;\frac{\lambda}{\mu})};\quad \kappa=1,2.\\
\therefore E(x) &= \frac{\lambda}{\mu}\Lambda_1.\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= (\frac{\lambda}{\mu})^2\Lambda_2+\frac{\lambda}{\mu}\Lambda_1-(\frac{\lambda}{\mu})^2\Lambda_1^2\\
\therefore Var(X) &= \frac{\lambda}{\mu}\Lambda_1+(\frac{\lambda}{\mu})^2(\Lambda_2-\Lambda_1^2).
\end{aligned}
$$

3.3 Waiting lines for a finite number of channels

Assume that the number of channels is finite and equals to k. If all channels are busy, each new call joins a waiting line and wait until a channel is free.
We say that the system is in a state E_n if there are n persons either being served or in the waiting line. Such a line exists only when, $n > k$ and there are $n-k$ persons in it.
As long as at least one channel is free then,

$$\lambda_n = \lambda,$$

and

$$\mu_n = n\mu;\quad \text{for}\quad n<k.$$

However, if $n > k$, only k conversations are going on and $\lambda_n = k\mu$; for $n \geq k$.
The basic difference differential equations for the steady state for waiting lines for a finite number of channels are:

$$0 = \mu P_1 - \lambda P_0 \tag{3.14}$$

$$0 = (n+1)\mu P_{n+1} + \lambda P_{n-1} - (\lambda+n\mu)P_n;\quad \text{for}\quad n<k \tag{3.15}$$

$$0 = k\mu P_{n+1} + \lambda P_{n-1} - (\lambda+k\mu)P_n;\quad n\geq k \tag{3.16}$$

Proposition 3.3

$$P_n = \frac{(\frac{\lambda}{\mu})^n}{k!k^{n-1}} e^{-\frac{\lambda}{\mu}}; \quad \text{for} \quad n \geq k; \quad \text{for} \quad \frac{\lambda}{\mu} < k. \tag{3.17}$$

Proof
Solving the steady state equations iteratively, we have;
Equation (3.14) can be expressed as;

$$\begin{aligned} \lambda P_0 &= \mu P_1 \\ \Rightarrow \frac{\lambda}{\mu} P_0 &= P_1 \\ \therefore P_1 &= \frac{\lambda}{\mu} P_0 \end{aligned} \tag{3.18}$$

When n=1;
Equation (3.15) becomes;

$$0 = 2\mu P_2 + \lambda P_0 - (\lambda + \mu) P_1 \tag{3.19}$$

Substituting (3.18) in equation (3.19), we get;

$$\begin{aligned} 0 &= 2\mu P_2 + \lambda P_0 - (\lambda + \mu)\frac{\lambda}{\mu} P_0 \\ 0 &= 2\mu P_2 + \lambda P_0 - \lambda P_0 - \frac{\lambda^2}{\mu} P_0 \\ 0 &= 2\mu P_2 - \frac{\lambda^2}{\mu} P_0 \\ \Longleftrightarrow \frac{\lambda^2}{\mu} P_0 &= 2\mu P_2 \\ \Rightarrow \frac{\lambda^2}{2!\mu^2} P_0 &= P_2 \\ \therefore P_2 &= \frac{\lambda^2}{2!\mu^2} P_0. \end{aligned} \tag{3.20}$$

When n=2;
Equation (3.15) becomes;

$$0 = 3\mu P_3 + \lambda P_1 - (\lambda + 2\mu)P_2 \tag{3.21}$$

Substituting (3.18) and (3.20) in equation (3.21), we get;

$$\begin{aligned}
0 &= 3\mu P_3 + \lambda \frac{\lambda}{\mu} P_0 - (\lambda + 2\mu)\frac{\lambda^2}{2!\mu^2} P_0 \\
0 &= 3\mu P_3 + \frac{\lambda^2}{\mu} P_0 - \frac{\lambda^2}{\mu} P_0 - \frac{\lambda^3}{2\mu^2} P_0 \\
0 &= 3\mu P_3 - \frac{\lambda^3}{2\mu^2} P_0 \\
\iff \frac{\lambda^3}{2!\mu^2} P_0 &= 3\mu P_3 \\
\Rightarrow \frac{\lambda^3}{3!\mu^3} P_0 &= P_3 \\
\therefore P_3 &= \frac{\lambda^3}{3!\mu^3} P_0.
\end{aligned}$$

Hence,

$$\begin{aligned}
P_4 &= \frac{\lambda^4}{4!\mu^4} P_0, \\
P_5 &= \frac{\lambda^5}{5!\mu^5} P_0, \\
&. \\
&. \\
&. \\
P_n &= \frac{\lambda^n}{n!\mu^n} P_0. \qquad (3.22) \\
&. \\
&. \\
&.
\end{aligned}$$

Note that;

$$\sum_{n=0}^{\infty} P_n = 1$$
$$\Rightarrow P_0 + \frac{(\frac{\lambda}{\mu})^1}{1!}P_0 + \frac{(\frac{\lambda}{\mu})^2}{2!}P_0 + \frac{(\frac{\lambda}{\mu})^3}{3!}P_0 + \frac{(\frac{\lambda}{\mu})^4}{4!}P_0 + \ldots + \ldots = 1$$
$$P_0\left\{1 + \frac{(\frac{\lambda}{\mu})^1}{1!} + \frac{(\frac{\lambda}{\mu})^2}{2!} + \frac{(\frac{\lambda}{\mu})^3}{3!} + \frac{(\frac{\lambda}{\mu})^4}{4!} + \ldots + \ldots\right\} = 1$$
$$P_0(e^{\frac{\lambda}{\mu}}) = 1$$
$$\therefore P_0 = e^{-\frac{\lambda}{\mu}}. \tag{3.23}$$

Thus;

$$P_1 = \frac{\lambda^1}{1!\mu^1}e^{-\frac{\lambda}{\mu}},$$
$$P_2 = \frac{\lambda^2}{2!\mu^2}e^{-\frac{\lambda}{\mu}},$$
$$P_3 = \frac{\lambda^3}{3!\mu^3}e^{-\frac{\lambda}{\mu}},$$
$$P_4 = \frac{\lambda^4}{4!\mu^4}e^{-\frac{\lambda}{\mu}},$$
$$.$$
$$.$$
$$.$$
$$\therefore P_n = \frac{\lambda^n}{n!\mu^n}e^{-\frac{\lambda}{\mu}}.$$

Which is a Poisson distribution with parameter $\frac{\lambda}{\mu}$.
Next,
When n=1;
Equation (3.16) becomes;

$$0 = k\mu P_2 + \lambda P_0 - (\lambda + k\mu)P_1 \quad \text{for} \quad n \geq k \tag{3.24}$$

Substituting (3.18) in equation (3.24), we get;

$$
\begin{aligned}
0 &= k\mu P_2 + \lambda P_0 - (\lambda + k\mu)\frac{\lambda}{\mu}P_0 \\
0 &= k\mu P_2 + \lambda P_0 - k\lambda P_0 - \frac{\lambda^2}{\mu}P_0 \\
0 &= k\mu P_2 - \lambda(1-k)P_0 - \frac{\lambda^2}{\mu}P_0 \\
\iff \lambda(1-k)P_0 + \frac{\lambda^2}{\mu}P_0 &= k\mu P_2 \\
\Rightarrow \frac{1}{k}\Big(\frac{\lambda}{\mu}(1-k) + \frac{\lambda^2}{\mu^2}\Big)P_0 &= P_2 \\
\therefore P_2 &= \frac{1}{k}\Big(\frac{\lambda}{\mu}(1-k) + \frac{\lambda^2}{\mu^2}\Big)P_0. \qquad \textbf{(3.25)}
\end{aligned}
$$

When n=2;
Equation (3.16) becomes;

$$
0 = k\mu P_3 + \lambda P_1 - (\lambda + k\mu)P_2 \quad \text{for} \quad n \geq k \qquad \textbf{(3.26)}
$$

Substituting (3.18) and 3.25) in equation (3.26), we get;

$$
\begin{aligned}
0 &= k\mu P_3 + \frac{\lambda^2}{\mu}P_0 - (\lambda + k\mu)\frac{1}{k}\Big(\frac{\lambda}{\mu}(1-k) + \frac{\lambda^2}{\mu^2}\Big)P_0 \\
\iff \frac{\lambda^3}{k\mu^2}P_0 + \frac{\lambda^2}{k\mu}(k-1)P_0 + \lambda(k-1)P_0 &= k\mu P_3 \\
\Rightarrow \frac{1}{k^2}\Big(\frac{\lambda^3}{\mu^3} + \frac{\lambda^2}{\mu^2}(k-1) + \frac{\lambda}{\mu}k(k-1)\Big)P_0 &= P_3 \\
\therefore P_3 &= \frac{1}{k^2}\Big(\frac{\lambda^3}{\mu^3} + \frac{\lambda^2}{\mu^2}(k-1) + \frac{\lambda}{\mu}k(k-1)\Big)P_0.
\end{aligned}
$$

Hence,

$$P_4 = \frac{1}{k^3}\Big(\frac{\lambda^4}{\mu^4} + \frac{\lambda^3}{\mu^3}(k-1) + \frac{\lambda^2}{\mu^2}k(k-1) + \frac{\lambda}{\mu}K(k-1)(k-2)\Big)P_0,$$

.

.

.

$$P_n = \frac{1}{k^{n-k}}\Big(\frac{\lambda^n}{\mu^n} + \frac{\lambda^{n-1}}{\mu^{n-1}}(k-1) + \ldots\frac{\lambda}{\mu}k(k-1)(k-2)\ldots(k-n)\Big)P_0.$$

for $n \geq k$.

Assume that $\frac{\lambda}{\mu} < k$.
But,

$$\sum_{n=0}^{\infty} P_n = 1 \quad \text{for} \quad n \geq k.$$

$$\Rightarrow P_0\Big(1 + \frac{\lambda}{1!\mu} + (\frac{1}{k})^{-1}\frac{\lambda^2}{2!\mu^2} + \ldots + (\frac{1}{k^{n-1}})^{-1}\frac{\lambda^n}{k!\mu^n} + \ldots\Big) = 1$$

$$P_0\Big((\frac{1}{k^{n-1}})^{-1}e^{\lambda\mu}\Big) = 1$$

$$\Rightarrow P_0 k^{n-k} = \frac{1}{e^{\frac{\lambda}{\mu}}}$$

$$\therefore P_0 = \frac{1}{k^{n-k}}e^{-\frac{\lambda}{\mu}}.$$

Thus,

$$P_n = \frac{(\frac{\lambda}{\mu})^n}{k!k^{n-1}}P_0.$$

$$\therefore P_n = \frac{(\frac{\lambda}{\mu})^n}{k!k^{n-1}}e^{-\frac{\lambda}{\mu}}; \quad \text{for} \quad n \geq k; \quad \text{for} \quad \frac{\lambda}{\mu} < k.$$

Remark 3.1
The series $\sum(\frac{P_n}{P_0})$ converge only if $\frac{\lambda}{\mu} < k$ i.e., $\sum P_n = 1$. If $\frac{\lambda}{\mu} \geq k$, a limiting distribution P_n cannot exists.
In this case, $P_n = 0$ for all n, which means that gradually the waiting line grow over all bounds.

3.4 Servicing of machines by a single repairman

We consider ξ automatic machines which call for service only in the event of breakdown. Let $\lambda\Delta t + 0(\Delta t)$ be the probability of calling for service in time Δt if it is working at t (time) and $\mu\Delta t + 0(\Delta t)$ be the probability of machine reverting back to work if it is being serviced at t.
Let the system be in state E_n when n machines are working. A transition $E_n \to E_{n+1}$ is caused by breaking of one of the working machines, while a transition $E_n \to E_{n-1}$ is caused by the return to work of a machine being serviced.
Thus, we have a Birth and Death process with coefficients:

$$\lambda_n = (\xi - n)\lambda,$$

and

$$\mu_0 = 0, \quad \mu_1 = \mu_2 = ... = \mu_\xi = \mu; \quad \text{for} \quad 1 \leq n \leq \xi - 1.$$

The basic difference differential equations for the steady state problem for servicing of machines with only one repairman are:

$$0 = -\xi\lambda P_0 \tag{3.27}$$

$$0 = -\{(\xi - n)\lambda + \mu\}P_n + (\xi - n + 1)\lambda P_{n-1} + \mu P_{n+1} \tag{3.28}$$

$$0 = -\mu P_\xi + \lambda P_{\xi-1} \tag{3.29}$$

Hence;
Proposition 3.4

$$P_\xi = \left\{1 + \frac{1}{1!}(\frac{\lambda}{\mu})^1 + \frac{1}{2!}(\frac{\lambda}{\mu})^2 + \frac{1}{3!}(\frac{\lambda}{\mu})^3 + \frac{1}{4!}(\frac{\lambda}{\mu})^4 + ... + \frac{1}{\xi!}(\frac{\lambda}{\mu})^\xi\right\}^{-1} \tag{3.30}$$

Proof
Solving the steady state equations iteratively, we get;
Equation (3.27) can be expresses as;

$$\mu P_1 = \xi\lambda P_0$$
$$\Rightarrow P_1 = \xi\frac{\lambda}{\mu}P_0$$
$$\therefore P_1 = \xi\frac{\lambda}{\mu}P_0. \tag{3.31}$$

When n=1;
Equation (3.28) becomes;

$$0 = -\{(\xi-1)\lambda+\mu\}P_1 + \xi\lambda P_0 + \mu P_2 \tag{3.32}$$

Substituting (3.31) in equation (3.32), we get;

$$0 = -\{(\xi-1)\lambda+\mu\}\xi\frac{\lambda}{\mu}P_0 + \xi\lambda P_0 + \mu P_2$$
$$0 = -\frac{\xi^2\lambda^2}{\mu}P_0 + \frac{\xi\lambda^2}{\mu}P_0 - \xi\lambda P_0 + \xi\lambda P_0 + \mu P_2$$
$$0 = -\frac{\xi^2\lambda^2}{\mu}P_0 + \frac{\xi\lambda^2}{\mu}P_0 + \mu P_2$$
$$0 = -\frac{\lambda^2}{\mu}\xi(\xi-1)P_0 + \mu P_2$$
$$\Longleftrightarrow \mu P_2 = \frac{\lambda^2}{\mu}\xi(\xi-1)P_0$$
$$\Rightarrow P_2 = \frac{\lambda^2}{\mu^2}\xi(\xi-1)P_0$$
$$\therefore P_2 = \frac{\lambda^2}{\mu^2}\xi(\xi-1)P_0. \tag{3.33}$$

When n=2;
Equation (3.28) becomes;

$$0 = -\{(\xi-2)\lambda+\mu\}P_2 + (\xi-1)\lambda P_1 + \mu P_3 \tag{3.34}$$

Substituting (3.31) and (3.33) in equation (3.34), we get;

$$
\begin{aligned}
0 &= -\{(\xi-2)\lambda+\mu\}\frac{\lambda^2}{\mu^2}\xi(\xi-1)P_0+(\xi-1)\lambda\xi\frac{\lambda}{\mu}P_0+\mu P_3\\
0 &= -\frac{\lambda^3}{\mu^2}\xi(\xi-1)(\xi-2)P_0-\frac{\lambda^2}{\mu}\xi(\xi-1)P_0+\frac{\lambda^2}{\mu}\xi(\xi-1)P_0+\mu P_3\\
0 &= -\frac{\lambda^3}{\mu^2}\xi(\xi-1)(\xi-2)P_0+\mu P_3\\
\Longleftrightarrow \mu P_3 &= \frac{\lambda^3}{\mu^2}\xi(\xi-1)(\xi-2)P_0\\
\Rightarrow P_3 &= \frac{\lambda^3}{\mu^3}\xi(\xi-1)(\xi-2)P_0\\
\therefore P_3 &= \frac{\lambda^3}{\mu^3}\xi(\xi-1)(\xi-2)P_0.
\end{aligned}
$$

Hence;

$$
\begin{aligned}
P_4 &= \frac{\lambda^4}{\mu^4}\xi(\xi-1)(\xi-2)(\xi-3)P_0,\\
P_5 &= \frac{\lambda^5}{\mu^5}\xi(\xi-1)(\xi-2)(\xi-3)(\xi-4)P_0,\\
&.\\
&.\\
&.\\
\therefore P_n &= \frac{\lambda^n}{\mu^n}\xi(\xi-1)(\xi-2)(\xi-3)...(\xi-n)P_0. \qquad (3.35)
\end{aligned}
$$

Note that;

$$
\begin{aligned}
\xi(\xi-1)(\xi-2)(\xi-3)...(\xi-\xi) &= \{\frac{1}{\xi}\}^{-1}\\
\Rightarrow P_1 &= \{\frac{1}{1!}\}^{-1}(\frac{\lambda}{\mu})^1P_0,\\
P_2 &= \{\frac{1}{2!}\}^{-1}(\frac{\lambda}{\mu})^2P_0,\\
\therefore P_3 &= \{\frac{1}{3!}\}^{-1}(\frac{\lambda}{\mu})^3P_0,
\end{aligned}
$$

Therefore,

$$P_4 = \{\frac{1}{4!}\}^{-1}(\frac{\lambda}{\mu})^4 P_0,$$

.

.

.

$$P_\xi = \{\frac{1}{\xi!}\}^{-1}(\frac{\lambda}{\mu})^\xi P_0.$$

Therefore,

$$P_\xi = \left\{1 + \frac{1}{1!}(\frac{\lambda}{\mu})^1 + \frac{1}{2!}(\frac{\lambda}{\mu})^2 + \frac{1}{3!}(\frac{\lambda}{\mu})^3 + \frac{1}{4!}(\frac{\lambda}{\mu})^4 + \ldots + \frac{1}{\xi!}(\frac{\lambda}{\mu})^\xi\right\}^{-1}.$$

Which is Erlang's loss function.

$$P_0 = \frac{1}{\xi}(\frac{\mu}{\lambda})^\xi P_\xi.$$

Which gives the probability of a repairman being idle.
We find by Mathematical induction that;

$$P_0 = (\frac{1}{\xi})^{-1}(\frac{\mu}{\lambda})^\xi P_\xi.$$

Hence,

$$P_\xi = \left\{1 + \frac{1}{1!}(\frac{\lambda}{\mu})^1 + \frac{1}{2!}(\frac{\lambda}{\mu})^2 + \frac{1}{3!}(\frac{\lambda}{\mu})^3 + \frac{1}{4!}(\frac{\lambda}{\mu})^4 + \ldots + \frac{1}{\xi!}(\frac{\lambda}{\mu})^\xi\right\}^{-1}.$$

Thus, the limiting distribution is an Erlang distribution with parameter $\frac{\lambda}{\mu}$.
It is independent of the initial condition, or equivalently the sum of independent exponential distributions, or a gamma distribution of time $\Gamma(\xi)e^{-\frac{\mu}{\lambda}}$.
The expected number of machines in the waiting line are:

$$\omega = \sum_{n=1}^{\xi}(n-1)P_n$$
$$\therefore \omega = \sum_{n=1}^{\xi} nP_n - (1-P_0).$$

But,

$$(\xi - n)\lambda P_n = \mu P_{n+1},$$
$$\therefore P_n = \frac{\mu}{\lambda(\xi - n)}P_{n+1}.$$

3.5 Servicing of machines by several repairmen

Let ξ machines be serviced by r repairers $(r < \xi)$. Thus, for $n \leq r$ the state E_n means $r-n$ repair-men are idle while n machines are being serviced and no machines are in the waiting line for repairs. In the case of $n > r$, the state E_n means r machines are being served and $n-r$ machines are in the waiting line.
Then set up is:

$$\lambda_0 = \xi\lambda; \quad \mu_0 = 0$$
$$\lambda_n = (\xi - n)\lambda; \quad \mu_n = n\mu(1 \leq n < r)$$
$$\lambda_n = (\xi - n)\lambda; \quad \mu_n = r\mu(r \leq n < \xi).$$

The basic difference differential equations for the steady state for servicing of machines with several repairmen are:

$$0 = -\xi\lambda P_0 + \mu P_1, \quad \text{for} \quad n = 0 \tag{3.36}$$
$$0 = -\{(\xi - n)\lambda + n\mu\}P_n + (\xi - n + 1)\lambda P_{n-1} + (n+1)\mu P_{n+1}, \quad (1 \leq n < r) \tag{3.37}$$
$$0 = -\{(\xi - n)\lambda + r\mu\}P_n + (\xi - n + 1)\lambda P_{n-1} + r\mu P_{n+1}, \quad (r \leq n < \xi). \tag{3.38}$$

Hence,
Proposition 3.5

$$P_n = \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(\frac{\lambda}{\lambda+\mu}\Big)^n \Big(\frac{\mu}{\lambda+\mu}\Big)^{\xi-n}; \quad n = 0,1,2,...,\xi-1 \quad (1 \leq n < r). \tag{3.39}$$

Proof
Equation (3.36) can be expressed as;

$$\begin{aligned} \mu P_1 &= \xi \lambda P_0 \\ \Rightarrow P_1 &= \xi \frac{\lambda}{\mu} P_0 \\ \therefore P_1 &= \xi \frac{\lambda}{\mu} P_0. \end{aligned} \tag{3.40}$$

When n=1;
Equation (3.37) becomes;

$$0 = -\{(\xi-1)\lambda+\mu\}P_1 + \xi\lambda P_0 + 2\mu P_2 \tag{3.41}$$

Substituting (3.40) in equation (3.41), we get;

$$\begin{aligned} 0 &= -\{(\xi-1)\lambda+\mu\}\xi\frac{\lambda}{\mu}P_0 + \xi\lambda P_0 + 2\mu P_2 \\ 0 &= -\frac{\lambda^2}{\mu}\xi(\xi-1) - \xi\lambda P_0 + \xi\lambda P_0 + 2\mu P_2 \\ 0 &= -\frac{\lambda^2}{\mu}\xi(\xi-1) + 2\mu P_2 \\ \Longleftrightarrow 2\mu P_2 &= \frac{\lambda^2}{\mu}\xi(\xi-1) + 2\mu P_2 \\ \Rightarrow P_2 &= \frac{\lambda^2}{\mu^2}\xi(\xi-1) + 2\mu P_2 \\ \therefore P_2 &= \frac{\lambda^2}{\mu^2}\xi(\xi-1) + 2\mu P_2. \end{aligned} \tag{3.42}$$

When n=2;
Equation (3.36) becomes;

$$0 = -\{(\xi - 2)\lambda + 2\mu\}P_2 + (\xi - 1)\lambda P_1 + 3\mu P_3 \tag{3.43}$$

Substituting (3.40) and (3.42) in equation (3.43), we get;

$$\begin{aligned}
0 &= -\{(\xi-2)\lambda+2\mu\}\frac{\lambda^2}{\mu^2}\xi(\xi-1)+2\mu P_2+(\xi-1)\lambda\xi\frac{\lambda}{\mu}P_0+3\mu P_3\\
0 &= -\frac{\lambda^3}{2\mu^2}\xi(\xi-1)(\xi-2)P_0-\frac{\lambda^2}{\mu}\xi(\xi-1)P_0+\frac{\lambda^2}{\mu}\xi(\xi-1)P_0+3\mu P_3\\
0 &= -\frac{\lambda^3}{2\mu^2}\xi(\xi-1)(\xi-2)P_0+3\mu P_3\\
\Longleftrightarrow 3\mu P_3 &= \frac{\lambda^3}{2\mu^2}\xi(\xi-1)(\xi-2)P_0\\
\Rightarrow P_3 &= \frac{\lambda^3}{3!\mu^3}\xi(\xi-1)(\xi-2)P_0\\
\therefore P_3 &= \frac{\lambda^3}{3!\mu^3}\xi(\xi-1)(\xi-2)P_0.
\end{aligned}$$

Hence,

$$\begin{aligned}
P_4 &= \frac{\lambda^4}{4!\mu^4}\xi(\xi-1)(\xi-2)(\xi-3)P_0,\\
P_5 &= \frac{\lambda^5}{5!\mu^5}\xi(\xi-1)(\xi-2)(\xi-3)(\xi-4)P_0,\\
&.\\
&.\\
&.\\
P_n &= \frac{\lambda^n}{n!\mu^n}\xi(\xi-1)(\xi-2)(\xi-3)\ldots(r-n)P_0.
\end{aligned} \tag{3.44}$$

Let,

$$\sum P_n = 1.$$

$$\Rightarrow P_1 = \Big(\frac{\lambda}{\mu}\Big)^1 \begin{pmatrix} \xi \\ 1 \end{pmatrix} P_0,$$

$$P_2 = \Big(\frac{\lambda}{\mu}\Big)^2 \begin{pmatrix} \xi \\ 2 \end{pmatrix} P_0,$$

$$P_3 = \Big(\frac{\lambda}{\mu}\Big)^3 \begin{pmatrix} \xi \\ 3 \end{pmatrix} P_0,$$

$$\vdots$$

But;

$$\sum_{n=0}^{\infty} P_n = 1$$

$$P_0\Big\{1 + \Big(\frac{\lambda}{\mu}\Big)^1 \begin{pmatrix} \xi \\ 1 \end{pmatrix} + \Big(\frac{\lambda}{\mu}\Big)^2 \begin{pmatrix} \xi \\ 2 \end{pmatrix} + \Big(\frac{\lambda}{\mu}\Big)^3 \begin{pmatrix} \xi \\ 3 \end{pmatrix} + ...\Big\} = 1$$

$$\therefore P_0 = \Big(\frac{\mu}{\lambda+\mu}\Big)^n; \quad n = 0, 1, 2, ..., \xi.$$

P_0= The probability of machines not serviced.

$$P_n = \Big\{\Big(\frac{\lambda}{\lambda+\mu}\Big)^0\Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi} \begin{pmatrix} \xi \\ 0 \end{pmatrix} + \Big(\frac{\lambda}{\lambda+\mu}\Big)^1\Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-1} \begin{pmatrix} \xi \\ 1 \end{pmatrix} +$$
$$\Big(\frac{\lambda}{\lambda+\mu}\Big)^2\Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-2} \begin{pmatrix} \xi \\ 2 \end{pmatrix} + ... + \Big(\frac{\lambda}{\lambda+\mu}\Big)^n\Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-n} \begin{pmatrix} \xi \\ n \end{pmatrix}\Big\}.$$

$$\therefore P_n = \Big\{\begin{pmatrix} \xi \\ n \end{pmatrix}\Big(\frac{\lambda}{\lambda+\mu}\Big)^n\Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-n}\Big\}; \quad n = 0, 1, 2, ..., \xi - 1 \quad (1 \le n < r). \tag{3.45}$$

Hence the limiting probabilities are given by the Binomial Distribution with parameters ξ and $\frac{\lambda}{\lambda+\mu}$.

Using Proposition 2.1
We have;

$$P_n = \frac{\lambda_0\lambda_1\lambda_2\lambda_3...\lambda_{n-1}}{\mu_1\mu_2\mu_3...\mu_n}P_0$$

But, $\lambda_0 = \xi\lambda, \quad \lambda_1 = (\xi-1)\lambda, \quad \lambda_2 = (\xi-2)\lambda, \quad \lambda_3 = (\xi-3)\lambda, \quad ... = \lambda_{n-1} = (\xi-(n-1))\lambda.$

and

$$\mu_1 = \mu, \quad \mu_2 = 2\mu, \quad \mu_3 = 3\mu, \quad ..., \quad \mu_n = n\mu.$$

$$\begin{aligned}
\therefore P_n &= \frac{\xi\lambda(\xi-1)\lambda(\xi-2)\lambda(\xi-3)\lambda...(\xi-(n-1))\lambda}{\mu 2\mu 3\mu ... n\mu}P_0 \\
&= \frac{\xi\lambda(\xi-1)\lambda(\xi-2)\lambda(\xi-3)\lambda...(\xi-(n-1))\lambda}{n!\mu^n}P_0 \\
&= \frac{P_0}{n!\mu^n}\prod_{j=0}^{n-1}(\xi-j)\lambda \\
&= \frac{\lambda^n P_0}{n!\mu^n}\prod_{j=0}^{n-1}(\xi-j) \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\left\{\xi(\xi-1)(\xi-2)(\xi-3)...(\xi-(n-1))\right\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\left\{(\xi-(n-1))(\xi-(n-2))...(\xi-2)(\xi-1)\xi\right\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\left\{(\xi-(n-1))(\xi-(n-2))...(\xi-2)(\xi-1)\xi\frac{\Gamma(\xi)}{\Gamma(\xi)}\right\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\left\{(\xi-(n-1))(\xi-(n-2))...(\xi-2)(\xi-1)\frac{\Gamma(\xi-1)}{\Gamma(\xi)}\right\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\frac{1}{\Gamma(\xi)}\left\{(\xi-(n-1))(\xi-(n-2))...(\xi-2)\Gamma(\xi-2)\right\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\frac{1}{\Gamma(\xi)}\left\{(\xi-(n-1))\Gamma(\xi-(n-1))\right\} \\
&= P_0\left(\frac{\lambda}{\mu}\right)^n\frac{1}{n!\Gamma(\xi)}\Gamma(\xi-n) \\
&= P_0\left(\frac{\lambda}{\mu}\right)^n\frac{\Gamma(\xi-n)}{n!\Gamma(\xi)} \\
&= P_0\left(1-\frac{\lambda}{\lambda+\mu}\right)^{\xi-n}\begin{pmatrix}\xi\\ n\end{pmatrix}
\end{aligned}$$

$$\therefore P_n = \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-n} P_0 \quad n = 0,1,2,...$$

But,

$$\sum_{n=0}^{\infty} P_n = 1$$

i.e.,

$$P_0 \sum_{n=0}^{\infty} \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-n} = 1$$

$$P_0 \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} -\xi \\ n \end{pmatrix} \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-n} = 1$$

$$P_0 \sum_{n=0}^{\infty} \begin{pmatrix} -\xi \\ n \end{pmatrix} \Big(\frac{-\lambda}{\lambda+\mu}\Big)^{\xi-n} = 1$$

$$P_0 \Big(\frac{\lambda}{\lambda+\mu}\Big)^{-n} = 1$$

$$\therefore P_0 = \Big(\frac{\lambda}{\lambda+\mu}\Big)^{n}$$

$$\therefore P_n = \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-n} \Big(\frac{\lambda}{\lambda+\mu}\Big)^{n} \quad n = 0,1,2,...,\xi-1.$$

Which is a Binomial Distribution with parameters ξ and $\frac{\lambda}{\lambda+\mu}$.

Let $\xi = \xi$ and $\frac{\lambda}{\lambda+\mu}$=$\zeta$;

Thus, the Binomial distribution becomes;

$$P_n = \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(1 - \zeta\Big)^{\xi-n} \Big(\zeta\Big)^{n} \quad n = 0,1,2,...,\xi-1.$$

Then;
Gauss hypergeometric series given by;

$$
\begin{aligned}
{}_2F_1(\xi,\ell;\ell;\zeta) &= 1+\frac{\xi.\ell}{1}\frac{\zeta}{1!}+\frac{\xi(\xi+1)\ell(\ell+1)}{\ell(\ell+1)}\frac{(\zeta)^2}{2!}+\ldots+\frac{(\xi+n-1)(\ell+n-1)}{(\ell+n-1)}\frac{(\zeta)^n}{n!}\\
&=\sum_{n=0}^{\infty}\frac{\xi(\xi+1)\ldots(\xi+n-1)\ell(\ell+1)(\ell+2)\ldots(\ell+n-1)}{\ell(\ell+1)(\ell+2)\ldots(\ell+n-1)}\frac{(\zeta)^n}{n!}\\
&=\sum_{n=0}^{\infty}\frac{(\xi+n-1)\ldots(\xi+1)\xi(\ell+n-1)\ldots(\ell+2)(\ell+1)\ell}{(\ell+n-1)\ldots(\ell+2)(\ell+1)\ell}\frac{(\zeta)^n}{n!}\\
&=\sum_{n=0}^{\infty}\frac{\Gamma(\xi+n)\Gamma(\ell+n)}{\Gamma(\ell+n)}.\frac{\Gamma(\ell)}{\Gamma(\xi)\Gamma(\ell)}\frac{(\zeta)^n}{n!}
\end{aligned}
$$

Normalizing, we get;

$$
1=\sum_{n=0}^{\infty}\frac{\Gamma(\xi+n)\Gamma(\ell+n)}{\Gamma(\ell+n)}.\frac{\Gamma(\ell)}{\Gamma(\xi)\Gamma(\ell)}.\frac{1}{{}_2F_1(\xi,\ell;\ell;\zeta)}\frac{(\zeta)^n}{n!}
$$

Hence,

$$
\begin{aligned}
P_n &= \text{Prob.}(N=n)\\
&=\frac{\Gamma(\xi+n)\Gamma(\ell+n)}{\Gamma(\ell+n)}.\frac{\Gamma(\ell)}{\Gamma(\xi)\Gamma(\ell)}.\frac{1}{{}_2F_1(\xi,\ell;\ell;\zeta)}\frac{(\zeta)^n}{n!}
\end{aligned}
$$

The pgf. in hypergeometric terms is given by;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty}P_n z^n\\
\phi(z) &= \sum_{n=0}^{\infty}\frac{\Gamma(\xi+n)\Gamma(\ell+n)}{\Gamma(\ell+n)}.\frac{\Gamma(\ell)}{\Gamma(\xi)\Gamma(\ell)}\frac{(\zeta z)^n}{n!}.\frac{1}{{}_2F_1(\xi,\ell;\ell;\zeta)}\\
\therefore \phi(z) &= \frac{{}_2F_1(\xi,\ell;\ell;\zeta z)}{{}_2F_1(\xi,\ell;\ell;\zeta)}.\\
\phi^{'}(z) &= \xi\zeta\frac{{}_2F_1(\xi+1,\ell+1;\ell+1;\zeta z)}{{}_2F_1(\xi,\ell;\ell;\zeta)}\\
\phi^{''}(z) &= \xi(\xi+1)(\zeta)^2\frac{{}_2F_1(\xi+2,\ell+2;\ell+2;\zeta z)}{{}_2F_1(\xi,\ell;\ell;\zeta)}
\end{aligned}
$$

Let, $\Lambda_\kappa=\dfrac{{}_2F_1(\xi+\kappa,\ell+\kappa;\ell+\kappa;\zeta)}{{}_2F_1(\xi,\ell;\ell;\zeta)};\quad \kappa=1,2.$

$$
\begin{aligned}
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \xi\zeta\Lambda_1.\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2.\\
&= \xi(\xi+1)\zeta^2\Lambda_2+\xi\zeta\Lambda_1-[\xi\zeta\Lambda_1]^2\\
\therefore Var(X) &= \xi\zeta\Lambda_1+\xi\zeta^2\{(\xi+1)\Lambda_2-\xi\Lambda_1^2\}.
\end{aligned}
$$

Using Proposition 2.2
We have;

$$
\begin{aligned}
\frac{P_n}{P_{n-1}} &= \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0 \\
\text{But,} \quad \lambda_{n-1} &= (\xi-(n-1))\lambda, \\
\text{and} & \\
\mu_n &= n\mu. \\
\therefore \frac{P_n}{P_{n-1}} &= \frac{(\xi-(n-1))\lambda}{n\mu}. \\
\Rightarrow n\mu P_n &= (\xi-(n-1))\lambda P_{n-1} \qquad \textbf{(3.46)}
\end{aligned}
$$

Using the probability generating function technique, we multiply (3.46) by s^n and sum the results over n to obtain;

$$
\begin{aligned}
\mu\sum_{n=0}^{\infty} nP_n s^n &= \lambda\sum_{n=0}^{\infty}(\xi-(n-1))P_{n-1}s^n \\
\mu s\sum_{n=0}^{\infty} nP_n s^{n-1} &= \lambda\xi\sum_{n=0}^{\infty}P_{n-1}s^n - \lambda\sum_{n=0}^{\infty}(n-1))P_{n-1}s^n \\
\mu s\sum_{n=0}^{\infty} nP_n s^{n-1} &= \lambda\xi s\sum_{n=0}^{\infty}P_{n-1}s^{n-1} - \lambda s^2\sum_{n=0}^{\infty}(n-1))P_{n-1}s^{n-2} \\
\mu s\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \lambda\xi s\psi(s) - \lambda s^2\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \\
(\mu+\lambda s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \lambda\xi\psi(s) \\
\frac{\mathrm{d}\psi(s)}{\psi(s)} &= \frac{\lambda\xi}{(\mu+\lambda s)}\mathrm{d}s
\end{aligned}
$$

Taking the integral, we have;

$$
\begin{aligned}
\int\frac{\mathrm{d}\psi(s)(s)}{\psi(s)} &= \int\frac{\lambda\xi}{(\mu+\lambda s)}\mathrm{d}s \\
\ln\psi(s) &= \xi\ln\left(\frac{\lambda}{\lambda s+\mu}\right)+\ln c \\
\therefore \psi(s) &= c_1\left(\frac{\lambda}{\lambda s+\mu}\right)^{\xi}
\end{aligned}
$$

Putting s=1;

$$
\begin{aligned}
\psi(1) &= 1 = c_1\Big(\frac{\lambda}{\lambda+\mu}\Big)^{\xi} \\
\Rightarrow c_1 &= \Big(\frac{\lambda}{\lambda+\mu}\Big)^{-\xi}. \\
\therefore \psi(s) &= \Big(\frac{\lambda}{\lambda+\mu}\Big)^{-\xi}\Big(\frac{\lambda}{\lambda s+\mu}\Big)^{\xi}. \\
\therefore \psi(s) &= \frac{\Big(\frac{\lambda}{\lambda s+\mu}\Big)^{\xi}}{\Big(\frac{\lambda}{\lambda+\mu}\Big)^{\xi}}. \\
\psi^{'}(s) &= \xi\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda s+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\xi-1} \\
\psi^{''}(s) &= \xi(\xi-1)\Big(\frac{\lambda}{\lambda+\mu}\Big)^{2}.\Big(\frac{(\frac{\lambda}{\lambda s+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\xi-2} \\
E(X) &= \psi^{'}(1) \\
\therefore E(X) &= \xi\Big(\frac{\lambda}{\lambda+\mu}\Big). \\
Var(X) &= \psi^{''}(1)+\psi^{'}(1)-[\psi^{'}(1)]^2 \\
&= \xi(\xi-1)\Big(\frac{\lambda}{\lambda+\mu}\Big)^{2}.\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\xi-2}+\xi\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\xi-1}-[\xi\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\xi-1}]^2 \\
\therefore Var(X) &= \xi\Big(\frac{\lambda}{\lambda+\mu}\Big)\Big(1-\frac{\lambda}{\lambda+\mu}\Big).
\end{aligned}
$$

Next;
When n=2;
Equation (3.38) becomes;

$$0 = -\{(\xi-1)\lambda + r\mu\}P_1 + \xi\lambda P_0 + r\mu P_2$$

Note that;

$$r\mu P_{n+1} = (\xi-n)\lambda P_n \quad \text{for} \quad (n \geq r)$$

$$\Rightarrow P_0 = \frac{\xi\lambda}{r\mu}P_0. \qquad \textbf{(i)}$$

Substituting (i) we get;

$$0 = -\{(\xi-1)\lambda + r\mu\}\frac{\xi\lambda}{r\mu}P_0 + \xi\lambda P_0 + r\mu P_2$$
$$0 = -\frac{\xi^2\lambda^2}{r\mu}P_0 + \frac{\xi\lambda^2}{r\mu}P_0 - \xi\lambda P_0 + \xi\lambda P_0 + r\mu P_2$$
$$0 = -\frac{\xi^2\lambda^2}{r\mu}P_0 + \frac{\xi\lambda^2}{r\mu}P_0 + r\mu P_2$$
$$\Longleftrightarrow r\mu P_2 = \frac{1}{r}\{\frac{\lambda^2}{\mu}\xi(\xi-1)\}P_0$$
$$\Rightarrow P_2 = \frac{1}{r^2}\{\frac{\lambda^2}{\mu^2}\xi(\xi-1)\}P_0,$$
$$\therefore P_2 = \frac{1}{r^2}\{\frac{\lambda^2}{\mu^2}\xi(\xi-1)\}P_0.$$
$$P_3 = \frac{1}{r^3}\{\frac{\lambda^3}{\mu^3}\xi(\xi-1)(\xi-2)\}P_0,$$
$$.$$
$$.$$
$$.$$
$$P_\xi = \frac{1}{r^{\xi-r}}\Big\{\Big(\frac{1}{1!}\Big)^{-1}(\frac{\lambda}{\mu})^1 + \Big(\frac{1}{2!}\Big)^{-1}(\frac{\lambda}{\mu})^2 + \Big(\frac{1}{3!}\Big)^{-1}(\frac{\lambda}{\mu})^3 + \Big(\frac{1}{4!}\Big)^{-1}(\frac{\lambda}{\mu})^4 + \ldots + \Big(\frac{1}{\xi!}\Big)^{-1}(\frac{\lambda}{\mu})^\xi + \Big\}P_0.$$

P_0= The probability of repairmen being idle.

3.6 A power supply problem

A welder working independently draws current from some circuit. If at time t, a welder is using current, the probability that he ceases using it at time $t+\Delta t$ is $\mu\Delta t + 0(\Delta t)$. If at time t he does not require current, the probability that he call for current in next time interval Δt is $\lambda\Delta t + 0(\Delta t)$.
We say that the system is in a state E_n, if n welders are using current.
Thus, we have only a finite number of states $E_0, E_1, ..., E_\vartheta$.
If the system is in state E_n, $\vartheta - n$ welders are not using current and

$$\lambda_n = (\vartheta - n)\lambda,$$
and
$$\mu_n = n\mu. \quad \text{for} \quad 0 \leq n \leq \vartheta.$$

For a steady state, $t \to \infty$ implying that $P_n(t)$ is not dependent on t.

$$P_n(t) = P_n,$$
and
$$\lim_{t\to\infty} P'_n(t) = 0.$$

Thus, the basic difference differential equations for the steady-state for a power supply problem are:

$$0 = -\vartheta\lambda P_0 + \mu P_1 \tag{3.47}$$
$$0 = -\{(\vartheta - n)\lambda + n\mu\}P_n + \lambda\{\vartheta - (n-1)\}P_{n-1} + (n+1)\mu P_{n+1} \tag{3.48}$$
$$0 = -\vartheta\mu P_\vartheta + \lambda P_{\vartheta-1} \tag{3.49}$$

Hence,
Proposition 3.6

$$P_n = \begin{pmatrix} \vartheta \\ n \end{pmatrix} \Big(\frac{\lambda}{\lambda+\mu}\Big)^n \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\vartheta-n}, \quad n = 0, 1, 2, ..., \vartheta - 1. \tag{3.50}$$

Proof
Solving the steady state equations iteratively, we have;
Equation (3.47) can be expressed as;

$$
\begin{aligned}
\mu P_1 &= \vartheta\lambda P_0 \\
\Rightarrow P_1 &= \frac{\vartheta\lambda}{\mu}P_0 \\
\therefore P_1 &= \frac{\vartheta\lambda}{\mu}P_0. \qquad \textbf{(3.51)}
\end{aligned}
$$

When n=1;
Equation (3.48) becomes;

$$
0 = -\{(\vartheta-1)\lambda+\mu\}P_1 + \lambda\vartheta P_0 + 2\mu P_2 \qquad \textbf{(3.52)}
$$

Substituting (3.51) in equation (3.52), we get;

$$
\begin{aligned}
0 &= -\{(\vartheta-1)\lambda+\mu\}\frac{\vartheta\lambda}{\mu}P_0 + \lambda\vartheta P_0 + 2\mu P_2 \\
0 &= -\frac{\vartheta^2\lambda^2}{\mu} - \vartheta\lambda P_0 + \vartheta\lambda P_0 + \frac{\vartheta\lambda^2}{\mu}P_0 + 2\mu P_2 \\
0 &= -\frac{\vartheta^2\lambda^2}{\mu} + \frac{\vartheta\lambda^2}{\mu}P_0 + 2\mu P_2 \\
\Longleftrightarrow 2\mu P_2 &= \frac{\vartheta^2\lambda^2}{\mu} - \frac{\vartheta\lambda^2}{\mu}P_0 \\
\Rightarrow 2\mu P_2 &= \frac{\vartheta\lambda^2}{\mu}(\vartheta-1)P_0 \\
\Rightarrow P_2 &= \frac{\vartheta\lambda^2}{2\mu^2}(\vartheta-1)P_0 \\
\therefore P_2 &= \frac{\vartheta\lambda^2}{2!\mu^2}(\vartheta-1)P_0. \qquad \textbf{(3.53)}
\end{aligned}
$$

Thus,

$$P_3 = \frac{\lambda^3}{3!\mu^3}\vartheta(\vartheta-1)(\vartheta-2)P_0,$$
$$P_4 = \frac{\lambda^4}{4!\mu^4}\vartheta(\vartheta-1)(\vartheta-2)(\vartheta-3)P_0,$$
$$\vdots$$
$$\therefore P_1 = \frac{\lambda}{\mu}\binom{\vartheta}{1}P_0,$$
$$P_2 = \frac{\lambda^2}{\mu^2}\binom{\vartheta}{2}P_0,$$
$$P_3 = \frac{\lambda^3}{\mu^3}\binom{\vartheta}{3}P_0,$$
$$P_4 = \frac{\lambda^4}{\mu^4}\binom{\vartheta}{4}P_0,$$
$$\vdots$$
$$P_n = \frac{\lambda^n}{\mu^n}\binom{\vartheta}{n}P_0.$$

Let,

$$\sum_{n=0}^{\vartheta} P_n = 1$$
$$P_0 + P_1 + P_2 + P_3 + \ldots + P_n = 1$$
$$P_0 + \frac{\lambda}{\mu}\binom{\vartheta}{1}P_0 + \frac{\lambda^2}{\mu^2}\binom{\vartheta}{2}P_0 + \frac{\lambda^3}{\mu^3}\binom{\vartheta}{3}P_0 + \ldots + \frac{\lambda^n}{\mu^n}\binom{\vartheta}{n}P_0 = 1$$

Therefore,

$$P_0\Big(1+\frac{\lambda}{\mu}\begin{pmatrix}\vartheta\\1\end{pmatrix}+\frac{\lambda^2}{\mu^2}\begin{pmatrix}\vartheta\\2\end{pmatrix}+\frac{\lambda^3}{\mu^3}\begin{pmatrix}\vartheta\\3\end{pmatrix}+\ldots+\frac{\lambda^n}{\mu^n}\begin{pmatrix}\vartheta\\n\end{pmatrix}\Big)=1$$

$$\Big(1+\frac{\lambda}{\mu}\Big)P_0=1$$

$$\Rightarrow P_0=\Big(1-\frac{\lambda}{\lambda+\mu}\Big)$$

$$\therefore P_0=\Big(1-\frac{\lambda}{\lambda+\mu}\Big).$$

$$\text{for}\quad n=0,1,2,\ldots,\vartheta.$$

P_0= The probability of welder not using current.

$$P_n=\Big\{\begin{pmatrix}\vartheta\\0\end{pmatrix}\Big(\frac{\lambda}{\lambda+\mu}\Big)^0\Big(1-\frac{\lambda}{\lambda+\mu}\Big)^{\vartheta}+\begin{pmatrix}\vartheta\\1\end{pmatrix}\Big(\frac{\lambda}{\lambda+\mu}\Big)^1\Big(1-\frac{\lambda}{\lambda+\mu}\Big)^{\vartheta-1}$$
$$+\begin{pmatrix}\vartheta\\2\end{pmatrix}\Big(\frac{\lambda}{\lambda+\mu}\Big)^2\Big(1-\frac{\lambda}{\lambda+\mu}\Big)^{\vartheta-2}+\ldots+\begin{pmatrix}\vartheta\\n\end{pmatrix}\Big(\frac{\lambda}{\lambda+\mu}\Big)^n\Big(1-\frac{\lambda}{\lambda+\mu}\Big)^{\vartheta-n}\Big\}.$$

Thus, the limiting probability are given by the Binomial Distribution,

$$\therefore P_n=\begin{pmatrix}\vartheta\\n\end{pmatrix}\Big(\frac{\lambda}{\lambda+\mu}\Big)^n\Big(1-\frac{\lambda}{\lambda+\mu}\Big)^{\vartheta-n}.\quad n=0,1,2,\ldots,\vartheta-1.$$

with parameters ϑ and $\frac{\lambda}{\lambda+\mu}$, and ϑ is a total number of welders.

Using Proposition 2.1
We have;

$$P_n = \frac{\lambda_0\lambda_1\lambda_2\lambda_3...\lambda_{n-1}}{\mu_1\mu_2\mu_3...\mu_n}P_0$$

But, $\lambda_0 = \vartheta\lambda, \quad \lambda_1 = (\vartheta-1)\lambda, \quad \lambda_2 = (\vartheta-2)\lambda, \quad \lambda_3 = (\vartheta-3)\lambda, \quad ... = \lambda_{n-1} = (\vartheta-(n-1))\lambda.$

and

$$\mu_1 = \mu, \quad \mu_2 = 2\mu, \quad \mu_3 = 3\mu, \quad ..., \quad \mu_n = n\mu.$$

$$\begin{aligned}
\therefore P_n &= \frac{\vartheta\lambda(\vartheta-1)\lambda(\vartheta-2)\lambda(\vartheta-3)\lambda...(\vartheta-(n-1))\lambda}{\mu 2\mu 3\mu...n\mu}P_0 \\
&= \frac{\vartheta\lambda(\vartheta-1)\lambda(\vartheta-2)\lambda(\vartheta-3)\lambda...(\vartheta-(n-1))\lambda}{n!\mu^n}P_0 \\
&= \frac{P_0}{n!\mu^n}\prod_{j=0}^{n-1}(\vartheta-j)\lambda \\
&= \frac{\lambda^n P_0}{n!\mu^n}\prod_{j=0}^{n-1}(\vartheta-j) \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\Big\{\vartheta(\vartheta-1)(\vartheta-2)(\vartheta-3)...(\vartheta-(n-1))\Big\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\Big\{(\vartheta-(n-1))(\vartheta-(n-2))...(\vartheta-2)(\vartheta-1)\vartheta\Big\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\Big\{(\vartheta-(n-1))(\vartheta-(n-2))...(\vartheta-2)(\vartheta-1)\vartheta\frac{\Gamma(\vartheta)}{\Gamma(\vartheta)}\Big\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\Big\{(\vartheta-(n-1))(\vartheta-(n-2))...(\vartheta-2)(\vartheta-1)\frac{\Gamma(\vartheta-1)}{\Gamma(\vartheta)}\Big\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\frac{1}{\Gamma(\vartheta)}\Big\{(\vartheta-(n-1))(\vartheta-(n-2))...(\vartheta-2)\Gamma(\vartheta-2)\Big\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\frac{1}{\Gamma(\vartheta)}\Big\{(\vartheta-(n-1))\Gamma(\vartheta-(n-1))\Big\} \\
&= P_0\left(\frac{\lambda}{\mu}\right)^n\frac{1}{n!\Gamma(\vartheta)}\Gamma(\vartheta-n) \\
&= P_0\left(\frac{\lambda}{\mu}\right)^n\frac{\Gamma(\vartheta-n)}{n!\Gamma(\vartheta)} \\
&= P_0\left(1-\frac{\lambda}{\lambda+\mu}\right)^{\vartheta-n}\begin{pmatrix}\vartheta \\ n\end{pmatrix}
\end{aligned}$$

$$\therefore P_n = \begin{pmatrix}\vartheta \\ n\end{pmatrix}\left(1-\frac{\lambda}{\lambda+\mu}\right)^{\vartheta-n}P_0 \quad n = 0,1,2,...$$

But,

$$\sum_{n=0}^{\infty} P_n = 1$$

i.e.,

$$P_0 \sum_{n=0}^{\infty} \begin{pmatrix} \vartheta \\ n \end{pmatrix} \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\vartheta-n} = 1$$

$$P_0 \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} -\vartheta \\ n \end{pmatrix} \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\vartheta-n} = 1$$

$$P_0 \sum_{n=0}^{\infty} \begin{pmatrix} -\vartheta \\ n \end{pmatrix} \Big(\frac{-\lambda}{\lambda+\mu}\Big)^{\vartheta-n} = 1$$

$$P_0 \Big(\frac{\lambda}{\lambda+\mu}\Big)^{-n} = 1$$

$$\therefore P_0 = \Big(\frac{\lambda}{\lambda+\mu}\Big)^{n}$$

$$\therefore P_n = \begin{pmatrix} \vartheta \\ n \end{pmatrix} \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\vartheta-n} \Big(\frac{\lambda}{\lambda+\mu}\Big)^{n} \quad n = 0, 1, 2, ..., \vartheta - 1.$$

Which is a Binomial Distribution with parameters ϑ and $\frac{\lambda}{\lambda+\mu}$.

Let $\vartheta = \vartheta$ and $\frac{\lambda}{\lambda+\mu}$=$\zeta$;

Thus, the Binomial distribution becomes;

$$P_n = \begin{pmatrix} \vartheta \\ n \end{pmatrix} \Big(1 - \zeta\Big)^{\vartheta-n} \Big(\zeta\Big)^{n} \quad n = 0, 1, 2, ..., \vartheta - 1.$$

Then;
Gauss hypergeometric series given by;

$$
\begin{aligned}
{}_2F_1(\vartheta,\ell;\ell;\zeta) &= 1+\frac{\vartheta.\ell}{1}\frac{\zeta}{1!}+\frac{\vartheta(\vartheta+1)\ell(\ell+1)}{\ell(\ell+1)}\frac{(\zeta)^2}{2!}+\ldots+\frac{(\vartheta+n-1)(\ell+n-1)}{(\ell+n-1)}\frac{(\zeta)^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\vartheta(\vartheta+1)\ldots(\vartheta+n-1)\ell(\ell+1)(\ell+2)\ldots(\ell+n-1)}{\ell(\ell+1)(\ell+2)\ldots(\ell+n-1)}\frac{(\zeta)^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(\vartheta+n-1)\ldots(\vartheta+1)\vartheta(\ell+n-1)\ldots(\ell+2)(\ell+1)\ell}{(\ell+n-1)\ldots(\ell+2)(\ell+1)\ell}\frac{(\zeta)^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\vartheta+n)\Gamma(\ell+n)}{\Gamma(\ell+n)}.\frac{\Gamma(\ell)}{\Gamma(\vartheta)\Gamma(\ell)}\frac{(\zeta)^n}{n!}
\end{aligned}
$$

Normalizing, we get;

$$
1 = \sum_{n=0}^{\infty}\frac{\Gamma(\vartheta+n)\Gamma(\ell+n)}{\Gamma(\ell+n)}.\frac{\Gamma(\ell)}{\Gamma(\vartheta)\Gamma(\ell)}.\frac{1}{{}_2F_1(\vartheta,\ell;\ell;\zeta)}\frac{(\zeta)^n}{n!}
$$

Hence,

$$
\begin{aligned}
P_n &= \text{Prob.}(N=n)\\
&= \frac{\Gamma(\vartheta+n)\Gamma(\ell+n)}{\Gamma(\ell+n)}.\frac{\Gamma(\ell)}{\Gamma(\vartheta)\Gamma(\ell)}.\frac{1}{{}_2F_1(\vartheta,\ell;\ell;\zeta)}\frac{(\zeta)^n}{n!}
\end{aligned}
$$

The pgf. in hypergeometric terms is given by;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty}P_n z^n\\
\phi(z) &= \sum_{n=0}^{\infty}\frac{\Gamma(\vartheta+n)\Gamma(\ell+n)}{\Gamma(\ell+n)}.\frac{\Gamma(\ell)}{\Gamma(\vartheta)\Gamma(\ell)}\frac{(\zeta z)^n}{n!}.\frac{1}{{}_2F_1(\vartheta,\ell;\ell;\zeta)}\\
\therefore \phi(z) &= \frac{{}_2F_1(\vartheta,\ell;\ell;\zeta z)}{{}_2F_1(\vartheta,\ell;\ell;\zeta)}.\\
\phi^{'}(z) &= \vartheta\zeta\frac{{}_2F_1(\vartheta+1,\ell+1;\ell+1;\zeta z)}{{}_2F_1(\vartheta,\ell;\ell;\zeta)}\\
\phi^{''}(z) &= \vartheta(\vartheta+1)(\zeta)^2\frac{{}_2F_1(\vartheta+2,\ell+2;\ell+2;\zeta z)}{{}_2F_1(\vartheta,\ell;\ell;\zeta)}
\end{aligned}
$$

Let, $\Lambda_\kappa = \dfrac{{}_2F_1(\vartheta+\kappa,\ell+\kappa;\ell+\kappa;\zeta)}{{}_2F_1(\vartheta,\ell;\ell;\zeta)};\quad \kappa = 1,2.$

$$
\begin{aligned}
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \vartheta\zeta\Lambda_1.\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2.\\
&= \vartheta(\vartheta+1)\zeta^2\Lambda_2+\vartheta\zeta\Lambda_1-[\vartheta\zeta\Lambda_1]^2\\
\therefore Var(X) &= \vartheta\zeta\Lambda_1+\vartheta\zeta^2\{(\vartheta+1)\Lambda_2-\vartheta\Lambda_1^2\}.
\end{aligned}
$$

Using Proposition 2.2
We have;

$$\frac{P_n}{P_{n-1}} = \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0$$

$$\text{But,} \quad \lambda_{n-1} = (\vartheta - (n-1))\lambda,$$

and

$$\mu_n = n\mu.$$

$$\therefore \frac{P_n}{P_{n-1}} = \frac{(\vartheta - (n-1))\lambda}{n\mu}.$$

$$\Rightarrow n\mu P_n = (\vartheta - (n-1))\lambda P_{n-1} \tag{3.54}$$

Using the probability generating function technique, we multiply (3.54) by s^n and sum the results over n to obtain;

$$\mu \sum_{n=0}^{\infty} nP_n s^n = \lambda \sum_{n=0}^{\infty} (\vartheta - (n-1))P_{n-1}s^n$$

$$\mu s \sum_{n=0}^{\infty} nP_n s^{n-1} = \lambda \vartheta \sum_{n=0}^{\infty} P_{n-1}s^n - \lambda \sum_{n=0}^{\infty} (n-1))P_{n-1}s^n$$

$$\mu s \sum_{n=0}^{\infty} nP_n s^{n-1} = \lambda \vartheta s \sum_{n=0}^{\infty} P_{n-1}s^{n-1} - \lambda s^2 \sum_{n=0}^{\infty} (n-1))P_{n-1}s^{n-2}$$

$$\mu s \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \lambda \vartheta s \psi(s) - \lambda s^2 \frac{\mathrm{d}\psi(s)}{\mathrm{d}s}$$

$$(\mu + \lambda s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \lambda \vartheta \psi(s)$$

$$\frac{\mathrm{d}\psi(s)}{\psi(s)} = \frac{\lambda \vartheta}{(\mu + \lambda s)}\mathrm{d}s$$

Taking the integral, we have;

$$\int \frac{\mathrm{d}\psi(s)(s)}{\psi(s)} = \int \frac{\lambda \vartheta}{(\mu + \lambda s)}\mathrm{d}s$$

$$\ln \psi(s) = \vartheta \ln(\frac{\lambda}{\lambda s + \mu}) + \ln c$$

$$\therefore \psi(s) = c_1 \Big(\frac{\lambda}{\lambda s + \mu}\Big)^{\vartheta}$$

Putting s=1;

$$
\begin{aligned}
\psi(1) &= 1 = c_1\Big(\frac{\lambda}{\lambda+\mu}\Big)^{\vartheta}\\
\Rightarrow c_1 &= \Big(\frac{\lambda}{\lambda+\mu}\Big)^{-\vartheta}.\\
\therefore \psi(s) &= \Big(\frac{\lambda}{\lambda+\mu}\Big)^{-\vartheta}\Big(\frac{\lambda}{\lambda s+\mu}\Big)^{\vartheta}.\\
\therefore \psi(s) &= \frac{\Big(\frac{\lambda}{\lambda s+\mu}\Big)^{\vartheta}}{\Big(\frac{\lambda}{\lambda+\mu}\Big)^{\vartheta}}.\\
\psi^{'}(s) &= \vartheta\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda s+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\vartheta-1}\\
\psi^{''}(s) &= \vartheta(\vartheta-1)\Big(\frac{\lambda}{\lambda+\mu}\Big)^{2}.\Big(\frac{(\frac{\lambda}{\lambda s+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\vartheta-2}\\
E(X) &= \psi^{'}(1)\\
\therefore E(X) &= \vartheta\Big(\frac{\lambda}{\lambda+\mu}\Big).\\
Var(X) &= \psi^{''}(1)+\psi^{'}(1)-[\psi^{'}(1)]^2\\
&= \vartheta(\vartheta-1)\Big(\frac{\lambda}{\lambda+\mu}\Big)^{2}.\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\vartheta-2}+\vartheta\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\vartheta-1}\\
&-[\vartheta\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\vartheta-1}]^2\\
\therefore Var(X) &= \vartheta\Big(\frac{\lambda}{\lambda+\mu}\Big)\Big(1-\frac{\lambda}{\lambda+\mu}\Big).
\end{aligned}
$$

4 BIRTH AND DEATH PROCESSES AT EQUILIBRIUM WITH APPLICATION TO QUEUING THEORY

4.1 Introduction

Queueing ststems can be studied through Birth-and-Death processes. These processes are characterized by the property that whenever a transition occurs from one state to another, then this transition can be to a neighboring state only. With an arrival, there is a transition from state $n(\geq 0)$ to the state $(n+1)$, and with a service completion, there is a transition from the state m to the state $(m-1) \quad (m \geq 0)$, the state denoting the number in the system.

In this chapter, we will discuss queueing models in terms of Birth-and-Death equations, based on the rate-equality principle which holds for systems in steady state.

Rate-Equality Principle states that the rate at which a processes enters a state $n(\geq 0)$, equals the rate at which the processes leaves that state n. In other words, the rate of entering and the rate of leaving a particular state are the same for every state-that is, ratein=rate out.

For the general Birth-and-Death processes, the basic difference differential equations at a steady state are:

$$\begin{aligned} 0 &= \mu_{n+1}P_{n+1} + \lambda_{n-1}P_{n-1} - (\lambda_n + \mu_n)P_n, \quad n \geq 1 \\ &\text{and} \\ 0 &= \mu_1 P_1 - \lambda_0 P_0, \quad \text{for} \quad n = 0. \end{aligned} \tag{*}$$

We now use equation (*) to solve iteratively special cases of λ_n and μ_n in relation to queueing systems.

4.2 M|M|1|GD|∞|∞ Queuing Process

An $M|M|1|GD|\infty|\infty$ queueing syatem uses Poisson arrivals or exponential inter-arrivals and exponential service time distribution, one parallel server and the queue discipline is general.

The number of customers allowed in the system (in queue plus in service) and the size of the source from where the customers arrive are finite.

Thus, this implies that;

$$\lambda_n = \lambda,$$
and
$$\mu_n = \mu.$$

For a steady state to exist, we have;

$$P_n(t) = P_n,$$
and
$$\lim_{t\to\infty} P_n'(t) = 0.$$

Thus, the basic difference differential equations are:

$$0 = \mu_1 P_1 - \lambda P_0 \tag{4.1}$$
$$0 = \mu P_{n+1} + \lambda P_{n-1} - (\lambda + \mu) P_n, \quad n \geq 1 \tag{4.2}$$

Hence,
Proposition 4.1

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right), \quad \text{for} \quad \frac{\lambda}{\mu} < 1. \tag{4.3}$$

Proof
Equation (4.1) above can be expressed as;

$$\lambda P_0 = \mu P_1$$
$$\Rightarrow \frac{\lambda}{\mu} P_0 = P_1$$
$$\therefore P_1 = \frac{\lambda}{\mu} P_0. \tag{4.4}$$

Solving equation (4.2) iteratively, we have;
When n=1;
Equation (4.2) becomes;

$$0=\mu P_2+\lambda P_0-(\lambda+\mu)P_1 \tag{4.5}$$

Substituting (4.4) in equation (4.5), we get;

$$\begin{aligned}
0&=\mu P_2+\lambda P_0-(\lambda+\mu)\frac{\lambda}{\mu}P_0\\
0&=\mu P_2+\lambda P_0-\lambda P_0-\frac{\lambda^2}{\mu}P_0\\
\Longleftrightarrow \frac{\lambda^2}{\mu}P_0&=\mu P_2\\
\Rightarrow \frac{\lambda^2}{\mu^2}P_0&=P_2\\
\therefore P_2&=\frac{\lambda^2}{\mu^2}P_0. \qquad (4.6)
\end{aligned}$$

When n=2;
Equation (4.2) becomes;

$$0=\mu P_3+\lambda P_1-(\lambda+\mu)P_2 \tag{4.7}$$

Substituting (4.4) and (4.6) in equation (4.7), we get;

$$\begin{aligned}
0&=\mu P_3+\lambda\frac{\lambda}{\mu}P_0-(\lambda+\mu)\frac{\lambda^2}{\mu^2}P_0\\
0&=\mu P_3+\frac{\lambda^2}{\mu}P_0-\frac{\lambda^2}{\mu}P_0-\frac{\lambda^3}{\mu^2}P_0\\
0&=\mu P_3-\frac{\lambda^3}{\mu^2}P_0
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\iff \frac{\lambda^3}{\mu^2} P_0 &= \mu P_3 \\
\Rightarrow \frac{\lambda^3}{\mu^3} P_0 &= P_3 \\
\therefore P_3 &= \frac{\lambda^3}{\mu^3} P_0.
\end{aligned} \tag{4.8}
$$

When n=3;
Equation (4.2) becomes;

$$
0 = \mu P_4 + \lambda P_2 - (\lambda + \mu) P_3 \tag{4.9}
$$

Substituting (4.6) and (4.8) in equation (4.9), we get;

$$
\begin{aligned}
0 &= \mu P_4 + \frac{\lambda^3}{\mu^2} P_0 - (\lambda + \mu) \frac{\lambda^3}{\mu^3} P_0 \\
0 &= \mu P_4 - \frac{\lambda^3}{\mu^2} P_0 + \frac{\lambda^3}{\mu^2} P_0 - \frac{\lambda^4}{\mu^3} P_0 \\
0 &= \mu P_4 - \frac{\lambda^4}{\mu^3} P_0 \\
\iff \frac{\lambda^4}{\mu^3} P_0 &= \mu P_4 \\
\Rightarrow \frac{\lambda^4}{\mu^4} P_0 &= P_4 \\
\therefore P_4 &= \left(\frac{\lambda}{\mu}\right)^4 P_0.
\end{aligned}
$$

Thus,

$$P_5 = \left(\frac{\lambda}{\mu}\right)^5 P_0,$$
$$P_6 = \left(\frac{\lambda}{\mu}\right)^6 P_0,$$
$$\vdots$$
$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0.$$

Hence,

$$\sum_{n=0}^{\infty} P_n = 1$$
$$\Rightarrow P_0 + \left(\frac{\lambda}{\mu}\right)^1 P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \left(\frac{\lambda}{\mu}\right)^3 P_0 + \left(\frac{\lambda}{\mu}\right)^4 P_0 + \ldots + \left(\frac{\lambda}{\mu}\right)^n P_0 + \ldots = 1$$
$$P_0 \left\{1 + \left(\frac{\lambda}{\mu}\right)^1 + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \left(\frac{\lambda}{\mu}\right)^4 + \ldots + \left(\frac{\lambda}{\mu}\right)^n\right\} = 1$$

Note that,

$$\frac{1}{1-s} = 1 + s + s^2 + s^3 + s^4 + \ldots$$

Therefore,

$$P_0 \left\{\frac{1}{1-\frac{\lambda}{\mu}}\right\} = 1$$
$$\therefore P_0 = 1 - \frac{\lambda}{\mu}. \qquad \textbf{(4.10)}$$
$$\Rightarrow P_1 = \left(\frac{\lambda}{\mu}\right)^1 \left(1 - \frac{\lambda}{\mu}\right),$$
$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 \left(1 - \frac{\lambda}{\mu}\right),$$

Therefore,

$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 (1 - \frac{\lambda}{\mu}\Big),$$
$$P_4 = \left(\frac{\lambda}{\mu}\right)^4 \left(1 - \frac{\lambda}{\mu}\right),$$
$$\cdot$$
$$\cdot$$
$$\cdot$$

By Mathematical induction, we have that;

$$\therefore P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right), \quad \text{for} \quad \frac{\lambda}{\mu} < 1; \quad \text{or} \quad \lambda < \mu; \quad n = 0, 1, 2, \ldots$$

Which is a Geometric Distribution with parameter $\frac{\lambda}{\mu}$.
Using Proposition 2.1
We have;

$$\begin{aligned}
P_n &= \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \ldots \mu_n} P_0 \\
\text{But,} \quad \lambda_0 &= \lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} = \lambda, \\
&\text{and} \\
\mu_0 &= \mu_1 = \mu_2 = \mu_3 = \ldots \mu_n = \mu. \\
\therefore P_n &= \left(\frac{\lambda}{\mu}\right)^n P_0.
\end{aligned}$$

But from (4.10), we have found that;

$$\begin{aligned}
P_0 &= 1 - \frac{\lambda}{\mu}. \\
&\text{Therefore,} \\
P_n &= \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right).
\end{aligned}$$

Which is a Geometric Distribution with parameter $\frac{\lambda}{\mu}$.

Using Proposition 2.2

We have;

$$\frac{P_n}{P_{n-1}} = \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0$$

$$\text{But,} \quad \lambda_{n-1} = \lambda,$$

$$\text{and}$$

$$\mu_n = \mu.$$

$$\therefore \frac{P_n}{P_{n-1}} = \frac{\lambda}{n\mu}.$$

$$\Rightarrow \mu P_n = \lambda P_{n-1} \tag{4.11}$$

Using the probability generating function technique, we multiply (4.11) by s^n and sum the results over n to obtain;

$$\sum_{n=1}^{\infty} \mu P_n s^n = \sum_{n=1}^{\infty} \lambda P_{n-1} s^n$$

$$\mu \sum_{n=1}^{\infty} P_n s^n = \lambda s \sum_{n=1}^{\infty} P_{n-1} s^{n-1}$$

$$\Longleftrightarrow \mu \psi(s) = \lambda s \psi(s)$$

$$\Rightarrow (\mu - \lambda s)\psi(s) - P_0 = 0$$

$$\Rightarrow (1 - \frac{\lambda}{\mu} s)\psi(s) = P_0$$

$$\text{But,} \quad P_0 = 1 - \frac{\lambda}{\mu}.$$

$$\therefore \psi(s) = \frac{1 - \frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu} s}.$$

$$\psi'(s) = \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \frac{\frac{\lambda}{\mu}(1 - \frac{\lambda}{\mu})}{(1 - \frac{\lambda}{\mu} s)^2}$$

$$\psi''(s) = \frac{\mathrm{d}\psi'(s)}{\mathrm{d}s} = \frac{2(\frac{\lambda}{\mu})^2 1 - \frac{\lambda}{\mu}}{(1 - \frac{\lambda}{\mu} s)^3}$$

Therefore,

$$
\begin{aligned}
E(X) &= \psi^{'}(1) \\
\therefore E(X) &= \frac{\frac{\lambda}{\mu}}{(1-\frac{\lambda}{\mu})} \\
Var(X) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= \frac{2(\frac{\lambda}{\mu})^2(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu})^3} + \frac{\frac{\lambda}{\mu}(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu})^2} - [\frac{\frac{\lambda}{\mu}(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu})^2}]^2 \\
&= \frac{2(\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu})^2} + \frac{\frac{\lambda}{\mu}}{(1-\frac{\lambda}{\mu})} - [\frac{(\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu})^2}] \\
&= \frac{(\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu})^2} + \frac{\frac{\lambda}{\mu}}{(1-\frac{\lambda}{\mu})} \\
&= \frac{(\frac{\lambda}{\mu})^2 + \frac{\lambda}{\mu} - (\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu})^2} \\
\therefore Var(X) &= \frac{\frac{\lambda}{\mu}}{(1-\frac{\lambda}{\mu})^2}.
\end{aligned}
$$

From Kummer's confluent hypergeometric series, we have;

$$
\begin{aligned}
{}_1F_1(1;1;\frac{\lambda}{\mu}) &= 1 + \frac{1}{1}\frac{\frac{\lambda}{\mu}}{1!} + \frac{1(1+1)}{1(1+1)}\frac{(\frac{\lambda}{\mu})^2}{2!} + \ldots + \frac{(1+n-1)}{(1+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1(1+1)(1+2)\ldots(1+n-1)}{1(1+1)(1+2)\ldots(1+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(1+n-1)(1+n-2)\ldots(1+2)(1+1)1}{(1+n-1)(1+n-2)\ldots(1+2)(1+1)1}\frac{\Gamma(1)}{\Gamma(1)}.\frac{\Gamma(1)}{\Gamma(1)}.\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(1+n)}{\Gamma(1+n)}\frac{\Gamma(1)}{\Gamma(1)}.\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}
$$

Normalizing, we have;

$$
1 = \sum_{n=0}^{\infty} \frac{\Gamma(1+n)}{\Gamma(1+n)}\frac{\Gamma(1)}{\Gamma(1)}.\frac{1}{{}_1F_1(1;1;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}
$$

Therefore,

$$
\begin{aligned}
P_n &= \text{Prob.}(N=n)\\
&= \frac{\Gamma(1+n)}{\Gamma(1+n)}\frac{\Gamma(1)}{\Gamma(1)}.\frac{1}{{}_1F_1(1;1;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}
$$

In hypergeometric terms, its pgf. is given as;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty} P_n z^n\\
&= \frac{\Gamma(1+n)}{\Gamma(1+n)}\frac{\Gamma(1)}{\Gamma(1)}\frac{(\frac{\lambda}{\mu}z)^n}{n!}.\frac{1}{{}_1F_1(1;1;\frac{\lambda}{\mu})}\\
\therefore \phi(z) &= \frac{{}_1F_1(1;1;\frac{\lambda}{\mu}z)}{{}_1F_1(1;1;\frac{\lambda}{\mu})}.\\
\phi^{'}(z) &= \frac{\lambda}{\mu}\frac{{}_1F_1(1+1;1+1;\frac{\lambda}{\mu}z)}{{}_1F_1(1;1;\frac{\lambda}{\mu})}\\
\phi^{''}(z) &= (\frac{\lambda}{\mu})^2\frac{{}_1F_1(2+1;2+1;\frac{\lambda}{\mu}z)}{{}_1F_1(1;1;\frac{\lambda}{\mu})}\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_1F_1(1+\kappa;1+\kappa;\frac{\lambda}{\mu})}{{}_1F_1(1;1;\frac{\lambda}{\mu})};\quad \kappa = 1,2.\\
\therefore E(x) &= \frac{\lambda}{\mu}\Lambda_1.\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= (\frac{\lambda}{\mu})^2\Lambda_2+\frac{\lambda}{\mu}\Lambda_1-(\frac{\lambda}{\mu})^2\Lambda_1^2\\
\therefore Var(X) &= \frac{\lambda}{\mu}\Lambda_1+(\frac{\lambda}{\mu})^2(\Lambda_2-\Lambda_1^2).
\end{aligned}
$$

4.3 M|M|∞ Queueing System

Suppose that infinitely many trunks or channels are available, and that the probability of a conversation ending between t and $t+\Delta t$ is $\mu\Delta t + 0(\Delta t)$ (exponential holding time). The incoming calls constitute a traffic of the Poisson type with parameter λ. Assume that

the durations of the conversations are mutually independent.
If n lines are busy, the probability that one of them will be free within time Δt is then $n\mu\Delta t + 0(\Delta t)$.
The probability of a new call arriving is $\lambda\Delta t + 0(\Delta t)$. The probability of a combination of several calls, or of a call arriving and a conversation ending is $0(\Delta t)$.
Thus, we have;

$$\lambda_n = \lambda, \quad n = 0, 1, 2, 3, \ldots$$
and
$$\mu_n = n\mu, \quad n = 1, 2, 3, \ldots$$

At a steady state, $t \to \infty$ implying that $P_n(t)$ is idependent of t.
Thus,

$$P_n(t) = P_n,$$
and
$$\lim_{t\to\infty} P_n'(t) = 0.$$

Therefore, the basic difference differential equations for the steady state are given by;

$$0 = \mu P_1 - \lambda P_0, \tag{4.12}$$
$$0 = (n+1)\mu P_{n+1} + \lambda P_{n-1} - (\lambda + n\mu)P_n. \tag{4.13}$$

Hence,
Proposition 4.2

$$P_n = e^{-\frac{\lambda}{\mu}} \frac{(\frac{\lambda}{\mu})^n}{n!}, \quad n = 0, 1, 2, \ldots \tag{4.14}$$

Proof
Equation (4.12) can be expressed as;

$$\lambda P_0 = \mu P_1$$
$$\Rightarrow \frac{\lambda}{\mu} P_0 = P_1$$
$$\therefore P_1 = \frac{\lambda}{\mu} P_0. \tag{4.15}$$

Solving steady state equations iteratively, we have;
When n=1;
Equation (4.13) becomes;

$$0 = 2\mu P_2 + \lambda P_0 - (\lambda + \mu) P_1 \tag{4.16}$$

Substituting (4.15) in equation (4.16), we get;

$$0 = 2\mu P_2 + \lambda P_0 - (\lambda + \mu) \frac{\lambda}{\mu} P_0$$
$$0 = 2\mu P_2 + \lambda P_0 - \lambda P_0 - \frac{\lambda^2}{\mu} P_0$$
$$0 = 2\mu P_2 - \frac{\lambda^2}{\mu} P_0$$
$$\Longleftrightarrow \frac{\lambda^2}{\mu} P_0 = 2\mu P_2$$
$$\Rightarrow \frac{\lambda^2}{2!\mu^2} P_0 = P_2$$
$$\therefore P_2 = \frac{\lambda^2}{2!\mu^2} P_0. \tag{4.17}$$

When n=2;
Equation (4.13) becomes;

$$0 = 3\mu P_3 + \lambda P_1 - (\lambda + 2\mu) P_2 \tag{4.18}$$

Substituting (4.15) and (4.17) in equation (4.18), we get;

$$
\begin{aligned}
0 &= 3\mu P_3 + \frac{\lambda^2}{\mu}P_0 - (\lambda + 2\mu)\frac{\lambda^2}{2!\mu^2}P_0 \\
0 &= 3\mu P_3 + \frac{\lambda^2}{\mu}P_0 - \frac{\lambda^2}{\mu}P_0 - \frac{\lambda^3}{2!\mu^2}P_0 \\
0 &= 3\mu P_3 - \frac{\lambda^3}{2!\mu^2}P_0 \\
\iff \frac{\lambda^3}{2!\mu^2}P_0 &= 3\mu P_3 \\
\Rightarrow \frac{\lambda^3}{3!\mu^3}P_0 &= P_3 \\
\therefore P_3 &= \frac{\lambda^3}{3!\mu^3}P_0. \qquad (4.19)
\end{aligned}
$$

When n=3;
Equation (4.13) becomes;

$$
0 = 4\mu P_4 + \lambda P_2 - (\lambda + 3\mu)P_3 \qquad (4.20)
$$

Substituting (4.17) and (4.19) in equation (4.20), we get;

$$
\begin{aligned}
0 &= 4\mu P_4 + \frac{\lambda^3}{2!\mu^2}P_0 - (\lambda + 3\mu)\frac{\lambda^3}{3!\mu^3}P_0 \\
0 &= 4\mu P_4 + \frac{\lambda^3}{2!\mu^2}P_0 - \frac{\lambda^3}{2!\mu^2}P_0 - \frac{\lambda^4}{3!\mu^3}P_0 \\
0 &= 4\mu P_4 - \frac{\lambda^4}{3!\mu^3}P_0 \\
\iff \frac{\lambda^4}{3!\mu^3}P_0 &= 4\mu P_4 \\
\Rightarrow \frac{\lambda^4}{4!\mu^4}P_0 &= P_4 \\
\therefore P_4 &= \frac{\lambda^4}{4!\mu^4}P_0.
\end{aligned}
$$

Therefore;

$$
\begin{aligned}
P_5 &= \frac{\lambda^5}{5!\mu^5}P_0,\\
P_6 &= \frac{\lambda^6}{6!\mu^6}P_0,\\
&.\\
&.\\
&.\\
\therefore P_n &= \frac{\lambda^n}{n!\mu^n}P_0. \qquad \textbf{(4.21)}
\end{aligned}
$$

Note that;

$$
\begin{aligned}
\sum_{n=0}^{\infty} P_n &= 1\\
\Rightarrow P_0\Big(\sum_{n=0}^{\infty}\frac{(\frac{\lambda}{\mu})^n}{n!}\Big) &= 1\\
P_0\Big(1+\frac{(\frac{\lambda}{\mu})^1}{1!}+\frac{(\frac{\lambda}{\mu})^2}{2!}+\frac{(\frac{\lambda}{\mu})^3}{3!}+\frac{(\frac{\lambda}{\mu})^4}{4!}+...\Big) &= 1
\end{aligned}
$$

But,

$$
\begin{aligned}
e^x &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+...\\
\Rightarrow P_0\Big(e^{\frac{\lambda}{\mu}}\Big) &= 1\\
\therefore P_0 &= e^{-\frac{\lambda}{\mu}}. \qquad \textbf{(4.22)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P_1 &= \frac{(\frac{\lambda}{\mu})^1}{1!} e^{-\frac{\lambda}{\mu}}, \\
P_2 &= \frac{(\frac{\lambda}{\mu})^2}{2!} e^{-\frac{\lambda}{\mu}}, \\
P_3 &= \frac{(\frac{\lambda}{\mu})^3}{3!} e^{-\frac{\lambda}{\mu}}, \\
P_4 &= \frac{(\frac{\lambda}{\mu})^4}{4!} e^{-\frac{\lambda}{\mu}}, \\
&. \\
&. \\
&.
\end{aligned}
$$

We find by Mathematical induction that;

$$
\begin{aligned}
P_n &= \frac{(\frac{\lambda}{\mu})^n}{n!} P_0. \\
\text{Hence,} \quad P_n &= \frac{(\frac{\lambda}{\mu})^n}{n!} e^{-\frac{\lambda}{\mu}}, \quad n = 0, 1, 2, 3, \ldots
\end{aligned}
$$

Thus, the limiting distribution is a Poisson distribution with parameter $\frac{\lambda}{\mu}$. It is independent of the initial state.

Using Proposition (2.1)

We have;

$$
\begin{aligned}
P_n &= \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \ldots \mu_n} P_0 \\
\text{But,} \quad \lambda_0 &= \lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} = \lambda, \\
&\text{and} \\
\mu_1 &= \mu, \quad \mu_2 = 2\mu, \quad \mu_3 = 3\mu, \quad \ldots, \quad \mu_n = n\mu. \\
\therefore P_n &= \frac{\lambda \lambda \lambda \lambda \ldots \lambda}{\mu 2\mu 3\mu \ldots n\mu} P_0 \\
\therefore P_n &= \frac{\lambda^n}{n! \mu^n} P_0
\end{aligned}
$$

But, from (3.11) above;

$$
\begin{aligned}
P_0 &= e^{-\frac{\lambda}{\mu}} \\
\therefore P_n &= \frac{\lambda^n}{n!\mu^n} e^{-\frac{\lambda}{\mu}}.
\end{aligned}
$$

Which is a Poisson distribution with parameter $\frac{\lambda}{\mu}$.
Using Proposition (2.2)

$$
\begin{aligned}
\frac{P_n}{P_{n-1}} &= \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0 \\
\text{But,} \quad \lambda_{n-1} &= \lambda, \\
\text{and} & \\
\mu_n &= n\mu. \\
\therefore \frac{P_n}{P_{n-1}} &= \frac{\lambda}{n\mu}. \\
\Rightarrow n\mu P_n &= \lambda P_{n-1} \qquad \textbf{(4.23)}
\end{aligned}
$$

Multiplying (4.23) by s^n and sum the results over n we obtain;

$$
\begin{aligned}
\sum_{n=0}^{\infty} n\mu P_n s^n &= \sum_{n=0}^{\infty} \lambda P_{n-1} s^n \\
\mu s \sum_{n=0}^{\infty} n P_n s^{n-1} &= \lambda s \sum_{n=0}^{\infty} P_{n-1} s^{n-1} \\
s\mu \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= s\lambda \psi(s) \\
\frac{\mathrm{d}\psi(s)}{\psi(s)} &= \frac{\lambda}{\mu} \mathrm{d}s
\end{aligned}
$$

Integrating both sides, we have;

$$\begin{aligned}
\int \frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int \frac{\lambda}{\mu}\mathrm{d}s \\
\ln\psi(s) &= \frac{\lambda}{\mu}s + c \\
\psi(s) &= c_1 e^{\frac{\lambda}{\mu}s}
\end{aligned}$$

Putting s=1;

$$\begin{aligned}
\psi(1) = 1 &= c_1 e^{\frac{\lambda}{\mu}} \\
\Rightarrow c_1 &= e^{-\frac{\lambda}{\mu}}. \\
\therefore \psi(s) &= e^{-\frac{\lambda}{\mu}} e^{\frac{\lambda}{\mu}s} \\
\therefore \psi(s) &= e^{-\frac{\lambda}{\mu}(1-s)}.
\end{aligned}$$

Which is the pgf. for a Poisson distribution with parameter $\frac{\lambda}{\mu}$.

$$\begin{aligned}
\psi^{'}(s) &= \frac{\lambda}{\mu} e^{-\frac{\lambda}{\mu}(1-s)} \\
\psi^{''}(s) &= (\frac{\lambda}{\mu})^2 e^{-\frac{\lambda}{\mu}(1-s)} \\
E(X) &= \psi^{'}(1) \\
\therefore E(X) &= \frac{\lambda}{\mu}. \\
Var(x) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= (\frac{\lambda}{\mu})^2 + \frac{\lambda}{\mu} - [\frac{\lambda}{\mu}]^2 \\
\therefore Var(X) &= \frac{\lambda}{\mu}.
\end{aligned}$$

From Kummer's confluent hypergeometric series, we have;

$$
\begin{aligned}
{}_1F_1(\varepsilon;\varepsilon;\frac{\lambda}{\mu}) &= 1+\frac{\varepsilon}{\varepsilon}\frac{\frac{\lambda}{\mu}}{1!}+\frac{\varepsilon(\varepsilon+1)}{\varepsilon(\varepsilon+1)}\frac{(\frac{\lambda}{\mu})^2}{2!}+\ldots+\frac{(\varepsilon+n-1)}{(\varepsilon+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{\varepsilon(\varepsilon+1)(\varepsilon+2)\ldots(\varepsilon+n-1)}{\varepsilon(\varepsilon+1)(\varepsilon+2)\ldots(\varepsilon+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{(\varepsilon+n-1)(\varepsilon+n-2)\ldots(\varepsilon+2)(\varepsilon+1)\varepsilon}{(\varepsilon+n-1)(\varepsilon+n-2)\ldots(\varepsilon+2)(\varepsilon+1)\varepsilon}\frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon)}.\frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon)}.\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\varepsilon+n)}{\Gamma(\varepsilon+n)}\frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon)}.\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}
$$

Normalizing, we have;

$$
1 = \sum_{n=0}^{\infty}\frac{\Gamma(\varepsilon+n)}{\Gamma(\varepsilon+n)}\frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon)}.\frac{1}{{}_1F_1(\varepsilon;\varepsilon;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}
$$

Therefore,

$$
\begin{aligned}
P_n &= \text{Prob.}(N=n) \\
&= \frac{\Gamma(\varepsilon+n)}{\Gamma(\varepsilon+n)}\frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon)}.\frac{1}{{}_1F_1(\varepsilon;\varepsilon;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}
$$

In hypergeometric terms, its pgf. is given as;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty}P_n z^n \\
&= \frac{\Gamma(\varepsilon+n)}{\Gamma(\varepsilon+n)}\frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon)}\frac{(\frac{\lambda}{\mu}z)^n}{n!}.\frac{1}{{}_1F_1(\varepsilon;\varepsilon;\frac{\lambda}{\mu})} \\
\therefore \phi(z) &= \frac{{}_1F_1(\varepsilon;\varepsilon;\frac{\lambda}{\mu}z)}{{}_1F_1(\varepsilon;\varepsilon;\frac{\lambda}{\mu})}.
\end{aligned}
$$

Therefore,

$$\phi^{'}(z) = \frac{\lambda}{\mu}\frac{{}_1F_1(\varepsilon+1;\varepsilon+1;\frac{\lambda}{\mu}z)}{{}_1F_1(\varepsilon;\varepsilon;\frac{\lambda}{\mu})}$$

$$\phi^{''}(z) = (\frac{\lambda}{\mu})^2\frac{{}_1F_1(\varepsilon+2;\varepsilon+2;\frac{\lambda}{\mu}z)}{{}_1F_1(\varepsilon;\varepsilon;\frac{\lambda}{\mu})}$$

$$\text{Let,}\quad \Lambda_\kappa = \frac{{}_1F_1(\varepsilon+\kappa;\varepsilon+\kappa;\frac{\lambda}{\mu})}{{}_1F_1(\varepsilon;\varepsilon;\frac{\lambda}{\mu})};\quad \kappa = 1,2.$$

$$\therefore E(x) = \frac{\lambda}{\mu}\Lambda_1.$$

$$Var(X) = \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2$$

$$= (\frac{\lambda}{\mu})^2\Lambda_2+\frac{\lambda}{\mu}\Lambda_1-(\frac{\lambda}{\mu})^2\Lambda_1^2$$

$$\therefore Var(X) = \frac{\lambda}{\mu}\Lambda_1+(\frac{\lambda}{\mu})^2(\Lambda_2-\Lambda_1^2).$$

4.4 M|M|S|GD|∞ Queuing Process

An M|M|S|GD|∞ Queuing Process uses Poisson arrivals and exponential service time distribution with s parallel servers.
The queue discipline is general and the number of customers in the system and the size of their source or where the customers come from are infinite.
In this case,

$$\lambda_n = \lambda,\quad \text{for}\quad n = 0,1,2,3,\ldots$$

$$\mu_n = \min(n,s)\quad \text{for}\quad n = 1,2,3,\ldots$$

$$\Rightarrow \mu_n = n\mu,\quad (n < s)$$

$$\mu_n = s\mu,\quad (n \geq s).$$

The basic difference differential equations for the steady state are:

$$0 = \mu P_1 - \lambda P_0 \tag{4.24}$$
$$0 = (n+1)\mu P_{n+1} + \lambda P_{n-1} - (\lambda + n\mu)P_n, \quad \text{for} \quad n < s \tag{4.25}$$
$$0 = s\mu P_{n+1} + \lambda P_{n-1} - (\lambda + s\mu)P_n, \quad \text{for} \quad n \geq s \tag{4.26}$$

Hence,
Proposition 4.3

$$P_n = \frac{\lambda^n}{n!\mu^n} e^{-\frac{\lambda}{\mu}}, \quad \text{for} \quad n < s$$

Proof
Solving the steady state equations iteratively, we have;
Equation (4.24) can be expressed as;

$$\lambda P_0 = \mu P_1$$
$$\Rightarrow \frac{\lambda}{\mu} P_0 = P_1$$
$$\therefore P_1 = \frac{\lambda}{\mu} P_0. \tag{4.27}$$

When n=1;
Equation (4.25) becomes;

$$0 = 2\mu P_2 + \lambda P_0 - (\lambda + \mu)P_1 \tag{4.28}$$

Substituting (4.27) in equation (4.28), we have;

$$
\begin{aligned}
0 &= 2\mu P_2 + \lambda P_0 - (\lambda+\mu)\frac{\lambda}{\mu}P_0 \\
0 &= 2\mu P_2 + \lambda P_0 - \lambda P_0 - \frac{\lambda^2}{\mu}P_0 \\
0 &= 2\mu P_2 - \frac{\lambda^2}{\mu}P_0 \\
\iff \frac{\lambda^2}{\mu}P_0 &= 2\mu P_2 \\
\Rightarrow \frac{\lambda^2}{2!\mu^2}P_0 &= P_2 \\
\therefore P_2 &= \frac{\lambda^2}{2!\mu^2}P_0. \qquad (4.29)
\end{aligned}
$$

When n=2;
Equation (4.25) becomes;

$$
0 = 3\mu P_3 + \lambda P_1 - (\lambda + 2\mu)P_2 \qquad (4.30)
$$

Substituting (4.27) and (4.29) in equation (4.30), we get;

$$
\begin{aligned}
0 &= 3\mu P_3 + \frac{\lambda^2}{\mu}P_0 - (\lambda+2\mu)\frac{\lambda^2}{2!\mu^2}P_0 \\
0 &= 3\mu P_3 + \frac{\lambda^2}{\mu}P_0 - \frac{\lambda^2}{\mu}P_0 - \frac{\lambda^3}{2!\mu^2}P_0 \\
0 &= 3\mu P_3 - \frac{\lambda^3}{2!\mu^2}P_0 \\
\iff \frac{\lambda^3}{2!\mu^2}P_0 &= 3\mu P_3 \\
\Rightarrow \frac{\lambda^3}{3!\mu^3}P_0 &= P_3 \\
\therefore P_3 &= \frac{\lambda^3}{3!\mu^3}P_0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
P_4 &= \frac{\lambda^4}{4!\mu^4}P_0, \\
P_5 &= \frac{\lambda^5}{5!\mu^5}P_0, \\
&. \\
&. \\
&. \\
P_n &= \frac{\lambda^n}{n!\mu^n}P_0. \qquad \textbf{(4.31)}
\end{aligned}
$$

Let,

$$
\begin{aligned}
\sum_{n=0}^{\infty} P_n &= 1 \\
P_0\Big(\sum_{n=0}^{\infty}\frac{\lambda^n}{n!\mu^n}\Big) &= 1 \\
P_0\Big(1+\frac{\lambda^1}{1!\mu^1}+\frac{\lambda^2}{2!\mu^2}+\frac{\lambda^3}{3!\mu^3}+\frac{\lambda^4}{4!\mu^4}+...\Big) &= 1 \\
P_0\Big(e^{\frac{\lambda}{\mu}}\Big) &= 1 \\
\therefore P_0 &= e^{-\frac{\lambda}{\mu}}. \qquad \textbf{(4.32)}
\end{aligned}
$$

From (4.31) and (4.32) above, we have;

$$
\begin{aligned}
P_n &= \frac{\lambda^n}{n!\mu^n}P_0. \\
\therefore P_n &= \frac{\lambda^n}{n!\mu^n}e^{-\frac{\lambda}{\mu}}. \quad \text{for} \quad n < s
\end{aligned}
$$

Thus, the limiting distribution is a Poisson with parameter $\frac{\lambda}{\mu}$.
Using Proposition (2.1)
We have that;

$$P_n = \frac{\lambda_0\lambda_1\lambda_2\lambda_3...\lambda_{n-1}}{\mu_1\mu_2\mu_3...\mu_n}P_0$$

But, $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = ... = \lambda_{n-1} = \lambda,$

and

$$\mu_1 = \mu, \quad \mu_2 = 2\mu, \quad \mu_3 = 3\mu, \quad ..., \quad \mu_n = n\mu.$$

$$\therefore P_n = \frac{\lambda\lambda\lambda\lambda...\lambda}{\mu 2\mu 3\mu...n\mu}P_0$$

$$= \frac{\lambda^n}{n!\mu^n}P_0$$

But, from (4.32) above;

$$P_0 = e^{-\frac{\lambda}{\mu}}$$

$$\therefore P_n = \frac{\lambda^n}{n!\mu^n}e^{-\frac{\lambda}{\mu}}.$$

Which is a Poisson distribution with parameter $\frac{\lambda}{\mu}$.

Using Proposition (2.2)

$$\frac{P_n}{P_{n-1}} = \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0$$

But, $\lambda_{n-1} = \lambda,$

and

$$\mu_n = n\mu.$$

$$\therefore \frac{P_n}{P_{n-1}} = \frac{\lambda}{n\mu}.$$

$$\Rightarrow n\mu P_n = \lambda P_{n-1} \tag{4.33}$$

Multiplying (4.33) by s^n and sum the results over n we obtain;

$$\sum_{n=0}^{\infty} n\mu P_n s^n = \sum_{n=0}^{\infty} \lambda P_{n-1} s^n$$
$$\mu s \sum_{n=0}^{\infty} nP_n s^{n-1} = \lambda s \sum_{n=0}^{\infty} P_{n-1} s^{n-1}$$
$$s\mu \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = s\lambda\,\psi(s)$$
$$\frac{\mathrm{d}\psi(s)}{\psi(s)} = \frac{\lambda}{\mu}\mathrm{d}s$$

Integrating both sides, we have;

$$\int \frac{\mathrm{d}\psi(s)}{\psi(s)} = \int \frac{\lambda}{\mu}\mathrm{d}s$$
$$\ln\psi(s) = \frac{\lambda}{\mu}s + c$$
$$\psi(s) = c_1 e^{\frac{\lambda}{\mu}s}$$

Putting s=1;

$$\psi(1) = 1 = c_1 e^{\frac{\lambda}{\mu}}$$
$$\Rightarrow c_1 = e^{-\frac{\lambda}{\mu}}.$$
$$\therefore \psi(s) = e^{-\frac{\lambda}{\mu}} e^{\frac{\lambda}{\mu}s}$$
$$\therefore \psi(s) = e^{-\frac{\lambda}{\mu}(1-s)}.$$

Which is a Poisson distribution with parameter $\frac{\lambda}{\mu}$.

$$\begin{aligned}
\psi^{'}(s) &= \frac{\lambda}{\mu}e^{-\frac{\lambda}{\mu}(1-s)} \\
\psi^{''}(s) &= (\frac{\lambda}{\mu})^2 e^{-\frac{\lambda}{\mu}(1-s)} \\
E(X) &= \psi^{'}(1) \\
\therefore E(X) &= \frac{\lambda}{\mu}. \\
Var(x) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= (\frac{\lambda}{\mu})^2 + \frac{\lambda}{\mu} - [\frac{\lambda}{\mu}]^2 \\
\therefore Var(X) &= \frac{\lambda}{\mu}.
\end{aligned}$$

From Kummer's confluent hypergeometric series, we have;

$$\begin{aligned}
{}_1F_1(\omega;\omega;\frac{\lambda}{\mu}) &= 1 + \frac{\omega}{\omega}\frac{\frac{\lambda}{\mu}}{1!} + \frac{\omega(\omega+1)}{\omega(\omega+1)}\frac{(\frac{\lambda}{\mu})^2}{2!} + ... + \frac{(\omega+n-1)}{(\omega+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\omega(\omega+1)(\omega+2)...(\omega+n-1)}{\omega(\omega+1)(\omega+2)...(\omega+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(\omega+n-1)(\omega+n-2)...(\omega+2)(\omega+1)\omega}{(\omega+n-1)(\omega+n-2)...(\omega+2)(\omega+1)\omega}\frac{\Gamma(\omega)}{\Gamma(\omega)}.\frac{\Gamma(\omega)}{\Gamma(\omega)}.\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(\omega+n)}{\Gamma(\omega+n)}\frac{\Gamma(\omega)}{\Gamma(\omega)}.\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}$$

Normalizing, we have;

$$1 = \sum_{n=0}^{\infty} \frac{\Gamma(\omega+n)}{\Gamma(\omega+n)}\frac{\Gamma(\omega)}{\Gamma(\omega)}.\frac{1}{{}_1F_1(\omega;\omega;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}$$

Therefore,

$$\begin{aligned}
P_n &= \text{Prob.}(N=n) \\
&= \frac{\Gamma(\omega+n)}{\Gamma(\omega+n)}\frac{\Gamma(\omega)}{\Gamma(\omega)}.\frac{1}{{}_1F_1(\omega;\omega;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}$$

In hypergeometric terms, its pgf. is given as;

$$\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty} P_n z^n \\
&= \frac{\Gamma(\omega+n)}{\Gamma(\omega+n)} \frac{\Gamma(\omega)}{\Gamma(\omega)} \frac{(\frac{\lambda}{\mu}z)^n}{n!} . \frac{1}{{}_1F_1(\omega;\omega;\frac{\lambda}{\mu})} \\
\therefore \phi(z) &= \frac{{}_1F_1(\omega;\omega;\frac{\lambda}{\mu}z)}{{}_1F_1(\omega;\omega;\frac{\lambda}{\mu})}. \\
\phi^{'}(z) &= \frac{\lambda}{\mu} \frac{{}_1F_1(\omega+1;\omega+1;\frac{\lambda}{\mu}z)}{{}_1F_1(\omega;\omega;\frac{\lambda}{\mu})} \\
\phi^{''}(z) &= (\frac{\lambda}{\mu})^2 \frac{{}_1F_1(\omega+2;\omega+2;\frac{\lambda}{\mu}z)}{{}_1F_1(\omega;\omega;\frac{\lambda}{\mu})} \\
\text{Let,} \quad \Lambda_\kappa &= \frac{{}_1F_1(\omega+\kappa;\omega+\kappa;\frac{\lambda}{\mu})}{{}_1F_1(\omega;\omega;\frac{\lambda}{\mu})}; \quad \kappa = 1,2. \\
\therefore E(x) &= \frac{\lambda}{\mu}\Lambda_1. \\
Var(X) &= \phi^{''}(1) + \phi^{'}(1) - [\phi^{'}(1)]^2 \\
&= (\frac{\lambda}{\mu})^2\Lambda_2 + \frac{\lambda}{\mu}\Lambda_1 - (\frac{\lambda}{\mu})^2\Lambda_1^2 \\
\therefore Var(X) &= \frac{\lambda}{\mu}\Lambda_1 + (\frac{\lambda}{\mu})^2(\Lambda_2 - \Lambda_1^2).
\end{aligned}$$

Next,
When n=1;
Equation (4.26) becomes;

$$0 = s\mu P_2 + \lambda P_0 - (\lambda + s\mu)P_1 \tag{4.34}$$

Substituting (4.27) in equation (4.34), we have;

$$
\begin{aligned}
0 &= s\mu P_2 + \lambda P_0 - (\lambda + s\mu)\frac{\lambda}{\mu}P_0 \\
0 &= s\mu P_2 + \lambda P_0 - s\lambda P_0 - \frac{\lambda^2}{\mu}P_0 \\
0 &= s\mu P_2 - \lambda(s-1)P_0 - \frac{\lambda^2}{\mu}P_0
\end{aligned}
$$

$$\iff \frac{\lambda^2}{\mu}P_0 + \lambda(s-1)P_0 = s\mu P_2$$

$$\Rightarrow \frac{1}{s}\Big(\frac{\lambda^2}{\mu^2} + \frac{\lambda}{\mu}(s-1)\Big)P_0 = P_2$$

$$\therefore P_2 = \frac{1}{s}\Big(\frac{\lambda^2}{\mu^2} + \frac{\lambda}{\mu}(s-1)\Big)P_0. \tag{4.35}$$

When n=2;
Equation (4.26) becomes;

$$0 = s\mu P_3 + \lambda P_1 - (\lambda + s\mu)P_2 \tag{4.36}$$

Substituting (4.27) and (4.35) in equation (4.36), we get;

$$
\begin{aligned}
0 &= s\mu P_3 + \frac{\lambda^2}{\mu}P_0 - (\lambda + s\mu)\frac{1}{s}\Big(\frac{\lambda^2}{\mu^2} + \frac{\lambda}{\mu}(s-1)\Big)P_0 \\
0 &= s\mu P_3 + \frac{\lambda^2}{\mu}P_0 - \lambda(s-1)P_0 - \frac{\lambda^2}{\mu}P_0 - \frac{\lambda^2}{s\mu}(s-1)P_0 - \frac{\lambda^3}{s\mu^2}P_0 \\
0 &= s\mu P_3 - \lambda(s-1)P_0 - \frac{\lambda^2}{s\mu}(s-1)P_0 - \frac{\lambda^3}{s\mu^2}P_0
\end{aligned}
$$

$$\iff \lambda(s-1)P_0 + \frac{\lambda^2}{s\mu}(s-1)P_0 + \frac{\lambda^3}{s\mu^2}P_0 = s\mu P_3$$

$$\Rightarrow \frac{1}{s^2}\Big(\frac{\lambda}{\mu}s(s-1) + \frac{\lambda^2}{\mu^2}(s-1) + \frac{\lambda^3}{\mu^3}\Big)P_0 = P_3$$

$$\therefore P_3 = \frac{1}{s^2}\Big(\frac{\lambda^3}{\mu^3} + \frac{\lambda^2}{\mu^2}(s-1) + \frac{\lambda}{\mu}s(s-1)\Big)P_0.$$

Thus,

$$P_4 = \frac{1}{s^3}\Big(\frac{\lambda^4}{\mu^4} + \frac{\lambda^3}{\mu^3}(s-1) + \frac{\lambda^2}{\mu^2}s(s-1) + \frac{\lambda}{\mu}s(s-1)(s-2)\Big)P_0,$$

$$\vdots$$

$$P_n = \frac{1}{s^{n-s}}\Big(\frac{\lambda^n}{\mu^n} + \frac{\lambda^{n-1}}{\mu^{n-1}}(s-1) + \ldots + \frac{\lambda}{\mu}s(s-1)(s-2)\ldots(s-n)\Big)P_0. \quad \text{for} \quad n \geq s$$

Assume that $\frac{\lambda}{\mu} < s$, we have;
Then,

$$\sum_{n=0}^{\infty} P_n = 1$$

$$P_0\Big(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!\mu^n}\Big) = 1$$

$$P_0\Big(1 + \frac{\lambda^1}{1!\mu^1} + \frac{\lambda^2}{2!\mu^2} + \frac{\lambda^3}{3!\mu^3} + \frac{\lambda^4}{4!\mu^4} + \ldots\Big) = 1$$

$$P_0\Big(e^{\frac{\lambda}{\mu}}\Big) = 1$$

$$\therefore P_0 = e^{-\frac{\lambda}{\mu}}.$$

We find that;

$$P_n = \frac{(\frac{\lambda}{\mu})^n}{s!s^{n-s}}e^{-\frac{\lambda}{\mu}}. \quad \text{for} \quad n \geq s \quad (\frac{\lambda}{\mu} < s)$$

Remark 4.2
(1) If the series $\sum(\frac{P_n}{P_0})$ converges only in $\frac{\lambda}{\mu} < s$. i.e., $\sum P_n = 1$.
(2) If $\frac{\lambda}{\mu} \geq s$, a limiting distribution P_n cannot exists. In this case $P_n = 0$ for all n, which means that gradually the waiting line grows over all bounds.

4.5 M|M|1|GD|c|∞ Queueing System

The M|M|1|GD|c|∞ queuing process uses a Poisson arrivals or exponential intervals exponential service time distribution with one server, the queueing discipline is general, c customers are allowed in the system (in queue plus in service) and the size of the source from where the customers arrive is finite.
In this case, the system can accomodate only c units so that a unit/customer which arrives when the system is in state E_c is not allowed into the system.
System is said to be in state E_n if there are exactly n units/persons either being served or in the waiting line, $(n \leq c-1)$.
Here,

$$\begin{aligned}
&\lambda_n = \lambda, \quad \text{for} \quad n = 0,1,2,3,...,c-1 \\
&\text{and} \\
&\mu_n = n\mu, \quad \text{for} \quad n \leq c-1 \\
&\mu_n = \mu, \quad \text{for} \quad n = 1,2,3,...,c.
\end{aligned}$$

For the steady state problem, $t \to \infty$ implying that $P_n(t)$ is independent of t.
Thus,

$$\begin{gathered}
P_n(t) = P_n, \\
\text{and} \\
\lim_{t\to\infty} P_n'(t) = 0.
\end{gathered}$$

Hence the basic difference differential equations for the steady state, at state $E_n(n \leq c-1)$, are:

$$0 = \mu P_1 - \lambda P_0, \quad \text{for} \quad n = 0 \tag{4.37}$$

$$0 = (n+1)\mu P_{n+1} + \lambda P_{n-1} - (\lambda + n\mu)P_n; \quad \text{for} \quad n \leq c-1. \tag{4.38}$$

Hence,
Proposition 4.4

$$P_n = e^{-\frac{\lambda}{\mu}} \frac{(\frac{\lambda}{\mu})^n}{n!} \tag{4.39}$$

Proof
Equation (4.37) can be expressed as;

$$\begin{aligned} \lambda P_0 &= \mu P_1 \\ \Rightarrow \frac{\lambda}{\mu} P_0 &= P_1 \\ \therefore P_1 &= \frac{\lambda}{\mu} P_0. \end{aligned} \tag{4.40}$$

Solving steady state equation iteratively, we have;
When n=1;
Equation (4.38) becomes;

$$0 = 2\mu P_2 + \lambda P_0 - (\lambda + \mu) P_1 \tag{4.41}$$

Substituting (4.40) in equation (4.41), we get;

$$\begin{aligned} 0 &= 2\mu P_2 + \lambda P_0 - (\lambda + \mu) \frac{\lambda}{\mu} P_0 \\ 0 &= 2\mu P_2 + \lambda P_0 - \lambda P_0 - \frac{\lambda^2}{\mu} P_0 \\ 0 &= 2\mu P_2 - \frac{\lambda^2}{\mu} P_0 \\ \iff \frac{\lambda^2}{\mu} P_0 &= 2\mu P_2 \\ \Rightarrow \frac{\lambda^2}{2!\mu^2} P_0 &= P_2 \\ \therefore P_2 &= \frac{\lambda^2}{2!\mu^2} P_0. \end{aligned} \tag{4.42}$$

When n=2;
Equation (4.38) becomes;

$$0 = 3\mu P_3 + \lambda P_1 - (\lambda + 2\mu) P_2 \tag{4.43}$$

Substituting (4.40) and (4.42) in equation (4.43), we get;

$$
\begin{aligned}
0 &= 3\mu P_3 + \frac{\lambda^2}{\mu}P_0 - (\lambda + 2\mu)\frac{\lambda^2}{2!\mu^2}P_0 \\
0 &= 3\mu P_3 + \frac{\lambda^2}{\mu}P_0 - \frac{\lambda^2}{\mu}P_0 - \frac{\lambda^3}{2!\mu^2}P_0 \\
0 &= 3\mu P_3 - \frac{\lambda^3}{2!\mu^2}P_0 \\
\Longleftrightarrow \frac{\lambda^3}{2!\mu^2}P_0 &= 3\mu P_3 \\
\Rightarrow \frac{\lambda^3}{3!\mu^3}P_0 &= P_3 \\
\therefore P_3 &= \frac{\lambda^3}{3!\mu^3}P_0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
P_4 &= \frac{\lambda^4}{4!\mu^4}P_0, \\
P_5 &= \frac{\lambda^5}{5!\mu^5}P_0, \\
&. \\
&. \\
&. \\
P_n &= \frac{(\frac{\lambda}{\mu})^n}{n!}P_0. \qquad (4.44)
\end{aligned}
$$

Let,

$$\sum_{n=0}^{c-1} P_n = 1$$

As $c - 1 \to \infty$.
Then,

$$
\begin{aligned}
P_0 + P_1 + P_2 + P_3 + ... + P_{c-1} &= 1 \\
P_0\Big(1 + \frac{(\frac{\lambda}{\mu})^1}{1!} + \frac{(\frac{\lambda}{\mu})^2}{2!} + \frac{(\frac{\lambda}{\mu})^3}{3!} + ...\Big) &= 1 \\
P_0\Big(e^{\frac{\lambda}{\mu}}\Big) &= 1 \\
\therefore P_0 &= e^{-\frac{\lambda}{\mu}}. \qquad \textbf{(4.45)}
\end{aligned}
$$

Hence, from (4.44) and (4.45), we have;

$$
\begin{aligned}
P_n &= \frac{(\frac{\lambda}{\mu})^n}{n!} P_0 \\
\therefore P_n &= \frac{(\frac{\lambda}{\mu})^n}{n!} e^{-\frac{\lambda}{\mu}}.
\end{aligned}
$$

Thus, the limiting distribution is a Poisson with parameter $\frac{\lambda}{\mu}$. It is independent of the initial condition.

Using Proposition (2.1)

We have that;

$$
\begin{aligned}
P_n &= \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3 ... \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 ... \mu_n} P_0 \\
\text{But,} \quad \lambda_0 &= \lambda_1 = \lambda_2 = \lambda_3 = ... = \lambda_{n-1} = \lambda, \\
&\text{and} \\
\mu_1 &= \mu, \quad \mu_2 = 2\mu, \quad \mu_3 = 3\mu, \quad ..., \quad \mu_n = n\mu. \\
\therefore P_n &= \frac{\lambda\lambda\lambda\lambda ... \lambda}{\mu 2\mu 3\mu ... n\mu} P_0 \\
\therefore P_n &= \frac{(\frac{\lambda}{\mu})^n}{n!} P_0
\end{aligned}
$$

But, from (4.45) above;

$$
\begin{aligned}
P_0 &= e^{-\frac{\lambda}{\mu}} \\
\therefore P_n &== \frac{(\frac{\lambda}{\mu})^n}{n!} e^{-\frac{\lambda}{\mu}}.
\end{aligned}
$$

Which is a Poisson distribution with parameter $\frac{\lambda}{\mu}$.
Using Proposition (2.2)

$$\frac{P_n}{P_{n-1}} = \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0$$

$$\text{But,} \quad \lambda_{n-1} = \lambda,$$

$$\text{and}$$

$$\mu_n = n\mu.$$

$$\therefore \frac{P_n}{P_{n-1}} = \frac{\lambda}{n\mu}.$$

$$\Rightarrow n\mu P_n = \lambda P_{n-1} \tag{4.46}$$

Multiplying (4.46) by s^n and sum the results over n we obtain;

$$\sum_{n=0}^{\infty} n\mu P_n s^n = \sum_{n=0}^{\infty} \lambda P_{n-1} s^n$$

$$\mu s \sum_{n=0}^{\infty} nP_n s^{n-1} = \lambda s \sum_{n=0}^{\infty} P_{n-1} s^{n-1}$$

$$s\mu \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = s\lambda \psi(s)$$

$$\frac{\mathrm{d}\psi(s)}{\psi(s)} = \frac{\lambda}{\mu}\mathrm{d}s$$

Integrating both sides, we have;

$$\int \frac{\mathrm{d}\psi(s)}{\psi(s)} = \int \frac{\lambda}{\mu}\mathrm{d}s$$

$$\ln \psi(s) = \frac{\lambda}{\mu}s + c$$

$$\psi(s) = c_1 e^{\frac{\lambda}{\mu}s}$$

Putting s=1;

$$\psi(1) = 1 = c_1 e^{\frac{\lambda}{\mu}}$$
$$\Rightarrow c_1 = e^{-\frac{\lambda}{\mu}}.$$
$$\therefore \psi(s) = e^{-\frac{\lambda}{\mu}} e^{\frac{\lambda}{\mu}s}$$
$$\therefore \psi(s) = e^{-\frac{\lambda}{\mu}(1-s)}.$$

Which is a Poisson distribution with parameter $\frac{\lambda}{\mu}$.

$$\psi^{'}(s) = \frac{\lambda}{\mu} e^{-\frac{\lambda}{\mu}(1-s)}$$
$$\psi^{''}(s) = (\frac{\lambda}{\mu})^2 e^{-\frac{\lambda}{\mu}(1-s)}$$
$$E(X) = \psi^{'}(1)$$
$$\therefore E(X) = \frac{\lambda}{\mu}.$$
$$Var(x) = \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2$$
$$= (\frac{\lambda}{\mu})^2 + \frac{\lambda}{\mu} - [\frac{\lambda}{\mu}]^2$$
$$\therefore Var(X) = \frac{\lambda}{\mu}.$$

From Kummer's confluent hypergeometric series, we have;

$$ {}_1F_1(\upsilon;\upsilon;\frac{\lambda}{\mu}) = 1 + \frac{\upsilon}{\upsilon}\frac{\frac{\lambda}{\mu}}{1!} + \frac{\upsilon(\upsilon+1)}{\upsilon(\upsilon+1)}\frac{(\frac{\lambda}{\mu})^2}{2!} + ... + \frac{(\upsilon+n-1)}{(\upsilon+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{\upsilon(\upsilon+1)(\upsilon+2)...(\upsilon+n-1)}{\upsilon(\upsilon+1)(\upsilon+2)...(\upsilon+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{(\upsilon+n-1)(\upsilon+n-2)...(\upsilon+2)(\upsilon+1)\upsilon}{(\upsilon+n-1)(\upsilon+n-2)...(\upsilon+2)(\upsilon+1)\upsilon}\frac{\Gamma(\upsilon)}{\Gamma(\upsilon)}.\frac{\Gamma(\upsilon)}{\Gamma(\upsilon)}.\frac{(\frac{\lambda}{\mu})^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma(\upsilon+n)}{\Gamma(\upsilon+n)}\frac{\Gamma(\upsilon)}{\Gamma(\upsilon)}.\frac{(\frac{\lambda}{\mu})^n}{n!}$$

Normalizing, we have;

$$1=\sum_{n=0}^{\infty}\frac{\Gamma(\upsilon+n)}{\Gamma(\upsilon+n)}\frac{\Gamma(\upsilon)}{\Gamma(\upsilon)}.\frac{1}{{}_1F_1(\upsilon;\upsilon;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}$$

Therefore,

$$\begin{aligned}P_n &= \text{Prob.}(N=n)\\ &= \frac{\Gamma(\upsilon+n)}{\Gamma(\upsilon+n)}\frac{\Gamma(\upsilon)}{\Gamma(\upsilon)}.\frac{1}{{}_1F_1(\upsilon;\upsilon;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}\end{aligned}$$

In hypergeometric terms, its pgf. is given as;

$$\begin{aligned}\phi(z) &= \sum_{n=0}^{\infty}P_n z^n\\ &= \frac{\Gamma(\upsilon+n)}{\Gamma(\upsilon+n)}\frac{\Gamma(\upsilon)}{\Gamma(\upsilon)}\frac{(\frac{\lambda}{\mu}z)^n}{n!}.\frac{1}{{}_1F_1(\upsilon;\upsilon;\frac{\lambda}{\mu})}\\ \therefore\phi(z) &= \frac{{}_1F_1(\upsilon;\upsilon;\frac{\lambda}{\mu}z)}{{}_1F_1(\upsilon;\upsilon;\frac{\lambda}{\mu})}.\\ \phi^{'}(z) &= \frac{\lambda}{\mu}\frac{{}_1F_1(\upsilon+1;\upsilon+1;\frac{\lambda}{\mu}z)}{{}_1F_1(\upsilon;\upsilon;\frac{\lambda}{\mu})}\\ \phi^{''}(z) &= (\frac{\lambda}{\mu})^2\frac{{}_1F_1(\upsilon+2;\upsilon+2;\frac{\lambda}{\mu}z)}{{}_1F_1(\upsilon;\upsilon;\frac{\lambda}{\mu})}\\ \text{Let,}\quad\Lambda_\kappa &= \frac{{}_1F_1(\upsilon+\kappa;\upsilon+\kappa;\frac{\lambda}{\mu})}{{}_1F_1(\upsilon;\upsilon;\frac{\lambda}{\mu})};\quad\kappa=1,2.\\ \therefore E(x) &= \frac{\lambda}{\mu}\Lambda_1.\\ Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\ &= (\frac{\lambda}{\mu})^2\Lambda_2+\frac{\lambda}{\mu}\Lambda_1-(\frac{\lambda}{\mu})^2\Lambda_1^2\\ \therefore Var(X) &= \frac{\lambda}{\mu}\Lambda_1+(\frac{\lambda}{\mu})^2(\Lambda_2-\Lambda_1^2).\end{aligned}$$

4.6 M|M|1|§ Queuing Preocess

In this queueing process, the assumption is that the system accommodates finite § number of units, the one being served included if any. Here, customers uses a Poisson arrivals process with a rate λ say; the customer only join the system when there are less than § units/customers and when a customer arrives and find § in the system will leave the system and be lost. The service time is exponential with μ rate.
When the system is at state $E_{\S}$ or when there are § customers in the system, it means that no more customers allowed in the system and hence the number of customers cannot exceed § in the system. This implies that when the system is at state §, departures only is possible.
Thus, this implies that;

$$\begin{aligned} \lambda_n &= \mu_n \\ \Rightarrow \lambda &= \mu. \end{aligned}$$

Therfore, the basic difference differential equations for the steady state are:

$$0 = \mu P_1 - \lambda P_0, \tag{4.47}$$

and

$$0 = \mu P_{n+1} - \lambda P_n, \quad \text{for} \quad n = 0, 1, 2, 3, ..., \S - 1. \tag{4.48}$$

Hence,
Proposition 4.5

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right). \tag{4.49}$$

Proof
Equation (4.47) can be expressed as;

$$\begin{aligned} \lambda P_0 &= \mu P_1 \\ \Rightarrow \frac{\lambda}{\mu} P_0 &= P_1 \\ \therefore P_1 &= \frac{\lambda}{\mu} P_0. \end{aligned} \tag{4.50}$$

Solving steady state equations iteratively, we have;
When n=1;
Equation (4.48) becomes;

$$0 = \mu P_2 - \lambda P_1 \tag{4.51}$$

Substituting (4.50) in equation (4.51), we get;

$$\begin{aligned} 0 &= \mu P_2 - \frac{\lambda^2}{\mu} P_0 \\ \iff \frac{\lambda^2}{\mu} P_0 &= \mu P_2 \\ \Rightarrow \frac{\lambda^2}{\mu^2} P_0 &= P_2 \\ \therefore P_2 &= \frac{\lambda^2}{\mu^2} P_0. \end{aligned} \tag{4.52}$$

When n=2;
Equation (4.48) becomes;

$$0 = \mu P_3 - \lambda P_2 \tag{4.53}$$

Substituting (4.52) in equation (4.53), we get;

$$\begin{aligned} 0 &= \mu P_3 - \frac{\lambda^3}{\mu^2} P_0 \\ \iff \frac{\lambda^3}{\mu^2} P_0 &= \mu P_3 \\ \Rightarrow \frac{\lambda^3}{\mu^3} P_0 &= P_3 \\ \therefore P_3 &= \frac{\lambda^3}{\mu^3} P_0. \end{aligned}$$

Thus,

$$\begin{aligned}
P_4 &= \frac{\lambda^4}{\mu^4} P_0, \\
P_5 &= \frac{\lambda^5}{\mu^5} P_0, \\
&. \\
&. \\
&. \\
P_n &= (\frac{\lambda}{\mu})^n P_0.
\end{aligned} \tag{4.54}$$

When applying the normalizing conditions, we have;

$$\begin{aligned}
\sum_{n=0}^{\S} P_n &= 1 \\
P_0 \Big(\sum_{n=0}^{\S} (\frac{\lambda}{\mu})^n \Big) &= 1 \\
P_0 \Big(1 + (\frac{\lambda}{\mu})^1 + (\frac{\lambda}{\mu})^2 + (\frac{\lambda}{\mu})^3 + (\frac{\lambda}{\mu})^4 + \ldots \Big) &= 1
\end{aligned}$$

Note that;

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \ldots \\
\therefore P_0 \Big(\frac{1}{1 - \frac{\lambda}{\mu}} \Big) &= 1 \\
\therefore P_0 &= 1 - \frac{\lambda}{\mu}. \qquad (4.55) \\
\Rightarrow P_1 &= \Big(\frac{\lambda}{\mu} \Big)^1 \Big(1 - \frac{\lambda}{\mu} \Big), \\
P_2 &= \Big(\frac{\lambda}{\mu} \Big)^2 \Big(1 - \frac{\lambda}{\mu} \Big),
\end{aligned}$$

Therefore,

$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 \left(1 - \frac{\lambda}{\mu}\right),$$
$$P_4 = \left(\frac{\lambda}{\mu}\right)^4 \left(1 - \frac{\lambda}{\mu}\right),$$
$$\vdots$$

We find by Mathematical induction that;

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right). \quad \text{for} \quad n = 0, 1, 2, 3, ..., \S - 1. \tag{4.56}$$

When n=0,1,2,3,4, . . . ,§ we have that;

$$P_n = \frac{(1 - \frac{\lambda}{\mu})(\frac{\lambda}{\mu})^n}{1 - (\frac{\lambda}{\mu})^{\S+1}}; \quad \lambda \neq \mu.$$
$$\text{and} \tag{4.57}$$
$$P_n = \frac{1}{\S + 1}; \quad \lambda = \mu.$$

Therefore, the limiting distribution is Uniform distribution when $\frac{\lambda}{\mu} = 1$ and truncated Geometric distribution when $\frac{\lambda}{\mu} \neq 1$.

Using Proposition (2.1)

We have that;

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3 ... \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 ... \mu_n} P_0$$
$$\text{But,} \quad \lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = ... = \lambda_{n-1} = \lambda,$$
$$\text{and}$$
$$\mu_1 = \mu, \quad \mu_2 = \mu, \quad \mu_3 = \mu, \quad ..., \quad \mu_n = \mu.$$

$$\therefore P_n = \frac{\lambda\lambda\lambda\lambda...\lambda}{\mu\mu\mu...\mu}P_0$$
$$\therefore P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

But, from (4.55) above;

$$P_0 = \left(1-\frac{\lambda}{\mu}\right).$$
$$\therefore P_n = \left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right), \quad \text{for} \quad n = 0,1,2,3,...,\S-1.$$

Which is a Geometric distribution with parameter $\frac{\lambda}{\mu}$.

Using Proposition (2.2)

$$\frac{P_n}{P_{n-1}} = \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0$$
$$\text{But,} \quad \lambda_{n-1} = \lambda,$$
$$\text{and}$$
$$\mu_n = \mu.$$
$$\therefore \frac{P_n}{P_{n-1}} = \frac{\lambda}{\mu}.$$
$$\Rightarrow \mu P_n = \lambda P_{n-1} \tag{4.58}$$

Multiplying (4.58) by s^n and sum the results over n we obtain;

$$\sum_{n=0}^{\infty}\mu P_n s^n = \sum_{n=0}^{\infty}\lambda P_{n-1}s^n$$
$$\mu\sum_{n=0}^{\infty}P_n s^n = \lambda s\sum_{n=0}^{\infty}P_{n-1}s^{n-1}$$
$$(1-\frac{\lambda}{\mu}s)\psi(s) = P_0$$
$$\text{But,} \quad P_0 = 1-\frac{\lambda}{\mu}.$$
$$\therefore \psi(s) = \frac{(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu}s)}.$$

Which is a Geometric distribution with parameter $\frac{\lambda}{\mu}$.

$$
\begin{aligned}
\psi^{'}(s) &= \frac{\frac{\lambda}{\mu}(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu}s)^2} \\
\psi^{''}(s) &= \frac{2(\frac{\lambda}{\mu})^2(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu}s)^3} \\
E(X) &= \psi^{'}(1) \\
&= \frac{\frac{\lambda}{\mu}(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu})^2} \\
\therefore E(X) &= \frac{\frac{\lambda}{\mu}}{(1-\frac{\lambda}{\mu})}. \\
Var(x) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= \frac{2(\frac{\lambda}{\mu})^2(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu})^3} + \frac{\frac{\lambda}{\mu}(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu})^2} - [\frac{\frac{\lambda}{\mu}(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu}s)^2}]^2 \\
&= \frac{2(\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu})^2} + \frac{\frac{\lambda}{\mu}}{(1-\frac{\lambda}{\mu})} - \frac{(\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu}s)^2} \\
&= \frac{(\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu})^2} + \frac{\frac{\lambda}{\mu}}{(1-\frac{\lambda}{\mu})}
\end{aligned}
$$

$$
\begin{aligned}
\therefore Var(X) &= \frac{(\frac{\lambda}{\mu})^2 + \frac{\lambda}{\mu}(1-\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu})^2} \\
\therefore Var(X) &= \frac{\frac{\lambda}{\mu}}{(1-\frac{\lambda}{\mu})^2}.
\end{aligned}
$$

From Kummer's confluent hypergeometric series, we have that;

$$
\begin{aligned}
{}_1F_1(\iota;\iota;\frac{\lambda}{\mu}) &= 1+\frac{\iota}{\iota}\frac{\frac{\lambda}{\mu}}{1!}+\frac{\iota(\iota+1)}{\iota(\iota+1)}\frac{(\frac{\lambda}{\mu})^2}{2!}+\ldots+\frac{(\iota+n-1)}{(\iota+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{\iota(\iota+1)(\iota+2)\ldots(\iota+n-1)}{\iota(\iota+1)(\iota+2)\ldots(\iota+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{(\iota+n-1)(\iota+n-2)\ldots(\iota+2)(\iota+1)\iota}{(\iota+n-1)(\iota+n-2)\ldots(\iota+2)(\iota+1)\iota}\frac{\Gamma(\rho)}{\Gamma(\iota)}.\frac{\Gamma(\iota)}{\Gamma(\iota)}.\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\iota+n)}{\Gamma(\iota+n)}\frac{\Gamma(\iota)}{\Gamma(\iota)}.\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}
$$

Normalizing, we have;

$$
1=\sum_{n=0}^{\infty}\frac{\Gamma(\iota+n)}{\Gamma(\iota+n)}\frac{\Gamma(\iota)}{\Gamma(\iota)}.\frac{1}{{}_1F_1(\iota;\iota;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}
$$

Therefore,

$$
\begin{aligned}
P_n &= \text{Prob.}(N=n) \\
&= \frac{\Gamma(\iota+n)}{\Gamma(\iota+n)}\frac{\Gamma(\iota)}{\Gamma(\iota)}.\frac{1}{{}_1F_1(\iota;\iota;\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}
$$

In hypergeometric terms, its pgf. is given as;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty}P_n z^n \\
&= \frac{\Gamma(\iota+n)}{\Gamma(\iota+n)}\frac{\Gamma(\iota)}{\Gamma(\iota)}\frac{(\frac{\lambda}{\mu}z)^n}{n!}.\frac{1}{{}_1F_1(\iota;\iota;\frac{\lambda}{\mu})} \\
\therefore \phi(z) &= \frac{{}_1F_1(\iota;\iota;\frac{\lambda}{\mu}z)}{{}_1F_1(\iota;\iota;\frac{\lambda}{\mu})}. \\
\phi'(z) &= \frac{\lambda}{\mu}\frac{{}_1F_1(\iota+1;\iota+1;\frac{\lambda}{\mu}z)}{{}_1F_1(\iota;\iota;\frac{\lambda}{\mu})}
\end{aligned}
$$

Therefore,

$$\phi^{''}(z) = (\frac{\lambda}{\mu})^2 \frac{{}_1F_1(\iota+2;\iota+2;\frac{\lambda}{\mu}z)}{{}_1F_1(\iota;\iota;\frac{\lambda}{\mu})}$$

$$\text{Let,} \quad \Lambda_\kappa = \frac{{}_1F_1(\iota+\kappa;\iota+\kappa;\frac{\lambda}{\mu})}{{}_1F_1(\iota;\iota;\frac{\lambda}{\mu})}; \quad \kappa = 1,2.$$

$$\therefore E(x) = \frac{\lambda}{\mu}\Lambda_1.$$

$$Var(X) = \phi^{''}(1) + \phi^{'}(1) - [\phi^{'}(1)]^2$$

$$= (\frac{\lambda}{\mu})^2\Lambda_2 + \frac{\lambda}{\mu}\Lambda_1 - (\frac{\lambda}{\mu})^2\Lambda_1^2$$

$$\therefore Var(X) = \frac{\lambda}{\mu}\Lambda_1 + (\frac{\lambda}{\mu})^2(\Lambda_2 - \Lambda_1^2).$$

4.7 M|M|c|c Queueing Process

Suppose we consider a c-server model with Poisson input and exponential service time such that when all the c-channels are busy, a customer leaves the system without waiting for service.
In this case:

$$\lambda_n = \lambda; \quad \text{for} \quad n = 0,1,2,3,...,c-1$$

and

$$\mu_n = n\mu; \quad \text{for} \quad n = 0,1,2,3,...,c-1$$

$$\mu_n = c\mu; \quad \text{for} \quad n \geq c.$$

The basic difference differential equations for the steady state are:

$$0 = \mu P_1 - \lambda P_0 \tag{4.59}$$

$$0 = (n+1)\mu P_{n+1} + \lambda P_{n-1} - (\lambda + n\mu)P_n; \quad \text{for} \quad n = 0,1,2,...,c-1 \tag{4.60}$$

$$0 = c\mu P_{n+1} + \lambda P_{n-1} - (\lambda + c\mu)P_n; \quad \text{for} \quad n = c+1, c+2, ... \tag{4.61}$$

Hence,
Proposition 4.6

$$P_n = \frac{(\frac{\lambda}{\mu})^n}{n!} e^{-\frac{\lambda}{\mu}}, \quad \text{for} \quad n = 0, 1, 2, ..., c-1.$$
$$P_n = \frac{(\frac{\lambda}{\mu})^n}{c!c^{n-c}} e^{-\frac{\lambda}{\mu}}. \quad \text{for} \quad n \geq c \quad (\frac{\lambda}{\mu} < c)$$

Proof
Solving steady state equations iteratively, we have;
Equation (4.59) can be expressed as;

$$\begin{aligned} \lambda P_0 &= \mu P_1 \\ \Rightarrow \frac{\lambda}{1\mu} P_0 &= P_1 \\ \therefore P_1 &= \frac{(\frac{\lambda}{\mu})^1}{1!} P_0. \end{aligned} \tag{4.62}$$

When n=1;
Equation (4.60) becomes;

$$0 = 2\mu P_2 + \lambda P_0 - (\lambda + \mu) P_1 \tag{4.63}$$

Substituting (4.62) in equation (4.63), we get;

$$\begin{aligned} 0 &= 2\mu P_2 + \lambda P_0 - (\lambda + \mu) \frac{\lambda}{\mu} P_0 \\ 0 &= 2\mu P_2 + \lambda P_0 - \lambda P_0 - \frac{\lambda^2}{\mu} P_0 \\ 0 &= 2\mu P_2 - \frac{\lambda^2}{\mu} P_0 \\ \Longleftrightarrow \frac{\lambda^2}{\mu} P_0 &= 2\mu P_2 \\ \Rightarrow \frac{\lambda^2}{2.1\mu^2} P_0 &= P_2 \\ \therefore P_2 &= \frac{(\frac{\lambda}{\mu})^2}{2!} P_0. \end{aligned} \tag{4.64}$$

When n=1;
Equation (4.60) becomes;

$$0 = 3\mu P_3 + \lambda P_1 - (\lambda + 2\mu)P_2 \tag{4.65}$$

Substituting (4.62) and (4.64) in equation (4.65), we get;

$$\begin{aligned}
0 &= 3\mu P_3 + \frac{\lambda^2}{\mu}P_0 - (\lambda + 2\mu)\frac{\lambda^2}{2!\mu^2}P_0 \\
0 &= 3\mu P_3 + \frac{\lambda^2}{\mu}P_0 - \frac{\lambda^2}{\mu}P_0 - \frac{\lambda^3}{2.1\mu^2}P_0 \\
0 &= 3\mu P_3 - \frac{\lambda^3}{2.1\mu^2}P_0 \\
\iff \frac{\lambda^3}{2.1\mu^2}P_0 &= 3\mu P_3 \\
\Rightarrow \frac{\lambda^3}{3.2.1\mu^3}P_0 &= 3P_3 \\
\therefore P_3 &= \frac{(\frac{\lambda}{\mu})^3}{3!}P_0.
\end{aligned}$$

Thus,

$$\begin{aligned}
P_4 &= \frac{(\frac{\lambda}{\mu})^4}{4!}P_0, \\
P_5 &= \frac{(\frac{\lambda}{\mu})^5}{5!}P_0, \\
&. \\
&. \\
&. \\
P_n &= \frac{(\frac{\lambda}{\mu})^n}{n!}P_0, \quad \text{for} \quad n = 0, 1, 2, 3, ..., c-1.
\end{aligned} \tag{4.66}$$

Suppose that $c - 1 \to \infty$.
Then,

$$\sum_{n=0}^{\infty} P_n = 1$$

$$P_0\Big\{\sum_{n=0}^{\infty}\frac{(\frac{\lambda}{\mu})^n}{n!}\Big\} = 1$$

$$P_0\Big\{1+\frac{(\frac{\lambda}{\mu})^1}{1!}+\frac{(\frac{\lambda}{\mu})^2}{2!}+\frac{(\frac{\lambda}{\mu})^3}{3!}+\frac{(\frac{\lambda}{\mu})^4}{4!}+...\Big\} = 1$$

$$P_0\Big\{e^{\frac{\lambda}{\mu}}\Big\} = 1$$

$$\therefore P_0 = e^{-\frac{\lambda}{\mu}}. \quad \textbf{(4.67)}$$

From (4.66) and (4.67) above, we have that;

$$P_n = \frac{(\frac{\lambda}{\mu})^n}{n!}e^{-\frac{\lambda}{\mu}}. \quad \text{for} \quad n = 0,1,2,3,...,c-1.$$

Which is a truncated Poisson distribution with parameter $\frac{\lambda}{\mu}$.
Using Proposition (2.1)
We have that;

$$P_n = \frac{\lambda_0\lambda_1\lambda_2\lambda_3...\lambda_{n-1}}{\mu_1\mu_2\mu_3...\mu_n}P_0$$

But, $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = ... = \lambda_{n-1} = \lambda,$

and

$$\mu_1 = \mu, \quad \mu_2 = 2\mu, \quad \mu_3 = 3\mu, \quad ..., \quad \mu_n = n\mu.$$

$$\therefore P_n = \frac{\lambda\lambda\lambda\lambda...\lambda}{\mu 2\mu 3\mu...n\mu}P_0$$

$$= \frac{(\frac{\lambda}{\mu})^n}{n!}P_0$$

But, from (4.67) above;

$$P_0 = e^{-\frac{\lambda}{\mu}}$$

$$\therefore P_n = \frac{(\frac{\lambda}{\mu})^n}{n!}e^{-\frac{\lambda}{\mu}}.$$

Which is a Poisson distribution with parameter $\frac{\lambda}{\mu}$.
Using Proposition (2.2)

$$\frac{P_n}{P_{n-1}} = \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0$$

$$\text{But,} \quad \lambda_{n-1} = \lambda,$$

and

$$\mu_n = n\mu.$$

$$\therefore \frac{P_n}{P_{n-1}} = \frac{\lambda}{n\mu}.$$

$$\frac{P_n}{P_{n-1}} = \frac{\lambda}{n\mu}$$

$$\Rightarrow n\mu P_n = \lambda P_{n-1} \tag{4.68}$$

Multiplying (4.68) by s^n and sum the results over n we obtain;

$$\sum_{n=0}^{\infty} n\mu P_n s^n = \sum_{n=0}^{\infty} \lambda P_{n-1} s^n$$

$$\mu s \sum_{n=0}^{\infty} nP_n s^{n-1} = \lambda s \sum_{n=0}^{\infty} P_{n-1} s^{n-1}$$

$$s\mu \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = s\lambda \psi(s)$$

$$\frac{\mathrm{d}\psi(s)}{\psi(s)} = \frac{\lambda}{\mu}\mathrm{d}s$$

Integrating both sides, we have;

$$\int \frac{\mathrm{d}\psi(s)}{\psi(s)} = \int \frac{\lambda}{\mu}\mathrm{d}s$$

$$\ln \psi(s) = \frac{\lambda}{\mu}s + c$$

$$\psi(s) = c_1 e^{\frac{\lambda}{\mu}s}$$

Putting s=1;

$$
\begin{aligned}
\psi(1) = 1 = c_1 e^{\frac{\lambda}{\mu}} \\
\Rightarrow c_1 = e^{-\frac{\lambda}{\mu}}. \\
\therefore \psi(s) = e^{-\frac{\lambda}{\mu}} e^{\frac{\lambda}{\mu}s} \\
\therefore \psi(s) = e^{-\frac{\lambda}{\mu}(1-s)}.
\end{aligned}
$$

Which is the pgf. for a Poisson distribution with parameter $\frac{\lambda}{\mu}$.

$$
\begin{aligned}
\psi^{'}(s) &= \frac{\lambda}{\mu} e^{-\frac{\lambda}{\mu}(1-s)} \\
\psi^{''}(s) &= (\frac{\lambda}{\mu})^2 e^{-\frac{\lambda}{\mu}(1-s)} \\
E(X) &= \psi^{'}(1) \\
\therefore E(X) &= \frac{\lambda}{\mu}. \\
Var(x) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= (\frac{\lambda}{\mu})^2 + \frac{\lambda}{\mu} - [\frac{\lambda}{\mu}]^2 \\
\therefore Var(X) &= \frac{\lambda}{\mu}.
\end{aligned}
$$

From Kummer's confluent hypergeometric series, we have that;

$$
\begin{aligned}
{}_1F_1(\natural;\natural;\frac{\lambda}{\mu}) &= 1 + \frac{\natural}{\natural}\frac{\frac{\lambda}{\mu}}{1!} + \frac{\natural(\natural+1)}{\natural(\natural+1)}\frac{(\frac{\lambda}{\mu})^2}{2!} + ... + \frac{(\natural+n-1)}{(\natural+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\natural(\natural+1)(\natural+2)...(\natural+n-1)}{\natural(\natural+1)(\natural+2)...(\natural+n-1)}\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(\natural+n-1)(\natural+n-2)...(\natural+2)(\natural+1)\natural}{(\natural+n-1)(\natural+n-2)...(\natural+2)(\natural+1)\natural}\frac{\Gamma(\natural)}{\Gamma(\natural)}.\frac{\Gamma(\natural)}{\Gamma(\natural)}.\frac{(\frac{\lambda}{\mu})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(\natural+n)}{\Gamma(\natural+n)}\frac{\Gamma(\natural)}{\Gamma(\natural)}.\frac{(\frac{\lambda}{\mu})^n}{n!}
\end{aligned}
$$

Normalizing, we have;

$$1 = \sum_{n=0}^{\infty} \frac{\Gamma(\natural+n)}{\Gamma(\natural+n)} \frac{\Gamma(\natural)}{\Gamma(\natural)} . \frac{1}{{}_1F_1(\natural;\natural;\frac{\lambda}{\mu})} \frac{(\frac{\lambda}{\mu})^n}{n!}$$

Therefore,

$$\begin{aligned} P_n &= \text{Prob.}(N=n) \\ &= \frac{\Gamma(\natural+n)}{\Gamma(\natural+n)} \frac{\Gamma(\natural)}{\Gamma(\natural)} . \frac{1}{{}_1F_1(\natural;\natural;\frac{\lambda}{\mu})} \frac{(\frac{\lambda}{\mu})^n}{n!} \end{aligned}$$

In hypergeometric terms, its pgf. is given as;

$$\begin{aligned} \phi(z) &= \sum_{n=0}^{\infty} P_n z^n \\ &= \frac{\Gamma(\natural+n)}{\Gamma(\natural+n)} \frac{\Gamma(\natural)}{\Gamma(\natural)} \frac{(\frac{\lambda}{\mu}z)^n}{n!} . \frac{1}{{}_1F_1(\natural;\natural;\frac{\lambda}{\mu})} \\ \therefore \phi(z) &= \frac{{}_1F_1(\natural;\natural;\frac{\lambda}{\mu}z)}{{}_1F_1(\natural;\natural;\frac{\lambda}{\mu})}. \\ \phi^{'}(z) &= \frac{\lambda}{\mu} \frac{{}_1F_1(\natural+1;\natural+1;\frac{\lambda}{\mu}z)}{{}_1F_1(\natural;\natural;\frac{\lambda}{\mu})} \\ \phi^{''}(z) &= (\frac{\lambda}{\mu})^2 \frac{{}_1F_1(\natural+2;\natural+2;\frac{\lambda}{\mu}z)}{{}_1F_1(\natural;\natural;\frac{\lambda}{\mu})} \\ \text{Let,} \quad \Lambda_\kappa &= \frac{{}_1F_1(\natural+\kappa;\natural+\kappa;\frac{\lambda}{\mu})}{{}_1F_1(\natural;\natural;\frac{\lambda}{\mu})}; \quad \kappa = 1,2. \\ \therefore E(x) &= \frac{\lambda}{\mu}\Lambda_1. \\ Var(X) &= \phi^{''}(1) + \phi^{'}(1) - [\phi^{'}(1)]^2 \\ &= (\frac{\lambda}{\mu})^2\Lambda_2 + \frac{\lambda}{\mu}\Lambda_1 - (\frac{\lambda}{\mu})^2\Lambda_1^2 \\ \therefore Var(X) &= \frac{\lambda}{\mu}\Lambda_1 + (\frac{\lambda}{\mu})^2(\Lambda_2 - \Lambda_1^2). \end{aligned}$$

Next,
When n=1;
Equation (4.61) becomes;

$$0 = c\mu P_2 + \lambda P_0 - (\lambda + c\mu)P_1 \tag{4.69}$$

Substituting (4.62) in equation (4.69), we get;

$$\begin{aligned}
0 &= c\mu P_2 + \lambda P_0 - (\lambda + c\mu)\frac{\lambda}{\mu}P_0 \\
0 &= c\mu P_2 + \lambda P_0 - c\lambda P_0 - \frac{\lambda^2}{\mu}P_0 \\
0 &= c\mu P_2 - \lambda(c-1)P_0 - \frac{\lambda^2}{\mu}P_0 \\
\iff \frac{\lambda^2}{\mu}P_0 + \lambda(c-1)P_0 &= c\mu P_2 \\
\Rightarrow \frac{1}{c}\Big(\frac{\lambda^2}{\mu^2} + \frac{\lambda}{\mu}(c-1)\Big)P_0 &= P_2 \\
\therefore P_2 &= \frac{1}{c}\Big(\frac{\lambda^2}{\mu^2} + \frac{\lambda}{\mu}(c-1)\Big)P_0.
\end{aligned} \tag{4.70}$$

When n=1;
Equation (4.61) becomes;

$$0 = c\mu P_3 + \lambda P_1 - (\lambda + c\mu)P_2 \tag{4.71}$$

Substituting (4.61) and (4.70) in equation (4.71), we get;

$$\begin{aligned}
0 &= c\mu P_3 + \frac{\lambda^2}{\mu}P_0 - (\lambda + c\mu)\frac{1}{c}\Big(\frac{\lambda^2}{\mu^2} + \frac{\lambda}{\mu}(c-1)\Big)P_0 \\
0 &= -\frac{\lambda^3}{c\mu^2}P_0 - \frac{\lambda^2}{c\mu}(c-1)P_0 - \frac{\lambda^2}{\mu}P_0 - \lambda(c-1)P_0 + \frac{\lambda^2}{\mu}P_0 + c\mu P_3 \\
0 &= -\frac{\lambda^3}{c\mu^2}P_0 - \frac{\lambda^2}{c\mu}(c-1)P_0 - \lambda(c-1)P_0 + c\mu P_3
\end{aligned}$$

Therefore,

$$\begin{aligned}\iff c\mu P_3 &= \frac{1}{c}\Big(\frac{\lambda^3}{\mu^2}+\frac{\lambda^2}{\mu}(c-1)+\lambda c(c-1)\Big)P_0\\ \Rightarrow P_3 &= \frac{1}{c^2}\Big(\frac{\lambda^3}{\mu^3}+\frac{\lambda^2}{\mu^2}(c-1)+\frac{\lambda}{\mu}c(c-1)\Big)P_0\\ \therefore P_3 &= \frac{1}{c^2}\Big(\frac{\lambda^3}{\mu^3}+\frac{\lambda^2}{\mu^2}(c-1)+\frac{\lambda}{\mu}c(c-1)\Big)P_0.\end{aligned}$$

Therefore,

$$P_4 = \frac{1}{c^3}\Big(\frac{\lambda^4}{\mu^4}+\frac{\lambda^3}{\mu^3}(c-1)+\frac{\lambda^2}{\mu^2}c(c-1)+\frac{\lambda}{\mu}c(c-1)(c-2)\Big)P_0,$$

.

.

.

$$P_n = \frac{1}{c^{n-c}}\Big(\frac{\lambda^n}{\mu^n}+\frac{\lambda^{n-1}}{\mu^{n-1}}(c-1)+\frac{\lambda^{n-2}}{\mu^{n-2}}c(c-1)+\ldots+\frac{\lambda}{\mu}c(c-1)\ldots(c-n)\Big)P_0, \quad \text{for} \quad n \geq c.$$

Supposed that $\frac{\lambda}{\mu} < c$,
Then,

$$\begin{aligned}\sum P_n &= 1\\ \Rightarrow P_0+P_1+P_2+P_3+\ldots+P_n &= 1\\ \frac{1}{c^{n-c}}P_0\Big\{1+\frac{(\frac{\lambda}{\mu})^1}{1!}+\frac{(\frac{\lambda}{\mu})^2}{2!}+\frac{(\frac{\lambda}{\mu})^3}{3!}+\frac{(\frac{\lambda}{\mu})^4}{4!}+\ldots\Big\} &= 1\\ P_0\Big(e^{\frac{\lambda}{\mu}}\Big) &= 1\\ \therefore P_0 &= e^{-\frac{\lambda}{\mu}}.\\ \therefore P_n &= \frac{(\frac{\lambda}{\mu})^n}{c!c^{n-c}}e^{-\frac{\lambda}{\mu}}. \quad \text{for} \quad n \geq c \quad (\frac{\lambda}{\mu} < c)\end{aligned}$$

Note that;

$$P_n = \frac{\frac{(\frac{\lambda}{\mu})^n}{n!}}{\sum_{\S=0}^{c} \frac{(\frac{\lambda}{\mu})^{\S}}{\S!}}, \quad \text{for} \quad n = 0, 1, 2, ..., c.$$

Which is the Erlang's distribution/formula.
Suppose that the unit that arrived in the system is lost because upon arrival, it found that all the channels in the system are busy.
Then, the probability of this event will be given by;

$$P_c = \frac{\frac{(\frac{\lambda}{\mu})^c}{c!}}{\sum_{\S=0}^{c} \frac{(\frac{\lambda}{\mu})^{\S}}{\S!}}. \tag{4.72}$$

Equation (4.72) is called Erlang's loss function and can be denoted by $L(c, \frac{\lambda}{\mu})$.

4.8 M|M|c||m ($m > c$) Queueing Process

Suppose the source of arrivals is from a finite population of size ξ. Let the unit or a customer be either in a system or outside the system. Let also the system be consits of parallel servers, say c were $c < \xi$. In this case, any unit arriving in the system will start receiving a service from any of the parallel servers if there is any that is free and if there is none a vailable, it will join the queue.
The distribution of the service time of c servers is exponentially distibuted and independently identically distributed with parameter μ. When n servers are busy, the total service rate is given as $n\mu$, for $n \leq c$ and $c\mu$, for $n \geq c.$ If at any given time there are n in the queue either being served or $n - c(\geq 0)$ in the system while c are being served, then there will be $\xi - n$ outside from where arrivals to the system occur, hence the average rate becomes $\lambda(\xi - n)$; the inter-arrival time distribution is exponential and is also independent identically distributed with parameter λ.
In this case;

$$\lambda_n = (\xi - n)\lambda, \quad n = 0,1,2,3,...,\xi - 1$$
and
$$\mu_n = n\mu, \quad n = 1,2,3,...,c-1$$
$$\mu_n = c\mu, \quad n \geq c.$$

The basic difference differential equations for the steady state are given by:

$$0 = \mu P_1 - \lambda \xi P_0 \tag{4.73}$$
$$0 = (n+1)\mu P_{n+1} + \lambda[\xi - (n-1)]P_{n-1} - [(\xi - n)\lambda + n\mu]P_n \tag{4.74}$$
$$0 = c\mu P_{n+1} + \lambda[\xi - (n-1)]P_{n-1} - [(\xi - n)\lambda + c\mu]P_n \tag{4.75}$$

Hence,
Proposition 4.7

$$P_n = \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(\frac{\lambda}{\lambda+\mu}\Big)^n \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-n}, \quad n = 1,2,3,...,\xi-1. \tag{4.76}$$

Proof
Solving the steady state equations iteratively, we have;
Equation (4.73) can be expressed as;

$$\mu P_1 = \lambda \xi P_0$$
$$\Rightarrow P_1 = \xi \frac{\lambda}{\mu} P_0$$
$$\therefore P_1 = \xi \frac{\lambda}{\mu} P_0. \tag{4.77}$$

When n=1;
Equation (4.74) becomes;

$$0 = -[(\xi - 1)\lambda + \mu]P_1 + \lambda \xi P_0 + 2\mu P_2 \tag{4.78}$$

Substituting (4.77) in equation (4.78), we get;

$$
\begin{aligned}
0 &= -[(\xi-1)\lambda+\mu]\xi\frac{\lambda}{\mu}P_0+\lambda\xi P_0+2\mu P_2\\
0 &= -\xi^2\frac{\lambda^2}{\mu}P_0+\xi\frac{\lambda^2}{\mu}P_0-\xi\lambda P_0+\xi\lambda P_0+2\mu P_2\\
0 &= -\xi^2\frac{\lambda^2}{\mu}P_0+\xi\frac{\lambda^2}{\mu}P_0+2\mu P_2\\
\Longleftrightarrow 2\mu P_2 &= \xi^2\frac{\lambda^2}{\mu}P_0-\xi\frac{\lambda^2}{\mu}P_0\\
\Rightarrow 2\mu P_2 &= \xi\frac{\lambda^2}{\mu}(\xi-1)P_0\\
\therefore P_2 &= \frac{\lambda^2}{2!\mu^2}\xi(\xi-1)P_0.
\end{aligned} \tag{4.79}
$$

When n=2;
Equation (4.74) becomes;

$$
0=-[(\xi-2)\lambda+2\mu]P_2+\lambda(\xi-1)P_1+3\mu P_3 \tag{4.80}
$$

Substituting (4.77) and (4.79) in equation (4.80), we get;

$$
\begin{aligned}
0 &= -[(\xi-2)\lambda+2\mu]\frac{\lambda^2}{2!\mu^2}\xi(\xi-1)P_0+\lambda(\xi-1)\xi\frac{\lambda}{\mu}P_0+3\mu P_3\\
0 &= -\frac{\lambda^3}{2\mu^2}\xi^2(\xi-1)P_0+\frac{\lambda^3}{\mu^2}\xi(\xi-1)P_0-\frac{\lambda^2}{\mu}\xi(\xi-1)P_0+\frac{\lambda^2}{\mu}\xi(\xi-1)P_0+3\mu P_3\\
0 &= -\frac{\lambda^3}{2\mu^2}\xi^2(\xi-1)P_0+\frac{\lambda^3}{\mu^2}\xi(\xi-1)P_0+3\mu P_3\\
\Longleftrightarrow 3\mu P_3 &= \frac{\lambda^3}{2\mu^2}\xi^2(\xi-1)P_0-\frac{\lambda^3}{\mu^2}\xi(\xi-1)P_0\\
\Rightarrow 3\mu P_3 &= \frac{\lambda^3}{2\mu^2}\xi(\xi-1)(\xi-2)P_0\\
\therefore P_3 &= \frac{\lambda^3}{3!\mu^3}\xi(\xi-1)(\xi-2)P_0.
\end{aligned}
$$

Hence,

$$P_4 = \frac{\lambda^4}{4!\mu^4}\xi(\xi-1)(\xi-2)(\xi-3)P_0,$$
$$P_5 = \frac{\lambda^5}{5!\mu^5}\xi(\xi-1)(\xi-2)(\xi-3)(\xi-4)P_0,$$
$$\vdots$$
$$P_n = \frac{\lambda^n}{n!\mu^n}\xi(\xi-1)(\xi-2)...(\xi-n)P_0.$$

Note that,

$$\sum_{n=0}^{\xi} P_n = 1$$

But,

$$P_1 = \left(\frac{\lambda}{\mu}\right)^1 \begin{pmatrix} \xi \\ n \end{pmatrix} P_0,$$
$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 \begin{pmatrix} \xi \\ n \end{pmatrix} P_0,$$
$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 \begin{pmatrix} \xi \\ n \end{pmatrix} P_0,$$
$$P_4 = \left(\frac{\lambda}{\mu}\right)^4 \begin{pmatrix} \xi \\ n \end{pmatrix} P_0,$$
$$\vdots$$
$$\therefore P_n = \left(\frac{\lambda}{\mu}\right)^n \begin{pmatrix} \xi \\ n \end{pmatrix} P_0.$$

$$\Rightarrow P_0 + \Big(\frac{\lambda}{\mu}\Big)^1 \begin{pmatrix} \xi \\ n \end{pmatrix} P_0 + \Big(\frac{\lambda}{\mu}\Big)^2 \begin{pmatrix} \xi \\ n \end{pmatrix} P_0 + \Big(\frac{\lambda}{\mu}\Big)^3 \begin{pmatrix} \xi \\ n \end{pmatrix} P_0 + ... + \Big(\frac{\lambda}{\mu}\Big)^n \begin{pmatrix} \xi \\ n \end{pmatrix} P_0 = 1$$

$$P_0\Big\{1 + \Big(\frac{\lambda}{\mu}\Big)^1 \begin{pmatrix} \xi \\ n \end{pmatrix} + \Big(\frac{\lambda}{\mu}\Big)^2 \begin{pmatrix} \xi \\ n \end{pmatrix} + \Big(\frac{\lambda}{\mu}\Big)^3 \begin{pmatrix} \xi \\ n \end{pmatrix} + ... + \Big(\frac{\lambda}{\mu}\Big)^n \begin{pmatrix} \xi \\ n \end{pmatrix}\Big\} = 1$$

$$\therefore \Big(1 + \frac{\lambda}{\mu}\Big)^n P_0 = 1$$

$$\therefore P_0 = \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^n, \quad \text{for} \quad n = 0, 1, 2, ..., \xi.$$

$$\therefore P_n = \Big\{ \begin{pmatrix} \xi \\ 0 \end{pmatrix} \Big(\frac{\lambda}{\lambda+\mu}\Big)^0 \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-0} + \begin{pmatrix} \xi \\ 1 \end{pmatrix} \Big(\frac{\lambda}{\lambda+\mu}\Big)^1 \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-1} + ...$$

$$+ \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(\frac{\lambda}{\lambda+\mu}\Big)^n \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-n} \Big\}$$

Hence, the limiting distribution is the Binomial Distribution with parameter $\frac{\lambda}{\lambda+\mu}$ and ξ.

$$P_n = \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(\frac{\lambda}{\lambda+\mu}\Big)^n \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-n}, \quad n = 1, 2, 3, ..., \xi - 1.$$

Letting $\dfrac{\lambda}{\lambda+\mu} = \rho$, we have;

$$\therefore P_n = \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(\rho\Big)^n \Big(1 - \rho\Big)^{\xi-n}, \quad n = 1, 2, 3, ..., \xi - 1.$$

Using Proposition 2.1

We have;

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3 ... \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 ... \mu_n} P_0$$

But, $\lambda_0 = \xi\lambda, \quad \lambda_1 = (\xi - 1)\lambda, \quad \lambda_2 = (\xi - 2)\lambda, \quad \lambda_3 = (\xi - 3)\lambda, \quad ... = \lambda_{n-1} = (\xi - (n-1))\lambda.$

and

$$\mu_1 = \mu, \quad \mu_2 = 2\mu, \quad \mu_3 = 3\mu, \quad ..., \quad \mu_n = n\mu.$$

$$
\begin{aligned}
\therefore P_n &= \frac{\xi\lambda(\xi-1)\lambda(\xi-2)\lambda(\xi-3)\lambda\ldots(\xi-(n-1))\lambda}{\mu 2\mu 3\mu\ldots n\mu}P_0 \\
&= \frac{\xi\lambda(\xi-1)\lambda(\xi-2)\lambda(\xi-3)\lambda\ldots(\xi-(n-1))\lambda}{n!\mu^n}P_0 \\
&= \frac{P_0}{n!\mu^n}\prod_{j=0}^{n-1}(\xi-j)\lambda \\
&= \frac{\lambda^n P_0}{n!\mu^n}\prod_{j=0}^{n-1}(\xi-j) \\
&= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{\xi(\xi-1)(\xi-2)(\xi-3)\ldots(\xi-(n-1))\Big\} \\
&= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{(\xi-(n-1))(\xi-(n-2))\ldots(\xi-2)(\xi-1)\xi\Big\} \\
&= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{(\xi-(n-1))(\xi-(n-2))\ldots(\xi-2)(\xi-1)\xi\frac{\Gamma(\xi)}{\Gamma(\xi)}\Big\} \\
&= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\Big\{(\xi-(n-1))(\xi-(n-2))\ldots(\xi-2)(\xi-1)\frac{\Gamma(\xi-1)}{\Gamma(\xi)}\Big\} \\
&= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\frac{1}{\Gamma(\xi)}\Big\{(\xi-(n-1))(\xi-(n-2))\ldots(\xi-2)\Gamma(\xi-2)\Big\} \\
&= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n\frac{1}{\Gamma(\xi)}\Big\{(\xi-(n-1))\Gamma(\xi-(n-1))\Big\} \\
&= P_0\Big(\frac{\lambda}{\mu}\Big)^n\frac{1}{n!\Gamma(\xi)}\Gamma(\xi-n) \\
&= P_0\Big(\frac{\lambda}{\mu}\Big)^n\frac{\Gamma(\xi-n)}{n!\Gamma(\xi)} \\
&= P_0\Big(1-\frac{\lambda}{\lambda+\mu}\Big)^{\xi-n}\begin{pmatrix}\xi\\ n\end{pmatrix} \\
\therefore P_n &= P_0\Big(1-\rho\Big)^{\xi-n}\begin{pmatrix}\xi\\ n\end{pmatrix} \\
\therefore P_n &= \begin{pmatrix}\xi\\ n\end{pmatrix}\Big(1-\rho\Big)^{\xi-n}P_0 \quad n=0,1,2,\ldots
\end{aligned}
$$

But,

$$\sum_{n=0}^{\infty} P_n = 1$$

i.e.,

$$P_0 \sum_{n=0}^{\infty} \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-n} = 1$$

$$P_0 \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} -\xi \\ n \end{pmatrix} \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\xi-n} = 1$$

$$P_0 \sum_{n=0}^{\infty} \begin{pmatrix} -\xi \\ n \end{pmatrix} \Big(\frac{-\lambda}{\lambda+\mu}\Big)^{\xi-n} = 1$$

$$P_0 \Big(\frac{\lambda}{\lambda+\mu}\Big)^{-n} = 1$$

$$\therefore P_0 = \Big(\frac{\lambda}{\lambda+\mu}\Big)^{n}$$

$$\therefore P_0 = \Big(\rho\Big)^{n}$$

$$\therefore P_n = \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(1-\rho\Big)^{\xi-n} \Big(\rho\Big)^{n} \quad n = 0, 1, 2, ..., \xi - 1.$$

Which is a Binomial Distribution with parameters ξ and ρ.

Let $\xi = \xi$ and $\frac{\lambda}{\lambda+\mu}$=$\rho$;

Thus, the Binomial distribution becomes;

$$P_n = \begin{pmatrix} \xi \\ n \end{pmatrix} \Big(1-\rho\Big)^{\xi-n} \Big(\rho\Big)^{n} \quad n = 0, 1, 2, ..., \xi - 1.$$

Then;
Gauss hypergeometric series given by;

$$
\begin{aligned}
{}_2F_1(\xi,\gimel;\gimel;\rho) &= 1+\frac{\xi.\gimel}{1}\frac{\rho}{1!}+\frac{\xi(\xi+1)\gimel(\gimel+1)}{\gimel(\gimel+1)}\frac{(\rho)^2}{2!}+\ldots+\frac{(\xi+n-1)(\gimel+n-1)}{(\gimel+n-1)}\frac{(\rho)^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\xi(\xi+1)\ldots(\xi+n-1)\gimel(\gimel+1)(\gimel+2)\ldots(\gimel+n-1)}{\gimel(\gimel+1)(\gimel+2)\ldots(\gimel+n-1)}\frac{(\rho)^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(\xi+n-1)\ldots(\xi+1)\xi(\gimel+n-1)\ldots(\gimel+2)(\gimel+1)\gimel}{(\gimel+n-1)\ldots(\gimel+2)(\gimel+1)\gimel}\frac{(\rho)^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\xi+n)\Gamma(\gimel+n)}{\Gamma(\gimel+n)}.\frac{\Gamma(\gimel)}{\Gamma(\xi)\Gamma(\gimel)}\frac{(\rho)^n}{n!}
\end{aligned}
$$

Normalizing, we get;

$$
1 = \sum_{n=0}^{\infty}\frac{\Gamma(\xi+n)\Gamma(\gimel+n)}{\Gamma(\gimel+n)}.\frac{\Gamma(\gimel)}{\Gamma(\xi)\Gamma(\gimel)}.\frac{1}{{}_2F_1(\xi,\gimel;\gimel;\rho)}\frac{(\rho)^n}{n!}
$$

Hence,

$$
\begin{aligned}
P_n &= \text{Prob.}(N=n)\\
&= \frac{\Gamma(\xi+n)\Gamma(\gimel+n)}{\Gamma(\gimel+n)}.\frac{\Gamma(\gimel)}{\Gamma(\xi)\Gamma(\gimel)}.\frac{1}{{}_2F_1(\xi,\gimel;\gimel;\rho)}\frac{(\rho)^n}{n!}
\end{aligned}
$$

The pgf. in hypergeometric terms is given by;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty}P_n z^n\\
\phi(z) &= \sum_{n=0}^{\infty}\frac{\Gamma(\xi+n)\Gamma(\gimel+n)}{\Gamma(\gimel+n)}.\frac{\Gamma(\gimel)}{\Gamma(\xi)\Gamma(\gimel)}\frac{(\rho z)^n}{n!}.\frac{1}{{}_2F_1(\xi,\gimel;\gimel;\rho)}\\
\therefore \phi(z) &= \frac{{}_2F_1(\xi,\gimel;\gimel;\rho z)}{{}_2F_1(\xi,\gimel;\gimel;\rho)}.\\
\phi^{'}(z) &= \xi\rho\frac{{}_2F_1(\xi+1,\gimel+1;\gimel+1;\rho z)}{{}_2F_1(\xi,\gimel;\gimel;\rho)}\\
\phi^{''}(z) &= \xi(\xi+1)(\rho)^2\frac{{}_2F_1(\xi+2,\gimel+2;\gimel+2;\rho z)}{{}_2F_1(\xi,\gimel;\gimel;\rho)}\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_2F_1(\xi+\kappa,\gimel+\kappa;\gimel+\kappa;\rho)}{{}_2F_1(\xi,\gimel;\gimel;\rho)};\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \xi\rho\Lambda_1.\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2.\\
&= \xi(\xi+1)\rho^2\Lambda_2+\xi\rho\Lambda_1-[\xi\rho\Lambda_1]^2\\
\therefore Var(X) &= \xi\rho\Lambda_1+\xi\rho^2\{(\xi+1)\Lambda_2-\xi\Lambda_1^2\}.
\end{aligned}
$$

Using Proposition 2.2
We have;

$$
\begin{aligned}
\frac{P_n}{P_{n-1}} &= \frac{\lambda_{n-1}}{\mu_n}, \quad \mu_n \neq 0 \\
\text{But,} \quad \lambda_{n-1} &= (\xi - (n-1))\lambda, \\
&\text{and} \\
\mu_n &= n\mu. \\
\therefore \frac{P_n}{P_{n-1}} &= \frac{(\xi - (n-1))\lambda}{n\mu}. \\
\Rightarrow n\mu P_n &= (\xi - (n-1))\lambda P_{n-1} \qquad (4.81)
\end{aligned}
$$

Using the probability generating function technique, we multiply (4.81) by s^n and sum the results over n to obtain;

$$
\begin{aligned}
\mu \sum_{n=0}^{\infty} nP_n s^n &= \lambda \sum_{n=0}^{\infty} (\xi - (n-1))P_{n-1}s^n \\
\mu s \sum_{n=0}^{\infty} nP_n s^{n-1} &= \lambda \xi \sum_{n=0}^{\infty} P_{n-1}s^n - \lambda \sum_{n=0}^{\infty} (n-1))P_{n-1}s^n \\
\mu s \sum_{n=0}^{\infty} nP_n s^{n-1} &= \lambda \xi s \sum_{n=0}^{\infty} P_{n-1}s^{n-1} - \lambda s^2 \sum_{n=0}^{\infty} (n-1))P_{n-1}s^{n-2} \\
\mu s \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \lambda \xi s \psi(s) - \lambda s^2 \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \\
(\mu + \lambda s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \lambda \xi \psi(s) \\
\frac{\mathrm{d}\psi(s)}{\psi(s)} &= \frac{\lambda \xi}{(\mu + \lambda s)}\mathrm{d}s
\end{aligned}
$$

Taking the integral, we have;

$$
\begin{aligned}
\int \frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int \frac{\lambda \xi}{(\mu + \lambda s)}\mathrm{d}s \\
\ln \psi(s) &= \xi \ln(\frac{\lambda}{\lambda s + \mu}) + \ln c \\
\therefore \psi(s) &= c_1 \Big(\frac{\lambda}{\lambda s + \mu}\Big)^{\xi}
\end{aligned}
$$

Putting s=1;

$$
\begin{aligned}
\psi(1) &= 1 = c_1\Big(\frac{\lambda}{\lambda+\mu}\Big)^{\xi}\\
\Rightarrow c_1 &= \Big(\frac{\lambda}{\lambda+\mu}\Big)^{-\xi}.\\
\therefore \psi(s) &= \Big(\frac{\lambda}{\lambda+\mu}\Big)^{-\xi}\Big(\frac{\lambda}{\lambda s+\mu}\Big)^{\xi}.\\
\therefore \psi(s) &= \frac{\Big(\frac{\lambda}{\lambda s+\mu}\Big)^{\xi}}{\Big(\frac{\lambda}{\lambda+\mu}\Big)^{\xi}}.\\
\psi^{'}(s) &= \xi\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda s+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\xi-1}\\
\psi^{''}(s) &= \xi(\xi-1)\Big(\frac{\lambda}{\lambda+\mu}\Big)^{2}.\Big(\frac{(\frac{\lambda}{\lambda s+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\xi-2}\\
E(X) &= \psi^{'}(1)\\
&= \xi\Big(\frac{\lambda}{\lambda+\mu}\Big).\\
\therefore E(X) &= \xi\rho.\\
Var(X) &= \psi^{''}(1)+\psi^{'}(1)-[\psi^{'}(1)]^2\\
&= \xi(\xi-1)\Big(\frac{\lambda}{\lambda+\mu}\Big)^{2}.\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\xi-2}+\xi\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\xi-1}-[\xi\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\xi-1}]^2\\
&= \xi\Big(\frac{\lambda}{\lambda+\mu}\Big)\Big(1-\frac{\lambda}{\lambda+\mu}\Big).\\
\therefore Var(X) &= \xi\rho(1-\rho).
\end{aligned}
$$

Next,
When n=1;
Equation (4.75) becomes;

$$0 = [(\xi-1)\lambda + c\mu]P_1 + \lambda\xi P_0 + c\mu P_2 \tag{4.82}$$

We note that,

$$\mu c P_{n+1} = (\xi-n)\lambda P_n \quad (n \geq c)$$

$$\therefore P_1 = \frac{\xi\lambda}{c\mu}P_0. \tag{4.83}$$

Substituting (4.83) in equation (4.82), we get;

$$
\begin{aligned}
0 &= [(\xi-1)\lambda + c\mu]\frac{\xi\lambda}{c\mu}P_0 + \lambda\xi P_0 + c\mu P_2 \\
0 &= -\{\frac{\xi^2\lambda^2}{c\mu}P_0 - \frac{\xi\lambda^2}{c\mu}P_0 + \lambda\xi P_0\} + \lambda\xi P_0 + c\mu P_2 \\
0 &= -\frac{\xi^2\lambda^2}{c\mu}P_0 + \frac{\xi\lambda^2}{c\mu}P_0 - \lambda\xi P_0 + \lambda\xi P_0 + c\mu P_2 \\
0 &= -\frac{\xi^2\lambda^2}{c\mu}P_0 + \frac{\xi\lambda^2}{c\mu}P_0 + c\mu P_2 \\
\Longleftrightarrow c\mu P_2 &= \frac{\xi\lambda^2}{c\mu}(\xi-1)P_0 \\
\Rightarrow P_2 &= \frac{1}{c^2}\{\frac{\lambda^2}{\mu^2}\xi(\xi-1)\}P_0 \\
\therefore P_2 &= \frac{1}{c^2}\{\frac{\lambda^2}{\mu^2}\xi(\xi-1)\}P_0. \qquad \textbf{(4.84)}
\end{aligned}
$$

When n=2;
Equation (4.75) becomes;

$$
0 = -[(\xi-2)\lambda + c\mu]P_2 + \lambda(\xi-1)P_1 + c\mu P_3 \qquad \textbf{(4.85)}
$$

Substituting (4.83) and (4.84) in equation (4.85), we get;

$$
\begin{aligned}
0 &= -[(\xi-2)\lambda + c\mu]\frac{1}{c^2}\{\frac{\lambda^2}{\mu^2}\xi(\xi-1)\}P_0 + \lambda(\xi-1)\frac{\xi\lambda}{c\mu}P_0 + c\mu P_3 \\
0 &= -\frac{1}{c^2}\frac{\lambda^3}{\mu^2}\xi^2(\xi-1)P_0 + \frac{2\lambda^3}{c^2\mu^2}\xi(\xi-1)P_0 - \frac{\lambda^2}{c\mu}P_0 + \frac{\lambda^2}{c\mu}P_0 + c\mu P_3 \\
0 &= -\frac{1}{c^2}\frac{\lambda^3}{\mu^2}\xi^2(\xi-1)P_0 + \frac{2\lambda^3}{c^2\mu^2}\xi(\xi-1)P_0 + c\mu P_3 \\
\Longleftrightarrow c\mu P_3 &= \frac{1}{c^2}\frac{\lambda^3}{\mu^2}\xi^2(\xi-1)P_0 - \frac{2\lambda^3}{c^2\mu^2}\xi(\xi-1)P_0 \\
\Rightarrow c\mu P_3 &= \frac{\lambda^3}{c^2\mu^2}\{\xi(\xi-1)(\xi-2)\}P_0 \\
\therefore P_3 &= \frac{1}{c^3}\{\frac{\lambda^3}{\mu^3}\xi(\xi-1)(\xi-2)\}P_0.
\end{aligned}
$$

Hence,

$$P_4 = \frac{1}{c^4}\{\frac{\lambda^4}{\mu^4}\xi(\xi-1)(\xi-2)(\xi-3)\}P_0,$$
$$P_5 = \frac{1}{c^5}\{\frac{\lambda^5}{\mu^5}\xi(\xi-1)(\xi-2)(\xi-3)(\xi-4)\}P_0,$$
$$.$$
$$.$$
$$.$$
$$\therefore P_n = \frac{1}{c^{n-c}}\{\frac{\lambda^n}{\mu^n}\frac{\xi!}{c!(\xi-n)!}\}P_0.$$

But,

$$\sum_{n=0}^{\xi} P_n = 1$$

Which will give,

$$P_0 = \Big\{\sum_{n=0}^{c-1}\begin{pmatrix}\xi\\ n\end{pmatrix}\Big(\frac{\lambda}{\mu}\Big)^{\xi} + \sum_{n=c}^{\xi}\frac{\xi!}{(\xi-n)!}\frac{1}{c!c^{n-c}}\Big(\frac{\lambda}{\mu}\Big)^{n}\Big\}^{-1}.$$

Remark 4.3

The distribution of τ servers that are busy in the queueing system will be given by;
P_τ=Prob (τ=number of busy servers), $\tau = 0,1,2,3,\ldots,c$.

$$\therefore P_\tau = \begin{pmatrix}\xi\\ \tau\end{pmatrix}\Big(\frac{\lambda}{\mu}\Big)^{\tau}P_0.$$
$$P_0 = \Big[\sum_{i=0}^{c}\Big(\frac{\lambda}{\mu}\Big)^{i}\Big]^{-1}.$$
$$\therefore P_\tau = \begin{pmatrix}\xi\\ \tau\end{pmatrix}\Big(\frac{\lambda}{\mu}\Big)^{\tau}\Big[\sum_{i=0}^{c}\Big(\frac{\lambda}{\mu}\Big)^{i}\Big]^{-1}.$$

Which is called Engset Distribution.

4.9 M|M|c||m ($m \leq c$) Queueing Process

Let the number of busy servers be n. When n servers are busy, the total rate is $n\mu$. This means that there are $(\gamma - j)$ outside the system where the arrivals occurs with an average rate of arrival being $\lambda(\gamma - j)$; the distribution of inter-arrival time is exponential and is also independent and identically distributed with parameter λ.
In this case, we have;

$$\lambda_j = \lambda(\gamma - j), \quad \text{for} \quad 0 \leq k \leq \gamma$$
and
$$\mu_j = j\mu, \quad \text{for} \quad j = 0, 1, 2, 3, ..., \gamma.$$

The basic difference differential equations for the steady state are given by:

$$0 = -\lambda\gamma P_0 + \mu P_1 \tag{4.86}$$
$$0 = -[(\gamma - j)\lambda + j\mu]P_j + \lambda[\gamma - (j-1)]P_{j-1} + (j+1)\mu P_{j+1} \tag{4.87}$$

Hence,
Proposition 4.8

$$P_n = \begin{pmatrix} \gamma \\ n \end{pmatrix} \Big(\frac{\lambda}{\lambda + \mu}\Big)^n \Big(1 - \frac{\lambda}{\lambda + \mu}\Big)^{\gamma - n}, \quad n = 0, 1, 2, 3, ..., \gamma. \tag{4.88}$$

Proof
Solving the steady state equations iteratively, we have;
Equation (4.86) can be expressed as;

$$\mu P_1 = \lambda\gamma P_0$$
$$\Rightarrow P_1 = \gamma\frac{\lambda}{\mu}P_0$$
$$\therefore P_1 = \gamma\frac{\lambda}{\mu}P_0. \tag{4.89}$$

When j=1;
Equation (4.87) becomes;

$$0 = -[(\gamma-1)\lambda+\mu]P_1+\lambda\gamma P_0+2\mu P_2 \tag{4.90}$$

Substituting (4.89) in equation (4.90), we get;

$$\begin{aligned}
0 &= -[(\gamma-1)\lambda+\mu]\gamma\frac{\lambda}{\mu}P_0+\lambda\gamma P_0+2\mu P_2\\
0 &= -\gamma^2\frac{\lambda^2}{\mu}P_0+\gamma\frac{\lambda^2}{\mu}P_0-\gamma\lambda P_0+\gamma\lambda P_0+2\mu P_2\\
0 &= -\gamma^2\frac{\lambda^2}{\mu}P_0+\gamma\frac{\lambda^2}{\mu}P_0+2\mu P_2\\
\Longleftrightarrow 2\mu P_2 &= \gamma^2\frac{\lambda^2}{\mu}P_0-\gamma\frac{\lambda^2}{\mu}P_0\\
\Rightarrow 2\mu P_2 &= \gamma\frac{\lambda^2}{\mu}(\gamma-1)P_0\\
\therefore P_2 &= \frac{\lambda^2}{2!\mu^2}\gamma(\gamma-1)P_0.
\end{aligned} \tag{4.91}$$

When j=2;
Equation (4.87) becomes;

$$0 = -[(\gamma-2)\lambda+2\mu]P_2+\lambda(\gamma-1)P_1+3\mu P_3 \tag{4.92}$$

Substituting (4.89) and (4.91) in equation (4.92), we get;

$$\begin{aligned}
0 &= -[(\gamma-2)\lambda+2\mu]\frac{\lambda^2}{2!\mu^2}\gamma(\gamma-1)P_0+\lambda(\gamma-1)\gamma\frac{\lambda}{\mu}P_0+3\mu P_3\\
0 &= -\frac{\lambda^3}{2\mu^2}\gamma^2(\gamma-1)P_0+\frac{\lambda^3}{\mu^2}\gamma(\gamma-1)P_0-\frac{\lambda^2}{\mu}\gamma(\gamma-1)P_0+\frac{\lambda^2}{\mu}\gamma(\gamma-1)P_0+3\mu P_3\\
0 &= -\frac{\lambda^3}{2\mu^2}\gamma^2(\gamma-1)P_0+\frac{\lambda^3}{\mu^2}\gamma(\gamma-1)P_0+3\mu P_3
\end{aligned}$$

$$\iff 3\mu P_3 = \frac{\lambda^3}{2\mu^2}\gamma^2(\gamma-1)P_0 - \frac{\lambda^3}{\mu^2}\gamma(\gamma-1)P_0$$
$$\Rightarrow 3\mu P_3 = \frac{\lambda^3}{2\mu^2}\gamma(\gamma-1)(\gamma-2)P_0$$
$$\therefore P_3 = \frac{\lambda^3}{3!\mu^3}\gamma(\gamma-1)(\gamma-2)P_0.$$

Hence,

$$P_4 = \frac{\lambda^4}{4!\mu^4}\gamma(\gamma-1)(\gamma-2)(\gamma-3)P_0,$$
$$P_5 = \frac{\lambda^5}{5!\mu^5}\gamma(\gamma-1)(\gamma-2)(\gamma-3)(\gamma-4)P_0,$$
$$\vdots$$
$$P_n = \frac{\lambda^n}{n!\mu^n}\gamma(\gamma-1)(\gamma-2)...(\gamma-n)P_0.$$

But,

$$\sum_{n=0}^{\gamma} P_n = 1$$

Thus,

$$P_1 = \left(\frac{\lambda}{\mu}\right)^1 \begin{pmatrix} \gamma \\ n \end{pmatrix} P_0,$$
$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 \begin{pmatrix} \gamma \\ n \end{pmatrix} P_0,$$
$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 \begin{pmatrix} \gamma \\ n \end{pmatrix} P_0,$$

Therefore,

$$P_4 = \Big(\frac{\lambda}{\mu}\Big)^4 \begin{pmatrix} \gamma \\ n \end{pmatrix} P_0,$$

$$\vdots$$

$$P_n = \Big(\frac{\lambda}{\mu}\Big)^n \begin{pmatrix} \gamma \\ n \end{pmatrix} P_0.$$

$$\Rightarrow P_0 + \Big(\frac{\lambda}{\mu}\Big)^1 \begin{pmatrix} \gamma \\ n \end{pmatrix} P_0 + \Big(\frac{\lambda}{\mu}\Big)^2 \begin{pmatrix} \gamma \\ n \end{pmatrix} P_0 + \Big(\frac{\lambda}{\mu}\Big)^3 \begin{pmatrix} \gamma \\ n \end{pmatrix} P_0 + \ldots + \Big(\frac{\lambda}{\mu}\Big)^n \begin{pmatrix} \gamma \\ n \end{pmatrix} P_0 = 1$$

$$P_0 \Big\{ 1 + \Big(\frac{\lambda}{\mu}\Big)^1 \begin{pmatrix} \gamma \\ n \end{pmatrix} + \Big(\frac{\lambda}{\mu}\Big)^2 \begin{pmatrix} \xi \\ n \end{pmatrix} + \Big(\frac{\lambda}{\mu}\Big)^3 \begin{pmatrix} \gamma \\ n \end{pmatrix} + \ldots + \Big(\frac{\lambda}{\mu}\Big)^n \begin{pmatrix} \gamma \\ n \end{pmatrix} \Big\} = 1$$

$$\therefore \Big(1 + \frac{\lambda}{\mu}\Big)^n P_0 = 1$$

$$\therefore P_0 = \Big(1 - \frac{\lambda}{\lambda + \mu}\Big)^n, \quad \text{for} \quad n = 0, 1, 2, \ldots, \gamma.$$

$$\therefore P_n = \Big\{ \begin{pmatrix} \gamma \\ 0 \end{pmatrix} \Big(\frac{\lambda}{\lambda+\mu}\Big)^0 \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\gamma-0} + \begin{pmatrix} \gamma \\ 1 \end{pmatrix} \Big(\frac{\lambda}{\lambda+\mu}\Big)^1 \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\gamma-1} + \ldots$$
$$+ \begin{pmatrix} \gamma \\ n \end{pmatrix} \Big(\frac{\lambda}{\lambda+\mu}\Big)^n \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\gamma-n} \Big\}$$

Which is the Binomial Distribution with parameter $\frac{\lambda}{\lambda+\mu}$ and γ.

$$P_n = \begin{pmatrix} \gamma \\ n \end{pmatrix} \Big(\frac{\lambda}{\lambda+\mu}\Big)^n \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{\gamma-n}, \quad n = 0, 1, 2, 3, \ldots, \gamma.$$

$$\text{Letting} \quad \frac{\lambda}{\lambda+\mu} = \rho, \quad \text{we have;}$$

$$\therefore P_n = \begin{pmatrix} \gamma \\ n \end{pmatrix} \left(\rho\right)^n \left(1-\rho\right)^{\gamma-n}, \quad n = 0,1,2,3,...,\gamma.$$

Using Proposition 2.1

We have;

$$P_n = \frac{\lambda_0\lambda_1\lambda_2\lambda_3...\lambda_{n-1}}{\mu_1\mu_2\mu_3...\mu_n}P_0$$

But, $\lambda_0 = \gamma\lambda, \quad \lambda_1 = (\gamma-1)\lambda, \quad \lambda_2 = (\gamma-2)\lambda, \quad \lambda_3 = (\gamma-3)\lambda, \quad ... = \lambda_{n-1} = (\gamma-(n-1))\lambda.$

and

$\mu_1 = \mu, \quad \mu_2 = 2\mu, \quad \mu_3 = 3\mu, \quad ..., \quad \mu_n = n\mu.$

$$\begin{aligned}
\therefore P_n &= \frac{\gamma\lambda(\gamma-1)\lambda(\gamma-2)\lambda(\gamma-3)\lambda...(\gamma-(n-1))\lambda}{\mu 2\mu 3\mu ... n\mu}P_0 \\
&= \frac{\gamma\lambda(\gamma-1)\lambda(\gamma-2)\lambda(\gamma-3)\lambda...(\gamma-(n-1))\lambda}{n!\mu^n}P_0 \\
&= \frac{P_0}{n!\mu^n}\prod_{j=0}^{n-1}(\gamma-j)\lambda \\
&= \frac{\lambda^n P_0}{n!\mu^n}\prod_{j=0}^{n-1}(\gamma-j) \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\Big\{\gamma(\gamma-1)(\gamma-2)(\gamma-3)...(\gamma-(n-1))\Big\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\Big\{(\gamma-(n-1))(\gamma-(n-2))...(\gamma-2)(\gamma-1)\gamma\Big\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\Big\{(\gamma-(n-1))(\gamma-(n-2))...(\gamma-2)(\gamma-1)\gamma\frac{\Gamma(\gamma)}{\Gamma(\gamma)}\Big\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\Big\{(\gamma-(n-1))(\gamma-(n-2))...(\gamma-2)(\gamma-1)\frac{\Gamma(\gamma-1)}{\Gamma(\gamma)}\Big\} \\
&= \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n\frac{1}{\Gamma(\gamma)}\Big\{(\gamma-(n-1))(\gamma-(n-2))...(\gamma-2)\Gamma(\gamma-2)\Big\}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
P_n &= \frac{P_0}{n!}\Big(\frac{\lambda}{\mu}\Big)^n \frac{1}{\Gamma(\gamma)}\Big\{(\gamma-(n-1))\Gamma(\gamma-(n-1))\Big\} \\
&= P_0\Big(\frac{\lambda}{\mu}\Big)^n \frac{1}{n!\Gamma(\gamma)}\Gamma(\gamma-n) \\
&= P_0\Big(\frac{\lambda}{\mu}\Big)^n \frac{\Gamma(\gamma-n)}{n!\Gamma(\gamma)} \\
&= P_0\Big(1-\frac{\lambda}{\lambda+\mu}\Big)^{\gamma-n}\begin{pmatrix}\gamma\\ n\end{pmatrix} \\
\therefore P_n &= P_0\Big(1-\rho\Big)^{\gamma-n}\begin{pmatrix}\gamma\\ n\end{pmatrix} \\
\therefore P_n &= \begin{pmatrix}\gamma\\ n\end{pmatrix}\Big(1-\rho\Big)^{\gamma-n} P_0 \quad n=0,1,2,\ldots
\end{aligned}
$$

But,

$$\sum_{n=0}^{\infty} P_n = 1$$

i.e.,

$$
\begin{aligned}
P_0\sum_{n=0}^{\infty}\begin{pmatrix}\xi\\ n\end{pmatrix}\Big(1-\frac{\lambda}{\lambda+\mu}\Big)^{\gamma-n} &= 1 \\
P_0\sum_{n=0}^{\infty}(-1)^n\begin{pmatrix}-\gamma\\ n\end{pmatrix}\Big(1-\frac{\lambda}{\lambda+\mu}\Big)^{\gamma-n} &= 1 \\
P_0\sum_{n=0}^{\infty}\begin{pmatrix}-\gamma\\ n\end{pmatrix}\Big(\frac{-\lambda}{\lambda+\mu}\Big)^{\gamma-n} &= 1 \\
P_0\Big(\frac{\lambda}{\lambda+\mu}\Big)^{-n} &= 1 \\
\therefore P_0 &= \Big(\frac{\lambda}{\lambda+\mu}\Big)^n \\
\therefore P_0 &= \Big(\rho\Big)^n
\end{aligned}
$$

$$\therefore P_n = \begin{pmatrix} \gamma \\ n \end{pmatrix} \left(1-\rho\right)^{\gamma-n} \left(\rho\right)^{n} \quad n = 0, 1, 2, ..., \gamma.$$

Which is a Binomial Distribution with parameters γ and ρ.

Where, $\gamma = \gamma$ and $\frac{\lambda}{\lambda+\mu}$=$\rho$;

Thus, the Binomial distribution becomes;

$$P_n = \begin{pmatrix} \gamma \\ n \end{pmatrix} \left(1-\rho\right)^{\gamma-n} \left(\rho\right)^{n} \quad n = 0, 1, 2, ..., \gamma.$$

Then;
Using the Gauss hypergeometric series we have;

$$\begin{aligned}
{}_2F_1(\gamma, \iota; \iota; \rho) &= 1 + \frac{\gamma.\iota}{1}\frac{\rho}{1!} + \frac{\gamma(\gamma+1)\iota(\iota+1)}{\iota(\iota+1)}\frac{(\rho)^2}{2!} + ... + \frac{(\gamma+n-1)(\iota+n-1)}{(\iota+n-1)}\frac{(\rho)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\gamma(\gamma+1)...(\gamma+n-1)\iota(\iota+1)(\iota+2)...(\iota+n-1)}{\iota(\iota+1)(\iota+2)...(\iota+n-1)}\frac{(\rho)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(\gamma+n-1)...(\gamma+1)\gamma(\iota+n-1)...(\iota+2)(\iota+1)\iota}{(\iota+n-1)...(\iota+2)(\iota+1)\iota}\frac{(\rho)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\Gamma(\iota+n)}{\Gamma(\iota+n)}.\frac{\Gamma(\iota)}{\Gamma(\gamma)\Gamma(\iota)}\frac{(\rho)^n}{n!}
\end{aligned}$$

Normalizing, we get;

$$1 = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\Gamma(\iota+n)}{\Gamma(\iota+n)}.\frac{\Gamma(\iota)}{\Gamma(\gamma)\Gamma(\iota)}.\frac{1}{{}_2F_1(\gamma, \iota; \iota; \rho)}\frac{(\rho)^n}{n!}$$

Hence,

$$
\begin{aligned}
P_n &= \text{Prob.}(N=n)\\
&= \frac{\Gamma(\gamma+n)\Gamma(\iota+n)}{\Gamma(\iota+n)}.\frac{\Gamma(\iota)}{\Gamma(\gamma)\Gamma(\iota)}.\frac{1}{{}_2F_1(\gamma,\iota;\iota;\rho)}\frac{(\rho)^n}{n!}
\end{aligned}
$$

The probability generating function in hypergeometric terms is given by;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty} P_n z^n\\
\phi(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\Gamma(\iota+n)}{\Gamma(\iota+n)}.\frac{\Gamma(\iota)}{\Gamma(\gamma)\Gamma(\iota)}\frac{(\rho z)^n}{n!}.\frac{1}{{}_2F_1(\gamma,\iota;\iota;\rho)}\\
\therefore \phi(z) &= \frac{{}_2F_1(\gamma,\iota;\iota;\rho z)}{{}_2F_1(\gamma,\iota;\iota;\rho)}.\\
\phi^{'}(z) &= \gamma\rho\frac{{}_2F_1(\gamma+1,\iota+1;\gimel+1;\rho z)}{{}_2F_1(\gamma,\iota;\iota;\rho)}\\
\phi^{''}(z) &= \gamma(\gamma+1)(\rho)^2\frac{{}_2F_1(\gamma+2,\iota+2;\iota+2;\rho z)}{{}_2F_1(\gamma,\iota;\iota;\rho)}\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_2F_1(\gamma+\kappa,\iota+\kappa;\iota+\kappa;\rho)}{{}_2F_1(\gamma,\iota;\iota;\rho)};\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \gamma\rho\Lambda_1.\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2.\\
&= \gamma(\gamma+1)\rho^2\Lambda_2+\gamma\rho\Lambda_1-[\gamma\rho\Lambda_1]^2\\
\therefore Var(X) &= \gamma\rho\Lambda_1+\gamma\rho^2\{(\gamma+1)\Lambda_2-\gamma\Lambda_1^2\}.
\end{aligned}
$$

Using Proposition 2.2

We have;

$$
\begin{aligned}
\frac{P_n}{P_{n-1}} &= \frac{\lambda_{n-1}}{\mu_n},\quad \mu_n\neq 0\\
\text{But,}\quad \lambda_{n-1} &= (\gamma-(n-1))\lambda,\\
&\text{and}\\
\mu_n &= n\mu.\\
\therefore \frac{P_n}{P_{n-1}} &= \frac{(\gamma-(n-1))\lambda}{n\mu}.\\
\Rightarrow n\mu P_n &= (\gamma-(n-1))\lambda P_{n-1} \qquad \textbf{(4.93)}
\end{aligned}
$$

Using the probability generating function technique, we multiply (4.93) by s^n and sum the results over n to obtain;

$$
\begin{aligned}
\mu \sum_{n=0}^{\infty} nP_n s^n &= \lambda \sum_{n=0}^{\infty} (\gamma - (n-1))P_{n-1}s^n \\
\mu s \sum_{n=0}^{\infty} nP_n s^{n-1} &= \lambda \gamma \sum_{n=0}^{\infty} P_{n-1}s^n - \lambda \sum_{n=0}^{\infty} (n-1))P_{n-1}s^n \\
\mu s \sum_{n=0}^{\infty} nP_n s^{n-1} &= \lambda \gamma s \sum_{n=0}^{\infty} P_{n-1}s^{n-1} - \lambda s^2 \sum_{n=0}^{\infty} (n-1))P_{n-1}s^{n-2} \\
\mu s \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \lambda \gamma s \psi(s) - \lambda s^2 \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \\
(\mu + \lambda s) \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \lambda \gamma \psi(s) \\
\frac{\mathrm{d}\psi(s)}{\psi(s)} &= \frac{\lambda \gamma}{(\mu + \lambda s)} \mathrm{d}s
\end{aligned}
$$

Integral both sides, we have;

$$
\begin{aligned}
\int \frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int \frac{\lambda \gamma}{(\mu + \lambda s)} \mathrm{d}s \\
\ln \psi(s) &= \gamma \ln(\frac{\lambda}{\lambda s + \mu}) + \ln c \\
\therefore \psi(s) &= c_1 \Big(\frac{\lambda}{\lambda s + \mu}\Big)^{\gamma}
\end{aligned}
$$

Putting s=1;

$$
\begin{aligned}
\psi(1) &= 1 = c_1\Big(\frac{\lambda}{\lambda+\mu}\Big)^{\gamma}\\
\Rightarrow c_1 &= \Big(\frac{\lambda}{\lambda+\mu}\Big)^{-\gamma}.\\
\therefore \psi(s) &= \Big(\frac{\lambda}{\lambda+\mu}\Big)^{-\gamma}\Big(\frac{\lambda}{\lambda s+\mu}\Big)^{\gamma}.\\
\therefore \psi(s) &= \frac{\Big(\frac{\lambda}{\lambda s+\mu}\Big)^{\gamma}}{\Big(\frac{\lambda}{\lambda+\mu}\Big)^{\gamma}}.\\
\psi^{'}(s) &= \gamma\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda s+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\gamma-1}\\
\psi^{''}(s) &= \gamma(\gamma-1)\Big(\frac{\lambda}{\lambda+\mu}\Big)^{2}.\Big(\frac{(\frac{\lambda}{\lambda s+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\gamma-2}\\
E(X) &= \psi^{'}(1)\\
&= \gamma\Big(\frac{\lambda}{\lambda+\mu}\Big).\\
\therefore E(X) &= \gamma\rho.\\
Var(X) &= \psi^{''}(1)+\psi^{'}(1)-[\psi^{'}(1)]^2\\
&= \gamma(\gamma-1)\Big(\frac{\lambda}{\lambda+\mu}\Big)^{2}.\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\gamma-2}+\gamma\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\gamma-1}-[\xi\Big(\frac{\lambda}{\lambda+\mu}\Big).\Big(\frac{(\frac{\lambda}{\lambda+\mu})}{(\frac{\lambda}{\lambda+\mu})}\Big)^{\gamma-1}]^2\\
&= \gamma\Big(\frac{\lambda}{\lambda+\mu}\Big)\Big(1-\frac{\lambda}{\lambda+\mu}\Big).\\
\therefore Var(X) &= \gamma\rho(1-\rho).
\end{aligned}
$$

5 KATZ FAMILY OF RECURSIVE MODELS IN BIRTH AND DEATH PROCESSES AT EQUILIBRIUM

5.1 Introduction

Suppose P_n, the distribution of population size n is proportional to $\Lambda_n/(M_n n!)$, where $\Lambda_n = \lambda_0\lambda_1\lambda_2...\lambda_{n-1}$ and $M_n = \mu_1\mu_2\mu_3...\mu_n; \lambda_n(\mu_n)$ is the birth (death) rateb when the population size is n.
In this case, the Pearson's differential equation given by;

$$\frac{P_{n+1}}{P_n} = \frac{P(n)}{Q(n)} \tag{5.1}$$

where $P(.)$ is the probability mass function; $P(n)$ and $Q(n)$ are polynomials.
(5.1) above will be used to determine some special cases and properties of some recursive models.

5.2 Katz (1965) Recursive Model

Katz considered the relationship;

$$\frac{P_{n+1}}{P_n} = \frac{\alpha+\beta n}{1+n}; \quad n = 0,1,2,... \tag{5.2}$$

$$\text{where,} \quad \sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow P_{n+1} = P_n \frac{\alpha+\beta n}{n+1}$$

$$\therefore P_{n+1}(n+1) = P_n(\alpha+\beta n) \tag{5.3}$$

Proposition 5.1

$$\frac{P_{n+1}}{P_n} = \frac{\alpha+\beta n}{1+n}; \quad n = 0,1,2,...$$

Proof
Solving equation (5.3) iteratively, we have;
When n=0;
Equation (5.3) becomes;

$$\begin{aligned} P_1 &= \alpha P_0 \\ \therefore P_1 &= \alpha P_0. \end{aligned} \tag{5.4}$$

When n=1;
Equation (5.3) becomes;

$$2P_2 = (\alpha+\beta)P_1 \tag{5.5}$$

Substituting (5.4) in equation (5.5), we have;

$$\begin{aligned} 2P_2 &= (\alpha+\beta)\alpha P_0 \\ \Rightarrow P_2 &= \frac{\alpha(\alpha+\beta)}{1.2}P_0 \\ \therefore P_2 &= \frac{\alpha(\alpha+\beta)}{2!}P_0. \end{aligned} \tag{5.6}$$

When n=2;
Equation (5.3) becomes;

$$3P_3 = (\alpha+2\beta)P_2 \tag{5.7}$$

Substituting (5.6) in equation (5.7), we have;

$$\begin{aligned} 3P_3 &= (\alpha+2\beta)\frac{\alpha(\alpha+\beta)}{1.2}P_0 \\ \Rightarrow P_3 &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{1.2.3}P_0 \\ \therefore P_3 &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{3!}P_0. \end{aligned}$$

By Mathematical induction;

$$\begin{aligned}
P_4 &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)}{4!}P_0,\\
P_5 &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)(\alpha+4\beta)}{5!}P_0,\\
&.\\
&.\\
&.\\
P_n &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)...(\alpha+(n-2)\beta)(\alpha+(n-1)\beta)}{n!}P_0. \qquad (5.8)
\end{aligned}$$

But,

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n &= 1\\
\Rightarrow P_0+P_1+P_2+P_3+...+P_n+... &= 1\\
P_0+\alpha P_0+\frac{\alpha(\alpha+\beta)}{2!}P_0+\frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{3!}P_0+... &= 1\\
P_0\left\{1+\frac{\alpha}{1!}+\frac{\alpha(\alpha+\beta)}{2!}+\frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{3!}+...\right\} &= 1\\
\therefore P_0 = \frac{1}{\left\{1+\frac{\alpha}{1!}+\frac{\alpha(\alpha+\beta)}{2!}+\frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{3!}+...\right\}}.
\end{aligned}$$

$$\therefore P_n = \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)...(\alpha+(n-2)\beta)(\alpha+(n-1)\beta)}{n!}.\frac{1}{\left\{1+\frac{\alpha}{1!}+\frac{\alpha(\alpha+\beta)}{2!}+\frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{3!}+...\right\}}.$$

From (5.8), we have that;

$$\begin{aligned}
P_{n+1} &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)...(\alpha+(n-1)\beta)(\alpha+\beta)}{(n+1)!}P_0.\\
\therefore \frac{P_{n+1}}{P_n} &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)...(\alpha+(n-1)\beta)(\alpha+\beta)}{n!(1+n)}.\frac{n!}{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)...(\alpha+(n-1)\beta)}\\
\therefore \frac{P_{n+1}}{P_n} &= \frac{n\beta+\alpha}{1+n}; \quad \text{as required.}
\end{aligned}$$

5.2.1 Special Cases and Properties

(i) When $\alpha \neq 0$ and $\beta = 0$ Then equation (5.3) becomes;

$$(1+n)P_{n+1} = \alpha P_n, \quad n = 0, 1, 2, \dots \tag{5.9}$$

When n=0;
Equation (5.9) become;

$$P_1 = \alpha P_0$$
$$\therefore P_1 = \alpha P_0.$$

When n=1;
Equation (5.9) become;

$$2P_2 = \alpha P_1$$
$$\Rightarrow P_2 = \frac{\alpha^2}{1.2} P_0$$
$$\therefore P_2 = \frac{\alpha^2}{2!} P_0.$$

When n=2;
Equation (5.9) become;

$$3P_3 = \alpha P_2$$
$$\Rightarrow P_3 = \frac{\alpha^3}{1.2.3} P_0$$
$$\therefore P_3 = \frac{\alpha^3}{3!} P_0.$$

Thus,
When n=$\tau - 1$;
Equation (5.9) become;

$$\begin{aligned} \tau P_\tau &= \alpha P_{\tau-1} \\ \Rightarrow P_\tau &= \frac{\alpha}{\tau} P_{\tau-1} \\ \therefore P_\tau &= \frac{\alpha^\tau}{\tau!} P_0, \quad \text{for} \quad \tau = 0,1,2,... \end{aligned} \tag{5.10}$$

Therefore (5.10) becomes;

$$\sum_{\tau=0}^{\infty} P_\tau = P_0 + \sum_{\tau=1}^{\infty} \frac{\alpha^\tau}{\tau!} P_0.$$

But,

$$\sum_{\tau=0}^{\infty} P_\tau = 1$$

Hence,

$$\begin{aligned} 1 &= \sum_{\tau=0}^{\infty} \frac{\alpha^\tau}{\tau!} P_0 \\ 1 &= P_0 \sum_{\tau=0}^{\infty} \frac{\alpha^\tau}{\tau!} \\ 1 &= P_0 e^\alpha \\ \therefore P_0 &= e^{-\alpha}. \end{aligned} \tag{5.11}$$

Substituting (5.11) in equation (5.10), we get;

$$\therefore P_\tau = \frac{\alpha^\tau}{\tau!} e^{-\alpha}, \quad \text{for} \quad \tau = 0,1,2,... \tag{5.12}$$

Which is the pgf. for a Poisson Distribution with parameter α.
Its pgf. in hypergeometric terms is given by;

$$
\begin{aligned}
\phi(z) &= \frac{{}_1F_1(1;1;\alpha z)}{{}_1F_1(1;1;\alpha)}. && (5.13)\\
\phi^{'}(z) &= \frac{1}{{}_1F_1(1;1;\alpha)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_1F_1(1;1;\alpha z)\}\\
&= \frac{1}{{}_1F_1(1;1;\alpha)}.\frac{\alpha}{1}.{}_1F_1(1+1;1+1;\alpha z)\\
&= \frac{\alpha}{1}\frac{{}_1F_1(2;2;\alpha z)}{{}_1F_1(1;1;\alpha)}.\\
\phi^{''}(z) &= \frac{1}{{}_1F_1(1;1;\alpha)}.\frac{\alpha}{1}.\frac{\mathrm{d}}{\mathrm{d}z}\{{}_1F_1(2;2;\alpha z)\}\\
&= \frac{1}{{}_1F_1(1;1;\alpha)}.\frac{\alpha}{1}.\frac{2\alpha}{2}.{}_1F_1(2+1;2+1;\alpha z)\\
&= \frac{2\alpha^2}{1.2}.\frac{{}_1F_1(3;3;\alpha z)}{{}_1F_1(1;1;\alpha)}.\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_1F_1(1+\kappa;1+\kappa;\alpha)}{{}_1F_1(1;1;\alpha)};\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\alpha}{1}\Lambda_1. && (5.14)\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= \frac{2\alpha^2}{1.2}\Lambda_2+\frac{\alpha}{1}\Lambda_1-[\frac{\alpha}{1}\Lambda_1]^2\\
\therefore Var(X) &= \frac{\alpha}{1}\Lambda_1+\frac{\alpha^2}{1}\{\frac{2}{2}\Lambda_2-\frac{1}{1}\Lambda_1^2\}. && (5.15)
\end{aligned}
$$

When multiplying equation (5.9) by s^n and sum the results over n, we obtain;

$$
\sum_{n=0}^{\infty}(1+n)P_{n+1}s^n = \sum_{n=0}^{\infty}\alpha P_n s^n \qquad (5.16)
$$

Define;

$$
\begin{aligned}
\psi(s) &= \sum_{n=0}^{\infty}P_n s^n,\\
\psi^{'}(s) &= \sum_{n=0}^{\infty}nP_n s^{n-1},\\
\psi^{'}(s) &= \sum_{n=0}^{\infty}(n+1)P_n s^n.
\end{aligned}
$$

Therefore, equation (5.16) becomes;

$$
\begin{aligned}
\psi^{'}(s) &= \sum_{n=0}^{\infty} \alpha P_n s^n \\
\psi^{'}(s) &= \alpha \psi(s) \\
\Rightarrow \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \alpha \psi(s) \\
\iff \int \frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int \alpha \mathrm{d}s \\
\ln \psi(s) &= \alpha s + c_1 \\
\psi(s) &= e^{\alpha s}.e^{c_1}
\end{aligned}
$$

Putting s=1;

$$
\begin{aligned}
\psi(1) &= 1 = e^{\alpha}.e^{c_1} \\
\Rightarrow e^{c_1} &= c = e^{-\alpha}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\psi(s) &= e^{\alpha s}.e^{-\alpha} \\
\therefore \psi(s) &= e^{-(1-s)}. \\
\psi^{'}(s) &= \alpha e^{-\alpha(1-s)} \\
\psi^{''}(s) &= \alpha^2 e^{-\alpha(1-s)}
\end{aligned}
$$

Putting s=1, we have;

$$
\begin{aligned}
\psi^{'}(s) &= \alpha \\
\psi^{''}(s) &= \alpha^2 \\
E(X) &= \psi^{'}(1) \\
\therefore E(X) &= \alpha. \\
Var(X) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= \alpha^2 + \alpha - \alpha^2 \\
\therefore Var(X) &= \alpha.
\end{aligned}
$$

Using Feller's method, we sum equation (5.9) over n to obtain;

$$\sum_{n=0}^{\infty} P_{n+1}(n+1) = \sum_{n=0}^{\infty} P_n \alpha \tag{5.17}$$

Define;

$$M_1 = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} P_{n+1}(1+n)$$

Thus, equation (5.17) becomes;

$$\begin{aligned} M_1 &= \alpha.1 \\ E(X) &= M_1 \\ \therefore E(X) &= \alpha. \end{aligned}$$

Next, multiply equation (5.9) by n and after summing the results over n, we obtain;

$$\sum_{n=0}^{\infty} n(1+n)P_{n+1} = \alpha \sum_{n=0}^{\infty} nP_n \tag{5.18}$$

Define;

$$M_2 = \sum_{n=0}^{\infty} n^2 P_n = \sum_{n=0}^{\infty} (1+n)^2 P_{n+1}$$

Thus, equation (5.18) becomes;

$$\begin{aligned} \sum_{n=0}^{\infty} (1-1+n)(1+n)P_{n+1} &= \alpha M_1 \\ \sum_{n=0}^{\infty} (1+n)^2 P_{n+1} - \sum_{n=0}^{\infty} (1+n)P_{n+1} &= \alpha M_1 \\ M_2 - M_1 &= \alpha M_1 \\ \Rightarrow M_2 &= \alpha M_1 + M_1 \end{aligned}$$

But,

$$
\begin{aligned}
M_1 &= \alpha \\
\text{Hence,} \quad M_2 &= \alpha^2 + \alpha \\
\text{Thus,} \quad Var(X) &= M_2 - [M_1]^2 \\
&= \alpha^2 + \alpha - \alpha^2 \\
\therefore Var(X) &= \alpha.
\end{aligned}
$$

(ii) When $\alpha \neq 0$ and $\beta \neq 0$
Then, equation (5.3) becomes;

$$
P_{n+1}(1+n) = P_n(n\beta + \alpha), \quad n = 0, 1, 2, 3, ... \tag{5.19}
$$

Solving equation (5.19) iteratively, we have;
When n=0;
Equation (5.19) becomes;

$$
\begin{aligned}
P_1 &= \alpha P_0 \\
\therefore P_1 &= \alpha P_0.
\end{aligned}
$$

When n=1;
Equation (5.19) becomes;

$$
\begin{aligned}
2P_2 &= (\beta + \lambda)P_1 \\
P_2 &= \frac{\alpha + \beta}{2} . \frac{\alpha}{1} P_0 \\
\therefore P_2 &= \frac{\alpha(\alpha + \beta)}{1.2} P_0.
\end{aligned}
$$

When n=2;
Equation (5.19) becomes;

$$
\begin{aligned}
3P_3 &= (\alpha+2\beta)P_2\\
P_3 &= \frac{(\alpha+2\beta)}{3}P_2\\
\Rightarrow P_3 &= \frac{(\alpha+2\beta)}{3}.\frac{\alpha(\alpha+\beta)}{1.2}P_0\\
\therefore P_3 &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{1.2.3}P_0.
\end{aligned}
$$

When n=3;
Equation (5.19) becomes;

$$
\begin{aligned}
4P_4 &= (\alpha+3\beta)P_3\\
\Rightarrow P_4 &= \frac{(\alpha+3\beta)}{4}P_3\\
\Rightarrow P_4 &= \frac{(\alpha+3\beta)}{4}.\frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{1.2.3}P_0\\
\therefore P_4 &= \frac{\alpha(\alpha+\beta)(\alpha+2\alpha)(\alpha+3\beta)}{1.2.3.4}P_0.
\end{aligned}
$$

Therefore,
When n=$\tau-1$;
Equation (5.19) becomes;

$$
\begin{aligned}
\tau P_{\tau-1} &= \{\alpha+(\tau-1)\beta\}P_{\tau-1}\\
\Rightarrow P_{\tau-1} &= \frac{\{\alpha+(\tau-1)\beta\}}{\tau}P_{\tau-1}\\
P_\tau &= \Big(\frac{\alpha+(\tau-1)\beta}{\tau}\Big)\Big(\frac{\alpha+(\tau-2)\beta}{\tau-1}\Big)...\Big(\frac{\alpha+3\beta}{4}\Big)\Big(\frac{\alpha+2\beta}{3}\Big)\Big(\frac{\alpha+\beta}{2}\Big)\Big(\frac{\alpha}{1}\Big)P_0\\
P_\tau &= \alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)(\alpha+4\beta)...(\alpha+(\tau-2)\beta)(\alpha+(\tau-1)\beta)\frac{P_0}{\tau!}\\
P_\tau &= \beta^\tau\frac{\alpha}{\beta}(\frac{\alpha}{\beta}+1)(\frac{\alpha}{\beta}+2)(\frac{\alpha}{\beta}+3)...(\frac{\alpha}{\beta}+\tau-2)(\frac{\alpha}{\beta}+\tau-1)\frac{P_0}{\tau!} \qquad \textbf{(5.20)}\\
P_\tau &= \frac{\beta^\tau(\frac{\alpha}{\beta}+\tau-1)!}{(\frac{\alpha}{\beta}-1)!\tau!}P_0.\\
\therefore P_\tau &= \beta^\tau\begin{pmatrix}\frac{\alpha}{\beta}+\tau-1\\ \tau\end{pmatrix}P_0, \quad \tau=0,1,2,3,... \qquad \textbf{(5.21)}
\end{aligned}
$$

Suppose (5.21) above is a probability mass function,
Then;

$$\sum_{\tau=0}^{\infty} P_\tau = 1$$

$$\Rightarrow P_0 + \sum_{\tau=1}^{\infty} P_\tau = 1$$

$$\text{That is,} \quad P_0 + \sum_{\tau=1}^{\infty} \beta^\tau \begin{pmatrix} \frac{\alpha}{\beta} + \tau - 1 \\ \tau \end{pmatrix} P_0 = 1$$

$$\text{Thus,} \quad P_0 \Big\{ 1 + \sum_{\tau=1}^{\infty} \beta^\tau \begin{pmatrix} \frac{\alpha}{\beta} + \tau - 1 \\ \tau \end{pmatrix} \Big\} = 1$$

$$\text{Hence,} \quad P_0 = \frac{1}{\Big\{ 1 + \sum_{\tau=1}^{\infty} \beta^\tau \begin{pmatrix} \frac{\alpha}{\beta} + \tau - 1 \\ \tau \end{pmatrix} \Big\}}$$

$$\text{But,} \quad P_\tau = \frac{\beta^\tau (\frac{\alpha}{\beta} + \tau - 1)!}{(\frac{\alpha}{\beta} - 1)! \tau!} P_0$$

$$\therefore P_\tau = \beta^\tau \begin{pmatrix} \frac{\alpha}{\beta} + \tau - 1 \\ \tau \end{pmatrix} P_0, \quad \tau = 0, 1, 2, 3, \ldots$$

$$\therefore P_\tau = \frac{\beta^\tau \begin{pmatrix} \frac{\alpha}{\beta} + \tau - 1 \\ \tau \end{pmatrix}}{\sum_{\tau=0}^{\infty} \beta^\tau \begin{pmatrix} \frac{\alpha}{\beta} + \tau - 1 \\ \tau \end{pmatrix}}, \quad \tau = 0, 1, 2, 3, \ldots \tag{5.22}$$

Case 1: Let $\xi = \frac{\alpha}{\beta}$ be a positive integer
Then,

$$\begin{pmatrix} \frac{\alpha}{\beta} + \tau - 1 \\ \tau \end{pmatrix} = \begin{pmatrix} \xi + \tau - 1 \\ \tau \end{pmatrix}$$

$$= (-1)^\tau \begin{pmatrix} -\xi \\ \tau \end{pmatrix}$$

$$\Rightarrow \sum_{\tau=0}^{\infty} \beta^{\tau} \begin{pmatrix} \frac{\alpha}{\beta}+\tau-1 \\ \tau \end{pmatrix} = \sum_{\tau=0}^{\infty} (-\beta)^{\tau} \begin{pmatrix} -\xi \\ \tau \end{pmatrix}$$
$$= (1-\beta)^{-\xi}, \quad 0<\beta<1.$$

Therefore, from (5.22) we have;

$$P_{\tau} = \frac{\beta^{\tau} \begin{pmatrix} \frac{\alpha}{\beta}+\tau-1 \\ \tau \end{pmatrix}}{(1-\beta)^{-\xi}}$$
$$P_{\tau} = \begin{pmatrix} \frac{\alpha}{\beta}+\tau-1 \\ \tau \end{pmatrix} \beta^{\tau}(1-\beta)^{\xi}$$
$$\therefore P_{\tau} = \begin{pmatrix} \xi+\tau-1 \\ \tau \end{pmatrix} \beta^{\tau}(1-\beta)^{\xi}; \quad 0<\beta<1; \quad \alpha>0; \quad \tau=0,1,2,... \tag{5.23}$$

Which is a Negative Binomial Distribution with parameters τ and $1-\beta$.
In hypergeometric terms, we note that;

$$\begin{aligned}
(1-\beta)^{-\xi} &= \sum_{\tau=0}^{\infty} \begin{pmatrix} -\xi \\ \tau \end{pmatrix} (-\beta)^{\tau}, \quad \xi>0 \\
&= \sum_{\tau=0}^{\infty} (-1)^{\tau} \begin{pmatrix} -\xi \\ \tau \end{pmatrix} \beta^{\tau} \\
&= \sum_{\tau=0}^{\infty} \begin{pmatrix} \xi+\tau-1 \\ \tau \end{pmatrix} \beta^{\tau} \\
&= \sum_{\tau=0}^{\infty} \frac{\Gamma(\xi)+\tau}{\Gamma(\xi)} . \frac{\beta^{\tau}}{\tau!} \\
&= \sum_{\tau=0}^{\infty} (\xi+\tau-1)(\xi+\tau-2)(\xi+\tau-3)...(\xi+\tau-\tau)\frac{\Gamma(\xi)}{\Gamma(\xi)} . \frac{\beta^{\tau}}{\tau!}
\end{aligned}$$

Therefore,

$$\begin{aligned}(1-\beta)^{-\xi} &= \sum_{\tau=0}^{\infty}(\xi+\tau-1)(\xi+\tau-2)(\xi+\tau-3)....\xi.\frac{\beta^{\tau}}{\tau!}\\ &= \sum_{\tau=0}^{\infty}\frac{\xi(\xi+1)...(\xi+\tau-1)\flat(\flat+1)...(\flat+\tau-1)}{\flat(\flat+1)...(\flat+\tau-1)}\frac{\beta^{\tau}}{\tau!}\\ \therefore (1-\beta)^{-\xi} &={}_2F_1(\xi,\flat;\flat;\beta).\end{aligned}$$

Its probability mass function is given by;

$$Pr(N=\tau)=\begin{pmatrix}\xi+\tau-1\\ \tau\end{pmatrix}\beta^{\tau}(1-\beta)^{\xi};\quad \tau=0,1,2,...$$

Hence, the pgf for a Negative Binomial Distribution is given as;

$$\begin{aligned}\phi(z) &= \frac{(1-\beta z)^{-\xi}}{(1-\beta)^{-\xi}}\\ \therefore \phi(z) &= \frac{{}_2F_1(\xi,\flat;\flat;\beta z)}{{}_2F_1(\xi,\flat;\flat;\beta)}.\\ \phi^{'}(z) &= \frac{1}{{}_2F_1(\xi,\flat;\flat;\beta)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(\xi,\flat;\flat;\beta z)\}\\ &= \frac{1}{{}_2F_1(\xi,\flat;\flat;\beta)}\frac{\xi\flat}{\flat}.\beta.{}_2F_1(\xi+1,\flat+1;\flat+1;\beta z)\\ &= \frac{\xi\flat}{\flat}\beta.\frac{{}_2F_1(\xi+1,\flat+1;\flat+1;\beta z)}{{}_2F_1(\xi,\flat;\flat;\beta)}.\\ \phi^{''}(z) &= \frac{1}{{}_2F_1(\xi,\flat;\flat;\beta)}\frac{\xi\flat}{\flat}\beta.\frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(\xi+1,\flat+1;\flat+1;\beta z)\}\\ &= \frac{1}{{}_2F_1(\xi,\flat;\flat;\beta)}\frac{\xi\flat}{\flat}\beta.\frac{(\xi+1)(\flat+1)}{(\flat+1)}.\beta.{}_2F_1(\xi+2,\flat+2;\flat+2;\beta z)\\ &= \frac{\xi(\xi+1)(\flat+1)}{\flat(\flat+1)}.\beta^2.\frac{{}_2F_1(\xi+2,\flat+2;\flat+2;\beta z)}{{}_2F_1(\xi,\flat;\flat;\beta)}.\end{aligned}$$

$$\text{Let,}\quad \Lambda_\kappa = \frac{{}_2F_1(\xi+\kappa,\flat+\kappa;\flat+\kappa;\beta)}{{}_2F_1(\xi,\flat;\flat;\beta)};\quad \kappa=1,2.$$

$$E(X) = \phi^{'}(1)$$

$$\therefore E(X) = \frac{\xi\flat}{\flat}\beta\Lambda_1.$$

and

$$Var(X) = \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2$$

$$= \frac{\xi(\xi+1)\flat(\flat+1)}{\flat(\flat+1)}\beta^2\Lambda_2 + \frac{\xi\flat}{\flat}\beta\Lambda_1 - [\frac{\xi\flat}{\flat}\beta\Lambda_1]^2$$

$$\therefore Var(X) = \frac{\xi\flat}{\flat}\beta\Lambda_1 + \frac{\xi\flat}{\flat}\beta^2\Big\{\frac{(\xi+1)(\flat+1)}{(\flat+1)}\Lambda_2 - \frac{\xi\flat}{\flat}\Lambda_1^2\Big\}.$$

(a) Note that, the pgf. for a Negative Binomial distribution can be given as;

$$P_n = \begin{pmatrix} \tau+n-1 \\ n \end{pmatrix} p^\tau q^n$$

Thus,

$$\psi(s) = E(S^N) = \sum_{n=0}^{\infty} P_n s^n$$

$$\psi(s) = \sum_{n=0}^{\infty} \begin{pmatrix} \tau+n-1 \\ n \end{pmatrix} p^\tau q^n s^n$$

$$= p^\tau \sum_{n=0}^{\infty} \begin{pmatrix} \tau+n-1 \\ n \end{pmatrix} (qs)^n$$

$$= p^\tau \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} -\tau \\ n \end{pmatrix} (qs)^n$$

$$= p^\tau (1-qs)^{-\tau}$$

$$\therefore \psi(s) = \Big(\frac{p}{1-qs}\Big)^\tau. \tag{5.24}$$

$$\psi^{'}(s) = \frac{\tau q p^\tau}{(1-qs)^{\tau+1}}$$

$$\psi^{''}(s) = \frac{\tau(\tau+1)q^2 p^\tau}{(1-qs)^{\tau+2}}$$

Putting s=1;

$$
\begin{aligned}
E(X) &= \psi^{'}(1) \\
&= \frac{\tau q p^{\tau}}{(1-q)^{\tau+1}} \\
\therefore E(X) &= \frac{\tau q}{p}. \\
\psi^{''}(1) &= \frac{\tau(\tau+1)q^2 p^{\tau}}{(1-q)^{\tau+2}} \\
&= \frac{\tau(\tau+1)q^2}{p^2} \\
Var(X) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= \frac{\tau(\tau+1)q^2}{p^2} + \frac{\tau q}{p} - \frac{\tau^2 q^2}{p^2} \\
&= \frac{\tau q(q+p)}{p^2} \\
\text{But,} \quad & p+q=1 \\
\therefore Var(X) &= \frac{\tau q}{p^2}.
\end{aligned}
$$

(b) Using Feller's Method, we sum (5.19) over n to obtain;

$$
\sum_{n=0}^{\infty}(1+n)P_{n+1} = \sum_{n=0}^{\infty}(\alpha+\beta n)P_n \tag{5.25}
$$

Define;

$$
M_1 = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty}(1+n)P_{n+1}
$$

and

$$
\sum_{n=0}^{\infty} P_n = 1.
$$

From (5.25), we have;

$$
\begin{aligned}
M_1 &= \alpha \sum_{n=0}^{\infty} P_n + \beta \sum_{n=0}^{\infty} nP_n \\
M_1 &= \alpha.1 + M_1\beta \\
\Rightarrow \alpha &= M_1.(1-\beta) \\
E(X) &= M_1 \\
\therefore E(X) &= \frac{\alpha}{1-\beta}.
\end{aligned}
$$

Next, we multiply (5.19) by n we get;

$$
\sum_{n=0}^{\infty} n(1+n)P_{n+1} = \sum_{n=0}^{\infty} n(\alpha+\beta n)P_n \tag{5.26}
$$

Define;

$$
\begin{aligned}
M_2 &= \sum_{n=0}^{\infty} n^2 P_n = \sum_{n=0}^{\infty} (1+n)^2 P_{n+1} \\
&\text{and} \\
\sum_{n=0}^{\infty} P_n &= \sum_{n=0}^{\infty} P_{n+1} = 1.
\end{aligned}
$$

From (5.26), we have;

$$
\begin{aligned}
\sum_{n=0}^{\infty} (1+n)^2 P_{n+1} - \sum_{n=0}^{\infty} (1+n)P_{n+1} &= M_1\alpha + M_2\beta \\
M_2 - M_1 &= \alpha M_1 + \beta M_2 \\
\Rightarrow (1-\beta)M_2 &= (\alpha+1)M_1
\end{aligned}
$$

But,

$$\frac{\alpha}{1-\beta} = M_1$$
$$\Rightarrow \frac{\alpha(\alpha+1)}{(1-\beta)^2} = M_2.$$
$$Var(X) = M_2 - [M_1]^2$$
$$= \frac{\alpha(\alpha+1)}{(1-\beta)^2} - \frac{\alpha^2}{(1-\beta)^2}$$
$$= \frac{\alpha^2+\alpha-\alpha^2}{(1-\beta)^2}$$
$$\therefore Var(X) = \frac{\alpha^2}{(1-\beta)^2}.$$

Case 2: Let $\xi = \frac{\alpha}{\beta}$ be a Negative integer
Then,
When $\xi > 0$,
Equation (5.20) becomes;

$$P_\tau = \beta^\tau(-\xi)(-\xi+1)(-\xi+2)...(-\xi+\tau-1)\frac{P_0}{\tau!}; \quad \tau = 1,2,3,...$$
$$P_\tau = (-\beta)^\tau \xi(\xi-1)(\xi-2)...(\xi-(\tau-1))\frac{P_0}{\tau!}$$
$$P_\tau = (-\beta)^\tau \begin{pmatrix} \xi \\ \tau \end{pmatrix} P_0; \quad \tau = 1,2,3,...,\xi \tag{5.27}$$

But, $\sum_{\tau=0}^{\infty} P_\tau = 1$

$$\Rightarrow 1 = \sum_{\tau=0}^{\infty} P_\tau = P_0 \sum_{\tau=0}^{\infty} (-\beta)^\tau \begin{pmatrix} \xi \\ \tau \end{pmatrix}$$
$$1 = P_0\{1+(-\beta)\}^\xi$$
$$\therefore P_0 = \frac{1}{(1-\beta)^\xi}. \tag{5.28}$$

Substituting (5.28) in equation (5.27), we have;

$$P_\tau = \frac{(-\beta)^\tau \begin{pmatrix} \xi \\ \tau \end{pmatrix}}{(1-\beta)^\xi}$$
$$\therefore P_\tau = \begin{pmatrix} \xi \\ \tau \end{pmatrix} \Big(\frac{-\beta}{1-\beta}\Big)^\tau \Big(\frac{1}{1-\beta}\Big)^{\xi-\tau}; \quad \tau = 0,1,2,3,...,\xi \qquad \textbf{(5.29)}$$

Which is a Binomial Distribution with parameters $\frac{-\alpha}{\beta} = \xi, \quad \frac{-\beta}{1-\beta}$; where $\frac{\alpha}{\beta}$ is a negative integer and $\beta < 0 \quad \Rightarrow \alpha > 0$.
Note that;

$$\begin{aligned}
(1+\beta)^\xi &= \sum_{\tau=0}^{\infty} \begin{pmatrix} \xi \\ \tau \end{pmatrix} \beta^\tau \\
&= 1 + \begin{pmatrix} \xi \\ 1 \end{pmatrix} \beta^1 + \begin{pmatrix} \xi \\ 2 \end{pmatrix} \beta^2 + \begin{pmatrix} \xi \\ 3 \end{pmatrix} \beta^3 + ... + \begin{pmatrix} \xi \\ \xi \end{pmatrix} \beta^\xi \\
&= 1 + \frac{\xi}{1!}\beta^1 + \frac{\xi(\xi-1)}{2!}\beta^2 + \frac{\xi(\xi-1)(\xi-2)}{3!}\beta^3 + ... + \xi(\xi-1)...2.1.\frac{\beta^\xi}{\xi!} \\
&= 1 + (-1)\xi(-1)\frac{\beta}{1!} + (-1)^2\xi(\xi-1)(-1)^2\frac{\beta^2}{2!} + (-1)^3\xi(\xi-1)(\xi-2)(-1)^3\frac{\beta^3}{3!} + ... \\
&+ (-1)^\xi\xi(\xi-1)...2.1.(-\xi)^\xi\frac{\beta^\xi}{\xi!} \\
&= 1 + \frac{(-\xi)}{1}\frac{(-\beta)}{1!} + \frac{(-\xi)(-\xi+1)1.2}{1.2}\frac{(-\beta)^\xi}{2!} + ... \\
&+ \frac{(-\xi)(-\xi+1)...(-\xi+\xi+1)}{1.2.3...\xi}1.2.3...\xi\frac{(-\beta)^\xi}{\xi!} \\
&= {}_2F_1(-\xi,1;1;-\beta).
\end{aligned}$$

Its probability mass function is given by;

$$
\begin{aligned}
Pr(N=\tau) &= \begin{pmatrix} \xi \\ \tau \end{pmatrix} \frac{\beta^{\tau}}{(1+\beta)^{\xi}} \\
&= \begin{pmatrix} \xi \\ \tau \end{pmatrix} \Big(\frac{\beta}{(1+\beta)}\Big)^{\tau}\Big(\frac{1}{(1+\beta)}\Big)^{\xi-\tau}; \quad \tau = 0,1,2,3,...,\xi.
\end{aligned}
$$

Which is a binomial distribution with parameters ξ and $\frac{\beta}{(1+\beta)}$.
The pgf. for a Binomial Distribution in hypergeometric terms is expressed as;

$$
\begin{aligned}
\phi(z) &= \frac{(1+\beta z)^{\xi}}{(1+\beta)^{\xi}} \\
\therefore \phi(z) &= \frac{{}_2F_1(-\xi;1;1;-\beta z)}{{}_2F_1(-\xi;1;1;-\beta)}. \\
\phi^{'}(z) &= \frac{1}{{}_2F_1(-\xi;1;1;-\beta)} \frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(-\xi;1;1;-\beta z)\} \\
&= \frac{1}{{}_2F_1(-\xi;1;1;-\beta)}.\xi.\beta.{}_2F_1(-\xi+1;1+1;1+1;-\beta z) \\
&= \xi\beta\frac{{}_2F_1(-\xi+1;2;2;-\beta z)}{{}_2F_1(-\xi;1;1;-\beta)}. \\
\phi^{''}(z) &= \frac{1}{{}_2F_1(-\xi;1;1;-\beta)}.\xi\beta.\frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(-\xi+1;2;2;-\beta z)\} \\
&= \frac{1}{{}_2F_1(-\xi;1;1;-\beta)}.\xi\beta.\frac{(-\xi+1)2}{2}.{}_2F_1(-\xi+2;2+1;2+1;-\beta z) \\
&= \frac{-\xi(-\xi+1)\beta^2}{2}\frac{{}_2F_1(-\xi+2;3;3;-\beta z)}{{}_2F_1(-\xi;1;1;-\beta)}. \\
\text{Let,} \quad \Lambda_{\kappa} &= \frac{{}_2F_1(-\xi+\kappa;1+\kappa;1+\kappa;-\beta)}{{}_2F_1(-\xi;1;1;-\beta)}; \quad \kappa = 1,2. \\
E(X) &= \phi^{'}(1) \\
\therefore E(X) &= \xi\beta\Lambda_1. \\
&\text{and} \\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2 \\
&= \frac{-\xi(-\xi+1)2\beta^2}{2}\Lambda_2+\xi\beta\Lambda_1-[\xi\beta\Lambda_1]^2 \\
\therefore Var(X) &= \xi\beta\Lambda_1+\xi\beta^2\{\frac{(\xi-1)2}{2}\Lambda_2-\xi\Lambda_1^2\}.
\end{aligned}
$$

5.2.2 Obtaining Means and Variance of Katz probability

Using the probability generating function technique, we multiply (5.19) by s^n and sum the results over n to obtain;

$$\sum_{n=0}^{\infty}(1+n)P_{n+1}s^n = \sum_{n=0}^{\infty}(\alpha+\beta n)P_n s^n \tag{5.30}$$

$$\text{Define;} \quad \psi(s) = \sum_{n=0}^{\infty} P_n s^n$$

$$\psi^{'}(s) = \sum_{n=0}^{\infty} nP_n s^n$$

$$\psi^{'}(s) = \sum_{n=0}^{\infty} (n+1)P_{n+1} s^n$$

From (5.30), we have;

$$\begin{aligned}
\psi^{'}(s) &= \alpha \sum_{n=0}^{\infty} P_n s^n + \beta \sum_{n=0}^{\infty} nP_n s^n \\
\psi^{'}(s) &= \alpha\psi(s) + \beta s \sum_{n=0}^{\infty} nP_n s^{n-1} \\
\psi^{'}(s) &= \alpha\psi(s) + \beta s\psi^{'}(s) \\
\Longleftrightarrow (1-\beta s)\psi^{'}(s) &= \alpha\psi(s) \\
\Rightarrow (1-\beta s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \alpha\psi(s) \\
\int \frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int \frac{\alpha}{1-\beta s}\mathrm{d}s \\
\ln\psi(s) &= \frac{\alpha}{-\beta}\ln(1-\beta s) + \ln c \\
\ln\psi(s) &= \ln(1-\beta s)^{\frac{\alpha}{-\beta}} + \ln c \\
\ln\psi(s) &= \ln c(1-\beta s)^{\frac{\alpha}{-\beta}} \\
\psi(s) &= c_1(1-\beta s)^{\frac{\alpha}{-\beta}}
\end{aligned} \tag{5.31}$$

Putting s=1;

$$
\begin{aligned}
\psi(s) = 1 = c_1(1-\beta)^{\frac{\alpha}{-\beta}} \\
\Rightarrow 1 = c_1(1-\beta)^{\frac{\alpha}{-\beta}} \\
\Rightarrow c_1 = \frac{1}{(1-\beta)^{\frac{\alpha}{-\beta}}} \\
\therefore c_1 = (1-\beta)^{\frac{\alpha}{\beta}} \\
\therefore \psi(s) = \Big(\frac{1-\beta s}{1-\beta}\Big)^{-\frac{\alpha}{\beta}} \\
\therefore \psi(s) = \Big(\frac{1-\beta}{1-\beta s}\Big)^{\frac{\alpha}{\beta}}.
\end{aligned} \tag{5.32}
$$

Let $\xi = \frac{\alpha}{\beta}$ be a positive integer,
Then,

$$
\begin{aligned}
\psi(s) = \Big(\frac{1-\beta s}{1-\beta}\Big)^{-\xi} \\
\therefore \psi(s) = \Big(\frac{1-\beta}{1-\beta s}\Big)^{\xi}.
\end{aligned}
$$

Let $1-\beta = p$ and $p+q=1$
Then,

$$
\psi(s) = \Big(\frac{p}{1-qs}\Big)^{\xi}.
$$

Which is the pgf. for a Negative binomial distribution with parameters $\xi > 0$ and $p = 1-\beta$.
From (5.32), the pgf. is given as;

$$
\psi(s) = \Big(\frac{1-\beta}{1-\beta s}\Big)^{\frac{\alpha}{\beta}}
$$

therefore;

$$\psi^{'}(s) = \frac{\alpha}{1-\beta s}\psi(s)$$
$$\psi^{''}(s) = \frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\alpha}{1-\beta s}\psi(s)\Big)$$
$$\Rightarrow \psi^{''}(s) = \frac{\alpha\psi^{'}(s)}{1-\beta s} + \frac{\alpha\beta\psi(s)}{(1-\beta s)^2}$$

Putting s=1;

$$\psi^{'}(1) = \frac{\alpha}{1-\beta}\psi(1)$$
$$\text{But,}\quad \psi(1) = 1$$
$$E(X) = \psi^{'}(1)$$
$$\therefore E(X) = \frac{\alpha}{1-\beta}.$$
$$\psi^{''}(1) = \frac{\alpha\psi^{'}(1)}{1-\beta} + \frac{\alpha\beta\psi(1)}{(1-\beta)^2}$$
$$= \frac{\alpha^2+\alpha\beta}{(1-\beta)^2}$$
$$Var(X) = \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2$$
$$= \frac{\alpha^2+\alpha\beta}{(1-\beta)^2} + \frac{\alpha}{1-\beta} - \frac{\alpha^2}{(1-\beta)^2}$$
$$\therefore Var(X) = \frac{\alpha}{(1-\beta)^2}.$$

5.3 Extension of Recursive Model by Tripathi - Gurland (1977)

Gurland and Tripathi (1975) and Tripathi and Gurland (1977) Extended Katz Family of distributions.
The relationship between the probabilities has the form:

$$\frac{P_{n+1}}{P_n} = \frac{\alpha+\beta n}{\gamma+n} \tag{**}$$
$$\Rightarrow (n+\gamma)P_{n+1} = (\beta n+\alpha)P_n \tag{5.33}$$

Proposition 5.2

$$\frac{P_{n+1}}{P_n} = \frac{\alpha+\beta n}{\gamma+n}; \quad n = 0, 1, 2, 3, \ldots$$

Proof
Solving equation (5.33) iteratively, we have;
When n=0;
Equation (5.33) becomes;

$$\begin{aligned}\gamma P_1 &= \alpha P_0 \\ \Rightarrow P_1 &= \frac{\alpha}{\gamma} P_0 \\ \therefore P_1 &= \frac{\alpha}{\gamma} P_0\end{aligned} \tag{5.34}$$

When n=1;
Equation (5.33) becomes;

$$\begin{aligned}(1+\gamma)P_2 &= (\beta+\alpha)P_1 \\ \Rightarrow P_2 &= \frac{\beta+\alpha}{1+\gamma} P_1 \\ \therefore P_2 &= \frac{\beta+\alpha}{1+\gamma} P_1.\end{aligned}$$

Substitution (5.34) in equation (5.35), we have;

$$\begin{aligned}P_2 &= \frac{\beta+\alpha}{1+\gamma} \frac{\alpha}{\gamma} P_0 \\ \therefore P_2 &= \frac{\alpha(\alpha+\beta)}{\gamma(\gamma+1)} P_0.\end{aligned} \tag{5.36}$$

When n=1;
Equation (5.33) becomes;

$$(\gamma+2)P_3 = (\alpha+2\beta)P_2 \tag{5.37}$$

Substituting (5.36) in equation (5.37), we have;

$$\begin{aligned}
(\gamma+2)P_3 &= (\alpha+2\beta)\frac{\alpha(\alpha+\beta)}{\gamma(\gamma+1)}P_0 \\
\Rightarrow P_3 &= \frac{(\alpha+2\beta)(\alpha+\beta)\alpha}{(\gamma+2)(\gamma+1)\gamma}P_0 \\
\therefore P_3 &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{\gamma(\gamma+1)(\gamma+2)}P_0.
\end{aligned}$$

By Mathematical Induction, we have;

$$\begin{aligned}
P_4 &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)}{\gamma(\gamma+1)(\gamma+2)(\gamma+3)}P_0, \\
P_5 &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)(\alpha+4\beta)}{\gamma(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}P_0, \\
&. \\
&. \\
&. \\
P_n &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)...(\alpha+n-1)}{\gamma(\gamma+1)(\gamma+2)(\gamma+3)...(\gamma+n-1)}P_0.
\end{aligned} \tag{5.38}$$

$$\begin{aligned}
\text{But,} \quad \sum_{n=0}^{\infty} P_n &= 1 \\
\Rightarrow P_0+P_1+P_2+P_3+...+P_n+... &= 1 \\
P_0\left\{1+\frac{\alpha}{\gamma}+\frac{\alpha(\alpha+\beta)}{\gamma(\gamma+1)}+...\right\} &= 1 \\
\therefore P_0 &= \frac{1}{\left\{1+\frac{\alpha}{\gamma}+\frac{\alpha(\alpha+\beta)}{\gamma(\gamma+1)}+...\right\}} \\
\therefore P_n &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)...(\alpha+n-1)}{\gamma(\gamma+1)(\gamma+2)(\gamma+3)...(\gamma+n-1)}.\frac{1}{\left\{1+\frac{\alpha}{\gamma}+\frac{\alpha(\alpha+\beta)}{\gamma(\gamma+1)}+...\right\}}
\end{aligned}$$

From (5.38), we have;

$$\begin{aligned}
P_{n+1} &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)...(\alpha+n-1)(\alpha+n\beta)}{\gamma(\gamma+1)(\gamma+2)...(\gamma+n-1)(\gamma+n)}P_0. \\
\therefore \frac{P_{n+1}}{P_n} &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)...(\alpha+n-1)(\alpha+n\beta)}{\gamma(\gamma+1)(\gamma+2)...(\gamma+n-1)(\gamma+n)}.\frac{\gamma(\gamma+1)(\gamma+2)...(\gamma+n-1)}{\alpha(\alpha+\beta)(\alpha+2\beta)...(\alpha+n-1)} \\
\therefore \frac{P_{n+1}}{P_n} &= \frac{\alpha+n\beta}{\gamma+n}; \quad \text{as required.}
\end{aligned}$$

Multiplying (5.33) by s^n and sum the results over n, we have;

$$\sum_{n=0}^{\infty}(\gamma+n)P_{n+1}s^n = \sum_{n=0}^{\infty}(\alpha+\beta n)P_n s^n$$

$$\gamma\sum_{n=0}^{\infty}P_{n+1}s^n + \sum_{n=0}^{\infty}nP_{n+1}s^n = \alpha\sum_{n=0}^{\infty}P_n s^n + \beta\sum_{n=0}^{\infty}nP_n s^n \tag{5.39}$$

$$\text{Define;}\quad \psi(s) = \sum_{n=0}^{\infty}P_n s^n$$

$$\psi^{'}(s) = \sum_{n=0}^{\infty}nP_n s^{n-1} = \sum_{n=0}^{\infty}(n+1)P_n s^n$$

From (5.39), we have;

$$\begin{aligned}
\gamma\psi(s) + \sum_{n=0}^{\infty}(n+1-1)P_{n+1}s^n &= \alpha\psi(s) + \beta s\psi^{'}(s) \\
\gamma\psi(s) + \sum_{n=0}^{\infty}(n+1)P_{n+1}s^n - \sum_{n=0}^{\infty}P_{n+1}s^n &= \alpha\psi(s) + \beta s\psi^{'}(s) \\
\gamma\psi(s) + \psi^{'}(s) - \psi(s) &= \alpha\psi(s) + \beta s\psi^{'}(s) \\
(1-\beta s)\psi^{'}(s) &= (\alpha+1-\gamma)\psi(s) \\
(1-\beta s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= (\alpha+1-\gamma)\psi(s) \\
\int\frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int\frac{\alpha+1-\gamma}{1-\beta s}\mathrm{d}s \\
\ln\psi(s) &= \int\frac{\alpha}{1-\beta s}\mathrm{d}s + \int\frac{1}{1-\beta s}\mathrm{d}s - \int\frac{\gamma}{1-\beta s}\mathrm{d}s \\
\ln\psi(s) &= \frac{\alpha}{-\beta}\ln(1-\beta s) + \frac{1}{-\beta}\ln(1-\beta s) - \frac{\gamma}{-\beta}\ln(1-\beta s) + \ln c
\end{aligned}$$

Therefore,

$$\begin{aligned}
\ln\psi(s) &= \ln(1-\beta s)^{\frac{\alpha}{-\beta}} + \ln(1-\beta s)^{\frac{1}{-\beta}} - \ln(1-\beta s)^{\frac{\gamma}{-\beta}} + \ln c \\
\ln\psi(s) &= \ln c_1 . \ln(1-\beta s)^{\frac{\alpha}{-\beta}} . \ln(1-\beta s)^{\frac{1}{-\beta}} . \frac{1}{\ln(1-\beta s)^{\frac{\gamma}{-\beta}}} \\
\therefore \psi(s) &= c_1 .(1-\beta s)^{\frac{\alpha}{-\beta}} .(1-\beta s)^{\frac{1}{-\beta}} . \frac{1}{(1-\beta s)^{\frac{\gamma}{-\beta}}}
\end{aligned}$$

Putting s=1;

$$\begin{aligned}
\psi(1) = 1 &= c_1 \frac{(1-\beta)^{\frac{\alpha}{-\beta}} .(1-\beta)^{\frac{1}{-\beta}}}{(1-\beta s)^{\frac{\gamma}{-\beta}}} \\
\Rightarrow c_1 &= \frac{(1-\beta s)^{\frac{\gamma}{-\beta}}}{(1-\beta)^{\frac{\alpha}{-\beta}} .(1-\beta)^{\frac{1}{-\beta}}} \\
\therefore \psi(s) &= \frac{(1-\beta s)^{\frac{-\alpha}{\beta}} (1-\beta s)^{\frac{-1}{\beta}} (1-\beta s)^{\frac{-\gamma}{\beta}}}{(1-\beta)^{\frac{-\alpha}{\beta}} (1-\beta)^{\frac{-1}{\beta}} (1-\beta)^{\frac{-\gamma}{\beta}}}
\end{aligned}$$

Thus, from gauss hypergeometric series, we have;

$$\begin{aligned}
{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta) &= 1 + \frac{\frac{\alpha}{\beta}.1}{\gamma} + \frac{\beta}{1!} + \frac{\frac{\alpha}{\beta}(\frac{\alpha}{\beta}+1)1(1+1)}{\gamma(\gamma+1)} \frac{\beta^2}{2!} + \ldots + \frac{(\frac{\alpha}{\beta}+n-1)(1+n-1)}{(\gamma+n-1)} \frac{\beta^n}{n!}; \quad \gamma > 0 \\
&= \sum_{n=0}^{\infty} \frac{\frac{\alpha}{\beta}(\frac{\alpha}{\beta}+1)\ldots(\frac{\alpha}{\beta}+n-1)1(1+1)(1+2)\ldots(1+n-1)}{\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)} \frac{\beta^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(\frac{\alpha}{\beta}+n-1)\ldots(\frac{\alpha}{\beta}+1)\frac{\alpha}{\beta}(1+n-1)\ldots(1+2)(1+1)1}{(\gamma+n-1)\ldots(\gamma+2)(\gamma+1)\gamma} \frac{\beta^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\alpha}{\beta}+n)\Gamma(1+n)}{\Gamma(\gamma+n)} . \frac{\Gamma(\gamma)}{\Gamma(\frac{\alpha}{\beta})\Gamma(1)} \frac{\beta^n}{n!}
\end{aligned}$$

Normalizing, we get;

$$1 = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\alpha}{\beta}+n)\Gamma(1+n)}{\Gamma(\gamma+n)} . \frac{\Gamma(\gamma)}{\Gamma(\frac{\alpha}{\beta})\Gamma(1)} . \frac{1}{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta)} \frac{\beta^n}{n!}$$

Hence,

$$\begin{aligned}P_n &= \text{Prob.}(N=n)\\ &= \sum_{n=0}^{\infty}\frac{\Gamma(\frac{\alpha}{\beta}+n)\Gamma(1+n)}{\Gamma(\gamma+n)}.\frac{\Gamma(\gamma)}{\Gamma(\frac{\alpha}{\beta})\Gamma(1)}.\frac{1}{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta)}\frac{\beta^n}{n!}\end{aligned}$$

The pgf. is given as;

$$\begin{aligned}\phi(z) &= \sum_{n=0}^{\infty}P_n z^n\\ \phi(z) &= \sum_{n=0}^{\infty}\frac{\Gamma(\frac{\alpha}{\beta}+n)\Gamma(1+n)}{\Gamma(\gamma+n)}.\frac{\Gamma(\gamma)}{\Gamma(\frac{\alpha}{\beta})\Gamma(1)}.\frac{(\beta z)^n}{n!}.\frac{1}{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta)}\\ \therefore \phi(z) &= \frac{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta z)}{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta)}.\end{aligned}$$

Which is the probability generating function of an extended hypergeometric function.

$$\begin{aligned}\phi^{'}(z) &= \frac{1}{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta z)\}\\ &= \frac{1}{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta)}\frac{\alpha}{\beta}.\frac{1}{\gamma}.\beta\{\frac{1}{{}_2F_1(\frac{\alpha}{\beta}+1,1+1;\gamma+1;\beta z)}\}\\ &= \frac{\alpha}{\beta}\frac{1}{\gamma}.\beta\frac{{}_2F_1(\frac{\alpha}{\beta}+1,2;\gamma+1;\beta z)}{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta)}\\ \phi^{''}(z) &= \frac{1}{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta)}\frac{\alpha}{\beta}\frac{1}{\gamma}\beta\frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(\frac{\alpha}{\beta}+1,2;\gamma+1;\beta z)\}\\ &= \frac{1}{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta)}\frac{\alpha}{\beta}\frac{1}{\gamma}\beta.\frac{(\frac{\alpha}{\beta}+1)2}{\gamma+1}\beta\{{}_2F_1(\frac{\alpha}{\beta}+1,2;\gamma+1;\beta z)\}\\ &= \frac{\frac{\alpha}{\beta}(\frac{\alpha}{\beta}+1)2}{\gamma(\gamma+1)}\beta^2\frac{{}_2F_1(\frac{\alpha}{\beta}+2,2+1;\gamma+2;\beta z)}{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta)}.\\ \text{Let,}\quad \Lambda_\kappa &= \frac{{}_2F_1(\frac{\alpha}{\beta}+\kappa,1+\kappa;\gamma+\kappa;\beta)}{{}_2F_1(\frac{\alpha}{\beta},1;\gamma;\beta)};\quad \kappa=1,2.\\ E(X) &= \phi^{'}(1)\\ \therefore E(X) &= \frac{\alpha}{\beta}\frac{1}{\gamma}\beta\Lambda_1.\end{aligned}$$

and

$$
\begin{aligned}
Var(X) &= \phi^{''}(1) + \phi^{'}(1) - [\phi^{'}(1)]^2 \\
&= \frac{\alpha}{\beta}(\frac{\alpha}{\beta}+1)\frac{2}{\gamma}\beta^2\Lambda_2 + \frac{\alpha}{\beta}\frac{1}{\gamma}\Lambda_1 - \{\frac{\alpha}{\beta}\frac{1}{\gamma}\Lambda_1\}^2 \\
\therefore Var(X) &= \frac{\alpha}{\beta}\frac{1}{\gamma}\beta\Lambda_1 + \frac{\alpha}{\beta}\frac{1}{\gamma}\beta^2\{(\frac{\alpha}{\beta}+1)\frac{2}{(\gamma+1)}\Lambda_2 - \frac{\alpha}{\beta}\frac{1}{\gamma}\beta\Lambda_1^2\}.
\end{aligned}
$$

5.4 Family of Distributions by Jewell and Sundt

Sundt and Jewell (1981) and Willmot (1988) have created the sundt and jewell family of distributions which is a modified form of the Katz family. Their recurrence relationship for the probability is given by;
Proposition 5.3

$$
\frac{P_{n+1}}{P_n} = \frac{a+b+an}{1+n}; \quad n = 0,1,2,... \tag{5.40}
$$

$$
\Rightarrow (1+n)P_{n+1} = (a+b+an)P_n \tag{5.41}
$$

Proof
Solving (5.41) iteratively, we have;
When n=0; Equation (5.41) becomes;

$$
\begin{aligned}
P_1 &= (a+b)P_0 \\
\therefore P_1 &= (a+b)P_0
\end{aligned} \tag{5.42}
$$

When n=1;
Equation (5.41) becomes;

$$
2P_2 = (2a+b)P_1 \tag{5.43}
$$

Substituting (5.42) in equation (5.43), we have;

$$
\begin{aligned}
2P_2 &= (2a+b)(a+b)P_0 \\
\Rightarrow P_2 &= \frac{(a+b)(2a+b)}{1.2}P_0 \\
\therefore P_2 &= \frac{(a+b)(2a+b)}{2!}P_0. \qquad (5.44)
\end{aligned}
$$

When n=2;
Equation (5.41) becomes;

$$
\begin{aligned}
3P_3 &= (3a+b)\frac{(a+b)(2a+b)}{1.2}P_0 \\
\Rightarrow P_3 &= \frac{(a+b)(2a+b)(3a+b)}{1.2.3}P_0 \\
\therefore P_3 &= \frac{(a+b)(2a+b)(3a+b)}{3!}P_0.
\end{aligned}
$$

Suppose, we let;

$$(a+b) = \Lambda_1, \quad (2a+b) = \Lambda_2, \quad (3a+b) = \Lambda_3, \quad ...(na+b) = \Lambda_n.$$

$$
\begin{aligned}
\text{Then,} \quad P_1 &= \frac{\Lambda_1}{1!}P_0, \\
P_2 &= \frac{\Lambda_1\Lambda_2}{2!}P_0, \\
P_3 &= \frac{\Lambda_1\Lambda_2\Lambda_3}{3!}P_0,
\end{aligned}
$$

By Mathematical Induction, we have;

$$
\begin{aligned}
P_4 &= \frac{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}{4!}P_0, \\
P_5 &= \frac{\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5}{5!}P_0, \\
&. \\
&. \\
&. \\
P_n &= \frac{\Lambda_1\Lambda_2\Lambda_3...\Lambda_{n-1}\Lambda_n}{n!}P_0. \qquad (5.45)
\end{aligned}
$$

Therefore, from (5.45) we have;

$$\begin{aligned}
P_{n+1} &= \frac{\Lambda_1\Lambda_2\Lambda_3 \ldots \Lambda_n\Lambda_{n+1}}{(n+1)!}P_0. \\
\frac{P_{n+1}}{P_n} &= \frac{\Lambda_1\Lambda_2\Lambda_3 \ldots \Lambda_{n-1}\Lambda_n\Lambda_{n+1}}{(n+1).n!}.\frac{n!}{\Lambda_1\Lambda_2\Lambda_3 \ldots \Lambda_{n-1}\Lambda_n} \\
\therefore \frac{P_{n+1}}{P_n} &= \frac{\Lambda_{n+1}}{n+1} \\
\therefore \frac{P_{n+1}}{P_n} &= \frac{a+b+an}{n+1}, \quad \text{as required.}
\end{aligned}$$

Multiplying (5.41) by s^n and sum the results over n to obtain;

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+1)P_{n+1}s^n &= \sum_{n=0}^{\infty}(a+b+an)P_n s^n \\
\psi'(s) &= a\sum_{n=0}^{\infty}P_n s^n + b\sum_{n=0}^{\infty}P_n s^n + as\sum_{n=0}^{\infty}nP_n s^{n-1} \\
\psi'(s) &= a\psi(s) + b\psi(s) + as\psi'(s) \\
\Rightarrow (1-as)\psi'(s) &= (a+b)\psi(s) \\
(1-as)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= (a+b)\psi(s) \\
\int \frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int \frac{a+b}{1-as}\mathrm{d}s \\
\ln\psi(s) &= \frac{a+b}{-a}\ln(1-as) + \ln c \\
\ln\psi(s) &= \ln c \ln(1-as)^{\frac{a+b}{-a}} \\
\psi(s) &= c_1(1-as)^{\frac{a+b}{-a}}
\end{aligned}$$

Putting s=1;

$$\begin{aligned}
\psi(1) = 1 &= c_1(1-a)^{\frac{a+b}{-a}} \\
\therefore c_1 &= (1-a)^{\frac{a+b}{a}}.
\end{aligned}$$

Therefore;

$$
\begin{aligned}
\psi(s) &= \Big(\frac{1-as}{1-a}\Big)^{\frac{-(a+b)}{a}} \\
\therefore \psi(s) &= \Big(\frac{1-a}{1-as}\Big)^{\frac{(a+b)}{a}}. \\
\psi^{'}(s) &= \frac{a+b}{1-as}.\Big(\frac{1-a}{1-as}\Big)^{\frac{(a+b)}{a}} \\
\psi^{''}(s) &= \frac{a+b}{1-as}.\frac{a+b}{1-as}\Big(\frac{1-a}{1-as}\Big)^{\frac{(a+b)}{a}} + \frac{(a+b)a}{(1-as)^2}\Big(\frac{1-a}{1-as}\Big)^{\frac{(a+b)}{a}} \\
E(X) &= \psi^{'}(1) \\
\therefore E(X) &= \frac{a+b}{1-a}. \\
Var(X) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= \Big[\frac{(a+b)^2}{(1-a)^2} + \frac{a(a+b)}{(1-a)^2}\Big] + \frac{a+b}{1-a} - \frac{(a+b)^2}{(1-a)^2} \\
&= \frac{(a+b)^2 + a(a+b)(a+b)(1-a) - (a+b)^2}{(1-a)^2} \\
&= \frac{a^2 + ab + a - a^2 + b - ab}{(1-a)^2} \\
\therefore Var(X) &= \frac{a+b}{(1-a)^2}.
\end{aligned}
$$

Using Feller's method, we sum (5.41) over n to obtain;

$$
\begin{aligned}
\sum_{n=0}^{\infty}(n+1)P_{n+1} &= \sum_{n=0}^{\infty}(a+b+an)P_n \\
M_1 &= a\sum_{n=0}^{\infty}P_n + b\sum_{n=0}^{\infty}P_n + a\sum_{n=0}^{\infty}nP_n \\
M_1 &= a+b+aM_1 \\
(1-a)M_1 &= (a+b) \\
\Rightarrow M_1 &= \frac{a+b}{1-a} \\
E(X) &= M_1 \\
\therefore E(X) &= \frac{a+b}{1-a}.
\end{aligned}
$$

Next, we multiply (5.41) by n and sum to obtain;

$$
\begin{aligned}
\sum_{n=0}^{\infty}(n+1)nP_{n+1} &= \sum_{n=0}^{\infty} n(a+b+an)P_n \\
\sum_{n=0}^{\infty}(n+1)(n+1-1)P_{n+1} &= a\sum_{n=0}^{\infty} nP_n + b\sum_{n=0}^{\infty} nP_n + a\sum_{n=0}^{\infty} + a\sum_{n=0}^{\infty} n^2P_n \\
\sum_{n=0}^{\infty}(1+n)^2P_{n+1} - \sum_{n=0}^{\infty}(1+n)P_{n+1} &= aM_1 + bM_1 + aM_2 \\
M_2 - M_1 &= aM_1 + bM_1 + aM_2 \\
(1-a)M_2 &= (a+b+1)M_1 \\
\therefore M_2 &= \frac{(a+b)(a+b+1)}{(1-a)^2} \\
Var(X) &= M_2 - \{M_1\}^2 \\
&= \frac{(a+b)(a+b+1)}{(1-a)^2} - \left\{\frac{a+b}{1-a}\right\}^2 \\
&= \frac{a^2+ab+ab+b^2+a+b-a^2-2ab-b^2}{(1-a)^2} \\
\therefore Var(X) &= \frac{a+b}{(1-a)^2}.
\end{aligned}
$$

5.5 Irwin (1963) Recursive Model

From equation (**) in section 5.3 above, when $\beta = 1$ and $\gamma > 1$, it belongs to Katz class family and has been presented by Irwin (1963) as;

Proposition 5.4

$$
\begin{aligned}
\frac{P_{n+1}}{P_n} &= \frac{n+\alpha}{n+\gamma} \\
(n+\gamma)P_{n+1} &= (n+\alpha)P_n
\end{aligned} \tag{5.46}
$$

Proof

Solving (5.46) iteratively, we have;

When n=0;

Equation (5.46) becomes;

$$\begin{aligned}\gamma P_1 &= \alpha P_0\\ \Rightarrow P_1 &= \frac{\alpha}{\gamma}P_0\\ \therefore P_1 &= \frac{\alpha}{\gamma}P_0. \end{aligned} \tag{5.47}$$

When n=1;
Equation (5.46) becomes;

$$(\gamma+1)P_2 = (\alpha+1)P_1 \tag{5.48}$$

Substituting (5.47) in equation (5.48), we have;

$$\begin{aligned}(\gamma+1)P_2 &= (\alpha+1)\frac{\alpha}{\gamma}P_0\\ \Rightarrow P_2 &= \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}P_0\\ \therefore P_2 &= \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}P_0. \end{aligned} \tag{5.49}$$

When n=2;
Equation (5.46) becomes;

$$(\gamma+2)P_3 = (\alpha+2)P_2 \tag{5.50}$$

Substituting (5.49) in equation (5.50), we have;

$$\begin{aligned}(\gamma+2)P_3 &= (\alpha+2)\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}P_0\\ \Rightarrow P_3 &= \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}P_0\\ \therefore P_3 &= \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}P_0. \end{aligned}$$

Suppose we let $\Lambda_1 = \frac{\alpha}{\gamma}, \quad \Lambda_2 = \frac{\alpha+1}{\gamma+1}, \quad \Lambda_3 = \frac{\alpha+2}{\gamma+2}, \quad \ldots, \quad \Lambda_n = \frac{\alpha+n-1}{\gamma+n-1}.$
Then;

$$\begin{aligned}
P_1 &= \Lambda_1 P_0, \\
P_2 &= \Lambda_1 \Lambda_2 P_0, \\
P_3 &= \Lambda_1 \Lambda_2 \Lambda_3 P_0.
\end{aligned}$$

Therefore, by Mathematical Induction, we have;

$$\begin{aligned}
P_4 &= \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 P_0, \\
P_5 &= \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \Lambda_5 P_0, \\
&. \\
&. \\
&. \\
P_n &= \Lambda_1 \Lambda_2 \Lambda_3 \ldots \Lambda_{n-1} \Lambda_n P_0. \qquad (5.51)
\end{aligned}$$

But,

$$\begin{aligned}
P_0 + P_1 P_2 + P_3 + \ldots &= 1 \\
\Rightarrow P_0 + \Lambda_1 P_0 + \Lambda_1 \Lambda_2 P_0 + \Lambda_1 \Lambda_2 \Lambda_3 P_0 + \ldots + \Lambda_1 \Lambda_2 \Lambda_3 \ldots \Lambda_{n-1} \Lambda_n P_0 &= 1 \\
P_0 \left\{ 1 + \Lambda_1 + \Lambda_1 \Lambda_2 + \Lambda_1 \Lambda_2 \Lambda_3 + \ldots + \Lambda_1 \Lambda_2 \Lambda_3 \ldots \Lambda_{n-1} \Lambda_n \right\} &= 1 \\
\therefore P_0 &= \frac{1}{\left\{ 1 + \Lambda_1 + \Lambda_1 \Lambda_2 + \Lambda_1 \Lambda_2 \Lambda_3 + \ldots \right\}}. \\
\therefore P_n &= \Lambda_1 \Lambda_2 \Lambda_3 \ldots \Lambda_n . \frac{1}{\left\{ 1 + \Lambda_1 + \Lambda_1 \Lambda_2 + \ldots \right\}}.
\end{aligned}$$

From (5.51), we have;

$$P_{n+1} = \Lambda_1 \Lambda_2 \Lambda_3 \ldots \Lambda_n \Lambda_{n+1} P_0.$$

Therefore,

$$
\begin{aligned}
\frac{P_{n+1}}{P_n} &= \Lambda_1\Lambda_2\Lambda_3 \ldots \Lambda_n\Lambda_{n+1} \cdot \frac{1}{\Lambda_1\Lambda_2\Lambda_3 \ldots \Lambda_{n-1}\Lambda_n} \\
\therefore \frac{P_{n+1}}{P_n} &= \Lambda_{n+1} \\
\therefore \frac{P_{n+1}}{P_n} &= \frac{\alpha+n}{\gamma+n}, \quad \text{as required.}
\end{aligned}
$$

Using the probability generating function techniques, we multiply (5.46) by s^n and sum the results over n to obtain;

$$
\begin{aligned}
\sum_{n=0}^{\infty}(\gamma+n)P_{n+1}s^n &= \sum_{n=0}^{\infty}(\alpha+n)P_n s^n \\
\gamma\frac{1}{s}\sum_{n=0}^{\infty}P_{n+1}s^{n+1} + \sum_{n=0}^{\infty}nP_{n+1}s^n &= \alpha\sum_{n=0}^{\infty}P_n s^n + s\sum_{n=0}^{\infty}nP_n s^{n-1} \\
\frac{\gamma}{s}\psi(s) + \sum_{n=0}^{\infty}[(n+1)-1]P_{n+1}s^n &= \alpha\psi(s) + s\psi^{'}(s) \\
\frac{\gamma}{s}\psi(s) + \sum_{n=0}^{\infty}(n+1)P_{n+1}s^n - \frac{1}{s}\sum_{n=0}^{\infty}P_{n+1}s^{n+1} &= \alpha\psi(s) + s\psi^{'}(s) \\
\frac{\gamma}{s}\psi(s) + \psi^{'}(s) - \frac{1}{s}\psi(s) &= \alpha\psi(s) + s\psi^{'}(s) \\
\gamma\psi(s) + s\psi^{'}(s) - \psi(s) &= s\alpha\psi(s) + s^2\psi^{'}(s) \\
s(1-s)\psi^{'}(s) &= (\gamma+\alpha s+s)\psi(s) \\
s(1-s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= (\gamma+\alpha s+s)\psi(s) \\
\int\frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int\frac{(\gamma+\alpha s+s)}{s(1-s)}\mathrm{d}s \\
\ln\psi(s) &= \int\frac{\gamma}{s(1-s)}\mathrm{d}s + \int\frac{\alpha}{(1-s)}\mathrm{d}s + \int\frac{1}{(1-s)}\mathrm{d}s \\
\therefore \psi(s) &= (1-s)^{-\alpha}.(1-s)^{-\gamma}.(1-s)^{-1}.
\end{aligned}
$$

Its probability generating function in hypergeometric terms is given by;

$$
\begin{aligned}
\phi(z) &= \frac{{}_2F_1(\alpha,1;\gamma;z)}{{}_2F_1(\alpha,1;\gamma;1)}.\\
\phi^{'}(z) &= \frac{\alpha.1}{\gamma}.\frac{{}_2F_1(\alpha+1,2;\gamma+1;z)}{{}_2F_1(\alpha,1;\gamma;1)}\\
\phi^{''}(z) &= \frac{\alpha(\alpha+1).1(2)}{\gamma(\gamma+1)}.\frac{{}_2F_1(\alpha+2,3;\gamma+2;z)}{{}_2F_1(\alpha,1;\gamma;1)}\\
&= \frac{2\alpha(\alpha+1)}{\gamma(\gamma+1)}.\frac{{}_2F_1(\alpha+2,3;\gamma+2;z)}{{}_2F_1(\alpha,1;\gamma;1)}\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_2F_1(\alpha+\kappa,1+\kappa;\gamma+\kappa;1)}{{}_2F_1(\alpha,1;\gamma;1)};\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\alpha}{\gamma}\Lambda_1.\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= \frac{2\alpha(\alpha+1)}{\gamma(\gamma+1)}\Lambda_2+\frac{\alpha}{\gamma}\Lambda_1-[\frac{\alpha}{\gamma}\Lambda_1]^2\\
\therefore Var(X) &= \frac{\alpha}{\gamma}\Lambda_1+\frac{\alpha}{\gamma}\Big\{\frac{2(\alpha+1)}{(\gamma+1)}\Lambda_2-\frac{\alpha}{\gamma}\Lambda_1^2\Big\}.
\end{aligned}
$$

5.6 Yousry and Srivastava (1987) Recursive Model

The hyper-negative binomial distribution of Yousry and Srivastava (1987) belongs to the extended Katz family. The ratio of successive probabilities is;

Proposition 5.5

$$
\begin{aligned}
\frac{P_{n+1}}{P_n} &= \frac{(r+n)q}{\theta+n},\quad 0<q<1,\quad r>0,\quad \theta>0\\
\Rightarrow (\theta+n)P_{n+1} &= (r+n)qP_n \qquad \textbf{(5.52)}
\end{aligned}
$$

Proof

Solving (5.52) iteratively, we have;

When n=0;

Equation (5.52) becomes;

$$\begin{aligned} \theta P_1 &= rqP_0 \\ \Rightarrow P_1 &= \frac{rq}{\theta}P_0 \\ \therefore P_1 &= \frac{rq}{\theta}P_0. \end{aligned} \tag{5.53}$$

When n=1;
Equation (5.52) becomes;

$$(\theta+1)P_2 = (r+1)qP_1 \tag{5.54}$$

Substituting (5.53) in equation (5.54), we have;

$$\begin{aligned} (\theta+1)P_2 &= (r+1)q\frac{rq}{\theta}P_0 \\ \Rightarrow P_2 &= \frac{r(r+1)q^2}{\theta(\theta+1)}P_0 \\ \therefore P_2 &= \frac{r(r+1)q^2}{\theta(\theta+1)}P_0. \end{aligned} \tag{5.55}$$

When n=2;
Equation (5.52) becomes;

$$(\theta+2)P_3 = (r+2)qP_2 \tag{5.56}$$

Substituting (5.55) in equation (5.56), we have;

$$(\theta+2)P_3 = (r+2)q\frac{r(r+1)q^2}{\theta(\theta+1)}P_0$$

$$\text{Suppose we let,} \quad \Pi_1 = \frac{r}{\theta}$$

$$\text{Then,} \quad \Pi_2 = \frac{(r+1)}{(\theta+1)}, \quad \Pi_3 = \frac{(r+2)}{(\theta+2)}, \quad \ldots, \quad \Pi_n = \frac{(r+n-1)}{(\theta+n-1)}.$$

Therefore,

$$\begin{aligned}
P_1 &= \Pi_1 q P_0,\\
P_2 &= \Pi_1 \Pi_2 q^2 P_0,\\
P_3 &= \Pi_1 \Pi_2 \Pi_3 q^3 P_0,
\end{aligned}$$

By Mathematical Induction, we have;

$$\begin{aligned}
P_4 &= \Pi_1 \Pi_2 \Pi_3 \Pi_4 q^4 P_0,\\
P_5 &= \Pi_1 \Pi_2 \Pi_3 \Pi_4 \Pi_5 q^5 P_0,\\
&.\\
&.\\
&.\\
P_n &= \Pi_1 \Pi_2 \Pi_3 \ldots \Pi_{n-1} \Pi_n q^n P_0.
\end{aligned} \tag{5.57}$$

From (5.57), we have;

$$\begin{aligned}
P_{n+1} &= \Pi_1 \Pi_2 \Pi_3 \ldots \Pi_n \Pi_{n+1} q^{n+1} P_0.\\
\frac{P_{n+1}}{P_n} &= \Pi_1 \Pi_2 \Pi_3 \ldots \Pi_n \Pi_{n+1} q.q^n . \frac{1}{\Pi_1 \Pi_2 \Pi_3 \ldots \Pi_n q^n}\\
\therefore \frac{P_{n+1}}{P_n} &= q \Pi_{n+1}.\\
\therefore \frac{P_{n+1}}{P_n} &= \frac{(r+n)q}{(\theta+n)}, \quad \text{as required.}
\end{aligned}$$

Its probability generating function in terms of hyper-geometric function is given by;

$$\begin{aligned}
\phi(z) &= \frac{{}_2F_1(1, r; \theta; qz)}{{}_2F_1(1, r; \theta; q)}.\\
\phi^{'}(z) &= \frac{r.1}{\theta} q. \frac{{}_2F_1(1+1, r+1; \theta+1; qz)}{{}_2F_1(1, r; \theta; q)}\\
&= \frac{r}{\theta} q. \frac{{}_2F_1(2, r+1; \theta+1; qz)}{{}_2F_1(1, r; \theta; q)}\\
\phi^{''}(z) &= \frac{2r(r+1)}{\theta(\theta+1)} q^2 . \frac{{}_2F_1(2+1, r+2; \theta+2; qz)}{{}_2F_1(1, r; \theta; q)}
\end{aligned}$$

Let,

$$\begin{aligned}
\Lambda_\kappa &= \frac{{}_2F_1(1+\kappa, r+\kappa; \theta+\kappa; q)}{{}_2F_1(1, r; \theta; q)}; \quad \kappa = 1, 2. \\
E(X) &= \phi^{'}(1) \\
\therefore E(X) &= \frac{rq}{\theta}\Lambda_1. \\
Var(X) &= \phi^{''}(1) + \phi^{'}(1) - [\phi^{'}(1)]^2 \\
&= \frac{2r(r+1)q^2}{\theta(\theta+1)}\Lambda_2 + \frac{rq}{\theta}\Lambda_1 - [\frac{rq}{\theta}\Lambda_1]^2 \\
\therefore Var(X) &= \frac{rq}{\theta}\Lambda_1 + \frac{rq^2}{\theta}\left\{\frac{2(r+1)}{(\theta+1)}\Lambda_2 - \frac{r}{\theta}\Lambda_1^2\right\}.
\end{aligned}$$

6 CROW-BARDWELL RECURSIVE MODEL AND ITS EXTENSION IN BIRTH AND DEATH PROCESSES AT EQUILIBRIUM

6.1 Introduction

Let P_n be the distribution of population size n, λ_n and μ_n is the birth (death) rate of the population.
The Pearson's differential equation given by;

$$\frac{P_{n+1}}{P_n} = \frac{P(n)}{Q(n)}$$

where P(n) and Q(n) are polynomials.

The Bardwell-Crow Recursive relations is given by;

$$\frac{P_{n+1}}{P_n} = \frac{\alpha}{\gamma+n}; \quad n = 0,1,2,...$$

6.2 Bardwell-Crow (1964) Model

To obtain the hyper-Poisson Distribution of Bardwell and Crow (1964) and Crow and Bardwell (1965), we let $\beta \to \infty$, in equation (**) in section 5.3.
Hence, we have;
Proposition 6.1

$$\frac{P_{n+1}}{P_n} = \frac{\alpha}{n+\gamma}; \quad n = 0,1,2,3,4,...$$

$$\Rightarrow (n+\gamma)P_{n+1} = \alpha P_n \tag{6.1}$$

Proof
Solving (6.1) iteratively, we have;

When n=0;
Equation (6.1) becomes;

$$\begin{aligned}\gamma P_1 &= \alpha P_0\\ \Rightarrow P_1 &= \frac{\alpha}{\gamma}P_0\\ \therefore P_1 &= \frac{\alpha}{\gamma}P_0.\end{aligned} \tag{6.2}$$

When n=2;
Equation (6.1) becomes;

$$(\gamma+1)P_2 = \alpha P_1 \tag{6.3}$$

Substituting (6.2) in equation (6.3), we get;

$$\begin{aligned}(\gamma+1)P_2 &= \frac{\alpha^2}{\gamma}P_0\\ \Rightarrow P_2 &= \frac{\alpha^2}{\gamma(\gamma+1)}P_0\\ \therefore P_2 &= \frac{\alpha^2}{\gamma(\gamma+1)}P_0.\end{aligned} \tag{6.4}$$

When n=3;
Equation (6.1) becomes;

$$(\gamma+2)P_3 = \alpha P_2 \tag{6.5}$$

Substituting (6.4) in equation (6.5), we have;

$$\begin{aligned}(\gamma+2)P_3 &= \frac{\alpha^3}{\gamma(\gamma+1)}P_0\\ \therefore P_3 &= \frac{\alpha^3}{\gamma(\gamma+1)(\gamma+2)}P_0.\end{aligned}$$

Suppose we let $\frac{\alpha}{\gamma} = \Phi_1$.
Then,

$$\Phi_2 = \frac{\alpha}{\gamma+1}, \quad \Phi_3 = \frac{\alpha}{\gamma+2}, \quad \Phi_4 = \frac{\alpha}{\gamma+3} \quad \dots \quad \Phi_n = \frac{\alpha}{\gamma+n-1}.$$

Therefore,

$$\begin{aligned} P_1 &= \Phi_1 P_0, \\ P_2 &= \Phi_1 \Phi_2 P_0, \\ P_3 &= \Phi_1 \Phi_2 \Phi_3 P_0. \end{aligned}$$

By Mathematical Induction, we have;

$$\begin{aligned} P_4 &= \Phi_1 \Phi_2 \Phi_3 \Phi_4 P_0, \\ P_5 &= \Phi_1 \Phi_2 \Phi_3 \Phi_4 \Phi_5 P_0, \\ &. \\ &. \\ &. \\ P_n &= \Phi_1 \Phi_2 \Phi_3 \dots \Phi_{n-1} \Phi_n P_0. \end{aligned} \tag{6.6}$$

But,

$$\begin{aligned} \sum_{n=0}^{\infty} P_n &= 1 \\ \Rightarrow P_0 + P_1 + P_2 + P_3 + \dots &= 1 \\ P_0 + \Phi_1 P_0 + \Phi_1 \Phi_2 P_0 + \Phi_1 \Phi_2 \Phi_3 P_0 + \dots &= 1 \\ P_0 \left\{ 1 + \Phi_1 + \Phi_1 \Phi_2 + \Phi_1 \Phi_2 \Phi_3 + \dots \right\} &= 1 \\ &\therefore P_0 = \frac{1}{\left\{ 1 + \Phi_1 + \Phi_1 \Phi_2 + \Phi_1 \Phi_2 \Phi_3 + \dots \right\}}. \\ &\therefore P_n = \Phi_1 \Phi_2 \Phi_3 \dots \Phi_{n-1} \Phi_n . \frac{1}{\left\{ 1 + \Phi_1 + \Phi_1 \Phi_2 + \Phi_1 \Phi_2 \Phi_3 + \dots \right\}}. \end{aligned}$$

From (6.6), we get;

$$
\begin{aligned}
P_{n+1} &= \Phi_1\Phi_2\Phi_3\ldots\Phi_n\Phi_{n+1}P_0. \\
\frac{P_{n+1}}{P_n} &= \Phi_1\Phi_2\Phi_3\ldots\Phi_n\Phi_{n+1}\cdot\frac{1}{\Phi_1\Phi_2\Phi_3\ldots\Phi_n}. \\
\therefore \frac{P_{n+1}}{P_n} &= \Phi_{n+1}. \\
\therefore \frac{P_{n+1}}{P_n} &= \frac{\alpha}{n+\gamma}, \quad \text{as required.}
\end{aligned}
$$

Multiplying (6.1) by s^n and sum the results over n, we have;

$$
\begin{aligned}
\sum_{n=0}^{\infty}(\gamma+n)P_{n+1}s^n &= \alpha\sum_{n=0}^{\infty}P_n s^n \\
\gamma\sum_{n=0}^{\infty}P_{n+1}s^n + \sum_{n=0}^{\infty}nP_{n+1}s^n &= \alpha\psi(s) \\
\gamma\psi(s) + \sum_{n=0}^{\infty}(n+1-1)P_{n+1}s^n &= \alpha\psi(s) \\
\gamma\psi(s) + \sum_{n=0}^{\infty}(n+1)P_{n+1}s^n - \sum_{n=0}^{\infty}P_{n+1}s^n &= \alpha\psi(s) \\
\gamma\psi(s) + \psi^{'}(s) - \psi(s) &= \alpha\psi(s) \\
\psi^{'}(s) &= (\alpha-\gamma+1)\psi(s) \\
\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= (\alpha-\gamma+1)\psi(s) \\
\int\frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int(\alpha-\gamma+1)\mathrm{d}s \\
\ln\psi(s) &= \int\alpha\mathrm{d}s + \int\mathrm{d}s - \int\gamma\mathrm{d}s \\
\ln\psi(s) &= \alpha s + s - \gamma s + c \\
\psi(s) &= e^{\alpha s+s-\gamma s}.e^{c} \\
\text{Putting s=1;} \quad \psi(1) &= 1 = e^{\alpha+1-\gamma}.e^{c_1} \\
\Rightarrow c_1 &= e^{-(\alpha+1-\gamma)}. \\
\therefore \psi(s) &= \frac{e^{(\alpha+1-\gamma)s}}{e^{(\alpha+1-\gamma)}}.
\end{aligned}
$$

From Kummer's confluent hypergeometric series, we have;

$$
\begin{aligned}
{}_1F_1(1;\gamma;\alpha) &= 1+\frac{1}{\gamma}\frac{\alpha}{1!}+\frac{1(1+1)}{\gamma(\gamma+1)}\frac{\alpha^2}{2!}+\frac{1(1+1)(1+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{\alpha^3}{3!}+\ldots \\
&= \sum_{n=0}^{\infty}\frac{1(1+1)(1+2)\ldots(1+n-1)}{\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)}\frac{\alpha^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{(1+n-1)(1+n-2)\ldots(1+2)(1+1)1}{(\gamma+n-1)(\gamma+n-2)\ldots(\gamma+2)(\gamma+1)\gamma}\frac{\Gamma(1)}{\Gamma(\gamma)}.\frac{\Gamma(\gamma)}{\Gamma(1)}.\frac{\alpha^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(1+n)}{\Gamma(\gamma+n)}.\frac{\Gamma(\gamma)}{\Gamma(1)}.\frac{\alpha^n}{n!}
\end{aligned}
$$

Normalizing, we get;

$$
1=\sum_{n=0}^{\infty}\frac{\Gamma(1+n)}{\Gamma(\gamma+n)}.\frac{\Gamma(\gamma)}{\Gamma(1)}.\frac{1}{{}_1F_1(1;\gamma;\alpha)}.\frac{\alpha^n}{n!}
$$

Thus,

$$
\begin{aligned}
P_n &= \text{Prob}(N=n) \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(1+n)}{\Gamma(\gamma+n)}.\frac{\Gamma(\gamma)}{\Gamma(1)}.\frac{1}{{}_1F_1(1;\gamma;\alpha)}.\frac{\alpha^n}{n!}; \quad n=0,1,2,\ldots \quad \gamma>0, \quad \alpha>0.
\end{aligned}
$$

The probability generating function is given by;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty}P_n z^n \\
\phi(z) &= \sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(1+n)}{\Gamma(\gamma+n)}.\frac{\Gamma(\gamma)}{\Gamma(1)}.\frac{(\alpha z)^n}{n!}.\frac{1}{{}_1F_1(1;\gamma;\alpha)} \\
\therefore \phi(z) &= \frac{{}_1F_1(1;\gamma;\alpha z)}{{}_1F_1(1;\gamma;\alpha)}.
\end{aligned}
$$

This implies that;

$$
\begin{aligned}
\phi^{'}(z) &= \frac{1}{{}_1F_1(1;\gamma;\alpha)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_1F_1(1;\gamma;\alpha z)\} \\
&= \frac{1}{{}_1F_1(1;\gamma;\alpha)}\frac{\alpha}{\gamma}.{}_1F_1(1+1;\gamma+1;\alpha z) \\
&= \frac{\alpha}{\gamma}\frac{{}_1F_1(2;\gamma+1;\alpha z)}{{}_1F_1(1;\gamma;\alpha)}. \\
\phi^{''}(z) &= \frac{1}{{}_1F_1(1;\gamma;\alpha)}.\frac{\alpha}{\gamma}.\frac{\mathrm{d}}{\mathrm{d}z}\{{}_1F_1(2;\gamma+1;\alpha z)\} \\
&= \frac{1}{{}_1F_1(1;\gamma;\alpha)}.\frac{\alpha}{\gamma}.\frac{2\alpha}{\gamma+1}.{}_1F_1(2+1;\gamma+2;\alpha z) \\
&= \frac{2\alpha^2}{\gamma(\gamma+1)}\frac{{}_1F_1(3;\gamma+2;\alpha z)}{{}_1F_1(1;\gamma;\alpha)}. \\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_1F_1(1+\kappa;\gamma+\kappa;\alpha)}{{}_1F_1(1;\gamma;\alpha)},\quad \kappa = 1,2. \\
E(X) &= \phi^{'}(1) \\
\therefore E(X) &= \frac{\alpha}{\gamma}\Lambda_1. \\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2 \\
&= \frac{2\alpha^2}{\gamma(\gamma+1)}\Lambda_2+\frac{\alpha}{\gamma}\Lambda_1-[\frac{\alpha}{\gamma}\Lambda_1]^2 \\
\therefore Var(X) &= \frac{\alpha}{\gamma}\Lambda_1+\frac{\alpha^2}{\gamma}\left\{\frac{2}{\gamma+1}\Lambda_2-\frac{1}{\gamma}\Lambda_1^2\right\}.
\end{aligned}
$$

6.3 Special Cases and Properties

Case1: When $\alpha = \theta$ and $\gamma = r$

Staff (1964, 1967) defined the displaced Poisson distribution as a left-truncated Poisson distribution that has been "displaced" which corresponds to the displaced distribution;

$$
\text{Prob}(N=n) = \frac{\theta^{n+r+1}}{(n+r+1)!}\left\{\sum_{j=r+1}^{\infty}\frac{\alpha^j}{j!}\right\}^{-1},\quad n = 0,1,2,\ldots \qquad \textbf{(6.7)}
$$

Using recurrence relation, Staff also obtained the distribution;

$$\text{Prob}[N=n+1]=\frac{\theta}{r+n}\text{Prob}[N=n],\quad n=1,2,...$$

where r is a positive integer.

Suppose we let the partial sum be given by;

$$\begin{aligned}
z &= \frac{\theta^{r+1}}{(r+1)!}+\frac{\theta^{r+2}}{(r+2)!}+\frac{\theta^{r+3}}{(r+3)!}+\cdots \\
&= \sum_{n=0}^{\infty}\frac{\theta^{n+r+1}}{(n+r+1)!} \\
\therefore 1 &= \sum_{n=0}^{\infty}\frac{\theta^{n+r+1}}{(n+r+1)!\,z} \\
\therefore f(n) &= \frac{\theta^{n+r+1}}{(n+r+1)!\sum_{n=0}^{\infty}\frac{\theta^{n+r+1}}{(n+r+1)!}} \text{ for } n=0,1,2,\cdots;\theta>0 \qquad \textbf{(6.8)} \\
f(n) &= \frac{\theta^{n+r+1}}{(n+r+1)!\sum_{n=0}^{\infty}\frac{\theta^{n+r+1}}{(n+r+1)!}} \\
&= \frac{\theta^{n+r+1}}{\Gamma(n+r)}\frac{1}{\sum_{n=0}^{\infty}\frac{\theta^{n+r+1}}{\Gamma(n+r)}} \\
&= \frac{\theta^{n}}{\Gamma(n+r)}\frac{1}{\sum_{n=0}^{\infty}\frac{\theta^{n}}{\Gamma(n+r)}} \\
&= \frac{\theta^{n}}{\Gamma(n+r)}\frac{1}{\sum_{n=0}^{\infty}\frac{\theta^{n}}{n!}\frac{n!}{\Gamma(n+r)}} \\
&= \frac{\theta^{n}}{\Gamma(n+r)}\frac{1}{\sum_{n=0}^{\infty}\frac{\Gamma(1+n)}{\Gamma(n+r)}\frac{\theta^{n}}{n!}} \\
&= \frac{\theta^{n}}{\Gamma(n+r)}\frac{1}{\sum_{n=0}^{\infty}\frac{\Gamma(1+n)}{\Gamma(n+r)}\frac{\theta^{n}}{n!}} \\
&= \frac{\theta^{n}}{\Gamma(n+r)}\frac{\Gamma(r)}{\Gamma(r)}\frac{1}{\sum_{n=0}^{\infty}\frac{\Gamma(1+n)}{\Gamma(n+r)}\frac{\Gamma(r)}{\Gamma(r)}\frac{\theta^{n}}{n!}} \\
&= \frac{\theta^{n}\Gamma(r)}{\Gamma(n+r)}\frac{1}{{}_1F_1(1;r;\theta)} \\
&= n!\frac{\Gamma(r)}{\Gamma(n+r)}\frac{1}{{}_1F_1(1;r;\theta)}\frac{\theta^{n}}{n!}
\end{aligned}$$

$$\therefore f(n) = \frac{\Gamma(1+n)}{\Gamma(r+n)} \frac{\Gamma(r)}{\Gamma(1)} \frac{1}{{}_1F_1(1;r;\theta)} \frac{\theta^n}{n!} \text{ for } n = 0,1,2,\cdots;\theta > 0 \tag{6.9}$$

Its probability generating function in hypergeometric terms is given by;

$$\begin{aligned}
\phi(z) &= \frac{{}_1F_1(1;r;\theta z)}{{}_1F_1(1;r;\theta)}. \qquad (6.10)\\
\phi^{'}(z) &= \frac{1}{{}_1F_1(1;r;\theta)} \frac{\mathrm{d}}{\mathrm{d}z}\{{}_1F_1(1;r;\theta z)\}\\
&= \frac{1}{{}_1F_1(1;r;\theta)} \frac{\theta}{r}\,{}_1F_1(1+1;r+1;\theta z)\\
&= \frac{\theta}{r} \frac{{}_1F_1(2;r+1;\theta z)}{{}_1F_1(1;r;\theta)}.\\
\phi^{''} &= \frac{1}{{}_1F_1(1;r;\theta)} \frac{\theta}{r} \frac{\mathrm{d}}{\mathrm{d}z}.{}_1F_1(2;r+1;\theta z)\\
&= \frac{1}{{}_1F_1(1;r;\theta)} \frac{\theta}{r} \frac{2\theta}{r+1}.{}_1F_1(2+1;r+2;\theta z)\\
&= \frac{2\theta^2}{r(r+1)} \frac{{}_1F_1(3;r+2;\theta z)}{{}_1F_1(1;r;\theta)}.\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_1F_1(1+\kappa;r+\kappa;\theta)}{{}_1F_1(1;r;\theta)}; \quad \kappa = 1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\theta}{r}\Lambda_1.\\
&\text{and}\\
Var(X) &= \phi^{''}(1) + \phi^{'}(1) - [\phi^{'}(1)]^2\\
&= \frac{2\theta^2}{r(r+1)}\Lambda_2 + \frac{\theta}{r}\Lambda_1 - [\frac{\theta}{r}\Lambda_1]^2\\
\therefore Var(X) &= \frac{\theta}{r}\Lambda_1 + \frac{\theta^2}{r}\Big\{\frac{2}{r+1}\Lambda_2 - \frac{1}{r}\Lambda_1^2\Big\}.
\end{aligned}$$

Case II: When $\alpha = \theta$ and $\gamma = \lambda$

In this case the "displaced Poisson" becomes a special case of the general hyper-Poisson distribution.

Then, the probability generating function becomes;

$$\phi(z) = \frac{{}_1F_1(1;\lambda;\theta z)}{{}_1F_1(1;\lambda;\theta)}. \tag{6.11}$$

where λ is non-negative real.

$$\begin{aligned}
\phi^{'}(z) &= \frac{1}{{}_1F_1(1;\lambda;\theta)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_1F_1(1;\lambda;\theta z)\} \\
&= \frac{1}{{}_1F_1(1;\lambda;\theta)}\frac{\theta}{\lambda}{}_1F_1(1+1;\lambda+1;\theta z) \\
&= \frac{\theta}{\lambda}\frac{{}_1F_1(2;\lambda+1;\theta z)}{{}_1F_1(1;\lambda;\theta)}. \\
\phi^{''} &= \frac{1}{{}_1F_1(1;\lambda;\theta)}\frac{\theta}{\lambda}\frac{\mathrm{d}}{\mathrm{d}z}.{}_1F_1(2;\lambda+1;\theta z) \\
&= \frac{1}{{}_1F_1(1;\lambda;\theta)}\frac{\theta}{\lambda}\frac{2\theta}{\lambda+1}.{}_1F_1(2+1;\lambda+2;\theta z) \\
&= \frac{2\theta^2}{\lambda(\lambda+1)}\frac{{}_1F_1(3;\lambda+2;\theta z)}{{}_1F_1(1;\lambda;\theta)}. \\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_1F_1(1+\kappa;\lambda+\kappa;\theta)}{{}_1F_1(1;\lambda;\theta)}; \quad \kappa = 1,2. \\
E(X) &= \phi^{'}(1) \\
\therefore E(X) &= \frac{\theta}{\lambda}\Lambda_1. \\
&\text{and} \\
Var(X) &= \phi^{''}(1) + \phi^{'}(1) - [\phi^{'}(1)]^2 \\
&= \frac{2\theta^2}{\lambda(\lambda+1)}\Lambda_2 + \frac{\theta}{\lambda}\Lambda_1 - [\frac{\theta}{\lambda}\Lambda_1]^2 \\
\therefore Var(X) &= \frac{\theta}{\lambda}\Lambda_1 + \frac{\theta^2}{\lambda}\Big\{\frac{2}{\lambda+1}\Lambda_2 - \frac{1}{\lambda}\Lambda_1^2\Big\}.
\end{aligned}$$

Note that, the distribution was termed sub-Poisson for $\lambda < 1$ and super-Poisson for $\lambda > 1$ by Bardwell and Crow (1964).

6.4 Hyper-Poisson as a Special Case of Bhattacharya (1966) and Hall (1956) Distribution

The hyper-Poisson distribution can be expressed as a special case of Bhattacharya (1966) and Hall (1956) confluent hypergeometric distribution give by;

$$
\begin{aligned}
{}_1F_1(\flat;\lambda;\theta) &= 1+\frac{\flat}{\lambda}\frac{\theta}{1!}+\frac{\flat(\flat+1)}{\lambda(\lambda+1)}\frac{\theta^2}{2!}+\frac{\flat(\flat+1)(\flat+2)}{\lambda(\lambda+1)(\lambda+2)}\frac{\theta^3}{3!}+\cdots \\
&= \sum_{n=0}^{\infty}\frac{\flat(\flat+1)(\flat+2)\cdots(\flat+n-1)}{\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}\frac{\theta^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{(\flat+n-1)(\flat+n-2)\cdots(\flat+2)(\flat+1)\flat}{(\lambda+n-1)(\lambda+n-2)\cdots(\lambda+2)(\lambda+1)\lambda}\frac{\Gamma(\flat)}{\Gamma(\lambda)}\frac{\Gamma(\lambda)}{\Gamma(\flat)}\frac{\theta^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\flat+n)}{\Gamma(\lambda+n)}\frac{\Gamma(\lambda)}{\Gamma(\flat)}\frac{\theta^n}{n!} \qquad (6.12)
\end{aligned}
$$

For normalizing, we have;

$$
1=\sum_{n=0}^{\infty}\frac{\Gamma(\flat+n)}{\Gamma(\lambda+n)}\frac{\Gamma(\lambda)}{\Gamma(\flat)}\frac{1}{{}_1F_1(\flat;\lambda;\theta)}\frac{\theta^n}{n!}
$$

Therefore,
P(n)=Prob(N=n)

$$
=\sum_{n=0}^{\infty}\frac{\Gamma(\flat+n)}{\Gamma(\lambda+n)}\frac{\Gamma(\lambda)}{\Gamma(\flat)}\frac{1}{{}_1F_1(\flat;\lambda;\theta)}\frac{\theta^n}{n!};\quad \text{for}\quad n=0,1,2,\cdots;\quad \flat>0,\quad \lambda>0,\quad \theta>0 \qquad (6.13)
$$

Its pgf. is given by;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty}P(n)z^n \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\flat+n)}{\Gamma(\lambda+n)}\frac{\Gamma(\lambda)}{\Gamma(\flat)}\frac{(\theta z)^n}{n!}\frac{1}{{}_1F_1(\flat;\lambda;\theta)} \\
\therefore \phi(z) &= \frac{{}_1F_1(\flat;\lambda;\theta z)}{{}_1F_1(\flat;\lambda;\theta)}. \qquad (6.14)
\end{aligned}
$$

This implies that;

$$
\begin{aligned}
\phi^{'}(z) &= \frac{1}{{}_1F_1(\flat;\lambda;\theta)}\frac{\mathrm{d}}{\mathrm{d}z}.{}_1F_1(\flat;\lambda;\theta z)\\
&= \frac{1}{{}_1F_1(\flat;\lambda;\theta)}\frac{\flat}{\lambda}\theta.{}_1F_1(\flat+1;\lambda+1;\theta z)\\
\therefore \phi^{'}(z) &= \frac{\flat\theta}{\lambda}\frac{{}_1F_1(\flat+1;\lambda+1;\theta z)}{{}_1F_1(\flat;\lambda;\theta)}.\\
\phi^{''}(z) &= \frac{1}{{}_1F_1(\flat;\lambda;\theta)}\frac{\flat\theta}{\lambda}\frac{(\flat+1)\theta}{\lambda+1}.{}_1F_1(\flat+2;\lambda+2;\theta z)\\
\therefore \phi^{''}(z) &= \frac{\flat(\flat+1)\theta^2}{\lambda(\lambda+1)}\frac{{}_1F_1(\flat+2;\lambda+2;\theta z)}{{}_1F_1(\flat;\lambda;\theta)}.\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_1F_1(\flat+\kappa;\lambda+\kappa;\theta)}{{}_1F_1(\flat;\lambda;\theta)};\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\theta\flat}{\lambda}\Lambda_1.\\
&\text{and}\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= \theta^2\frac{\flat(\flat+1)}{\lambda(\lambda+1)}\Lambda_2+\frac{\theta\flat}{\lambda}\Lambda_1-\Big(\frac{\theta\flat}{\lambda}\Big)^2\\
\therefore Var(X) &= \frac{\theta\flat}{\lambda}\Lambda_1+\frac{\theta^2\flat}{\lambda}\Big(\frac{\flat+1}{\lambda+1}\Lambda_2-\frac{\flat}{\lambda}\Lambda_1^2\Big).
\end{aligned}
$$

6.5 Hyper-Poisson Distribution in an Extended Form

Gurland and Tripathi (1975) and Tripathi and Gurland (1977, 1979) gave an extended form of the hyper-Poisson distribution as;

Proposition 6.2

$$\frac{P_{n+1}}{P_n} = \frac{\alpha(n+\rho)}{(1+n)(n+\gamma)};\quad n=0,1,2,\cdots \tag{6.15}$$

$$\Rightarrow (1+n)(\gamma+n)P_{n+1} = \alpha(\rho+n)P_n \tag{6.16}$$

Proof

Solving (6.16) iteratively, we have;

When n=0;

Equation (6.16) becomes;

$$\gamma P_1 = \alpha\rho P_0$$
$$\Rightarrow P_1 = \frac{\alpha\rho}{\gamma}P_0$$
$$\therefore P_1 = \frac{\alpha\rho}{\gamma}P_0. \tag{6.17}$$

When n=1;
Equation (6.16) becomes;

$$2(\gamma+1)P_2 = \alpha(\rho+1)P_1 \tag{6.18}$$

Substituting (6.17) in equation (6.18), we have;

$$2(\gamma+1)P_2 = \alpha(\rho+1)\frac{\alpha\rho}{\gamma}P_0$$
$$\Rightarrow P_2 = \frac{\rho(\rho+1)}{\gamma(\gamma+1)}\frac{\alpha^2}{1.2}P_0$$
$$\therefore P_2 = \frac{\rho(\rho+1)}{\gamma(\gamma+1)}\frac{\alpha^2}{2!}P_0. \tag{6.19}$$

When n=2;
Equation (6.16) becomes;

$$3(\gamma+2)P_3 = \alpha(\rho+2)P_2 \tag{6.20}$$

Substituting (6.19) in equation (6.20), we have;

$$3(\gamma+2)P_3 = \alpha(\rho+2)\frac{\rho(\rho+1)}{\gamma(\gamma+1)}\frac{\alpha^2}{1.2}P_0$$
$$\Rightarrow P_3 = \frac{\rho(\rho+1)(\rho+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{\alpha^3}{3.2.1}P_0$$
$$\therefore P_3 = \frac{\rho(\rho+1)(\rho+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{\alpha^3}{3!}P_0.$$

By Mathematical Induction;

$$P_n = \frac{\rho(\rho+1)(\rho+2)\cdots(\rho+n-2)(\rho+n-1)}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)(\gamma+n-1)}\frac{\alpha^n}{n!}P_0 \tag{6.21}$$

But;

$$\sum_{n=0}^{\infty} P_n = 1$$

$$P_0\left\{1+\frac{\alpha\rho}{\gamma}+\frac{\alpha^2}{2!}\frac{\rho(\rho+1)}{\gamma(\gamma+1)}+\frac{\alpha^3}{3!}\frac{\rho(\rho+1)(\rho+2)}{\gamma(\gamma+1)(\gamma+2)}\cdots\right\} = 1$$

$$\therefore P_0 = \frac{1}{\left\{1+\frac{\alpha\rho}{\gamma}+\frac{\alpha^2}{2!}\frac{\rho(\rho+1)}{\gamma(\gamma+1)}+\frac{\alpha^3}{3!}\frac{\rho(\rho+1)(\rho+2)}{\gamma(\gamma+1)(\gamma+2)}\cdots\right\}}.$$

$$\therefore P_n = \frac{\rho(\rho+1)(\rho+2)\cdots(\rho+n-2)(\rho+n-1)}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)(\gamma+n-1)}\frac{\alpha^n}{n!}$$

$$x\frac{1}{\left\{1+\frac{\alpha\rho}{\gamma}+\frac{\alpha^2}{2!}\frac{\rho(\rho+1)}{\gamma(\gamma+1)}+\frac{\alpha^3}{3!}\frac{\rho(\rho+1)(\rho+2)}{\gamma(\gamma+1)(\gamma+2)}\cdots\right\}}.$$

From (6.21), we have;

$$P_{n+1} = \frac{\rho(\rho+1)(\rho+2)\cdots(\rho+n-1)(\rho+n)}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)(\gamma+n)}\frac{\alpha^{n+1}}{(n+1)!}P_0$$

$$\frac{P_{n+1}}{P_n} = \frac{\rho(\rho+1)(\rho+2)\cdots(\rho+n-1)(\rho+n)}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)(\gamma+n)}\frac{\alpha^1.\alpha^n}{(n+1)n!}.\frac{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)(\gamma+n-1)}{\rho(\rho+1)(\rho+2)\cdots(\rho+n-2)(\rho+n-1)}\frac{n!}{\alpha^n}$$

$$\therefore \frac{P_{n+1}}{P_n} = \frac{\alpha(\rho+n)}{(n+1)(\gamma+n)}, \quad \text{as required.}$$

Multiplying (6.16) by s^n and sum the results over n, we get;

$$\sum_{n=0}^{\infty}(n+1)(\gamma+n)P_{n+1}s^n = \sum_{n=0}^{\infty}(\alpha\rho+\alpha n)P_n s^n$$

Thus;

$$
\begin{aligned}
\gamma \sum_{n=0}^{\infty}(n+1)P_{n+1}s^n + \sum_{n=0}^{\infty} n(n+1)P_{n+1}s^n &= \alpha\rho \sum_{n=0}^{\infty} P_n s^n + \alpha \sum_{n=0}^{\infty} nP_n s^n \\
\gamma\psi^{'}(s) + \psi^{'}(s) - \psi^{'}(s) &= \alpha\rho\,\psi(s) + \alpha s\psi^{'}(s) \\
(\gamma - \alpha s)\psi^{'}(s) &= \alpha\rho\,\psi(s) \\
(\gamma - \alpha s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \alpha\rho\,\psi(s) \\
\int \frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int \frac{\alpha\rho}{\gamma - \alpha s} ds \\
\ln\psi(s) &= -\rho\ln(\gamma - \alpha s) + \ln c \\
\ln\psi(s) &= \ln(\gamma - \alpha s)^{-\rho} + \ln c \\
\psi(s) &= c_1(\gamma - \alpha s)^{-\rho}
\end{aligned}
$$

Putting s=1;

$$
\begin{aligned}
\psi(1) = 1 &= c_1(\gamma - \alpha)^{-\rho} \\
\Rightarrow c_1 &= \frac{1}{(\gamma - \alpha)^{-\rho}} \\
\therefore c_1 &= (\gamma - \alpha)^{\rho}. \\
\therefore \psi(s) &= \left(\frac{\gamma - \alpha}{\gamma - \alpha s}\right)^{\rho}.
\end{aligned}
$$

From Kummer's confluent hypergeometric series, we have;

$$
\begin{aligned}
{}_1F_1(\rho;\gamma;\alpha) &= 1 + \frac{\rho}{\gamma}\frac{\alpha}{1!} + \frac{\rho(\rho+1)}{\gamma(\gamma+1)}\frac{\alpha^2}{2!} + \frac{\rho(\rho+1)(\rho+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{\alpha^3}{3!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{\rho(\rho+1)(\rho+2)\cdots(\rho+n-1)}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)}\frac{\alpha^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(\rho+n-1)(\rho+n-2)\cdots(\rho+2)(\rho+1)\rho}{(\gamma+n-1)(\gamma+n-2)\cdots(\gamma+2)(\gamma+1)\gamma}\frac{\Gamma(\rho)}{\Gamma(\gamma)}\frac{\Gamma(\gamma)}{\Gamma(\rho)}\frac{\alpha^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(\rho+n)}{\Gamma(\gamma+n)}\frac{\Gamma(\gamma)}{\Gamma(\rho)}\frac{\alpha^n}{n!}
\end{aligned}
$$

For normalizing, we have;

$$1=\sum_{n=0}^{\infty}\frac{\Gamma(\rho+n)}{\Gamma(\gamma+n)}\frac{\Gamma(\gamma)}{\Gamma(\rho)}\frac{1}{{}_1F_1(\rho;\gamma;\alpha)}\frac{\alpha^n}{n!}$$

Therefore,
P(n)=Prob(N=n)

$$=\sum_{n=0}^{\infty}\frac{\Gamma(\rho+n)}{\Gamma(\gamma+n)}\frac{\Gamma(\gamma)}{\Gamma(\rho)}\frac{1}{{}_1F_1(\rho;\gamma;\alpha)}\frac{\alpha^n}{n!};\quad \text{for}\quad n=0,1,2,\cdots;\quad \rho>0,\quad \gamma>0,\quad \alpha>0$$

Its pgf. is given by;

$$\begin{aligned}
\phi(z)&=\sum_{n=0}^{\infty}P(n)z^n\\
&=\sum_{n=0}^{\infty}\frac{\Gamma(\rho+n)}{\Gamma(\gamma+n)}\frac{\Gamma(\gamma)}{\Gamma(\rho)}\frac{(\alpha z)^n}{n!}\frac{1}{{}_1F_1(\rho;\gamma;\alpha)}\\
\therefore \phi(z)&=\frac{{}_1F_1(\rho;\gamma;\alpha z)}{{}_1F_1(\rho;\gamma;\alpha)}.\\
\phi^{'}(z)&=\frac{1}{{}_1F_1(\rho;\gamma;\alpha)}\frac{\mathrm{d}}{\mathrm{d}z}.{}_1F_1(\rho;\gamma;\alpha z)\\
&=\frac{1}{{}_1F_1(\rho;\gamma;\alpha)}\frac{\rho}{\gamma}\alpha.{}_1F_1(\rho+1;\gamma+1;\alpha z)\\
\therefore \phi^{'}(z)&=\frac{\rho\alpha}{\gamma}\frac{{}_1F_1(\rho+1;\gamma+1;\alpha z)}{{}_1F_1(\rho;\gamma;\alpha)}.\\
\phi^{''}(z)&=\frac{1}{{}_1F_1(\rho;\gamma;\alpha)}\frac{\rho\alpha}{\gamma}\frac{(\rho+1)\alpha}{\gamma+1}.{}_1F_1(\rho+2;\gamma+2;\alpha z)\\
\therefore \phi^{''}(z)&=\frac{\rho(\rho+1)\alpha^2}{\gamma(\gamma+1)}\frac{{}_1F_1(\rho+2;\gamma+2;\alpha z)}{{}_1F_1(\rho;\gamma;\alpha)}.
\end{aligned}$$

Let,

$$\begin{aligned}
\Lambda_\kappa &= \frac{{}_1F_1(\rho+\kappa;\gamma+\kappa;\alpha)}{{}_1F_1(\rho;\gamma;\alpha)}; \quad \kappa = 1,2. \\
E(X) &= \phi^{'}(1) \\
\therefore E(X) &= \frac{\alpha\rho}{\gamma}\Lambda_1. \\
&\text{and} \\
Var(X) &= \phi^{''}(1) + \phi^{'}(1) - [\phi^{'}(1)]^2 \\
&= \alpha^2\frac{\rho(\rho+1)}{\gamma(\gamma+1)}\Lambda_2 + \frac{\alpha\rho}{\gamma}\Lambda_1 - \left(\frac{\alpha\rho}{\gamma}\right)^2 \\
\therefore Var(X) &= \frac{\alpha\rho}{\gamma}\Lambda_1 + \frac{\alpha^2\rho}{\gamma}\left(\frac{\rho+1}{\gamma+1}\Lambda_2 - \frac{\rho}{\gamma}\Lambda_1^2\right).
\end{aligned}$$

7 PANJER'S RECURSIVE MODEL AND ITS EXTENSION IN BIRTH-AND-DEATH PROCESSES AT EQUILIBRIUM

7.1 Introduction

In this chapter, a recursive evaluation of the distribution of total claims is developed for a family of claim number distributions and arbitrary claim amount distribution. For a discrete claim amount, the recursive definition is used to compute the distribution of total amount.

The family of claim number distributions satisfies the Pearson's differential equations as a ratio of polynomials given as;

$$\frac{P_{n-1}}{P_n} = \frac{Q(n)}{P(n)}.$$

where Q(n) and P(n) are polynomials in n.

7.2 Panjer's (1981) Recursive Model

Consider the family of claim number distribution satisfying the recursion;

$$\frac{P_n}{P_{n-1}} = (\xi + \frac{\lambda}{n}), \quad n = 1,2,3,\cdots \tag{7.1}$$

and

$$P_0 > 0$$

$$\Rightarrow nP_n = \xi(n-1)P_{n-1} + (\xi + \lambda)P_{n-1}; \quad n = 1,2,3,\cdots \tag{7.2}$$

Proof

Solving (7.2) iteratively, we have;

When n=0;

Equation (7.2) becomes;

$$P_1 = (\xi + \lambda)P_0 \tag{7.3}$$

When n=1;
Equation (7.2) becomes;

$$2P_2 = \xi P_1 + (\xi + \lambda)P_1 \tag{7.4}$$

Substituting (7.3) in equation (7.4), we have;

$$\begin{aligned} 2P_2 &= \xi(\xi + \lambda)P_0 + (\xi + \lambda)(\xi + \lambda)P_0 \\ \Rightarrow 2P_2 &= (\xi + \lambda)\{\xi + (\xi + \lambda)\}P_0 \\ \therefore P_2 &= \frac{(\xi + \lambda)(\xi + (\xi + \lambda))}{2!}P_0. \end{aligned} \tag{7.5}$$

When n=2;
Equation (7.2) becomes;

$$3P_3 = 2\xi P_2 + (\xi + \lambda)P_2 \tag{7.6}$$

Substituting (7.5) in equation (7.6), we have;

$$\begin{aligned} 3P_3 &= 2\xi \frac{(\xi + \lambda)\{\xi + (\xi + \lambda)\}}{2}P_0 + (\xi + \lambda)\frac{(\xi + \lambda)\{\xi + (\xi + \lambda)\}}{2}P_0 \\ \Rightarrow 3P_3 &= \frac{(\xi + \lambda)\{\xi + (\xi + \lambda)\}}{2} + (2\xi + (\xi + \lambda))P_0 \\ \therefore P_3 &= \frac{(\xi + \lambda)(\xi + (\xi + \lambda)(2\xi + (\xi + \lambda)))}{3!}P_0. \end{aligned}$$

By Mathematical Induction, we have;

$$
\begin{aligned}
P_4 &= \frac{(\xi+\lambda)\{\xi+(\xi+\lambda)(2\xi+(\xi+\lambda))(3\xi+(\xi+\lambda))\}}{4!}P_0,\\
P_5 &= \frac{(\xi+\lambda)(\xi+(\xi+\lambda))(2\xi+(\xi+\lambda))(3\xi+(\xi+\lambda))(4\xi+(\xi+\lambda))}{5!}P_0,\\
&.\\
&.\\
&.\\
P_n &= \frac{(\xi+\lambda)(\xi+(\xi+\lambda))(2\xi+(\xi+\lambda))\cdots((n-2)\xi+(\xi+\lambda))((n-1)\xi+(\xi+\lambda))}{n!}P_0.
\end{aligned}
\tag{7.7}
$$

But,

$$
\begin{aligned}
\sum_{n=0}^{\infty} P_n &= 1\\
\Rightarrow P_0+P_1+P_2+P_3+\cdots &= 1\\
\Rightarrow P_0+(\xi+\lambda)P_0+\frac{(\xi+\lambda)(\xi+(\xi+\lambda))}{2!}P_0+\frac{(\xi+\lambda)(\xi+(\xi+\lambda)(2\xi+(\xi+\lambda)))}{3!}P_0+\cdots &= 1\\
P_0\Big(1+\frac{(\xi+\lambda)}{1!}+\frac{(\xi+\lambda)(\xi+(\xi+\lambda))}{2!}+\frac{(\xi+\lambda)(\xi+(\xi+\lambda)(2\xi+(\xi+\lambda)))}{3!}+\cdots\Big) &= 1\\
\therefore P_0 = \frac{1}{\Big(1+\frac{(\xi+\lambda)}{1!}+\frac{(\xi+\lambda)(\xi+(\xi+\lambda))}{2!}+\cdots\Big)}&.\\
\therefore P_n = \frac{(\xi+\lambda)(\xi+(\xi+\lambda))(2\xi+(\xi+\lambda))\cdots((n-2)\xi+(\xi+\lambda))((n-1)\xi+(\xi+\lambda))}{n!}&\\
x\frac{1}{\Big(1+\frac{(\xi+\lambda)}{1!}+\frac{(\xi+\lambda)(\xi+(\xi+\lambda))}{2!}+\cdots\Big)}&.
\end{aligned}
$$

From (7.7), we have;

$$
P_{n-1} = \frac{(\xi+\lambda)(\xi+(\xi+\lambda))(2\xi+(\xi+\lambda))\cdots((n-3)\xi+(\xi+\lambda))((n-2)\xi+(\xi+\lambda))}{(n-1)!}P_0.
$$

Therefore,

$$\frac{P_n}{P_{n-1}} = \frac{(\xi+\lambda)(\xi+(\xi+\lambda))(2\xi+(\xi+\lambda))\cdots((n-2)\xi+(\xi+\lambda))((n-1)\xi+(\xi+\lambda))}{n.(n-1)!}$$
$$x\frac{(n-1)!}{(\xi+\lambda)(\xi+(\xi+\lambda))(2\xi+(\xi+\lambda))\cdots((n-3)\xi+(\xi+\lambda))((n-2)\xi+(\xi+\lambda))}$$
$$\therefore \frac{P_n}{P_{n-1}} = \frac{n\xi+\lambda}{n}, \text{as required.}$$

Multiplying (7.2) by s^n and sum the results over n to obtain;

$$\begin{aligned}
\sum_{n=1}^{\infty} nP_n s^n &= \xi\sum_{n=1}^{\infty}(n-1)P_{n-1}s^n + (\xi+\lambda)\sum_{n=1}^{\infty}P_{n-1}s^n \\
s\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \xi s^2\sum_{n=1}^{\infty}(n-1)P_{n-1}s^{n-2} + (\xi+\lambda)s\sum_{n=1}^{\infty}P_{n-1}s^{n-1} \\
\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \xi s\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} + (\xi+\lambda)\psi(s) \\
\therefore (1-\xi s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= (\xi+\lambda)\psi(s) \qquad \textbf{(7.8)}
\end{aligned}$$

7.3 Special Cases of Panjer's Recursive Model

7.3.1 Case (i) When $\xi = 0$ and $\lambda = 0$

Equation (7.8) becomes;

$$\begin{aligned}
\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= 0 \\
\Rightarrow \psi(s) &= c \\
\Rightarrow \psi(1) &= 1 = c \\
\text{Hence,} \quad \psi(1) &= 1 \\
\therefore P_n &= \begin{cases} 1, & \text{if n=0} \\ 0, & \text{if n >1} \end{cases}
\end{aligned}$$

7.3.2 Case (ii) When $\xi = 0$ and $\lambda \neq 0$

1. Poisson Distribution

Equation (7.8) becomes;

$$
\begin{aligned}
\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \lambda\,\psi(s) \\
\Rightarrow \frac{1}{\psi(s)}\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \lambda \\
\frac{\mathrm{d}}{\mathrm{d}s}\log\psi(s) &= \lambda \\
\Rightarrow \log\psi(s) &= \lambda s + c \\
\text{Thus,}\quad \psi(s) &= c_1 e^{\lambda s} \\
\Rightarrow \psi(1) = 1 &= c_1 e^{\lambda} \\
\Rightarrow c_1 &= e^{-\lambda} \\
\therefore \psi(s) &= e^{-\lambda(1-s)}.
\end{aligned}
$$

Which is the pgf. for Poisson distribution with parameter λ.
Its pgf. in hypergeometric terms is given as;

$$
\begin{aligned}
\phi(z) &= \frac{{}_1F_1(\flat;\flat;\lambda z)}{{}_1F_1(\flat;\flat;\lambda}. \\
\phi^{'}(z) &= \frac{\flat}{\flat}\lambda\frac{{}_1F_1(\flat+1;\flat+1;\lambda z)}{{}_1F_1(\flat;\flat;\lambda} \\
\phi^{''}(z) &= \frac{\flat(\flat+1)}{\flat(\flat+1)}\lambda^2\frac{{}_1F_1(\flat+2;\flat+2;\lambda z)}{{}_1F_1(\flat;\flat;\lambda} \\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_1F_1(\flat+\kappa;\flat+\kappa;\lambda z)}{{}_1F_1(\flat;\flat;\lambda}, \quad \kappa = 1,2. \\
E(X) &= \psi^{'}(1). \\
\therefore E(X) &= \frac{\lambda\flat}{\flat}\Lambda_1. \\
Var(X) &= \phi^{''}(1) + \phi^{'}(1) - \{\phi^{'}(1)\}^2 \\
&= \frac{\lambda^2\flat(\flat+1)}{\flat(\flat+1)}\Lambda_2 + \frac{\lambda\flat}{\flat}\Lambda_1 - \{\frac{\lambda\flat}{\flat}\Lambda_1\}^2 \\
\therefore Var(X) &= \frac{\lambda\flat}{\flat}\Lambda_1 + \frac{\lambda^2\flat}{\flat}\Big\{\frac{\flat+1}{\flat+1}\Lambda_2 - \frac{\flat}{\flat}\Lambda_1^2\Big\}.
\end{aligned}
$$

Recursion and Moments

Given that;

$$P_n = \frac{e^{-\lambda}\lambda^n}{n!}, \quad \text{for} \quad n = 0,1,2,3,\cdots; \quad \lambda > 0.$$

and

$$P_0 = e^{-\lambda}.$$

Then,

$$P_{n-1} = \frac{e^{-\lambda}\lambda^{n-1}}{(n-1)!}, \quad \text{for} \quad n = 1,2,3,\cdots.$$

Proposition 7.2

$$\therefore \frac{P_n}{P_{n-1}} = \frac{\lambda}{n} \qquad \textbf{(7.9)}$$

$$\Rightarrow nP_n = \lambda P_{n-1}, \quad \text{for} \quad n = 1,2,3,\cdots \qquad \textbf{(7.10)}$$

Proof

Solving (7.10) iteratively, we have;

When n=1;

Equation (7.10) becomes;

$$P_1 = \lambda P_0$$

$$\therefore P_1 = \lambda P_0. \qquad \textbf{(7.11)}$$

When n=2;

Equation (7.10) becomes;

$$2P_2 = \lambda P_1 \qquad \textbf{(7.12)}$$

Substituting (7.11) in equation (7.12), we have;

$$\begin{aligned} 2P_2 &= \lambda^2 P_0 \\ \Rightarrow P_2 &= \frac{\lambda^2}{1.2} P_0 \\ \therefore P_2 &= \frac{\lambda^2}{2!} P_0. \end{aligned} \tag{7.13}$$

When n=3;
Equation (7.10) becomes;

$$3P_3 = \lambda P_2 \tag{7.14}$$

Substituting (7.13) in equation (7.14), we have;

$$\begin{aligned} 3P_3 &= \frac{\lambda^3}{1.2} P_0 \\ \Rightarrow P_3 &= \frac{\lambda^3}{1.2.3} P_0 \\ \therefore P_3 &= \frac{\lambda^3}{3!} P_0. \end{aligned}$$

By Mathematical Induction;

$$\begin{aligned} P_4 &= \frac{\lambda^4}{4!} P_0, \\ P_5 &= \frac{\lambda^5}{5!} P_0, \\ &. \\ &. \\ &. \\ P_n &= \frac{\lambda^n}{n!} P_0. \end{aligned} \tag{7.15}$$

But;

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow P_0\left(1+\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right) = 1$$

$$\therefore P_0 = \frac{1}{\left(1+\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right)}.$$

$$\therefore P_n = \frac{\lambda^n}{n!}.\frac{1}{\left(1+\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right)}.$$

From (7.15);

$$P_{n-1} = \frac{\lambda^{n-1}}{(n-1)!}.$$

$$\frac{P_n}{P_{n-1}} = \frac{\lambda.\lambda^{n-1}}{n.(n-1)!}.\frac{(n-1)!}{\lambda^{n-1}}$$

$$\therefore \frac{P_n}{P_{n-1}} = \frac{\lambda}{n}, \quad \text{as required.}$$

Using the pgf. we multiply equation (7.10) by s^n and sum the results over n to get;

$$\sum_{n=0}^{\infty} nP_n s^n = \lambda \sum_{n=0}^{\infty} P_{n-1} s^n$$

$$s\sum_{n=0}^{\infty} nP_n s^{n-1} = \lambda s \sum_{n=0}^{\infty} P_{n-1} s^{n-1}$$

$$s\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \lambda s \psi(s) \tag{7.16}$$

$$\text{where,} \quad \psi(s) = \sum_{n=0}^{\infty} P_n s^n = \sum_{n=0}^{\infty} P_{n-1} s^{n-1}$$

$$\psi^{'}(s) = \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \sum_{n=0}^{\infty} nP_n s^{n-1} = \sum_{n=0}^{\infty} (n-1)P_n s^{n-2}$$

Therefore, from equation (7.16), we have;

$$\frac{1}{\psi(s)}\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \lambda$$

Integrating both sides with respect to s, we get;

$$\begin{aligned}
\int \frac{\mathrm{d}\psi(s)}{\psi(s)} &= \lambda \int \mathrm{d}s \\
\Rightarrow \log \psi(s) &= \lambda s + c \\
\psi(s) &= e^{\lambda s + c} \\
\text{Putting s=1,} \quad \psi(1) &= e^{\lambda}.e^{c} \\
\Rightarrow e^{c} &= c_1 = e^{-\lambda} \\
\therefore \psi(s) &= e^{-\lambda(1-s)}. \\
\psi^{'}(s) &= \lambda e^{-\lambda(1-s)} \\
\psi^{''}(s) &= \lambda^2 e^{-\lambda(1-s)} \\
E(x) &= \psi^{'}(1) \\
\therefore E(X) &= \lambda. \\
Var(X) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
\therefore Var(X) &= \lambda.
\end{aligned}$$

Using Feller's Method, we sum equation (7.10) over n to get;

$$\begin{aligned}
\sum_{n=1}^{\infty} nP_n &= \lambda \sum_{n=1}^{\infty} P_{n-1} \\
M_1 &= \lambda \\
E(X) &= M_1 \\
\therefore E(X) &= \lambda.
\end{aligned}$$

Next, we multiply equation (7.10) by n^2 and sum it over n to obtain;

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2 P_n &= \lambda \sum_{n=1}^{\infty} nP_{n-1} \\
M_2 &= \lambda \sum_{n=1}^{\infty} (n+1-1)P_{n-1} \\
M_2 &= \lambda \sum_{n=1}^{\infty} (n-1)P_{n-1} + \lambda \sum_{n=1}^{\infty} P_{n-1} \\
M_2 &= \lambda M_1 + \lambda \\
&= \lambda^2 + \lambda, \quad \text{since} \quad M_1 = \lambda. \\
Var(X) &= M_2 - (M_1)^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
\therefore Var(X) &= \lambda.
\end{aligned}$$

Remark 7.1

The recursion for the compound Poisson distribution with discrete claim amount distribution was originally given by Adelson (1966) in an inventory problem by using generating functions to obtain the results.

2. Zero-truncated Poisson distribution

For the zero-truncated Poisson distribution;
Let,

$$P_n = (e^{-\lambda} - 1)^{-1} \frac{\lambda^n}{n!}; \quad n = 1, 2, 3, \cdots$$

and

$$P_1 = \frac{\lambda}{e^{-\lambda} - 1}.$$

Proposition 7.3

$$\frac{P_n}{P_{n_1}} = \frac{\lambda}{n} \qquad \textbf{(7.17)}$$

$$\Rightarrow nP_n = \lambda P_{n-1}; \quad n = 2, 3, 4, \cdots \qquad \textbf{(7.18)}$$

Proof

Solving (7.18) iteratively, we have;

When n=1;

Equation (7.18) becomes;

$$P_1 = \lambda P_0$$
$$\therefore P_1 = \frac{\lambda^1}{1!} P_0.$$

When n=2;

Equation (7.18) becomes;

$$2P_2 = \lambda P_1$$
$$P_2 = \frac{\lambda^2}{2} P_0$$
$$\therefore P_2 = \frac{\lambda^2}{2!} P_0.$$

When n=3;

Equation (7.18) becomes;

$$3P_3 = \lambda P_2$$
$$\Rightarrow P_3 = \frac{\lambda^3}{1.2.3} P_0$$
$$\therefore P_3 = \frac{\lambda^3}{3!} P_0.$$

By Mathematical Induction, we have;

$$
\begin{aligned}
P_4 &= \frac{\lambda^4}{4!}P_0,\\
P_5 &= \frac{\lambda^5}{5!}P_0,\\
&.\\
&.\\
&.\\
P_n &= \frac{\lambda^n}{n!}P_0.\\
\Rightarrow P_{n-1} &= \frac{\lambda^{n-1}}{(n-1)!}P_0.\\
\frac{P_n}{P_{n-1}} &= \frac{\lambda.\lambda^{n-1}}{n.(n-1)!}.\frac{(n-1)!}{\lambda^{n-1}}\\
\therefore \frac{P_n}{P_{n-1}} &= \frac{\lambda}{n}, \quad \text{as required.}
\end{aligned}
$$

Using the pgf. technique, we multiply equation (7.18) by s^n and sum the results over n to obtain;

$$
\begin{aligned}
\sum_{n=2}^{\infty} nP_n s^n &= \lambda \sum_{n=2}^{\infty} P_{n-1}s^n\\
s\sum_{n=2}^{\infty} nP_n s^{n-1} &= \lambda s^2 \sum_{n=2}^{\infty} P_{n-1}s^{n-1}\\
s\Big(\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} - P_1\Big) &= \lambda s^2 \frac{\mathrm{d}\psi(s)}{\mathrm{d}s}\\
\Longleftrightarrow (1-\lambda s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= P_1\\
\Rightarrow (1-\lambda s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \frac{\lambda}{e^{-\lambda}-1}\\
\text{Let,} \quad \frac{1}{e^{-\lambda}-1} &= \pi\\
\text{Then,} \quad (1-\lambda s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \frac{\lambda}{\pi}. \qquad \textbf{(a)}
\end{aligned}
$$

Integrating both side of (a) above with respect to s, we have;

$$\int \mathrm{d}\psi(s) = \pi \int \frac{\lambda}{1-\lambda s}\mathrm{d}s$$
$$\psi(s) = \frac{-\ln(1-\lambda s)}{\pi} + c$$

Putting s=1;

$$\psi(1) = 1 = \frac{-\ln(1-\lambda)}{\pi} + c_1$$
$$\Rightarrow c_1 = 1 - \frac{-\ln(1-\lambda)}{\pi}$$
$$\therefore \psi(s) = \frac{-\ln(1-\lambda s)}{\pi} + \frac{-\ln(1-\lambda)}{\pi} + 1$$
$$\psi^{'}(s) = \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \frac{\lambda}{(1-\lambda s)\pi}$$
$$\psi^{''}(s) = \frac{\mathrm{d}\psi'(s)}{\mathrm{d}s} = \frac{\lambda^2}{(1-\lambda s)^2\pi}$$
$$E(X) = \psi^{'}(1)$$
$$\therefore E(x) = \frac{\lambda}{(1-\lambda)\pi}.$$
$$Var(X) = \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2$$
$$= \frac{\lambda^2}{(1-\lambda)^2\pi} + \frac{\lambda}{(1-\lambda)\pi} - [\frac{\lambda}{(1-\lambda)\pi}]^2$$
$$= \frac{\lambda^2}{(1-\lambda)^2\pi} + \frac{\lambda}{(1-\lambda)\pi} - \frac{\lambda^2}{(1-\lambda)^2\pi^2}$$
$$= \frac{\lambda^2\pi + \lambda\pi(1-\lambda) - 1}{(1-\lambda)^2\pi^2}$$
$$= \frac{\lambda^2\pi + \lambda\pi - \lambda^2\pi - 1}{(1-\lambda)^2\pi^2}$$
$$\therefore Var(X) = \frac{\lambda\pi - 1}{(1-\lambda)^2\pi^2}.$$

Using Feller's Method, we sum equation (7.18) over n to obtain;

$$\begin{aligned}
\sum_{n=2}^{\infty} nP_n &= \lambda \sum_{n=2}^{\infty} P_{n-1} \\
M_1 - P_1 &= \lambda \\
\Rightarrow M_1 &= \lambda + P_1 \\
E(X) &= M_1 \\
\therefore E(X) &= \lambda + \frac{\lambda}{e^{-\lambda} - 1} \\
\therefore E(X) &= \lambda + \frac{\lambda}{\pi} \\
\therefore E(X) &= \frac{\lambda(\pi + 1)}{\pi}.
\end{aligned}$$

Next, we multiply (7.18) by n and sum the results over n to obtain;

$$\begin{aligned}
\sum_{n=2}^{\infty} n^2 P_n &= \lambda \sum_{n=2}^{\infty} nP_{n-1} \\
\sum_{n=2}^{\infty} n^2 P_n &= \lambda \sum_{n=2}^{\infty} (n-1+1)P_{n-1} \\
M_2 - P_1 &= \lambda \sum_{n=2}^{\infty} (n-1)P_{n-1} - \sum_{n=2}^{\infty} P_{n-1} \\
M_2 - P_1 &= \lambda M_1 + \lambda \\
M_2 &= \lambda M_1 + \lambda + P_1 \\
\Rightarrow M_2 &= \lambda^2 + \frac{\lambda^2}{\pi} + \lambda + \frac{\lambda}{\pi} \\
Var(X) &= M_2 - [M_1]^2 \\
&= \lambda^2 + \frac{\lambda^2}{\pi} + \lambda + \frac{\lambda}{\pi} - \lambda^2 - \frac{\lambda^2}{\pi^2} \\
&= \frac{\lambda^2}{\pi} + \lambda + \frac{\lambda}{\pi} - \frac{\lambda^2}{\pi^2} \\
\therefore Var(X) &= \lambda + \frac{\lambda^2}{\pi} + \frac{\lambda}{\pi} - \frac{\lambda^2}{\pi^2}.
\end{aligned}$$

7.3.3 Case (iii) When $\xi \neq 0$ and $\lambda = 0$

From equation (7.18), we have;

$$
\begin{aligned}
(1-\xi s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \xi\,\psi(s)\\
\Longleftrightarrow \frac{1}{\psi(s)}\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \frac{\xi}{1-\xi s}\\
\Rightarrow \frac{\mathrm{d}s}{\mathrm{d}}\log\psi(s) &= \frac{\xi}{1-\xi s}
\end{aligned}
$$

Taking the integral on both sides, we have;

$$
\begin{aligned}
\int \log\psi(s) &= \int \frac{\xi \mathrm{d}s}{1-\xi s}\\
\log\psi(s) &= \int \frac{\xi \mathrm{d}s}{1-\xi s}\\
\log\psi(s) &= -\log c(1-\xi s)\\
\Rightarrow \psi(s) &= \frac{c_1}{1-\xi s}
\end{aligned}
$$

Putting s=1;

$$
\begin{aligned}
\Rightarrow \psi(1) = 1 &= \frac{c_1}{1-\xi}\\
\therefore c_1 &= 1-\xi.\\
\therefore \psi(s) &= \frac{1-\xi}{1-\xi s}. \qquad \textbf{(7.19)}
\end{aligned}
$$

Suppose we let $\xi = p$, then we have;

$$
\Rightarrow \psi(s) = \frac{1-p}{1-ps}.
$$

Which is a pgf. of a geometric distribution with parameter $1-\xi$.
From (7.19), we have;

$$
\psi(s) = \frac{(1-\xi s)^{-1}}{(1-\xi)^{-1}}.
$$

The probability generating function in hypergeometric term is given by;

$$
\begin{aligned}
\phi(z) &= \frac{{}_1F_1(1;1;\xi z)}{{}_1F_1(1;1;\xi)}.\\
\phi^{'}(z) &= \frac{1.\xi}{1}\frac{{}_1F_1(2;2;\xi z)}{{}_1F_1(1;1;\xi)}\\
\phi^{''}(z) &= \frac{2\xi^2}{2}\frac{{}_1F_1(3;3;\xi z)}{{}_1F_1(1;1;\xi)}\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_1F_1(1+\kappa;1+\kappa;\xi)}{{}_1F_1(1;1;\xi)},\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \xi\Lambda_1.\\
Var(X) &= \phi^{''}+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= \xi^2\Lambda_2+\xi\Lambda_1-\xi^2\Lambda_1^2\\
\therefore Var(X) &= \xi\Lambda_1+\xi^2\{\Lambda_2-\Lambda_1^2\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P_n &= \xi^n(1-\xi);\quad n=0,1,2,3,\cdots\\
\text{i.e.}\quad P_n &= p^n q;\quad n=0,1,2,3,\cdots\\
\text{were,}\quad q &= 1-p.
\end{aligned}
$$

3. Geometric Distribution
Geometric distribution is given as;

$$
\begin{aligned}
P_n &= p^n q,\quad n=0,1,2,\cdots;\quad q=1-p.\\
&\text{and}\\
P_0 &= q.\\
\text{Then,}\quad P_{n-1} &= p^{n-1}q,\quad n=1,2,3,\cdots
\end{aligned}
$$

Proposition 7.4

$$\frac{P_n}{P_{n-1}} = \frac{p^n q}{p^{n-1} q} = p. \tag{7.20}$$

$$\Longleftrightarrow P_n = pP_{n-1}, \quad n = 1,2,3,\cdots \tag{7.21}$$

Proof
Solving (7.21) iteratively, we have;
When n=1;
Equation (7.21) becomes;

$$P_1 = pP_0$$
$$\therefore P_1 = pP_0.$$

When n=2;
Equation (7.21) becomes;

$$P_2 = pP_1$$
$$\Rightarrow P_2 = p.pP_0$$
$$\therefore P_2 = p^2 P_0.$$

When n=3;
Equation (7.21) becomes;

$$P_3 = pP_2$$
$$\Rightarrow P_3 = p.p^2 P_0$$
$$\therefore P_3 = p^3 P_0.$$
$$\therefore P_4 = p^4 P_0,$$
$$\therefore P_5 = p^5 P_0,$$
$$.$$
$$.$$
$$.$$
$$\therefore P_n = p^n P_0.$$

Therefore,

$$
\begin{aligned}
P_{n-1} &= p^{n-1}P_0. \\
\Rightarrow \frac{P_n}{P_{n-1}} &= \frac{p.p^{n-1}}{p^{n-1}} \\
\therefore \frac{P_n}{P_{n-1}} &= p, \quad \text{as required.}
\end{aligned}
$$

Multiplying both sides of equation (7.21) by s^n and sum the results over n, we have;

$$
\begin{aligned}
\sum_{n=1}^{\infty} P_n s^n &= p\sum_{n=1}^{\infty} P_{n-1}s^n \\
\iff \psi(s) - P_0 &= ps\psi(s) \\
\Rightarrow (1-ps)\psi(s) &= P_0 = q \\
\therefore \psi(s) &= \frac{q}{1-ps}. \\
\psi^{'}(s) &= \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \frac{qp}{(1-ps)^2} \\
\psi^{''}(s) &= \frac{\mathrm{d}\psi^{'}(s)}{\mathrm{d}s} = \frac{2qp^2}{(1-ps)^3} \\
E(X) &= \psi^{'}(1) \\
&= \frac{qp}{(1-p)^2} \\
&= \frac{qp}{q^2} \\
\therefore E(X) &= \frac{p}{q}. \quad \text{Since} \quad q = 1-p \\
\psi^{''}(1) &= \frac{2qp^2}{(1-p)^3} \\
&= \frac{2p^2}{q^2} \\
Var(X) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= \frac{2p^2}{q^2} + \frac{p}{q} - \frac{p^2}{q^2} \\
\therefore Var(X) &= \frac{p}{q^2}. \quad \text{Since} \quad p+q = 1
\end{aligned}
$$

Using Feller's Method, we sum equation (7.21) over n to get;

$$\begin{aligned}
\sum_{n=1}^{\infty} nP_n &= p\sum_{n=1}^{\infty} nP_{n-1}\\
M_1 &= p\Big(\sum_{n=1}^{\infty}(n-1)P_{n-1} + \sum_{n=1}^{\infty}P_{n-1}\Big)\\
M_1 &= pM_1 + p\\
(1-p)M_1 &= p\\
qM_1 &= p\\
\therefore M_1 &= \frac{p}{q}.\\
E(X) &= M_1\\
\therefore E(X) &= \frac{p}{q}.
\end{aligned}$$

Next, we multiply both sides of (7.21) by n^2 and sum it over to obtain;

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2P_n &= p\sum_{n=1}^{\infty} n^2P_{n-1}\\
\sum_{n=1}^{\infty} n^2P_n &= p\sum_{n=1}^{\infty}(n-1+1)^2P_{n-1}\\
M_2 &= p\sum_{n=1}^{\infty}[(n-1)^2+2(n-1)+1]P_{n-1}\\
M_2 &= p\sum_{n=1}^{\infty}(n-1)^2P_{n-1} + 2p\sum_{n=1}^{\infty}P_{n-1} + p\sum_{n=1}^{\infty}P_{n-1}\\
M_1 &= 2pM_2 + p\\
\text{But,}\quad M_1 &= \frac{p}{q}\\
\Rightarrow (1-p)M_2 &= 2p\frac{p}{q} + p\\
M_2 &= \frac{2p^2+qp}{q^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
Var(X) &= M_2 - [M_1]^2 \\
&= \frac{2p^2+qp}{q^2} - \frac{p^2}{q^2} \\
&= \frac{2p^2+qp-p^2}{q^2} \\
&= \frac{p^2+qp}{q^2} \\
&= \frac{p(p+q)}{q^2} \\
\therefore Var(X) &= \frac{p}{q^2}. \quad \text{Since} \quad p+q=1
\end{aligned}$$

Support we assume that;

$$\begin{aligned}
&P_n = p^{n-1}q; \quad n=1,2,3,\cdots; \quad q=1-p \\
&\text{and} \\
&P_1 = q. \\
\text{Then,} \quad &P_{n-1} = p^{n-2}q; \quad n=2,3,\cdots \\
\text{Thus,} \quad &\frac{P_n}{P_{n-1}} = \frac{p.p^{n-2}q}{p^{n-2}q} = p. \\
&\Rightarrow P_n = pP_{n-1}, \quad \text{for} \quad n=2,3,\cdots \qquad \textbf{(7.22)}
\end{aligned}$$

Using the pgf. technique, we multiply equation (7.22) by s^n and sum the results over n to obtain;

$$\begin{aligned}
\sum_{n=2}^{\infty} P_n s^n &= p\sum_{n=2}^{\infty} P_{n-1}s^n \\
\sum_{n=2}^{\infty} P_n s^n &= ps\sum_{n=2}^{\infty} P_{n-1}s^{n-1} \\
\psi(s) - P_1 s - P_0 &= ps(\psi(s) - P_0) \\
\text{But,} \quad P_1 = q \quad &\text{and} \quad P_0 = 0
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow (1-ps)\psi(s) &= sP_1 \\
\Rightarrow (1-ps)\psi(s) &= qs \\
\therefore \psi(s) &= \frac{qs}{1-ps}. \\
\psi'(s) &= \frac{d\psi(s)}{ds} = \frac{qp}{(1-ps)^2} \\
\psi''(s) &= \frac{d\psi'(s)}{ds} = \frac{2qp^2}{(1-ps)^3} \\
E(X) &= \psi^{'}(1) \\
&= \frac{qp}{(1-p)^2} \\
&= \frac{qp}{q^2} \\
\therefore E(X) &= \frac{p}{q}. \quad \text{Since} \quad 1-p=q. \\
\psi''(1) &= \frac{2qp^2}{(1-p)^3} \\
&= \frac{2p^2}{q^2} \\
Var(X) &= \psi^{''}(1)+\psi^{'}(1)-[\psi^{'}(1)]^2 \\
&= \frac{2p^2}{q^2}+\frac{p}{q}-\frac{p^2}{q^2} \\
&= \frac{p^2}{q^2}+\frac{p}{q} \\
&= \frac{(p^2+pq)}{q^2} \\
&= \frac{p(p+q)}{q^2} \\
\therefore Var(X) &= \frac{p}{q^2}. \quad \text{Since} \quad p+q=1
\end{aligned}
$$

Using Feller's Method, we multiply equation (7.22) by n and sum the results over n to obtain;

$$
\begin{aligned}
\sum_{n=1}^{\infty} nP_n &= p\sum_{n=1}^{\infty} nP_{n-1} \\
\sum_{n=1}^{\infty} nP_n &= p\sum_{n=1}^{\infty} (n+1-1)P_{n-1}
\end{aligned}
$$

$$\Rightarrow M_1 = p\Big[\sum_{n=1}^{\infty}(n-1)P_{n-1} + \sum_{n=1}^{\infty}P_{n-1}\Big]$$

$$M_1 = pM_1 + p$$

$$(1-p)M_1 = p$$

$$M_1 = \frac{p}{1-p}$$

$$\Rightarrow M_1 = \frac{p}{q}$$

$$E(X) = M_1$$

$$\therefore E(X) = \frac{p}{q}.$$

Next, we multiply (7.22) by n^2 and sum the results over n to obtain;

$$\sum_{n=1}^{\infty} n^2 P_n = p\sum_{n=1}^{\infty} n^2 P_{n-1}$$

$$M_2 = p\sum_{n=1}^{\infty}(n-1+1)^2 P_{n-1}$$

$$M_2 = p\Big[\sum_{n=1}^{\infty}(n-1)^2 P_{n-1} + 2\sum_{n=1}^{\infty}(n-1)P_{n-1} + \sum_{n=1}^{\infty}P_{n-1}\Big]$$

$$M_2 = p\sum_{n=1}^{\infty}(n-1)^2 P_{n-1} + 2p\sum_{n=1}^{\infty}(n-1)P_{n-1} + p\sum_{n=1}^{\infty}P_{n-1}$$

$$M_2 = pM_2 + 2pM_1 + p$$

But, $M_1 = \frac{p}{q}$

$$\Rightarrow (1-p)M_2 = \frac{2p^2}{q} + p$$

$$M_2 = \frac{2p^2 + qp}{q^2}.$$

$$
\begin{aligned}
Var(X) &= M_2 - [M_1]^2 \\
&= \frac{2p^2+qp}{q^2} - \frac{p^2}{q^2} \\
&= \frac{2p^2+qp-p^2}{q^2} \\
&= \frac{p^2+qp}{q^2} \\
&= \frac{p(p+q)}{q^2} \\
\therefore Var(X) &= \frac{p}{q^2}. \quad \text{Since} \quad p+q=1
\end{aligned}
$$

4. Truncated Geometric Distribution

For a truncated geometric distribution;

$$
\begin{aligned}
P_n &= p^{n=\tau}q; \quad n=\tau,\tau+1,\tau+2,\cdots; \quad q=1-p \\
P_\tau &= q. \\
\text{Thus,} \quad P_{n-1} &= p^{(n-1)-\tau}q; \quad n=\tau+1,\tau+2,\cdots \\
\text{Therefore,} \quad \frac{P_n}{P_{n-1}} &= \frac{p^{n-\tau}q}{p^{(n-1)-\tau}q} = p \qquad \textbf{(7.23)} \\
\Rightarrow P_n &= pP_{n-1}; \quad n=\tau+1,\tau+2,\cdots \qquad \textbf{(7.24)}
\end{aligned}
$$

Using probability generating function technique, we multiply both sides of equation (7.24) by s^n and sum the results over n to obtain;

$$
\begin{aligned}
\sum_{n=\tau+1}^{\infty} P_n s^n &= p \sum_{n=\tau+1} P_{n-1} s^n \\
\sum_{n=\tau+1}^{\infty} P_n s^n &= ps \sum_{n=\tau+1} P_{n-1} s^{n-1} \\
\psi(s) - P_\tau s^\tau &= ps\psi(s) \\
\text{But,} \quad P_\tau &= q
\end{aligned}
$$

Therefore;

$$
\begin{aligned}
\psi(s) &= \frac{qs^{\tau}}{1-ps}.\\
\psi^{'}(s) &= \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \frac{\tau qs^{\tau-1}}{(1-ps)} + \frac{qps^{\tau}}{(1-ps)^2}\\
\psi^{''}(s) &= \frac{\mathrm{d}\psi^{'}(s)}{\mathrm{d}s} = \frac{\tau(\tau-1)qs^{\tau-1}}{(1-ps)} + \frac{2qp\tau s^{\tau-1}}{(1-ps)^2} + \frac{2qp^2\tau s^{\tau}}{(1-ps)^3}\\
E(X) &= \psi^{'}(1)\\
&= \frac{\tau q}{(1-p)} + \frac{qp}{(1-p)^2}\\
&= \frac{\tau q^2 + qp}{q^2}\\
&= \frac{q(\tau+p)}{q^2}\\
\therefore E(X) &= \tau + \frac{p}{q}.\\
\psi^{''}(1) &= \frac{\tau(\tau-1)q}{(1-p)} + \frac{2qp\tau}{(1-p)^2} + \frac{2qp^2\tau}{(1-p)^3}
\end{aligned}
$$

But, $q = 1-p$

$$
\begin{aligned}
Var(X) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2\\
&= \tau(\tau-1) + \frac{2p\tau}{q} + \frac{2p^2}{q^2} + \tau + \frac{p}{q} - (\tau + \frac{p}{q})^2\\
&= \tau^2 - \tau + \frac{2p\tau}{q} + \frac{2p^2}{q^2} + \tau + \frac{p}{q} - \tau^2 - \frac{2p\tau}{q} - \frac{p^2}{q^2}\\
&= \frac{2p^2}{q^2} + \frac{p}{q} - \frac{p^2}{q^2}\\
&= \frac{p^2 + pq}{q^2}\\
&= \frac{p(p+q)}{q^2}\\
\therefore Var(X) &= \frac{p}{q^2}. \quad \text{Since} \quad p+q=1
\end{aligned}
$$

Using Feller's Method, we multiply equation (7.24) by n and sum the results over n to obtain;

$$\sum_{n=\tau+1}^{\infty} nP_n = \sum_{n=\tau+1}^{\infty} nP_{n-1}$$

$$\sum_{n=\tau+1}^{\infty} nP_n = \sum_{n=\tau+1}^{\infty} [(n-1)+1]P_{n-1}$$

$$M_1 - \tau P_\tau = p\sum_{n=\tau+1}^{\infty}(n-1)P_{n-1} + p\sum_{n=\tau+1}^{\infty} P_{n-1}$$

$$P_\tau = pM_1 + p$$

$$\text{But,} \quad P_\tau = q$$

$$\Rightarrow M_1 - \tau q = pM_1 + p$$

$$(1-p)M_1 = p + \tau q$$

$$qM_1 = p + \tau q$$

$$M_1 = \frac{p+\tau q}{q}$$

$$E(X) = M_1$$

$$\therefore E(X) = \tau + \frac{p}{q}.$$

Next, we multiply equation (7.24) by n^2 and sum the results over n to obtain;

$$\sum_{n=\tau+1}^{\infty} n^2P_n = p\sum_{n=\tau+1}^{\infty} n^2P_{n-1}$$

$$\sum_{n=\tau+1}^{\infty} n^2P_n = p\sum_{n=\tau+1}^{\infty} [(n-1)+1]^2P_{n-1}$$

$$M_2 - \tau^2P_\tau = p\sum_{n=\tau+1}^{\infty} [(n-1)^2 + 2(n-1)+1]P_{n-1}$$

$$M_2 - \tau^2P_\tau = p\sum_{n=\tau+1}^{\infty} (n-1)^2P_{n-1} + 2p\sum_{n=\tau+1}(n-1)P_{n-1} + p\sum_{n=\tau+1}^{\infty} P_{n-1}$$

$$M_2 - \tau^2P_\tau = pM_2 + 2pM_1 + p$$

$$\text{But,} \quad P_\tau = q \quad \text{and} \quad M_1 = \tau + \frac{p}{q}$$

$$M_2 - \tau^2 q = pM_2 + 2p\tau + \frac{2p^2}{q} + p$$

$$(1-p)M_2 = \tau^q + 2\tau p + \frac{2p^2}{q} + p$$

$$
\begin{aligned}
\therefore M_2 &= \tau^2 + \frac{2\tau p}{q} + \frac{2p^2}{q^2} + \frac{p}{q}. \\
Var(X) &= M_2 - [M_1]^2 \\
&= \tau^2 + \frac{2\tau p}{q} + \frac{2p^2}{q^2} + \frac{p}{q} - (\tau + \frac{p}{q})^2 \\
&= \tau^2 + \frac{2\tau p}{q} + \frac{2p^2}{q^2} + \frac{p}{q} - \tau^2 - \frac{2\tau p}{q} - \frac{p^2}{q^2} \\
&= \frac{p^2}{q^2} + \frac{p}{q} \\
&= \frac{p^2 + pq}{q^2} \\
&= \frac{p(p+q)}{q^2} \\
\therefore Var(X) &= \frac{p}{q^2}. \quad \text{Since} \quad p + q = 1
\end{aligned}
$$

7.3.4 Case (iv) When $\xi \neq 0$ and $\lambda \neq 0$

From equation (7.8), we have;

$$
\begin{aligned}
(1 - \xi s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= (\xi - \lambda)\psi(s) \\
\text{Assuming that} \quad \xi + \lambda &> 0; \\
\text{Then,} \\
\frac{1}{\psi(s)}\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \frac{\xi + \lambda}{1 - \xi s} \\
\int \mathrm{d}\log\psi(s) &= (\xi + \lambda)\int \frac{\mathrm{d}s}{1 - \xi s} \\
\log\psi(s) &= \frac{\xi + \lambda}{-\xi}\log c(1 - \xi s) \\
\log\psi(s) &= -\alpha \log c(1 - \xi s) \\
\text{where,} \quad \alpha &= \frac{\xi + \lambda}{\xi}. \\
\Rightarrow \psi(s) &= c_1(1 - \xi s)^{-\alpha}
\end{aligned}
$$

Putting s=1, we have;

$$\begin{aligned} \psi(1) &= 1 = c_1(1-\xi)^{-\alpha} \\ \Rightarrow c_1 &= (1-\xi)^{\alpha} \\ \therefore \psi(s) &= \left(\frac{1-\xi}{1-\xi s}\right)^{\alpha}. \end{aligned} \tag{*}$$

If α is a positive integer;
Then;

$$\begin{aligned} \psi(s) &= (1-\xi)^{\alpha}(1-\xi)^{-\alpha} \\ &= (1-\xi)^{\alpha}\sum_{n=0}^{\infty}\begin{pmatrix} -\alpha \\ n \end{pmatrix}(-\xi s)^{n} \\ &= (1-\xi)^{\alpha}\sum_{n=0}^{\infty}(-1)^{n}\begin{pmatrix} -\alpha \\ n \end{pmatrix}\xi^{n}s^{n} \\ &= (1-\xi)^{\alpha}\sum_{n=0}^{\infty}\begin{pmatrix} \alpha+n-1 \\ n \end{pmatrix}\xi^{n}s^{n}. \end{aligned}$$

Therefore;

$$P_n = \begin{pmatrix} \alpha+n-1 \\ n \end{pmatrix}\xi^{n}(1-\xi)^{\alpha} \quad \text{for} \quad 0<\xi<1, \quad \alpha>0$$

Let, $\xi = p$

$$\Rightarrow P_n = \begin{pmatrix} \alpha+n-1 \\ n \end{pmatrix}\mathfrak{p}^{n}(1-p)^{\alpha} \quad \text{for} \quad 0<p<1, \quad \alpha>0; \quad n=0,1,2,\cdots \tag{7.25}$$

Which is a Negative Binomial Distribution.
Its probability generating function in hypergeometric terms is given by;

$$\phi(z) = \frac{{}_2F_1(\alpha,\flat;\flat;\xi z)}{{}_2F_1(\alpha,\flat;\flat;\xi)}.$$

This implies that;

$$
\begin{aligned}
\phi^{'}(z) &= \frac{\alpha\flat}{\flat}\xi.\frac{{}_2F_1(\alpha+1,\flat+1;\flat+1;\xi z)}{{}_2F_1(\alpha,\flat;\flat;\xi)}\\
\phi^{''}(z) &= \frac{\alpha(\alpha+1)\flat(\flat+1)}{\flat(\flat+1)}\xi^2\frac{{}_2F_1(\alpha+2,\flat+2;\flat+2;\xi z)}{{}_2F_1(\alpha,\flat;\flat;\xi)}\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_2F_1(\alpha+\kappa,\flat+\kappa;\flat+\kappa;\xi z)}{{}_2F_1(\alpha,\flat;\flat;\xi)},\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\alpha\xi\flat}{\flat}\Lambda_1.\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-\{\phi^{'}(1)\}^2\\
&= \frac{\alpha(\alpha+1)\flat(\flat+1)\xi^2}{\flat(\flat+1)}\Lambda_2+\frac{\alpha\xi\flat}{\flat}\Lambda_1-\{\frac{\alpha\xi\flat}{\flat}\Lambda_1\}^2\\
\therefore Var(X) &= \frac{\alpha\xi\flat}{\flat}\Lambda_1+\frac{\alpha\xi^2\flat}{\flat}\Big\{\frac{(\alpha+1)(\flat+1)}{(\flat+1)}\Lambda_2-\frac{\alpha\flat}{\flat}\Lambda_1^2\Big\}.
\end{aligned}
$$

Note that;

$$
\begin{aligned}
P_n &= \left(\begin{array}{c}\alpha+n-1\\ n\end{array}\right)\mathfrak{p}^n(1-p)^\alpha;\quad \text{for}\quad n=0,1,2,\cdots\\
\Rightarrow P_{n-1} &= \left(\begin{array}{c}\alpha+n-2\\ n-1\end{array}\right)\mathfrak{p}^{n-1}(1-p)^\alpha\\
\Rightarrow \frac{P_n}{P_{n-1}} &= (\alpha+n-1)\frac{p}{n}\\
\text{and}\quad P_0 &= (1-p)^\alpha.\\
\text{But,}\quad 1-p &= q\\
\Rightarrow P_0 &= q^\alpha.
\end{aligned}
$$

5. Negative Binomial Distribution

<u>Case I:</u>

Suppose n is the number of failures before πth success;

Then;

$$
\begin{aligned}
P_n &= \begin{pmatrix} n+\pi-1 \\ \pi-1 \end{pmatrix} p^n q^{\pi}; \quad n = 0,1,2,\cdots; \quad q = 1-p \\
P_0 &= q^{\pi}. \\
\Rightarrow P_{n-1} &= \begin{pmatrix} (n-1)+\pi-1 \\ \pi-1 \end{pmatrix} p^{n-1} q^{\pi}; \quad n = 0,1,2,\cdots
\end{aligned}
$$

Hence;

Proposition 7.5

$$
\begin{aligned}
\frac{P_n}{P_{n-1}} &= (n+\pi-1)\frac{p}{n} \\
\Rightarrow nP_n &= p(n+\pi-1)P_{n-1}; \quad n = 1,2,3,\cdots \qquad (7.26)
\end{aligned}
$$

Proof

Solving (7.26) iteratively, we have;

When n=1;

Equation (7.26) becomes;

$$
\begin{aligned}
P_1 &= p\pi P_0 \\
\therefore P_1 &= p\pi P_0.
\end{aligned}
$$

When n=2;

Equation (7.26) becomes;

$$
\begin{aligned}
2P_2 &= p(\pi+1)P_1 \\
\Rightarrow 2P_2 &= p(\pi+1)p\pi P_0 \\
\therefore P_2 &= \frac{\pi(\pi+1)p^2}{2!}P_0.
\end{aligned}
$$

When n=3;

Equation (7.26) becomes;

$$
\begin{aligned}
3P_3 &= p(\pi+2)P_2 \\
\Rightarrow 3P_3 &= p(\pi+2).\frac{\pi(\pi+1)p^2}{1.2}P_0 \\
\therefore P_3 &= \frac{\pi(\pi+1)(\pi+2)p^3}{3!}P_0.
\end{aligned}
$$

By Mathematical Induction, we have;

$$
\begin{aligned}
P_4 &= \frac{\pi(\pi+1)(\pi+2)(\pi+3)p^4}{4!}P_0, \\
P_5 &= \frac{\pi(\pi+1)(\pi+2)(\pi+3)(\pi+4)p^5}{5!}P_0, \\
&. \\
&. \\
&. \\
P_n &= \frac{\pi(\pi+1)(\pi+2)\cdots(\pi+n-2)(\pi+n-1)p^n}{n!}P_0. \\
\Rightarrow P_{n-1} &= \frac{\pi(\pi+1)(\pi+2)\cdots(\pi+n-3)(\pi+n-2)p^{n-1}}{(n-1)!}P_0. \\
\frac{P_n}{P_{n-1}} &= \frac{\pi(\pi+1)(\pi+2)\cdots(\pi+n-2)(\pi+n-1)p.p^{n-1}}{n.(n-1)!}.\frac{(n-1)!}{\pi(\pi+1)(\pi+2)\cdots(\pi+n-2)p^{n-1}} \\
\therefore \frac{P_n}{P_{n-1}} &= (\pi+n-1)\frac{p}{n}, \text{as required.}
\end{aligned}
$$

Using the pgf. technique, we multiply equation (7.26) by s^n and sum the results over n to obtain;

$$
\begin{aligned}
\sum_{n=1}^{\infty} nP_n s^n &= p\sum_{n=1}^{\infty}[\pi+(n-1)]P_{n-1}s^n \\
\sum_{n=1}^{\infty} nP_n s^n &= p\pi\sum_{n=1}^{\infty}P_{n-1}s^n + p\sum_{n=1}^{\infty}(n-1)P_{n-1}s^n \\
s\sum_{n=1}^{\infty} nP_n s^{n-1} &= p\pi s\sum_{n=1}^{\infty}P_{n-1}s^{n-1} + ps^2\sum_{n=1}^{\infty}(n-1)P_{n-1}s^{n-2}
\end{aligned}
$$

Thus;

$$\begin{aligned} s\frac{d\psi(s)}{ds} &= p\pi s\psi(s) + ps^2\frac{d\psi(s)}{ds} \\ \iff (1-ps)\frac{d\psi(s)}{ds} &= p\pi\psi(s) \\ \Rightarrow \frac{1}{\psi(s)}d\psi(s) &= \frac{p\pi}{1-ps}ds \end{aligned}$$

Taking the integral, we have;

$$\begin{aligned} \int \frac{d\psi(s)}{\psi(s)} &= \int \frac{p\pi ds}{(1-ps)} \\ \ln\psi(s) &= -\pi\ln(1-ps) + \ln c \\ \Rightarrow \psi(s) &= c_1(1-ps)^{-\pi} \end{aligned}$$

Putting s=1;

$$\begin{aligned} \psi(1) &= 1 = c_1(1-p)^{-\pi} \\ \Rightarrow c_1 &= q^{\pi}. \\ \Rightarrow \psi(s) &= q^{\pi}(1-ps)^{-\pi} \\ \therefore \psi(s) &= \left(\frac{q}{1-ps}\right)^{\pi}. \\ \psi'(s) &= \frac{p\pi q^{\pi}}{(1-ps)^{\pi+1}} \\ \psi''(s) &= \frac{p^2\pi(\pi+1)q^{\pi}}{(1-ps)^{\pi+2}} \\ &= \frac{p^2\pi^2 q^{\pi} + p^2\pi q^{\pi}}{(1-ps)^{\pi+2}} \\ E(X) &= \psi'(1) \\ &= \frac{p\pi q^{\pi}}{q^{\pi+1}} \\ \therefore E(X) &= \frac{p\pi}{q}. \end{aligned}$$

and,

$$\psi''(1) = \frac{p^2\pi^2 q^\pi + p^2\pi q^\pi}{q^{\pi+2}}$$
$$= \frac{p^2\pi^2 + p^2\pi}{q^2}$$
$$Var(X) = \psi''(1) + \psi'(1) - [\psi'(1)]^2$$
$$= \frac{p^2\pi^2 + p^2\pi}{q^2} + \frac{p\pi}{q} - [\frac{p\pi}{q}]^2$$
$$= \frac{p^2\pi^2 + p^2\pi}{q^2} + \frac{p\pi}{q} - \frac{p^2\pi^2}{q^2}$$
$$= \frac{p^2\pi}{q^2} + \frac{p\pi}{q}$$
$$= \frac{p^2\pi + qp\pi}{q^2}$$
$$= \frac{p\pi(p+q)}{q^2}$$
$$\therefore Var(X) = \frac{p\pi}{q^2}. \quad \text{Since} \quad p+q=1$$

Using Feller's method, we sum equation (7.26) over n, to get;

$$\sum_{n=1}^{\infty} nP_n = p\sum_{n=1}^{\infty}(n+\pi-1)P_{n-1}$$
$$\sum_{n=1}^{\infty} nP_n = p\sum_{n=1}^{\infty}[(n-1)+\pi]P_{n-1}$$
$$M_1 = p\sum_{n=1}^{\infty}(n-1)P_{n-1} + p\pi\sum_{n=1}^{\infty}P_{n-1}$$
$$M_1 = pM_1 + p\pi$$
$$(1-p)M_1 = p\pi$$
$$M_1 = \frac{p\pi}{q}. \quad \text{Since} \quad 1-p=q$$
$$E(X) = M_1$$
$$\therefore E(X) = \frac{p\pi}{q}.$$

Next, we multiply equation (7.26) by n and sum the results over n to get;

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^2 P_n &= p \sum_{n=1}^{\infty} n(n+\pi-1)P_{n-1} \\
\sum_{n=1}^{\infty} n^2 P_n &= p \sum_{n=1}^{\infty} [(n-1)+1][(n-1)+\pi]P_{n-1} \\
M_2 &= p \sum_{n=1}^{\infty} (n-1)^2 P_{n-1} + p\pi \sum_{n=1}^{\infty} (n-1)P_{n-1} + p \sum_{n=1}^{\infty} (n-1)P_{n-1} + p\pi \sum_{n=1}^{\infty} P_{n-1} \\
M_2 &= pM_2 + p\pi M_1 + pM_1 + p\pi \\
(1-p)M_2 &= p\pi . \frac{p\pi}{q} + p . \frac{p\pi}{q} + p\pi \\
qM_2 &= \frac{p^2\pi^2}{q} + \frac{p^2\pi}{q} + p\pi \\
\Rightarrow M_2 &= \frac{p^2\pi^2 + p^2\pi + qp\pi}{q^2} \\
Var(X) &= M_2 - [M_1]^2 \\
&= \frac{p^2\pi^2 + p^2\pi + qp\pi}{q^2} - \frac{p^2\pi^2}{q^2} \\
&= \frac{p^2\pi + pq\pi}{q^2} \\
&= \frac{p\pi(p+q)}{q^2} \\
\therefore Var(X) &= \frac{p\pi}{q^2}. \quad \text{Since} \quad p+q=1
\end{aligned}
$$

<u>Case II:</u>

Suppose n is the number of trials required to achieve the π th success;
Then,

$$
P_n = \begin{pmatrix} n-1 \\ \pi-1 \end{pmatrix} p^{n-\pi} q^{\pi}, \quad n = \pi, \pi+1, \pi+2, \cdots; \quad q = 1-p
$$

and

$$
P_{\pi} = q^{\pi}.
$$

$$
P_{n-1} = \begin{pmatrix} (n-1)-1 \\ \pi-1 \end{pmatrix} p^{(n-1)-\pi} q^{\pi}, \quad n = \pi+1, \pi+2, \cdots.
$$

Thus,

Proposition 7.6

$$\frac{P_n}{P_{n-1}} = \frac{n-1}{n-\pi}p \tag{7.27}$$

$$\Rightarrow (n-\pi)P_n = p(n-1)P_{n-1}; \quad n = \pi+1, \pi+2, \cdots \tag{7.28}$$

Proof

Solving (7.28) iteratively, we have;

When n=π+1;

Equation (7.28) becomes;

$$P_{\pi+1} = p\pi P_\pi$$
$$\therefore P_{\pi+1} = p\pi P_\pi.$$

When n=π+2;

Equation (7.28) becomes;

$$2P_{\pi+2} = p(\pi+1)P_{\pi+1}$$
$$\Rightarrow 2P_{\pi+2} = p(\pi+1).p\pi P_\pi$$
$$\therefore P_{\pi+2} = \frac{\pi(\pi+1)p^2}{2!}P_\pi$$

When n=π+3;

Equation (7.28) becomes;

$$3P_{\pi+3} = p(\pi+2)P_{\pi+2}$$
$$\Rightarrow 3P_{\pi+3} = p(\pi+2).\frac{\pi(\pi+1)p^2}{1.2}P_\pi$$
$$\therefore P_{\pi+3} = \frac{\pi(\pi+1)(\pi+2)p^3}{3!}P_\pi$$

By Mathematical Induction, we have;

$$
\begin{aligned}
P_{\pi+4} &= \frac{\pi(\pi+1)(\pi+2)(\pi+3)p^4}{4!}P_\pi,\\
P_{\pi+5} &= \frac{\pi(\pi+1)(\pi+2)(\pi+3)(\pi+4)p^5}{5!}P_\pi,\\
&\ \ .\\
&\ \ .\\
&\ \ .\\
P_n &= \frac{\pi(\pi+1)(\pi+2)\cdots(\pi+n-2)(\pi+n-1)p^n}{n!}P_\pi.\\
\Rightarrow P_{n-1} &= \frac{\pi(\pi+1)(\pi+2)\cdots(\pi+n-3)(\pi+n-2)p^{n-1}}{(n-1)!}P_\pi.\\
\frac{P_n}{P_{n-1}} &= \frac{\pi(\pi+1)(\pi+2)\cdots(\pi+n-2)(\pi+n-1)p.p^{n-1}}{n.(n-1)!}.\frac{(n-1)!}{\pi(\pi+1)(\pi+2)\cdots(\pi+n-2)p^{n-1}}\\
\therefore \frac{P_n}{P_{n-1}} &= \frac{n-1}{n-\pi}p. \quad \text{as required}
\end{aligned}
$$

Using the probability generating function technique, we multiply equation (7.28) by s^n and sum the results over n to obtain;

$$
\begin{gathered}
\sum_{n=\pi+1}^{\infty} nP_n s^n - \pi \sum_{n=\pi+1}^{\infty} P_n s^n = p \sum_{n=\pi+1}^{\infty} (n-1)P_{n-1}s^n\\
s\sum_{n=\pi+1}^{\infty} nP_n s^{n-1} - \pi \sum_{n=\pi+1}^{\infty} P_n s^n = ps^2 \sum_{n=\pi+1}^{\infty} (n-1)P_{n-1}s^{n-2}\\
\text{But,} \quad P_\pi = q^\pi\\
s\Big[\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} - \pi q^\pi s^{\pi-1}\Big] - \pi\Big[\psi(s) - q^\pi s^\pi\Big] = ps^2\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}
\end{gathered}
$$

Therefore,

$$
\begin{aligned}
s(1-ps)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \pi\psi(s)\\
\Longleftrightarrow \frac{1}{\psi(s)} &= \frac{\pi \mathrm{d}s}{s(1-ps)}
\end{aligned}
$$

Integrating both sides with respect to s, we have;

$$\begin{aligned}\int \frac{\mathrm{d}\psi(s)}{\psi(s)} &= \pi \int \frac{\mathrm{d}s}{s(1-ps)} \\ \ln\psi(s) &= \pi \ln s - \pi \ln(1-ps) + \ln c \\ \ln\psi(s) &= \pi \ln\left[\frac{s}{1-ps}\right]^{\pi} + \ln c\end{aligned}$$

Hence,

$$\psi(s) = c_1\left[\frac{s}{1-ps}\right]^{\pi}$$

Putting s=1;

$$\begin{aligned}\psi(1) = 1 &= c_1\left[\frac{s}{1-p}\right]^{\pi} \\ \Rightarrow c_1 &= (1-p)^{\pi} = q^{\pi}. \\ \psi(s) &= q^{\pi}\left[\frac{s}{1-ps}\right]^{\pi} \\ \therefore \psi(s) &= \left[\frac{qs}{1-ps}\right]^{\pi}. \\ \psi^{'}(s) &= \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \frac{\pi q^{\pi} s^{\pi-1}}{(1-ps)^{\pi}} + \frac{p\pi q^{\pi} s^{\pi}}{(1-ps)^{\pi+1}} \\ \psi^{''}(s) &= \frac{\pi(\pi-1)q^{\pi}s^{\pi-2}}{(1-ps)^{\pi}} + \frac{2p\pi^2 q^{\pi} s^{\pi-1}}{(1-ps)^{\pi+1}} + \frac{p^2\pi(\pi+1)q^{\pi}s^{\pi}}{(1-ps)^{\pi+2}}\end{aligned}$$

Putting s=1;

$$\begin{aligned}\psi^{'}(1) &= \frac{\pi q^{\pi}}{(1-p)^{\pi}} + \frac{p\pi q^{\pi}}{(1-p)^{\pi+1}} \\ &= \pi + \frac{p\pi}{q}\end{aligned}$$

Therefore,

$$
\begin{aligned}
\psi^{'}(1) &= \frac{q\pi + p\pi}{q} \\
&= \frac{\pi(q+p)}{q} \\
\therefore \psi^{'}(1) &= \frac{\pi}{q}. \quad \text{Since} \quad q+p=1 \\
E(X) &= \psi^{'}(1) \\
\therefore E(X) &= \frac{\pi}{q}. \\
\psi^{'}(1) &= \frac{\pi(\pi-1)q^{\pi}}{q^{\pi}} + \frac{2p\pi^2 q^{\pi}}{q^{\pi+1}} + \frac{p^2\pi(\pi+1)q^{\pi}}{q^{\pi+2}} \\
&= \pi(\pi-1) + \frac{2p\pi^2}{q} + \frac{p^2\pi(\pi+1)}{q^2} \\
Var(X) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= \pi(\pi-1) + \frac{2p\pi^2}{q} + \frac{p^2\pi(\pi+1)}{q^2} + \frac{\pi}{q} - \frac{\pi^2}{q^2} \\
&= \frac{q^2\pi^2 - q^2\pi + 2pq\pi^2 + p^2\pi^2 + p^2\pi + q\pi - \pi^2}{q^2} \\
&= \frac{\pi^2(q+p)^2 - \pi q^2 + \pi p^2 + \pi p - \pi^2}{q^2} \\
&= \frac{\pi p^2 + \pi p - \pi q^2}{q^2} \\
\therefore Var(X) &= \frac{\pi p}{q^2}.
\end{aligned}
$$

Using Feller's method, we sum equation (7.28) over n to obtain;

$$
\begin{aligned}
\sum_{n=\pi+1}^{\infty} nP_n - \pi \sum_{n=\pi+1}^{\infty} P_n &= p \sum_{n=\pi+1}^{\infty} (n-1)P_{n-1} \\
\Longleftrightarrow M_1 - \pi P_{\pi} &= pM_1 + \pi - \pi P_{\pi} \\
\text{But,} \quad P_{\pi} &= q^{\pi} \\
M_1 - \pi q^{\pi} &= pM_1 + \pi - \pi q^{\pi} \\
(1-p)M_1 &= \pi \\
\Rightarrow M_1 &= \frac{\pi}{q}. \quad \text{Since} \quad 1-p=q
\end{aligned}
$$

Therefore,

$$E(X) = M_1$$
$$\therefore E(X) = \frac{\pi}{q}.$$

Multiplying equation (7.28) by n and sum the results over n, we get;

$$\sum_{n=\pi+1}^{\infty} n^2 P_n - \pi \sum_{n=\pi+1}^{\infty} nP_n = p \sum_{n=\pi+1}^{\infty} [(n-1)+1](n-1)P_{n-1}$$
$$M_2 - \pi^2 P_\pi - [\pi M_1 - \pi^2 P_\pi] = p \sum_{n=\pi+1}^{\infty} (n-1)^2 P_{n-1} + p \sum_{n=\pi+1}^{\infty} (n-1)P_{n-1}$$
$$M_2 - \pi M_1 = pM_2 + pM_1$$
$$\text{But} \quad M_1 = \frac{\pi}{q}.$$
$$\Rightarrow (1-p)M_2 = \frac{\pi^2}{q} + \frac{\pi p}{q}$$
$$\therefore M_2 = \frac{\pi^2 + \pi p}{q^2}. \quad \text{Since} \quad 1-p = q$$
$$Var(X) = M_2 - [M_1]^2$$
$$= \frac{\pi^2 + \pi p}{q^2} - \frac{\pi^2}{q^2}$$
$$\therefore Var(X) = \frac{\pi p}{q^2}.$$

6. Truncated Negative Binomial Distribution
For a truncated Negative Binomial Distribution;
Let,

$$P_n = \frac{q^\pi}{1-q^\pi} \begin{pmatrix} n+\pi-1 \\ \pi-1 \end{pmatrix} p^n q^\pi; \quad n = 1,2,3,\cdots; \quad q = 1-p$$
$$\text{and} \quad P_1 = \frac{p\pi q^\pi}{1-q^\pi}$$
$$P_{n-1} = \frac{q^\pi}{1-q^\pi} \begin{pmatrix} (n-1)+\pi-1 \\ \pi-1 \end{pmatrix} p^{(n-1)} q^\pi; \quad n = 1,2,3,\cdots$$

Proposition 7.7

$$\frac{P_n}{P_{n-1}} = \frac{(n+\pi-1)p}{n} \tag{7.29}$$

$$\Rightarrow nP_n = p(n+\pi-1)P_{n-1}; \quad n = 2,3,\cdots \tag{7.30}$$

Proof
Solving (7.30) iteratively, we have;
When n=1;
Equation (7.30) becomes;

$$P_1 = p\pi P_0$$
$$\therefore P_1 = p\pi P_0.$$

When n=2;
Equation (7.30) becomes;

$$2P_2 = p(\pi+1)P_1$$
$$\Rightarrow 2P_2 = p(\pi+1)p\pi P_0$$
$$\therefore P_2 = \frac{\pi(\pi+1)p^2}{2!}P_0.$$

When n=3;
Equation (7.30) becomes;

$$3P_3 = p(\pi+2)P_2$$
$$\Rightarrow 3P_3 = p(\pi+2)\frac{\pi(\pi+1)p^2}{1.2}P_0$$
$$\therefore P_3 = \frac{\pi(\pi+1)(\pi+2)p^3}{3!}P_0.$$

By Mathematical Induction, we have;

$$
\begin{aligned}
P_4 &= \frac{\pi(\pi+1)(\pi+2)(\pi+3)p^4}{4!}P_0,\\
P_5 &= \frac{\pi(\pi+1)(\pi+2)(\pi+3)(\pi+4)p^5}{5!}P_0,\\
&.\\
&.\\
&.\\
P_n &= \frac{\pi(\pi+1)(\pi+2)\cdots(\pi+n-2)(\pi+n-1)p^n}{n!}P_0.\\
\Rightarrow P_{n-1} &= \frac{\pi(\pi+1)(\pi+2)\cdots(\pi+n-2)p^{n-1}}{(n-1)!}P_0.\\
\frac{P_n}{P_{n-1}} &= \frac{\pi(\pi+1)(\pi+2)\cdots(\pi+n-2)(\pi+n-1)p.p^{n-1}}{n.(n-1)!}.\frac{(n-1)!}{\pi(\pi+1)(\pi+2)\cdots(\pi+n-2)p^{n-1}}\\
\therefore \frac{P_n}{P_{n-1}} &= \frac{p(\pi+n-1)}{n}. \quad \text{as required}
\end{aligned}
$$

Using the probability generating function technique, we multiply equation (7.30) by s^n and sum the results over n to obtain;

$$
\begin{aligned}
\sum_{n=2}^{\infty} nP_n s^n &= p\sum_{n=2}^{\infty}(n+\pi-1)P_{n-1}s^n\\
s\sum_{n=2}^{\infty} nP_n s^{n-1} &= p\sum_{n=2}^{\infty}[(n-1)+\pi]P_{n-1}s^n\\
s\sum_{n=2}^{\infty} nP_n s^{n-1} &= p\sum_{n=2}^{\infty}(n-1)P_{n-1}s^n + p\pi\sum_{n=2}^{\infty}P_{n-1}s^n\\
s\sum_{n=2}^{\infty} nP_n s^{n-1} &= ps^2\sum_{n=2}^{\infty}(n-1)P_{n-1}s^{n-1} + ps\pi\sum_{n=2}^{\infty}P_{n-1}s^{n-1}\\
s\Big[\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} - P_1\Big] &= ps^2\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} + p\pi s\Big[\psi(s) - P_0\Big]\\
\text{But,}\quad P_0 &= 0\\
\text{and}\quad P_1 &= \frac{p\pi q^{\pi}}{1-q^{\pi}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
s\left[\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}-\frac{p\pi q^{\pi}}{1-q^{\pi}}\right] &= ps^2\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}+p\pi s\Big[\psi(s)-P_0\Big] \\
s(1-ps)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= p\pi\psi(s)+\frac{p\pi q^{\pi}}{1-q^{\pi}}
\end{aligned} \tag{7.31}
$$

Note that, equation (7.31) cannot be solved explicitly.

Using Feller's method, we sum equation (7.30) over n to obtain;

$$
\begin{aligned}
\sum_{n=2}^{\infty} nP_n &= p\sum_{n=2}^{\infty}(n+\pi-1)P_{n-1} \\
\sum_{n=2}^{\infty} nP_n &= p\sum_{n=2}^{\infty}[(n-1)+\pi]P_{n-1} \\
M_1-P_1 &= p\sum_{n=2}^{\infty}(n-1)P_{n-1}+p\pi\sum_{n=2}^{\infty}P_{n-1} \\
M_1-P_1 &= pM_1+p\pi \\
\text{But,}\quad P_1 &= \frac{p\pi q^{\pi}}{1-q^{\pi}} \\
\Rightarrow M_1 &= \frac{p\pi}{q}+\frac{p\pi q^{\pi}}{q(1-q^{\pi})} \\
E(X) &= M_1 \\
&= \frac{p\pi(1-q^{\pi})+p\pi q^{\pi}}{q(1-q^{\pi})} \\
&= \frac{p\pi[1-q^{\pi}+q^{\pi}]}{q(1-q^{\pi})} \\
\therefore E(X) &= \frac{p\pi}{q(1-q^{\pi})}.
\end{aligned}
$$

Also, multiplying equation (7.30) by n and sum the results over n, we get;

$$
\begin{aligned}
\sum_{n=2}^{\infty} n^2 P_n &= p\sum_{n=2}^{\infty} n(n+\pi-1)P_{n-1} \\
\sum_{n=2}^{\infty} n^2 P_n &= p\sum_{n=2}^{\infty} [(n-1)+1][(n-1)+\pi]P_{n_1} \\
M_2 - P_1 &= p\sum_{n=2}^{\infty}(n-1)^2 P_{n-1} + p\pi\sum_{n=2}^{\infty}(n-1)P_{n-1} + p\sum_{n=2}^{\infty}(n-1)P_{n-1} + p\pi\sum_{n=2}^{\infty}P_{n-1} \\
M_2 - P_1 &= pM_2 + p\pi M_1 + pM_1 + p\pi \\
(1-p)M_2 &= (p\pi - p)\frac{p\pi}{q(1-q^{\pi})} + p\pi + \frac{p\pi q^{\pi}}{1-q^{\pi}} \\
\text{Since} \quad P_1 &= \frac{p\pi q^{\pi}}{1-q^{\pi}}, \quad \text{and} \quad M_1 = \frac{p\pi}{q(1-q^{\pi})}. \\
\Rightarrow qM_2 &= \frac{p^2\pi^2}{q(1-q^{\pi})} + \frac{p^2\pi}{q(1-q^{\pi})} + p\pi + \frac{p\pi q^{\pi}}{1-q^{\pi}} \\
M_2 &= \frac{p^2\pi^2}{q^2(1-q^{\pi})} + \frac{p^2\pi}{q^2(1-q^{\pi})} + \frac{p\pi}{q} + \frac{p\pi q^{\pi}}{q(1-q^{\pi})} \\
&= \frac{p^2\pi^2 + p^2\pi + p\pi q - p\pi q^{\pi} + p\pi q^{\pi+1}}{q^2(1-q^{\pi})} \\
&= \frac{p^2\pi^2 + p\pi(p+q) + p\pi q^{\pi}(-1+q)}{q^2(1-q^{\pi})} \\
&= \frac{p^2\pi^2 + p\pi - p\pi q^{\pi}}{q^2(1-q^{\pi})} \\
\Rightarrow M_2 &= \frac{p^2\pi^2 + p^2\pi + p\pi}{q} + \frac{p\pi q^{\pi}}{q(1-q^{\pi})} \\
Var(X) &= M_2 - \{M_1\}^2 \\
&= \frac{p^2\pi^2 + p^2\pi + p\pi}{q} + \frac{p\pi q^{\pi}}{q(1-q^{\pi})} - \{\frac{p\pi}{q(1-q^{\pi})}\} \\
\therefore Var(X) &= \frac{p\pi}{q^2(1-q^{\pi})} - \frac{p^2\pi^2 q^{\pi}}{q^2(1-q^{\pi})^2}.
\end{aligned}
$$

From Equation (*) above, we have;

$$\psi(s) = \left(\frac{1-\xi}{1-\xi s}\right)^{\alpha}$$

If α is a negative integer;
Let $\alpha = -\omega$ where, ω is a positive integer,
Then,

$$\psi(s) = (1-\xi)^{-\omega}(1-\xi s)^{\omega}$$
$$= (1-\xi)^{-\omega}\sum_{n=0}^{\omega}\begin{pmatrix}\omega\\ n\end{pmatrix}(-\xi s)^n$$
$$\Rightarrow \psi(s) = \sum_{n=0}^{\omega}\begin{pmatrix}\omega\\ n\end{pmatrix}\frac{(-\xi s)^n}{(1-\xi)^{\omega}}$$
$$= \sum_{n=0}^{\omega}\begin{pmatrix}\omega\\ n\end{pmatrix}\frac{(-\xi s)^n}{(1-\xi)^n(1-\xi)^{\omega-n}}$$
$$= \sum_{n=0}^{\omega}\begin{pmatrix}\omega\\ n\end{pmatrix}\left(\frac{-\xi}{1-\xi}\right)^n\left(\frac{1}{1-\xi}\right)^{\omega-n}s^n$$
$$\text{Let,}\quad p = \frac{-\xi}{1-\xi}$$

Thus,

$$1-p = 1+\frac{\xi}{1-\xi}$$
$$= \frac{1}{1-\xi}$$
$$\therefore \psi(s) = \sum_{n=0}^{\omega}\begin{pmatrix}\omega\\ n\end{pmatrix}p^n(1-p)^{\omega-n}s^n; \quad \xi < 0$$

Therefore,

$$P_n = \begin{pmatrix}\omega\\ n\end{pmatrix}p^n(1-p)^{\omega-n}, \quad \text{for} \quad n = 0,1,2,\cdots,\omega.$$

Which is a Binomial distribution.

$$\text{Let,}\quad P_n = \begin{pmatrix}\omega\\ n\end{pmatrix}p^n(1-p)^{\omega-n}, \quad \text{for} \quad n = 0,1,2,\cdots,\omega; \quad q = 1-p$$
$$\Rightarrow P_{n-1} = \begin{pmatrix}\omega\\ n-1\end{pmatrix}p^{n-1}(1-p)^{\omega-(n-1)}, \quad \text{for} \quad n = 1,2,\cdots,\omega.$$

Hence,
Proposition 7.8

$$\frac{P_n}{P_{n-1}} = \frac{(\omega - n + 1)}{nq}.$$
$$\iff nP_n = (\omega - n + 1)\frac{p}{q}P_{n-1}; \quad \text{for} \quad n = 1, 2, 3, \cdots, \omega \tag{7.32}$$

Proof
Solving (7.32) iteratively, we have;
When n=1;
Equation (7.32) becomes;

$$P_1 = \frac{\omega p}{q}P_0$$
$$\therefore P_1 = \frac{\omega p}{q}P_0.$$

When n=2;
Equation (7.32) becomes;

$$2P_2 = (\omega - 1)\frac{p}{q}P_1$$
$$\Rightarrow 2P_2 = (\omega - 1)\frac{p}{q}\frac{\omega p}{q}P_0$$
$$\therefore P_2 = \frac{\omega(\omega - 1)\left(\frac{p}{q}\right)^2}{2!}P_0.$$

When n=3;
Equation (7.32) becomes;

$$3P_3 = (\omega - 2)\frac{p}{q}P_2$$
$$\Rightarrow 3P_3 = (\omega - 2)\frac{p}{q}\frac{\omega(\omega - 1)\left(\frac{p}{q}\right)^2}{1.2}P_0$$
$$\therefore P_3 = \frac{\omega(\omega - 1)(\omega - 2)\left(\frac{p}{q}\right)^3}{3!}P_0.$$

Therefore, by Mathematical Induction, we have;

$$P_4 = \frac{\omega(\omega-1)(\omega-2)(\omega-3)\left(\frac{p}{q}\right)^4}{4!} P_0,$$

$$P_4 = \frac{\omega(\omega-1)(\omega-2)(\omega-3)\left(\frac{p}{q}\right)^4}{4!} P_0,$$

$$P_5 = \frac{\omega(\omega-1)(\omega-2)(\omega-3)(\omega-4)\left(\frac{p}{q}\right)^5}{5!} P_0,$$

.

.

.

$$P_n = \frac{\omega(\omega-1)(\omega-2)\cdots(\omega-n+2)(\omega-n+1)\left(\frac{p}{q}\right)^n}{n!} P_0.$$

$$\Rightarrow P_{n-1} = \frac{\omega(\omega-1)(\omega-2)\cdots(\omega-n+2)\left(\frac{p}{q}\right)^{n-1}}{(n-1)!} P_0.$$

$$\frac{P_n}{P_{n-1}} = \frac{\omega(\omega-1)(\omega-2)\cdots(\omega-n+2)(\omega-n+1)\left(\frac{p}{q}\right)^1 . \left(\frac{p}{q}\right)^{n-1}}{n.(n-1)!}$$

$$x \frac{(n-1)!}{\omega(\omega-1)(\omega-2)\cdots(\omega-n+2)\left(\frac{p}{q}\right)^{n-1}}$$

$$\therefore \frac{P_n}{P_{n-1}} = \frac{(\omega-n+1)(\frac{p}{q})}{n}. \quad \text{as required}$$

Its pgf. in hypergeometric terms is given by;

$$
\begin{aligned}
\phi(z) &= \frac{{}_2F_1(\alpha,1;1;\xi z)}{{}_2F_1(\alpha,1;1;\xi)}.\\
\phi^{'}(z) &= \frac{1.\alpha\xi}{1}\frac{{}_2F_1(\alpha+1,2;2;\xi z)}{{}_2F_1(\alpha,1;1;\xi)}\\
\phi^{''}(z) &= \frac{1(2)\alpha(\alpha+1)\xi^2}{1(2)}\frac{{}_2F_1(\alpha+2,3;3;\xi z)}{{}_2F_1(\alpha,1;1;\xi)}\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_2F_1(\alpha+\kappa,1+\kappa;1+\kappa;\xi)}{{}_2F_1(\alpha,1;1;\xi)},\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \alpha\xi\Lambda_1.
\end{aligned}
$$

and,

$$
\begin{aligned}
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= \alpha(\alpha+1)\xi^2\Lambda_2+\alpha\xi\Lambda_1-[\alpha\xi\Lambda_1]^2\\
\therefore Var(X) &= \alpha\xi\Lambda_1+\alpha\xi^2\{(\alpha+1)\Lambda_2-\alpha\Lambda_1^2\}.
\end{aligned}
$$

Using the probability generating function technique, we multiply equation (7.32) by s^n and sum the results over n to obtain;

$$
\begin{aligned}
q\sum_{n=1}^{\omega} nP_n s^n &= p\sum_{n=1}^{\omega}[\omega-(n-1)]P_{n-1}s^n\\
qs\sum_{n=1}^{\infty} nP_{n-1}s^{n-1} &= \omega p\sum_{n=1}^{\infty}P_{n-1}s^n - p\sum_{n=1}^{\infty}(n-1)P_{n-1}s^n\\
qs\sum_{n=1}^{\infty} nP_{n-1}s^{n-1} &= \omega ps\sum_{n=1}^{\infty}P_{n-1}s^{n-1} - ps^2\sum_{n=1}^{\infty}(n-1)P_{n-1}s^{n-2}\\
qs\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \omega ps\psi(s) - ps^2\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}\\
\frac{1}{\psi(s)}\mathrm{d}\psi(s) &= \frac{\omega p}{q+ps}\mathrm{d}s
\end{aligned}
$$

Integrating both sides with respect to s, we have;

$$\int \frac{\mathrm{d}\psi(s)}{\psi(s)} = \omega \int \frac{p\mathrm{d}s}{(q+ps)}$$

Therefore,

$$\psi(s) = c_1(q+ps)^{\omega}$$

Putting s=1;

$$\begin{aligned}
\psi(1) &= 1 = c_1(q+p)^{\omega} \\
&\Rightarrow c_1 = 1. \\
\therefore \psi(s) &= (q+ps)^{\omega}.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \psi^{'}(s) &= \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \omega p(q+ps)^{\omega-1} \\
\psi^{''}(s) &= \frac{\mathrm{d}\psi^{'}(s)}{\mathrm{d}s} = \omega(\omega-1)p^2(q+ps)^{\omega-2} \\
E(X) &= \psi^{'}(1) \\
\therefore E(X) &= \omega p. \\
\psi^{''}(1) &= \omega(\omega-1)p^2 \\
\Rightarrow \psi^{''}(1) &= \omega^2 p^2 - \omega p^2 \\
Var(X) &= \psi^{''}(1) + \psi^{'}(1) - \{\psi^{'}(1)\}^2 \\
&= \omega^2 p^2 - \omega p^2 + \omega p - \omega^2 p^2 \\
&= \omega p(1-p) \\
\therefore Var(X) &= \omega pq.
\end{aligned}$$

Using Feller's method, we sum equation (7.32) over n to obtain;

$$\sum_{n=1}^{\omega} nP_n = \sum_{n=1}^{\omega} (\omega - n + 1)\frac{p}{q}P_{n-1}$$

$$\sum_{n=1}^{\omega} nP_n = \frac{p}{q}\sum_{n=1}^{\omega}[\omega - (n-1)]P_{n-1}$$

$$M_1 = \frac{p}{q}\omega\sum_{n=1}^{\omega} P_{n-1} - \frac{p}{q}\sum_{n=1}^{\omega}(n-1)P_{n-1}$$

$$M_1 = \frac{p}{q}[\omega(1-P_\omega)] - \frac{p}{q}[M_1 - \omega P_\omega]$$

Therefore,

$$M_1 = \frac{p}{q}\omega - \frac{p}{q}M_1$$

$$\iff qM_1 + pM_1 = \omega p$$

$$(q+p)M_1 = \omega p$$

$$\therefore M_1 = \omega p. \quad \text{Since} \quad q+p=1$$

$$E(X) = M_1$$

$$\therefore E(X) = \omega p.$$

Again, multiplying equation (7.32) by n and sum the results over n, we get;

$$\begin{aligned}
\sum_{n=1}^{\omega} n^2 P_n &= \sum_{n=1}^{\omega} n(\omega - n + 1)\frac{p}{q} P_{n-1} \\
\sum_{n=1}^{\omega} n^2 P_n &= \frac{p}{q} \sum_{n=1}^{\omega} [(n-1)+1][\omega - (n-1)] P_{n-1} \\
M_2 &= \frac{p}{q} \Big[\omega \sum_{n=1}^{\omega} (n-1) P_{n-1} - \sum_{n=1}^{\omega} (n-1)^2 P_{n-1} + \omega \sum_{n=1}^{\omega} P_{n-1} - \sum_{n=1}^{\omega} (n-1) P_{n-1} \Big] \\
M_2 &= \frac{p}{q} \Big[\omega (M_1 - \omega P_\omega) - (M_2 - \omega^2 P_\omega) + \omega (1 - P_\omega) - (M_1 - \omega P_\omega) \Big] \\
M_2 &= \frac{p}{q} \Big[\omega M_1 - M_2 + \omega - M_1 \Big] \\
\text{But,} \quad M_1 &= \omega p \\
\Rightarrow M_2 &= \frac{p}{q} \Big[\omega^2 p - M_2 + \omega - \omega p \Big] \\
\Longleftrightarrow q M_2 &= \omega p^2 - p M_2 + \omega p - \omega p^2 \\
\Rightarrow (q+p) M_2 &= \omega^2 p^2 + \omega p - \omega p^2 \\
\therefore M_2 &= \omega^2 p^2 + \omega p - \omega p^2. \\
Var(X) &= M_2 - \{M_1\}^2 \\
&= \omega^2 p^2 + \omega p - \omega p^2 - \omega^2 p^2 \\
&= \omega p (1-p) \\
\therefore Var(X) &= \omega p q.
\end{aligned}$$

7.4 Sundt - Jewell (1981) Class

Remark 7.2

Sundt, B. and Jewell, W.S.(1981) have shown that: Poisson, Binomial, Negative Binomial and Geometric distributions are the only members of the family of claim number distributions.

7.5 Panjer's Class of order k

Sundt and Jewell (1981) generalized Panjer's (1981) recursive algorithm to the class of counting distributions with discrete density $\{P_n\}_{n=0}^{\infty}$ satisfying the recursion;

$$P_n = (\xi + \frac{\lambda}{n})P_{n-1}; \quad n = \pi + 1, \pi + 2, \cdots \tag{7.33}$$

7.6 Special Cases of Panjer's Class of Order k

7.6.1 When $\pi = 0$, we obtain Panjer's Class

$$P_n = (\xi + \frac{\lambda}{n})P_{n-1}; \quad n = 1, 2, 3, \cdots$$

$$\text{and} \quad P_1 > 0, \quad P_0 = 0$$

$$\begin{aligned} nP_n &= (\xi n + \lambda)P_{n-1} \\ &= [\xi(n-1+1) + \lambda]P_{n-1} \\ \therefore nP_n &= \xi(n-1)P_{n-1} + (\xi + \lambda)P_{n-1}; \quad n = 1, 2, 3, \cdots \end{aligned}$$

Proposition 7.9

$$nP_n = (\xi n + \lambda)P_{n-1} \tag{7.34}$$

Proof

Solving (7.34) iteratively, we have;

When n=1;

Equation (7.34) becomes;

$$\begin{aligned} P_1 &= (\xi + \lambda)P_0 \\ \Rightarrow P_1 &= (\xi + \lambda)P_0 \\ \therefore P_1 &= \frac{(\xi + \lambda)}{1!}P_0. \end{aligned}$$

When n=2;

Equation (7.34) becomes;

$$\begin{aligned}
2P_2 &= (\xi+\lambda)(2\xi+\lambda)P_0 \\
\Rightarrow P_2 &= \frac{(\xi+\lambda)(2\xi+\lambda)}{1.2}P_0 \\
\therefore P_2 &= \frac{(\xi+\lambda)(2\xi+\lambda)}{2!}P_0.
\end{aligned}$$

When n=3;
Equation (7.34) becomes;

$$\begin{aligned}
3P_3 &= (3\xi+\lambda)P_2 \\
\Rightarrow 3P_3 &= (3\xi+\lambda)\frac{(\xi+\lambda)(2\xi+\lambda)}{1.2}P_0 \\
\therefore P_3 &= \frac{(\xi+\lambda)(2\xi+\lambda)(3\xi+\lambda)}{3!}P_0.
\end{aligned}$$

By Mathematical Induction, we have;

$$\begin{aligned}
P_4 &= \frac{(\xi+\lambda)(2\xi+\lambda)(3\xi+\lambda)(4\xi+\lambda)}{4!}P_0, \\
P_5 &= \frac{(\xi+\lambda)(2\xi+\lambda)(3\xi+\lambda)(4\xi+\lambda)(5\xi+\lambda)}{5!}P_0, \\
&. \\
&. \\
&. \\
P_n &= \frac{(\xi+\lambda)(2\xi+\lambda)(3\xi+\lambda)\cdots((n-1)\xi+\lambda)(n\xi+\lambda)}{n!}P_0. \\
\Rightarrow P_{n-1} &= \frac{(\xi+\lambda)(2\xi+\lambda)\cdots((n-1)\xi+\lambda)}{(n-1)!}P_0. \\
\frac{P_n}{P_{n-1}} &= \frac{(\xi+\lambda)(2\xi+\lambda)\cdots((n-1)\xi+\lambda)(n\xi+\lambda)}{n.(n-1)!}\cdot\frac{(n-1)!}{(\xi+\lambda)(2\xi+\lambda)(3\xi+\lambda)\cdots((n-1)\xi+\lambda)} \\
\therefore \frac{P_n}{P_{n-1}} &= \frac{(n\xi+\lambda)}{n}. \quad \text{as required}
\end{aligned}$$

Multiplying equation (7.34) by s^n and summing over n, we obtain;

$$\begin{aligned}
\sum_{n=1}^{\infty} nP_n s^n &= \xi \sum_{n=1}^{\infty}(n-1)P_{n-1}s^n + (\xi+\lambda)\sum_{n=1}^{\infty} P_{n-1}s^n \\
s\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \xi s^2 \sum_{n=1}^{\infty}(n-1)P_{n-1}s^{n-2} + (\xi+\lambda)s\sum_{n=1}^{\infty} P_{n-1}s^{n-1} \\
\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \xi s\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} + (\xi+\lambda)\psi(s) \\
(1-\xi s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= (\xi+\lambda)\psi(s) \qquad \textbf{(7.35)}
\end{aligned}$$

When $\xi \neq 0, \quad \lambda = 0$ **and** $(\xi+\lambda) \geq 0$
Then,

$$(1-\xi s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = (\xi+\lambda)\psi(s) = P_1$$

When $(\xi+\lambda) = 0$, **then** $\xi \neq 0$
Thus,

$$\begin{aligned}
(1-\xi s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= P_1 \\
\Rightarrow \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= \frac{P_1}{1-\xi s} \\
\Rightarrow \psi(s) &= \int \frac{P_1 \mathrm{d}s}{1-\xi s} \\
&= \frac{P_1}{\xi}\int \frac{-\xi \mathrm{d}s}{1-\xi s} \\
&= \frac{P_1}{\xi}\log(1-\xi s) + c_1
\end{aligned}$$

Putting s=1;

$$\begin{aligned}\psi(1) = 1 &= \frac{P_1}{\xi}\log(1-\xi) + c_1 \\ &= \frac{P_1}{\xi}c_1\log(1-\xi) \\ \Rightarrow c_1 &= \frac{\xi}{P_1\log(1-\xi)}\end{aligned}$$

Therefore,

$$\psi(s) = \frac{\log(1-\xi s)}{\log(1-\xi)}.$$

Which is the probability generating function for a logarithmic distribution with parameter ξ.

$$P_n = \frac{\xi^n}{-n\log(1-\xi)}; \quad \text{for} \quad n = 1,2,3,\cdots; \quad 0 < \xi < 1.$$

Suppose we let $p = \xi$;

$$\begin{aligned}\Rightarrow P_n &= \frac{p^n}{-n\log q}; \quad \text{for} \quad n = 1,2,3,\cdots; \quad q = 1-p. \\ \text{and} \quad P_1 &= \frac{p}{\log q}\end{aligned}$$

Thus,

$$\begin{aligned}P_{n-1} &= \frac{p^{n-1}}{-(n-1)\log q}; \quad \text{for} \quad n = 2,3,4,\cdots \\ \Rightarrow \frac{P_n}{P_{n-1}} &= \frac{(n-1)p}{n}. \\ \Longleftrightarrow nP_n &= p(n-1)P_{n-1}; \quad n = 2,3,4,\cdots \qquad (7.36)\end{aligned}$$

Solving (7.36) iteratively, we have;
When n=1;

Equation (7.36) becomes;

$$P_1 = P_0$$
$$\therefore P_1 = P_0.$$

When n=2;
Equation (7.36) becomes;

$$2P_2 = pP_1$$
$$\Rightarrow 2P_2 = pP_0$$
$$\therefore P_2 = \frac{p}{2!}P_0.$$

When n=3;
Equation (7.36) becomes;

$$3P_3 = 2pP_2$$
$$\Rightarrow 3P_3 = 2p\frac{p}{1.2}P_0$$
$$\therefore P_3 = \frac{2p^2}{3!}P_0.$$

Therefore, by Mathematical Induction, we have;

$$P_4 = \frac{3p^3}{4!}P_0,$$
$$P_5 = \frac{4p^4}{5!}P_0,$$
$$.$$
$$.$$
$$.$$
$$P_n = \frac{(n-1)p^{n-1}}{n!}P_0.$$
$$\Rightarrow P_{n-1} = \frac{(n-2)p^{n-2}}{(n-1)!}P_0.$$

Therefore,

$$\frac{P_n}{P_{n-1}} = \frac{(n-2).(n-1)p.p^{n-2}}{n.(n-1)!}.\frac{(n-1)!}{(n-2)p^{n-2}}$$
$$\therefore \frac{P_n}{P_{n-1}} = \frac{p(n-1)}{n}. \quad \text{as required}$$

Using the probability generating function technique, we multiply equation (7.36) by s^n and sum the results over n to obtain;

$$\sum_{n=2}^{\infty} nP_n s^n = p\sum_{n=2}^{\infty}(n-1)P_{n-1}s^n$$
$$s\sum_{n=2}^{\infty} nP_n s^{n-1} = ps^2\sum_{n=2}^{\infty}(n-1)P_{n-1}s^{n-2}$$
$$s\left[\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} - P_1\right] = ps^2\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}$$
$$(1-ps)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \frac{-p}{-\log q}; \quad \text{Since} \quad P_1 = \frac{-p}{\log q}$$

Therefore,

$$\int \mathrm{d}\psi(s) = \frac{1}{\log q}\int \frac{-p\mathrm{d}s}{1-ps}$$

Thus,

$$\psi(s) = \frac{\log(1-ps)}{\log q}.$$
$$\psi^{'}(s) = \frac{-p}{(1-ps)\log q}$$
$$\psi^{''}(s) = \frac{\mathrm{d}\psi^{'}(s)}{\mathrm{d}s} = \frac{-p^2}{(1-ps)^2\log q}$$
$$E(X) = \psi^{'}(1)$$
$$= \frac{-p}{(1-p)\log q}$$
$$\therefore E(X) = \frac{-p}{q\log q}.$$
$$\psi^{''}(1) = \frac{-p^2}{(1-p)^2\log q}$$
$$= \frac{-p^2}{q^2\log q}$$

and,

$$\begin{aligned}
Var(X) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= \frac{-p^2}{q^2 \log q} + \frac{-p}{q \log q} - \frac{-p^2}{q^2 (\log q)^2} \\
\therefore Var(X) &= \frac{-p \log q - p^2}{q^2 (\log q)^2}.
\end{aligned}$$

Using Feller's method, we sum equation (7.36) over n to obtain;

$$\begin{aligned}
\sum_{n=2}^{\infty} nP_n &= p \sum_{n=2}^{\infty} (n-1) P_{n-1} \\
\Rightarrow M_1 - P_1 &= pM_1 \\
\Rightarrow (1-p)M_1 &= P_1 \\
\text{But,} \quad P_1 &= \frac{-p}{\log q} \\
\text{Therefore,} \quad (1-p)M_1 &= \frac{-p}{\log q} \\
\therefore M_1 &= \frac{-p}{q \log q}. \quad \text{Since} \quad 1-p = q \\
E(X) &= M_1 \\
\therefore E(X) &= \frac{-p}{q \log q}.
\end{aligned}$$

Multiplying equation (7.36) by n and sum the results over n, we get;

$$\begin{aligned}
\sum_{n=2}^{\infty} n^2 P_n &= p \sum_{n=2}^{\infty} n(n-1) P_{n-1} \\
M_2 - P_1 &= p \sum_{n=2}^{\infty} [(n-1)+1](n-1) P_{n-1} \\
M_2 - P_1 &= p[\sum_{n=2}^{\infty} (n-1)^2 P_{n-1} + \sum_{n=2}^{\infty} (n-1) P_{n-1}] \\
M_2 - P_1 &= p[M_2 - M_1] \\
\Rightarrow (1-ps)M_2 &= P_1 + pM_1
\end{aligned}$$

But,

$$
\begin{aligned}
M_1 &= \frac{-p}{q\log q} \quad \text{and} \quad P_1 = \frac{-p}{\log q} \\
\Rightarrow (1-ps)M_2 &= \frac{-p}{\log q} + \frac{-p^2}{q\log q} \\
\therefore M_2 &= \frac{-p}{q\log q} + \frac{-p^2}{q^2\log q}. \\
Var(X) &= M_2 - \{M_1\}^2 \\
&= \frac{-p}{q\log q} + \frac{-p^2}{q^2\log q} - \left\{\frac{-p}{q\log q}\right\}^2 \\
\therefore Var(X) &= \frac{-p\log q - p^2}{q^2(\log q)^2}.
\end{aligned}
$$

Remark (7.3)

7.6.2 When $\pi = 1$, we obtain a Willmot (1988) Class

From equation (7.33), we have;

$$P_n = (\xi + \frac{\lambda}{n})P_{n-1}; \quad n = 2,3,\cdots \tag{**}$$

Remark (7.4)
Schroter (1990) generalized Panjer's recursive algorithm to the class of counting distributions satisfying the class;

$$P_n = (\xi + \frac{\lambda}{n})P_{n-1} + \frac{c}{n}P_{n-2}; \quad n = 1,2,3,\cdots$$

with $P_{-1} = 0$ for some constants ξ, λ and c.

Case I: Let $\xi = p$ and $\lambda = -p$
From (**), we have;

$$
\begin{aligned}
P_n &= \left(p + \frac{-p}{n}\right)P_{n-1} \\
\Rightarrow P_n &= \frac{p(n-1)}{n}P_{n-1} \\
\iff nP_n &= p(n-1)P_{n-1}; \quad n = 2,3,4,\cdots
\end{aligned}
\tag{7.37}
$$

Which is the same as equation (7.36) and it will generate the probability generating function for a logarithmic distribution whose properties have been discussed.

Case II: Let $\xi = p$ and $\lambda = 0$

Then,

Proposition 7.10

$$
\begin{aligned}
P_n &= pP_{n-1}; \quad \text{for} \quad n = 2,3,\cdots \\
\text{and} \quad P_1 &= q
\end{aligned}
\tag{7.38}
$$

Proof

Solving (7.38) iteratively, we have;

When n=1;

Equation (7.38) becomes;

$$
\begin{aligned}
P_1 &= pP_0 \\
\therefore P_1 &= pP_0.
\end{aligned}
$$

When n=2;

Equation (7.38) becomes;

$$
\begin{aligned}
P_2 &= pP_1 \\
\Rightarrow P_2 &= p.pP_0 \\
\therefore P_2 &= p^2P_0.
\end{aligned}
$$

When n=3;

Equation (7.38) becomes;

$$
\begin{aligned}
P_3 &= pP_2 \\
\Rightarrow P_3 &= p.p^2P_0 \\
\therefore P_3 &= p^3P_0. \\
\Rightarrow P_4 &= p^4P_0, \\
P_5 &= p^5P_0, \\
&. \\
&. \\
&. \\
P_n &= p^nP_0. \\
\Rightarrow P_{n-1} &= p^{n-1}P_0. \\
\frac{P_n}{P_{n-1}} &= \frac{p.p^{n-1}}{p^{n-1}} \\
\therefore \frac{P_n}{P_{n-1}} &= p. \quad \text{as required}
\end{aligned}
$$

Which is the probability generating function for Geometric Distribution Type II.

Multiplying equation (7.38) by s^n and sum the results over n to obtain;

$$
\begin{aligned}
\sum_{n=2}^{\infty} P_n s^n &= p\sum_{n=2}^{\infty} P_{n-1}s^n \\
\psi(s) - P_1 &= ps\psi(s) \\
\Rightarrow (1-ps)\psi(s) &= P_1 \\
\text{But,} \quad P_1 &= q \\
\therefore \psi(s) &= \frac{q}{1-ps}. \\
\psi^{'}(s) &= \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \frac{pq}{(1-ps)^2} \\
\psi^{''}(s) &= \frac{\mathrm{d}\psi^{'}(s)}{\mathrm{d}s} = \frac{2p^2q}{(1-ps)^3}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E(X) &= \psi^{'}(1) \\
&= \frac{pq}{(1-p)^2} \\
&= \frac{pq}{q^2} \\
\therefore E(X) &= \frac{p}{q}. \\
Var(X) &= \psi^{''}(1) + \psi^{'}(1) - [\psi^{'}(1)]^2 \\
&= \frac{2p^2q}{(1-p)^3} + \frac{p}{q} - \frac{p^2}{q^2} \\
&= \frac{2p^2 + pq - p^2}{q^2} \\
&= \frac{p^2 + pq}{q^2} \\
&= \frac{p(p+q)}{q^2} \\
\therefore Var(X) &= \frac{p}{q^2}.
\end{aligned}
$$

Its probability generation function in hypergeometric term is given by;

$$
\begin{aligned}
\phi(z) &= \frac{{}_1F_1(1;1;pz)}{{}_1F_1(1;1;p)}. \\
\phi^{'}(z) &= p.\frac{{}_1F_1(2;2;pz)}{{}_1F_1(1;1;p)} \\
\phi^{''}(z) &= p^2\frac{{}_1F_1(3;3;pz)}{{}_1F_1(1;1;p)} \\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_1F_1(1+\kappa;1+\kappa;p)}{{}_1F_1(1;1;p)}, \quad \kappa = 1,2. \\
E(X) &= \phi^{'}(1) \\
\therefore E(X) &= p\Lambda_1. \\
Var(X) &= \phi^{''}(1) + \phi^{'}(1) - \{\phi^{'}(1)\}^2 \\
&= p^2\Lambda_2 + p\Lambda_1 - p^2\Lambda_1^2 \\
\therefore Var(X) &= p\Lambda_1 + p^2\{\Lambda_2 - \Lambda_1^2\}.
\end{aligned}
$$

7.7 Panjer's - Willmot (1982) Class

Consider the counting distributions whose discrete density $\{P_n\}_{n=0}^{\infty}$ satisfying a class in the form;

$$P_n = \sum_{\iota=1}^{k}\left(\xi_\iota + \frac{\lambda_\iota}{n}\right); \quad n = 1, 2, 3, \cdots \tag{7.39}$$

with $P_n = 0$ for $n < 0$.

Therefore,
Proposition 7.11

$$nP_n = \sum_{\iota=1}^{k}(n\xi_\iota + \lambda_\iota)P_{n-1} \tag{7.40}$$

$$= P_{n-1}\sum_{\iota=1}^{k}\left\{\xi_\iota(n-1+1) + \lambda_\iota\right\}$$

$$\therefore nP_n = P_{n-1}\sum_{\iota=1}^{k}\xi_\iota(n-1) + P_{n-1}\sum_{\iota}^{k}(\xi_\iota + \lambda_\iota); \quad n = 1, 2, 3, \cdots \tag{7.41}$$

Proof
Solving (7.40) iteratively, we have;
When n=1;
Equation (7.40) becomes;

$$P_1 = \sum_{\iota=1}^{k}(\xi_\iota + \lambda_\iota)P_0$$

$$\therefore P_1 = \frac{\sum_{\iota=1}^{k}(\xi_\iota + \lambda_\iota)}{1!}P_0.$$

When n=2;
Equation (7.40) becomes;

$$
\begin{aligned}
2P_2 &= \sum_{\iota=1}^{k}(2\xi_\iota+\lambda_\iota)P_1 \\
\Rightarrow 2P_2 &= \sum_{\iota=1}^{k}(2\xi_\iota+\lambda_\iota)\frac{\sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)}{1}P_0 \\
\therefore P_2 &= \frac{\sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)(2\xi_\iota+\lambda_\iota)}{2!}P_0.
\end{aligned}
$$

When n=3;
Equation (7.40) becomes;

$$
\begin{aligned}
3P_3 &= \sum_{\iota=1}^{k}(3\xi_\iota+\lambda_\iota)P_2 \\
\Rightarrow 3P_3 &= \sum_{\iota=1}^{k}(3\xi_\iota+\lambda_\iota)\frac{\sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)(2\xi_\iota+\lambda_\iota)}{1.2}P_0 \\
P_3 &= \frac{\sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)(2\xi_\iota+\lambda_\iota)(3\xi_\iota+\lambda_\iota)}{3!}P_0.
\end{aligned}
$$

By Mathematical Induction, we have;

$$
\begin{aligned}
P_4 &= \frac{\sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)(2\xi_\iota+\lambda_\iota)(3\xi_\iota+\lambda_\iota)(4\xi_\iota+\lambda_\iota)}{4!}P_0, \\
P_5 &= \frac{\sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)(2\xi_\iota+\lambda_\iota)(3\xi_\iota+\lambda_\iota)(4\xi_\iota+\lambda_\iota)(5\xi_\iota+\lambda_\iota)}{5!}P_0, \\
&. \\
&. \\
&. \\
P_n &= \frac{\sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)(2\xi_\iota+\lambda_\iota)\cdots((n-1)\xi_\iota+\lambda_\iota)(n\xi_\iota+\lambda_\iota)}{n!}P_0. \\
\Rightarrow P_{n-1} &= \frac{\sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)(2\xi_\iota+\lambda_\iota)\cdots((n-1)\xi_\iota+\lambda_\iota)}{(n-1)!}P_0. \\
\frac{P_n}{P_{n-1}} &= \frac{\sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)(2\xi_\iota+\lambda_\iota)\cdots((n-1)\xi_\iota+\lambda_\iota)(n\xi_\iota+\lambda_\iota)}{n.(n-1)!}.\frac{(n-1)!}{\sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)(2\xi_\iota+\lambda_\iota)\cdots((n-1)\xi_\iota+\lambda_\iota)} \\
\therefore \frac{P_n}{P_{n-1}} &= \sum_{\iota=1}^{k}\frac{(n\xi_\iota+\lambda_\iota)}{n}. \quad \text{as required}
\end{aligned}
$$

Multiplying equation (7.41) by s^n and sum the results over n, we obtain;

$$s\sum_{n=1}^{\infty} nP_n s^{n-1} = \sum_{\iota=1}^{k} \xi_\iota s^2 \sum_{n=1}^{\infty}(n-1)P_{n-1}s^{n-2} + \sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)s\sum_{n=1}^{\infty} P_{n-1}s^{n-1}$$

$$s\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \sum_{\iota=1}^{k} \xi_\iota s^2 \sum_{n=1}^{\infty}(n-1)P_{n-1}s^{n-2} + \sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)s\sum_{n=1}^{\infty} P_{n-1}s^{n-1}$$

$$\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \sum_{\iota=1}^{k} \xi_\iota s\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} + \sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)\psi(s)$$

Therefore,

$$(1-s\sum_{\iota=1}^{k}\xi_\iota)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \sum_{\iota=1}^{k}(\xi_\iota+\lambda_\iota)\psi(s) \tag{7.42}$$

7.8 Special Cases of Panjer's - Willmot (1982) Class and their Properties

7.8.1 When k=1, we obtain Panjer's Class

Equation (7.39) becomes Panjer's Class;

$$P_n = \Big(\xi + \frac{\lambda}{n}\Big)P_{n-1}; \quad n = 1,2,3,\cdots$$

and equation (7.42) becomes;

$$\Big(1-\xi s\Big)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \Big(\xi+\lambda\Big)\psi(s).$$

Which is as discussed in section (7.2) - Panjer's (1981) class.

7.8.2 When k=2, we obtain the pgf. which does not satisfy existing probability distributions

$$
\begin{aligned}
P_n &= \sum_{\iota=1}^{k}\Big(\xi_\iota + \frac{\lambda_\iota}{n}\Big)P_{n-1}; \quad n = 1,2,3,\cdots \\
\Rightarrow P_n &= \Big[(\xi_1 + \frac{\lambda_1}{n}) + (\xi_2 + \frac{\lambda_2}{n})\Big]P_{n-1}; \quad n = 1,2,3,\cdots
\end{aligned}
$$

Which does not satisfy existing probability distributions.

7.8.3 When k=4, we obtain the pgf. for Hyper-Geometric distribution

Equation (7.42) becomes;

$$
\Big(1 - s\varphi\Big)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \Phi\psi(s)
$$
$$
\text{where, } \varphi = \sum_{\iota=1}^{4}\xi_\iota \text{ and } \Phi = \sum_{\iota=1}^{4}(\xi_\iota + \lambda_\iota)
$$

Therefore;

$$
\frac{1}{\psi(s)}\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = \frac{\Phi}{1 - \varphi s}
$$

Integrating both sides, we have;

$$
\begin{aligned}
\int \mathrm{d}\ln\psi &= \int \frac{\Phi \mathrm{d}s}{(1-\varphi s)} \\
\ln\psi(s) &= -\ln c_1(1-\varphi s) \\
\psi(s) &= \frac{c_1}{1-\varphi s}
\end{aligned}
$$

Putting s=1;

$$
\begin{aligned}
\psi(1) = 1 &= \frac{c_1}{1-\varphi} \\
\Rightarrow c_1 &= 1-\varphi \\
\therefore \psi(s) &= \frac{1-\varphi}{1-\varphi s}.
\end{aligned}
$$

Which is the probability generating function for a hypergeometric distribution with parameter $1-\varphi$.

The pgf. in hypergeometric term is given by;

$$
\begin{aligned}
\phi(z) &= \frac{{}_1F_1(1;1;\varphi z)}{{}_1F_1(1;1;\varphi)}. \\
\phi^{'}(z) &= \frac{1.\varphi}{1}\frac{{}_1F_1(2;2;\varphi z)}{{}_1F_1(1;1;\varphi)} \\
\phi^{''}(z) &= \frac{2\varphi^2}{2}\frac{{}_1F_1(3;3;\varphi z)}{{}_1F_1(1;1;\varphi)} \\
\text{Let,} \quad \Lambda_\kappa &= \frac{{}_1F_1(1+\kappa;1+\kappa;\varphi)}{{}_1F_1(1;1;\varphi)}, \quad \kappa = 1,2. \\
E(X) &= \phi^{'}(1) \\
\therefore E(X) &= \varphi\Lambda_1. \\
Var(X) &= \phi^{''} + \phi^{'}(1) - [\phi^{'}(1)]^2 \\
&= \varphi^2\Lambda_2 + \varphi\Lambda_1 - \varphi^2\Lambda_1^2 \\
\therefore Var(X) &= \varphi\Lambda_1 + \varphi^2\{\Lambda_2 - \Lambda_1^2\}.
\end{aligned}
$$

8 KAPUR'S RECURSIVE MODEL IN BIRTH AND DEATH PROCESSES AT EQUILIBRIUM

8.1 Introduction

In this chapter, we study the generalized cases of birth and death processes in its steady state using difference differential equations.
Expression for the probability distributions, probability generating function, and moments in terms of hypergeometric functions are derived.
The relations between hypergeometric functions will be used to give their interpretation in terms of probability of extintions for these generalized birth and death processes.

8.2 Kapur's generalized birth and death processes

Kapur's generalized birth and death processes can be expressed as a ratio of polynomials P_ι and $P_{\iota-1}$ in ι, such that the equilibrium solution is given by;

$$\frac{P_\iota}{P_{\iota-1}} = \frac{\lambda_{\iota-1}}{\mu_\iota}, \quad \mu_\iota \neq 0 \tag{8.1}$$

which can be re-written as;

$$\frac{P_\iota}{P_{\iota-1}} = \frac{(\iota+a_1-1)(\iota+a_2-1)\cdots(\iota+a_p-1)\rho}{(\iota+b_1-1)(\iota+b_2-1)\cdots(\iota+b_q-1)}. \tag{8.2}$$

Hence, the pgf. in hypergeometric terms is expressed as;

$$\phi(z) = \frac{{}_{p+1}F_q(1,a_1,\cdots,a_p;b_1,\cdots,b_q;\rho z)}{{}_{p+1}F_q(1,a_1,\cdots,a_p;b_1,\cdots,b_q;\rho)}.$$

Let $a_1,\cdots,a_p = a_\iota$ for ι=1,2,3,$\cdots$,p and $b_1,\cdots,b_q = b_\tau$ for τ=1,2,3,$\cdots$,q.
Therefore;

$$\phi(z) = \frac{{}_{p+1}F_q(1,a_\iota;b_\tau;\rho z)}{{}_{p+1}F_q(1,a_\iota;b_\tau;\rho)}.$$

This implies that;

$$\begin{aligned}
\phi^{'}(z) &= \frac{1}{{}_{p+1}F_q(1,a_\iota;b_\tau;\rho)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_{p+1}F_q(1,a_\iota;b_\tau;\rho z)\}\\
&= \frac{1}{{}_{p+1}F_q(1,a_\iota;b_\tau;\rho)}\frac{a_\iota\rho}{b_\tau}\cdot{}_{p+1}F_q(1,a_\iota+1;b_\tau+1;\rho z)\\
&= \frac{a_\iota\rho}{b_\tau}\frac{{}_{p+1}F_q(2,a_\iota+1;b_\tau+1;\rho z)}{{}_{p+1}F_q(1,a_\iota;b_\tau;\rho)}.\\
\phi^{''}(z) &= \frac{1}{{}_{p+1}F_q(1,a_\iota;b_\tau;\rho)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_{p+1}F_q(2,a_\iota+1;b_\tau+1;\rho z)\}\\
&= \frac{1}{{}_{p+1}F_q(1,a_\iota;b_\tau;\rho)}\frac{a_\iota(a_\iota+1)\rho^2}{b_\tau(b_\tau+1)}\{{}_{p+1}F_q(2+1,a_\iota+2;b_\tau+2;\rho z)\}\\
&= \frac{a_\iota(a_\iota+1)\rho^2}{b_\tau(b_\tau+1)}\frac{{}_{p+1}F_q(3,a_\iota+2;b_\tau+2;\rho z)}{{}_{p+1}F_q(1,a_\iota;b_\tau;\rho)}.\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_{p+1}F_q(1+\kappa,a_\iota+\kappa;b_\tau+\kappa;\rho)}{{}_{p+1}F_q(1,a_\iota;b_\tau;\rho)};\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{a_\iota}{b_\tau}\rho\Lambda_1.
\end{aligned}$$

and

$$\begin{aligned}
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= \frac{a_\iota(a_\iota+1)\rho^2}{b_\tau(b_\tau+1)}\Lambda_2+\frac{a_\iota}{b_\tau}\rho\Lambda_1-\frac{a_\iota^2}{b_\tau^2}\rho^2\Lambda_1^2\\
\therefore Var(X) &= \frac{a_\iota}{b_\tau\rho}\Lambda_1+\frac{a_\iota\rho^2}{b_\tau}\Big[\frac{a_\iota+1}{b_\tau+1}\Lambda_2-\frac{a_\iota}{b_\tau}\Lambda_1^2\Big].
\end{aligned}$$

Moments generating function

The moment generating function is given by;

$$\begin{aligned}
\varphi(\theta) &= \sum_{\iota=0}^{\rho}P_\iota e^{\iota\theta}\\
\therefore \varphi(\theta) &= \frac{{}_{p=1}F_q(1,a_\iota;b_\tau;\rho e^\theta)}{{}_{p=1}F_q(1,a_\iota;b_\tau;\rho)}.
\end{aligned}$$

Which, on replacing e^θ by z, yields the probability generating function $\phi(\ln z)$.
The factorial moment of order δ is given by;

$$\mu_\delta^{'} = \sum_{\iota=\delta}^{\rho} \iota(\iota-1)(\iota-2)\cdots(\iota-\delta+1)P_\iota$$
$$= \sum_{\iota=0}^{\rho} \frac{(\iota+\delta)!}{\iota} P_{\iota+\delta}$$
$$\therefore \mu_\delta^{'} = \delta! \frac{(a_\iota)_\delta}{(b_\tau)_\delta} \rho^\delta . \frac{{}_{p+1}F_q(1+\delta, a_\iota+\delta; b_\tau+\delta; \rho)}{{}_{p+1}F_q(1, a_\iota; b_\tau; \rho)}; \quad \iota = 1,2,3,\cdots,p; \quad \tau = 1,2,3,\cdots,q.$$

8.3 Special Cases of Kapur (1978 a) generalized birth and death processes

8.3.1 Case 1: $\lambda_\iota = (\iota a_1 + b_1)$, $\mu_\iota = (\iota c_1 + d_1)$

For $\lambda_\iota = (\iota a_1 + b_1)$, $\mu_\iota = (\iota c_1 + d_1)$, the basic difference differential equations at equilibrium are:

$$0 = (c_1 + d_1)P_1 - b_1 P_0 \tag{8.3}$$

and

$$0 = ((\iota+1)c_1 + d_1)P_{\iota+1} + ((\iota-1)a_1 + b_1)P_{\iota-1} - \{(\iota a_1 + b_1) + (\iota c_1 + d_1)\}P_\iota; \quad \iota \geq 1 \tag{8.4}$$

Hence,

Proposition 8.1

$$\frac{P_\iota}{P_{\iota-1}} = \frac{(\iota-1)a_1 + b_1}{\iota c_1 + d_1}; \quad \iota \geq 1 \tag{8.5}$$

Proof

Equation (8.3) can be expressed as;

$$\begin{aligned}(c_1+d_1)P_1 &= b_1P_0\\ \Rightarrow P_1 &= \frac{b_1}{c_1+d_1}P_0\\ \therefore P_1 &= \frac{b_1}{c_1+d_1}P_0. \end{aligned} \tag{8.6}$$

Solving (8.4) iteratively, we have;
When ι=1;
Equation (8.4) becomes;

$$0=(2c_1+d_1)P_2-\{(a_1+b_1)+(c_1+d_1)\}P_1+b_1P_0 \tag{8.7}$$

Substituting (8.6) in equation (8.7), we have;

$$\begin{aligned}0 &= (2c_1+d_1)P_2-\{(a_1+b_1)+(c_1+d_1)\}.\frac{b_1}{c_1+d_1}P_0+b_1P_0\\ 0 &= (2c_1+d_1)P_2-\frac{b_1(a_1+b_1)}{(c_1+d_1)}P_0-b_1P_0+b_1P_0\\ 0 &= (2c_1+d_1)P_2-\frac{b_1(a_1+b_1)}{(c_1+d_1)}P_0\\ \Longleftrightarrow (2c_1+d_1)P_2 &= \frac{b_1(a_1+b_1)}{(c_1+d_1)}P_0\\ \Rightarrow P_2 &= \frac{b_1(a_1+b_1)}{(c_1+d_1)(2c_1+d_1)}P_0\\ \therefore P_2 &= \frac{b_1(a_1+b_1)}{(c_1+d_1)(2c_1+d_1)}P_0. \end{aligned} \tag{8.8}$$

When ι=2;
Equation (8.4) becomes;

$$0=(3c_1+d_1)P_3-\{(2a_1+b_1+2c_1+d_1)\}P_2+(a_1+b_1)P_1 \tag{8.9}$$

Substituting (8.6) and (8.8) in equation (8.9), we have;

$$
\begin{aligned}
0 &= (3c_1+d_1)P_3 - \{(2a_1+b_1+2c_1+d_1)\}.\frac{b_1(a_1+b_1)}{(c_1+d_1)(2c_1+d_1)}P_0 \\
&+ (a_1+b_1).\frac{b_1}{c_1+d_1}P_0 \\
0 &= (3c_1+d_1)P_3 - \frac{b_1(a_1+b_1)(2a_1+b_1)}{(c_1+d_1)(2c_1+d_1)} - \frac{b_1(a_1+b_1)}{(c_1+d_1)}P_0 + \frac{b_1(a_1+b_1)}{(c_1+d_1)}P_0 \\
0 &= (3c_1+d_1)P_3 - \frac{b_1(a_1+b_1)(2a_1+b_1)}{(c_1+d_1)(2c_1+d_1)} \\
\Longleftrightarrow (3c_1+d_1)P_3 &= \frac{b_1(a_1+b_1)(2a_1+b_1)}{(c_1+d_1)(2c_1+d_1)} \\
\Rightarrow P_3 &= \frac{b_1(a_1+b_1)(2a_1+b_1)}{(c_1+d_1)(2c_1+d_1)(3c_1+d_1)} \\
\therefore P_3 &= \frac{b_1(a_1+b_1)(2a_1+b_1)}{(c_1+d_1)(2c_1+d_1)(3c_1+d_1)}.
\end{aligned}
$$

Hence, by Mathematical Induction, we have;

$$
\begin{aligned}
P_4 &= \frac{b_1(a_1+b_1)(2a_1+b_1)(3a_1+b_1)(4a_1+b_1)}{(c_1+d_1)(2c_1+d_1)(3c_1+d_1)(4c_1+d_1)}, \\
P_5 &= \frac{b_1(a_1+b_1)(2a_1+b_1)(3a_1+b_1)(4a_1+b_1)(5a_1+b_1)}{(c_1+d_1)(2c_1+d_1)(3c_1+d_1)(4c_1+d_1)(5c_1+d_1)}, \\
&. \\
&. \\
&. \\
P_\imath &= \frac{b_1(a_1+b_1)(2a_1+b_1)\cdots((\imath-2)a_1+b_1)(\imath a_1+b_1)}{(c_1+d_1)(2c_1+d_1)\cdots((\imath-2)c_1+d_1)(\imath c_1+d_1)}P_0. \quad \imath \geq 1 \qquad \textbf{(8.10)}
\end{aligned}
$$

But,

$$\sum_{\imath=0}^{\infty} P_\imath = 1$$

$$P_0 + P_1 + P_2 + P_3 + \cdots = 1$$

$$P_0 \left\{ 1 + \frac{b_1}{c_1 + d_1} + \frac{b_1(a_1 + b_1)}{(c_1 + d_1)(2c_1 + d_1)} + \cdots \right\} = 1$$

$$\therefore P_0 = \frac{1}{\left\{ 1 + \frac{b_1}{c_1+d_1} + \frac{b_1(a_1+b_1)}{(c_1+d_1)(2c_1+d_1)} + \cdots \right\}}.$$

is the probability of ultimate extinction.
Therefore,

$$P_\imath = \frac{b_1(a_1+b_1)(2a_1+b_1)\cdots((\imath-2)a_1+b_1)((\imath-1)a_1+b_1)}{(c_1+d_1)(2c_1+d_1)\cdots((\imath-2)c_1+d_1)(\imath c_1+d_1)} . \frac{1}{1 + \frac{b_1}{c_1+d_1} + \frac{b_1(a_1+b_1)}{(c_1+d_1)(2c_1+d_1)} + \cdots}.$$

From (8.10);

$$P_{\imath-1} = \frac{b_1(a_1+b_1)(2a_1+b_1)\cdots((\imath-2)a_1+b_1)}{(c_1+d_1)(2c_1+d_1)\cdots((\imath-1)c_1+d_1)} P_0.$$

$$\frac{P_\imath}{P_{\imath-1}} = \frac{b_1(a_1+b_1)(2a_1+b_1)\cdots((\imath-2)a_1+b_1)(\imath a_1+b_1)}{(c_1+d_1)(2c_1+d_1)\cdots((\imath-2)c_1+d_1)(\imath c_1+d_1)} . \frac{(c_1+d_1)(2c_1+d_1)\cdots((\imath-1)c_1+d_1)}{b_1(a_1+b_1)(2a_1+b_1)\cdots((\imath-2)a_1+b_1)}$$

$$\therefore \frac{P_\imath}{P_{\imath-1}} = \frac{(\imath-1)a_1+b_1}{\imath c_1+d_1}. \quad \text{as required}$$

(8.10) can be rewritten as;

$$\frac{P_\iota}{P_{\iota-1}} = \frac{(\alpha_1)_\iota}{(\gamma_1)_\iota} = \sum_{\iota=0}^{\infty} \frac{(\alpha_1)_\iota}{(\gamma_1)_\iota}.$$

$$\text{where,} \quad (\alpha_1)_\iota = \alpha_1(\alpha_1+1)(\alpha_1+2)\cdots(\alpha_1+\iota-1), \quad \iota \geq 1; \quad (\alpha_1)_0 = 1,$$
$$= \frac{\Gamma(\alpha_1+\iota)}{\Gamma\alpha_1}.$$

and

$$(\gamma_1)_\iota = \gamma_1(\gamma_1+1)(\gamma_1+2)\cdots(\gamma_1+\iota-1), \quad \iota \geq 1; \quad (\gamma_1)_0 = 1,$$
$$= \frac{\Gamma(\gamma_1+\iota)}{\Gamma\gamma_1}.$$
$$\alpha_1 = \frac{b_1}{c_1}, \quad \gamma_1 = \frac{d_1}{c_1}+1$$

Its probability generating function is given by;

$$\phi(z) = \frac{{}_2F_1(1,\alpha_1;\gamma_1;\rho z)}{{}_2F_1(1,\alpha_1;\gamma_1;\rho)}.$$
$$\phi^{'}(z) = \frac{1}{{}_2F_1(1,\alpha_1;\gamma_1;\rho)} \frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(1,\alpha_1;\gamma_1;\rho z)\}$$

Therefore,

$$\begin{aligned}
\phi^{'}(z) &= \frac{1}{{}_2F_1(1,\alpha_1;\gamma_1;\rho)}\frac{\alpha_1\rho}{\gamma_1}.{}_2F_1(1+1,\alpha_1+1;\gamma_1+1;\rho z)\\
&= \frac{\alpha_1\rho}{\gamma_1}\frac{{}_2F_1(2,\alpha_1+1;\gamma_1+1;\rho z)}{{}_2F_1(1,\alpha_1;\gamma_1;\rho)}.\\
\phi^{''}(z) &= \frac{1}{{}_2F_1(1,\alpha_1;\gamma_1;\rho)}\frac{\alpha_1\rho}{\gamma_1}.\frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(2,\alpha_1+1;\gamma_1+1;\rho z)\}\\
&= \frac{1}{{}_2F_1(1,\alpha_1;\gamma_1;\rho)}\frac{\alpha_1\rho}{\gamma_1}.\frac{(2\alpha_1+1)\rho}{(\gamma_1+1)}\{{}_2F_1(2+1,\alpha_1+2;\gamma_1+2;\rho z)\}\\
&= \frac{2\alpha_1(\alpha_1+1)\rho^2}{\gamma_1(\gamma_1+1)}.\frac{{}_2F_1(3,\alpha_1+2;\gamma_1+2;\rho z)}{{}_2F_1(1,\alpha_1;\gamma_1;\rho)}.\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_2F_1(1+\kappa,\alpha_1+\kappa;\gamma_1+\kappa;\rho)}{{}_2F_1(1,\alpha_1;\gamma_1;\rho)};\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\alpha_1\rho}{\gamma_1}\Lambda_1.
\end{aligned}$$

and

$$\begin{aligned}
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= \frac{2\alpha_1(\alpha_1+1)\rho^2}{\gamma_1(\gamma_1+1)}\Lambda_2+\frac{\alpha_1\rho}{\gamma_1}\Lambda_1-\frac{\alpha_1^2\rho^2}{\gamma_1^2}\Lambda_1^2\\
&= \frac{\alpha_1\rho}{\gamma_1}\Lambda_1+\frac{\alpha_1\rho^2}{\gamma_1}\Big(\frac{2\alpha_1+1}{\gamma_1+1}\Lambda_2-\frac{\alpha_1}{\gamma_1}\Lambda_1^2\Big).
\end{aligned}$$

Moment's generating function

The moment generating function is given by;

$$\varphi(\theta) = \sum_{\iota=0}^{\rho} P_\iota e^\theta$$

$$\therefore \varphi(\theta) = \frac{{}_2F_1(1,\alpha_1;\gamma_1;\rho e^\theta)}{{}_2F_1(1,\alpha_1;\gamma_1;\rho)}.$$

The factorial moment of order δ is given by;

$$\mu_\delta^{'} = \delta!\frac{(\alpha_1)_\delta}{(\gamma_1)_\delta}\rho^\delta\frac{{}_2F_1(1+\delta,\alpha_1+\delta;\gamma_1+\delta;\rho)}{{}_2F_1(1,\alpha_1;\gamma_1;\rho)}.$$

8.3.2 Case 2: $\lambda_\iota = (\iota a_1 + b_1)$, $\mu_\iota = (\iota c_1 + d_1)(\iota c_2 + d_2)$

For $\lambda_\iota = (\iota a_1 + b_1)$, $\mu_\iota = (\iota c_1 + d_1)(\iota c_2 + d_2)$, the basic difference differential equations at equilibrium are:

$$0 = (c_1 + d_1)(c_2 + d_2)P_1 - b_1 P_0 \tag{8.11}$$

$$\begin{aligned} 0 &= \{((\iota+1)c_1 + d_1)((\iota+1)c_2 + d_2)\}P_{\iota+1} + ((\iota-1)a_1 + b_1)P_{\iota-1} \\ &\quad - \{(\iota a_1 + b_1) + (\iota c_1 + d_1)(\iota c_2 + d_2)\}P_\iota; \quad \iota \geq 1 \end{aligned} \tag{8.12}$$

Hence,
Proposition 8.2

$$\frac{P_\iota}{P_{\iota-1}} = \frac{((\iota-1)a_1 + b_1)}{(\iota c_1 + d_1)(\iota c_2 + d_2)}; \quad \iota \geq 1 \tag{8.13}$$

Proof
Equation (8.11) can be expressed as;

$$\begin{aligned} (c_1 + d_1)(c_2 + d_2)P_1 &= b_1 P_0 \\ \Rightarrow P_1 &= \frac{b_1}{(c_1 + d_1)(c_2 + d_2)} P_0 \\ \therefore P_1 &= \frac{b_1}{(c_1 + d_1)(c_2 + d_2)} P_0. \end{aligned} \tag{8.14}$$

Solving (8.12) iteratively, we have;
When $\iota = 1$;
Equation (8.12) becomes;

$$0 = -\{(a_1 + b_1) + (c_1 + d_1)(c_2 + d_2)\}P_1 + (2c_1 + d_1)(2c_2 + d_2)P_2 + b_1 P_0 \tag{8.15}$$

Substituting (8.14) in equation (8.15), we have;

$$0 = -\{(a_1 + b_1) + (c_1 + d_1)(c_2 + d_2)\} . \frac{b_1}{(c_1 + d_1)(c_2 + d_2)} P_0 + (2c_1 + d_1)(2c_2 + d_2)P_2 + b_1 P_0$$

Therefore;

$$
\begin{aligned}
0 &= (2c_1+d_1)(2c_2+d_2)P_2 - \frac{(a_1+b_1)b_1}{(c_1+d_1)(c_2+d_2)}P_0 - b_1P_0 + b_1P_0 \\
0 &= (2c_1+d_1)(2c_2+d_2)P_2 - \frac{(a_1+b_1)b_1}{(c_1+d_1)(c_2+d_2)}P_0 \\
\iff (2c_1+d_1)(2c_2+d_2)P_2 &= \frac{(a_1+b_1)b_1}{(c_1+d_1)(c_2+d_2)}P_0 \\
\Rightarrow P_2 &= \frac{(a_1+b_1)b_1}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)}P_0 \\
\therefore P_2 &= \frac{(a_1+b_1)b_1}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)}P_0. \qquad \textbf{(8.16)}
\end{aligned}
$$

Thus, by Mathematical Induction, we have;

$$
\begin{aligned}
P_3 &= \frac{b_1(a_1+b_1)(2a_1+b_1)}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)(3c_1+d_1)(3c_2+d_2)}P_0, \\
&. \\
&. \\
&. \\
P_\iota &= \frac{b_1(a_1+b_1)(2a_1+b_1)\cdots((\iota-1)a_1+b_1)}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)\cdots(\iota c_1+d_1)(\iota c_2+d_2)}P_0. \qquad \textbf{(8.17)}
\end{aligned}
$$

But;

$$
\begin{aligned}
\sum_{\iota=0}^{\infty} P_\iota &= 1 \\
p_0\Big\{1 + \frac{b_1}{(c_1+d_1)(c_2+d_2)} + \frac{(a_1+b_1)b_1}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)} + \cdots\} &= 1 \\
\therefore P_0 = \frac{1}{1+\frac{b_1}{(c_1+d_1)(c_2+d_2)} + \frac{(a_1+b_1)b_1}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)} + \cdots}&.
\end{aligned}
$$

which is the probability of ultimate extinction.

$$\therefore P_\imath = \frac{b_1(a_1+b_1)(2a_1+b_1)\cdots((\imath-1)a_1+b_1)}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)\cdots(\imath c_1+d_1)(\imath c_2+d_2)}$$
$$x\frac{1}{1+\frac{b_1}{(c_1+d_1)(c_2+d_2)}+\frac{(a_1+b_1)b_1}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)}+\cdots}.$$

From (8.17);

$$P_{\imath-1} = \frac{b_1(a_1+b_1)(2a_1+b_1)\cdots((\imath-2)a_1+b_1)}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)\cdots((\imath-1)c_1+d_1)((\imath-1)c_2+d_2)}P_0.$$
$$\therefore \frac{P_\imath}{P_{\imath-1}} = \frac{((\imath-1)a_1+b_1)}{(\imath c_1+d_1)(\imath c_2+d_2)}. \quad \text{as required}$$

Let,

$$P_\imath = \frac{(\alpha_1)_\imath}{(\gamma_1)_\imath(\gamma_2)_\imath}P_0.$$

where, $(\alpha_1)_\imath = \alpha_1(\alpha_1+1)(\alpha_1+2)\cdots(\alpha_1+\imath-1), \quad \imath \geq 1; \quad (\alpha_1)_0 = 1,$

$$= \frac{\Gamma(\alpha_1+\imath)}{\Gamma\alpha_1}.$$

and

$$(\gamma_1)_\imath = \gamma_1(\gamma_1+1)(\gamma_1+2)\cdots(\gamma_1+\imath-1), \quad \imath \geq 1; \quad (\gamma_1)_0 = 1,$$
$$= \frac{\Gamma(\gamma_1+\imath)}{\Gamma\gamma_1},$$
$$(\gamma_2)_\imath = \frac{\Gamma(\gamma_2+\imath)}{\Gamma\gamma_2}.$$
$$\alpha_1 = \frac{b_1}{c_1}, \quad \gamma_1 = \frac{d_1}{c_1}+1, \quad \gamma_2 = \frac{d_2}{c_2}+1.$$

The pgf. in hypergeometric term is given by;

$$\phi(z) = \frac{{}_2F_1(1,\alpha_1;\gamma_1;\gamma_2;\rho z)}{{}_2F_1(1,\alpha_1;\gamma_1;\gamma_2;\rho)}.$$

This implies that;

$$
\begin{aligned}
\phi^{'}(z) &= \frac{1}{{}_2F_1(1,\alpha_1;\gamma_1;\gamma_2;\rho)}\frac{\alpha_1\rho}{\gamma_1\gamma_2}.{}_2F_1(1+1,\alpha_1+1;\gamma_1+1;\gamma_2+1;\rho z)\\
&= \frac{\alpha_1\rho}{\gamma_1\gamma_2}\frac{{}_2F_1(2,\alpha_1+1;\gamma_1+1;\gamma_2+1;\rho z)}{{}_2F_1(1,\alpha_1;\gamma_1;\gamma_2;\rho)}.\\
\phi^{''}(z) &= \frac{1}{{}_2F_1(1,\alpha_1;\gamma_1;\gamma_2;\rho)}\frac{\alpha_1\rho}{\gamma_1\gamma_2}.\frac{2(\alpha_1+1)\rho}{(\gamma_1+1)(\gamma_2+1)}.{}_2F_1(2+1,\alpha_1+2;\gamma_1+2;\gamma_2+2;\rho z)\\
&= \frac{2\alpha_1(\alpha_1+1)\rho^2}{\gamma_1(\gamma_1+1)\gamma_2(\gamma_2+1)}\frac{{}_2F_1(3,\alpha_1+2;\gamma_1+2;\gamma_2+2;\rho z)}{{}_2F_1(1,\alpha_1;\gamma_1;\gamma_2;\rho)}.\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_2F_1(1+\kappa,\alpha_1+\kappa;\gamma_1+\kappa;\gamma_2+\kappa;\rho z)}{{}_2F_1(1,\alpha_1;\gamma_1;\gamma_2;\rho)};\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\alpha_1\rho}{\gamma_1\gamma_2}\Lambda_1.\\
\text{and}&\\
Var(x) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= \frac{2\alpha_1(\alpha_1+1)\rho^2}{\gamma_1(\gamma_1+1)\gamma_2(\gamma_2+1)}\Lambda_2+\frac{\alpha_1\rho}{\gamma_1\gamma_2}\Lambda_1-[\frac{\alpha_1\rho}{\gamma_1\gamma_2}\Lambda_1]^2\\
\therefore Var(X) &= \frac{\alpha_1\rho}{\gamma_1\gamma_2}\Lambda_1+\frac{\alpha_1\rho^2}{\gamma_1\gamma_2}\Big\{\frac{2(\alpha_1+1)}{(\gamma_1+1)(\gamma_2+1)}\Lambda_2-\frac{\alpha_1}{\gamma_1\gamma_2}\Lambda_1^2\Big\}.
\end{aligned}
$$

Moments generating function

The moment generating function is given by;

$$
\begin{aligned}
\varphi(\theta) &= \sum_{\iota=0}^{\rho}P_\iota e^{\theta}\\
\therefore \varphi(\theta) &= \frac{{}_2F_1(1,\alpha_1;\gamma_1;\gamma_2;\rho e^{\theta})}{{}_2F_1(1,\alpha_1;\gamma_1;\gamma_2;\rho)}.
\end{aligned}
$$

The factorial moment of order δ is given by;

$$
\mu_\delta^{'} = \delta!\frac{(\alpha_1)_\delta}{(\gamma_1)_\delta(\gamma_2)_\delta}\rho^\delta\frac{{}_2F_1(1+\delta,\alpha_1+\delta;\gamma_1+\delta;\gamma_2+\delta;\rho)}{{}_2F_1(1,\alpha_1;\gamma_1;\gamma_2;\rho)}.
$$

8.3.3 Case 3: $\lambda_\imath = (\imath a_1 + b_1)$, $\mu_\imath = (\imath c_1 + d_1)$

For $\lambda_\imath = (\imath a_1 + b_1)$, $\mu_\imath = (\imath c_1 + d_1)$, the basic difference differential equations at equilibrium are:

$$0 = -b_1 P_0 + (c_1 + d_1) P_1 \tag{8.18}$$

$$0 = -\{(\imath a_1 + b_1) + \imath(\imath c_1 + d_1)\} P_\imath + (\imath + 1)\{(\imath + 1)c_1 + d_1\} P_{\imath+1} + ((\imath - 1)a_1 + b_1) P_{\imath-1}; \quad \imath \geq 1 \tag{8.19}$$

Hence,

$$\frac{P_\imath}{P_{\imath-1}} = \frac{(\imath)a_1 + b_1}{\imath(\imath c_1 + d_1)}; \quad \imath \geq 1 \tag{8.20}$$

Proof
Equation (8.31) can be expressed as;

$$\begin{aligned} (c_1 + d_1) P_1 &= b_1 P_0 \\ \Rightarrow P_1 &= \frac{b_1}{(c_1 + d_1)} P_0 \\ \therefore P_1 &= \frac{b_1}{(c_1 + d_1)} P_0. \end{aligned} \tag{8.21}$$

Solving (8.19) iteratively, we have;
When $\imath = 1$;
Equation (8.19) becomes;

$$0 = 2\{2c_1 + d_1\} P_2 - \{(a_1 + b_1) + (c_1 + d_1)\} P_1 + b_1 P_0 \tag{8.22}$$

Substituting (8.21) in equation (8.22), we have;

$$0 = -\{(a_1+b_1)+(c_1+d_1)\}.\frac{b_1}{(c_1+d_1)}P_0 + 2(2c_1+d_1)P_2 + b_1P_0$$

$$0 = 2(2c_1+d_1)P_2 - \frac{b_1(a_1+b_1)}{(c_1+d_1)}P_0 - b_1P_0 + b_1P_0$$

$$0 = 2(2c_1+d_1)P_2 - \frac{b_1(a_1+b_1)}{(c_1+d_1)}P_0$$

$$\Longleftrightarrow 2(2c_1+d_1)P_2 = \frac{b_1(a_1+b_1)}{(c_1+d_1)}P_0$$

$$\Rightarrow P_2 = \frac{b_1(a_1+b_1)}{(c_1+d_1)2(2c_1+d_1)}P_0$$

$$\therefore P_2 = \frac{b_1(a_1+b_1)}{(c_1+d_1)2(2c_1+d_1)}P_0.$$

$$\Rightarrow P_3 = \frac{b_1(a_1+b_1)(2a_1+b_1)}{(c_1+d_1)2(2c_1+d_1)3(3c_1+d_1)}P_0,$$

$$\vdots$$

$$P_\imath = \frac{b_1(a_1+b_1)(2a_1+b_1)\cdots((\imath-1)a_1+b_1)}{(c_1+d_1)2(2c_1+d_1)\cdots\imath(\imath c_1+d_1)}P_0; \quad \imath \geq 1 \tag{8.23}$$

But,

$$\sum_{\imath=0}^{\infty} P_\imath = 1$$

$$\Rightarrow P_0 + P_1 + P_2 \cdots = 1$$

$$P_0\left\{1 + \frac{b_1}{(c_1+d_1)} + \frac{b_1(a_1+b_1)}{(c_1+d_1)2(2c_1+d_1)}\cdots\right\} = 1$$

$$\therefore P_0 = \frac{1}{1+\frac{b_1}{(c_1+d_1)} + \frac{b_1(a_1+b_1)}{(c_1+d_1)2(2c_1+d_1)}\cdots}$$

which is the probability of ultimate extinction.

$$\therefore P_\imath = \frac{b_1(a_1+b_1)(2a_1+b_1)\cdots((\imath-1)a_1+b_1)}{(c_1+d_1)2(2c_1+d_1)\cdots\imath(\imath c_1+d_1)}.\frac{1}{1+\frac{b_1}{(c_1+d_1)} + \frac{b_1(a_1+b_1)}{(c_1+d_1)2(2c_1+d_1)}\cdots}.$$

From (8.23), we have;

$$P_{\imath-1} = \frac{b_1(a_1+b_1)(2a_1+b_1)\cdots((\imath-2)a_1+b_1)}{(c_1+d_1)2(2c_1+d_1)\cdots(\imath-1)((\imath-1)c_1+d_1)}P_0.$$
$$\therefore \frac{P_\imath}{P_{\imath-1}} = \frac{(\imath-1)a_1+b_1}{\imath(\imath c_1+d_1)}.\text{as required}$$

Its pgf. is given by;

$$\begin{aligned}
\phi(z) &= \frac{{}_1F_1(\alpha_1;\gamma_1;\rho z)}{{}_1F_1(\alpha_1;\gamma_1;\rho)}.\\
\phi^{'}(z) &= \frac{1}{{}_1F_1(\alpha_1;\gamma_1;\rho)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_1F_1(\alpha_1;\gamma_1;\rho z)\}\\
&= \frac{1}{{}_1F_1(\alpha_1;\gamma_1;\rho)}\frac{\alpha_1\rho}{\gamma_1}.{}_1F_1(\alpha_1;\gamma_1;\rho z)\\
&= \frac{\alpha_1\rho}{\gamma_1}\frac{{}_1F_1(\alpha_1+1;\gamma_1+1;\rho z)}{{}_1F_1(\alpha_1;\gamma_1;\rho)}.\\
\phi^{''}(z) &= \frac{1}{{}_1F_1(\alpha_1;\gamma_1;\rho)}\frac{\alpha_1\rho}{\gamma_1}.\frac{\mathrm{d}}{\mathrm{d}z}\{{}_1F_1(\alpha_1+1;\gamma_1+1;\rho z)\}\\
&= \frac{1}{{}_1F_1(\alpha_1;\gamma_1;\rho)}\frac{\alpha_1\rho}{\gamma_1}.\frac{(\alpha_1+2)\rho}{(\gamma_1+2)}.{}_1F_1(\alpha_1;\gamma_1;\rho z)\\
&= \frac{\alpha_1(\alpha_1+1)\rho^2}{\gamma_1(\gamma_1+1)}\frac{{}_1F_1(\alpha_1+2;\gamma_1+2;\rho z)}{{}_1F_1(\alpha_1;\gamma_1;\rho)}.\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_1F_1(\alpha_1+\kappa;\gamma_1+\kappa;\rho z)}{{}_1F_1(\alpha_1;\gamma_1;\rho)},\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\alpha_1\rho}{\gamma_1}\Lambda_1.\\
&\text{and}\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}]^2\\
&= \frac{\alpha_1(\alpha_1+1)\rho^2}{\gamma_1(\gamma_1+1)}\Lambda_2+\frac{\alpha_1\rho}{\gamma_1}\Lambda_1-[\frac{\alpha_1\rho}{\gamma_1}\Lambda_1]^2\\
\therefore Var(X) &= \frac{\alpha_1\rho}{\gamma_1}\Lambda_1+\frac{\alpha_1\rho^2}{\gamma_1}\Big(\frac{\alpha_1+1}{\gamma_1+1}\Lambda_2-\frac{\alpha_1}{\gamma_1}\Lambda_1^2\Big).
\end{aligned}$$

Moments generating function

The moment generating function is given by;

$$\varphi(\theta) = \sum_{\iota=0}^{\rho} P_\iota e^{\theta}$$
$$\therefore \varphi(\theta) = \frac{{}_1F_1(\alpha_1;\gamma_1;\rho e^{\theta})}{{}_1F_1(\alpha_1;\gamma_1;\rho)}.$$

The factorial moment of order δ is given by;

$$\mu_\delta^{'} = \delta!\frac{(\alpha_1)_\delta}{(\gamma_1)_\delta}\rho^\delta \frac{{}_2F_1(\alpha_1+\delta;\gamma_1+\delta;\rho)}{{}_2F_1(\alpha_1;\gamma_1;\rho)}.$$

8.3.4 Case 4: $\lambda_\iota = (\iota a_1 + b_1)(\iota a_2 + b_2)$, $\mu_\iota = \iota(\iota c_1 + d_1)$

For $\lambda_\iota = (\iota a_1 + b_1)(\iota a_2 + b_2)$, $\mu_\iota = \iota(\iota c_1 + d_1)$, the basic difference differential equations at equilibrium are:

$$0 = -(b_1 b_2)P_0 + (c_1 + d_1)P_1 \tag{8.24}$$
$$\begin{aligned} 0 = &-\{(\iota a_1 + b_1)(\iota a_2 + b_2) + \iota(\iota c_1 + d_1)\}P_\iota + (\iota+1)\{(\iota+1)c_1 + d_1\}P_{\iota+1} \\ &+ \{((\iota-1)a_1 + b_1)((\iota-1)a_2 + b_2)\}P_{\iota-1}; \quad \iota \geq 1 \end{aligned} \tag{8.25}$$

Hence,
Proposition 8.4

$$\frac{P_\iota}{P_{\iota-1}} = \frac{((\iota-1)a_1 + b_1)((\iota-1)a_2 + b_2)}{\iota(\iota c_1 + d_1)}; \quad \iota \geq 1 \tag{8.26}$$

Proof
Equation (8.24) can be expressed as;

$$\begin{aligned} (c_1 + d_1)P_1 &= (b_1 b_2)P_0 \\ \Rightarrow P_1 &= \frac{b_1 b_1}{c_1 + d_1}P_0 \\ \therefore P_1 &= \frac{b_1 b_1}{c_1 + d_1}P_0. \end{aligned} \tag{8.27}$$

Solving (8.25) iteratively, we have;
When ι=1;
Equation (8.25) becomes;

$$0 = 2(2c_1+d_1)P_2 - \{(a_1+b_1)(a_2+b_2)+(c_1+d_1)\}P_1 + b_1b_2P_0 \tag{8.28}$$

Substituting (8.27) in equation (8.28), we have;

$$\begin{aligned}
0 &= 2(2c_1+d_1)P_2 - \{(a_1+b_1)(a_2+b_2)+(c_1+d_1)\}.\frac{b_1b_1}{c_1+d_1}P_0 + b_1b_2P_0\\
0 &= 2(2c_1+d_1)P_2 - \frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)}P_0 - \frac{b_1b_2}{(c_1+d_1}P_0 + \frac{b_1b_2}{(c_1+d_1}P_0\\
0 &= 2(2c_1+d_1)P_2 - \frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)}P_0\\
\Longleftrightarrow 2(2c_1+d_1)P_2 &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)}P_0\\
\Rightarrow P_2 &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)2(2c_1+d_1)}P_0\\
\therefore P_2 &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)2(2c_1+d_1)}P_0.\\
\Rightarrow P_3 &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)(2a_1+b_1)(2a_2+b_2)}{(c_1+d_1)2(2c_1+d_1)3(3c_1+d_1)}P_0,\\
&.\\
&.\\
&.\\
P_\iota &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)\cdots((\iota-1)a_1+b_1)((\iota-1)a_2+b_2)}{(c_1+d_1)2(2c_1+d_1)\cdots\iota(\iota c_1+d_1)}P_0.
\end{aligned} \tag{8.29}$$

But,

$$\sum_{\iota=0}^{\infty} P_\iota = 1$$
$$\Rightarrow P_0\Big(1+\frac{b_1b_1}{c_1+d_1}+\frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)2(2c_1+d_1)}\cdots\Big)=1$$
$$\therefore P_0=\frac{1}{1+\frac{b_1b_1}{c_1+d_1}+\frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)2(2c_1+d_1)}\cdots}.$$

which is the probability of ultimate extinction.

From (8.29);

$$\Rightarrow P_{\iota-1}=\frac{b_1b_2(a_1+b_1)(a_2+b_2)\cdots((\iota-2)a_1+b_1)((\iota-2)a_2+b_2)}{(c_1+d_1)2(2c_1+d_1)\cdots(\iota-1)((\iota-1)c_1+d_1)}P_0.$$
$$\therefore \frac{P_\iota}{P_{\iota-1}}=\frac{((\iota-1)a_1+b_1)((\iota-1)a_2+b_2)}{\iota(\iota c_1+d_1)};\quad \iota\geq 1.\quad \text{as required}$$

Its pgf. is given by;

$$\phi(z)=\frac{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho z)}{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho)}.$$
$$\phi^{'}(z)=\frac{1}{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho z)\}$$
$$=\frac{1}{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho)}\frac{\flat_1\flat_2\rho}{\gamma_1}.{}_2F_1(\flat_1+1;\flat_2+1;\gamma_1+1;\rho z)$$
$$=\frac{\flat_1\flat_2\rho}{\gamma_1}\frac{{}_2F_1(\flat_1+1;\flat_2+1;\gamma_1+1;\rho z)}{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho)}.$$
$$\phi^{''}(z)=\frac{1}{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(\flat_1+1;\flat_2+1;\gamma_1+1;\rho z)\}$$
$$=\frac{1}{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho)}\frac{\flat_1\flat_2\rho}{\gamma_1}.\frac{(\flat_1+1)(\flat_2+1)\rho}{(\gamma_1+1)}.{}_2F_1(\flat_1+2;\flat_2+2;\gamma_1+2;\rho z)$$
$$=\frac{\flat_1(\flat_1+1)\flat_2(\flat_2+1)\rho^2}{\gamma_1(\gamma_1+1)}\frac{{}_2F_1(\flat_1+2;\flat_2+2;\gamma_1+2;\rho z)}{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho)}.$$

Let, $\Lambda_\kappa=\frac{{}_2F_1(\flat_1+\kappa;\flat_2+\kappa;\gamma_1+\kappa;\rho z)}{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho)};\quad \kappa=1,2.$

Therefore,

$$E(X) = \phi^{'}(1)$$
$$\therefore E(X) = \frac{\flat_1\flat_2\rho}{\gamma_1}\Lambda_1.$$
and
$$Var(X) = \phi^{''}(1) + \phi^{'}(1) - [\phi^{'}(1)]^2$$
$$= \frac{\flat_1(\flat_1+1)\flat_2(\flat_2+1)\rho^2}{\gamma_1(\gamma_1+1)}\Lambda_2 + \frac{\flat_1\flat_2\rho}{\gamma_1}\Lambda_1 - [\frac{\flat_1\flat_2\rho}{\gamma_1}\Lambda_1]^2$$
$$\therefore Var(X) = \frac{\flat_1\flat_2\rho}{\gamma_1}\Lambda_1 + \frac{\flat_1\flat_2\rho^2}{\gamma_1}\Big\{\frac{(\flat_1+1)(\flat_2+1)}{\gamma_1+1}\Lambda_2 - \frac{\flat_1\flat_2}{\gamma_1}\Lambda_1^2\}.$$

Moments generating function

The moment generating function is given by;

$$\varphi(\theta) = \sum_{\iota=0}^{\rho} P_\iota e^\theta$$
$$\therefore \varphi(\theta) = \frac{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho e^\theta)}{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho)}.$$

The factorial moment of order δ is given by;

$$\mu_\delta^{'} = \delta!\frac{(\flat_1)_\delta(\flat_2)_\delta}{(\gamma_1)_\delta}\rho^\delta\frac{{}_2F_1(\flat_1+\delta;\flat_2+\delta;\gamma_1+\delta;\rho)}{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho)}.$$

8.3.5 Case 5: $\lambda_\iota = (\iota a_1 + b_1)(\iota a_2 + b_2)$, $\mu_\iota = (\iota c_1 + d_1)(\iota c_2 + d_2)$

For $\lambda_\iota = (\iota a_1 + b_1)(\iota a_2 + b_2)$, $\mu_\iota = (\iota c_1 + d_1)(\iota c_2 + d_2)$, the basic difference differential equations at equilibrium are:

$$0 = -b_1 b_2 P_0 + (c_1 + d_1)(c_2 + d_2)P_1 \tag{8.30}$$
$$0 = -\{((\iota-1)a_1 + b_1)((\iota-1)a_2 + b_2)\}P_{\iota-1} - \{(\iota a_1 + b_1)(\iota a_2 + b_2) + (\iota c_1 + d_1)(\iota c_2 + d_2)\}P_\iota + \{((\iota+1)c_1 + d_1)((\iota+1)c_2 + d_2)\}P_{\iota+1}; \quad \iota \geq 1 \tag{8.31}$$

Hence,

Proposition 8.5

$$\frac{P_\iota}{P_{\iota-1}} = \frac{((\iota-1)a_1+b_1)((\iota-1)a_2+b_2)}{(\iota c_1+d_1)(\iota c_2+d_2)}; \quad \iota \geq 1 \tag{8.32}$$

Proof

Equation (8.30) can be expressed as;

$$\begin{aligned}(c_1+d_1)(c_2+d_2)P_1 &= b_1b_2P_0\\ \Rightarrow P_1 &= \frac{b_1b_2}{(c_1+d_1)(c_2+d_2)}P_0\\ \therefore P_1 &= \frac{b_1b_2}{(c_1+d_1)(c_2+d_2)}P_0.\end{aligned} \tag{8.33}$$

Solving (8.31) iteratively, we have;

When ι=1;

Equation (8.31) becomes;

$$0 = -b_1b_2P_0 - \{(a_1+b_1)(a_2+b_2)+(c_1+d_1)(c_2+d_2)\}P_1 + \{(2c_1+d_1)(2c_2+d_2)\}P_2 \tag{8.34}$$

Substituting (8.33) in equation (8.34), we have;

$$\begin{aligned}0 &= -b_1b_2P_0 - \{(a_1+b_1)(a_2+b_2)+(c_1+d_1)(c_2+d_2)\}\frac{b_1b_2}{(c_1+d_1)(c_2+d_2)}P_0\\ &\quad + \{(2c_1+d_1)(2c_2+d_2)\}P_2\\ 0 &= (2c_1+d_1)(2c_2+d_2)P_2 - \frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)(c_2+d_2)}P_0 - b_1b_2P_0 + b_1b_2P_0\\ 0 &= (2c_1+d_1)(2c_2+d_2)P_2 - \frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)(c_2+d_2)}P_0\end{aligned}$$

Therefore,

$$
\begin{aligned}
\Longleftrightarrow (2c_1+d_1)(2c_2+d_2)P_2 &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)(c_2+d_2)}P_0 \\
\Rightarrow P_2 &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)}P_0 \\
\therefore P_2 &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)}P_0. \\
\Rightarrow P_3 &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)(2a_1+b_1)(2a_2+b_2)}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)(3c_1+d_1)(3c_2+d_2)}P_0, \\
&. \\
&. \\
&. \\
P_\iota &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)\cdots((\iota-1)a_1+b_1)((\iota-1)a_2+b_2)}{(c_1+d_1)(c_2+d_2)\cdots(\iota c_1+d_1)(\iota c_2+d_2)}P_0. \quad \iota \geq 1
\end{aligned}
\tag{8.35}
$$

But,

$$
\begin{aligned}
\sum_{\iota=0}^{\infty} P_\iota &= 1 \\
\Rightarrow P_0\Big(1+\frac{b_1b_2}{(c_1+d_1)(c_2+d_2)}+\frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)}+\cdots\Big) &= 1 \\
\therefore P_0 = \frac{1}{1+\frac{b_1b_2}{(c_1+d_1)(c_2+d_2)}+\frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)}+\cdots}&.
\end{aligned}
$$

which is the probability of ultimate extinction.

Therefore,

$$
\begin{aligned}
P_\iota &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)\cdots((\iota-1)a_1+b_1)((\iota-1)a_2+b_2)}{(c_1+d_1)(c_2+d_2)\cdots(\iota c_1+d_1)(\iota c_2+d_2)} \\
&x\frac{1}{1+\frac{b_1b_2}{(c_1+d_1)(c_2+d_2)}+\frac{b_1b_2(a_1+b_1)(a_2+b_2)}{(c_1+d_1)(c_2+d_2)(2c_1+d_1)(2c_2+d_2)}+\cdots}.
\end{aligned}
$$

From (8.35), we have;

$$P_{\imath-1} = \frac{b_1 b_2 (a_1+b_1)(a_2+b_2)\cdots((\imath-2)a_1+b_1)((\imath-2)a_2+b_2)}{(c_1+d_1)(c_2+d_2)\cdots((\imath-1)c_1+d_1)((\imath-1)c_2+d_2)} P_0.$$
$$\therefore \frac{P_\imath}{P_{\imath-1}} = \frac{((\imath-1)a_1+b_1)((\imath-1)a_2+b_2)}{(\imath c_1+d_1)(\imath c_2+d_2)}; \quad \imath \geq 1. \quad \text{as required}$$

Its pgf. is given as;

$$\begin{aligned}
\phi(z) &= \frac{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\gamma_2;\rho z)}{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\gamma_2;\rho)}.\\
\phi^{'}(z) &= \frac{1}{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\gamma_2;\rho)} \frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\gamma_2;\rho z)\}\\
&= \frac{1}{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\gamma_2;\rho)} \frac{\flat_1\flat_2\rho}{\gamma_1\gamma_2} . {}_2F_1(1+1,\flat_1+1;\flat_2+1;\gamma_1+1;\rho z)\\
&= \frac{\flat_1\flat_2\rho}{\gamma_1\gamma_2} \frac{{}_2F_1(2,\flat_1+1;\flat_2+1;\gamma_1+1;\gamma_2+1;\rho z)}{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\gamma_2;\rho)}.\\
\phi^{''}(z) &= \frac{1}{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\gamma_2;\rho)} \frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_1(2,\flat_1+1;\flat_2+1;\gamma_1+1;\gamma_2+1;\rho z)\}\\
&= \frac{1}{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\rho)} \frac{\flat_1\flat_2\rho}{\gamma_1\gamma_2} . \frac{2(\flat_1+1)(\flat_2+1)\rho}{(\gamma_1+1)(\gamma_2+1)} . {}_2F_1(3,\flat_1+2;\flat_2+2;\gamma_1+2;\gamma_2+2;\rho z)\\
&= \frac{2\flat_1(\flat_1+1)\flat_2(\flat_2+1)\rho^2}{\gamma_1(\gamma_1+1)\gamma_2(\gamma_2+1)} \frac{{}_2F_1(3,\flat_1+2;\flat_2+2;\gamma_1+2;\gamma_2+2;\rho z)}{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\gamma_2;\rho)}.
\end{aligned}$$

Let, $\Lambda_\kappa = \dfrac{{}_2F_1(1+\kappa,\flat_1+\kappa;\flat_2+\kappa;\gamma_1+\kappa;\gamma_2+\kappa;\rho z)}{{}_2F_1(\flat_1;\flat_2;\gamma_1;\rho)}; \quad \kappa = 1,2.$

Therefore,

$$\begin{aligned}
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\flat_1\flat_2\rho}{\gamma_1\gamma_2}\Lambda_1.
\end{aligned}$$

and

$$\begin{aligned}
Var(X) &= \phi^{''}(1) + \phi^{'}(1) - [\phi^{'}(1)]^2\\
&= \frac{2\flat_1(\flat_1+1)\flat_2(\flat_2+1)\rho^2}{\gamma_1(\gamma_1+1)\gamma_2(\gamma_2+1)}\Lambda_2 + \frac{\flat_1\flat_2\rho}{\gamma_1\gamma_2}\Lambda_1 - [\frac{\flat_1\flat_2\rho}{\gamma_1\gamma_2}\Lambda_1]^2\\
\therefore Var(X) &= \frac{\flat_1\flat_2\rho}{\gamma_1\gamma_2}\Lambda_1 + \frac{\flat_1\flat_2\rho^2}{\gamma_1\gamma_2}\Big\{\frac{(\flat_1+1)(\flat_2+1)}{(\gamma_1+1)(\gamma_2+1)}\Lambda_2 - \frac{\flat_1\flat_2}{\gamma_1\gamma_2}\Lambda_1^2\}.
\end{aligned}$$

Moments generating function

The moment generating function is given by;

$$\varphi(\theta) = \sum_{\iota=0}^{\rho} P_\iota e^\theta$$
$$\therefore \varphi(\theta) = \frac{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\gamma_2;\rho e^\theta)}{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\gamma_2;\rho)}.$$

The factorial moment of order δ is given by;

$$\mu'_\delta = \delta! \frac{(\flat_1)_\delta(\flat_2)_\delta}{(\gamma_1)_\delta(\gamma_2)_\delta}\rho^\delta \frac{{}_2F_1(1,\flat_1+\delta;\flat_2+\delta;\gamma_1+\delta;\gamma_2+\delta;\rho)}{{}_2F_1(1,\flat_1;\flat_2;\gamma_1;\gamma_2;\rho)}.$$

8.3.6 Case 6: $\lambda_\iota = (\iota a_1+b_1)(\iota a_2+b_2)$, $\mu_\iota = \iota(\iota c_1+d_1)(\iota c_2+d_2)$

For $\lambda_\iota = (\iota a_1+b_1)(\iota a_2+b_2)$, $\mu_\iota = \iota(\iota c_1+d_1)(\iota c_2+d_2)$, the basic difference differential equations are:

$$0 = (c_1+d_1)(c_2+d_2)P_1 - b_1b_2P_0 \tag{8.36}$$
$$\begin{aligned}0 &= (\iota+1)\{((\iota+1)c_1+d_1)((\iota+1)c_2+d_2)\}P_{\iota+1} - \{(\iota a_1+b_1)(\iota a_2+b_2)+\iota(\iota c_1+d_1)(\iota c_2+d_2)\}P_\iota \\ &\quad + \{((\iota-1)a_1+b_1)((\iota-1)a_2+b_2)(\iota-1)[((\iota-1)c_1+d_1)((\iota-1)c_2+d_2)]\}P_{\iota-1}; \quad \iota \geq 1\end{aligned} \tag{8.37}$$

Hence,

Proposition 8.6

$$\frac{P_\iota}{P_{\iota-1}} = \frac{((\iota-1)a_1+b_1)((\iota-1)a_2+b_2)}{\iota(\iota c_1+d_1)\iota(\iota c_2+d_2)}; \quad \iota \geq 1 \tag{8.38}$$

Proof

Equation (8.36) can be expressed as;

$$
\begin{aligned}
(c_1+d_1)(c_2+d_2)P_1 &= b_1b_2P_0 \\
\Rightarrow P_1 &= \frac{b_1b_2}{(c_2+d_2)(c_1+d_1)}P_0 \\
\therefore P_1 &= \frac{b_1b_2}{(c_1+d_1)(c_2+d_2)}P_0. \qquad \textbf{(8.39)}
\end{aligned}
$$

Solving (8.37) iteratively, we have;
When $\iota = 1$;
Equation (8.37) becomes;

$$
0 = 2\{(2c_1+d_1)(2c_2+d_2)\}P_2 - \{(c_1+d_1)(c_2+d_2)+(a_1+b_1)(a_2+b_2)\}P_1 + b_1b_2P_0 \qquad \textbf{(8.40)}
$$

Substituting (8.39) in equation (8.40), we have;

$$
\begin{aligned}
0 &= 2\{(2c_1+d_1)(2c_2+d_2)\}P_2 \\
&\quad -\{(c_1+d_1)(c_2+d_2)+(a_1+b_1)(a_2+b_2)\}.\frac{b_1b_2}{(c_2+d_2)(c_1+d_1)}P_0 \\
&\quad +b_1b_2P_0 \\
0 &= 2\{(2c_1+d_1)(2c_2+d_2)\}P_2 - \frac{(a_2+b_2)b_2b_1(a_1+b_1)}{(c_2+d_2)(c_1+d_1)}P_0 \\
&\quad -b_1b_2P_0+b_1b_2P_0 \\
\iff 2\{(2c_1+d_1)(2c_2+d_2)\}P_2 &= \frac{(a_2+b_2)b_2b_1(a_1+b_1)}{(c_2+d_2)(c_1+d_1)}P_0 \\
\Rightarrow P_2 &= \frac{(a_2+b_2)b_2b_1(a_1+b_1)}{(c_2+d_2)(c_1+d_1)2(2c_1+d_1)2(2c_2+d_2)}P_0 \\
\therefore P_2 &= \frac{(a_2+b_2)b_2b_1(a_1+b_1)}{(c_2+d_2)(c_1+d_1)2(2c_1+d_1)2(2c_2+d_2)}P_0. \\
&. \\
&. \\
&. \\
P_\iota &= \frac{b_1b_2(a_1+b_1)(a_2+b_2)\cdots((\iota-1)a_1+b_1)((\iota-1)a_2+b_2)}{(c_1+d_1)2(2c_2+d_2)\cdots\iota(\iota c_1+d_1)\iota(\iota c_2+d_2)}P_0. \\
\therefore \frac{P_\iota}{P_{\iota-1}} &= \frac{((\iota-1)a_1+b_1)((\iota-1)a_2+b_2)}{\iota(\iota c_1+d_1)\iota(\iota c_2+d_2)}; \quad \iota \geq 1. \quad \text{as require}
\end{aligned}
$$

Its pgf. is given by;

$$\begin{aligned}
\phi(z) &= \frac{{}_2F_2(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho z)}{{}_2F_2(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho)}.\\
\phi^{'}(z) &= \frac{1}{{}_2F_2(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_2(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho z)\}\\
&= \frac{1}{{}_2F_2(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho)}\frac{\flat_1\flat_2\rho}{\zeta_1\zeta_2}.{}_2F_2(\flat_1+1;\flat_2+1;\zeta_1+1;\zeta_2+1;\rho z)\\
&= \frac{\flat_1\flat_2\rho}{\zeta_1\zeta_2}\frac{{}_2F_2(\flat_1+1;\flat_2+1;\zeta_1+1;\zeta_2+1;\rho z)}{{}_2F_2(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho)}.\\
\phi^{''}(z) &= \frac{1}{{}_2F_2(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho)}\frac{\flat_1(\flat_1+1)\flat_2(\flat_2+1)\rho^2}{\zeta_1(\zeta_1+1)\zeta_2(\zeta_2+1)}.{}_2F_2(\flat_1+2;\flat_2+2;\zeta_1+2;\zeta_2+2;\rho z)\\
&= \frac{\flat_1(\flat_1+1)\flat_2(\flat_2+1)\rho^2}{\zeta_1(\zeta_1+1)\zeta_2(\zeta_2+1)}\frac{{}_2F_2(\flat_1+2;\flat_2+2;\zeta_1+2;\zeta_2+2;\rho z)}{{}_2F_2(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho)}.\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_2F_2(\flat_1+\kappa;\flat_2+\kappa;\zeta_1+\kappa;\zeta_2+\kappa;\rho)}{{}_2F_2(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho)};\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\flat_1\flat_2\rho}{\zeta_1\zeta_2}\Lambda_1.\\
&\text{and}\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= \frac{\flat_1(\flat_1+1)\flat_2(\flat_2+1)\rho^2}{\zeta_1(\zeta_1+1)\zeta_2(\zeta_2+1)}\Lambda_2+\frac{\flat_1\flat_2\rho}{\zeta_1\zeta_2}\Lambda_1-\Big(\frac{\flat_1\flat_2\rho}{\zeta_1\zeta_2}\Lambda_1\Big)^2\\
\therefore Var(X) &= \frac{\flat_1\flat_2\rho}{\zeta_1\zeta_2}\Lambda_1+\frac{\flat_1\flat_2\rho^2}{\zeta_1\zeta_2}\Big(\frac{(\flat_1+1)(\flat_2+1)}{(\zeta_1+1)(\zeta_2+1)}\Lambda_2-\frac{\flat_1\flat_2}{\zeta_1\zeta_2}\Lambda_1^2\Big).
\end{aligned}$$

Moments generating function

The moment generating function is given by;

$$\begin{aligned}
\varphi(\theta) &= \sum_{\iota=0}^{\rho} P_\iota e^\theta\\
\therefore \varphi(\theta) &= \frac{{}_2F_1(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho e^\theta)}{{}_2F_1(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho)}.
\end{aligned}$$

The factorial moment of order δ is given by;

$$\mu_{\delta}^{'} = \delta! \frac{(\flat_1)_{\delta}(\flat_2)_{\delta}}{(\zeta_1)_{\delta}(\zeta_2)_{\delta}} \rho^{\delta} \frac{{}_2F_1(\flat_1+\delta;\flat_2+\delta;\zeta_1+\delta;\zeta_2+\delta;\rho)}{{}_2F_1(\flat_1;\flat_2;\zeta_1;\zeta_2;\rho)}.$$

8.3.7 Case 7: $\lambda_\imath = (\imath a_1+b_1)(\imath a_2+b_2)(\imath a_3+b_3)$, $\mu_\imath = \imath(\imath c_1+d_1)(\imath c_2+d_2)$

For $\lambda_\imath = (\imath a_1+b_1)(\imath a_2+b_2)(\imath a_3+b_3)$, $\mu_\imath = \imath(\imath c_1+d_1)(\imath c_2+d_2)$, the basic difference differential equations are:

$$0 = (c_1+d_1)(c_2+d_2)P_1 - b_1b_2b_3P_0 \tag{8.41}$$

$$\begin{aligned} 0 &= (\imath+1)\{((\imath+1)c_1+d_1)((\imath+1)c_2+d_2)\}P_{\imath+1} - \{(\imath a_1+b_1)(\imath a_2+b_2)(\imath a_3+b_3)+\imath(\imath c_1+d_1)(\imath c_2+d_2)\}P_\imath \\ &+\{((\imath-1)a_1+b_1)((\imath-1)a_2+b_2)((\imath-1)a_3+b_3)(\imath-1)[((\imath-1)c_1+d_1)((\imath-1)c_2+d_2)]\}P_{\imath-1}; \quad \imath \geq 1 \end{aligned} \tag{8.42}$$

Hence,

Proposition 8.7

$$\frac{P_\imath}{P_{\imath-1}} = \frac{((\imath-1)a_1+b_1)((\imath-1)a_2+b_2)((\imath-1)a_3+b_3)}{\imath(\imath c_1+d_1)\imath(\imath c_2+d_2)}; \quad \imath \geq 1 \tag{8.43}$$

Proof

Equation (8.41) can be expressed as;

$$\begin{aligned} (c_1+d_1)(c_2+d_2)P_1 &= b_1b_2b_3P_0 \\ \Rightarrow P_1 &= \frac{b_1b_2b_3}{(c_2+d_2)(c_1+d_1)}P_0 \\ \therefore P_1 &= \frac{b_1b_2}{(c_1+d_1)(c_2+d_2)}P_0. \end{aligned} \tag{8.44}$$

Solving (8.42) iteratively, we have;

When $\imath = 1$;

Equation (8.42) becomes;

$$0 = 2\{(2c_1+d_1)(2c_2+d_2)\}P_2 - \{(c_1+d_1)(c_2+d_2)+(a_1+b_1)(a_2+b_2)\}P_1 + b_1b_2P_0 \tag{8.45}$$

Substituting (8.44) in equation (8.45), we have;

$$\begin{aligned}
0 &= 2\{(2c_1+d_1)(2c_2+d_2)\}P_2 \\
&\quad -\{(c_1+d_1)(c_2+d_2)+(a_1+b_1)(a_2+b_2)(a_3+b_3)\} \\
&\quad x\frac{b_1b_2b_3}{(c_2+d_2)(c_1+d_1)}P_0 + b_1b_2b_3P_0 \\
0 &= 2\{(2c_1+d_1)(2c_2+d_2)\}P_2 - \frac{(a_3+b_3)(a_2+b_2)b_3b_2b_1(a_1+b_1)}{(c_2+d_2)(c_1+d_1)}P_0 \\
&\quad - b_1b_2b_3P_0 + b_1b_2b_3P_0 \\
\Longleftrightarrow 2\{(2c_1+d_1)(2c_2+d_2)\}P_2 &= \frac{(a_3+b_3)(a_2+b_2)b_3b_2b_1(a_1+b_1)}{(c_2+d_2)(c_1+d_1)}P_0 \\
\Rightarrow P_2 &= \frac{(a_3+b_3)(a_2+b_2)b_3b_2b_1(a_1+b_1)}{(c_2+d_2)(c_1+d_1)2(2c_1+d_1)2(2c_2+d_2)}P_0 \\
\therefore P_2 &= \frac{(a_3+b_3)(a_2+b_2)b_3b_2b_1(a_1+b_1)}{(c_2+d_2)(c_1+d_1)2(2c_1+d_1)2(2c_2+d_2)}P_0. \\
&. \\
&. \\
&. \\
P_\imath &= \frac{b_1b_2b_3(a_1+b_1)(a_2+b_2)\cdots((\imath-1)a_2+b_2)((\imath-1)a_3+b_3)}{(c_1+d_1)2(2c_2+d_2)\cdots\imath(\imath c_1+d_1)\imath(\imath c_2+d_2)}P_0. \\
\therefore \frac{P_\imath}{P_{\imath-1}} &= \frac{((\imath-1)a_1+b_1)((\imath-1)a_2+b_2)((\imath-1)a_3+b_3)}{\imath(\imath c_1+d_1)\imath(\imath c_2+d_2)}; \quad \imath \geq 1. \text{ as require}
\end{aligned}$$

Its pgf. is given by;

$$
\begin{aligned}
\phi(z) &= \frac{{}_2F_2(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho z)}{{}_2F_2(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho)}.\\
\phi^{'}(z) &= \frac{1}{{}_2F_2(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho)}\frac{\mathrm{d}}{\mathrm{d}z}\{{}_2F_2(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho z)\}\\
&= \frac{1}{{}_2F_2(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho)}\frac{\flat_1\flat_2\flat_3\rho}{\zeta_1\zeta_2}.{}_2F_2(\flat_1+1;\flat_2+1;\flat_3+1;\zeta_1+1;\zeta_2+1;\rho z)\\
&= \frac{\flat_1\flat_2\flat_3\rho}{\zeta_1\zeta_2}\frac{{}_2F_2(\flat_1+1;\flat_2+1;\flat_3+1;\zeta_1+1;\zeta_2+1;\rho z)}{{}_2F_2(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho)}.\\
\phi^{''}(z) &= \frac{1}{{}_2F_2(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho)}\frac{\flat_1(\flat_1+1)\flat_2(\flat_2+1)\flat_3(\flat_3+1)\rho^2}{\zeta_1(\zeta_1+1)\zeta_2(\zeta_2+1)}\\
&\quad .{}_2F_2(\flat_1+2;\flat_2+2;\flat_3+2;\flat_3+2;\zeta_1+2;\zeta_2+2;\rho z)\\
&= \frac{\flat_1(\flat_1+1)\flat_2(\flat_2+1)\flat_3(\flat_3+1)\rho^2}{\zeta_1(\zeta_1+1)\zeta_2(\zeta_2+1)}\frac{{}_2F_2(\flat_1+2;\flat_2+2;\flat_3+2;\zeta_1+2;\zeta_2+2;\rho z)}{{}_2F_2(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho)}.\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_2F_2(\flat_1+\kappa;\flat_2+\kappa;\flat_3+\kappa;\zeta_1+\kappa;\zeta_2+\kappa;\rho)}{{}_2F_2(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho)};\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\flat_1\flat_2\flat_3\rho}{\zeta_1\zeta_2}\Lambda_1.\\
&\text{and}\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= \frac{\flat_1(\flat_1+1)\flat_2(\flat_2+1)\flat_3(\flat_3+1)\rho^2}{\zeta_1(\zeta_1+1)\zeta_2(\zeta_2+1)}\Lambda_2+\frac{\flat_1\flat_2\flat_3\rho}{\zeta_1\zeta_2}\Lambda_1-\Big(\frac{\flat_1\flat_2\flat_3\rho}{\zeta_1\zeta_2}\Lambda_1\Big)^2\\
\therefore Var(X) &= \frac{\flat_1\flat_2\flat_3\rho}{\zeta_1\zeta_2}\Lambda_1+\frac{\flat_1\flat_2\flat_3\rho^2}{\zeta_1\zeta_2}\Big(\frac{(\flat_1+1)(\flat_2+1)(\flat_3+1)}{(\zeta_1+1)(\zeta_2+1)}\Lambda_2-\frac{\flat_1\flat_2\flat_3}{\zeta_1\zeta_2}\Lambda_1^2\Big).
\end{aligned}
$$

Moments generating function

The moment generating function is given by;

$$
\begin{aligned}
\varphi(\theta) &= \sum_{\iota=0}^{\rho} P_\iota e^\theta\\
\therefore \varphi(\theta) &= \frac{{}_2F_1(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho e^\theta)}{{}_2F_1(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho)}.
\end{aligned}
$$

The factorial moment of order δ is given by;

$$\mu_{\delta}^{'} = \delta! \frac{(\flat_1)_{\delta}(\flat_2)_{\delta}(\flat_3)_{\delta}}{(\zeta_1)_{\delta}(\zeta_2)_{\delta}} \rho^{\delta} \frac{{}_2F_1(\flat_1+\delta;\flat_2+\delta;\flat_3+\delta;\zeta_1+\delta;\zeta_2+\delta;\rho)}{{}_2F_1(\flat_1;\flat_2;\flat_3;\zeta_1;\zeta_2;\rho)}.$$

9 KEMP'S FAMILIES OF RECURSIVE MODEL IN BIRTH AND DEATH PROCESSES AT EQUILIBRIUM

9.1 Introduction

The family of distributions by Kemp can be interpreted in terms of birth-and-death processes. Suppose that $\frac{P_{n+1}}{P_n}$ is the ratio of polynomials in n with real roots,
Then;

$$\frac{P_{n+1}}{P_n} = \frac{(a_1+n)(a_2+n)\cdots(a_p+n)\lambda}{(b_1+n)(b_2+n)\cdots(b_{q+1}+n)}.$$

where, the coefficients of the highest powers of n in these polynomials can be assumed to be unity without loss of generality concerning the form of the resultant distribution.

9.2 A Ratio of a General Linear Recursive Model

Kemp's (1968) family of distributions is the family of generalized hypergeometric probability distributions have many useful properties.
The ratio of two polynomials is given by;
Proposition 9.1

$$\frac{P_{n+1}}{P_n} = \frac{(a_1+n)(a_2+n)\cdots(a_p+n)\lambda}{(b_1+n)(b_2+n)\cdots(b_{q+1}+n)} \tag{9.1}$$

$$\Rightarrow (b_1+n)(b_2+n)\cdots(b_{q+1}+n)P_{n+1} = (a_1+n)(a_2+n)\cdots(a_p+n)\lambda P_n$$

$$\Rightarrow (b_\tau+n)P_{n+1} = (a_\iota+n)\lambda P_n \tag{9.2}$$

$$\text{where,} \quad \iota = 1,2,\cdots,p; \quad \tau = 1,2,\cdots,q+1$$

Proof
Solving (9.2) iteratively, we have;
When n=0;
Equation (9.2) becomes;

$$
\begin{aligned}
b_\tau P_1 &= \lambda a_\iota P_0 \\
\Rightarrow P_1 &= \frac{a_\iota}{b_\tau}\lambda P_0 \\
\therefore P_1 &= \frac{a_\iota}{b_\tau}\lambda P_0.
\end{aligned} \tag{9.3}
$$

When n=1;
Equation (9.2) becomes;

$$
(b_\tau + 1)P_2 = (a_\iota + 1)\lambda P_1 \tag{9.4}
$$

Substituting (9.3) in equation (9.4), we have;

$$
\begin{aligned}
(b_\tau + 1)P_2 &= (a_\iota + 1)\lambda \frac{a_\iota}{b_\tau}\lambda P_0 \\
\Rightarrow P_2 &= \frac{a_\iota(a_\iota + 1)}{b_\tau(b_\tau + 1)}\lambda^2 P_0 \\
\therefore P_2 &= \frac{a_\iota(a_\iota + 1)}{b_\tau(b_\tau + 1)}\lambda^2 P_0.
\end{aligned} \tag{9.5}
$$

When n=2;
Equation (9.2) becomes;

$$
(b_\tau + 2)P_3 = (a_\iota + 2)\lambda P_2 \tag{9.6}
$$

Substituting (9.5) in equation (9.6), we have;

$$
\begin{aligned}
(b_\tau + 2)P_3 &= (a_\iota + 2)\lambda \frac{a_\iota(a_\iota + 1)}{b_\tau(b_\tau + 1)}\lambda^2 P_0 \\
\Rightarrow P_3 &= \frac{a_\iota(a_\iota + 1)(a_\iota + 2)}{b_\tau(b_\tau + 1)(b_\tau + 2)}\lambda^3 P_0 \\
\therefore P_3 &= \frac{a_\iota(a_\iota + 1)(a_\iota + 2)}{b_\tau(b_\tau + 1)(b_\tau + 2)}\lambda^3 P_0.
\end{aligned}
$$

By Mathematical Induction, we have that;

$$
\begin{aligned}
P_4 &= \frac{a_\iota(a_\iota+1)(a_\iota+2)(a_\iota+3)}{b_\tau(b_\tau+1)(b_\tau+2)(b_\tau+3)}\lambda^4 P_0,\\
P_5 &= \frac{a_\iota(a_\iota+1)(a_\iota+2)(a_\iota+3)(a_\iota+4)}{b_\tau(b_\tau+1)(b_\tau+2)(b_\tau+3)(b_\tau+4)}\lambda^5 P_0,\\
&.\\
&.\\
&.\\
P_n &= \frac{a_\iota(a_\iota+1)(a_\iota+2)\cdots(a_\iota+n-1)}{b_\tau(b_\tau+1)(b_\tau+2)\cdots(b_\tau+n-1)}\lambda^n P_0.
\end{aligned}
\tag{9.7}
$$

But;

$$
\begin{aligned}
P_0+P_1+P_2+P_3+\cdots &= 1\\
P_0+\frac{a_\iota}{b_\tau}\lambda P_0+\frac{a_\iota(a_\iota+1)}{b_\tau(b_\tau+1)}\lambda^2 P_0+\frac{a_\iota(a_\iota+1)(a_\iota+2)}{b_\tau(b_\tau+1)(b_\tau+2)}\lambda^3 P_0+\cdots &= 1\\
P_0\Big(1+\frac{a_\iota}{b_\tau}\lambda+\frac{a_\iota(a_\iota+1)}{b_\tau(b_\tau+1)}\lambda^2+\frac{a_\iota(a_\iota+1)(a_\iota+2)}{b_\tau(b_\tau+1)(b_\tau+2)}\lambda^3+\cdots\Big) &= 1\\
\therefore P_0 &= \frac{1}{\Big(1+\frac{a_\iota}{b_\tau}\lambda+\frac{a_\iota(a_\iota+1)}{b_\tau(b_\tau+1)}\lambda^2+\frac{a_\iota(a_\iota+1)(a_\iota+2)}{b_\tau(b_\tau+1)(b_\tau+2)}\lambda^3+\cdots\Big)}.\\
\therefore P_n = \frac{a_\iota(a_\iota+1)(a_\iota+2)\cdots(a_\iota+n-1)}{b_\tau(b_\tau+1)(b_\tau+2)\cdots(b_\tau+n-1)}\lambda^n. &\frac{1}{\Big(1+\frac{a_\iota}{b_\tau}\lambda+\frac{a_\iota(a_\iota+1)}{b_\tau(b_\tau+1)}\lambda^2+\frac{a_\iota(a_\iota+1)(a_\iota+2)}{b_\tau(b_\tau+1)(b_\tau+2)}\lambda^3+\cdots\Big)}.
\end{aligned}
$$

From (9.7), we have;

$$
\begin{aligned}
P_{n+1} &= \frac{a_\iota(a_\iota+1)(a_\iota+2)\cdots(a_\iota+n-1)(a_\iota+n)}{b_\tau(b_\tau+1)(b_\tau+2)\cdots(b_\tau+n-1)(b_\tau+n)}\lambda^{n+1} P_0.\\
\therefore \frac{P_{n+1}}{P_n} &= \frac{(a_\iota+n)\lambda}{(b_\tau+n)}, \quad \text{as required.}
\end{aligned}
\tag{9.8}
$$

Multiplying (9.2) by s^n and sum the results over n to obtain;

$$
\begin{aligned}
\sum_{n=0}^{\infty}(b_\tau+n)P_{n+1}s^n &= \sum_{n=0}^{\infty}(a_\iota+n)\lambda P_n s^n\\
b_\tau\sum_{n=0}^{\infty}P_{n+1}s^n+\sum_{n=0}^{\infty}(n+1-1)P_{n+1}s^n &= a_\iota\lambda\sum_{n=0}^{\infty}P_n s^n+\lambda\sum_{n=0}^{\infty}(n+1-1)P_n s^n\\
b_\tau\psi(s)+\sum_{n=0}^{\infty}(n+1)P_{n+1}s^n-\sum_{n=0}^{\infty}P_{n+1}s^n &= a_\iota\lambda\psi(s)+\lambda\psi^{'}(s)-\lambda\psi(s)\\
b_\tau\psi(s)+\psi^{'}(s)-\psi(s) &= a_\iota\lambda\psi(s)+\lambda\psi^{'}(s)-\lambda\psi(s)\\
(1-\lambda)\psi^{'}(s) &= (a_\iota\lambda-\lambda+1-b_\tau)\psi(s)\\
(1-\lambda)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} &= (a_\iota\lambda-\lambda+1-b_\tau)\psi(s)\\
\int\frac{\mathrm{d}\psi(s)}{\psi(s)} &= \int\frac{(a_\iota\lambda-\lambda+1-b_\tau)}{(1-\lambda)}\mathrm{d}s\\
\ln\psi(s) &= \int\frac{1}{1-\lambda}.\lambda(a_\iota-1)+1(1-b_\tau)\mathrm{d}s\\
\ln\psi(s) &= \frac{1}{1-\lambda}(\lambda+1)\int(a_\iota-1)(1-b_\tau)\mathrm{d}s\\
\ln\psi(s) &= -(a_\iota-1)(1-b_\tau)s+c\\
\therefore\psi(s) &= e^{(a_\iota-1)(b_\tau-1)s}.e^c
\end{aligned}
$$

Putting s=1;

$$
\begin{aligned}
\psi(1) &= 1 = c_1 e^{(a_\iota-1)(b_\tau-1)}\\
\Rightarrow c_1 &= e^{-\{(a_\iota-1)(b_\tau-1)\}}\\
\therefore\psi(s) &= \frac{e^{(a_\iota-1)(b_\tau-1)s}}{e^{(a_\iota-1)(b_\tau-1)}}.
\end{aligned}
$$

But, ${}_{p+1}F_{q+1}(1,a_1,\cdots,a_p;b_1,\cdots,b_{q+1};\lambda) = {}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda)$

for, $\iota = 1,2,\cdots,p;\quad \tau = 1,2,\cdots,q+1$

Therefore,

$$
\begin{aligned}
{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda) &= 1+\frac{a_\iota}{b_\tau}\frac{\lambda}{1!}+\frac{1(1+1)a_{a\iota}(a_\iota+1)}{b_\tau(b_\tau+1)}\frac{\lambda^2}{2!}+\cdots+\frac{(1+n-1)(a_\iota+n-1)}{(b_\tau+n-1)}\frac{\lambda^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{a_\iota(a_\iota+1)\cdots(a_\iota+n-1)1(1+1)\cdots(1+n-1)}{b_\tau(b_\tau+1)\cdots(b_\tau+n-1)}\frac{\lambda^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(a_\iota+n-1)\cdots(a_\iota+1)a_\iota(1+n-1)1}{(b_\tau+n-1)\cdots(b_\tau+1)b_\tau}\frac{\lambda^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\Gamma(1+n)\Gamma(a_\iota+n)}{\Gamma(b_\tau+n)}\frac{\Gamma(b_\tau)}{\Gamma(a_\iota)\Gamma(1)}\frac{\lambda^n}{n!}
\end{aligned}
$$

Normalizing, we have;

$$
1=\sum_{n=0}^{\infty}\frac{\Gamma(1+n)\Gamma(a_\iota+n)}{\Gamma(b_\tau+n)}\frac{\Gamma(b_\tau)}{\Gamma(a_\iota)\Gamma(1)}.\frac{1}{{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda)}.\frac{\lambda^n}{n!}
$$

Thus, P_n=Prob (N=n)

$$
=\sum_{n=0}^{\infty}\frac{\Gamma(1+n)\Gamma(a_\iota+n)}{\Gamma(b_\tau+n)}\frac{\Gamma(b_\tau)}{\Gamma(a_\iota)\Gamma(1)}.\frac{1}{{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda)}.\frac{\lambda^n}{n!}
$$

Hence, the pgf. is given by;

$$
\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty}P_n z^n\\
&= \sum_{n=0}^{\infty}\frac{\Gamma(1+n)\Gamma(a_\iota+n)}{\Gamma(b_\tau+n)}\frac{\Gamma(b_\tau)}{\Gamma(a_\iota)\Gamma(1)}\frac{(\lambda z)^n}{n!}.\frac{1}{{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda)}\\
\therefore \phi(z) &= \frac{{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda z)}{{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda)},\quad \iota=1,2,\cdots,p;\quad \tau=1,2,\cdots,q+1;\quad \lambda>0. \qquad \textbf{(9.9)}
\end{aligned}
$$

Thus;

$$
\begin{aligned}
\phi^{'}(z) &= \frac{1}{{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda)}\frac{\mathrm{d}}{\mathrm{d}z}.{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda z)\\
&= \frac{1}{{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda)}\frac{a_\iota}{b_\tau}\lambda.{}_{p+1}F_{q+1}(1+1,a_\iota+1;b_\tau+1;\lambda z)\\
\therefore \phi^{'}(z) &= \frac{\lambda a_\iota}{b_\tau}\frac{{}_{p+1}F_{q+1}(2,a_\iota+1;b_\tau+1;\lambda z)}{{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda)}.\\
\phi^{''}(z) &= \frac{1}{{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda)}\frac{2a_\iota(a_\iota+1)}{b_\tau(b_\tau+1)}\lambda^2.{}_{p+1}F_{q+1}(2+1,a_\iota+2;b_\tau+2;\lambda z)\\
\therefore \phi^{''}(z) &= \frac{2a_\iota(a_\iota+1)}{b_\tau(b_\tau+1)}\lambda^2\frac{{}_{p+1}F_{q+1}(3,a_\iota+2;b_\tau+2;\lambda z)}{{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda)}.\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_{p+1}F_{q+1}(1+\kappa,a_\iota+\kappa;b_\tau+\kappa;\lambda)}{{}_{p+1}F_{q+1}(1,a_\iota;b_\tau;\lambda)},\quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\lambda a_\iota}{b_\tau}\Lambda_1.\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-[\phi^{'}(1)]^2\\
&= \frac{2\lambda^2a_\iota(a_\iota+1)}{b_\tau(b_\tau+1)}\Lambda_2+\frac{\lambda a_\iota}{b_\tau}\Lambda_1-[\frac{\lambda a_\iota}{b_\tau}\Lambda_1]^2\\
\therefore Var(X) &= \frac{\lambda a_\iota}{b_\tau}\Lambda_1+\frac{\lambda^2a_\iota}{b_\tau}\Big[\frac{2(a_\iota+1)}{(b_\tau+1)}\Lambda_2-\frac{a_\iota}{b_\tau}\Lambda_1^2\Big].
\end{aligned}
$$

9.3 Special Cases and Properties

9.3.1 When $b_{q+1}=1$

Suppose that in equation (9.2), one of the denominator parameter is equal to unity, say, $b_{q+1}=1$;
Then, we have;
Proposition 9.2

$$
\begin{aligned}
\frac{P_{n+1}}{P_n} &= \frac{(a_1+n)\cdots(a_p+n)\lambda}{(b_1+n)\cdots(b_q+n)(n+1)}\\
\Rightarrow (b_1+n)\cdots(b_q+n)(n+1)P_{n+1} &= (a_1+n)\cdots(a_p+n)\lambda P_n\\
\Rightarrow (b_\tau+n)(n+1)P_{n+1} &= (a_\iota+n)\lambda P_n
\end{aligned}
\tag{9.10}
$$

Proof
Solving (9.10) iteratively, we have;
When n=0;
Equation (9.10) becomes;

$$\begin{aligned} b_\tau P_1 &= a_\iota \lambda P_0 \\ \Rightarrow P_1 &= \frac{a_\iota}{b_\tau}\lambda P_0 \\ \therefore P_1 &= \frac{a_\iota}{b_\tau}\lambda P_0. \end{aligned} \tag{9.11}$$

When n=1;
Equation (9.10) becomes;

$$(b_\tau + 1)2P_2 = (a_\iota + 1)\lambda P_1 \tag{9.12}$$

Substituting (9.11) in equation (9.12), we have;

$$\begin{aligned} (b_\tau + 1)2P_2 &= (a_\iota + 1)\lambda \frac{a_\iota}{b_\tau}\lambda P_0 \\ \Rightarrow P_2 &= \frac{a_\iota(a_\iota + 1)}{b_\tau(b_\tau + 1)}\frac{\lambda^2}{1.2}P_0 \\ \therefore P_2 &= \frac{a_\iota(a_\iota + 1)}{b_\tau(b_\tau + 1)}\frac{\lambda^2}{2!}P_0. \end{aligned} \tag{9.13}$$

When n=2;
Equation (9.10) becomes;

$$(b_\tau + 2)3P_3 = (a_\iota + 2)\lambda P_2 \tag{9.14}$$

Substituting (9.13) in equation (9.14), we have;

$$(b_\tau+2)3P_3=(a_\iota+2)\lambda\frac{a_\iota(a_\iota+1)}{b_\tau(b_\tau+1)}\frac{\lambda^2}{2!}P_0$$
$$\Rightarrow P_3=\frac{a_\iota(a_\iota+1)(a_\iota+2)}{b_\tau(b_\tau+1)(b_\tau+2)}\frac{\lambda^3}{1.2.3}P_0$$
$$\therefore P_3=\frac{a_\iota(a_\iota+1)(a_\iota+2)}{b_\tau(b_\tau+1)(b_\tau+2)}\frac{\lambda^3}{3!}P_0.$$

By Mathematical Induction, we have;

$$P_4=\frac{a_\iota(a_\iota+1)(a_\iota+2)(a_\iota+3)}{b_\tau(b_\tau+1)(b_\tau+2)(b_\tau+3)}\frac{\lambda^4}{4!}P_0,$$
$$P_5=\frac{a_\iota(a_\iota+1)(a_\iota+2)(a_\iota+3)(a_\iota+4)}{b_\tau(b_\tau+1)(b_\tau+2)(b_\tau+3)(b_\tau+4)}\frac{\lambda^5}{5!}P_0,$$
$$\vdots$$
$$P_n=\frac{a_\iota(a_\iota+1)(a_\iota+2)\cdots(a_\iota+n-2)(a_\iota+n-1)}{b_\tau(b_\tau+1)(b_\tau+2)\cdots(b_\tau+n-2)(b_\tau+n-1)}\frac{\lambda^n}{n!}P_0. \quad \textbf{(9.15)}$$

From (9.15), we have;

$$P_{n+1}=\frac{a_\iota(a_\iota+1)(a_\iota+2)\cdots(a_\iota+n-1)(a_\iota+n)}{b_\tau(b_\tau+1)(b_\tau+2)\cdots(b_\tau+n-1)(b_\tau+n)}\frac{\lambda^{n+1}}{(n+1)!}P_0.$$
$$\frac{P_{n+1}}{P_n}=\frac{a_\iota(a_\iota+1)(a_\iota+2)\cdots(a_\iota+n-1)(a_\iota+n)}{b_\tau(b_\tau+1)(b_\tau+2)\cdots(b_\tau+n-1)(b_\tau+n)}\frac{\lambda^n.\lambda^1}{(n+1).n!}$$
$$x\frac{b_\tau(b_\tau+1)(b_\tau+2)\cdots(b_\tau+n-2)(b_\tau+n-1)}{a_\iota(a_\iota+1)(a_\iota+2)\cdots(a_\iota+n-2)(a_\iota+n-1)}\frac{n!}{\lambda^n}$$
$$\therefore\frac{P_{n+1}}{P_n}=\frac{(a_\iota+n)\lambda}{(b_\tau+n)(n+1)}, \quad \text{as required.}$$

The probability generating function in hypergeometric terms is given by;

$$
\begin{aligned}
\phi(z) &= \frac{{}_pF_q(1,a_\iota;b_\tau;\lambda z)}{{}_pF_q(1,a_\iota;b_\tau;\lambda)}, \quad \iota=1,2,\cdots,p; \quad \tau=1,2,\cdots,q; \quad \lambda>0.\\
\phi^{'}(z) &= \frac{1}{{}_pF_q(1,a_\iota;b_\tau;\lambda)}\frac{\mathrm{d}}{\mathrm{d}z}.{}_pF_q(1,a_\iota;b_\tau;\lambda z)\\
&= \frac{1}{{}_pF_q(1,a_\iota;b_\tau;\lambda)}\frac{a_\iota}{b_\tau}\lambda.{}_pF_q(1+1,a_\iota+1;b_\tau+1;\lambda z)\\
\therefore \phi^{'}(z) &= \frac{\lambda a_\iota}{b_\tau}\frac{{}_pF_q(2,a_\iota+1;b_\tau+1;\lambda z)}{{}_pF_q(1,a_\iota;b_\tau;\lambda)}.\\
\phi^{''}(z) &= \frac{1}{{}_pF_q(1,a_\iota;b_\tau;\lambda)}\frac{2a_\iota(a_\iota+1)}{b_\tau(b_\tau+1)}\lambda^2.{}_pF_q(2+1,a_\iota+2;b_\tau+2;\lambda z)\\
\therefore \phi^{''}(z) &= \frac{2a_\iota(a_\iota+1)}{b_\tau(b_\tau+1)}\lambda^2\frac{{}_pF_q(3,a_\iota+2;b_\tau+2;\lambda z)}{{}_pF_q(1,a_\iota;b_\tau;\lambda)}.\\
\text{Let,}\quad \Lambda_\kappa &= \frac{{}_pF_q(1+\kappa,a_\iota+\kappa;b_\tau+\kappa;\lambda)}{{}_pF_q(1,a_\iota;b_\tau;\lambda)}, \quad \kappa=1,2.\\
E(X) &= \phi^{'}(1)\\
\therefore E(X) &= \frac{\lambda a_\iota}{b_\tau}\Lambda_1.\\
Var(X) &= \phi^{''}(1)+\phi^{'}(1)-\{\phi^{'}(1)\}^2\\
&= \frac{2\lambda^2 a_\iota(a_\iota+1)}{b_\tau(b_\tau+1)}\Lambda_2+\frac{\lambda a_\iota}{b_\tau}\Lambda_1-\{\frac{\lambda a_\iota}{b_\tau}\Lambda_1\}^2\\
\therefore Var(X) &= \frac{\lambda a_\iota}{b_\tau}\Lambda_1+\frac{\lambda^2 a_\iota}{b_\tau}\Big\{\frac{2(a_\iota+1)}{(b_\tau+1)}\Lambda_2-\frac{a_\iota}{b_\tau}\Lambda_1^2\Big\}.
\end{aligned}
$$

10 CONFLUENT HYPERGEOMETRIC DISTRIBUTIONS AND THEIR GENERALIZATIONS

10.1 Introduction

In this chapter, a confluent hypergeometric series distribution has been constructed using Kummer's series.
Its generalization is based on a compound distribution; i.e.,a distribution of a random sum of independent and identically distributed random variables.
An integral representation of Kummer's series has been used to construct a confluent hypergeometric continuous distribution.
Its generalization is based on a confluent hypergeometric series with two variables.Some properties and special cases of theses distributions have also been discussed.
The results in this chapter will help us in solving difference-differential equations of birth and death processes at equilibrium.

10.2 Bhattacharya's Confluent Hypergeometric Series Distribution

10.2.1 Construction and Properties

Kummer's confluent hypergeometric series is given by:

$$\begin{aligned}
{}_1F_1(\nu;\lambda;\eta) &= 1+\frac{\nu}{\lambda}\,\frac{\eta}{1!}+\frac{\nu(\nu+1)}{\lambda(\lambda+1)}\,\frac{\eta^2}{2!}+\frac{\nu(\nu+1)(\nu+2)}{\lambda(\lambda+1)(\lambda+2)}\,\frac{\eta^3}{3!}+\cdots\\
&=\sum_{x=0}^{\infty}\frac{\nu(\nu+1)(\nu+2)\cdots(\nu+x-1)}{\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+x-1)}\,\frac{\eta^x}{x!}\\
&=\sum_{x=0}^{\infty}\frac{(\nu+x-1)(\nu+x-2)\cdots(\nu+2)(\nu+1)\nu}{(\lambda+x-1)(\lambda+x-2)\cdots(\lambda+2)(\lambda+1)\lambda}\,\frac{\Gamma(\nu)}{(\lambda)}\,\frac{\Gamma(\lambda)}{\Gamma(\nu)}\,\frac{\eta^x}{x!}\\
&=\sum_{x=0}^{\infty}\frac{\Gamma(\nu+x)}{(\lambda+x)}\,\frac{\Gamma(\lambda)}{\Gamma(\nu)}\,\frac{\eta^x}{x!} \qquad (10.1)
\end{aligned}$$

For normalizing,we have;

$$1=\sum_{x=0}^{\infty}\frac{\Gamma(\nu+x)}{(\lambda+x)}\,\frac{\Gamma(\lambda)}{\Gamma(\nu)}\,\frac{1}{{}_1F_1(\nu;\lambda;\eta)}\frac{\eta^x}{x!}$$

Therefore,

$$f(x) = \text{Prob}(X = x)$$
$$= \sum_{x=0}^{\infty} \frac{\Gamma(\nu+x)}{(\lambda+x)} \frac{\Gamma(\lambda)}{\Gamma(\nu)} \frac{1}{{}_1F_1(\nu;\lambda;\eta)} \frac{\eta^x}{x!} \quad \text{for } x = 0,1,2,\cdots; \quad \nu > 0, \quad \lambda > 0, \quad \eta > 0 \tag{10.2}$$

This result is due to Bhattacharya (1966).The probability generating function (pgf) is given by:

$$\begin{aligned}
F(s) &= \sum_{x=0}^{\infty} f(x)s^x \\
&= \sum_{x=0}^{\infty} \frac{\Gamma(\nu+x)}{(\lambda+x)} \frac{\Gamma(\lambda)}{\Gamma(\nu)} \frac{(\eta s)^x}{x!} \frac{1}{{}_1F_1(\nu;\lambda;\eta)} \\
&= \frac{{}_1F_1(\nu;\lambda;\eta s)}{{}_1F_1(\nu;\lambda;\eta)} \qquad (10.3) \\
\therefore F'(s) &= \frac{1}{{}_1F_1(\nu;\lambda;\eta)} \frac{d}{ds}({}_1F_1(\nu;\lambda;\eta s)) \\
&= \frac{1}{{}_1F_1(\nu;\lambda;\eta)} \frac{\nu}{\lambda}\eta *_1 F_1(\nu+1;\lambda+1;\eta s) \\
&\text{and} \\
F''(s) &= \frac{1}{{}_1F_1(\nu;\lambda;\eta)} \frac{\nu}{\lambda}\eta * \frac{\nu+1}{\lambda+1} * \eta *_1 F_1(\nu+2;\lambda+2;\eta s) \\
&= \frac{\eta^2}{{}_1F_1(\nu;\lambda;\eta)} \frac{\nu(\nu+1)}{\lambda(\lambda)+1} *_1 F_1(\nu+2;\lambda+2;\eta s)
\end{aligned}$$

Let

$$\Lambda_j = \frac{{}_1F_1(\nu+j;\lambda+j;\eta)}{{}_1F_1(\nu;\lambda;\eta)}; \text{ for } \quad j = 1,2. \tag{10.4}$$

$$\begin{aligned}
\therefore E(X) = F'(1) &= \frac{\eta\nu}{\lambda} \frac{{}_1F_1(\nu+1;\lambda+1;\eta)}{{}_1F_1(\nu;\lambda;\eta)} \\
&= \frac{\eta\nu}{\lambda}\Lambda_1 \qquad (10.5) \\
&\text{and} \\
Var(X) &= F''(1) + F'(1) - [F'(1)]^2 \\
&= \eta^2 \frac{\nu(\nu+1)}{\lambda(\lambda+1)}\Lambda_2 + \frac{\eta\nu}{\lambda}\Lambda_1 - \left(\frac{\eta\nu}{\lambda}\Lambda_1\right)^2 \\
&= \frac{\eta\nu}{\lambda}\Lambda_1 + \frac{\eta^2\nu}{\lambda}\left[\frac{\nu+1}{\lambda+1}\Lambda_2 - \frac{\nu}{\lambda}\Lambda_1^2\right] \qquad (10.6)
\end{aligned}$$

10.2.2 Special Cases

(i) Poisson Distribution

$$
\begin{aligned}
e^{\theta} &= \sum_{x=0}^{\infty}\frac{\theta^x}{x!} = \sum_{x=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+x-1)}{a(a+1)(a+2)\cdots(a+x-1)}\ \frac{\theta^x}{x!}\\
\therefore &=_1 F_1(a;a;\theta)\\
\therefore &= e^{-\theta}\frac{\theta^x}{x!} = \frac{1}{e^{\theta}}\frac{\theta^x}{x!}, \quad x=0,1,2,\cdots;\theta>0\\
\therefore f(x) &= \frac{1}{{}_1F_1(a;a;\theta)}\ \frac{\theta^x}{x!}\\
&= \frac{\Gamma(a+x)}{\Gamma(a+x)}\ \frac{\Gamma(a)}{\Gamma(a)}\ \frac{1}{{}_1F_1(a;a;\theta)}\ \frac{\theta^x}{x!} \text{ for } x=0,1,2,\cdots;\theta>0 \qquad \textbf{(10.7)}
\end{aligned}
$$

Thus, $\nu = \lambda = a$ and $\eta = \theta$ in (10.2)
The pgf is;

$$
F(s) = \frac{{}_1F_1(a;a;\theta s)}{{}_1F_1(a;a;\theta)} \qquad \textbf{(10.8)}
$$

$$
E(X) = \theta\Lambda_1 = \theta\frac{{}_1F_1(a;a;\theta s)}{{}_1F_1(a;a;\theta)} = \theta \qquad \textbf{(10.9)}
$$

and

$$
\begin{aligned}
Var(X) &= \theta\Lambda_1 + \theta^2(\Lambda_2 - \Lambda_1^2)\\
&= \theta + \theta^2\Lambda_2 - \theta^2\\
&= \theta + \theta^2 - \theta^2\\
&= \theta \qquad \textbf{(10.10)}
\end{aligned}
$$

(ii) Displaced Poisson Distribution
From the expansion of e^{θ}, let us have a partial sum;

$$
\begin{aligned}
S &= \frac{\theta^{r+1}}{(r+1)!} + \frac{\theta^{r+2}}{(r+2)!} + \frac{\theta^{r+3}}{(r+3)!} + \cdots\\
&= \sum_{x=0}^{\infty}\frac{\theta^{x+r+1}}{(x+r+1)!}\\
\therefore 1 &= \sum_{x=0}^{\infty}\frac{\theta^{x+r+1}}{(x+r+1)!\,S}\\
\therefore f(x) &= \frac{\theta^{x+r+1}}{(x+r+1)!\sum_{x=0}^{\infty}\frac{\theta^{x+r+1}}{(x+r+1)!}} \text{ for } x=0,1,2,\cdots;\theta>0 \qquad \textbf{(10.11)}
\end{aligned}
$$

and r= positive integer.
Which is called Displaced Poisson distribution introduced by Staff (1964).

$$
\begin{aligned}
f(x) &= \frac{\theta^{x+r+1}}{(x+r+1)!\sum_{x=0}^{\infty}\frac{\theta^{x+r+1}}{(x+r+1)!}} \\
&= \frac{\theta^{x+r+1}}{\Gamma(x+r)}\frac{1}{\sum_{x=0}^{\infty}\frac{\theta^{x+r+1}}{\Gamma(x+r)}} \\
&= \frac{\theta^{x}}{\Gamma(x+r)}\frac{1}{\sum_{x=0}^{\infty}\frac{\theta^{x}}{\Gamma(x+r)}} \\
&= \frac{\theta^{x}}{\Gamma(x+r)}\frac{1}{\sum_{x=0}^{\infty}\frac{\theta^{x}}{x!}\frac{x!}{\Gamma(x+r)}} \\
&= \frac{\theta^{x}}{\Gamma(x+r)}\frac{1}{\sum_{x=0}^{\infty}\frac{\Gamma(1+x)}{\Gamma(x+r)}\frac{\theta^{x}}{x!}} \\
&= \frac{\theta^{x}}{\Gamma(x+r)}\frac{1}{\sum_{x=0}^{\infty}\frac{\Gamma(1+x)}{\Gamma(x+r)}\frac{\theta^{x}}{x!}} \\
&= \frac{\theta^{x}}{\Gamma(x+r)}\,\frac{\Gamma(r)}{\Gamma(r)}\frac{1}{\sum_{x=0}^{\infty}\frac{\Gamma(1+x)}{\Gamma(x+r)}\frac{\Gamma(r)}{\Gamma(r)}\frac{\theta^{x}}{x!}} \\
&= \frac{\theta^{x}\Gamma(r)}{\Gamma(x+r)}\,\frac{1}{{}_1F_1(1;r;\theta)} \\
&= x!\frac{\Gamma(r)}{\Gamma(x+r)}\,\frac{1}{{}_1F_1(1;r;\theta)}\frac{\theta^{x}}{x!} \\
&= \frac{\Gamma(1+x)}{\Gamma(r+x)}\frac{\Gamma(r)}{\Gamma(1)}\,\frac{1}{{}_1F_1(1;r;\theta)}\frac{\theta^{x}}{x!} \text{ for } x=0,1,2,\cdots;\theta>0
\end{aligned} \tag{10.12}
$$

and r=postive integer.
This is $f(x)$ given in (10.2) where $v=1,\lambda=1,$ a positive integer and $\eta=\theta>0$

$$F(s) = \frac{{}_1F_1(1;r;\theta s)}{{}_1F_1(1;r;\theta)} \tag{10.13}$$

$$E(X) = \frac{\theta}{r}\Lambda_1 \tag{10.14}$$

and

$$Var(X) = \frac{\theta}{r}\Lambda_1 + \frac{\theta^2}{r}[\frac{2}{r+1}\Lambda_2 - \frac{1}{r}\Lambda_1^2] \tag{10.15}$$

where

$$\Lambda_j = \frac{{}_1F_1(1+j;r+j;\theta)}{{}_1F_1(1;r;\theta)} \text{ for } j=1,2 \tag{10.16}$$

(iii) Hyper-Poisson Distribution

In the displaced Poisson distribution,we have $\lambda=r$ is a positive integer.

For any $\lambda > 0$, we have a hyper Poisson distribution.
Bardwell and Crow (1964) termed the distribution Sub-Poisson for $\lambda < 1$, Super-Poisson for $\lambda > 1$.
Thus,the hyper-Poisson distribution is given by;

$$f(x) = \frac{\Gamma(1+x)}{\Gamma(\lambda+x)}\frac{\Gamma(\lambda)}{\Gamma(1)}\frac{1}{{}_1F_1(1;\lambda;\theta)}\frac{\theta^x}{x!} \quad \text{for } x = 0,1,2,\cdots;\lambda > 0, \quad \theta > 0 \tag{10.17}$$

$$\therefore F(s) = \frac{{}_1F_1(1;\lambda;\theta s)}{{}_1F_1(1;\lambda;\theta)} \tag{10.18}$$

$$E(X) = \frac{\theta}{\lambda}\Lambda_1 \tag{10.19}$$

and

$$Var(X) = \frac{\theta}{\lambda}\Lambda_1 + \frac{\theta^2}{\lambda}[\frac{2}{\lambda+1}\Lambda_2 - \frac{1}{\lambda}\Lambda_1^2] \tag{10.20}$$

where

$$\Lambda_j = \frac{{}_1F_1(1+j;\lambda+j;\theta)}{{}_1F_1(1;\lambda;\theta)} \quad \text{for } j = 1,2 \tag{10.21}$$

Remark (10.1) Barton (1966) point out that, a hyper-Poisson distribution can be obtained by considering a truncated Pearson type III mixture of Poisson distribution as shown below:
We shall first construct the truncated Pearson type III as a mixing distribution.
The Pearson differential equation is given by;

$$\frac{1}{y}\frac{dy}{dx} = \frac{-(a+x)}{c_0 + c_1 x + c_2 x^2}$$

Where $y = f(x)$, is the pdf of a random variable X and a, c_0, c_1, c_2 are parameters.
Pearson type III corresponds to the case of $c_2 = 0$.

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{-(a+x)}{c_1x+c_0} = -\frac{1}{c_1}\left[\frac{x+a}{x+\frac{c_0}{c_1}}\right]$$
$$= -\frac{1}{c_1}\left[1+\frac{a-\frac{c_0}{c_1}}{x+\frac{c_0}{c_1}}\right]$$
$$= -\frac{1}{c_1} - \frac{(a-\frac{c_0}{c_1})}{c_1x+c_0}$$
$$= -\frac{1}{c_1} + \frac{\frac{c_0}{c_1}-a}{c_1x+c_0}$$
$$\therefore \frac{d}{dx}\log y = -\frac{1}{c_1} + \frac{\frac{c_0}{c_1}-a}{c_1x+c_0}$$

$$\therefore \log y = \int\{-\frac{1}{c_1}+\frac{\frac{c_0}{c_1}-a}{c_1x+c_0}\}dx$$
$$= -\frac{x}{c_1} + -\frac{1}{c_1} + \frac{\frac{c_0}{c_1}-a}{c_1}\log(c_1x+c_0)+\log k$$
$$= -\frac{x}{c_1} + \log k(c_1x+c_0)^{\alpha}$$

where

$$\alpha = \frac{\frac{c_0}{c_1}-a}{c_1}$$
$$\therefore y = e^{-\frac{x}{c_1}}k(c_1x+c_0)^{\alpha}; \quad c_1 \neq 0$$

If $c_1 > 0$, then $c_1x+c_0 > 0 \Rightarrow x > -\frac{c_0}{c_1}$.
If $c_1 < 0$, let $c_1 = -\frac{1}{\theta}$ where $\frac{1}{\theta} > 0$
Then,

$$c_1x+c_0 > 0$$
$$\Rightarrow -\frac{x}{\theta}+c_0 > 0$$
$$\Rightarrow -\frac{x}{\theta} > -c_0$$
$$\Rightarrow \frac{x}{\theta} < c_0$$
$$\therefore c_1 < 0 \Rightarrow < \theta c_0$$

Which is the case we wish to consider.

$$\therefore y = e^{\theta x} k(c_0 - \frac{x}{\theta})^{\alpha}$$
$$= e^{\theta x} k(\frac{\theta c_0 - x}{\theta})^{\alpha}$$
$$= \frac{k}{\theta^{\alpha}} e^{\theta x}(1-x)^{\alpha}, \quad 0 < x < 1$$
$$\text{where } \theta c_0 = 1$$
$$\text{But } y = 1$$
$$\therefore \int_0^1 f(x)dx = \frac{k}{\theta^{\alpha}} \int_0^1 e^{\theta x}(1-x)^{\alpha} dx$$
$$i.e$$
$$1 = \frac{k}{\theta^{\alpha}} \int_0^1 e^{\theta x}(1-x)^{\alpha} dx$$
$$\therefore \frac{\theta^{\alpha}}{k} = \int_0^1 x^{1-1}(1-x)^{\alpha+2-1-1} e^{\theta x} dx$$
$$= B(1, \alpha+1) \int_0^1 \frac{x^{1-1}(1-x)^{\alpha+2-1-1} e^{\theta x}}{B(1, \alpha+1)} dx$$
$$= B(1, \alpha+1)\ {}_1F_1(1; \alpha+2; \theta)$$
$$\therefore \frac{k}{\theta^{\alpha}} = \frac{1}{B(1, \alpha+1)\ {}_1F_1(1; \alpha+2; \theta)}$$
$$\therefore f(x) = \frac{e^{\theta x}(1-x)^{\alpha}}{B(1, \alpha+1)\ {}_1F_1(1; \alpha+2; \theta)}, \quad 0 < x < 1$$

Let $\alpha + 2 = \beta \Rightarrow \alpha = \beta - 2$

$$\therefore f(x) = \frac{e^{\theta x}(1-x)^{\beta-2}}{B(1, \beta-1)\ {}_1F_1(1; \beta; \theta)}, \quad 0 < x < 1; \quad \beta > 0$$

as given by Johnson et al (2005 page 370).As a mixing distribution,we shall use the notations;

$$g(t) = \frac{e^{\theta t}(1-t)^{\lambda-2}}{B(1, \lambda-1)\ {}_1F_1(1; \lambda; \theta)}, \quad \text{for} \quad t > 0 \tag{10.23}$$

which is the truncated Pearson type III.
The mixture is given by

$$\begin{aligned}
f(x) &= \int_0^1 e^{-\theta t}\frac{(\theta t)^x}{x!}g(t)dt \\
&= \int_0^1 e^{-\theta t}\frac{(\theta t)^x}{x!}\ \frac{e^{\theta t}(1-t)^{\lambda-2}}{B(1,\lambda-1)\ {}_1F_1(1;\lambda;\theta)}dt \\
&= \int_0^1 \frac{\theta^x}{x!}\ \frac{t^x(1-t)^{\lambda-2}}{B(1,\lambda-1)\ {}_1F_1(1;\lambda;\theta)}dt \\
&= \frac{\theta^x}{x!}\ \frac{1}{B(1,\lambda-1)}\ \frac{1}{{}_1F_1(1;\lambda;\theta)}\int_0^1 t^x(1-t)^{\lambda-1-1}dt \\
&= \frac{\theta^x}{x!}\ \frac{1}{B(1,\lambda-1)}\ \frac{1}{{}_1F_1(1;\lambda;\theta)}B(x+1,\lambda-1) \\
&= \frac{\theta^x}{x!}\ \frac{\Gamma(\lambda)}{\Gamma(1)\Gamma(\lambda-1)}\ \frac{1}{{}_1F_1(1;\lambda;\theta)}\ \frac{\Gamma(x+1)\Gamma(\lambda-1)}{\lambda+x} \\
\therefore f(x) &= \frac{\Gamma(1+x)}{\Gamma(\lambda+x)}\ \frac{\Gamma(\lambda)}{\Gamma(1)}\ \frac{1}{{}_1F_1(1;\lambda;\theta)}\ \frac{\theta^x}{x!} \quad \text{for } x=0,1,2,\cdots;\ \theta>0
\end{aligned}$$

Which is the hyper-Poisson distribution.

Remark (10.2)

The truncated Pearson type III distribution can also be derived from a gamma distribution truncated from above as follows:

A two-parameter gamma distribution is given by;

$$h(y) = \frac{\theta^b}{\Gamma b}e^{-\theta y}y^{b-1}, \quad y>0; \quad \theta,b>0.$$

Consider the integral;

$$\begin{aligned}
I &= \int_0^p e^{-\theta y}y^{b-1}dy \\
\text{Let } \theta y = x \Rightarrow y &= \frac{x}{\theta} \text{ and } dy = \frac{dx}{\theta} \\
\therefore I &= \int_0^{\theta p} e^{-x}(\frac{x}{\theta})^{b-1}\frac{dx}{\theta} \\
&= \frac{1}{\theta^b}\int_0^{\theta p} e^{-x}x^{b-1}dx \\
&= \frac{1}{\theta^b}\gamma(b,\theta p) \\
\therefore \int_0^{\theta p} e^{-\theta y}y^{b-1}dy &= \frac{\gamma(b,\theta p)}{\theta^b} \\
\therefore \int_0^{\theta p} \frac{\gamma(b,\theta p)}{\theta^b}e^{-\theta y}y^{b-1}dy &= 1
\end{aligned}$$

$$\therefore g(y) = \int_0^{\theta p} \frac{\gamma(b,\theta p)}{\theta^b} e^{-\theta y} y^{b-1}, \quad 0 < y < p; \quad p > 0; \quad \theta > 0; \quad b > 0$$

But

$$\gamma(a,x) = \frac{x^a}{a} e^{-x} ({}_1F_1(1; a+1; x))$$

$$\therefore g(y) = \frac{\theta^b e^{-\theta y} y^{b-1}}{\frac{(\theta p)^b}{b}\, {}_1F_1(1; b+1; \theta p) e^{-\theta p}}$$

Put $p = 1$

$$\therefore g(y) = \frac{b e^{-\theta y} y^{b-1}}{{}_1F_1(1; b+1; \theta) e^{-\theta}}$$

Let,

$$y = 1 - t \Rightarrow \frac{dy}{dt} = |-1| = 1$$

$$\therefore g(t) = \frac{b e^{-\theta(1-t)} (1-t)^{b-1}}{{}_1F_1(1; b+1; \theta) e^{-\theta}}$$

$$= \frac{b e^{-\theta} e^{\theta t} (1-t)^{b-1}}{{}_1F_1(1; b+1; \theta) e^{-\theta}}$$

$$= \frac{b e^{\theta t} (1-t)^{b-1}}{{}_1F_1(1; b+1; \theta)}$$

$$= \frac{e^{\theta t} (1-t)^{b-1}}{B(1,b)\, {}_1F_1(1; b+1; \theta)}$$

Let, $b + 1 = \lambda \Rightarrow b = \lambda - 1$

$$\therefore g(t) = \frac{e^{\theta t} (1-t)^{\lambda - 2}}{B(1, \lambda - 1)\, {}_1F_1(1; \lambda; \theta)}, \quad 0 < t < 1; \; \lambda > 0$$

Which is a truncated Pearson type III distribution.

10.3 Extended Confluent Hypergeometric Series (ECHS) Distribution

10.3.1 Compound Distribution

Let $S_Y = X_1 + X_2 + \cdots + X_Y$
Where the $X_{i's}$ are iid random variables.
Further,let Y also be a random variable independent of $X_{i's}$.
If
$H(s)$=the probability generating function of $S_Y = E[S^{S_Y}]$
$F(s)$=the probability generating function of $Y = E[S^Y]$
$G(s)$=the probability generating function of $X_i = E[S^{X_i}]$
Then;

$$\begin{aligned}
H(s) &= E[S^{S_Y}] \\
&= EE[S^{S_Y}/Y=y] \\
&= E\{E[S^{X_1+X_{2+\cdots+X_Y}}]\} \\
&= E\{E(S^{X_1})E(S^{X_2})\cdots E(S^{X_Y})\} \\
&= E\{G(s)\}^Y \\
\therefore H(s) &= FG(s)
\end{aligned} \tag{10.25}$$

Suppose
X_i=1 with probability $\alpha = \frac{\eta_1}{\eta}$
and
$X_i = m$ with probability $1-\alpha = \frac{\eta_2}{\eta}$
Where $\eta_1 > 0,\ \eta_2 > 0, \eta = \eta_1 + \eta_2$ and $m = 1,2,3,\cdots$ Then the pgf of X_i is;

$$\begin{aligned}
G(s) &= \sum_{x_i} g(x_i)s^{x_i} \\
&= \text{Prob}(X_i = 1)s + \text{Prob}(X_i = 1)s^m \\
\therefore G(s) &= \alpha s + (1-\alpha)s^m
\end{aligned} \tag{10.26}$$

From (10.3)

$$\begin{aligned}
F(s) &= \frac{{}_1F_1(\nu;\lambda;\eta s)}{{}_1F_1(\nu;\lambda;\eta)} \\
\therefore H(s) &= F[G(s)] \\
&= \frac{{}_1F_1(\nu;\lambda;\eta G(s))}{{}_1F_1(\nu;\lambda;\eta)} \\
&= \frac{{}_1F_1(\nu;\lambda;\eta[\alpha s + (1-\alpha)s^m])}{{}_1F_1(\nu;\lambda;\eta)} \\
&= \frac{{}_1F_1(\nu;\lambda;\eta\alpha s + \eta(1-\alpha)s^m)}{{}_1F_1(\nu;\lambda;\eta)} \\
\therefore H(s) &= \frac{{}_1F_1(\nu;\lambda;\eta_1 s + \eta_2 s^m)}{{}_1F_1(\nu;\lambda;\eta)}
\end{aligned} \tag{10.27}$$

We define a distribution with pgf (10.27) as the "Extended Confluent Hypergeometric Series (ECHS)" distribution has five parameters namely:
$\nu, \quad \lambda, \quad \eta_1, \quad \eta_2, \quad m.$

$$\therefore H'(s) = \frac{1}{{}_1F_1(\nu;\lambda;\eta)} \frac{d}{ds} {}_1F_1(\nu;\lambda;\eta_1 s + \eta_2 s^m)$$

$$= \frac{1}{{}_1F_1(\nu;\lambda;\eta)} \frac{\nu}{\lambda}(\eta_1 + m\eta_2 s^{m-1}) \, {}_1F_1(\nu+1;\lambda+1;\eta_1 s + \eta_2 s^m)$$

$$H''(s) = \frac{1}{{}_1F_1(\nu;\lambda;\eta)} \frac{\nu}{\lambda}\{m(m-1)\eta_2 s^{m-2} \, {}_1F_1(\nu+1;\lambda+1;\eta_1 s + \eta_2 s^m)$$

$$+(\eta_1 + m\eta_2 s^{m-1})^2 \frac{\nu+1}{\lambda+1} \, {}_1F_1(\nu+2;\lambda+2;\eta_1 s + \eta_2 s^m)\}$$

Let

$$\therefore \Lambda_j = \frac{{}_1F_1(\nu+j;\lambda+j;\eta_1)}{{}_1F_1(\nu;\lambda;\eta)}, \quad \text{for} \quad j = 1,2 \tag{10.28}$$

$$\therefore E[S_Y] = H'(1) = \frac{\nu}{\lambda}(\eta_1 + m\eta_2) \frac{{}_1F_1(\nu+1;\lambda+1;\eta_1+\eta_2)}{{}_1F_1(\nu;\lambda;\eta)}$$

$$E[S_Y] = \frac{\nu}{\lambda}(\eta_1 + m\eta_2)\Lambda_1 \tag{10.29}$$

$$Var[S_Y] = H''(1) + H'(1) - [H'(1)]^2$$

$$= \frac{1}{{}_1F_1(\nu;\lambda;\eta)} \frac{\nu}{\lambda}\Big\{m(m-1)\eta_2 \, {}_1F_1(\nu+1;\lambda+1;\eta_1+\eta_2)$$

$$+(\eta_1 + m\eta_2)^2 \frac{\nu+1}{\lambda+1} \frac{{}_1F_1(\nu+2;\lambda+2;\eta)}{{}_1F_1(\nu;\lambda;\eta)}\Big\}$$

$$\frac{\nu}{\lambda}(\eta_1 + m\eta_2)\Lambda_1 - \frac{\nu^2}{\lambda^2}(\eta_1 + m\eta_2)^2\Lambda_1^2$$

$$= \frac{\nu}{\lambda}m(m-1)\eta_2\Lambda_2 + (\eta_1 + m\eta_2)^2 \frac{\nu}{\lambda} \frac{\nu+1}{\lambda+1}\Lambda_1$$

$$+ \frac{\nu}{\lambda}(\eta_1 + m\eta_2)\Lambda_1 - \frac{\nu^2}{\lambda^2}(\eta_1 + m\eta_2)^2\Lambda_1^2$$

$$\therefore Var[S_Y] = \frac{\nu}{\lambda}[m^2\eta_2 - m\eta_2 + \eta_1 + m\eta_1]\Lambda_1 + \frac{\nu}{\lambda}(\eta_1 + m\eta_2)^2[\frac{\nu+1}{\lambda+1}\Lambda_2 - \frac{\nu}{\lambda}\Lambda_1^2]$$

$$\therefore Var[S_Y] = \frac{\nu}{\lambda}(\eta_1 + m^2\eta_2)\Lambda_1 + \frac{\nu}{\lambda}(\eta_1 + m\eta_2)^2(\frac{\nu+1}{\lambda+1}\Lambda_2 - \frac{\eta}{\lambda}\Lambda_1^2) \tag{10.30}$$

10.3.2 Recursion in Extended Confluent Hypergeometric Series Distributions

(a) A recursive formula for probabilities

The random variable under consideration is S_Y

Let

$h_j = \text{Prob}\ \{S_Y = j\} \equiv h_j(\nu,\lambda)$

Hence the pgf is:

$$H(s)=\sum_{j=0}^{\infty}h_j(\nu,\lambda)s^j \tag{10.31}$$

i.e

$$\frac{{}_1F_1(\nu;\lambda;\eta_1 s+\eta_2 s^m)}{{}_1F_1(\nu;\lambda;\eta)}=\sum_{j=0}^{\infty}h_j(\nu,\lambda)s^j \tag{10.32}$$

Differentiating (10.32) with respect to s, we have;

$$\begin{aligned}\frac{\nu}{\lambda}\,\frac{[{}_1F_1(\nu+1;\lambda+1;\eta_1 s+\eta_2 s^m)](\eta_1+m\eta_2 s^{m-1})}{{}_1F_1(\nu;\lambda;\eta)}&=\sum_{j=0}^{\infty}jh_j(\nu,\lambda)s^{j-1}\\&=\sum_{j=1}^{\infty}jh_j(\nu,\lambda)s^{j-1}\\&=\sum_{j=0}^{\infty}(j+1)h_{j+1}(\nu,\lambda)s^j \qquad (10.33)\end{aligned}$$

Replacing $\nu=\nu+1, \quad \lambda=\lambda+1$ in (10.32) we have;

$$\frac{{}_1F_1(\nu+1;\lambda+1;\eta_1 s+\eta_2 s^m)}{{}_1F_1(\nu+1;\lambda+1;\eta)}=\sum_{j=0}^{\infty}h_j(\nu+1,\lambda+1)s^j \tag{10.34}$$

Divide (10.33) by (10.34), we have;

$$\frac{\nu}{\lambda}(\eta_1+m\eta_2 s^{m-1})\frac{{}_1F_1(\nu+1;\lambda+1;\eta)}{{}_1F_1(\nu;\lambda;\eta)}=\frac{\sum_{j=0}^{\infty}(j+1)h_{j+1}(\nu,\lambda)s^j}{\sum_{j=0}^{\infty}h_j(\nu+1,\lambda+1)s^j}$$

Let $\quad \dfrac{{}_1F_1(\nu+1;\lambda+1;\eta)}{{}_1F_1(\nu;\lambda;\eta)}=\Lambda_1$

$$\therefore \frac{\nu}{\lambda}(\eta_1+m\eta_2 s^{m-1})\Lambda_1=\frac{\sum_{j=0}^{\infty}(j+1)h_{j+1}(\nu,\lambda)s^j}{\sum_{j=0}^{\infty}h_j(\nu+1,\lambda+1)s^j}$$

$$\therefore \frac{\nu}{\lambda}\Lambda_1\{\sum_{j=0}^{\infty}\eta_1 h_j(\nu+1,\lambda+1)s^j+\sum_{j=0}^{\infty}m\eta_2 h_j(\nu+1,\lambda+1)s^{m+j-1}\}=\sum_{j=0}^{\infty}(j+1)h_{j+1}(\nu,\lambda)s^j \tag{10.35}$$

Comparing the coefficient of s^r in (6.35) we have:
For the Right Hand Side,the coefficient=$(r+1)h_{r+1}(\nu,\lambda)$ by putting $j=r$

For the Left Hand Side,put $j = s$ for the first term to get;

$$\frac{\nu}{\lambda}\Lambda_1\eta_1 h_r(\nu+1,\lambda+1).$$

Put $m+j-1 = r \Rightarrow j = r-m+1$ for the second term to get;

$$m\eta_2\frac{\nu}{\lambda}\Lambda_1 h_{r-m+1}(\nu+1,\lambda+1)$$

Thus we have;

$$(r+1)h_{r+1}(\nu,\lambda) = \frac{\nu}{\lambda}\Lambda_1\{\eta_1 h_r(\nu+1,\lambda+1) + m\eta_2 h_{r-m+1}(\nu+1,\lambda+1)\} \quad \textbf{(10.36)}$$

when $r \geq m$

and

$$(r+1)h_{r+1}(\nu,\lambda) = \frac{\nu}{\lambda}\Lambda_1\eta_1 h_r(\nu+1,\lambda+1) \quad \textbf{(10.37)}$$

when $r < m$.

(b) A recursive formula for factorial moments

Consider;

$$H(1+s) = \sum_{j=0}^{\infty} h_j(\nu,\lambda)(1+s)^j \quad \textbf{(10.38)}$$

i.e

$$\begin{aligned}
\frac{{}_1F_1(\nu;\lambda;\eta_1(1+s)+\eta_2(1+s)^m)}{{}_1F_1(\nu;\lambda;\eta)} &= \sum_{j=0}^{\infty} h_j(\nu,\lambda)(1+s)^j \\
&= \sum_{j=0}^{\infty}\left\{h_j(\nu,\lambda)\sum_{n=0}^{j}\binom{j}{n}(s)^n\right\} \\
&= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} h_j(\nu,\lambda)\binom{j}{n}(s)^n \\
&= \sum_{n=0}^{\infty}\left\{[\sum_{j=0}^{\infty}\frac{j!}{(j-n)!}h_j(\nu,\lambda)]\frac{s^n}{n!}\right\} \\
&= \sum_{n=0}^{\infty}\left\{[\sum_{j=0}^{\infty} j(j-1)(j-2)\cdots(j-n+1)h_j(\nu,\lambda)]\frac{s^n}{n!}\right\} \\
&= \sum_{n=0}^{\infty}\mu_n(\nu,\lambda)\frac{s^n}{n!}
\end{aligned} \quad \textbf{(10.39)}$$

Differentiating (10.39) with respect to s, we get;

$$\frac{\nu}{\lambda}\frac{[{}_1F_1(\nu+1;\lambda+1;\eta_1(1+s)+\eta_2 s^m)](\eta_1+m\eta_2(1+s)^{m-1})}{{}_1F_1(\nu;\lambda;\eta)}=\sum_{n=0}^{\infty}\mu_n(\nu,\lambda)\frac{s^{n-1}}{(n-1)!}$$
$$=\sum_{n=0}^{\infty}\mu_{n+1}(\nu,\lambda)\frac{s^n}{n!} \tag{10.40}$$

Replacing $\nu=\nu+1,\quad \lambda=\lambda+1$ in (10.39) we get;

$$\frac{{}_1F_1(\nu+1;\lambda+1;\eta_1(1+s)+\eta_2(1+s)^m)}{{}_1F_1(\nu+1;\lambda+1;\eta)}=\sum_{n=0}^{\infty}\mu_n(\nu+1,\lambda+1)\frac{s^n}{n!} \tag{10.41}$$

Divide (10.40) by (10.41) to get;

$$\begin{aligned}
\frac{\nu}{\lambda}\Lambda_1[\eta_1+m\eta_2(1+s)^{m-1}] &= \frac{\sum_{n=0}^{\infty}\mu_{n+1}(\nu,\lambda)\frac{s^n}{n!}}{\sum_{n=0}^{\infty}\mu_n(\nu+1,\lambda+1)\frac{s^n}{n!}}\\
\therefore\sum_{n=0}^{\infty}\mu_{n+1}(\nu,\lambda)\frac{s^n}{n!} &= \frac{\nu}{\lambda}\Lambda_1\sum_{n=0}^{\infty}\eta_1\mu_n(\nu+1,\lambda+1)\frac{s^n}{n!}\\
&+\frac{\nu}{\lambda}\Lambda_1\sum_{n=0}^{\infty}m\eta_2(1+s)^{m-1}\mu_n(\nu+1,\lambda+1)\frac{s^n}{n!}\\
&=\frac{\nu}{\lambda}\Lambda_1\sum_{n=0}^{\infty}\eta_1\mu_n(\nu+1,\lambda+1)\frac{s^n}{n!}\\
&+\frac{\nu}{\lambda}\Lambda_1\sum_{n=0}^{\infty}\left\{m\eta_2\sum_{j=0}^{m-1}\binom{m-1}{j}s^j\mu_n(\nu+1,\lambda+1)\frac{s^n}{n!}\right\}\\
&=\frac{\nu}{\lambda}\Lambda_1\sum_{n=0}^{\infty}\eta_1\mu_n(\nu+1,\lambda+1)\frac{s^n}{n!}\\
&+\frac{\nu}{\lambda}\Lambda_1\sum_{n=0}^{\infty}\left\{m\eta_2\sum_{j=0}^{m-1}\binom{m-1}{j}s^j\mu_n(\nu+1,\lambda+1)\frac{s^{j+n}}{n!}\right\}
\end{aligned} \tag{10.42}$$

Comparing the coefficient of s^n in (10.42) we have;
For the LHS put $n=r$ to get,

$$\frac{\mu_{[r+1]}(\nu,\lambda)}{r!}$$

For the RHS,put $n = r$ for the first term to get,

$$\frac{\frac{\nu}{\lambda}\Lambda_1 \eta_1 \mu_{[r]}(\nu+1,\lambda+1)}{r!}$$

Put $\quad j+n=r \Rightarrow n=r-j$ to get;

$$\frac{\nu}{\lambda}\Lambda_1 m\eta_2 \sum_{j=0}^{m-1} \binom{m-1}{j} \frac{\mu_{[r-j]}(\nu+1,\lambda+1)}{(r-j)!}$$

$$\therefore \frac{\mu_{[r+1]}(\nu,\lambda)}{r!} = \frac{\nu}{\lambda}\Lambda_1 \eta_1 \mu_{[r]}(\nu+1,\lambda+1) + \frac{\nu}{\lambda}\Lambda_1 m\eta_2 \sum_{j=0}^{m-1} \binom{m-1}{j} \frac{\mu_{[r-j]}(\nu+1,\lambda+1)}{(r-j)!}$$

$$\therefore \mu_{[r+1]}(\nu,\lambda) = \frac{\nu}{\lambda}\Lambda_1 \left\{ \eta_1 \mu_{[r]}(\nu+1,\lambda+1) + m\eta_2 \sum_{j=0}^{m-1} r^j \binom{m-1}{j} \mu_{[r-j]}(\nu+1,\lambda+1) \right\} \tag{10.43}$$

where $\quad r^j = r(r-1)(r-2)\cdots(r-j+1)$

(b) Recursive formula for the rth moments

The characteristic function of S_Y is;

$$\begin{aligned} Q_{S_Y}(t) &= E[e^{itS_Y}] \\ &= E[e^{(it)S_Y}] \\ &= H[e^{it}] \end{aligned}$$

$$\therefore Q_{S_Y}(t) = \frac{{}_1F_1(\nu;\lambda;\eta_1 e^{it} + \eta_2 e^{mit})}{{}_1F_1(\nu;\lambda;\eta)} \tag{10.44}$$

And since $H(s) = \sum_j (\nu,\lambda) s^j$

Then,

$$\begin{aligned} H(e^{it}) &= \sum_j (\nu,\lambda) e^{it\,j} \\ &= \sum_{j=0}^{\infty} \{ h_j(\nu,\lambda) \sum_{n=0}^{\infty} \frac{(it\ j)^n}{n!} \} \\ &= \sum_{n=0}^{\infty} \{ [\sum_{j=0}^{\infty} j^n h_j(\nu,\lambda)] \frac{(it)^n}{n!} \} \end{aligned}$$

$$\therefore H(e^{it}) = \sum_{n=0}^{\infty} \mu_n'(\nu,\lambda) \frac{(it)^n}{n!} \tag{10.45}$$

Therefore,

$$\frac{{}_1F_1(\nu;\lambda;\eta_1 e^{it}+\eta_2 e^{mit})}{{}_1F_1(\nu;\lambda;\eta)}=\sum_{n=0}^{\infty}\mu_n'(\nu,\lambda)\frac{(it)^n}{n!} \tag{10.46}$$

Differentiating (10.46) with respect to t,we get;

$$\frac{\frac{\nu}{\lambda}\,{}_1F_1(\nu+1;\lambda+1;\eta_1 e^{it}+\eta_2 e^{mit})[i\eta_1 e^{it}+mi\eta_2 e^{mit}]}{{}_1F_1(\nu;\lambda;\eta)}=\sum_{n=1}^{\infty}\mu_n'(\nu,\lambda)\frac{int^{n-1}}{(n-1)!} \tag{10.47}$$

Replacing $\nu=\nu+1, \quad \lambda=\lambda+1$ in (10.46) we get;

$$\frac{{}_1F_1(\nu+1;\lambda+1;\eta_1 e^{it}+\eta_2 e^{mit})}{{}_1F_1(\nu+1;\lambda+1;\eta)}=\sum_{n=0}^{\infty}\mu_n'(\nu+1,\lambda+1)\frac{(it)^n}{n!} \tag{10.48}$$

Next divide (10.47) by (10.48) to get;

$$\begin{aligned}
\frac{\nu}{\lambda}\Lambda_1[i\eta_1 e^{it}+mi\eta_2 e^{mit}] &= \frac{\sum_{n=1}^{\infty}\mu_n'(\nu,\lambda)\frac{i^n t^{n-1}}{(n-1)!}}{\sum_{n=0}^{\infty}\mu_n'(\nu+1,\lambda+1)\frac{(it)^n}{n!}}\\
&= \frac{\sum_{n=0}^{\infty}\mu_{n+1}'(\nu,\lambda)\frac{i^{n+1}t^n}{(n)!}}{\sum_{n=0}^{\infty}\mu_n'(\nu+1,\lambda+1)\frac{(it)^n}{n!}}\\
\therefore \frac{\nu}{\lambda}\Lambda_1[\eta_1 e^{it}+m\eta_2 e^{mit}] &= \frac{\sum_{n=0}^{\infty}\mu_{n+1}'(\nu,\lambda)\frac{(it)^n}{(n)!}}{\sum_{n=0}^{\infty}\mu_n'(\nu+1,\lambda+1)\frac{(it)^n}{n!}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=0}^{\infty}\mu_{n+1}'(\nu,\lambda)\frac{(it)^n}{(n)!} &= \frac{\nu}{\lambda}\Lambda_1\{\sum_{n=0}^{\infty}\eta_1 e^{it}\mu_n'(\nu+1,\lambda+1)\frac{(it)^n}{n!}\\
&+\sum_{n=0}^{\infty}m\eta_2 e^{mit}\mu_n'(\nu+1,\lambda+1)\frac{(it)^n}{n!}\}\\
&= \frac{\nu}{\lambda}\Lambda_1\{\sum_{n=0}^{\infty}[\eta_1\mu_n'(\nu+1,\lambda+1)\sum_{k=0}^{\infty}\frac{(it)^k}{k!}\frac{(it)^n}{n!}]\\
&+\sum_{n=0}^{\infty}[m\eta_2\mu_n'(\nu+1,\lambda+1)\sum_{k=0}^{\infty}\frac{(mit)^k}{k!}\frac{(it)^n}{n!}]\}
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty}\mu'_{n+1}(\nu,\lambda)\frac{(it)^n}{(n)!}=\frac{\nu}{\lambda}\Lambda_1\{\sum_{n=0}^{\infty}[\eta_1\mu'_n(\nu+1,\lambda+1)\sum_{k=0}^{\infty}\frac{(it)^{k+n}}{k!n!}]$$
$$+\sum_{n=0}^{\infty}[m\eta_2\mu'_n(\nu+1,\lambda+1)\sum_{k=0}^{\infty}\frac{m^k(it)^{k+n}}{k!n!}]\}$$
$$\sum_{n=0}^{\infty}\mu'_{n+1}(\nu,\lambda)\frac{(it)^n}{(n)!}=\frac{\nu}{\lambda}\Lambda_1\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\{(\eta_1+m\eta_2m^k)\frac{\mu'_n(\nu+1,\lambda+1)(it)^{k+n}}{k!n!}\} \tag{10.49}$$

Comparing the coefficient of $(it)^r$, we have;
From LHS by putting $n=r$

$$\frac{\mu'_{r+1}(\nu,\lambda)}{r!}$$

Putting $k+n=r,\Rightarrow n=r-k$ we have;
From RHS:

$$\frac{\nu}{\lambda}\Lambda_1\sum_{k=0}^{\infty}(\eta_1+m\eta_2m^k)\frac{\mu'_{r-k}(\nu+1,\lambda+1)}{k!(r-k)!}$$

Therefore,

$$\frac{\mu'_{r+1}(\nu,\lambda)}{r!}=\frac{\nu}{\lambda}\Lambda_1\sum_{k=0}^{\infty}(\eta_1+m\eta_2m^k)\frac{\mu'_{r-k}(\nu+1,\lambda+1)}{k!(r-k)!}$$
$$\therefore\mu'_{r+1}(\nu,\lambda)=\frac{\nu}{\lambda}\Lambda_1\sum_{k=0}^{\infty}\binom{r}{k}(\eta_1+m\eta_2m^k)\mu'_{r-k}(\nu+1,\lambda+1)$$
$$\therefore\mu'_{r+1}(\nu,\lambda)=\frac{\nu}{\lambda}\Lambda_1\sum_{k=0}^{\infty}\binom{r}{k}(\eta_1+\eta_2m^{k+1})\mu'_{r-k}(\nu+1,\lambda+1)$$

10.3.3 Probability mass function of the Extended Confluent Hypergeometric Series (ECHS) Distributions

Again, let

$$H(s)=\sum_{x=0}^{\infty}h_x(\nu,\lambda)s^x$$

i.e

$$\frac{{}_1F_1(\nu;\lambda;\eta_1 s+\eta_2 s^m)}{{}_1F_1(\nu;\lambda;\eta)}=\sum_{x=0}^{\infty}h_x(\nu,\lambda)s^x$$

i.e

$$\frac{1}{{}_1F_1(\nu;\lambda;\eta)}\sum_{x=0}^{\infty}\frac{\nu(\nu+1)(\nu+2)\cdots(\nu+x-1)}{\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+x-1)}\frac{(\eta_1 s+\eta_2 s^m)^x}{x!}=\sum_{x=0}^{\infty}h_x(\nu,\lambda)s^x$$

$$\therefore\frac{1}{{}_1F_1(\nu;\lambda;\eta)}\sum_{x=0}^{\infty}\frac{(\nu)_x}{(\lambda)_x}\frac{(\eta_1 s+\eta_2 s^m)^x}{x!}=\sum_{x=0}^{\infty}h_x(\nu,\lambda)s^x \quad \textbf{(10.51)}$$

Where $(a)_0=1$ and $(a)_n=a(a+1)(a+2)\cdots(a+n-1)$ for $n\geq 1$
We can rewrite (10.51) as;

$$\sum_{x=0}^{\infty}h_x(\nu,\lambda)s^x=\frac{1}{{}_1F_1(\nu;\lambda;\eta)}\sum_{x=0}^{\infty}\{\frac{(\nu)_x}{(\lambda)_x}\sum_{n=0}^{\infty}\binom{x}{n}\frac{(\eta_1 s)^{x-n}}{x!}(\eta_2 s^m)^n\}$$

$$\therefore\sum_{x=0}^{\infty}h_x(\nu,\lambda)s^x=\frac{1}{{}_1F_1(\nu;\lambda;\eta)}\sum_{x=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\nu)_x}{(\lambda)_x}\frac{\eta_1^{x-n}\eta_2^n s^{x-n+mn}}{(x-n)!n!}$$

Comparing the coefficient of S^r,
Put $x=r$ on the LHS to get $h_r(\nu,\lambda)$
Put $x-n+mn=r$ on the RHS .
Therefore, $x=r-(m-1)n$.
THerefore,the coefficient of S^r on the RHS is;

$$\frac{1}{{}_1F_1(\nu;\lambda;\eta)}\sum_{n=0}^{\infty}\frac{(\nu)_{r-(m-1)n}}{(\lambda)_{r-(m-1)n}}\frac{\eta_1^{r-mn}\eta_2^n}{(r-mn)!n!}$$

Therefore,

$$h_r(\nu,\lambda)=\frac{1}{{}_1F_1(\nu;\lambda;\eta)}\sum_{n=0}^{\infty}\frac{(\nu)_{r-(m-1)n}}{(\lambda)_{r-(m-1)n}}\frac{\eta_1^{r-mn}\eta_2^n}{(r-mn)!n!} \quad \textbf{(10.52)}$$

10.4 Special Cases of ECHS (ν,m,η_1,η_2) Distribution

For each case,we determine the pgf,mean and the variance using formulae (10.27),(10.29) and (10.30) respectively.

10.4.1 When $\eta_2 = 0$

Then

$$H(s) = \frac{{}_1F_1(\nu;\lambda;\eta_1 s)}{{}_1F_1(\nu;\lambda;\eta_1)}$$

Which is the pgf of Bhattacharya's Confluent Hypergeometric Series distribution given by (10.3)

$$E[S_Y] = \frac{\nu}{\lambda}\eta_1\Lambda_1$$

and

$$Var[S_Y] = \frac{\nu}{\lambda}\eta_1\Lambda_1 + \frac{\nu}{\lambda}\eta_1(\frac{\nu+1}{\lambda+1}\Lambda_2 - \frac{\nu}{\lambda}\Lambda_1^2)$$

$$\text{Where, } \Lambda_j = \frac{{}_1F_1(\nu+j;\lambda+j;\eta_1)}{{}_1F_1(\nu;\lambda;\eta_1)}$$

10.4.2 When $\nu = \lambda, m = 2; \eta_1 > 0, \eta_2 > 0 = 0$

THen,

$$H(s) = \frac{{}_1F_1(\nu;\lambda;\eta_1 s + \eta_2 s^2)}{{}_1F_1(\nu;\lambda;\eta)}$$

Which is the pgf of the Hermite distribution due to Kemp and Kemp (1965).

$$E[S_Y] = (\eta_1 + 2\eta_2)\Lambda_1$$

and

$$Var[S_Y] = (\eta_1 + 4\eta_2)\Lambda_1 + (\eta_1 + 2\eta_2)^2(\Lambda_2 - \Lambda_1^2)$$

$$\text{Where, } \Lambda_j = \frac{{}_1F_1(\lambda+j;\lambda+j;\eta)}{{}_1F_1(\lambda;\lambda;\eta)}$$

Remark (10.3)

The pgf of a Hermite distribution is a Compound Poisson distribution obtained by considering
$S_N = X_1 + X_2 + \cdots + X_N$

Where X_i^s are iid binomial random variables with parameters 2 and p.
N is poisson with parameter λ.

$$\therefore H(s) = e^{\lambda[G(s)-1]} = e^{\lambda[(q+ps)^2-1]}$$
$$\therefore H(s) = exp\{\lambda[q^2+p^2s^2+2qps-1]\}$$
$$= exp\{\lambda[q^2+p^2-p^2+p^2s^2+2qps-1]\}$$
$$= exp\{\lambda[(q+p)^2+p^2(s^2-1)+2qps-1]\}$$
$$= exp\{\lambda[(q+p)^2-2qp+p^2(s^2-1)+2qps-1]\}$$
$$= exp\{\lambda[1-2qp+p^2(s^2-1)+2qps-1]\}$$
$$= exp\{\lambda[2qp(s-1)+p^2(s^2-1)]\}$$
$$\therefore H(s) = e^{\lambda[2qps-2qp+p^2s^2-p^2]}$$
$$= e^{\lambda[(2qps+p^2s^2)-(2qp+p^2)]}$$
$$= e^{(2\lambda qps+\lambda p^2s^2)}e^{-\lambda(2qp+p^2)}$$
$$\therefore H(s) = \frac{e^{2\lambda qps+\lambda p^2s^2}}{e^{2\lambda qp+\lambda p^2}}$$

From section (10.2.2),we found tha;

$$e^{\theta} =_1 F_1(a;a;\theta)$$
$$\therefore H(s) = \frac{{}_1F_1(a;a;2\lambda qps+\lambda p^2s^2)}{{}_1F_1(a;a;2\lambda qp+\lambda p^2)}$$
$$= \frac{{}_1F_1(\lambda;\lambda;2\lambda qps+\lambda p^2s^2)}{{}_1F_1(\lambda;\lambda;2\lambda qp+\lambda p^2)}$$
$$= \frac{{}_1F_1(\lambda;\lambda;\eta_1 s+\eta_2 s^2)}{{}_1F_1(\lambda;\lambda;\eta_1+\eta_2)}$$
$$= \frac{{}_1F_1(\lambda;\lambda;\eta_1 s+\eta_2 s^2)}{{}_1F_1(\lambda;\lambda;\eta)}$$

Where, $\eta_1 = 2\lambda qp, \quad \eta_2 = \lambda p^2; \quad \eta = \eta_1+\eta_2, \quad \nu = \lambda, \quad m = 2$

Remark (10.4)

A Hermite distribution is a discrete Poisson mixture as shown below:

$$
\begin{aligned}
\text{Let,}\quad h_j &= \text{Prob}\{S_N = j\} \\
&= \sum_n \text{Prob}\{S_N = j, N = n\} \\
&= \sum_n \text{Prob}\{S_N = j/N = n\}\text{Prob}\{N = n\} \\
&= \sum_n \text{Prob}\{X_1 + X_2 + \cdots + X_n = j\}\text{Prob}\{N = n\} \\
&= \sum_n \{f_j\}^{n*}\text{P}_n
\end{aligned}
$$

Where $\{f_j\}^{n*}$ is an n-th fold convolution X_i
In terms of pgf;

$$
H(s) = \sum_{j=0}^{\infty} h_j s^j = \sum_n [G(s)]^n \text{P}_n
$$

Where $G(s)$ is the pgf of $X_i s$
If $X_i \sim Bin(2, p)$ and $N \sim Poiss(\lambda)$
THen,

$$
\begin{aligned}
H(s) &= \sum_{n=0}^{\infty} [(q+ps)^2]^n \frac{e^{-\lambda\lambda^n}}{n!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{[(q+ps)^2]^n}{n!} \\
&= e^{-\lambda} e^{\lambda(q+ps)2} \\
&= exp\{-\lambda + \lambda q^2 + 2\lambda pqs + \lambda p^2 s^2\} \\
&= exp\{-\lambda + \lambda q^2 + 2\lambda pqs + \lambda p^2 s^2 - \lambda p^2 + \lambda p^2\} \\
&= exp\{-\lambda + \lambda q^2 + 2\lambda pqs + \lambda p^2(s^2 - 1)\} \\
&= exp\{-\lambda + 2\lambda pqs + \lambda(p^2 + q^2) + \lambda p^2(s^2 - 1)\} \\
\therefore H(s) &= exp\{-\lambda + 2\lambda pqs + \lambda(p^2 + q^2 + 2pq - 2pq) + \lambda p^2(s^2 - 1)\} \\
&= exp\{-\lambda + 2\lambda pqs + \lambda[(p+q)^2 - 2pq] + \lambda p^2(s^2 - 1)\} \\
&= exp\{-\lambda + 2\lambda pqs + \lambda - 2\lambda pq + \lambda p^2(s^2 - 1)\} \\
\therefore H(s) &= exp\{2\lambda pq(s-1) + \lambda p^2(s^2 - 1)\}
\end{aligned}
$$

Which is a pgf of a Hermite distribution that can be written as;

$$\begin{aligned} H(s) &= e^{2\lambda pqs - 2\lambda pq + \lambda p^2 s^2 - \lambda p^2} \\ &= \frac{e^{2\lambda pqs + \lambda p^2 s^2}}{e^{2\lambda pq + \lambda p^2}} \\ \therefore H(s) &= \frac{{}_1F_1(\lambda;\lambda;2\lambda pqs + \lambda p^2 s^2)}{{}_1F_1(\lambda;\lambda;2\lambda pq + \lambda p^2)} \end{aligned}$$

Remark (10. 5)

Kemp and Kemp (1966) found that if mixing is treated purely as a formal process with the Poisson parameter θ taking negative values,then a Hermite distribution can be derived as a Poisson-Normal mixture as follows:

$$
\begin{aligned}
H(s) &= \int_{-\infty}^{\infty} e^{-\theta(1-s)\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(\theta-\mu)^2}{2\sigma^2}}}\, d\theta \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} exp\{-\theta(1-s) - \frac{(\theta-\mu)^2}{2\sigma^2}\} d\theta \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} exp\{\frac{-2\theta(1-s)\sigma^2 - (\theta-\mu)^2}{2\sigma^2}\} d\theta \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} exp\{\frac{-2\theta(1-s)\sigma^2 - [\theta^2 - 2\mu\theta + \mu^2]}{2\sigma^2}\} d\theta \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} exp\{-\frac{1}{2\sigma^2}[2\sigma^2(1-s)\theta + \theta^2 - 2\mu\theta + \mu^2]\} d\theta \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} exp\{-\frac{1}{2\sigma^2}[\theta^2 - 2\{\mu - \sigma^2(1-s)\}\theta + \mu^2]\} d\theta \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} exp\{-\frac{1}{2\sigma^2}[\{\theta - \mu + \sigma^2(1-s)\}^2 - \{\mu - \sigma^2(1-s)\}^2 + \mu^2]\} d\theta \\
&= exp\{-\frac{1}{2\sigma^2}[\mu^2 - \{\mu - \sigma^2(1-s)\}^2]\}.1 \\
&= exp\{-\frac{1}{2\sigma^2}[2\mu - \sigma^2(1-s)][\sigma^2(1-s)]\} \\
&= exp\{-\frac{1}{2}[2\mu - \sigma^2(1-s)](1-s)\} \\
&= exp\{-\frac{1}{2}[2\mu(1-s) - \sigma^2(1-s)^2]\} \\
&= exp\{-\frac{1}{2}[2\mu(1-s) - \sigma^2(1 - 2s + s^2 - 1 + 1)]\} \\
&= exp\{-\frac{1}{2}[2\mu(1-s) - \sigma^2(2 - 2s + s^2 - 1)]\} \\
&= exp\{-\frac{1}{2}[2\mu(1-s) - 2\sigma^2(1-s) - \sigma^2(s^2 - 1)]\} \\
&= exp\{-\frac{1}{2}[2(\mu - \sigma^2)(1-s) - \sigma^2(s^2 - 1)]\} \\
\therefore H(s) &= exp\{(\mu - \sigma^2)(s-1) + \frac{\sigma^2}{2}(s^2 - 1)\}
\end{aligned}
$$

Which is a pgf of a Hermite distribution.

10.4.3 When $\nu = \lambda, \quad m > 0; \quad \eta_1 > 0, \quad \eta_2 > 0$

Then,

$$
H(s) = \frac{{}_1F_1(\lambda; \lambda; \eta_1 s + \eta_2 s^m)}{{}_1F_1(\lambda; \lambda; \eta)}
$$

which is the pgf of the generalized Hermite distribution due to Gupta and Jain (1974).

$$E[S_Y] = (\eta_1 + m\eta_2)\Lambda_1$$
and
$$Var[S_Y] = (\eta_1 + m^2\eta_2)\Lambda_1 + (\eta_1 + m\eta_2)(\Lambda_2 - \Lambda_1^2)$$
$$\text{where} \quad \Lambda_j = \frac{{}_1F_1(\lambda + j; \lambda + j; \eta)}{{}_1F_1(\lambda; \lambda; \eta)}, \quad j = 1, 2$$

Remark(10.6)

In constructing a generalized Hermite distribution,Gupta and Jain (1974) considered the variable x given by;
$X = X_1 + mX_2$
Where $X_1 + mX_2$ are independent Poisson random variable with parameter η_1 and η_2 respectively. The pgf of X is;

$$\begin{aligned}
H(s) &= E[S^{X_1+mX_2}] \\
&= E(S^{X_1})E(S^{mX_2}) \\
&= e^{-\eta_1(1-s)}e^{-\eta_2(1-s^m)} \\
&= e^{-\eta_1+\eta_1 s-\eta_2+\eta_2 s^m} \\
&= e^{\eta_1 s+\eta_1 s^m-\eta_1-\eta_2} \\
&= \frac{e^{\eta_1 s+\eta_1 s^m}}{e^{\eta_1+\eta_2}} \\
&= \frac{{}_1F_1(\lambda; \lambda; \eta_1 s + \eta_2 s^m)}{{}_1F_1(\lambda; \lambda; \eta_1 + \eta_2)} \\
\therefore H(s) &= \frac{{}_1F_1(\lambda; \lambda; \eta_1 s + \eta_2 s^m)}{{}_1F_1(\lambda; \lambda; \eta)}
\end{aligned}$$

10.4.4 When $\nu = 1, \quad m = 2 \quad and \quad \lambda = \lambda$

a positive integer,
Then

$$H(s) = \frac{{}_1F_1(\lambda; \lambda; \eta_1 s + \eta_2 s^2)}{{}_1F_1(\lambda; \lambda; \eta)}$$

which is the pgf of the extended displaced Poisson distribution of type I with parameters $\lambda, \quad \eta_1 > 0 \quad and \quad \eta_2 > 0$

$$E[S_Y] = \frac{1}{\lambda}(\eta_1 + 2\eta_2)\Lambda_1$$

and

$$Var[S_Y] = \frac{1}{\lambda}(\eta_1 + 4\eta_2)\Lambda_1 + \frac{1}{\lambda}(\eta_1 + 2\eta_2)^2(\frac{2}{\lambda+1}\Lambda_1 - \frac{1}{\lambda}\Lambda_1^2)$$

$$\text{where} \quad \Lambda_j = \frac{{}_1F_1(1+j;\lambda+j;\eta)}{{}_1F_1(1;\lambda;\eta)}, \quad j = 1,2$$

10.4.5 When $\nu = 1, \quad \lambda = rand \quad m \geq 2$

where r is a positive integer,
Then

$$H(s) = \frac{{}_1F_1(1;r;\eta_1 s + \eta_2 s^m)}{{}_1F_1(1;r;\eta)}$$

which is the pgf of the extended displaced Poisson distribution of type II with parameters $\lambda, \quad m, \quad \eta_1 > 0 \quad and \quad \eta_2 > 0$

$$E[S_Y] = \frac{1}{r}(\eta_1 + m\eta_2)\Lambda_1$$

and

$$Var[S_Y] = \frac{1}{r}(\eta_1 + m\eta_2)\Lambda_1 + \frac{1}{r}(\eta_1 + m\eta_2)^2(\frac{2}{r+1}\Lambda_2 - \frac{1}{r}\Lambda_1^2)$$

$$\text{where} \quad \Lambda_j = \frac{{}_1F_1(1+j;r+j;\eta)}{{}_1F_1(1;r;\eta)}, \quad j = 1,2$$

10.4.6 When $m = 2$

Then

$$H(s) = \frac{{}_1F_1(\nu;\lambda;\eta_1 s + \eta_2 s^2)}{{}_1F_1(\nu;\lambda;\eta)}$$

which is the pgf of the extended Bardwell and Crow family of distributions of type III with parameters $\nu > 0, \quad \lambda > 0, \quad \eta_1 > 0 \quad and \quad \eta_2 > 0$

$$E[S_Y] = \frac{\nu}{\lambda}(\eta_1 + 2\eta_2)\Lambda_1$$

and

$$Var[S_Y] = \frac{\nu}{\lambda}(\eta_1 + 4\eta_2)\Lambda_1 + \frac{\nu}{\lambda}(\eta_1 + 2\eta_2)^2(\frac{\nu+1}{\lambda+1}\Lambda_2 - \frac{\nu}{\lambda}\Lambda_1^2)$$

$$\text{where} \quad \Lambda_j = \frac{{}_1F_1(\nu + j; \lambda + j; \eta)}{{}_1F_1(\nu; \lambda; \eta)}, \quad j = 1,2$$

10.4.7 When $\nu = 1$

Then

$$H(s) = \frac{{}_1F_1(1; \lambda; \eta_1 s + \eta_2 s^2)}{{}_1F_1(1; \lambda; \eta)}$$

which is the pgf of the extended Crow-Bardwell family of distributions of type II with parameters $m > 0, \quad ,\lambda > 0, \quad \eta_1 > 0 \quad and \quad \eta_2 > 0$

$$E[S_Y] = \frac{1}{\lambda}(\eta_1 + m\eta_2)\Lambda_1$$

and

$$Var[S_Y] = \frac{1}{\lambda}(\eta_1 + m^2\eta_2)\Lambda_1 + \frac{1}{\lambda}(\eta_1 + m\eta_2)^2(\frac{2}{\lambda+1}\Lambda_2 - \frac{1}{\lambda}\Lambda_1^2)$$

$$\text{where} \quad \Lambda_j = \frac{{}_1F_1(1 + j; \lambda + j; \eta)}{{}_1F_1(1; \lambda; \eta)}, \quad j = 1,2$$

10.4.8 When $\nu = 1, \quad m = 2$

Then

$$H(s) = \frac{{}_1F_1(1; \lambda; \eta_1 s + \eta_2 s^2)}{{}_1F_1(1; \lambda; \eta)}$$

which is the pgf of the extended Crow-Bardwell family of distributions of type I with parameters $\lambda > 0, \quad \eta_1 > 0 \quad and \quad \eta_2 > 0$

$$E[S_Y] = \frac{1}{\lambda}(\eta_1 + 2\eta_2)\Lambda_1$$

and

$$Var[S_Y] = \frac{1}{\lambda}(\eta_1 + 4\eta_2)\Lambda_1 + \frac{1}{\lambda}(\eta_1 + 2\eta_2)^2(\frac{2}{\lambda+1}\Lambda_2 - \frac{1}{\lambda}\Lambda_1^2)$$

where $\Lambda_j = \frac{{}_1F_1(1+j;\lambda+j;\eta)}{{}_1F_1(1;\lambda;\eta)}, \quad j = 1, 2$

11 CONCLUSIONS AND RECOMMENDATIONS

11.1 Conclusion

Stochastic processes specifically Birth and Death processes at equilibrium can be studied based on recursive models of P_{n+1} as a function of P_n and P_{n-1}. Queues are birth and death processes at equilibrium where the process is characterised by the property that whenever a transition occurs from one state to another, then this transition can be to a neighboring state only.With an arrival there is a transition from the state $n \geq 0$ to the state $(n+1)$, and with a service completion there is a transition from state m to the state $(m-1)(m \geq 0)$, the state denoting the number in the system.
Birth and Death processes at equilibrium are generated using Pearson's differential equations based on Statistical and Actuarial literature. Using Pearson‘s difference equation;

$$\frac{P_{n+1}}{P_n} = \frac{P_n}{Q_n}$$

where $P_{.}$ is the pmf; P_n and Q_n are polynomials.
Some special cases and properties of some recursive models are determined and their pgf‘s in terms of hypergeometric function derived. The hypergeometric series distribution is used as a tool for constructing the pgf for birth and death process at equilibrium.
Generalized birth and death processes at equilibrium as ratios of polynomials are derived using Kapur (198a) general case in its steady state and study of its special cases, their properties are also determined.
In all these cases studied, it demonstrated that the hypergeometric distribution is useful in statistics; specifically in stochastic processes-birth and death processes at equilibrium.

11.2 Recommendation

In this study the basic difference differential equation in general is not easy to solve in most cases, but can be solved at steady state; solving difference equation for birth and death processes as $t \to \infty$.
In all these cases studied, it demonstrated that the hyper-geometric function is useful in Mathematical Statistics; specifically in stochastic processes-birth and death processes at equilibrium.

11.3 Future Research

Determine birth-and-death processes at equilibrium in terms of other special functions other than hypergeometric function.

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