# METRIC-DEPENDENT DIMENSION FUNCTIONS

BY

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This thesis is my original work and has not been presented for a degree in any other University.

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TABLE OF CONTENTS

SUMMARY OF CONTENTS

Section 0 is a review of results in general topology and basic dimension theory which are used in the sequel.

In section 1, we study the relationships between the various dimension functions. We give a proof of a result mentioned by Nagami and Roberts (Nagami and Roberts, 1967) to the effect that on locally compact metric spaces, all the dimension functions studied here coincide. We prove a lemma (lemma 1.3) which shortens the proofs of a number of results.

In section 2 we study examples which show that different dimension functions can have different values on the same metric space. We give an example of a connected subset of $I^2$ which is a union of countably many (an more than one) disjoint non-empty closed sets which shows that a lemma used by Nagami and Roberts (lemma 2.3) cannot be extended to normal (infact metric) spaces. Nagami and Roberts also show that if $A_i$, $i \in N$ is a disjoint sequence of closed sets of $I^n$ at least two of which are non-empty, then dim $(I^n - \bigcup_{i=1}^{\infty} A_i) \geq n-1$. They give a sketch of a Cantor 2-manifold for which this result is not true. We give a rigorous proof of this. Nagami and Roberts have given an example of a metric space $(X,\ell)$ with $d_2(X,\ell) = 2$, $d_3(X,\ell) = \mu\text{-dim}(X,\ell) = 3$ and dim $(X,\ell) = 4$. This has been the only known example where $d_2$ and $d_3$ differ. We generalize this to examples with $d_2 \leq n-2$, $d_3 = \mu\text{-dim} = n-1$ and dim $= n$ for any n, $n \geq 4$.

In section 3 we study results which show that a given metric-dependent dimension function can give different values for equivalent metrics on a set. We then study realization theorems, i.e. theorems to the effect that there exist equivalent metrics to a given metric that make a given dimension function realize given values. We prove a lemma (lemma 3.4) which generalizes a similar lemma by Goto (Goto, lemma 1).

In section 4 we study more characterizations of metric-dependent dimension functions, notably Lebesgue cover characterizations. We study a weak sum theorem for some metric-dependent dimension functions.

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# LIST OF SYMBOLS.

| Symbol | Meaning |
|---|---|
| s.t. | such that |
| $\forall$ | for all |
| $\exists$ | there exists |
| iff | if and only if |
| W.L.G. | without loss of generality |
| w.r.t | with respect to |
| clopen | closed and open |
| $\sigma$.l.f. | $\sigma$-locally finite |
| l.f | locally finite |
| nbhd | neighbourhood |
| Int A | interior of A |
| bdry A | boundary of A |
| $B(x,\varepsilon)$ | the open ball of radius $\varepsilon$ about x. |
| $\ell(U)$ | diameter of U in a metric space $(X,\ell)$. |
| $\mathcal{U}\|_A$ | The restriction of $\mathcal{U}$ to A where $\mathcal{U}$ is a collection of sets, i.e. $\{C\cap A,\ C\in\mathcal{U}\}$. |
| $\mathcal{V}<\mathcal{U}$ | $\mathcal{V}$ refines $\mathcal{U}$ where $\mathcal{V}$, $\mathcal{U}$ are collections of subsets of a set X. |
| $[x]$ | the integral part of x. |
| N | The set of natural numbers 1, 2, 3,... |
| Q | The set of rational numbers |
| R | The real line |
| $R^n$ | Euclidean n-space. |
| I | The unit interval $[0, 1]$ (or sometimes $[-1, 1]$) |
| $I^n$ | The n-cube IxIx...xI n-times. |

## INTRODUCTION:

Dimension is a basic notion in geometry. A curve is one-dimensional, a surface two-dimensional, e.t.c. It is a basic fact of nature that space-time is four-dimensional.

Certain mathematical discoveries in the nineteenth century, e.g. that the unit interval can be continuously mapped onto the unit square revealed that the intuitive notion of dimension is insufficient. Mathematical concepts, however, if they are not clear enough to be taken as primitive ideas, must be rigorously defined. Dimension theory results from a successful attempt, in the latter half of the nineteenth century, to give rigorous definitions to the vague notion of dimension expressed above.

## LITERATURE REVIEW:

Dimension theory as a subject had its beginnings in certain publications by Poincaré (Poincaré) and Lebesgue (Lebesgue). Poincaré considered curves as boundaries of surfaces, surfaces as boundaries of volumes e.t.c. Thus to separate a space of n dimensions one needs a space of n-1 dimensions. Poincaré's idea of dimension was given a rigorous topologically invariant definition by Brouwer (Brouwer) leading to the definition of the small inductive dimension ind and the large inductive dimension Ind on the class of topological spaces. Lebesgue's idea of dimension, on the other hand, lead to the definition of the covering dimension dim on the class of topological spaces and the metric dimension μ-dim on the class of metric spaces.

Ind, ind, dim and μ-dim are refered to as dimension functions. The dimension function μ-dim was defined by Alexandroff in 1935. μ-dim differs from the other three dimension functions in that it is defined on the class of metric spaces and its definition involves the metric. It is what we call a metric-dependent dimension function. Many other metric-dependent dimension functions have been defined to date. We thus have the metric-dependent dimension functions $d_1$, $d_2$ (Nagami and Roberts, 1965), $d_3$, $d_4$ (Nagami and Roberts, 1967), $d_5$ (Hodel, 1967), $d_6$ and $d_7$ (Smith, 1968).

Dimension functions, by requirement, must have a value of n on $R^n$, i.e. if d is a dimension function, then we must have $d(R^n) = n$. By convention, $d(\Phi) = -1$.

This thesis is a study of the metric-dependent dimension functions $d_1$, $d_2$, $d_3$, $d_4$, $d_5$, $d_6$, $d_7$, μ-dim and their relations with the covering dimension function dim which is the most widely used dimension function.

## SECTION 0

In this section we review results in general topology and basic dimensions theory. The proofs of the results in general topology can be found in "General Topology" by J.L. Kelly while the proofs of the results in dimension theory can be found in "Dimension Theory" by R. Engelkin.

**Theorem 0.1** (Urysohn's lemma)

Let X be a normal topological space and C, C' be two disjoint closed sets of X. Then ∃ a continuous function $f: X \longrightarrow I$ s.t. $f(C) = \{0\}$ and $f(C') = \{1\}$

**Theorem 0.2** (Tietze's extension theorem)

Let X be a normal topological space and F a closed subset of X. If $f: F \longrightarrow I$ is a continuous function, then f has a continuous extension $f^*: X \longrightarrow I$. I may be replaced in this theorem by R, $I^n$ or $R^n$.

**Defn 0.1**

A topological space X is said to be completely normal if every subspace of X is normal.

**Theorem 0.3**

Let X be a completely normal topological space and Y a subspace of X. then if U, U' are disjoint open sets of Y, $\exists$ disjoint open sets V, V' of X s.t. $V \cap Y = U$ and $V' \cap Y = U'$.

Defn 0.2

Let X be a set and $\mathcal{U} = \{U_\alpha, \alpha \in \mathcal{A}\}$ an indexed collection of subsets of X. Let n = -1, 0, 1, 2, 3.... we say $\mathcal{U}$ has order not exceeding n and write ord $\mathcal{U} \leq n$ if for any n+2 distinct members $\alpha_1, \alpha_2, \ldots, \alpha_{n+2}$ of $\mathcal{A}$ we have $\bigcap_{i=1}^{n+2} U_{\alpha_i} = \Phi$ (some authors define the order to be $\leq n$ if $\bigcap_{i=1}^{n+1} U_{\alpha_i} = \Phi$ for distinct $\alpha_i$).

We say ord $\mathcal{U} = n$ if ord $\mathcal{U} \leq n$ and ord $\mathcal{U} \not\leq n-1$. Note that the order depends on the indexing so, strictly speaking we should write something like $\underset{\alpha \in \mathcal{A}}{\text{ord}}$ $\{U_\alpha\}$ but this has not been the tradition. No confusion will arise over the indexing. Something like ord {bdry U, $U \in \mathcal{U}$} will mean if $U_1, U_2, \ldots U_{n+2}$ are distinct members of $\mathcal{U}$ then $\bigcap_{i=1}^{n+2}$ bdry $U_i = \Phi$.

Every set indexes itself so when we merely talk of a collection $\upsilon$ without giving an indexing, ord $\upsilon \leq n$ will mean $\bigcap_{i=1}^{n+2} U_i = \Phi$ for any n+2 distinct members $U_i$, $1 \leq i \leq n+2$, of $\upsilon$. Likewise, when we say a collection $\{U_\alpha, \alpha \in \Delta\}$ is locally finite, we shall mean that for each x, x has a nbhd intersecting $U_\alpha$ for only finitely many indices $\alpha$. The same will apply for point finiteness, point-boundedness and other such properties.

If $x \in X$, we say <u>the order of $\mathcal{U}$ at x does not exceed n</u> and write $\text{ord}_x \mathcal{U} \leq n$ if there are no n+2 distinct indices $\alpha_1, \alpha_2 \cdots \alpha_{n+2}$ s.t. $x \in U_{\alpha_i}$, $1 \leq i \leq n+2$. $\text{Ord}_x \mathcal{U} = n$ if $\text{ord}_x \mathcal{U} \leq n$ and $\text{ord}_x \mathcal{U} \not\leq n-1$. If Y is a subset of X, then $\text{ord} \mathcal{U}|_Y$ will always be with reference to the indexing $\{U \cap Y, U \in \mathcal{U}\}$.

<u>Defn 0.3</u>

Let X be a topological space and C, C' be two subsets of X. We say a subset Y of X separates C and C' if X-Y is the union of two disjoint relatively open sets one containing C and the other containing C'.

Three dimension functions, the <u>small inductive dimension ind</u>, the <u>large inductive dimension Ind</u> and the covering dimension dim are defined on the class of topological spaces as follows:-

Let X be a topological space.

<u>Defn 0.4</u>

- $\text{ind } X \leq -1$ iff $X = \Phi$
- for n = 0, 1, 2,....., $\text{ind } X \leq n$ if for any point $x \in X$ and closed set C of X s.t. $x \notin C$, $\exists$ a closed set B of X s.t. B separates $\{x\}$ and C and $\text{ind } B \leq n-1$. (Note the inductive nature of the definition).
- $\text{ind } X = n$ if $\text{ind } X \leq n$ and $\text{ind } X \not\leq n-1$.
- $\text{ind } X = \infty$ if $\text{ind } X \not\leq n$ for n = -1, 0, 1, 2,

Defn 0.5

- Ind X $\leq$ -1 iff X = $\Phi$
- for n = 0, 1, 2, Ind X $\leq$ n if for any pair C, C' of disjoint closed sets of X $\exists$ a closed set B of X s.t. B separates C and C' and Ind B $\leq$ n-1.
- Ind X = n if Ind X $\leq$ n and Ind X $\nleq$ n-1.
- Ind X = $\infty$ if Ind X $\nleq$ n for n = -1, 0, 1, 2,

Defn 0.6

For n = -1, 0, 1, 2,....dim X $\leq$ n if for any finite open cover $\upsilon$ of X, $\upsilon$ has an open refinement $\Theta$ s.t. ord $\Theta \leq$ n.

Theorem 0.4 (Otto-Eilenberg theorem).

Let X be a normal space. Then the following are equivalent (n $\geq$ 0):-

(i) dim X $\leq$ n

(ii) For any n+1 pairs $(C_i, C'_i$ $1 \leq i \leq n+1$ of disjoint closed sets of X $\exists$ closed sets $B_i$, $1 \leq i \leq n+1$ s.t. $B_i$ separates $C_i$ and $C'_i$ and $\bigcap_{i=1}^{n+1} B_i = \Phi$

(iii) For any n+1 pairs $(C_i, C'_i)$ $1 \leq i \leq n+1$ of disjoint closed sets of X $\exists$ pairs of disjoint closed sets $(E_i, E'_i)$ $1 \leq i \leq n+1$ s.t. $C_i \subset E_i$, $C'_i \subset E'_i$ and $\bigcup_{i=1}^{n+1} (E_i \cup E'_i) = X$

(iv) For any n+1 pairs $(C_i, C'_i)$ $1 \leq i \leq n+1$ of disjoint closed sets of X $\exists$ pairs $(U_i, U'_i)$ of open sets of X, $1 \leq i \leq n+1$ s.t. $\overline{U_i} \cap \overline{U'_i} = \Phi$, $C_i \subset U_i$, $C'_i \subset U'_i$, $1 \leq i \leq n+1$ and $\bigcup_{i=1}^{n+1} (U_i \cup U'_i) = X$.

Theorem 0.5

Let X be a normal space. Then the following are equivalent $(n \geq 0)$:-

(i) dim X = n

(ii) If F is a closed set of X and $f: F \longrightarrow S^n$ is a continuous function then f has an extension $f^*: X \longrightarrow S^n$.

Theorem 0.6

Let X be any topological space and F a closed set of X. Then if d is any of the dimension function ind, Ind or dim, we have $d(F) \leq d(X)$.
This is also clearly true for all the dimension functions discussed below except $d_6$ and $d_7$ and will be assumed without mention.

Theorem 0.7 (Countable sum theorem)

Let X be a normal topological space. Let X = $\bigcup_{\alpha\varepsilon\Delta} F_\alpha$ where $F_\alpha$ is a closed set of X and dim $F_\alpha$ $\leq n$ for each $\alpha$ then if $\Delta$ is countable or $\{F_\alpha, \alpha\varepsilon\Delta\}$ is l.f, then dim X $\leq$ n.

Theorem 0.8 (Urysohn's inequality)

Let X be a completely normal topological space Then if $X = \bigcup_{i=1}^{n+1} X_i$ we have

$$\text{Ind } X \leq n + \sum_{i=1}^{n+1} \text{Ind } X_i .$$

Defn 0.7

A subset A of a topological space X is said to be an $F_\sigma$ set if A is a countable union of closed sets of X. A is said to be a $G_\delta$ set if A is a countable intersection of open sets of X.

Defn 0.8

A topological space X is said to be perfectly normal if X is normal and every open subset of X is an $F\sigma$ set.

Theorem 0.9

Let X be a perfectly normal topological space. Then if Y is a subspace of X, then dim Y $\leq$ dim X.

Theorem 0.10

A Hausdorff topological space X is metrizable iff X is regular and has a $\sigma$-locally discrete base.

Theorem 0.11

A metrizable topological space is completely normal and perfectly normal.

Defn 0.9

Let $(X, \ell)$ be a metric space. Let $\upsilon$ be a collection of subsets of X. Then $\ell$-mesh $\upsilon$ (or just mesh $\upsilon$ if $\ell$ is understood) is defined to be $\sup\{\ell(U), U \in \upsilon\}$.

Theorem 0.12

For a metric space $(X,\rho)$ the following conditions are equivalent:-

(a) $\dim X \leq n$

(b) $\exists$ a sequence of l.f. open covers $\mathcal{U}_i$, $i \in N$, of X s.t. mesh $\mathcal{U}_i \leq 1/i$, ord $\{\bar{U}, U \in \mathcal{U}_i\} \leq n$, and $\mathcal{U}_{i+1} < \mathcal{U}_i$ $\forall i \in N$.

(c) $\exists$ a sequence of l.f. closed covers $\Omega_i$, $i \in N$, of X s.t. mesh $\Omega_i \leq 1/i$, ord $\Omega_i \leq n$ and $\Omega_{i+1} < \Omega_i$ $\forall$ $i \in N$,

(d) X has a $\sigma$.l.f. base $\chi$ s.t. ord $\{\text{bdry}\, U, U \in \chi\} \leq n-1$.

(e) X has a $\sigma$.f.l. base consisting of open sets with boundaries of dim $\leq n-1$.

(f) $X = X_1 \cup X_2$ with Ind $X_1 \leq 0$, Ind $X_2 \leq n-1$.

(g) Ind $X \leq n$.

Theorem 0.13

If X is a separable metric space then ind X = Ind X = dim X.

Theorem 0.14

ind $R^n$ = Ind $R^n$ = dim $R^n$ = n.

Theorem 0.15

If M is a subset of $R^n$, then dim M = n iff the interior of M in $R^n$ is non-empty.

SECTION 1

In this section we define the various metric dependent dimension functions and study relations between them.

The dimension function $d_1$ is defined inductively on metric spaces as follows:-

Def 1.1 (Nagami and Roberts, 1965).

Let $(X, \ell)$ be a metric space

- $d_1(X,\ell) \leq -1$ iff $X = \Phi$
- for $n \geq 0$, $d_1(X,\ell) \leq n$ iff for each pair $C$, $C'$ of closed sets of $X$ s.t. $\ell(C, C') > 0$ $\exists$ a closed set $B$ of $X$ s.t. $B$ separates $C$ and $C'$ and $d_1(B,\ell|_B) \leq n-1$ where $\ell|_B$ is the metric $\ell$ restricted to $B$.
- $d_1(X,\ell) = n$ iff $d_1(X,\ell) \leq n$ and $d_1(X,\ell) \not\leq n-1$.
- $d_1(X,\ell) = \infty$ iff $d_1(X,\ell) \not\leq n$ for $n = -1, 0, 1, 2...$

The metric-dependent dimension functions $d_2$, $d_3$, $d_4$, $d_5$, $d_6$, $d_7$ and $\mu$-dim are defined as follows:-

$(X,\ell)$ is always a metric space.

Def 1.2 (Nagami and Roberts, 1965).

- $d_2(X,\ell) \leq -1$ iff $X = \Phi$

- for $n \geq 0$, $d_2(X,\ell) \leq n$ iff for any $n+1$ pairs $(C_i, C'_i)$ $1 \leq i \leq n+1$, of closed sets of $X$ s.t. $\ell(C_i, C'_i) > 0$ $\exists$ closed sets $B_i$, $1 \leq i \leq n+1$ s.t. $B_i$ separates $C_i$ and $C'_i$ and $\bigcap_{i=1}^{n+1} B_i = \Phi$

- For $n = -1, 0, 1, 2, \ldots\ldots d_2 (X,\ell) = n$ if $d_2 (X,\ell) \leq n$ and $d_2 (X,\ell) \not\leq n-1$.
- $d_2 (X,\ell) = \infty$ if $d_2 (X,\ell) \not\leq n$ for $n = -1, 0, 1\ldots\ldots$

Def. 1.3 (Nagami and Roberts, 1967).

- $d_3 (X,\ell) \leq -1$ iff $X = \Phi$
- for $n \geq 0$, $d_3 (X,\ell) \leq n$ iff the following condition is satisfied:- Given any positive integer k and k pairs $(C_i, C'_i), 1 \leq i \leq k$, of closed sets of X such that $\ell(C_i, C'_i) > 0$, $\exists$ closed sets $B_i$, $1 \leq i \leq k$, of X s.t. $B_i$ separates $C_i$ and $C'_i$ and ord $\{B_i, 1 \leq i \leq k\} \leq n-1$.
- for $n = -1, 0, 1, 2\ldots d_3 (X,\ell) = n$ if $d_3 (X,\ell) \leq n$ and $d_3 (X,\ell) \not\leq n-1$.
- $d_3 (X,\ell) = \infty$ if $d_3 (X,\ell) \not\leq n$ for $n = -1, 0, 1, 2,\ldots.$

Def. 1.4 (Nagami and Roberts, 1967).

- $d_4 (X,\ell) \leq -1$ iff $X = \Phi$
- for $n \geq 0$, $d_4 (X,\ell) \leq$ iff X satisfies the following condition:- Given any sequence $(C_i, C'_i) i\varepsilon N$ of closed sets of X s.t. $\ell(C_i, C'_i) > 0 \ \forall i$, $\exists$ a sequence $B_i$, $i\varepsilon N$, of closed sets of X s.t. $B_i$ separates $C_i$ and $C'_i$ and ord$\{B_i, i = 1, 2, \ldots.\} \leq n-1$.

- for $n = -1, 0, 1, 2, \ldots\ldots, d_4(X,\ell) = n$ iff $d_4(X,\ell) \leq n$ and $d_4(X,\ell) \nleq n-1$.

- $d_4(X,\ell) = \infty$ if $d_4(X,\ell) \nleq n$ for $n = -1, 0, 1, 2, \ldots\ldots$

<u>Def. 1.5</u> (Hodel)

- $d_5(X,\ell) \leq -1$ iff $X = \Phi$
- for $n \geq 0$, $d_5(X,\ell) \leq n$ iff $(X,\ell)$ satisifes the following condition:- given any sequence $(C_i, C'_i)$, $i\varepsilon N$, of pairs of closed sets of X such that for some real number $\varepsilon$, $\varepsilon > 0$, $\ell(C_i, C'_i) \geq \varepsilon \forall i\varepsilon N$, $\exists$ a sequence $B_i$, $i\varepsilon N$, of closed sets of X s.t. $B_i$ separates $C_i$ and $C'_i$ and ord $\{B_i, i\varepsilon N\} \leq n-1$.

- for $n = -1, 0, 1, \ldots. d_5(X,\ell) = n$ if $d_5(X,\ell) \leq n$ and $d_5(X,\ell) \nleq n-1$.
- $d_5(X,\ell) = \infty$ if $d_5(X,\ell) \nleq n$ for $n = -1, 0, 1\ldots.$

<u>Defn. 1.6</u> (Smith, 1968).

- $d_6(X,\ell) \leq -1$ iff $X = \Phi$
- for $n \geq 0$, $d_6(X,\ell) \leq n$ iff for each sequence $(C_i, C'_i)$ of pairs of closed sets of X s.t. for some $\varepsilon > 0$, $\ell(C_i, C'_i) \geq \varepsilon \forall i$ and $\{X-C'_i, i\varepsilon N\}$ is locally finite, $\exists$ a sequence $B_i$, $i\varepsilon N$, of closed sets of X s.t. $B_i$ separates $C_i$ and $C'_i$ and ord $\{B_i, i\varepsilon N\} \leq n-1$.

- for $n = -1, 0, 1, \ldots\ldots d_6(X,\ell) = n$ iff $d_6(X,\ell) \leq n$ and $d_6(X,\ell) \not\leq n-1$.
- $d_6(X,\ell) = \infty$ if $d_6(X,\ell) \not\leq n$, $n = -1, 0, 1, 2, \ldots\ldots$

<u>Defn. 1.7</u> (Smith, 1968)

- $d_7(X,\ell) \leq -1$ iff $X = \Phi$
- for $n \geq 0$, $d_7(X,\ell) \leq n$ iff given any collection $(C_\alpha, C'_\alpha)$, $\alpha\varepsilon\Delta$, of pairs of closed sets of X s.t. for some $\varepsilon > 0$ $\ell(C_\alpha, C'_\alpha) \geq \varepsilon \forall \alpha\varepsilon\Delta$ and $\{X-C'_\alpha, \alpha\varepsilon\Delta\}$ is locally finite, then $\exists$ a collection $\{B_\alpha, \alpha\varepsilon\Delta\}$ of closed sets of X s.t. $B_\alpha$ separates $C_\alpha$ and $C'_\alpha$ for each $\alpha$ and ord $\{B_\alpha, \alpha\varepsilon\Delta\} \leq n-1$.

<u>Defn 1.8</u> (Alexandroff)

For $n = -1, 0, 1\ldots$: $\mu$-dim $(X,\ell) \leq n$ iff for any $\varepsilon > 0$ $\exists$ an open cover $\mathcal{U}$ of $(X,\ell)$ s.t. ord $\mathcal{U} \leq n$ and mesh $\mathcal{U} \leq \varepsilon$.

Evidently, $d_2 \leq d_3 \leq d_6 \leq d_5 \leq d_4$ and $d_6 \leq d_7$. We shall show that for any metric space $(X,\ell)$, $d_1(X,\ell) = d_4(X,\ell) = \dim X$.

<u>Theorem 1.1.</u> (Nagami and Roberts, 1965).

For any metric space $(X,\ell)$, $d_1(X,\ell) = \dim X$.

Proof: It is clear from a trivial induction that $d_1(X,\ell) \leq \operatorname{Ind} X = \dim X$. We show that $\dim X \leq d_1(X,\ell)$. The proof is by induction. Assume that for some n, $n = -1, 0, 1, 2, \ldots\ldots d_1(X,\ell) \leq n \Rightarrow \dim X \leq n$.

Suppose that $d_1(X,\ell) \leq n + 1$. Let $C$, $C'$ be disjoint closed sets of $X$. Let $E_i = \{x \in X: \ell(x, C) + \ell(x, C') \geq 1/i\}$, $i = 1, 2, \ldots\ldots$

Then $\ell(E_i, X\text{-Int } E_{i+1}) > 0 \ \forall \ i \in N$.

So for each $i \in N \ \exists$ an open set $G_i$ s.t. $E_i \subset G_i \subset \overline{G}_i \subset \text{Int } E_{i+1}$ and $d_1(\text{bdry } G_i) \leq n$. By the induction hypothesis, dim bdry $G_i \leq n \ \forall \ i \in N$. Clearly $G_i$, $i \in N$, satisfy:-

(i) $X = \bigcup_{i=1}^{\infty} G_i$

(ii) $\ell(C \cap \overline{G}_i, C' \cap \overline{G}_i) > 0$.

From (ii), and since $d_1(X,\ell) \leq n+1$, $\exists$ open sets $U_i$, $U'_i$ s.t.:-

(a1) $C \cap \overline{G}_i \subset U_i$, $C' \cap \overline{G}_i \subset U'_i$

(a2) $U_i \cap U'_i = \phi$

(a3) $d_1(X - (U_i \cup U'_i)) \leq n \ \forall \ i \in N$.

From (a3) and the induction hypothesis, we have:-

(a4) $\dim (X - (U_i \cup U'_i)) \leq n \ \forall \ i \in N$.

Let $V_i = U_i \cap G_i$, $V'_i = U'_i \cap G_i$.

Claim:-

(b1) dim bdry $V_i \leq n$, dim bdry $V'_i \leq n$.

(b2) $C \cap G_i \subset V_i$, $C' \cap G_i \subset V'_i$

(b3) $\overline{V}_i \cap C' = \overline{V'_i} \cap C = \phi$

(b4) $V_i \cap V'_i = \phi$

(b1) follows since bdry $V_i \subset$ bdry $U_i \cup$ bdry $G_i \subset [X-(U_i \cup U'_i)] \cup$ bdry $G_i$. And similarly for $V'_i$.

(b2) (b4) are clear and (b3) follows from (a1) and (a2).

Let $W_i = V_i - \bigcup_{j<i} \overline{V'}_j$, $W'_i = V'_i - \bigcup_{j<i} \overline{V}_j$ ($\bigcup_{j\in\phi} \overline{V}_j = \phi$).

Let $W = \bigcup_{i=1}^{\infty} W_i$, $W' = \bigcup_{i=1}^{\infty} W'_i$

Then W, W' are open, $C \subset W$, $C' \subset W'$ (from (i), (b2), and (b3)) and dim $(X - (W \cup W')) \leq n$. To see the last part, let $x \in X-(W \cup W')$. Either $x \notin (V_i \cup V'_i)$ $\forall$ $i \in N$ or $x \in (\text{bdry } V_i \cup \text{bdry } V'_i)$ for some i. If $x \notin (V_i \cup V'_i)$ $\forall$ $i \in N$, then, since $x \in G_{i_0}$ for some $i_0$, then $x \notin U_{i_0} \cup U'_{i_0}$ whence $x \in X-(U_{i_0} \cup U_{i_0})$. Thus $X-(W \cup W') \subset \bigcup_{i=1}^{\infty} [X-(U_i \cup U'_i)] \cup \bigcup_{i=1}^{\infty} [\text{bdry } (V_i) \cup \text{bdry } (V'_i]$. From the countable sum theorem, dim $X-(W \cup W') \leq n$, so Ind $(X-(W \cup W')) \leq n$. Thus dim X = Ind X $\leq n+1$. The result is trivial when n = -1 or $\infty$ and this completes the induction.

<u>Theorem 1.2</u> (Nagami and Roberts, 1967)
If X is a normal space with dim X $\leq$ n, then X satisfies the following condition:- If $(C_j, C'_j)$ $j \in N$ is a sequence of pairs of disjoint closed sets of X, then $\exists$ closed sets $B_j$, $j \in N$, s.t. $B_j$ separates $C_j$ and $C'_j$ and ord$\{B_j, j \in N\} \leq n-1$.

Proof: The collection of subsets of N containing precisely n+1 elements is countable. Denote these subsets by $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots$ $\exists$ open sets $U_{ij}$, $U'_{ij}$, i, $j \in N$, satisfying the following conditions:-

(i) $C_j \subset U_{ij}$, $C'_j \subset U'_{ij}$, $\overline{U}_{ij} \cap \overline{U'}_{ij} = \phi$ $\forall i, j \in N$

(ii) $U_{ij} \subset U_{i+1j}$, $U'_{ij} \subset U'_{i+1j}$ $\forall$ i, $j \in N$.

(iii) $\bigcup_{j \in \alpha_i} (U_{ij} \cup U'_{ij}) = X$

The construction is by induction on i.

Assume the construction achieved upto $i = k$.

By the Otto-Eilenberg theorem, $\exists$ open sets $(V_j, V'_j)$ $j \in \alpha_{k+1}$ s.t. $\bar{U}_{kj} \subset V_j$, $\bar{U}'_{kj} \subset V'_j$, $\bar{V}_j \cap \bar{V}'_j = \phi \ \forall \ j \in \alpha_{k+1}$ and $X = \bigcup_{j \in \alpha_{k+1}} (V_j \cup V'_j)$. Let $U_{k+1j} = U_{kj}$ if $j \notin \alpha_{k+1}$

$U_{k+1j} = V_j$ if $j \in \alpha_{k+1}$, $U'_{k+1j} = U'_{kj}$ if $j \notin \alpha_{k+1}$ and $U'_{k+1j} = V'_j$ if $j \in \alpha_{k+1}$. $U_{1i}$, $U'_{1i}$ are constructed by replacing $\bar{U}_{kj}$, $\bar{U}'_{kj}$ above with $C_j$, $C'_j$ if $j \in \alpha_1$, and letting $U_{1j}$, $U'_{1j}$ be open sets with disjoint closures containing $C_j$ and $C'_j$ respectively if $j \notin \alpha_1$. Clearly, $U_{ij}$, $U'_{ij}$ satisfy conditions (i), (ii), (iii).

Let $U_j = \bigcup_{i=1}^{\infty} U_{ij}$, $U'_j = \bigcup_{i=1}^{\infty} U'_{ij}$. Let $B_j = X - (U_j \cup U'_j)$. Then $B_j$, $j \in N$ are as required.

Theorem 1.2.5 (Nagami and Roberts, 1967).

For any metric space $(X, \rho)$, $d_4(X, \rho) = \dim X$.

Proof.

From theorem 1.2., $d_4(X, \rho) \leq \dim X$.

We show that $\dim X \leq d_4(X, \rho)$. It is enough to assume $d_4(X, \rho) \leq n$ and show $\dim X \leq n$. Suppose $d_4(X, \rho) \leq n$. X has a $\sigma$-locally discrete base $\mathcal{U} = \bigcup_{i \in N} \mathcal{U}_i$ where $\mathcal{U}_i$ is locally discrete. Let $V_i = \bigcup_{U \in \mathcal{U}_i} U$ and $E_{ik} =$

$\{x \in X: \rho(x, X - V_i) \geq 1/k$ . $d_4$ $(X, \rho) \leq n$ implies that $\exists$ open sets $G_{ik}$ i, $k \in N$ s.t. $E_{ik} \subset G_{ik} \subset V_i$ and ord

$\{\text{bdry } G_{ik}, \ i, k, \in N\} \leq n-1$. Then $\mathcal{G} = \{G_{ik} \cap U, \ i, k \in N,$ $U \in \mathcal{U}_i\}$ is a $\sigma$-locally discrete base of X with ord $\{\text{bdry}$ $G, \ G \in \mathcal{G}\} \leq n-1$ (after noting that bdry $(G_{ik} \cap U) \subset$ bdry $G_{ik}$ and bdry $(G_{ik} \cap U) \cap$ bdry $(G_{ik} \cap V) = \phi$ for $U \neq V$, $U, V \in \mathcal{U}_i$). It follows from theorem 0.12 that dim X $\leq$ n. Having shown that $d_1$ and $d_4$ are equal to the covering dimension dim, we now concentrate on the dimension functions $d_2$, $d_3$, $d_5$, $d_6$, $d_7$, $\mu$-dim, and dim.

It is clear from theorem 0.12 that $\mu$-dim $(X,\rho) \leq$ dim X for any metric space $(X,\rho)$.

Lemma 1.1.

If X is a paracompact topological space and $\mathcal{U}$ is an open cover of X with ord $\mathcal{U} \leq n$, $n = -1, 0, 1, 2, \ldots,$ then $\mathcal{U}$ has an open locally finite refinement $\mathcal{V}$ with ord $\mathcal{V} \leq n$.

Proof. Let $\mathcal{U}$ be an open cover of X with ord $\mathcal{U} \leq n$. Since X is paracompact, $\exists$ an open locally finite refinement $\mathcal{V}'$ of $\mathcal{U}$. $\exists$ a function $f: \mathcal{V}' \longrightarrow \mathcal{U}$ s.t. for each $W \in \mathcal{V}'$, $W \subset f(W)$. For each $U \in \mathcal{U}$, let $g(U) = \bigcup_{\substack{W \in \mathcal{V}' \\ f(W) = U}} W$.
Let $\mathcal{V} = \{g(U), \ U \in \mathcal{U}\}$.

Then clearly ord $\mathcal{V} \leq$ ord $\mathcal{U} \leq n$. It is also easy to see that $\mathcal{V}$ is locally finite and the lemma is proved.

Theorem 1.3 (Hodel)

For any metric space $(X,\rho)$, $d_5(X,\rho) \leq \mu$-dim $(X,\rho)$.

Proof: The proof is trivial if $\mu$-dim $(X,\mathcal{C}) = -1$, Assume $\mu$-dim $(X,\mathcal{C}) \leq n \geq 0$. Let $(C_i, C'_i)$ be a sequence of pairs of closed sets with $\mathcal{C}(C_i, C'_i) \geq \epsilon > 0$ $\forall$ $i \in N$ for some $\epsilon$. Since $\mu$-dim $(X,\mathcal{C}) \leq n$, $\exists$ an open cover $\mathcal{U}'$ of X s.t. ord $\mathcal{U}' \leq n$ and mesh $\mathcal{U}' < \epsilon$. From lemma 1.1, $\exists$ a l.f. open cover $\mathcal{U}$ of X s.t. ord $\mathcal{U}$ $\leq n$ and mesh $\mathcal{U} \leq \epsilon$. Because $\mathcal{U}$ is l.f. and X is normal, we can find a closed cover $\{E_u, U \in \mathcal{U}\}$ of X s.t. $E_u \subset U$ for all U . Using normality, we can construct a sequence $G_{iu}$ (for each U) of open sets of X s.t. $E_u \subset G_{1u} \subset \overline{G}_{1u} \subset G_{2u} \subset \overline{G}_{2u} \ldots\ldots \subset U$. If we let $\mathcal{U}_i = \{G_{iu}, U \in \mathcal{U}\}$ then mesh $\mathcal{U}_i < \epsilon$, $\mathcal{U}_i$ is l.f., and $\mathcal{U}_i$ covers X for each i.

Let $H_i = \bigcup_{\substack{U \in \mathcal{U} \\ G_{iu} \cap C_i \neq \phi}} G_{iu}$, $F_i = \bigcup_{\substack{U \in \mathcal{U} \\ G_{iu} \cap C_i \neq \phi}} \overline{G}_{iu}$.

Since mesh $\mathcal{U}_i < \epsilon$ , $F_i \cap C'_i = \phi$. Also, $F_i$ is closed because $\mathcal{U}_i$ is l.f. $H_i$ is an open set containing $C_i$ and $H_i \subset F_i$ so if we set $B_i = F_i - H_i$, then $B_i$ is a closed set separating $C_i$ and $C'_i$.

We show that ord $\{B_i, i = 1, 2, \ldots\ldots\} \leq n-1$ Suppose $x \in \bigcap_{k=1}^{m} B_{i_k}$ where $i_k$, $i \leq k \leq m$ are distrrict.

Then for each k, $1 \leq k \leq m$, $\exists$ $U_k \in \mathcal{U}$ s.t. $x \in \overline{G}_{i_k U_k} - G_{i_k U_k}$. $U_k$ $1 \leq k \leq m$ are distinct. For suppose $U_k = U_{k'}$ with $i_k < i_{k'}$. Then we would have $x \in \overline{G}_{i_k U_k}$ and $x \notin G_{i_{k'} U_k}$ a contradiction since $\overline{G}_{i_k U_k} \subset G_{i_{k'} U_k}$

So $x \in \overline{G}_{i_k U_k} \subset U_k$ for $1 \le k \le m$ with $U_k$ distinct. Also, if $i_0 = \min \{i_k, 1 \le k \le m\}$, then $x \notin G_{i_0 U_k}$ $1 \le k \le m$. But $\mathcal{U}_{i_0}$ is a cover of X so $x \in G_{i_0 U_0}$ for some $U_0$. Of course $U_0 \neq U_k$ for $1 \le k \le m$. So $x \in \bigcap_{k=0}^{m} U_k$ with $U_k$ $0 \le k \le m$ distinct. If we put m=n+1, then we see that $\bigcap_{k=1}^{m} B_{i_k} = \phi$ since ord $\mathcal{U} \le n$. So ord $\{B_i, i = 1, 2, \ldots\} \le n-1$ as required. Thus $d_5(X, \ell) \le n$ and it follows that $d_5 (X, \ell) \le \mu\text{-dim} (X, \ell)$.

We can summarize the results so far obtained in the following proposition.

Proposition 1.1.

For a metric space $(X, \ell)$, $d_2(X, \ell) \le d_3(X, \ell) \le d_6(X, \ell) \le d_5 (X, \ell) \le \mu\text{-dim} (X, \ell) \le \dim X$ and $d_6 (X, \ell) \le d_7 (X, \ell)$.

Remark 1.1.

It is also true that $d_7 (X, \ell) \le \mu\text{-dim} (X, \ell)$. This will be proved in Chapter 4 after we have developed the theory of Lebesgue cover characterizations of metric dependent dimension functions.

To qualify as dimension functions, the above functions should have a value of n or $R^n$, euclidean n-space. To that end we prove:-

Theorem 1.4
If $(X,\varrho)$ is a locally compact metric space, then

$d_2(X,\varrho) = d_3(X,\varrho) = d_6(X,\varrho) = d_7(X,\varrho) = d_5(X,\varrho)$
$= \mu\text{-dim}\,(X,\varrho) = \dim X$.

Proof: Let X be a locally compact metric space. In view of prop. 1.1. and remark 1.1., it suffices to prove $d_2(X,\varrho) \geq \dim X$. This is obvious if $d_2(X,\varrho) = -1, \infty$. Assume $d_2(X,\varrho) \leq n \geq 0$. Every point of X has a compact, hence closed, nbhd. Since in a compact metric space $E \cap F = \phi \Longrightarrow \varrho(E, F) > 0$ for E, F closed, we immediately have $\dim Y \leq n$ if Y is a compact subspace of X. Thus each $x \in X$ has a nbhd, and hence an open nbhd of dim $\leq n$. So X has an open cover $\mathcal{W}$ s.t. for $W \in \mathcal{W}$ dim $W \leq n$. It follows, since X is normal and paracompact, that X has a l.f. open cover $\mathcal{U}$ s.t. $\{\bar{U}, U \in \mathcal{U}\}$ refines $\mathcal{W}$. Thus dim $\bar{U}$ $\leq n$ for $U \in \mathcal{U}$. From theorem 0.12 $\bar{U}$ has a $\sigma$.l.f. (in $\bar{U}$) base $\mathcal{B}'_u$ consisting of sets with boundaries (in $\bar{U}$) of dim $\leq n-1$. Since $\bar{U}$ is closed in X, it follows that $\mathcal{B}'_u$ is $\sigma$.l.f. in X.

Let $\mathcal{B}_u$ be the collection of those members of $\mathcal{B}'_u$ whose closures in X are contained in U. Then $\mathcal{B}_u$ is a $\sigma$.l.f. (in X) base for U whose members have boundaries in X of dim $\leq n-1$. Let $\mathcal{B} = \bigcup_{U \in \mathcal{U}} \mathcal{B}_u$. Then $\mathcal{B}$ is a $\sigma$.l.f. base for X with boundaries of dim $\leq n-1$. It follows from theorem 0.12 that dim $X \leq n$.

To see that $\mathcal{B}$ is $\sigma$.l.f., let $\mathcal{B}_u = \bigcup_{i=1}^{\infty} \mathcal{B}^i_u$, where $\mathcal{B}^i_u$ is l.f. Let $x \in X$ and let i be fixed. $\exists$ a nbhd $V_0$ of x which intersects only a finite number of the members of $\mathcal{U}$, say $U_1, U_2, \ldots, U_k$. For each j, $1 \leq j \leq k$, $\exists$ a nbhd $V_j$ of x which intersects only finitely many members of $\mathcal{B}^i_{U_j}$. Let $V = \bigcap_{j=0}^{k} V_j$. Then V is a nbhd of x intersecting only finitely many members of $\bigcup_{U \in \mathcal{U}} \mathcal{B}^i_u$. Thus $\bigcup_{U \in \mathcal{U}} \mathcal{B}^i_u$ is l.f. for each i. Since $\mathcal{B} = \bigcup_{i=1}^{\infty} [\bigcup_{U \in \mathcal{U}} \mathcal{B}^i_u]$, $\mathcal{B}$ is $\sigma$-l.f.

The equality of the various dimension functions does not, however, appear to be a strong condition on a metric space. We give an example of a non-locally compact non-complete metric space X where the above dimension functions coincide. We note that if $d_2(X,\mathcal{C}) \leq 0$ then $d_1(X,\mathcal{C}) \leq 0$ so dim $X \leq 0$. It is obvious that in that case all the function coincide. Also if dim X=1 then, from the above observation we cannot have $d_2(X,\mathcal{C}) \leq 0$ so we must have $d_2(X,\mathcal{C})=1$ and hence $d(X,\mathcal{C}) = 1$ where d is any of the functions $d_2$, $d_3$, $d_5$, $d_6$, $d_7$ or $\mu$-dim. In view of this, we would like the example we give of a non-locally compact non-complete space where the dimension functions coincide to have dim = 2.

<u>Example 1.1.</u> (Nagami and Roberts, 1967)
Let A be the subset $\{(x_1, x_2, x_3): x_1 = 0\}$ of $I^3$. Let B be the subset $\{(x_1, x_2, x_3): x_1, x_2, x_3$ are rational$\}$. Let $X = A \cup B$. We have dim $X \leq 2$ from

the countable sum theorem. Also $d_2(X,\varrho) \geq d_2(A,\varrho')$ $= 2$ ($d_2$ satisfies an obvious subset theorem).
The metrics $\varrho$ and $\varrho'$ are the euclidean metric and its restriction to A. $d_2(A,\varrho') = 2$ from theorem 1.4.
We have, from proposition 1.1. that $d_2(X,\varrho'') = d_3(X,\varrho'')$ $= d_5(X,\varrho'') = d_6(X,\varrho'') = d_7(X,\varrho'') = \mu\text{-dim}(X,\varrho'')$ $= \dim X$ where $\varrho''$ is the restriction to X of the euclidean metric. X is not locally compact at any point, because for $x \in X$, assume U is a compact nbhd of x. Then for some open subset V of $I^3$, $Q^3 \cap V \subset U$ (where $Q^3 = Q \times Q \times Q$). Since U is compact, it is closed in $I^3$ so $V \subset \overline{Q^3 \cap V} \subset U$ a contradiction since the 'irrationals' in V which are not on bdry $I^3$ are not contained in X. Obviously X is not complete since it is not closed in $I^3$.

We will need the following lemma to prove the next theorem.

<u>Lemma 1.2</u>

If $\{F_\alpha, \alpha \in \mathcal{A}\}$ is a l.f. collection of closed sets of a paracompact Hausdorff topological space X and $\{U_\alpha, \alpha \in \mathcal{A}\}$ is a collection of open sets of X s.t. $F_\alpha \subset U_\alpha$ $\forall \alpha \in \mathcal{A}$, then $\exists$ a collection $\{V_\alpha, \alpha \in \mathcal{A}\}$ of open sets of X s.t. $F_\alpha \subset V_\alpha \subset U_\alpha$ and $\{V_\alpha, \alpha \in \mathcal{A}\}$ is of the same <u>type</u> as $\{F_\alpha, \alpha \in \mathcal{A}\}$, i.e. for any subset $\mathcal{B}$ of $\mathcal{A}$, $\bigcap_{\alpha \in \mathcal{B}} V_\alpha = \phi$ iff $\bigcap_{\alpha \in \mathcal{B}} F_\alpha = \phi$.

For a proof of this lemma, see Nagami "Dimension Theory" prop 9.2 pp 47.

Lemma 1.3

Let X be a paracompact Hausdorff space and let $\{U_\alpha, \alpha\in\mathcal{A}\}$ be a l.f. or countable open cover of X s.t. $\{U_\alpha, \alpha\in\mathcal{A}\}$ has a refinement $\{G_\alpha, \alpha\in\mathcal{A}\}$ with $\overline{G}_\alpha \subset U_\alpha$ and ord $\{\text{bdry } G_\alpha, \alpha\in\mathcal{A}\} \leq n-1$. Then $\{U_\alpha, \alpha\in\mathcal{A}\}$ has an open refinement of order $\leq n$.

Proof: First take the case where $\{U_\alpha, \alpha\in\mathcal{A}\}$ is l.f. Let $\leq$ be a well ordering on $\mathcal{A}$ and $<$ the associated strict partial order i.e. $\alpha<\beta$ iff $\alpha\leq\beta$ and $\alpha\neq\beta$. For each $\alpha\in\mathcal{A}$, let $E_\alpha = \overline{G_\alpha - \bigcup_{\beta<\alpha} G_\beta}$ $(\bigcup_{\beta\in\phi} G_\beta = \phi)$.

Then $E_\alpha \subset \overline{G}_\alpha \subset U_\alpha$. Claim:- ord $\{E_\alpha, \alpha\in\mathcal{A}\} \leq n$. For suppose $\alpha_1, \alpha_2, \ldots\ldots, \alpha_{n+2}$ are n+2 distinct members of $\mathcal{A}$. W.L.G. assume $\alpha_1<\alpha_2<\ldots\ldots<\alpha_{n+2}$. Suppose $x\in \bigcap_{i=1}^{n+2} E_{\alpha_i}$. That $x\in E_{\alpha_{n+2}}$ implies $x\notin G_{\alpha_i}$, $i<n+2$, because the $G_\alpha$'s are open. But $E_{\alpha_i}\subset \overline{G}_{\alpha_i}$ so $x\in\overline{G}_{\alpha_i}$ $1\leq i\leq n+2$. Thus $x\in \text{bdry } G_{\alpha_i}$ $1\leq i\leq n+1$. (note that the condition of the theorem implies $n\geq 0$). This is impossible because ord $\{\text{bdry } G_\alpha, \alpha\in\mathcal{A}\}\leq n-1$. Since $E_\alpha\subset U_\alpha$, $\{E_\alpha, \alpha\in\mathcal{A}\}$ is l.f. From lemma 1.2, $\exists$ open sets $V_\alpha, \alpha\in\mathcal{A}$ s.t. $E_\alpha \subset V_\alpha \subset U_\alpha$ and ord $\{V_\alpha, \alpha\in\mathcal{A}\}$ $\leq n$. Because $\leq$ is a well ordering on $\mathcal{A}$, $\{E_\alpha, \alpha\in\mathcal{A}\}$, and therefore $\{V_\alpha, \alpha\in\mathcal{A}\}$, covers X. Thus $\{V_\alpha, \alpha\in\mathcal{A}\}$ is the required refinement. If $\{U_\alpha, \alpha\in\mathcal{A}\}$ is countable then it may be taken as $U_i$, $i\in N$ in which case $\{E_i, i\in N\}$ is still l.f. and the result follows.

Theorem 1.5 (Hodel)

If $(X,\rho)$ is a separable metric space, then $d_5(X,\rho) = \mu\text{-dim}\,(X,\rho)$.

Proof: It suffices to show that $\mu\text{-dim}\,(X,\rho) \leq d_5(X,\rho)$ (in view of prop. 1.1.). Leaving out the trivial cases $d_5(X,\rho) = -1, \infty$, assume $d_5(X,\rho) \leq n \geq 0$. Let $\epsilon > 0$ be given, and let $x_i$, $i \in N$ be a dense subset of X. Let $E_i = \overline{B(x_i, \epsilon)}$ and $U_i = B(x_i, 2\epsilon)$. Then $\{E_i, i \in N\}$ is a cover of X since any $x \in X$ must satisfy $\rho(x, x_i) < \epsilon$ for some i. We have $\rho(E_i, X-U_i) \geq \epsilon \,\forall\, i \in N$ so $d_5(X,\rho) \leq n$ implies $\exists$ open sets $V_i$, $i \in N$ s.t. $E_i \subset V_i \subset \overline{V}_i \subset U_i$ and ord $\{\text{bdry } V_i, i \in N\} \leq n-1$. From lemma 1.3, $U_i$ has an open refinement of order $\leq n$. This refinement also has mesh $\leq 4\epsilon$. Since $\epsilon$ is arbitrary, this shows that $\mu\text{-dim}\,(X,\rho) \leq n$.

Theorem 1.6 (Nagami and Roberts, 1967)

If $(X,\rho)$ is a totally bounded metric space, then $d_3(X,\rho) = d_6(X,\rho) = d_5(X,\rho) = d_7(X,\rho) = \mu\text{-dim}\,(X,\rho)$.

Proof: In view of prop. 1.1. and remark 1.1., we need only sow that $\mu\text{-dim}\,(X,\rho) \leq d_3(X,\rho)$. Leaving out the cases $d_3(X,\rho) = -1, \infty$, assume $d_3(X,\rho) \leq n \geq 0$. Let $\epsilon > 0$ be given. Since $(X,\rho)$ is totally bounded, $\exists$ a finite cover $\{B(x_i, \epsilon), 1 \leq i \leq k\}$ of X by open balls of radius $\epsilon$. Let $E_i = \overline{B(x_i, \epsilon)}$ and $U_i = B(x_i, 2\epsilon)$. Proceeding as in the proof of theorem 1.5 we obtain the result.

So far we have seen conditions under which certain dimension functions coincide. While the various dimension function do not always coincide, any two of them, say d and d' may only differ within the limits of the inequality $d(X,\mathcal{E}) \leq 2d'(X,\mathcal{E})$. We shall now prove this inequality but first we prove a lemma.

Lemma 1.4 (Roberts)

Let X be any topological space. Let $G_j$, j = 0, 1, 2, 3,.... be open sets of X s.t. $G_0 = \phi$, $\overline{G}_j \subset G_{j+1}$, j = 0, 1, 2, .... and $X = \bigcup_{j=1}^{\infty} G_j$. Let $F_j = \overline{G}_j - G_{j-1}$, j=1, 2,.... Suppose C and C' are disjoint closed sets of X and $B_j$, j = 1, 2,..... are closed subsets of $F_j$ s.t. $B_j$ separates $C \cap F_j$ and $C' \cap F_j$ in $F_j$. Then $\exists$ a closed set B of X separating C and C' and s.t. $B \subset \bigcup_{j=1}^{\infty} (B_j \cup \text{bdry } G_j)$.

Proof: Let $F_j - B_j = U_j \cup V_j$ where $C \cap F_j \subset U_j$, $C' \cap F_j \subset V_j$, and $U_j$, $V_j$ are disjoint relatively open subsets of $F_j$. Set $B = \bigcup_{j=1}^{\infty} [B_j \cup (U_j \cap V_{j+1}) \cup (U_{j+1} \cap V_j)]$. (Fig 1.1)

We have:-

(a1) $B \cap (C \cup C') = \phi$. For, obviously, $B_j \cap (C \cup C') = \phi$. On the other hand if $x \in U_j \cap V_{j+1}$, then $x \in F_{j+1}$ and $x \notin U_{j+1} \supset C \cap F_{j+1}$ so $x \notin C$. Similarly $x \in F_j$ and $x \notin C' \cap F_j$ so $x \notin C'$. Similarly for $x \in U_{j+1} \cap V_j$. Let $U = (\bigcup_{j=1}^{\infty} U_j) - B$, $V = (\bigcup_{j=1}^{\infty} V_j) - B$. In view of (a1) and the fact that $C \subset \bigcup_{j=1}^{\infty} U_j$, $C' \subset \bigcup_{j=1}^{\infty} V_j$, we have (a2) $C \subset U$, $C' \subset V$.

Because $B \supset X - (\bigcup_{j=1}^{\infty} U_j \cup \bigcup_{j=1}^{\infty} V_j)$, we have (a3) X-B $= U \cup V$. (a4) B is closed.

For suppose x is a limit point of B. $x \in G_r$ for some r which means x must be a limit point to $\bigcup_{j=1}^{r} [B_j \cup (U_j \cap V_{j+1}) \cup (U_{j+1} \cap V_j)]$ which in turn means x is a limit point to $B_k$ or $U_k \cap V_{k+1}$ or $U_{k+1} \cap V_k$ for some $1 \leq k \leq r$. If $x \notin \bigcup_{j=1}^{\infty} B_j$, then x is a limit point to $U_k \cap V_{k+1}$ or $U_{k+1} \cap V_k$. Suppose $x \in \overline{U_k \cap V_{k+1}}$; then $x \in F_k$, $x \in \overline{U}_k$ and, because $U_k$, $V_k$ are relatively open and disjoint, $x \notin V_k$. Since also $x \notin B_k$, $x \in U_k$. Similarly $x \in V_{k+1}$ so $x \in U_k \cap V_{k+1}$. Similarly if $x \in \overline{U_{k+1} \cap V_k}$ ($x \notin \bigcup_{j=1}^{\infty} B_j$) then $x \in U_{k+1} \cap V_k$. Thus for $x \in \overline{B}$, we must have $x \in B$ so B is closed.

(a5) U, V are disjoint. For suppose $x \in U$. Then $x \in U_r - B$ for some r. Since $V_j \subset F_j$, $U_j \subset F_j$, and $F_i \cap F_j = \phi$ if $|i-j| > 1$, we only need to show that $x \notin V_{r+1}$ and $x \notin V_{r-1}$. Of course $x \notin V_r$. But $x \in V_{r+1} \Rightarrow x \in U_r \cap V_{r+1} \subset B$ contradicting $x \in U_r - B$. Similarly for $x \in V_{r-1}$ so $U \cap V = \phi$.

(a6) U, V are open. For let x be a limit point of U. Because $x \in G_r$ for some r, we must have x being a limit point of some $U_k$, $1 \leq k \leq r$. If $x \in V$, we must have $x \in V_{k+1} - B$ or $x \in V_{k-1} - B$ (because $x \in F_k$). We would then have x being a limit point of $U_k \cap V_{k+1}$ or $U_k \cap V_{k-1}$ and hence of B contradicting the fact that B is closed and $x \in V_{k+1} - B$ or $x \in V_{k-1} - B$. So $x \notin V$. Similarly, Fig 1.1

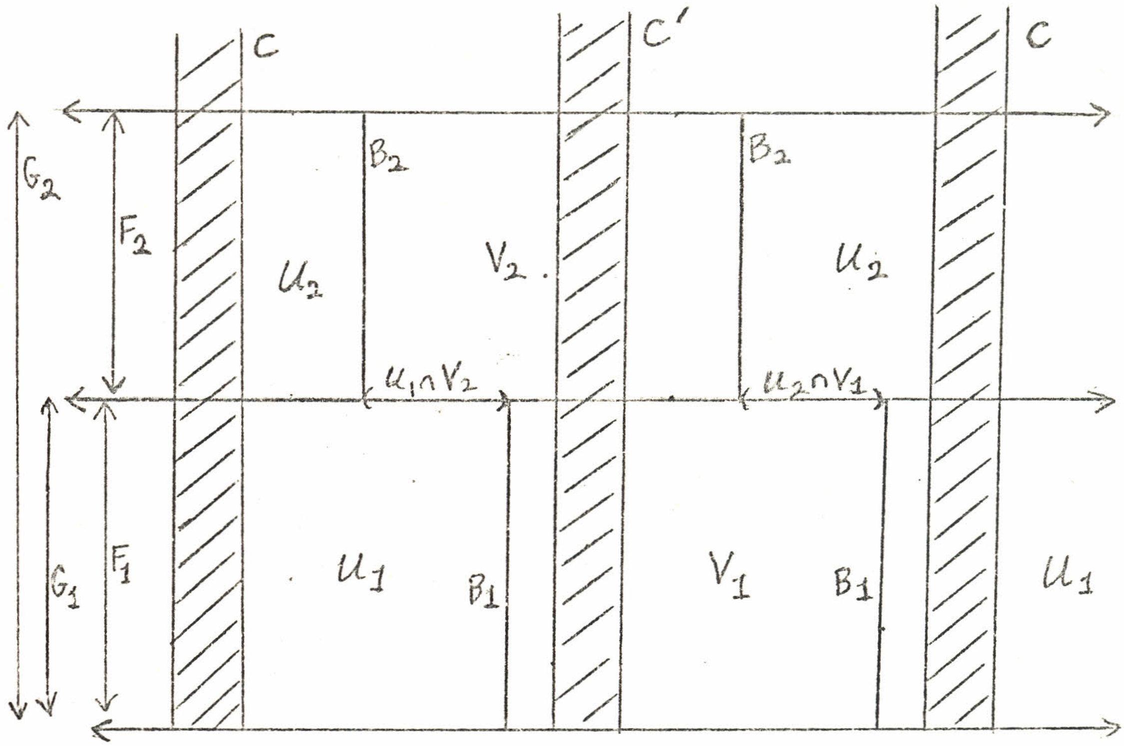
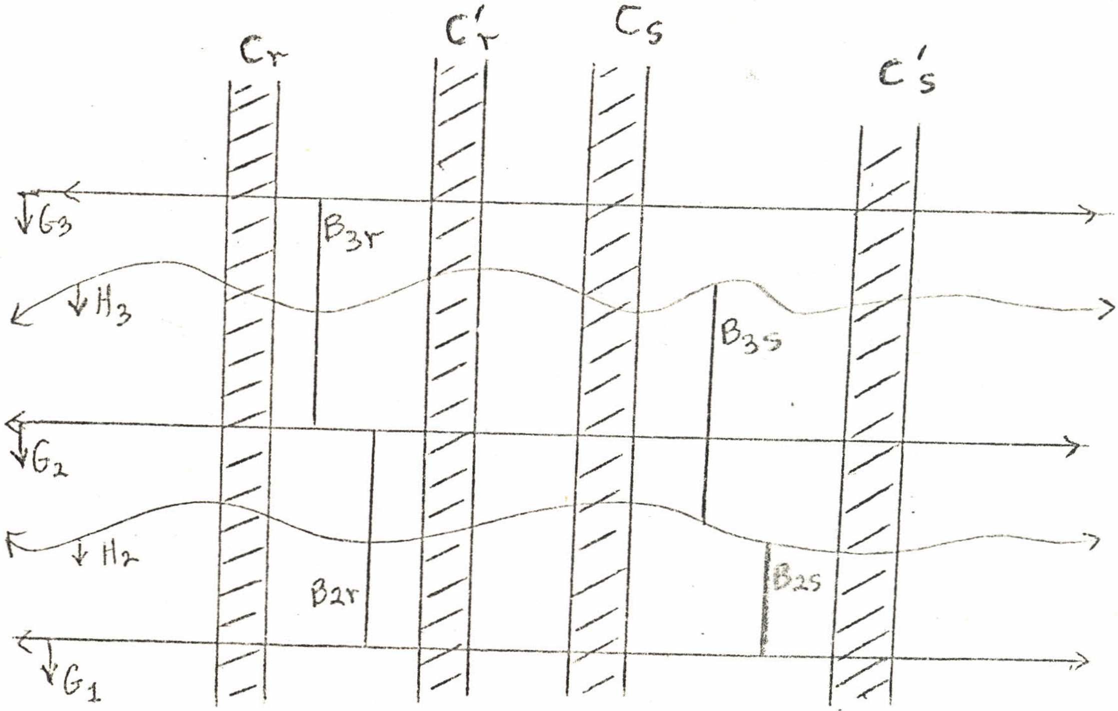



Fig 1.2




$1 \leq r \leq n, \quad n+1 \leq s \leq 2n+1$

U does not contain points of $\overline{V}$. So U, V are disjoint closed sets of $U \cup V$. Since $U \cup V = X-B$ is open, U, V are open. Conditions (a2), (a4), (a5), (a6) mean that B is as required. (Of course $U_j \cap V_{j+1}$ and $U_{j+1} \cap V_j$ are contained in $F_j \cap F_{j+1}$ which is equal to bdry $G_j$).

Theorem 1.7 (Roberts)

Let $(X, \rho)$ be a metric space. Let d, d' be any of the dimension functions $d_2$, $d_3$, $d_5$, $d_6$, $d_7$, μ-dim, or dim; then $d(X,\rho) \leq 2d'(X,\rho)$.

Proof. In view of proposition 1.1 and remark 1.1, it suffices to show that $\dim X \leq 2d_2(X,\rho)$. Assume $d_2(X,\rho) \leq n$. Let $(C_j, C'_j)$ $1 \leq j \leq 2n+1$ be pairs of disjoint closed sets of X. We want to construct closed sets $B_j$ $1 \leq j \leq 2n+1$ s.t. $B_j$ separates $C_j$ and $C'_j$ $\forall_j$, $1 \leq j \leq 2n+1$ and $\bigcap_{j=1}^{2n+1} B_j = \phi$.

$\exists$ open sets $G_i$, $i \in N$ s.t.:-

(a1) $\bigcup_{i \in N} G_i = X$

(a2) $\rho(C_j \cap \overline{G}_i, C'_j \cap \overline{G}_i) > 0$ $\forall$ $i \in N$, $1 \leq j \leq 2n+1$

(a3) $\rho(\overline{G}_i, X-G_{i+1}) > 0$ $\forall$ $i \in N$.

Infact, let $G_i = \bigcap_{j=1}^{2n+1} \{x \in X: \rho(x, C_j) + \rho(x, C'_j) > 1/i\}$. It is clear that (a1) to (a3) aresatisfied.

Let $F_i = \overline{G}_i - G_{i-1}$ $(G_0 = \phi)$. Then $\rho(C_j \cap F_i, C'_j \cap F_i) > 0$ $\forall$ $i \in N$, $1 \leq j \leq 2n+1$. Since $d_2(X,\rho) \leq n$, and from (a3), $\exists$, for each $i \in N$, closed sets $B'_{ij}$ $1 \leq j \leq n$ and an open set $H_i$ s.t. $B'_{ij}$ separates $C_j \cap F_i$ and $C'_j \cap F_i$ $\forall$ j, $1 \leq j \leq n$ $\overline{G}_{i-1} \subset H_i \subset \overline{H}_i \subset G_i$ $(G_0 = \phi)$ and $(\bigcap_{j=1}^{n} B'_{ij})$

$\cap$ bdry $H_i = \phi$. Let $B_{ij} = B'_{ij} \cap F_i$ $1 \leq j \leq n$.

Then $B_{ij}$, $1 \leq j \leq n$ and $H_i$ satisfy:-

(b1) $B_{ij}$ separates $C_j \cap F_i$ and $C'_j \cap F_i$ in $F_i$ for $1 \leq j \leq n$

(b2) $\overline{G}_{i-1} \subset H_i \subset \overline{H}_i \subset G_i$ $\forall$ $i \in N$.

(b3) $(\bigcap_{j=1}^{n} B_{ij}) \cap$ bdry $H_i = \phi$.

It is clear from (b2) that $F_i \cap$ bdry $H_{i'} = \phi$ if $i \neq i'$. Combining with (b3), and since $B_{ij} \subset F_i$, we obtain:-

(b4) $(\bigcap_{j=1}^{n} B_{ij}) \cap (\bigcup_{k \in N}$ bdry $H_k) = \phi$ $\forall$ $i$.

It is also clear that:-

(b5) $B_{ij} \cap B_{i'j'} \subset F_i \cap F_{i'} \subset$ bdry $G_i \cup$ bdry $G_{i'}$ if $i \neq i'$.
$1 \leq j \leq n$

From lemma 1.4, for each j, $1 \leq j \leq n$, $\exists$ a closed set $B_j$ s.t. $B_j$ separates $C_j$ and $C'_j$ and:-

(b6) $B_j \subset \bigcup_{i \in N} (B_{ij} \cup$ bdry $G_i)$

We now turn to the case where $n+1 \leq j \leq 2n+1$.

From (a1), (a2) and b2), it is clear that:-

(c1) $\bigcup_{i \in N} H_i = X$

(c2) $\overline{H}_i \subset H_{i+1}$ $\forall$ $i \in N$

(c3) $\varrho(C_j \cap \overline{H}_i, C'_j \cap \overline{H}_i) > 0$ $\forall$ $i \in N$, $1 \leq j \leq 2n+1$.

(c4) $(\bigcup_{i \in N}$ bdry $G_i) \cap (\bigcup_{i \in N}$ bdry $H_i) = \phi$.

Let $F'_i = \overline{H}_i - H_{i-1}$ $i \in N$ $(H_0 = \phi)$. As in the case for

$G_i$, we obtain closed sets $B_{ij}$ $n+1 \leq j \leq 2n+1$ s.t. $B_{ij}$, separates $C_j \cap F'_i$ and $C'_j \cap F'_i$ in $F'_i$ and :-

(d1) $\bigcap_{j=n+1}^{2n+1} B_{ij} = \phi$.

From lemma 1.4 $\exists$ closed sets $B_j$ $n+1 \leq j \leq 2n+1$ s.t. $B_j$ separates $C_j$ and $C'_j$ in X, $n+1 \leq j \leq 2n+1$ and:-

(d2) $B_j \subset \bigcup_{i \in N} (B_{ij} \cup \text{bdry } H_i)$. We have:-

(d3) $B_{ij} \cap B_{i'j'} \subset F'_i \cap F'_{i'} \subset \text{bdry } H_i \cup \text{bdry } H_{i'}$ if $i \neq i'$, $n+1 \leq j \leq 2n+1$

Now $B_j$, $1 \leq j \leq 2n+1$ are closed sets s.t. $B_j$ separates $C_j$ and $C'_j$ in X.

Claim: $\bigcap_{j=1}^{2n+1} B_j = \phi$. For suppose $x \in \bigcap_{j=1}^{2n+1} B_j$.

Then $x \in \bigcap_{j=n+1}^{2n+1} B_j$. From (d1), (d2) and (d3), we have:-

(e1) $x \in \bigcup_{i \in N} \text{bdry } H_i$.

Also $x \in \bigcap_{j=1}^{n} B_j$. From (b5) and (b6), either:-

(e2) $x \in \bigcap_{j=1}^{n} B_{i_0 j}$ for some $i_0 \in N$ or:-

(e3) $x \in \bigcup_{i \in N} \text{bdry } G_i$.

Both (e2) and (e3) contradict (e1) in view of (b3) and (c4) respectively. So $\bigcap_{j=1}^{2n+1} B_j = \phi$ and the proof is complete.

Historical Notes:

The relation $\dim X \leq 2d_2(X, \rho)$ obtained by J.H. Roberts (Roberts) is the last in a series of results each of which generalizes the previous one. Katětov (Katětov) proved in 1958 that dim $X \leq 2$ μ-dim $(X, \mathcal{E})$. In 1967 Hodel (Hodel) sharpened this result to dim $X \leq 2d_3(X, \mathcal{E})$. Finally Roberts (Roberts) proved in 1970 that dim $X \leq 2d_2(X, \mathcal{E})$.

## SECTION TWO

In the last chapter, we saw that for any two of the above mentioned dimension functions, say d and d', we have $d \leq 2d'$. We shall now give examples to show that if d is any of the above dimension functions different from the covering dimension dim, then there exists a metric space where d and dim differ by the maximum amount allowed by the inequality in theorem 1.7.

Lemma 2.1 (Nagami and Roberts, 1967)

Let X be a completely normal space and Y a subset of X with dim (X-Y) < n. Then for any n pairs $(C_i, C'_i)$ $1 \leq i \leq n$, of disjoint closed sets of X $\exists$ closed sets $B_i$, $1 \leq i \leq n$, of X, s.t. $B_i$ separates $C_i$ and $C'_i$ and $\bigcap_{i=1}^{n} B_i \subset Y$.

Proof: Let X, Y, $C_i$, $C'_i$ $1 \leq i \leq n$ be as in the statement of the theorem. Let $U_i$, $U'_i$ be open sets of X s.t. $C_i \subset U_i$, $C'_i \subset U'_i$ and $\overline{U}_i \cap \overline{U}'_i = \phi$ for $1 \leq i \leq n$. Because dim X-Y <n, $\exists$, by theorem 0.4 open sets $0_i$, $0'_i$ of X-Y s.t. $\overline{U}_i - Y \subset 0_i$, $\overline{U}'_i - Y \subset 0'_i$, $0_i \cap 0'_i = \phi$ and $X-Y = \bigcup_{i=1}^{n} 0_i \cup 0'_i$. Because X is completely normal, $\exists$ disjoint open sets $V_i$, $V'_i$ of X s.t. $0_i \subset V_i$, $0'_i \subset V'_i$.

Let $W_i = U_i \cup (0_i - \overline{U}'_i)$, $W'_i = U'_i \cup (0'_i - \overline{U}_i)$.

Let $B_i = X - (W_i \cup W'_i)$. Then $B_i$, $1 \leq i \leq n$, satisfy the required condition.

Lemma 2.2.

Let X be a compact Hausdorff space and let H and K be disjoint closed sets of X such that no connected set of X intersects both H and K. Then the empty set separates H and K.

For a proof of this lemma, see Nagami "Dimension Theory" corollary 6-8 pp 41.

Lemma 2.3

A connected compact Hausdorff space cannot be the disjoint union of a countable collection of more than one non-empty closed sets.

For a proof of this lemma see Nagami "Dimension Theory" theorem 6-10 pp 41.

We would like to give an example to show that lemma 2.3 cannot be extended to normal (infact metric) spaces.

Example 2.1 A connected subset of $I^2$ that is a union of a countable collection of more than one none-empty disjoint closed sets.

Let $q_1$, $q_2$, $q_3$...... be the rational numbers in I.

Let $X = I \times \{0\} \cup (\bigcup_{i=1}^{\infty} \{q_i\} \times [1/i, 1])$.

Let $A_i = \{q_i\} \times [1/i, 1]$ and $B = I \times \{0\}$. Then X is the union of the non-empty closed sets B, $A_1$, $A_2$, $A_3$......

$X$ is connected; for suppose not, and assume $X = U \cup V$ where $U$, $V$ are disjoint non-empty open sets of $X$. Since $I^2$ is completely normal, $\exists$ disjoint open sets $G$, $H$ of $I^2$ s.t. $G \cap X = U$, $H \cap X = V$. Since $B$ is connected, $B$ is wholly contained in either $G$ or $H$. Assume without loss of generality that $B \subset G$. Since $H \cap X \neq \phi$, $H \cap A_i \neq \phi$ for some $i$, say $i_0$. Since $A_{i_0}$ is connected, $A_{i_0} \subset H$. We have, because $I$ is compact, that $I \times [0, \epsilon) \subset G$ for some $\epsilon$. Also $V \times \{a\} \subset H$ for some nbhd $V$ of $q_{i_0}$ and $a \in [1/i_0, 1]$. $V$ contains infinitely many rationals so it contains some rational $q_j$ with $j \geq i_0$ and $1/j < \epsilon$. But then $(q_j, 1/j) \in A_j \cap G \neq \phi$ and $(q_j, a) \in A_j \cap H \neq \phi$ contradicting the fact that $A_j$ is connected.

Defn 2.1

Let $X$ be a normal space. A collection of $n$ pairs $(C_i, C'_i)$ $1 \leq i \leq n$ of subsets of $X$ is said to be an essential family if (i) $C_i$, $C'_i$ are disjoint closed sets of $X$, $1 \leq i \leq n$

(ii) for any $n$ closed sets $B_i$, $1 \leq i \leq n$ s.t. $B_i$ separates $C_i$ and $C'_i$ we have $\bigcap_{i=1}^{n} B_i \neq \phi$.

Lemma 2.4 (Nagami and Roberts, 1967)

Let $X$ be a normal space, $F$ a closed set of $X$ and $f: F \longrightarrow S^{n-1}$ a continuous function. Considering $S^{n-1}$ as the boundary of $J^n$ where $J = [-1, 1]$, let $C_i = \{(x_1, x_2, \ldots x_n) \in J^n : x_i = -1\}$ $C'_i = \{(x_1, x_2, \ldots x_n) \in J^n : x_i = 1\}$ for $1 \leq i \leq n$.

If the collection $(f^{-1}(C_i),\ f^{-1}(C'_i))$ $1\leq i\leq n$ is not an essential family, then f has an extension f*: $X \longrightarrow S^{n-1}$.

Proof: First we construct a function $g: X \longrightarrow J^n$ which extends f and does not assume the value $\overline{0}$ $(= (0, 0, \ldots, 0))$ in $J^n$. Since $(f^{-1}(C_i),\ f^{-1}(C'_i))$ $1\leq i\leq n$ is not an essential family, $\exists$ pairs $U_i$, $U'_i$ of disjoint open sets, $1\leq i\leq n$, s.t. $f^{-1}(C_i) \subset U_i$, $f^{-1}(C'_i) \subset U'_i$ and $\bigcup_{i=1}^{n} (U_i \cup U'_i) = X$. Since X is normal, $\exists$ closed sets $E_i$, $E'_i$ s.t. $E_i \subset U_i$, $E'_i \subset U'_i$ and $\bigcup_{i=1}^{n} (E_i \cup E'_i) = X$. Let $F_i = E_i \cup f^{-1}(C_i)$, $F'_i = E'_i \cup F^{-1}(C'_i)$. Then $F_i$, $F'_i$ are disjoint closed sets with $f^{-1}(C_i) \subset F_i$, $f^{-1}(C'_i) \subset F'_i$ and $\bigcup_{i=1}^{n} (F_i \cup F'_i) = X$.

By Urysohn's lemma, $\exists$ for each i, $1\leq i\leq n$, a continuous function $h_i: X \longrightarrow J$ s.t. $h_i(F_i) = -1$, $h_i(F'_i) = 1$.

Let $h: X \longrightarrow J^n$ be the function.

$h(x) = (h_1(x), h_2(x), \ldots h_n(x))$. Then h is continuous.

By Tietze's extension theorem there is a continuous extension $\overline{f}: X \longrightarrow J^n$ of f. Let U be the set $\{x \in X: \overline{f}_i(x)h_i(x) > 0 \text{ for some } i, 1\leq i\leq n\}$ where $\overline{f}_i$ is the ith coordinate function of $\overline{f}$. If U = X then set $g = \overline{f}$. Otherwise, we note that U is open and U contains F. Since $X-U \neq \phi$, by Urysohn's lemma $\exists$ a continuous function $\phi: X \longrightarrow I$ s.t. $\phi(F) = 1$ and $\phi(X-U) = 0$. Let $g(x) = \overline{f}(x)\phi(x) + h(x)(1 - \phi(x))$. Then if $x \in U$ then for some i, $\overline{f}_i(x)h_i(x) > 0$

whence $\overline{f}_i(x)\phi(x) + h_i(x)(1-\phi(x)) \neq 0$ whence $g_i(x) \neq 0$ whence $g(x) \neq \overline{0}$. If $x \notin U$ then $g(x) = h(x) \neq \overline{0}$, (clear) and for $x \in F$, $g(x) = f(x)$. So $g$ is as required.

Now let $\psi : (J^n - \overline{0}) \longrightarrow S^{n-1}$ be the projection $\psi(a) = \frac{a}{\|a\|}$ where $\|a\|$ is the sup norm of a i.e. $\|a\| = \sup\{|a_i|, 1 \leq i \leq n\}$ where $a = (a_1, \ldots a_n)$. Put $f^* = \psi \circ g$. Then $f^*$ is the required extension of f.

Theorem 2.1 (Nagami 1967)

Let X be a compact completely normal space with dim $X \geq n \geq 0$. Let $A_i$, $i = 1, 2, \ldots$ be disjoint closed sets of X s.t. dim $A_i \leq n-1$. Then dim $(X - \bigcup_{i=1}^{\infty} A_i) \geq n-1$.

Proof: We omit the trivial case $n = 0$ so assume $n \geq 1$. Since dim $X \geq n$, by theorem 0.5 $\exists$ a closed set F of X and a continuous function $f: F \longrightarrow S^{n-1}$ s.t. f does not extend to X.

Step 1. $\exists$ a continuous function $h: X \longrightarrow I^n$ s.t. h extends f, $\overline{0} \notin h(\bigcup_{i=1}^{\infty} A_i \cup F)$.

We construct h as follows:- Since dim $A_i \leq n-1$ f extends to $F \cup A_1$, and hence to $\overline{U}_1$ where $U_1$ is an open set containing $F \cup A_1$. Similarly, f extends to $\overline{U}_1 \cup A_2$ and hence to $\overline{U}_2$ where $U_2$ is open and contains $\overline{U}_1 \cup A_2$ (These extensions are into $S^{n-1}$). We thus define recursively a continuous function $g: U \longrightarrow S^{n-1}$ where $U = \bigcup_{i=1}^{\infty} U_i$ is an open set containing $F \cup (\bigcup_{i=1}^{\infty} A_i)$.

Let $\phi_i : X \longrightarrow [0, 1/2^i]$ be s.t. $\phi_i(F \cup A_i) = 1/2^i$, $\phi_i(X-U) = \overline{0}$. Then $\phi = \sum_{i=1}^{\infty} \phi_i$ is a continuous function into [0., 1] s.t. $\phi(F) = \{1\}$, $\phi(A_i) \subset (0,1]$. Define $h$ by $h(x) = \phi(x)g(x)$ for $x \in U$, $h(x) = \overline{0}$ for $x \notin U$. Then $h$ satisfies the given conditions.

Step 2. Assume $\dim (X - \bigcup_{i=1}^{\infty} A_i) < n-1$. Then, because $X - \bigcup_{i=1}^{\infty} A_i$ is normal, and $h^{-1}(\overline{0})$ is $G_\delta$, $\dim ((X-h^{-1}(\overline{0})) - \bigcup_{i=1}^{\infty} A_i) < n-1$.

For a set $S$ in $R^n$ and a set $J$ in $R$, let $JS$ denote the set $js$, $j \in J$, $s \in S$.

We take $I$ to be the interval $[-1, 1]$.
Let $B = \{x \in I^n : x_n = 1\}$ i.e. one face of $I^n$ (see fig. 2.1). Let $P$ be the pyramid $[0, 1]B$
For $1 < i < n-1$, let $S_i = \{x \in B : x_i = -1\}$, $S'_i = \{x \in B : xi = 1\}$.
Let $T_i = (0, 1]S_i$, $T'_i = (0, 1]S'_i$.
Then $h^{-1}(T_i)$, $h^{-1}(T'_i)$ $1 \leq i \leq n-1$ are disjoint closed sets of $X - h^{-1}(\overline{0})$. By lemma 2.1, $\exists$ closed sets $B_i$ $1 \leq i \leq n-1$ of $X - h^{-1}(\overline{0})$ s.t. $B_i$ separates $h^{-1}(T_i)$ and $h^{-1}(T'_i)$ in $X - h^{-1}(\overline{0})$ and $\bigcap_{i=1}^{n-1} B_i \subset \bigcup_{i=1}^{\infty} A_i$. Let $H = \bigcap_{i=1}^{n-1} B_i$.

$H \cup h^{-1}(0)$ is closed in $X$ and is therefore compact.
Assume $H \cap h^{-1}(B) \neq \phi$. Suppose some connected set $J$ of $H \cup h^{-1}(\overline{0})$ intersects both $H \cap h^{-1}(B)$ and $h^{-1}(\overline{0})$. Then $\overline{J}$ is a connected compact set of $H \cup h^{-1}(\overline{0})$ intersecting both $H \cap h^{-1}(B)$ and $h^{-1}(\overline{0})$. Then $\overline{J}$ has a non-empty intersection with $h^{-1}(\overline{0})$ and $\bigcup_{i=1}^{\infty} A_i$ (which is disjoint from $h^{-1}(\overline{0})$).

Thus $\overline{J}$ is the union of a disjoint countable collection of more than one closed set contrary to lemma 2.3.

Thus no connected subset of $H \cup h^{-1}(\overline{0})$ touches both $h^{-1}(\overline{0})$ and $H \cap h^{-1}(B)$. By lemma 2.2 $H \cup h^{-1}(\overline{0})$ is a union of two disjoint closed sets one containing $H \cap h^{-1}(B)$ and the other containing $h^{-1}(\overline{0})$.

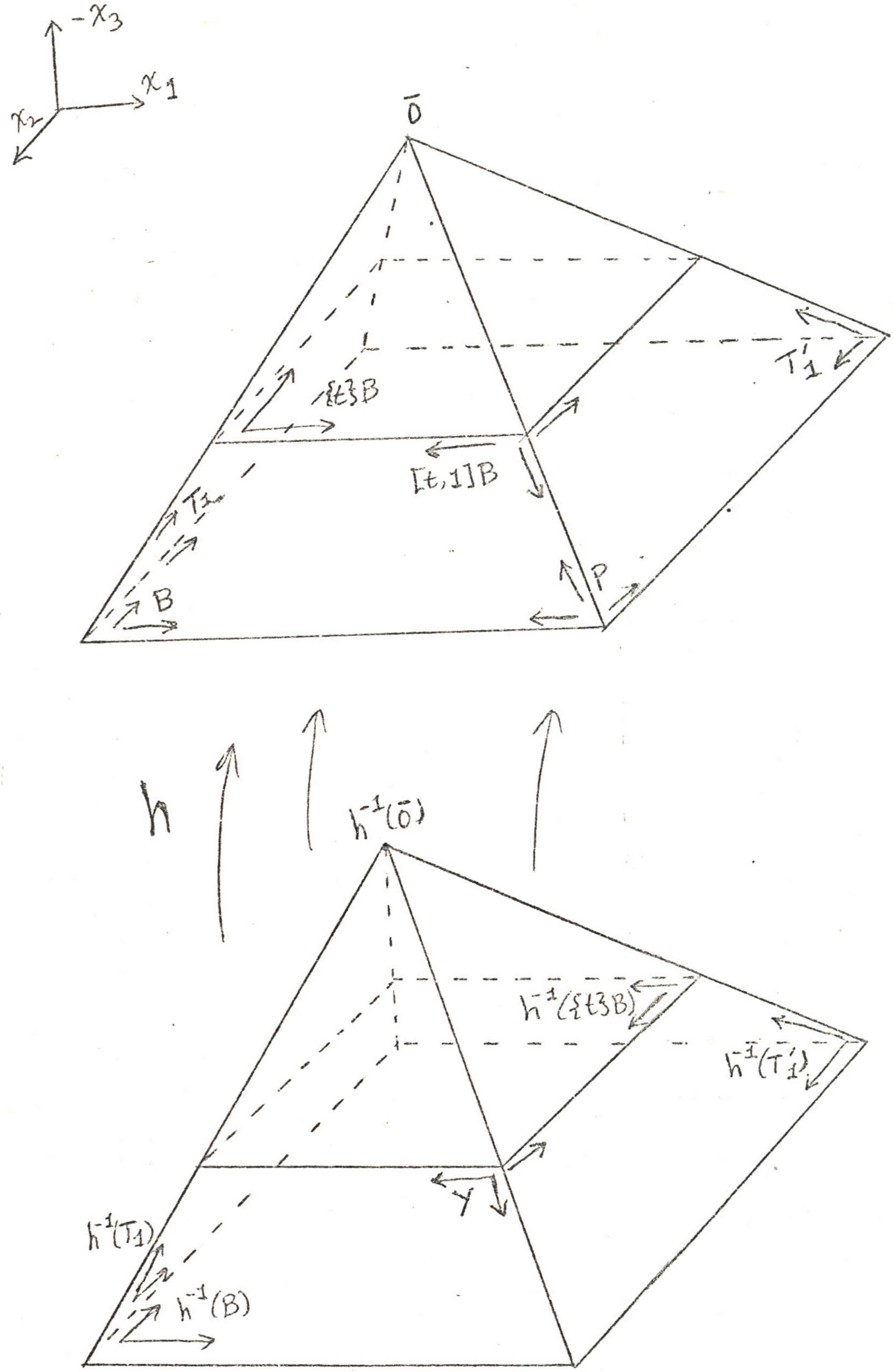
Claim:- $H \cup h^{-1}(\overline{0}) \cup h^{-1}(B)$ is a union of two disjoint closed sets one containing $h^{-1}(B)$ and the other containing $h^{-1}(\overline{0})$.

One of these closed sets is formed by uniting $h^{-1}(B)$ to the one of the two closed sets of $H \cup h^{-1}(\overline{0})$ which contains $H \cap h^{-1}(B)$ (this is assuming $H \cap h^{-1}(B) \neq \phi$ because otherwise the result is obvious). The other closed set is just the closed set of $H \cup h^{-1}(0)$ which does not contain $H \cap h^{-1}(B)$.

By extending to disjoint open sets of X we obtain a closed set $B_n$ of X separating $h^{-1}(B)$ and $h^{-1}(\overline{0})$ without touching H. Because of the compactness of X, and considering that $h^{-1}(\overline{0}) = \bigcap_{i=1}^{\infty} h^{-1}([0, 1/i]B)$,

We see that $B_n$ also separates $h^{-1}(B)$ and $h^{-1}(\{t\}B)$ for some t, $0<t<1$. Restricting attention to the space $Y = h^{-1}([t, 1]B)$, if $B'_i = B_i \cap Y$, then $B'_i$ separates $h^{-1}([t, 1]S_i)$ and $h^{-1}([t, 1]S'_i)$($= h^{-1}(T_i) \cap Y$ and $h^{-1}(T'_i) \cap Y$) in Y. $B'_i$ is closed in Y since $Y \subset X - h^{-1}(\overline{0})$. That, so far, is for $1 \leq i \leq n-1$. If $i = n$, then again $B'_n = B_n \cap Y$ separates $h^{-1}(\{t\}B)$ and $h^{-1}(B)$ in Y. Now $\bigcap_{i=1}^{n} B_i = \phi$ by the construction of $B_i$. Thus the system $h^{-1}([t, 1]S_i)$, $h^{-1}([t, 1]S'_i)$

Fig. 2.1



$1 \leq i \leq n-1$ and $h^{-1}(\{t\}B)$, $h^{-1}(B)$ is not an essential family in Y. Let C = boundary in $R^n$ of $[t, 1]B = B \cup \{t\}B \cup \bigcup_{i=1}^{n-1} ([t, 1]S_i \cup [t,1]S'_i)$.

C is homoemorphic to $S^{n-1}$ with $(B, \{t\}B)$, $([t,1]S_i, [t, 1]S'_i)$ $1 \leq i \leq n-1$ corresponding to pairs of opposite faces so, in view of lemma 2.4 $\exists$ a map $\psi: Y \longrightarrow C$ s.t. $\psi$ extends $h|h^{-1}(C)$. If we define $\theta: X \longrightarrow I^n$ by

$$\theta(x) = \begin{cases} \psi(x) & \text{for } x \in Y \\ h(x) & \text{for } x \notin Y \end{cases}$$

then $\theta$ is a continuous map which does not assume values in the interior (in $R^n$) of $[t, 1]B$ ($\theta$ is continuous because it coincides with $\psi$ on Y and it coincides with h on $\overline{X-Y}$). If we compose $\theta$ with the projection from an interior (in $R^n$) point of $[t, 1]B$ to $S^{n-1}$, we obtain an extension of f contrary to the choice of f. So we cannot have dim $X - \bigcup_{i=1}^{\infty} A_i < n-1$.

<u>Corrollary 2.1</u> (Nagami 1967)

Let $A_i$, $i \in N$, be a sequence of disjoint closed sets of $I^n$ at least two of which are non-empty. Then dim $I^n - \bigcup_{i=1}^{\infty} A_i \geq n-1$.

Proof: With the notation introduced in theorem 2.1, if $\{t\}I^n$ does not meet two $A_i$'s for any $0<t<1$ then $\exists$ $i_0$ s.t. $A_i \subset S^{n-1}$ if $i \neq i_0$. Since $(0, 1)I^n \not\subset A_{i_0}$, we have dim $I^n - \bigcup_{i=1}^{\infty} A_i \geq$ dim $(0, 1)I^n - \bigcup_{i=1}^{\infty} A_i$ = dim

$(0,1)I^n - A_{i_0} = n$. Otherwise, for some $t \in (0,1)$, $\{t\}I^n$ intersects $A_i$, $A_j$ for $i \neq j$ and by lemma 2.3, $\exists\, x \in \{t\}I^n$ s.t. $x \notin \bigcup_{i=1}^{\infty} A_i$. We may assume $x = 0$. We let $F = S^{n-1}$, $f$ = identity, $h$ = identity and proceed as in theorem 3.1.

Corrollary 2.2. (Nagami 1967).

Let $X$ be a connected metric space s.t. every point has a nbhd homeomorphic to $I^n$. Let $A_i$ be a disjoint sequence of closed sets of $X$ at least two of which are non-empty. Then $\dim (X - \bigcup_{i=1}^{\infty} A_i) \geq n-1$.

Proof: Let $A_i$ be as in the corrollary.
Let $I_x$ be a nbhd of $X$ homeomorphic to $I^n$ for $x \in X$.
If each $I_x$ is contained in some $A_i$, then each $A_i$ is clopen contradicting the connectedness of $X$.
We cannot have $I_x \subset \bigcup_{i=1}^{\infty} A_i$ for each $x \in X$ because, in view of the above observation, this would contradict lemma 2.3.

So for some $x_0$, $I_{x_0} \not\subset \bigcup_{i=1}^{\infty} A_i$. If $I_{x_0}$ intersects at most one $A_i$ then $\dim X - \bigcup_{i=1}^{\infty} A_i \geq \dim I_{x_0} - \bigcup_{i=1}^{\infty} A_i = n$ (since it is open in $I_x$ ). If $I_x$ intersects two $A_i$'s then by corrollary 2.1, $\dim X - \bigcup_{i=1}^{\infty} A_i \geq \dim I_{x_0} - \bigcup_{i=1}^{\infty} A_i \geq n-1$.

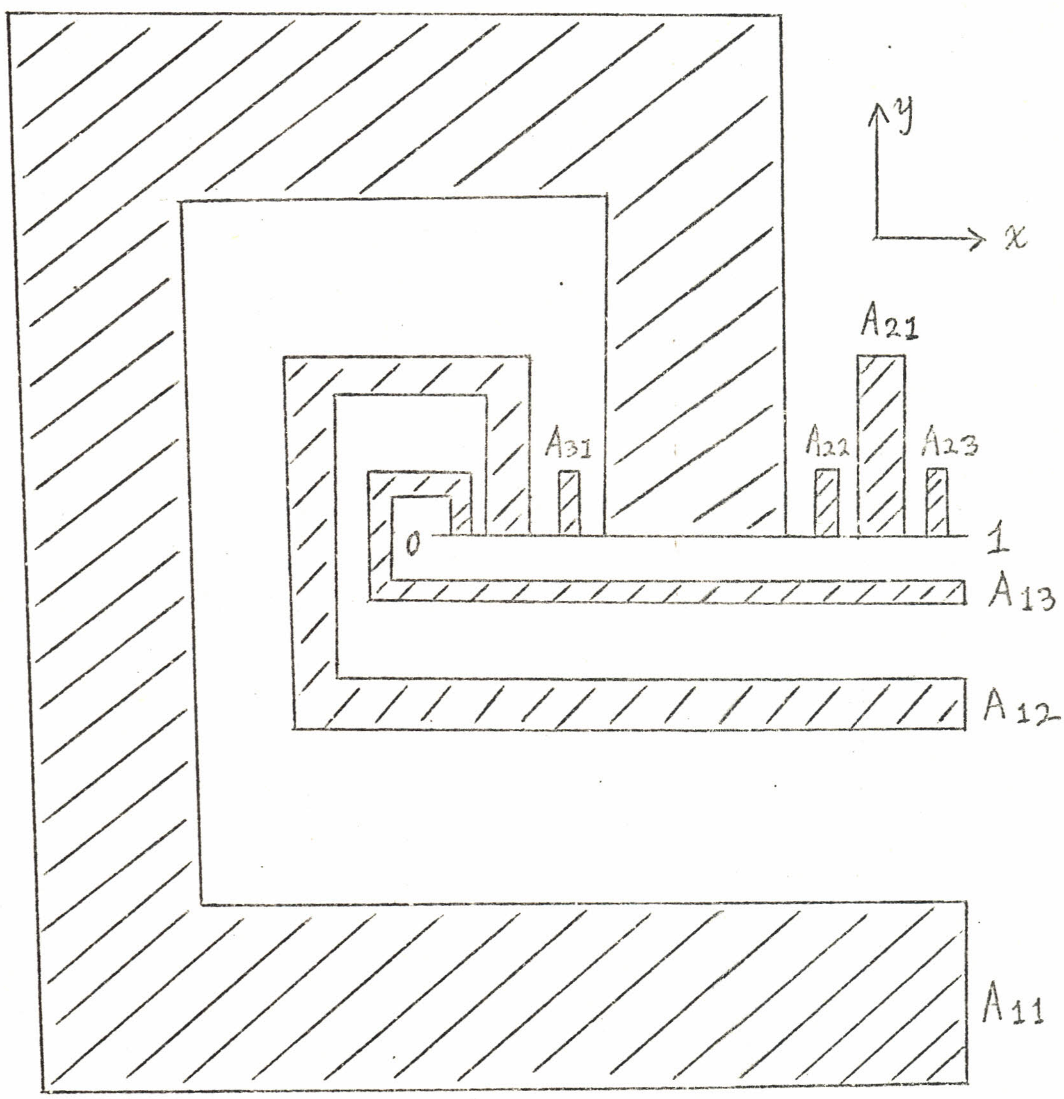
Defn 2.2.

A compact Hansdorff space of dimension $n$, $n \geq 1$ is called a Cantor n-manifold if it cannot be separated by a closed subset of dimension less than $n-1$.

Example 2.2.

Fig 2.2. gives a cantor 2-manifold X s.t. a proposition for X analogous to corrollary 2.1 fails. dim x = 2 but dim $X - \bigcup_{i=1}^{\infty} A_i = 0$ since $X - \bigcup_{i=1}^{\infty} A_i$ is a subset of Cantor's discontinuum.

Fig. 2.2.



We would like to give an explanation (omitted from Nagami and Roberts 1967) as to why X is a Cantor 2-manifold.

First we note that the sets $A_{ij}$ ($i=1, 2, \ldots 1 \leq j \leq n_i$ are so chosen that for any $\varepsilon > 0$, only finitely many of them have a span exceeding $\varepsilon$ in the y-direction. This ensures that X is closed in $R^2$. Since it is bounded, it is compact.

The sets $A_{ij}$ of fig 2.2. have the following properties.

(i) $A_{ij}$ is homeomorphic to $I^2$.

(ii) $\bigcup_{i,j} A_{ij}$ is dense in X

(iii) $\exists$ an infinite subcollection $\mathcal{A}$ of $\{A_{ij}\}$ s.t. for any infinite subcollection $\mathcal{B}$ of $\mathcal{A}$, $\overline{\bigcup_{A \in \mathcal{B}} A}$ intersects each $A_{ij}$ on a set of dim 1. $\mathcal{A}$ is the collection $A_{11}$, $A_{12}$, $A_{13}$.....

(i) implies that $A_{ij}$ is a cantor 2-manifold, (see Engelkin pp 77). Suppose X is separated by a closed set B, with $X-B = U \cup V$, U, V disjoint, $U \neq \phi \neq V$, $\dim B \leq 0$. Because $A_{ij}$ are cantor 2-manifolds, we must have $A_{ij} \cap U = \phi$ or $A_{ij} \cap V = \phi$ for each i, j. Let $\mathcal{B}_1 = \{A_{ij} \in \mathcal{A}: A_{ij} \cap V = \phi\}$, $\mathcal{B}_2 = \{A_{ij} \in \mathcal{A}: A_{ij} \cap U = \phi\}$. Then one of $\mathcal{B}_1$, $\mathcal{B}_2$ must be infinite. Assume it is $\mathcal{B}_1$. Because $\bigcup_{i,j} A_{ij}$ is dense, at least one $A_{ij}$, say $A_{i_0 j_0}$ intersects V. So $A_{i_0 j_0} \cap U = \phi$. Then $\overline{\bigcup_{A \in \mathcal{B}_1} A} \cap A_{i_0 j_0} \subset B$.

But $\dim(\overline{\bigcup_{A \in \mathcal{B}_1} A} \cap A_{i_0 j_0}) = 1$ (from (iii)) so $\dim B \geq 1$, a contradiction.

So X cannot be separated by a zero-dimensional closed set and is therefore a cantor 2-manifold, (of course dim X = 2).

<u>Lemma 2.5</u> (Nagami and Roberts, 1967).

Let $(X, \ell)$ be a metric space and $C_i$ $i=1, 2, \ldots$ be a sequence of closed subsets of X s.t. $\dim C_i \leq n_i$. Let $\mathcal{U}$ be any open cover of X, and r any positive integer. Then $\exists$ r l.f. open covers $\mathcal{U}_1, \mathcal{U}_2, \ldots \mathcal{U}_r$ and r l.f. closed covers $\xi_1, \xi_2, \ldots \xi_r$. s.t.

(i) $\xi_1$ refines $\mathcal{U}$ (we will write $\xi_1 < \mathcal{U}$) and $\xi_{i+1} < \mathcal{U}_i < \xi_i$ for $1 \leq i \leq r$. ($i<r$ for the first inequality)

(ii) If $E_1, E_2, \ldots E_s$ are s distinct members of $\xi_{i+1}$ for some positive integer s, then $\exists$ s distinct members $U_1, U_2, \ldots U_s$ of $\mathcal{U}_i$ s.t. $E_1 \subset U_1, E_2 \subset U_2, \ldots E_s \subset U_s$ and similarly for $\mathcal{U}_i$ and $\xi_i$.

(iii) ord $\xi_i | C_i \leq n_i$ $1 \leq i \leq r$.

(iv) ord $\xi_r | C_i \leq n_i$ for $1 \leq i \leq r$

Proof: The proof is by induction. Assume the result true for r-1. Then we can obtain covers $\xi_i, \mathcal{U}_i$ $1 \leq i \leq r-1$ satisfying (i) - (iii) with r replaced by r-1. $\mathcal{V} = \{U \cap C_r, U \in \mathcal{U}_{r-1}\}$ is an open cover of $C_r$. Since $\dim C_r \leq n_r$ and from lemma 1.1, $\mathcal{V}$ has an open l.f. refinement of order $\leq n_r$. This in turn has a closed l.f. refinement of order $\leq n_r$. Furthermore, using a technique as in the proof of lemma 1.1, this closed refinement may be assumed to be of the form

$\{H_U, U\in\mathcal{U}_{r-1}\}$ where $H_U \subset U \cap C_r$ (and the order is indexwise, see defn. 0.2). By lemma 1.2, since $H_U$ is closed in X, $\exists$ a l.f. open collection $\{G_U, U\in\mathcal{U}_{r-1}\}$ s.t. $H_U \subset G_U \subset U \in \mathcal{U}_{r-1}$ and ord $\{G_U\} \leq n_r$.

Let $M_U = G_U \cup (U - C_r)$. $\{M_U, U\in\mathcal{U}_{r-1}\}$ is a l.f. (because $\mathcal{U}_{r-1}$ is) open cover of X s.t. its restriction to $C_r$ i.e. $\{M_U \cap C_r, U\in\mathcal{U}_{r-1}\}$ has order $\leq n_r$. Normality of X implies the existence of an open cover $\{W_U, U\in\mathcal{U}_{r-1}\}$ with $\overline{W}_U \subset M_U$. Let $\mathcal{U}_r = \{W_U, U\in\mathcal{U}_{r-1}\}$, $\xi_r = \{\overline{W}_U, U\in\mathcal{U}_{r-1}\}$. Conditions (i), (ii), (iii) are satisfied while (iv) follows from (ii) and (iii). The construction when r=1 is just as above, taking $\mathcal{U}$ instead of $\mathcal{U}_{r-1}$.

Theorem 2.2. (Nagami and Roberts, 1967).
Let $(X,\rho)$ be a metric space and $C_i$, $i = 1, 2, \ldots\ldots$ be a sequence of closed sets of X s.t. dim $C_i \leq n_i$. Let $\mathcal{U}$ be an open cover of X. $\exists$ a sequence $\mathcal{F}_i = \{F_\alpha, \alpha\in B_i\}$ of l.f. closed covers of X s.t.

(i) $\mathcal{F}_1$ refines $\mathcal{U}$.

(ii) mesh $\mathcal{F}_i \leq 1/i$.

(iii) ord $\mathcal{F}_i | C_j \leq n_j$

(iv) $\exists$ a system of functions $f_i^j : B_j \longrightarrow B_i$ $i \leq j$ s.t $f_i^i$ = identity, $f_i^j \circ f_j^k = f_i^k$ and for $\alpha \in B_i$ and $j \geq i$, $F_\alpha = \bigcup_{\beta\in(f_i^j)^{-1}(\alpha)} F_\beta$.

(v) For any positive integers i, j, k, if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are distinct elements of $B_j$, then dim

$(\bigcap_{r=1}^{k} F_{\alpha_r} \cap C_i \leq \overline{n_i - k + 1}$ where $\overline{x}$ means max $\{x, -1\}$

for $x \in R$.

Proof: We construct sequences $\mathcal{H}_i$ of l.f. closed covers and $\mathcal{G}_i$ of l.f. open covers of X satisfying:-

(i) $\mathcal{G}_{i+1} < \mathcal{H}_{i+1} < \mathcal{G}_i$ $\forall i$ and $\mathcal{H}_1 < \mathcal{U}$.

(ii) ord $\mathcal{H}_i | C_j \leq n_j$ for $j \leq i$.

(iii) mesh $\mathcal{H}_i \leq 1/i$.

The construction is by induction. Assume the sequence constructed upto $i=r-1$.

Then let $\mathcal{H}_r$ and $\mathcal{G}_r$ be $\xi_r$ and $\mathcal{U}_r$ (respectively) of lemma 2.5, replacing $\mathcal{U}$ of lemma 2.5. with a l.f. open refinement of $\mathcal{G}_{r-1}$ whose mesh is $\leq 1/i$. (i), (ii), (iii) are obviously satisfied. Let $\mathcal{H}_1$, $\mathcal{G}_1$ be $\xi_1$, $\mathcal{U}_1$ of lemma 2.5 with $\mathcal{U}$ replaced by a l.f. open refinement of $\mathcal{U}$ with mesh $\leq 1$.

Now write $\mathcal{H}_i = \{H_\alpha, \alpha \in B_i\}$ where $\alpha \neq \beta \Rightarrow H_\alpha \neq H_\beta$

Define $f_i^{i+1} : B_{i+1} \longrightarrow B_i$ s.t. for $\beta \in B_{i+1}$. $H_\beta \subset H_{f_i^{i+1}(\beta)}$.

For $i<j$ let $f_i^j = f_i^{i+1} \circ f_{i+1}^{i+2} \circ \ldots\ldots \circ f_{j-1}^j$ and let $f_i^i$ = identity. We note that $f_i^j \circ f_j^k = f_i^k$. Let $B$ = inv lim $\{B_i, f_i^j\}$ and $\pi_i : B \longrightarrow B_i$ be the projections. For each i, define a collection $\mathcal{K}_i$ as follows:- for $\alpha \in B_i$, let $K_\alpha = \bigcup_{a \in \pi_i^{-1}(\alpha)} [\bigcap_{j=i}^{\infty} H_{\pi_j(a)}]$ (we take $\bigcup_{A \in \phi} A = \phi$).

Let $\mathcal{K}_i = \{K_\alpha, \alpha \in B_i\}$ .

Claim:-

(i) $K_\alpha \subset H_\alpha$ for each $\alpha \in B_i$ for each i.

(ii) $\mathcal{K}_i$ is l.f. for each i.

(iii) For $i \leq j$ and $\alpha \in B_i$, $K_\alpha = \bigcup_{\beta \in (f_i^j)^{-1}(\alpha)} K_\beta$

(i) follows from the fact that if $a \in \pi_i^{-1}(\alpha)$ and $j \geq i$ then $f_i^j\,(\pi_j(a)) = f_i^j \circ \pi_j\,(a) = \pi_i(a) = \alpha$ so $H_{\pi_j(a)} \subset H_\alpha$ (the last part follows easily from the definition of $f_i^{i+1}$ and $f_i^j$ $i \leq j$)

(ii) follows from (i) because distinct members $K_{\alpha_1}$, ......, $K_{\alpha_r}$ of $\mathcal{K}_i$ are contained in distinct members $H_{\alpha_1}$, ......$H_{\alpha_r}$ of $\mathcal{H}_i$ $(\alpha_1, \ldots\ldots \alpha_r \in B_i)$.

To see (iii), let $i \leq i'$ and $\alpha \in B_i$.

$$K_\alpha = \bigcup_{a \in \pi_i^{-1}(\alpha)} [\bigcap_{j=i}^{\infty} H_{\pi_j(a)}] = \bigcup_{a \in (f_i^{i'} \circ \pi_{i'})^{-1}(\alpha)} [\bigcap_{j=i}^{\infty} H_{\pi_j(a)}]$$

$$= \bigcup_{a \in \pi_{i'}^{-1}((f_i^{i'})^{-1}(\alpha))} [\bigcap_{j=i}^{\infty} H_{\pi_j(a)}]$$

$$= \bigcup_{\beta \in (f_i^{i'})^{-1}(\alpha)} \left\{ \bigcup_{a \in \pi_{i'}^{-1}(\beta)} [\bigcap_{j=i}^{\infty} H_{\pi_j(a)}] \right\}$$

Now for $j \leq j'$ $\pi_j(a) = f_j^{j'} \circ \pi_{j'}\,(a) = f_j^{j'}\,(\pi_{j'}\,(a))$ and, as before, $H_{\pi_{j'}(a)} \subset H_{\pi_j(a)}$. So the sequence $H_{\pi_j(a)}$ is decreasing so

$$\bigcap_{j=i}^{\infty} H_{\pi_j(a)} = \bigcap_{j=i'}^{\infty} H_{\pi_j(a)}.$$

$$\text{So } K_\alpha = \bigcup_{\beta \in (f_i^{i'})^{-1}(\alpha)} \left\{ \bigcup_{a \in \pi_{i'}^{-1}(\beta)} [\bigcap_{j=i'}^{\infty} H_{\pi_j(a)}] \right.$$

$= \bigcup_{\beta \in (f_i^{i'})^{-1}(\alpha)} K_\beta$ as required

Now for each i, put $F_\alpha = \overline{K}_\alpha, \alpha\in B_i$ and let $\mathcal{F}_i = \{F_\alpha, \alpha\in B_i\}$.

Claim:-

(i) $F_\alpha \subset H_\alpha$

(ii) $\mathcal{F}_i$ is l.f. for each i

(iii) For $i\leq j$ and $\alpha\in B_i$, $F_\alpha = \bigcup_{\beta\in(f_i^j)^{-1}(\alpha)} F_\beta$.

(i) follows because $H_\alpha$ is closed so $K_\alpha \subset H_\alpha \Rightarrow \overline{K}_\alpha \subset H_\alpha$.

(ii) follows from the fact that $\{K_\alpha, \alpha\in B_i\}$ l.f. implies $\{\overline{K}_\alpha, \alpha\in B_i\}$ is l.f.

(iii) follows from condition (iii) and (ii) of the previous claim (i.e. the same conditions as above but for the $K_\alpha$).

Condition (iv) of the theorem has now been established (i) and (ii) follow immediately from the fact that $F_\alpha \subset H_\alpha, \alpha\in B_i$ for any i. To see (iii) i.e. that ord $\mathcal{F}_i | C_j \leq n_j$, take first the case where $j\leq i$, then the result follows from the fact that ord $\mathcal{H}_i | C_j \leq n_j$, since distinct members $F_{\alpha_1}, \ldots. F_{\alpha_r}$ of $\mathcal{F}_i$ are contained in the distinct members $H_{\alpha_1} \ldots\ldots H_{\alpha_r}$ of $\mathcal{H}_i$. Now take the case where $i<j$. The fact that for $\alpha\in B_i$, $F_\alpha = \bigcup_{\beta\in(f_i^j)^{-1}(\alpha)} F_\beta \subset \bigcup_{\beta\in(f_i^j)^{-1}(\alpha)} H_\beta$ together with the fact that $H_{\beta_1} \neq H_{\beta_2}$ if $\beta_1 \neq \beta_2$ imply that ord $(\mathcal{F}_i | C_j) \leq$ ord $\mathcal{H}_j | C_j \leq n_j$ as required. Condition (v) of the theorem follows from condition (ii), (iii) and (iv) as follows:-

Let i, j, k, $\alpha_1, \alpha_2, \ldots, \alpha_k$ be as in condition (v). Let $Z = (\bigcap_{r=1}^{k} F_{\alpha_r}) \cap C_i$. $Z \subset F_{\alpha_1}$, so for any $p > j$, $\{F_\alpha \cap Z, \alpha \in (f_j^p)^{-1}(\alpha_1)\} = \mathcal{L}_p$ is a l.f. closed cover of $Z$ with mesh $\mathcal{L}_p \leq 1/p$. Furthermore $\mathcal{L}_m < \mathcal{L}_p$ if p<m so if we can show that ord $\mathcal{L}_p \leq \overline{n_i-k+1}$ it will follow from theorem 0.12 that dim $Z \leq \overline{n_i-k+1}$. Let $q = \overline{n_i-k+1} + 2$ and let $L_{\beta_1} = F_{\beta_1} \cap Z$, $L_{\beta_2} = F_{\beta_2} \cap Z, \ldots, L_{\beta_q} = F_{\beta_q} \cap Z$ be q distinct members of $\mathcal{L}_p$ for some $p > j$ and $\beta_1, \ldots, \beta_q \in (f_j^p)^{-1}(\alpha_1)$. $L_{\beta_1} \cap L_{\beta_2} \ldots \cap L_{\beta_q} \subset (\bigcap_{t=2}^{k} F_{\alpha_t}) \cap C_i$. Since $L_{\beta_t} \subset F_{\beta_t}$ $1 \leq t \leq q$ and $F_{\alpha_t} = \bigcup_{\beta \in (f_j^p)^{-1}(\alpha_t)} F_\beta$ $2 \leq t \leq k$, we have

$$L_{\beta_1} \cap \ldots \cap L_{\beta_q} \subset \bigcup_{\substack{\beta'_2 \in (f_j^p)^{-1}(\alpha_2) \\ \beta'_3 \in (f_j^p)^{-1}(\alpha_3) \\ \vdots \\ \beta'_k \in (f_j^p)^{-1}(\alpha_k)}} F_{\beta_1} \cap \ldots \cap F_{\beta_q} \cap F_{\beta'_2} \cap \ldots \cap F_{\beta'_k}$$

$\beta_1, \beta_2, \ldots, \beta_q \in (f_j^p)^{-1}(\alpha_1)$ and $\beta'_t \in (f_j^p)^{-1}(\alpha_t)$ $2 \leq t \leq k$ imply that $\beta_1, \beta_2, \ldots, \beta_q, \beta'_2, \ldots, \beta'_k$ are all distinct (of course $\beta_1, \beta_2, \ldots \beta_q$ are distinct.) These are $q+k-1 \geq n_i+2$ which implies $F_{\beta_1} \cap \ldots \cap F_{\beta_q} \cap F_{\beta'_2} \cap \ldots \cap F_{\beta'_k} \cap C_i \subset H_{\beta_1} \cap \ldots \cap H_{\beta_q} \cap H_{\beta'_2} \cap \ldots \cap H_{\beta'_k} \cap C_i = \phi$ since $\alpha \neq \beta \Rightarrow H_\alpha \neq H_\beta$ and ord $\mathcal{H}_p | C_i \leq n_i$ (since $i \leq p$).

So $L_{\beta_1} \cap \ldots \cap L_{\beta_q} = \phi$. This shows that ord $\mathcal{L}_p \leq \overline{n_i-k+1}$ as required and this completes the proof of the theorem.

Lemma 2.6. (Nagami 1967).

If $(X, \ell)$ is a metric space, $\mu\text{-dim}\,(X, \ell) \leq n$ iff for each $\varepsilon > 0$, $\exists$ a l.f. closed cover $\mathcal{F}$ of X s.t.

(i) mesh $\mathcal{F} < \varepsilon$

(ii) ord $\mathcal{F} \leq n$.

Proof: This is obvious from lemma 1.2.

Lemma 2.7 (Nagami 1967)

Let $(X, \ell)$ be a metric space and $C_1, C_2, \ldots$ be a sequence of closed subsets of X with $\dim C_i \leq n_i$. Let $\varepsilon > 0$ be given. Then $\exists$ a l.f. closed cover $\mathcal{F} = \{F_\alpha, \alpha \in \mathcal{A}\}$ of X s.t.

(i) mesh $\mathcal{F} \leq \varepsilon$

(ii) ord $\mathcal{F} | C_i \leq n_i$

(iii) if $F_{\alpha_1} \ldots\ldots F_{\alpha_t}$ are t distinct members of $\mathcal{F}$ then $\dim \left( \bigcap_{r=1}^{t} F_{\alpha_t} \right) \cap C_i \leq \overline{n_i - t + 1}$ for any i, t.

Proof:

The lemma is a direct consequence of theorem 2.2.

Example 2.3 (Nagami and Roberts, 1967)

Construction of totally bounded metric spaces $(Y_n, \ell_n)$ with $\mu\text{-dim}\,(Y_n, \ell_n) = [\frac{n}{2}]$, $\dim Y_n \geq n-1$.

Let $(X, \ell)$ be a compact metric space with $\dim X = n$ for $n \geq 3$.

We want to construct a sequence $B_i$, $i=1, 2, 3, \ldots$ of closed sets of X and a sequence $\mathcal{F}_i$ $i=1, 2, \ldots$

of l.f. closed covers of X satisfying.

(i) dim Bi $\leq n - [\frac{n}{2}] - 1$

(ii) $Bi \cap Bj = \phi$ for $i \neq j$.

(iii) mesh $\mathcal{F}i \leq 1/i$

(iv) ord $\mathcal{F}i| X-Bi \leq [\frac{n}{2}]$.

The construction is by induction.

Assume Bi and $\mathcal{F}i$ have been constructed for $1 \leq i \leq k$.

Let $m = [^n/_2] + 2$

From lemma 2.7 with $C_1, C_2, \ldots\ldots$ replaced by X, $B_1$, $B_2, \ldots B_k, \Phi, \Phi \ldots\ldots$ and $\varepsilon = {}^1/(k+1)$, we obtain a l.f. closed cover $\mathcal{F}$ of X s.t.

(a) mesh $\mathcal{F} \leq {}^1/(k+1)$

(b) if $F_1, F_2, \ldots F_t$ are t distinct members of $\mathcal{F}$ for any positive integer t, then if C is any of X, $B_1$, $B_2$, .....we have $\dim (\bigcap_{j=1}^{t} Fj) \cap C \leq \overline{\dim C - t + 1}$

Let $B = \{ x: \text{ord}_x \mathcal{F} > m-2 \}$

Then $B = \bigcup_{\gamma \varepsilon \Gamma} F_\gamma$ where $F_\gamma$ is an intersection of at least m members of $\mathcal{F}$ and the collection $\{F_\gamma, \gamma \varepsilon \Gamma\}$ is l.f. (A collection consisting of arbitrary intersections of members of a l.f. collection is l.f., the gist of the proof being that only a finite number of intersections can be formed from a finite number of members.) From condition (b) above, dim $F_\gamma \leq n-m+1$. Since F is closed and $\{F_\gamma\}$ is l.f. we have from theorem 0.7 that $\dim B \leq n-m+1 = n-[\frac{n}{2}]-1$. Again from condition (b) above we have that if $i \leq k$, then, putting C = Bi, $\dim B \cap Bi \leq \overline{\dim Bi - m + 1}$

$\leq \overline{n-[\frac{n}{2}]-1 \; -([\frac{n}{2}]+2)+1} = \overline{n-2([\frac{n}{2}]+1)} = -1$ so $B \cap Bi = \phi$ .

From the construction of B, $\operatorname{ord}_x \mathfrak{F} \leq m-2 = [\frac{n}{2}]$ if $x \in X-B$ which means $\operatorname{ord} \mathfrak{F}|X-B \leq [\frac{n}{2}]$. Thus if we let $\mathfrak{F}k+1 = \mathfrak{F}$ and $Bk+1 = B$ then conditions (i) to (iv) are satisfied. $B_1$ is constructed as above with $C_1$, $C_2$,....replaced with X, $\phi$, $\phi$, .......

Now let $Yn = X - \bigcup_{i=1}^{\infty} Bi$.

Since $\dim Bi \leq n - [\frac{n}{2}] - 1 \leq n-1$, we have from theorem 2.1 that $\dim Yn \geq n-1$.

Condition (iv) above implies that $\operatorname{ord} \mathfrak{F}i|Yn \leq [\frac{n}{2}]$ for each i. Combining this with the fact that $\mathfrak{F}i|Yn$ is l.f. with mesh $\leq {}^1/i$, we have from lemma 2.6 that μ-dim $(Yn, \ell n) \leq [\frac{n}{2}]$ ($\ell n$ is the inherited metric of Yn).

It follows from proposition 1.1. that if d is any of the dimension functions $d_2$, $d_3$, $d_5$, $d_6$, $d_7$ or μ-dim, then $d(Yn, \ell n) \leq [\frac{n}{2}]$, and if n is odd dim Yn = n-1. It is obvious that $(Yn, \ell n)$ is totally bounded.

NOTE. If we start with $X = I^n$, then dim Yn = n-1. This is because μ-dim $(Yn, \ell n) \leq [\frac{n}{2}]$ implies Int $Yn(\text{in } I^n) = \phi$, which inturn implies, by theorem 0.15 that $\dim Yn \leq n-1$.

These examples show that $d_2$, $d_3$, $d_5$, $d_6$, $d_7$, and μ-dim do not always coincide with dim. We next give an example to show that $d_2$ and μ-dim and $d_2$ and $d_3$

do not always coincide.

Lemma 2.8

Let $(X, \ell)$ be a compact metric space. Then for any positive integer m, $\exists$ a collection $(C_{ik}, C'_{ik})$ $1 \leq k \leq m$, $i \in N$ of disjoint closed sets of $(X, \ell)$ s.t. if $(C_k, C'_k)$ $1 \leq k \leq m$ are any m pairs of disjoint closed sets of $(X, \ell)$, then $\exists i \in N$ s.t. $C_k \subset C_{ik}$ and $C'_k \subset C'_{ik}$ for $1 \leq k \leq m$.

Proof: Let $\mathcal{U}_i$ be a finite covering of $(X, \ell)$ by open balls of radius $1/i$.

For each $i \in N$, $\exists$ positive integers $t_i$ and open sets $U_{ijk}$, $U'_{ijk}$ $1 \leq j \leq t_i$, $1 \leq k \leq m$ s.t. as j varies, we obtain all possible m pairs $(U_{ij1}, U'_{ij1})$, $(U_{ij2}, U'_{ij2})$,..., $(U_{ijm}, U'_{ijm})$ s.t. $U_{ijk}$, $U'_{ijk}$ are unions of members of $\mathcal{U}_i$ and $U_{ijk} \cap U'_{ijk} = \phi$ for all $1 \leq k \leq m$.
Let $C_{sijk} = \{x \in X: \ell(x, X-U_{ijk}) \geq 1/s\}$ and $C'_{sijk} = \{x \in X: \ell(x, X-U'_{ijk}) \geq 1/s\}$ for $s \in N$. Let $(C_1, C'_1)$,....., $(C_m, C'_m)$ be any m pairs of disjoint closed sets of X. Because X is compact, $\exists\ \varepsilon > 0$ s.t. $\ell(C_k, C'_k) > \varepsilon\ \forall\ k$, $1 \leq k \leq m$. Choose i s.t. $1/i < \frac{1}{2}\varepsilon$. If for each k we let $U_k$ be the union of members of $\mathcal{U}_i$ which intersect $C_k$ and $U'_k$ be the union of members of $\mathcal{U}_i$ which intersect $C'_k$, then $U_k \cap U'_k = \phi$. So for some j, $U_k = U_{ijk}$, $U'_k = U'_{ijk}$ for $1 \leq k \leq m$. So $C_k \subset U_{ijk}$, $C'_k \subset U'_{ijk}$, $1 \leq k \leq m$. Again because X is compact, $\ell(C_k, X-U_{ijk}) > \delta > 0$ and $\ell(C'_k, X - U'_{ijk}) > \delta > 0$ for $1 \leq k \leq m$

for some $\delta$. Choose $s \in N$ s.t. $1/s < \delta$.

Then $C_k \subset C_{sijk}$, $C'_k \subset C'_{sijk}$ for $1 \leq k \leq m$.

So the collection $(C_{sijk}, C'_{sijk})$ $s, i \in N$, $1 \leq j \leq t_i$, $1 \leq k \leq m$ is the required collection (the tuples $(s,i,j)$ are countable.).

<u>Lemma 2.9</u>

If $C$, $C'$ are disjoint closed sets of a completely normal topological space $X$ and $\mathcal{Z}$, $A$ are closed sets of $X$ s.t. $A \subset \mathcal{Z}$ and $A$ separates $C \cap \mathcal{Z}$ and $C' \cap \mathcal{Z}$ in $\mathcal{Z}$, then $\exists$ a closed set $A'$ of $X$ s.t. $A'$ separates $C$ and $C'$ in $X$ and $A' \cap \mathcal{Z} \subset A$.

Proof. $\mathcal{Z} - A = K \cup K'$ where $K$, $K'$ are open sets of $\mathcal{Z}$, $K \cap K' = \phi$, and $C \cap \mathcal{Z} \subset K$, $C' \cap \mathcal{Z} \subset K'$. Then $\overline{K} \cap C' = \overline{K'} \cap C = \phi$ (closures in $X$). This, together with the fact that $X$ is completely normal implies that we can obtain open sets $G(K)$, $G(K')$ of $X$ s.t. $G(K) \cap \mathcal{Z} = K$, $G(K') \cap \mathcal{Z} = K'$, $G(K) \cap G(K') = \phi$, $\overline{G(K)} \cap C' = \overline{G(K')} \cap C = \phi$. $\exists$ disjoint open sets $H(C)$, $H(C')$ of $X$ containing $C$ and $C'$ respectively.

Put $U = G(K) \cup (H(C) - \overline{G(K')})$,

$U' = G(K') \cup (H(C') - \overline{G(K)})$

Then putting $A' = X - (U \cup U')$, we see that $A'$ is as required.

Def. 2.3

Let $C_i$, $i \in N$ be a sequence of subsets of a topological space X. Then liminf $C_i$ is the set $\{x \in X$: for each nbhd U of x, $\exists\, m \in N$ s.t. $i \geq m \Rightarrow U \cap C_i \neq \phi\}$.

Limsup $C_i$ is the set $\{x \in X$: for each nbhd U of x and each $j \in N$, $\exists\, i \geq j$ s.t. $U \cap C_i \neq \phi$ .

Clearly, liminf $C_i$ and limsup $C_i$ are always closed sets of X and liminf $C_i \subset$ limsup $C_i$.

Lemma 2.10.

Let X be a compact, normal topological space. If liminf $C_i \neq \phi$ , and each $C_i$ is connected, then limsup $C_i$ is connected.

Proof. Let X, $C_i$ be as above with $x \in$ liminf $C_i$ and assume limsup $C_i$ is not connected. Then limsup $C_i$ is the union of disjoint, closed, non-empty sets E, F. Since limsup $C_i$ is closed in X, E, F are closed in X. $\exists$ disjoint open sets U, V of X with $E \subset U$, $F \subset V$. W.L.G. assume $x \in U$. Then for some $m \in N$, $i \geq m \Rightarrow C_i \cap U \neq \phi$. Let $y \in F \subset V$. Then for $i \in N$, $\exists\, r_i$ s.t. $r_i \geq i$, $C_{r_i} \cap V \neq \phi$ (because $y \in$ limsup $C_i$). Then for $i \geq m$, we have $C_{r_i} \cap U \neq \phi \neq C_{r_i} \cap V$. Since $C_{r_i}$ is connected, $W_i = C_{r_i} \cap [X-(U \cup V)] \neq \phi$. Let $x_i \in W_i$. Then, since $X-(U \cup V)$ is compact, the sequence $\{x_i\}$ has a convergent subsequence converging to say z, $z \in X-(U \cup V)$. But then $z \in$ limsup $C_i$ contrary to the fact that limsup $C_i =$ $E \cup F \subset U \cup V$.

Example 2.4

For any integer n, $n \geq 4$, we construct a metric space $(Xn, \ell n)$ with $d_2(Xn, \ell n) \leq n-2$, $d_3(Xn, \ell n) = \mu$-dim$(Xn, \ell n) = n-1$, and dim $Xn = n$. This generalizes on the example given by Nagami and Roberts (Nagami and Roberts, 1967 pp 430) of a metric space $(X, \ell)$ with $d_2(X, \ell) = 2$, $d_3(X, \ell) = \mu$-dim $(x, \ell) = 3$, and dim $(X, \ell) = 4$. Note that the inequality dim $X \leq 2d_2(X, \ell)$ implies that in our example, when n=4 we must have $d_2(Xn, \ell n) = n-2$. The main sets discussed are subsets of $I^n$, so when we talk of hyperplanes e.t.c. we shall mean their intersection with $I^n$. In addition, boundaries, closures, interiors e.t.c. of subsets of $I^n$ will be with reference to $I^n$. Boundaries, closures, interiors e.t.c. of subsets of I will be with reference to I.

First we construct a metric space $(Yn, \sigma n)$ with $d_2(Yn, \sigma n) \leq n-2$, $\mu$-dim$(Yn, \sigma n) =$ dim $Yn = n-1$. For a prime number $\pi$, $\pi \geq 5$, let $\mathfrak{D}(\pi)$ be the collection of overlapping intervals

$$\{[0, \frac{2}{\pi}), (\frac{\pi-2}{\pi}, 1], (\frac{2k-1}{\pi}, \frac{2k+2}{\pi}), k = 1, 2, \ldots \frac{\pi-3}{2}\}$$

Let $\overline{\mathfrak{D}}(\pi)$ be the collection of closures in I of the intervals of $\mathfrak{D}(\pi)$.

Let $\xi(\pi) = \{D_1 \times D_2 \times \ldots \times D_n;\ D_1, D_2, \ldots D_n\ \mathfrak{D}(\pi)\}$.
$\xi(\pi)$ is an open cover of $I^n$.

From lemma 2.8, $\exists$ a collection of disjoint pairs of closed sets of $I^n$ $Cij$, $1 \leq j \leq n-1$, $i \in N$ such that if $(Cj, C'j)$ $1 \leq j \leq n-1$ are any n-1 pairs of disjoint closed sets of $I^n$ then for some i, $Cj \subset Cij$, $C'j \subset C'ij$ $1 \leq j \leq n-1$.

Let $\pi_{ij}$, $i = 1, 2, 3, \ldots$ $1 \leq j \leq n-1$ be distinct prime numbers s.t. $\pi_{ij} \geq 5$ and for each i $\max_j$ mesh $\xi(\pi_{ij}) < \min_j \{d(C_{ij}, C'_{ij})\ 1 \leq j \leq n-1\}$ where d is the euclidean metric ($I^n$ is compact so $C_{ij} \cap C'_{ij} = \phi$ $\Rightarrow d(C_{ij}, C'_{ij}) > 0$).

Let $B_{ij} = \text{bdry}\{\bigcup_{\substack{E \in \xi(\pi_{ij}) \\ E \cap C_{ij} \neq \phi}} E\}$.

Then $B_{ij}$ separates $C_{ij}$ and $C'_{ij}$. Let $B_i = \bigcap_{j=1}^{n-1} B_{ij}$.

<u>Proposition 2.1.</u> If p,q are distinct positive prime numbers and a,b are integers s.t. $1 \leq a \leq p-1$ and $1 \leq b \leq q-1$ then $a/p \neq b/q$.

<u>Whenever we talk of $a/\pi_{ij}$ in the rest of this discussion, we shall have $a \in \{1, 2, \ldots \pi_{ij}-1\}$, unless otherwise stated.</u>

Let $\xi'(\pi_{ij})$ be the collection of the closures of those members of $\xi(\pi_{ij})$ which intersect with $C_{ij}$.

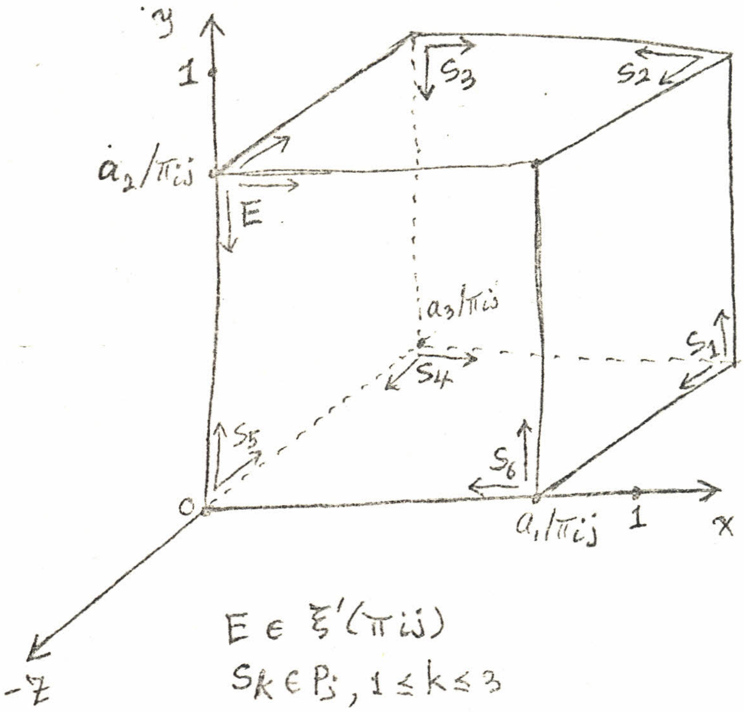
Let $F_{ij} = \bigcup_{E \in \xi'(\pi_{ij})} E$. Then $B_{ij} = \text{bdry } F_{ij}$. Let i be fixed (until after the proof of assertion 3).

Let $P_j$ the collection of faces of members of $\xi'(\pi_{ij})$ which faces intersect with the interior of $I^n$, i.e. faces of the form $D_1 \times \ldots \times \{a/\pi_{ij}\} \times \ldots \times D_n$ and not $D_1 \times \ldots \times \{0\} \times \ldots \times D_n$ or $D_1 \times \ldots \times \{1\} \times \ldots \times D_n$, $D_r \in \mathcal{D}(\pi_{ij})$.

For a member S of $P_j$ where $S = D_1 \times \ldots \times D_{r-1} \times \{a/\pi_{ij}\} \times \ldots \times D_n$, we say S has normal vector $e_r$ and write $N(S) = e_r$ ($e_1 = (1,0,\ldots,0)$ $e_2 = (0, 1, 0, \ldots, 0)$ e.t.c.).

We note that $B_{ij} \subset \bigcup_{S \in P_j} S$ $\forall j$, $1 \leq j \leq n-1$

Diagram 2.1




$E \in \xi'(\pi_{ij})$

$S_k \in P_j, \; 1 \le k \le 3$

$S_k \notin P_j, \; 4 \le k \le 6$

$N(S_k) = e_k, \; 1 \le k \le 3$

Proposition 2.2.

(i) If $x \in B_i$, then for any j, $1 \le j \le n-1$, there is at least one and at most two integers r, $1 \le r \le n$, s.t. x is contained in a member of $P_j$ with normal vector $e_r$. Furthermore, if for some $j_o$ there are two integers r s.t, x is contained in a member of $P_{j_o}$ with normal vector $e_r$, then for any j, $j \ne j_o$, there is only one integer r s.t. x is contained in a member of $P_j$ with normal vector $e_r$.

(ii) $\exists$ n-1 distinct integers $r_j$, $1 \le j \le n-1$ s.t. $x_{r_j} = a_j/\pi_{ij}$ (for $x \in B_i$).

Proof:

if $x \in S \in P_j$ and $N(S) = e_r$, then $x_r = a/\pi ij$. Since $Bij \subset \bigcup_{S \in P_j} S$ and $Bi \subset Bij$, $1 \leq j \leq n-1$, we have for each $j$, that $x \in S \in P_j$ for some S with $N(S) = e_{r_j}$ and $x_{r_j} = a_j/\pi ij$ for some $r_j$. From proposition 1, $a_j/\pi ij$, $1 \leq j \leq n-1$ are distinct, and therefore $r_j$, $1 \leq j \leq n-1$ are distinct. This proves part (ii). If in addition for some $j_o$ we have two more integers $r'_{j_o}$ and $r''_{j_o}$ with $r_{j_o}, r'_{j_o}, r''_{j_o}$ distinct and $x \in S' \in P_{j_o}$ with $N(S') = e_{r'_{j_o}}$, $x \in S'' \in P_{j_o}$ with $N(S'') = e_{r''_{j_o}}$, then we would have $x_{r'_{j_o}} = a'_{j_o}/\pi ij_o$, $x_{r''_{j_o}} = a''_{j_o}/\pi ij_o$. Since x has only n coordinates, this would force, for some j, $j \neq j_o$, $r_j \in \{r'_{j_o}, r''_{j_o}\}$ and $a/\pi ij \in \{a'_{j_o}/\pi i j_o, a''_{j_o}/\pi i j_o\}$ which is impossible in view of prop. 2.1. Similarly, we cannot have two integers j, $1 \leq j \leq n-1$ for each of which there are two integers r s.t. $x \in S \in P\,j$ with $N(S) = e_r$ for some S.

<u>Assertion 1.</u> If $k \neq i$, then $Bk \cap Bi = \phi$.

This follows immediately from prop. 2.2. (ii) and prop. 2.1. since $x \in Bi \cap Bk$ would imply $\exists$ n-1 integers $r_1, \ldots, r_{n-1}$ and n-1 integers $r'_1 \ldots r'_{n-1}$ s.t. $x_{r_j} = a_j/\pi_{ij}$ $1 \leq j \leq n-1$ and $x_{r'_j} = a'_j/\pi_{kj}$, $1 \leq j \leq n-1$.

<u>Assertion 2</u>

(i) Bi does not meet the n-2-dimensional edge of $I^n$ and (ii) Bi meets the surface of $I^n$ at only finitely many points.

To see this, we note that prop. 2.2. (ii) implies Bi is contained in a union of line segments of the form $\{y \in I^n : y_{r_j} = a_j/\pi_{ij}, 1 \leq j \leq n-1\}$ with $r_j$, $1 \leq j \leq n-1$ distinct. Since $0 < a_j/\pi_{ij} < 1$, any such segment meets the surface of $I^n$ at only two points. There are only a finite number of them for each i, hence the assertion.

Assertion 3

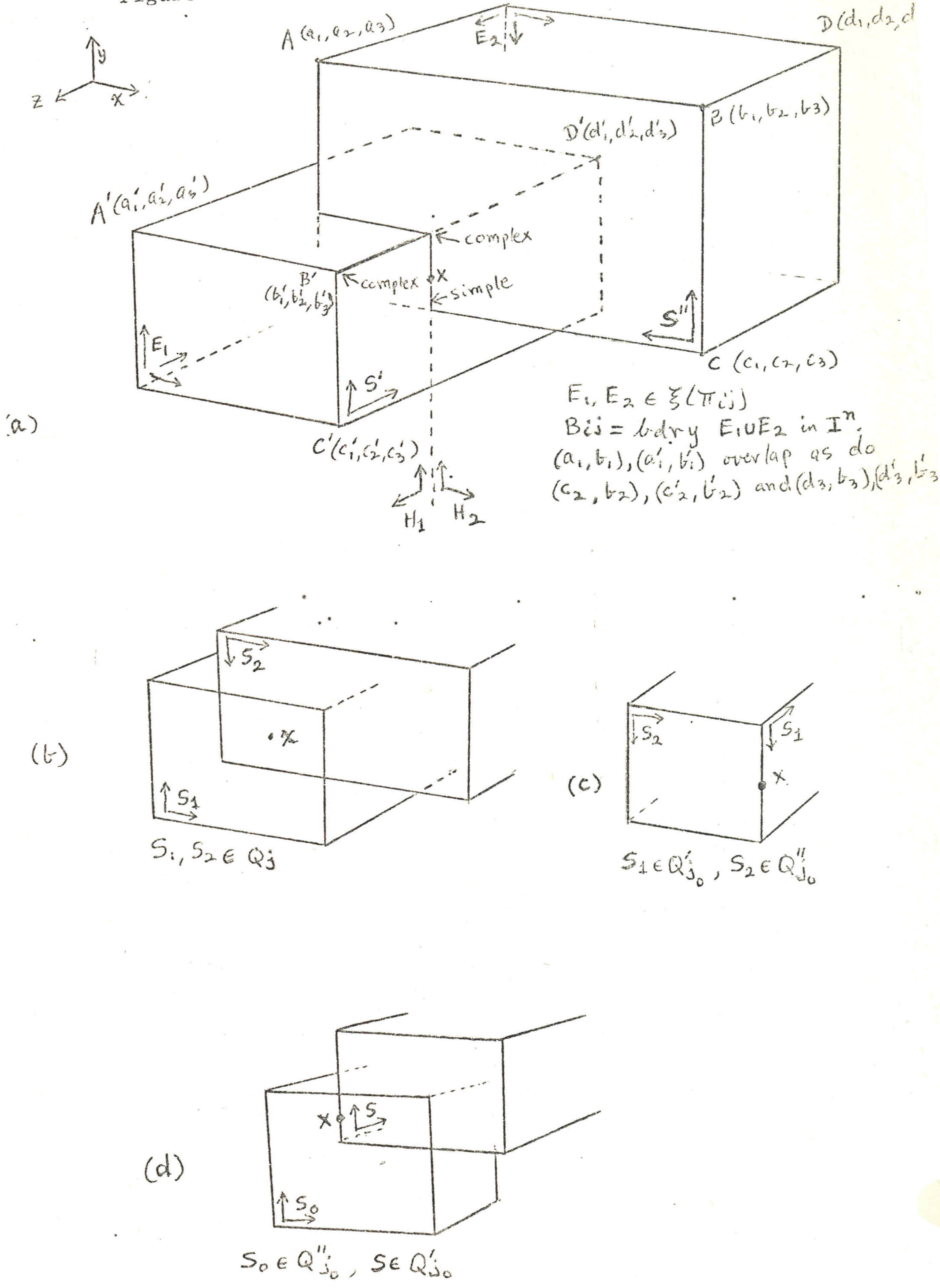
(i) Bi is a finite union of line segments of the form $\{y \in I^n : y_{r_j} = a_j/\pi_{ij}, 1 \leq j \leq n-1, y_{r_0} \in [\frac{a}{\pi_{ij'}}, \frac{b}{\pi_{ij''}}]\}$

$r_j$, $0 \leq j \leq n-1$ distinct, $a \in \{0, 1, \ldots. \pi_{ij'}\}$,
$b \in \{0, \ldots, \pi_{ij''}\}$.

(ii) Bi is the disjoint union of a finite number of simple closed curves and a finite number of simple arcs (i.e. Bi does not contain something like this $\perp$), the curves and arcs being closed sets of $I^n$.

We first give an intuitive argument. If $x \in$ Bi, then, with the exception of at most one j, x is not contained in a 'corner' of $B_{ij}$, i.e. on some nbhd of x, $B_{ij}$ coincides with a hyperplane. Furthermore, when x is contained in a 'corner' of $B_{ij}$ it must be a 'simple corner', i.e. one that involves the intersection of only two faces. This is because for each j, at least one coordinate of x is determined at a value of the form $\frac{a_j}{\pi_{ij}}$ and to be contained in a 'corner' of $B_{ij}$ means at least two coordinates of x are determined at

Figure 2.3.



y
z
x
A $(a_1, a_2, a_3)$
$E_2$
$D(d_1, d_2, d$
$B(b_1, b_2, b_3)$
$D'(d_1', d_2', d_3')$
$A'(a_1', a_2', a_3')$
complex
$B'$
$(b_1', b_2', b_3')$
complex
X
simple
$S''$
$E_1$
$S'$
$C(c_1, c_2, c_3)$
$E_1, E_2 \in \xi(\pi_{ij})$
$B_{ij}$ = bdry $E_1 \cup E_2$ in $I^n$.
$(a_1, b_1), (a_1', b_1')$ overlap as do
$(c_2, b_2), (c_2', b_2')$ and $(d_3, b_3), (d_3', b_3'$
(a)
$C'(c_1', c_2', c_3')$
$H_1$
$H_2$
(b)
$S_2$
x
$S_1$
$S_1, S_2 \in Q_j$
(c)
$S_2$
$S_1$
x
$S_1 \in Q'_{j_0}$, $S_2 \in Q''_{j_0}$
(d)
x
S
$S_0$
$S_0 \in Q''_{j_0}$, $S \in Q'_{j_0}$

values of the form $\frac{a_j}{\pi_{ij}}$ while to be contained in a 'complex corner' means at least three coordinates of x are determined at values of the form $\frac{a_j}{\pi_{ij}}$. The claim now follows from proposition 2.1. Given also that the intervals of $\mathcal{D}(\pi_{ij})$ are either a positive distance apart or overlapping, it follows that when x is at a 'corner' of $B_{ij}$, then on some nbhd of x $B_{ij}$ coincides with the union of two half-hyperplanes $H_1$, $H_2$ intersecting along their edges both of which edges contain x. (see fig. 2.3a) . Thus at worst we could have a situation where on some nbhd of x, Bi coincides with the intersection of n-1 sets, n-2 of them being hyperplanes and the n-1th being a union of two half-hyperplanes intersecting along their edges, both edges containing x. The hyperplanes and half-hyperplanes furthermore, have distinct normal vectors. In this case Bi coincides, on the nbhd of x, with an arc having a corner at x. Otherwise Bi coincides on some nbhd of x with the intersection of n-1 hyperplanes having distinct normal vectors in which case Bi coincides, on the nbhd, with a line segment. This shows that Bi cannot contain something like this $\perp$. That Bi is a finite union of line segments is intuitively clear. The assertion then follows.

We now give the detailed proof.

First we show that Bi is a finite union of line segments. From prop. 2.2. (ii), $x \in B_i$ implies x is contained in a line segment of the form $\{y \in I^n:$

$y_{r_j} = a_j/\pi_{ij}$, $1 \le j \le n-1\}$ with $r_j$, $1 \le j \le n-1$ distinct.
Let $r_0$ be the unique integer s.t. $1 \le r_0 \le n$,

$r_0 \notin \{r_j, 1 \le j \le n-1\}$.

Let J be the set $\{a \in I: (x_1, x_2, \ldots x_{r_0-1}, a, \ldots\ldots, x_n) \in B_i\}$. Then $x_{r_0} \in J$. Let J' be the component of J containing $x_{r_0}$. Then J' is an interval (from a property of R). Let b = inf J' and b' = sup J'. J' is closed in J (being a component of J) and therefore in I since J is closed in I. So b, b' $\in$ J'. Thus $(x_1, x_2, \ldots x_{r_0-1}, b, \ldots, \ldots\ldots x_n)$ is contained in Bi and for any $\epsilon > 0$, $\exists\ 0 < \delta < \epsilon$ s.t. $(x_1, x_2, \ldots x_{r_0-1}, b'+\delta, \ldots x_n) \notin Bi$. Since $Bi = \bigcap_{j=1}^{n-1} Bij$, then for some

$j_1$, $(x_1, x_2, \ldots\ldots x_{r_0-1}, b', \ldots\ldots x_n) \in Bij_1$
and for any $\epsilon > 0, \exists\ 0 < \delta < \epsilon$, s.t.

$(x_1, x_2, \ldots x_{r_0-1}, b'+\delta, \ldots\ldots, x_n) \notin Bij_1$. The same statement then holds for some E in $\xi'(\pi ij_1)$ which makes it clear that $b' = \frac{a'}{\pi ij_1}$ for some $a' \in \{0, 1, \ldots \pi ij_1\}$. Similarly, $b = \frac{a}{\pi ij_2}$, $a \in \{0, \ldots, \pi ij_2\}$.
Since $J' \subset J$, the line segment $\{x_1\} \times \ldots\ldots \times \{x_{r_0-1}\} \times [\frac{a}{\pi ij_2}, \frac{a'}{\pi ij_1}] \times \ldots \times \{x_n\}$ is contained in Bi (and contains x). Thus Bi is a union of such line segments which are finite in number (recall $x_{r_j} = a_j/\pi_{ij}$, $r_j \ne r_0$ and $a_j$, a, a' $\in \{0, \ldots\ldots \pi ij'\}$ for the appropriate j').
This proves part (i).

We next prove the following:-

Let $x \in Bi$, then:-

Proposition 2.3: Either (i) $\exists$ a nbhd U of x s.t. $B_i \cap U \subset L = \bigcap_{j=1}^{n-1} H_j$ where $H_j$ is a hyperplane of the form $\{y \in I^n: y_{r_j} = x_{r_j} = a_j/\pi_{ij}\}$ or (ii) $\exists$ a nbhd U of x and an integer $j_o$, $1 \leq j_o \leq n-1$ s.t.

$B_i \cap U \subset L = [(\bigcap_{j=1, j \neq j_o}^{n-1} H_j) \cap H'_{j_o}] \cup [(\bigcap_{j=1, j \neq j_o}^{n-1} H_j) \cap H''_{j_o}]$ where $H_j$ is a hyperplane of the form $\{y \in I^n: y_{r_j} = x_{r_j} = a_j/\pi_{ij}\}$ $j \neq j_o$, $H'_{j_o}$ is a half-hyperplane of the form $\{y \in I^n: y_{r'_{j_o}} = x_{r'_{j_o}} = a'_{j_o}/\pi_{ij_o}, y_{r''_{j_o}} \leq a''_{j_o}/\pi_{ij_o} = x_{r''_{j_o}}\}$ (or $y_{r''_{j_o}} \geq a''_{j_o}/\pi_{ij_o} = x_{r''_{j_o}}$), and $H''_{j_o}$ is a half hyperplane of the form $\{y \in I^n: y_{r''_{j_o}} = x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o}, y_{r'_{j_o}} \leq a'_{j_o}/\pi_{ij_o} = x_{r'_{j_o}}\}$ (or $y_{r'_{j_o}} \geq a'_{j_o}/\pi_{ij_o} = x_{r'_{j_o}}$).

We see that in the first case, because $a_j/\pi_{ij}$, and therefore $x_{r_j}$, and therefore $r_j$, $1 \leq j \leq n-1$ are distinct (prop. 2.1), L is a line segment containing x while in the second case, for similar reasons, L is an arc containing x with a corner at x (the union of two line segments ending at x). This together with part (i) of assertion 3 gives us part (ii) of assertion 3.

We divide the argument into two cases (in view of prop. 2.2.).

Case 1. For all j, $1 \leq j \leq n-1$, $\exists$ one and only one $r_j$ s.t. $x \in S \in P_j$ with $N(S) = e_{r_j}$ (fig. 2.3 b)
Fix j. Let $Q_j = \{S \in P_j: x \in S\}$. Then, for $S \in Q_j$, since $N(S) = e_{r_j}$, we have $S = D_1 \times \ldots \times D_{r_{j-1}} \times \{a/\pi_{ij}\} \times \ldots \times D_n$, $D_r \in \mathcal{D}(\pi_{ij})$. Since $x \in S$ we have $a/\pi_{ij} = x_{r_j}$.

Thus $\bigcup_{S\in Q_j} S\subset H_j = \{y\in I^n: y_{r_j} = x_{r_j} = aj/\pi_{ij}\}$ (for some $aj$). The set $\bigcup_{S\in P_j - Q_j} S$ is closed so $\exists$ a nbhd $U_j$ of $x$ s.t. $(\bigcup_{S\in P_j} S)\cap U_j \subset \bigcup_{S\in Q_j} S$. Since $B_{ij}\subset \bigcup_{S\in P_j} S$,

$B_{ij}\cap U_j\subset (\bigcup_{S\in P_j} S)\cap U_j \subset \bigcup_{S\in Q_j} S\subset H_j$.

Now unfix $j$ and let $U = \bigcap_{j=1}^{n-1} U_j$.

Then $B_i\cap U\subset (\bigcap_{j=1}^{n-1} B_{ij})\cap U = \bigcap_{j=1}^{n-1} (B_{ij}\cap U)\subset \bigcap_{j=1}^{n-1} H_j$ as required.

Case 2. $\exists\ j_o$, $1\le j_o\le n$ and integers $r'_{j_o}$, $r''_{j_o}$ $1\le r'_{j_o} < r''_{j_o}\le n$ s.t. $x\in S'\in P_{j_o}$ and $N(S') = e_{r'_{j_o}}$ for some $S'$; $x\in S''\in P_{j_o}$ and $N(S'') = e_{r''_{j_o}}$ for some $S''$. Define $Q_j$ as in case 1.

Then from prop. 2.2., for each $j$, $j\ne j_o$, $\exists\ r_j$ s.t. $S\in Q_j \Rightarrow N(S) = e_{r_j}$. As in case 1, we have $\bigcup_{S\in Q_j} S\subset H_j = \{y\in I^n: y_{r_j} = x_{r_j} = ai/\pi_{ij}\}$ for $j\ne j_o$. Again from Prop. 2.2., $S\in Q_{j_o} \Rightarrow N(S) = e_{r'_{j_o}}$ or $N(S) = e_{r''_{j_o}}$

Let $Q'_{j_o} = \{S\in Q_{j_o} : N(S) = e_{r'_{j_o}}\}$, $Q''_{j_o} = \{S\in Q_{j_o} : N(S) = e_{r''_{j_o}}\}$

For $x\in S\in P_j$ where $S = D_1\times\ldots\times D_{r_o-1}\times\{x_{r_o}\}\times\ldots\times D_n$, we shall say $x$ is <u>within</u> $S$ if $x_r\in \text{Int } D_r$ for $r\ne r_o$ where $\text{Int } D_r$ is the interior of $D_r$ in $I$. Thus $x$ is within $S$ iff $x\in \text{Int } S$ where $\text{Int } S$ is the interior of $S$ in the hyperplane containing $S$.

<u>Proposition 2.4.</u>

If $x$ is not within $S$ for any $S$ in $Q'_{j_o}$, then

$\bigcup$ $S\subset\{y\in I^n:$ ...

$\bigcup_{S\in Q'_{j_o}} S\subset\{y\in I^n: y_{r''_{j_o}} \geq x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o}\}$

Likewise, if x is not within S for any S in $Q''_{j_o}$,

then $\bigcup_{S\in Q''_{j_o}} S\subset\{y\in I^n: y_{r'_{j_o}} \leq x_{r'_{j_o}} = a'_{j_o}/\pi_{ij_o}\}$ or

$\bigcup_{S\in Q''_{j_o}} S\subset\{y\in I^n: y_{r'_{j_o}} \geq x_{r'_{j_o}} = a'_{j_o}/\pi_{ij_o}\}$.

For, suppose x is not within S for some S in $Q'_{j_o}$.

Then $S = D_1 x \ldots x\, D_{r'_{j_o}-1} \times \{x_{r'_{j_o}}\} x \ldots x D_n$,

$D_r\in\mathcal{D}(\pi_{ij_o})$ with $a/\pi_{ij_o} = x_{r_1} = \inf D_{r_1}$ or $a/\pi_{ij_o} =$

$x_{r_1} = \sup D_{r_1}$ for some $r_1$, $1\leq r_1\leq n$, $r_1\neq r'_{j_o}$. But

this implies $x\in S'\in Q_{j_o}$ with $N(S') = e_{r_1}$ and since $r_1\neq$

$r'_{j_o}$. we must infact have $r_1 = r''_{j_o}$ (see a condition

above on $Q_{j_o}$). Thus either $S\subset\{y\in I^n: y_{r''_{j_o}} \leq x_{r''_{j_o}}$

$= a''_{j_o}/\pi_{ij_o}\}$ or $S\subset\{y\in I^n: y_{r''_{j_o}} \geq x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o}\}$. Suppose

now that x is not within S for any S in $Q'_{j_o}$. If

$S_1, S_2\in Q'_{j_o}$ with $S_1\subset\{y\in I^n: y_{r''_{j_o}} \leq x_{r''_{j_o}}\}$ and $S_2\subset\{y\in I^n:$

$y_{r''_{j_o}} \geq x_{r''_{j_o}}\}$, then $\inf D = \sup D' = x_{r''_{j_o}}$ for some $D, D'\in$

$\mathcal{D}(\pi_{ij_o})$. But the description of intervals in $\mathcal{D}(\pi_{ij_o})$ precludes this.

We must therefore have $\bigcup_{S\in Q'_{j_o}} S\subset\{y\in I^n: y_{r''_{j_o}} \leq x_{r''_{j_o}}\}$

or $\bigcup_{S\in Q'_{j_o}} S\subset\{y\in I^n: y_{r''_{j_o}} \geq x_{r''_{j_o}}\}$.

The second part of proposition 2.4 is proved in a similar manner.

We now divide case two into four situations.

Situation 1: x is not within S for any S in $Q'_{j_o}$ and x is not within S for any S in $Q''_{j_o}$ (fig. 2.3c). Of four possible cases here, we treat only one, the rest being similar. So we assume $\bigcup_{S\in Q'_{j_o}} S\subset\{y\in I^n: y_{r''_{j_o}} \le x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o}\}$ and $\bigcup_{S\in Q''_{j_o}} S\subset\{y\in I^n: y_{r'_{j_o}} \le x_{r'_{j_o}} = a'_{j_o}/\pi_{ij_o}\}$.

We have, as in case 1, that $\bigcup_{S\in Q'_{j_o}} S\subset\{y\in I^n: y_{r'_{j_o}} = x_{r'_{j_o}} = a'_{j_o}/\pi_{ij_o}\}$ and $\bigcup_{S\in Q''_{j_o}} S\subset\{y\in I^n: y_{r''_{j_o}} = x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o}\}$.

So if we let $H'_{j_o} = \{y\in I^n: y_{r'_{j_o}} = x_{r'_{j_o}} = a'_{j_o}/\pi_{ij_o},\ y_{r''_{j_o}} \le x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o}\}$ and $H''_{j_o} = \{y\in I^n: y_{r''_{j_o}} = x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o},\ y_{r'_{j_o}} \le x_{r'_{j_o}}\}$ we have $\bigcup_{S\in Q_{j_o}} S\subset H'_{j_o}\cup H''_{j_o}$. As before, $\exists$ a nbhd $U_{j_o}$ of x s.t. $B_{ij_o}\cap U_{j_o} \subset \bigcup_{S\in Q_{j_o}} S$. If we let $U = \bigcap_{j=1}^{n} U_j$ ($U_j$, $H_j$, $j\neq j_o$ have been treated earlier) then we end up with the situation in prop. 2.3 (ii).

Situation 2: (fig 2.3d) x is not within S for any S in $Q'_{j_o}$ and x is within $S_o$ for some $S_o$ in $Q''_{j_o}$. Again $j\neq j_o$ is treated as in case 1 so we have $U_j$, $H_j$, $r_j$ satisfying the same conditions as in case 1. For some $E_o\in\xi'(\pi_{ij_o})$, $S_o$ is a face of $E_o$ so we have $E_o = D_1 x \dots x D_n$, $D_r\in\mathcal{D}(\pi_{ij_o})$ and $S_o = D_1 x \dots x D_{r''_{j_o}-1} x \{x_{r''_{j_o}}\} x \dots x D_n$ with $x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o}$. Since x is within $S_o$, $x_r\in \operatorname{Int} D_r$ if $r \neq r''_{j_o}$. Either $x_{r''_{j_o}} = \inf D_{r''_{j_o}}$ or $x_{r''_{j_o}} = \sup D_{r''_{j_o}}$. We consider only the case where $x_{r''_{j_o}} = \sup D_{r''_{j_o}}$, the other case being similar. Then $E_o = D_1 x \dots x D_{r''_{j_o}-1} x [x_{r''_{j_o}} - 3/\pi_{ij_o}, x_{r''_{j_o}}] x \dots x D_n$.
From prop. 2.4 and the assumption of situation 2, we either have $\bigcup_{S\in Q'_{j_o}} S\subset\{y\in I^n: y_{r''_{j_o}} \le x_{r''_{j_o}}\}$ or $\bigcup_{S\in Q'_{j_o}} S\subset\{y\in I^n:$

$y_{r''_{j_o}} \geq x_{r''_{j_o}}\}$. The latter inclusion implies that $x_{r''_{j_o}} = \inf D$ for some $D \in \mathcal{D}(\pi_{ij_o})$ (recall $x \in \bigcup_{S \in Q'_{j_o}} S$ and $r'_{j_o} \neq r''_{j_o}$) but we already have $x_{r''_{j_o}} = \sup D^{j_o}_{r''_{j_o}}$ and we have seen earlier that this cannot happen. So $\bigcup_{S \in Q'_{j_o}} S \subset \{y \in I^n: y_{r''_{j_o}} \leq x_{r''_{j_o}}\}$. Let $U' = \text{Int } D_1 \times \text{Int } D_2 \times \ldots \times \text{Int } D_{r''_{j_o}-1} \times (x_{r''_{j_o}} - 3/\pi_{ij_o}, 1] \times \ldots \times \text{Int } D_n$. Then $U'$ is a nbhd of $x$ s.t. $(\{y \in I^n: y_{r''_{j_o}} \leq x_{r''_{j_o}}\} - S_o) \cap U' \subset \text{Int } E_o \subset \text{Int } F_{ij_o} \subset X - B_{ij_o}$. Thus $(\{y \in I^n: y_{r''_{j_o}} \leq x_{r''_{j_o}}\} \cap B_{ij_o}) \cap U' \subset S_o$. Since $\bigcup_{S \in Q'_{j_o}} S \subset \{y \in I^n: y_{r''_{j_o}} \leq x_{r''_{j_o}}\}$, we have $[(\bigcup_{S \in Q'_{j_o}} S) \cap B_{ij_o}] \cap U' \subset S_o$. As in case 1, $\bigcup_{S \in Q''_{j_o}} S \subset \{y \in I^n: y_{r''_{j_o}} = x_{r''_{j_o}}\}$. Again as in case 1, $\exists$ a nbhd $U'_{j_o}$ of $x$ s.t. $B_{ij_o} \cap U'_{j_o} \subset \bigcup_{S \in Q_{j_o}} S$. Let $U_{j_o} = U'_{j_o} \cap U'$. Then $B_{ij_o} \cap U_{j_o} \subset \bigcup_{S \in Q''_{j_o}} S \subset \{y \in I^n: y_{r''_{j_o}} = x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o}\}$

Put $H_{j_o} = \{y \in I^n: y_{r''_{j_o}} = x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o}\}$. Let $r_{j_o} = r''_{j_o}$. Then with $U = \bigcap_{j=1}^{n-1} U_j$, we have a situation as in prop. 2.3.(i).

Situation 3. $x$ is not within $S$ for any $S$ in $Q''_{j_o}$ and $x$ is within $S_o$ for some $S_o$ in $Q'_{j_o}$. This situation is similar to situation 2.

Situation 4. (fig 2.3a)

$x$ is within $S'$ for some $S'$ in $Q'_{j_o}$ and $x$ is within $S''$ for some $S''$ in $Q''_{j_o}$.

For some $E', E'' \in \xi'(\pi_{ij_o})$, $S'$ is a face of $E'$ and $S''$ is a face of $E''$. We have $E' = D'_1 \times \ldots \times D'n$, $E'' = D''_1 \times \ldots \times D''n$, $S' = D'_1 \times \ldots \times D'_{r_{j_o}-1} \times \{x_{r'_{j_o}}\} \times \ldots \times D'n$,

$S'' = D''_1 x \ldots\ldots x D''_{r''_{j_o}-1} x \{x_{r''_{j_o}}\} x \ldots x D''n.$ where $x_{r'_{j_o}}$ $= a'_{j_o}/\pi_{ij_o}$, $x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o}$, $x_r \in \text{Int } D'_r$ if $r \neq r'_{j_o}$, and $x_r \in \text{Int } D''_r$ if $r \neq r''_{j_o}$. Furthermore, $x_{r'_{j_o}} \in \{\sup D'_{r'_{j_o}}, \inf D'_{r'_{j_o}}\}$ and $x_{r''_{j_o}} \in \{\sup D''_{r''_{j_o}}, \inf D''_{r''_{j_o}}\}$. Of the four possible cases, we consider only one, i.e. $x_{r'_{j_o}} = \inf D'_{r'_{j_o}}$ and $x_{r''_{j_o}} = \sup D''_{r''_{j_o}}$.

Then $E' = D'_1 x \ldots x D'_{r'_{j_o}-1} x [x_{r'_{j_o}}, x_{r'_{j_o}} + 3/\pi_{ij_o}] x \ldots\ldots x$

$E'' = D''_1 x \ldots x D''_{r''_{j_o}-1} x [x_{r''_{j_o}} - 3/\pi_{ij_o}, x_{r''_{j_o}}] x \ldots x D''$

Let $U' = \text{Int } D'_1 \, x \ldots\ldots x \text{ Int } D'_{r'_{j_o}-1} \, x \, [0, x_{r'_{j_o}} + 3/\pi_{ij_o}) x$ $.x \text{Int} D'n.$ and $U'' = \text{Int } D''_1 \, x \ldots\ldots x \text{ Int } D''_{r''_{j_o}-1} \, x$ $(x_{r''_{j_o}} - 3/\pi_{ij_o}, 1] x \ldots x \text{Int} D''n$

and $U_o = U' \cap U''$. Then $U_o$ is a nbhd of x s.t. $U_o \cap \{y \in I^n : y_{r'_{j_o}} = x_{r'_{j_o}}\} \subset S'$, $\wedge$ $U_o \cap \{y \in I^n : y_{r''_{j_o}} = x_{r''_{j_o}}\} \subset S''$.

As in case 1, $\bigcup_{S \in Q'_{j_o}} S \subset \{y \in I^n : y_{r'_{j_o}} = x_{r'_{j_o}}\}$ and

$\bigcup_{S \in Q''_{j_o}} S \subset \{y \in I^n : y_{r''_{j_o}} = x_{r''_{j_o}}\}$

whence $(\bigcup_{S \in Q'_{j_o}} S) \cap U_o \subset S'$ and $(\bigcup_{S \in Q''_{j_o}} S) \cap U_o \subset S''$. As in case 1, obtain a nbhd V of $x$ s.t. $B_{ij_o} \cap V \subset \bigcup_{S \in Q_{j_o}} S$.

Let $U_{j_o} = U_o \cap V$. Then (*) $B_{ij_o} \cap U_{j_o} \subset S' \cup S'' \subset \{y \in I^n :$ $y_{r'_{j_o}} = x_{r'_{j_o}} = a'_{j_o}/\pi_{ij_o}\} \cup \{y \in I^n : y_{r''_{j_o}} = x_{r''_{j_o}} = a''_{j_o}/\pi_{ij_o}\}$.

As in a previous case (because $U_{j_o} \subset U'$) we have $\{y \in I^n :$ $y_{r'_{j_o}} > x_{r'_{j_o}}\} \cap U_{j_o} \subset \text{Int } E' \subset \text{Int } F_{ij_o} \subset I^n - B_{ij_o}$.

Similarly, $\{y \in I^n : y_{r''_{j_o}} < x_{r''_{j_o}}\} \cap U_{j_o} \subset I^n - B_{ij_o}$.

Thus $B_{ij_o} \cap U_{j_o} \subset \{y \in I^n : y_{r'_{j_o}} \leq x_{r'_{j_o}}\} \cap \{y \in I^n : y_{r''_{j_o}} \geq x_{r''_{j_o}}\}$.

Combining with (*) above, we have $Bi_{j_0} \cap U_{j_0} \subset H'_{j_0} \cup H''_{j_0}$. Where $H'_{j_0}$, $H''_{j_0}$ are as in prop. 2.3(ii).
Thus letting $U = \bigcap_{j=1}^{n-1} U_j$, we have a situation as in prop. 2.3 (ii).
Let $(Yn, \sigma n) = I^n - \bigcup_{i=1}^{\infty} Bi$ with the inherited (euclidean) metric. We show that $(Yn, \sigma n)$ satisfies the requirements mentioned earlier.

<u>Assertion 4</u> $d_2(Yn\ \sigma n) \leq n-2$.

Proof: Let $(C_j, C'_j)$ $1 \leq j \leq n-1$ be n-1 pairs of closed sets of $(Yn, \sigma n)$ s.t. $\sigma n(C_j, C'_j) > 0$ $\forall j$, $1 \leq j \leq n-1$. Then if $\bar{C}_j$ and $\bar{C}'_j$ are the closures of $C_j$ and $C'_j$ in $I^n$ we have $\bar{C}_j \cap \bar{C}'_j = \phi$ $1 \leq j \leq n-1$. So for some $i \in N$, $\bar{C}_j \subset Ci_j$, $\bar{C}'_j \subset C'i_j$ $1 \leq j \leq n-1$. Thus $Bi_j$ separates $\bar{C}_j$ and $\bar{C}'_j$ in $I^n$ for each $j$, $1 \leq j \leq n-1$. Let $B'_j = Bi_j \cap Yn$. Then $B'_j$ is a closed set of $(Yn, \sigma n)$ separating $C_j$ and $C'_j$ for each $j$. Furthermore $\bigcap_{j=1}^{n-1} B'_j = (\bigcap_{j=1}^{n-1} Bi_j) \cap Yn = Bi \cap Yn = \phi$. Thus $d_2(Yn, \sigma n) \leq n-2$.

<u>Assertion 5</u> $\mu$-dim $(Yn, \sigma n) \geq n-1$.

Proof. Assume $\mu$-dim $(Yn, \sigma n) \leq n-2$. For $1 \leq k \leq n$, let $Ak = \{y \in I^n : y_k = 0\}$, $A'k = \{y \in I^n : y_k = 1\}$.
We want to construct closed sets Mk, $0 \leq k \leq n$, of $I^n$ satisfying:-

(1) $M_0 = I^n$, $Mn = \phi$.

(2) $Mk \subset Mk-1$ $1 \leq k \leq n$

(3) Mk separates $Ak \cap Mk-1$ and $A'k \cap Mk-1$ in Mk-1.

The construction is by induction. Assume, for some $j$, $0 \leq j \leq n-2$, that we have constructed closed sets $M_k$ of $I^n$ and collections $\mathcal{F}_k$ of closed subsets of $Y_n$ satisfying:- (for $0 \leq k \leq j$)

(i) $M_0 = I^n$

(ii) $M_k \subset M_{k-1}$, $1 \leq k \leq j$

(iii) $M_k$ separates $A_k \cap M_{k-1}$ and $A'_k \cap M_{k-1}$ in $M_{k-1}$

(iv) $\mathcal{F}_k$ is finite

(v) $\overline{\mathcal{F}}_k = \{\overline{F}, F \in \mathcal{F}_k\}$ covers $M_k$.

(vi) Mesh $\mathcal{F}_k < 1$

(vii) If $x \in M_k \cap Y_n$, then $\mathrm{ord}_x \mathcal{F}_k \leq n-k-2$

Construct $\mathcal{F}_{j+1}$ and $M_{j+1}$ as follows.

Put $\mathcal{F}_{j+1} = \{F \in \mathcal{F}_j : \overline{F} \cap A_{j+1} \cap M_j \neq \phi\}$ .

Put $W = (\bigcup_{F \in \mathcal{F}_{j+1}} \overline{F}) \cap M_j$ and let $M_{j+1}$ be the boundary in $M_j$ of $W$. (i) and (ii) are obvious. To see that (iii) holds, note that $M_j - (\bigcup_{F \in \mathcal{F}_j - \mathcal{F}_{j+1}} \overline{F})$ is an open set of $M_j$ (condition iv) containing $A_{j+1} \cap M_j$ (by the construction of $\mathcal{F}_{j+1}$) and contained in $W$ (because of (v)). Furthermore, (vi) implies that $W \cap A'_{j+1} = \phi$ . This proves (iii).

(iv), (v) and (vi) are obvious. To see (vii), let $x \in M_{j+1} \cap Y_n$. Then $x \in \overline{M_j - W} \subset \overline{(\bigcup_{F \in \mathcal{F}_j, F \notin \mathcal{F}_{j+1}} \overline{F})} = \bigcup_{F \in \mathcal{F}_j, F \notin \mathcal{F}_{j+1}} \overline{F}$ (from iv) (closures are in $I^n$).

So for some $F$, $F \in \mathcal{F}_j$, $F \notin \mathcal{F}_{j+1}$, $x \in \overline{F}$.

Since $x \in Y_n$ and $F$ is a closed set of $Y_n$, $x \in F$. Thus $\mathrm{ord}_x \mathcal{F}_{j+1} \leq \mathrm{ord}_x \mathcal{F}_j - 1 \leq n-(j+1) - 2$

Put $M_0 = I^n$ and construct $\mathcal{F}_0$ as follows.

Let $\mathcal{B}$ be a finite open cover of $I^n$ by open balls of radius $\frac{1}{3}$. Since $I^n$ is compact, $\mathcal{B}$ has a Lebesque number $\epsilon$ so that any set of diameter not exceeding $\epsilon$ is contained in a member of $\mathcal{B}$. Since $\mu$-dim $Y_n \leq n-2$, $\exists$ a closed (in $Y_n$) l.f. (in $Y_n$) cover $\mathcal{F}'$ of $Y_n$ with ord $\mathcal{F}' \leq n-2$ and mesh $\mathcal{F}' < \epsilon$. Then $\exists$ a function $f: \mathcal{F}' \to \mathcal{B}$ s.t. $F \subset f(F)$. For $B \in \mathcal{B}$, let $g(B) = \bigcup_{\substack{F \in \mathcal{F}' \\ f(F)=B}} F$. Let $\mathcal{F}_0 = \{g(B), B \in \mathcal{B}\}$. Then $\mathcal{F}_0$ is a finite closed cover of Yn, with mesh $\mathcal{F}_0 < 1$, and ord $\mathcal{F}_0 \leq n-2$. Also $\bigcup_{F \in \mathcal{F}_0} \overline{F}$ is a closed set containing Yn. Since dim $\bigcup_{i=1}^{\infty} B_i \leq n-1$ (countable sum theorem), Yn is dense in $I^n$. Thus $\bigcup_{F \in \mathcal{F}_0} \overline{F} = I^n$. We have therefore shown that (i), (iv), (v), (vi) and (vii) are satisfied. The rest of the conditions are vacuously satisfied. We can therefore construct closed sets Mk, $0 \leq k \leq n-1$ satisfying conditions (i) to (vii). However, the empty set may not separate $A \cap M_{n-1}$ and $A' \cap M_{n-1}$ in $M_{n-1}$. We shall therefore refer to $M_{n-1}$ as M and construct the proper $M_{n-1}$ from it. From (vii) if $x \in M \cap Y_n$ then $\mathrm{ord}_x \mathcal{F}_{n-1} \leq -1$. But $M \subset \bigcup_{F \in \mathcal{F}_{n-1}} \overline{F}$ and further, if $x \in M \cap Y_n$ then $x \in \overline{F}$, $F \in \mathcal{F}_{n-1} \Rightarrow x \in F$ since F is a closed set of Yn. So $\mathcal{F}_{n-1}$ covers $M \cap Y_n$. Combining this with $\mathrm{ord}_x \mathcal{F}_{n-1} \leq -1$ for $x \in M \cap Y_n$, we see that $M \cap Y_n = \phi$.

Let $T = \{x \in M_{n-2} : \mathrm{ord}_x \overline{\mathcal{F}}_{n-2} \geq 1\}$. T is a closed set of $M_{n-2}$. $M \subset T \subset M_{n-2} \cap (\bigcup_{i=1}^{\infty} B_i)$. The first inclusion follows because $x \in M \Rightarrow x \in \overline{F}$ for some $F \in \mathcal{F}_{n-1}$ and $x \in \overline{F}$ for some $F \in \mathcal{F}_{n-2} - \mathcal{F}_{n-1}$ (as we have seen earlier). To see that the second inclusion holds, we recall that $x \in M_{n-2} \cap Y_n$ and $x \in \overline{F}$, $F \in \mathcal{F}_{n-2}$ implies $x \in F$ so for $x \in M_{n-2} \cap Y_n$ $\mathrm{ord}_x \overline{\mathcal{F}_{n-2}} = \mathrm{ord}_x \mathcal{F}_{n-2} \leq 0$ (from (vii)). So $T \subset M_{n-2} - Y_n = M_{n-2} \cap (\bigcup_{i=1}^{\infty} B_i)$

Let $P$, $P'$, $Q$, $Q'$ be the union of components of $T$ that intersect $A_{n-1}$, $A'_{n-1}$, $A_n$, $A'_n$ respectively. These sets are closed. Take $P$, for example. Let $\alpha \in T$ be a limit point of $P$. $\exists$ sequences $\alpha_i$ of points of $P$ and $C_i$ of components of $P$ s.t. $\lim_i \alpha_i = \alpha$ and $\alpha_i \in C_i$. Then $\alpha \in \liminf C_i$ (w.r.t.T). So, from lemma 2.10, $\limsup C_i$ (w.r.t.T) is connected. $\limsup C_i$ intersects $A_{n-1}$. This is because each $C_i$ intersects $A_{n-1}$ at, say, $\beta_i$. Exempting the trivial case where the $\beta_i$ are only a finite number, $\{\beta_i\}$ is an infinite subset of the compact $T \cap A_{n-1}$ and so has a limit point $\beta$. Then $\beta \in \limsup C_i$. So $\limsup C_i$ is a connected set of $T$ intersecting $A_{n-1}$ which implies $\limsup C_i \subset P$ so $\alpha \in \liminf C_i \subset \limsup C_i \subset P$. So $\alpha \in P$. So $P$ is closed in $T$. Similarly, $P'$, $Q$, $Q'$ are closed.

Claim: There is no connected set of $T$ intersecting both $P \cup P'$ and $Q \cap Q'$. For suppose there were. Then we could construct (by uniting with appropriate components of $T$ and taking the closure) a connected compact set of T intersecting An and A'n and one (or both) of An-1 and A'n-1. Since $T \subset \bigcup_{i=1}^{\infty} Bi$ and from assertion 1, assertion 3(ii), and lemma 2.3, this connected subset must be contained in some simple arc or some simple closed curve say $\Gamma$ of some Bi. This would imply that $\Gamma$ touches An-1 (or A'n-1), An, and A'n. From assertion 3(i) and assertion 2(ii) it follows that $\Gamma$ is a simple arc and $\Gamma$ meets the surface of $I^n$ only at its end points. But now assertion 2(i) implies $\Gamma$ has three end points, impossible for a simple arc.

It also follows that $P \cup P'$ and $Q \cap Q'$ are disjoint (a point is connected). So there exist, by lemma 2.2, disjoint clopen sets U, U' of T with $T = U \cup U'$, $P \cup P' \subset U$, and $Q \cap Q' \subset U'$.

Because $An\text{-}1 \cup A'n\text{-}1$ does not intersect U' (because $An\text{-}1 \cup A'n\text{-}1 \cap T \subset U$), we have $[(An\text{-}1 \cup A'n\text{-}1) \cap Mn\text{-}2] \cup U$ and U' are disjoint closed sets of $M_{n-2}$ We can therefore find an open set V of $M_{n-2}$ s.t. $V \cap T = U'$ and $\overline{V} \cap (An\text{-}1 \cup A'n\text{-}1 \cup U) = \phi$ . Define Mn-1 as follows.

Let $Mn\text{-}1 = (M-V) \cup (\overline{V}-V)$.

We recall that M separates $An\text{-}1 \cap Mn\text{-}2$ and $A'n\text{-}1 \cap Mn\text{-}2$ in Mn-2. Let G, G' be open sets of Mn-2 s.t.

$Mn\text{-}2 - M = G \cup G'$, $An\text{-}1 \cap Mn\text{-}2 \subset G$, $A'n\text{-}1 \cap Mn\text{-}2 \subset G'$.
and $G \cap G' = \phi$

Let $H = G-\overline{V}$, $H' = (G' \cup V)-(\overline{V}-V)$.

Then clearly H, H' are open sets of Mn-2 s.t.

$Mn\text{-}2-Mn\text{-}1 = H \cup H'$,

$An\text{-}1 \cap Mn\text{-}2 \subset H$, $A'n\text{-}1 \cap Mn\text{-}2 \subset H'$, and $H \cap H' = \phi$ .

So $M_{n-1}$ separates $A_{n-1} \cap M_{n-2}$ and $A'_{n-1} \cap M_{n-2}$ in $M_{n-2}$. We show that no component of $M_{n-1}$ meets both $A_n$ and $A'_n$. We first note that $(\overline{V}-V) \cap T = \phi$. For suppose $x \in T = U \cup U'$ (see above). If $x \in U' \subset V$ then $x \notin \overline{V}-V$. If $x \in U$ then we already have $\overline{V} \cap U = \phi$.

Since $M \subset T$, $M_{n-1}$ is a union of the disjoint clopen sets $M-V$ and $\overline{V}-V$. It suffices to show that no component of either of these sets meets both $A_n$ and $A'_n$. Suppose a component of $M-V$ meets both $A_n$ and $A'_n$. Then it is contained in a component of $T$ that meets both $A_n$ and $A'_n$. But such a component is contained in $Q \cap Q' \subset U' \subset V$, a contradiction. To see that no component of $\overline{V}-V$ touches both $A_n$ and $A'_n$, we recall that $(\overline{V}-V) \cap T = \phi$ and $T$ is the set $\{x \in M_{n-2} : \mathrm{ord}_x \overline{\mathfrak{F}}_{n-2} \geq 1\}$. Since $\overline{V}-V \subset M_{n-2}$, we have $\mathrm{ord}_x \overline{\mathfrak{F}}_{n-2} \leq 0$ if $x \in \overline{V}-V$. Thus $\overline{\mathfrak{F}}_{n-2} | \overline{V}-V$ is a finite disjoint clopen cover of $\overline{V}-V$. So any component of $\overline{V}-V$ must lie in a member of $\overline{\mathfrak{F}}_{n-2} | \overline{V}-V$. Since mesh $\mathfrak{F}_{n-2} < 1$, no such member, and therefore no component of $\overline{V}-V$ can touch both $A_n$ and $A'_n$. So no component of $M_{n-1}$ touches both $A_n$ and $A'_n$.

Let $J$ be the union of components of $M_{n-1}$ that touch $A_n$ and $J'$ the union of components of $M_{n-1}$ that touch $A_n'$. As in the case for $P$, $P'$, $Q$, $Q'$, $J$ and $J'$ are closed sets of $M_{n-1}$. There is no connected set of $M_{n-1}$ touching both $J$ and $J'$ since this would yield a component of $M_{n-1}$ touching both $A_n$ and $A_n'$. It follows that $J \cap J' = \phi$. By lemma 2.2. $J$, $J'$ are separated in $M_{n-1}$ by $\phi$. Since $A_n \cap M_{n-1} \subset J$, $A'_n \cap M_{n-1} \subset J'$,

$A_n \cap M_{n-1}$, $A'_n \cap M_{n-1}$ are separated in $M_{n-1}$ by $\phi$. The sets $M_k$, $0 \leq k \leq n$ then satisfy conditions (1), (2), (3) at the beginning of the proof. Now from lemma 2.9, $\exists$ for each $1 \leq k \leq n$, a closed set $N_k$ of $I^n$ s.t. $N_k$ separates $A_k$ and $A'_k$ in $I^n$ and $N_k \cap M_{k-1} \subset M_k$.

Suppose for $1 \leq j \leq n-1$ that $\bigcap_{k=1}^{j} N_k \subset M_j$.

Then $\bigcap_{k=1}^{j+1} N_k = (\bigcap_{k=1}^{j} N_k) \cap N_{j+1} \subset M_j \cap N_{j+1} \subset M_{j+1}$.

Since $N_1 = N_1 \cap I^n = N_1 \cap M_o \subset M_1$, we have $\bigcap_{k=1}^{n} N_k \subset M_n = \phi$.

Thus we have found closed sets $N_k$, $1 \leq k \leq n$ s.t. $N_k$ separates $A_k$ and $A'_k$ and $\bigcap_{k=1}^{n} N_k = \phi$. This, however, is impossible; the boundary of $I^n$ in $R^n$ is isomorphic to $S^{n-1}$ so we refer to it as $S^{n-1}$. Let $f: S^{n-1} \longrightarrow S^{n-1}$ be the function given by $f(x) = (1-x_1, 1-x_2, 1-x_3, \ldots\ldots, 1-x_n)$. Then $f$ is continuous and $f^{-1}(A_k) = A'_k$, $f^{-1}(A'_k) = A_k$, $1 \leq k \leq n$. From the condition satisfied by $N_k$, above, $f^{-1}(A_k)$, $f^{-1}(A'_k)$ $1 \leq k \leq n$ is not an essential family (see def 2.1). So from lemma 3.4 (after adjusting to using $I = [0, 1]$ instead of $J = [-1, 1]$), $f$ has an extension $f^*: I^n \longrightarrow S^{n-1}$. But then $f^*$ is a continuous function of $I^n$ into $I^n$ not having a fixed point contrary to Brouwer's theorem. This contradiction shows that $\mu$-dim $(Y_n, \sigma_n) \geq n-1$.

<u>Assertion 6.</u> dim $Y_n \leq n-1$.

The set $\bigcup_{i=1}^{\infty} B_i$ is dense in $I^n$. For, let $U$ be an open set of $I^n$, $U \neq \phi$. $\exists$ an open set $V$ of $I^n$ s.t. $\phi \neq V \subset \overline{V} \subset U$. Now dim $\overline{V} = n > n-2$. So $\exists$ $n-1$ pairs $(C_j, C'_j)$ of disjoint closed sets of $\overline{V}$ s.t. if $T_j$,

$1 \leq i \leq n-1$, are closed sets of $\overline{V}$ s.t. $T_i$ separates $C_i$ and $C'_i$ in $\overline{V}$ for $1 \leq i \leq n-1$, then $\bigcap_{i=1}^{n-1} T_i \neq \phi$. $C_i$ and $C'_i$ are also disjoint closed sets of $I^n$ for each $i$ so by the choice of $B_{ij}$, $\exists$ $i \in N$ s.t. $B_{ij}$ separates $C_i$ and $C'_i$ in $I^n$ for $1 \leq i \leq n-1$. Then $B_{ij} \cap \overline{V}$ is a closed set of $\overline{V}$ separating $C_i$ and $C'_i$ in $\overline{V}$, $1 \leq i \leq n-1$ Thus $\phi \neq \bigcap_{i=1}^{n-1} (B_{ij} \cap \overline{V}) = (\bigcap_{i=1}^{n-1} B_{ij}) \cap \overline{V} = B_i \cap \overline{V} \subset$

$B_i \cap U$. So for some $i \in N$ $B_i \cap U \neq \phi$. The assertion now follows from theorem 0.15.

We therefore have $\mu$-dim $(Y_n, \sigma_n) = n-1$

We now construct $(X_n, \rho_n)$.

Let $(Z, \psi)$ be a totally bounded and therefore bounded metric space as in example 2.3 with $\dim Z = n$ and $\mu$-dim $(Z, \psi) = d_2(Z, \psi) = [\frac{n+1}{2}] \leq n-2$ (remember $n \geq 4$). We may assume the diameter of $Y_n$ and $Z$ is 1.

Let $X_n$ be the disjoint union of $Y_n$ and $Z$ and define the metric $\rho_n$ on $X_n$ as follows:-

$\rho_n(x,y) = \sigma_n(x, y)$ if $x, y \in Y_n$,

$\rho_n(x,y) = \psi(x, y)$ if $x, y \in Z$

$\rho_n(x,y) = 1$ if $x \in Y_n$, $y \in Z$ or $x \in Z$, $y \in Y_n$.

Clearly, $\rho_n$ is a metric, $d_2(X_n, \rho_n) \leq n-2$, $\mu$-dim $(X_n, \rho_n) = n-1$ and $\dim X_n = n$. Furthermore, $(Y_n, \sigma_n)$ is clearly totally bounded and so, therefore, is $(X_n, \rho_n)$. From theorem 1.6, $d_3(X_n,\rho_n) = \mu$-dim $(X_n,\rho_n)$. Thus $(X_n,\rho_n)$ is as required.

## SECTION THREE

A natural question to ask about metric-dependent dimension functions is whether they actually depend on the metric as opposed to, say, the topology arising from the metric. That is, is the terminology 'metric-dependent' justified? We show below that it is. Infact for any integer n, $n \geq 3$, we shall exhibit a set Xn and equivalent metrics $\rho ni$, $[\frac{n}{2}] \leq i \leq n-1$, on Xn such that $d(Xn, \rho ni) = i$, where d is any of the metric-dependent dimension functions discussed above, and dim Xn = n-1.

Lemma 3.1 (Nagami and Roberts, 1967).
If X is any metrizable topological space with dim X = n, $\exists$ a metric $\rho$ on X giving the topology of X s.t. $(X,\rho)$ is bounded and $d(X,\rho) = n$ where d is any of the above metric-dependent dimension functions.

Proof: In view of proposition 1.1 and remark 1.1, we only need to find a metric $\rho$ s.t. $d_2(X,\rho) = n$. Since X is metrizable, $\exists$ a metric $\rho'$ on X giving the topology of X and s.t. $(X,\rho')$ is bounded. If n=0, we would necessarily have $d_2(X,\rho') = 0$ and we would be through. Assume n>0. Since dim X > n-1, $\exists$ n pairs (Ci, C'i), $1 \leq i \leq n$, of closed sets of X satisfying:-

(i) $Ci \cap C'i = \phi$ $1 \leq i \leq n$.

(ii) If Bi, $1 \leq i \leq n$ are n closed sets of X s.t. Bi separates Ci and C'i, then $\bigcap_{i=1}^{n} Bi \neq \phi$.

(This is because of theorem 0.4.) $\exists$, by Urysohn's lemma, continuous functions $f_i : X \longrightarrow I$ $1 \leq i \leq n$ s.t. $f_i(C_i) = \{0\}$, $f_i(C'_i) = \{1\}$ for $1 \leq i \leq n$. Define a metric $\rho$ on X by

$\rho(x, y) = \rho'(x,y) + \sum_{i=1}^{n} | f_i(x) - f_i(y) |$. It is clear that $\rho$ is an equivalent metric to $\rho'$ and so gives the topology of X. It is also clear that $\rho$ is bounded, since $\rho'$ is. The fact that $\rho(C_i, C'_i) \geq 1$ $\forall i$, $1 \leq i \leq n$ and the pairs $(C_i, C'_i)$, $1 \leq i \leq n$, satisfy condition (ii) implies that $d_2 (X, \rho) > n-1$. But $d_2 (X, \rho) \leq \dim X$ so $d_2(X, \rho) = n$ as desired.

<u>Example 3.1</u> (Nagami and Roberts, 1967)

Let $n \geq 3$. For $[\frac{n}{2}]+1 \leq j \leq n$, let $(Y_j, \sigma_j)$ be a bounded metric space with $d_2 (Y_j, \sigma_j) = \mu\text{-dim} (Y_j, \sigma_j) = [\frac{j}{2}]$ and $\dim Y_j = j-1$ as in example 2.3. From lemma 3.1, $\exists$, for each $j$, $[\frac{n}{2}]+1 \leq j \leq n$ a bounded metric $\sigma'_j$ on $Y_j$ which is equivalent to $\sigma_j$ and s.t. $d_2 (Y_j, \sigma'_j) = \dim Y_j = j-1$. Let $X_n$ be the disjoint union of the spaces $Y_j$, $[\frac{n}{2}]+1 \leq j \leq n$. We may assume the diameters of the spaces $(Y_j, \sigma_j)$, $(Y_j, \sigma'_j)$ are all less than 1. Define for each $i$, $[\frac{n}{2}] \leq i \leq n-1$ a metric $\rho_{ni}$ on $X_n$ as follows:-

$$\rho_{ni}(x,y) = \begin{cases} \sigma_j(x,y) & \text{if } x, y \in Y_j \text{ and } j \neq i+1 \\ \sigma'_j(x, y) & \text{if } x,y \in Y_j \text{ and } j = i+1 \\ 1 & \text{if } x \in Y_{j_1},\ y \in Y_{j_2} \text{ and } j_1 \neq j_2. \end{cases}$$

Clearly, $\rho_{ni}$ is a metric on $X_n$.

$\rho_{ni}|Y_j = \sigma_j$ if $j \neq i+1$, $\rho_{ni}|Y_{i+1} = \sigma'_{i+1}$.

It is clear that $d_2(Xn, \ell ni) = \mu\text{-dim}\,(Xn, \ell ni)$ $= i$ and $\dim Xn = n-1$.

Since $\sigma i$ and $\sigma' i$ are equivalent $[\frac{n}{2}]+1 \leq i \leq n$, it is clear that $\ell ni$, $[\frac{n}{2}] \leq i \leq n-1$ are all equivalent.

We now turn to the following question :-

If d is a metric-dependent dimension function and $d(X, \ell) = m < n = \dim X$, then do there exist metrics $\ell i$ for each i, $m \leq i \leq n$ s.t. $\ell i$ is equivalent to $\ell$ and $d(X, \ell i) = i$?

We answer this question in the affirmative for the metric-dependent dimension functions $\mu$-dim, $d_2$, $d_3$ and $d_5$.

Lemma 3.2 (Roberts and Slaughter)

Let X be a paracompact Hansdorff space and $\mathcal{U}$ an open cover of X s.t. ord $\mathcal{U} \leq n \geq 0$. Then $\mathcal{U}$ has an open l.f. refinement $\mathcal{V} = \bigcup_{i=0}^{n} \mathcal{V}i$ where each $\mathcal{V}i$ is a disjoint collection.

Proof: The proof is by induction on n. The result is obvious when $n = 0$. Now assume the result true for some non-negative integer n. Suppose $\mathcal{U}$ is an open cover of X s.t. ord $\mathcal{U} \leq n+1$. $\mathcal{U}$ has a l.f. open refinement of order $\leq n+1$ so we may assume $\mathcal{U}$ is l.f. Let $S(\mathcal{U})$ be the collection of all subcollections of $\mathcal{U}$ with n+2 members. For each $A \in S(\mathcal{U})$ let $V_A = \bigcap_{U \in A} U$. Then $\{V_A, A \in S(\mathcal{U})\}$ is a l.f. open disjoint collection of subsets of X. It is disjoint because ord $\mathcal{U} \leq n+1$. Let $Y = X - \bigcup_{A \in S(\mathcal{U})} V_A$.

Then Y is a closed subset of X and $\mathcal{U}|Y$ is an open (in Y), l.f. (in Y), cover of Y of order $\leq$ n. $\mathcal{U}|Y$ has an open (in Y), l.f (in Y) refinement $\mathcal{V}'$ s.t. $\mathcal{V}' = \bigcup_{i=0}^{n} \mathcal{V}'_i$. Where each $\mathcal{V}'_i$ is a disjoint collection. Because Y is normal, $\mathcal{V}'$ has a l.f. (in Y) closed refinement $\mathcal{F} = \bigcup_{i=0}^{n} \mathcal{F}_i$ where each $\mathcal{F}_i$ is disjoint. Since Y is closed, $\mathcal{F}$ is also l.f. and closed in X. Since X is paracompact and normal, $\exists$, by lemma 1.2 an open l.f. collection $\mathcal{V}''$ of subsets of X s.t. $\mathcal{V}'' = \bigcup_{i=0}^{n} \mathcal{V}_i$, each $\mathcal{V}_i$ disjoint, and $V \in \mathcal{V}'' \Rightarrow V \subset U \in \mathcal{U}$ for some U. Let $\mathcal{V}_{n+1} = \{V_A, A \in S(\mathcal{U})\}$. Then $\mathcal{V} = \bigcup_{i=0}^{n+1} \mathcal{V}_i$ is the required refinement of $\mathcal{U}$.

<u>Lemma 3.3.</u> (Roberts and Slaughter)

Given $\varepsilon > 0$ and a positive integer k, $\exists$ k finite open covers $\xi_1, \xi_2, \ldots\ldots, \xi_k$ of the unit interval I s.t.

(i) mesh $\xi_i < \varepsilon \; \forall i, 1 \leq i \leq k$.

(ii) ord $\xi_i \leq 1 \; \forall i, 1 \leq i \leq k$.

(iii) If $\text{ord}_x \; \xi_{i_0} = 1$ then $\text{ord}_x \; \xi_i \leq 0$ for $i \neq i_0$.

Proof: $\exists$ a set of k distinct prime numbers $q_1, q_2 \ldots q_k$ s.t. $q_i \geq 3$ and $1/q_i < \varepsilon/3$ $\forall i, 1 \leq i \leq k$. Let $\delta = \min \left\{ \left|\frac{r}{q_i} - \frac{s}{q_j}\right|, r = 1, 2, \ldots\ldots, q_i - 1, s = 1, 2, \ldots\ldots, q_j - 1 \text{ and } 1 \leq i, j \leq k, i \neq j \; r = s \right\}$. (We note that $\left|\frac{r}{q_i} - \frac{s}{q_j}\right| > 0$).

Let $\xi_i = [0, \frac{1}{q_i} + \frac{1}{2}\delta), (\frac{1}{q_i} - \frac{1}{2}\delta, \frac{2}{q_i} + \frac{1}{2}\delta), (\frac{2}{q_i} - \frac{1}{2}\delta, \frac{3}{q_i} + \frac{1}{2}\delta), \ldots\ldots (\frac{q_i - 1}{q_i} - \frac{1}{2}\delta, 1]$.

Then the covers $\xi_1, \xi_2, \ldots\ldots, \xi_k$ are as required.

Theorem 3.1 (Roberts and Slaughter)

If $(X,\rho)$ is a metric space with $\mu$-dim $(X,\rho) \leq r$ and $f: X \longrightarrow I$ is a continuous function, then $\sigma: X x X \longrightarrow R$ defined by $\sigma(x,y) = \rho(x,y) + |f(x)-f(y)|$ is an equivalent metric to $\rho$ and $\mu$-dim $(X,\sigma) \leq r+1$.

Proof: The facts that $\rho(x,y) \leq \sigma(x,y)\ \forall\ x,y \in X$ and $\sigma: X x X \longrightarrow R$ is continuous w.r.t. $\rho$ imply that $\rho$ and $\sigma$ are equivalent.

Let $\epsilon > 0$ be given. Since $\mu$-dim $(X,\rho) \leq r\ \exists$ an open cover $\mathcal{U}$ of X with $\rho$-mesh $\mathcal{U} < \frac{1}{2}\epsilon$ and ord $\mathcal{U} \leq r$.
By lemma 3.2, $\mathcal{U}$ has an open refinement $\mathcal{U}' = \bigcup_{i=0}^{r} \mathcal{U}i$ where each $\mathcal{U}i$ $0 \leq i \leq r$ is disjoint. By lemma 3.3. $\exists$ r+1 open covers $\xi 0, \xi_1, \ldots\ldots, \xi_r$ of I s.t. (i) mesh $\xi i < \frac{1}{2}\epsilon$, (ii) ord $\xi i \leq 1$ and (iii) if $\text{ord}_x\ \xi i_o = 1$ then $\text{ord}_x\ \xi i \leq 0$ if $i \neq i_o$

For each $0 \leq i \leq r$, let $\mathcal{V}i = \{U \cap f^{-1}(G),\ U \in \mathcal{U}i,\ G \in \xi i\}$.

Then $\mathcal{V}_i$ is is an open collection and $\bigcup_{V \in \mathcal{V}i} V = \bigcup_{U \in \mathcal{U}i} U$ so $\mathcal{V} = \bigcup_{i=0}^{r} \mathcal{V}i$ is a cover of X.

Claim 1: ord $\mathcal{V} \leq r+1$.

Let $x \in X$. Suppose that x is contained in r+3 distinct members of $\mathcal{V}$, say $V_o, V_1, \ldots\ldots, V_{r+2}$. Suppose three of these, say $V_o, V_1, V_2$ are members of $\mathcal{V}i_o$ for some $i_o$. Since $\mathcal{U}i_o$ is discrete, and $\bigcap_{j=0}^{2} V_j \neq \phi$, we must have $V_j = U \cap f^{-1}(G_j)$ $0 \leq j \leq 2$ for some $U \in \mathcal{U}i_o$ and $G_j \in \xi i_o$, $G_j$ distinct. This implies $x \in \bigcap_{j=0}^{2} f^{-1}(G_j)$ which implies $f(x) \in \bigcap_{j=0}^{2} G_j$ contradicting the fact that ord $\xi i_0 \leq 1$.

So we cannot have three members of $\{V_0, \ldots . V_{r+2}\}$ being in the same $\mathcal{V}i_0$. It follows that we must have two members, say $V_0$, $V_1$ in $\mathcal{V}i_0$ and two other members, say $V_2$, $V_3$ in $\mathcal{V}i_1$, $i_0 \neq i_1$. ($V_0$, $V_1$, $V_2$, $V_3$ are all distinct). From an argument analogous to the one above, we see that $f(x) \varepsilon G_0 \cap G_1$ $G_0$, $G_1$ being distinct members of $\xi i_0$ and $f(x) \varepsilon G_2 \cap G_3$, $G_2$, $G_3$ being distinct members of $\xi i_1$. Thus $\text{ord}_{f(x)} \xi i_0 = \text{ord}_{f(x)} \xi i_1 = 1$ contradicting condition (iii) above for the $\xi i$. So x cannot be contained in r+3 distinct members of $\mathcal{V}$. Since x is arbitrary, $\text{ord}\,\mathcal{V} \leq r+1$ as required.

Claim 2. $\sigma$-mesh $\mathcal{V} \leq \varepsilon$

For, if $V \varepsilon \mathcal{V}$, then $V = U \cap f^{-1}(G)$ for some $U \varepsilon \mathcal{U}'$ and $G \varepsilon \xi i_0$ for some $i_0$. Since $\mathcal{U}'$ refines $\mathcal{U}$ and mesh $\mathcal{U} < \frac{1}{2}\varepsilon$, diameter $U < \frac{1}{2}\varepsilon$. Also, since mesh $\xi i_0 < \frac{1}{2}\varepsilon$, $x, y \varepsilon f^{-1}(G) \Rightarrow |f(x) - f(y)| < \frac{1}{2}\varepsilon$. Thus $x, y \varepsilon V \Rightarrow \sigma(x, y) = \ell(x, y) + |f(x) - f(y)| < \varepsilon$. So $\sigma(V) \leq \varepsilon$ and $\sigma$ mesh $\mathcal{V} \leq \varepsilon$ as required.

So for $\varepsilon > 0$, $\exists$ an open cover $\mathcal{V}$ of X s.t. ord $\mathcal{V} \leq r + 1$ and $\sigma$-mesh $\mathcal{V} < \varepsilon$ which shows that $\mu$-dim $(X, \sigma) \leq r+1$.

<u>Theorem 3.2</u> (Roberts and Slaughter)

Let $(X, \ell)$ be a metric space with $\mu$-dim $(X, \ell) = r < n = \dim X$. Then for each i, $r \leq i \leq n$ $\exists$ a metric $\ell i$ on X s.t. $\ell i$ is equivalent to $\ell$ and $\mu$-dim $(X, \ell i) = i$.

Proof: Since dim X > n-1, $\exists$ n pairs (Ci, C'i) $1 \leq i \leq n$ of disjoint closed sets of X s.t. if $\{Bi, 1 \leq i \leq n\}$ is any collection of closed sets of X s.t. Bi separates Ci and C'i then $\bigcap_{i=1}^{n} Bi \neq \phi$.

By Urysohn's lemma, $\exists$ continuous functions fi: X $\longrightarrow$ I, $1 \leq i \leq n$, s.t. fi(Ci) = 0 and fi($C'_i$) = 1.

Let $\ell_j(x, y) = \ell(x, y) + \sum_{i=1}^{j} |fi(x) - fi(y)|$, $1 \leq j \leq n$, $\ell_0 = \ell$. As in the proof of lemma 3.1, $\ell_j$ is equivalent to $\ell$ for $1 \leq j \leq n$ and μ-dim $(X, \ell_n) = n$.

From theorem 3.1, μ-dim $(X, \ell_{j+1}) \leq$ μ-dim $(X, \ell_j)$ +1 for $0 \leq j \leq n-1$. It follows from these facts that for each i, $r \leq i \leq n$, $\exists j$, $1 \leq j \leq n$ s.t. μ-dim $(X, \ell_j) = i$. This proves the theorem.

<u>Lemma 3.4</u>

Let $(X, \ell)$ be a metric space, (C, C') be disjoint closed sets of X and Wi, i = 1, 2, ....be subsets of X s.t.

(i) $\ell(Wi+1, X-Wi) > 0$

(ii) $\ell(C-Wi, C'-Wi) > 0$ $\forall$i.

Then $\exists$ a continuous function f: X $\longrightarrow$ I s.t. f(C) = {0}= , f(C') = {1} and f|X-Wi is uniformly continuous w.r.t. $\ell$ for all i.

Proof: Define f by $f(x) = \dfrac{\ell(x, C)}{\ell(x,C) + \ell(x,C')}$

Then f(x) $\varepsilon$ I, f is continuous and f(C) = {0} , f(C') = {1} .

Claim: For each i, $\exists$ $\delta i > 0$ s.t.

$\ell(x, C) + \ell(x, C') \geq \delta i \; \forall \; x \in X - Wi$.

Infact, put $\delta i = \min \{\ell(C - Wi+1, C' - Wi+1), \ell(Wi+1, X - Wi)\}$. Suppose for some $y \in C$, $y' \in C'$ we have

$\ell(x, y) + \ell(x, y') < \delta i$ where $x \in X - Wi$.

Then $\ell(y, y') < \ell(C - Wi+1, C' - W_{i+1})$ so we must have either $y \in Wi+1$ or $y' \in Wi+1$. Assume W.L.G, that $y \in Wi+1$. Then, since $x \in X - Wi$, $\ell(x, y) \geq \delta i$, a contradiction.

So for $y \in C$, $y' \in C'$, we always have $\ell(x, y) + \ell(x, y') \geq \delta i$ if $x \notin Wi$. Fixing x and y' and letting y vary over C, we have $\ell(x, C) + \ell(x, y') \geq \delta_i$ and similarly,

$\ell(x, C) + \ell(x, C') \geq \delta i$.

Let $g(x) = \ell(x, C)$ and $h(x) = \ell(x, C) + \ell(x, C')$.

Since $|\ell(a, A) - \ell(b, A)| \leq \ell(a, b)$ for a, b, $\varepsilon$ X and A a subset of X, g(x) and h(x) are uniformly continuous functions. We have seen above that $h(x) \geq \delta i$ for $x \in X - Wi$. For $x, y \in X - Wi$, $|f(x) - f(y)| =$

$$\frac{|g(x)\,h(y) - h(x)\,g(y)|}{h(x)\,h(y)} = \frac{|(g(x)h(y) - g(x)h(x)) - (h(x)g(y) - h(x)g(x))|}{h(x)\,h(y)}$$

$$\leq \frac{g(x)\,|h(x) - h(y)| + h(x)\,|g(x) - g(y)|}{h(x)\,h(y)}$$

$$= \frac{1}{h(y)}\left[\frac{g(x)}{h(x)}\,|h(x) - h(y)| + |g(x) - g(y)|\right]$$

Since $\frac{1}{h(y)} \leq \frac{1}{\delta i} \; \forall \; y \in X - Wi$, $\frac{g(x)}{h(x)} \leq 1$, and h and g are uniformly continuous functions, it is now clear that f is uniformly continuous on X-Wi.

Notation: For a set X and a collection $\mathcal{U}$ of subsets of X, if $x \in X$, we denote by St $(x, \mathcal{U})$ the set $\bigcup_{\substack{U \in \mathcal{U} \\ x \in U}} U$.

If A is a subset of X, we denote by $St(A,\mathcal{U})$ the set $\bigcup_{\substack{U \in \mathcal{U} \\ U \cap A \neq \phi}} U$. If $(X, \ell)$ is a metric space we denote by $U(\ell,\varepsilon)$ the collection of all open balls of radius $\varepsilon$ w.r.t. $\ell$.

<u>Def 3.1.</u> A cover $\mathcal{V}$ of X is said to be a star refinement of a cover $\mathcal{U}$ of X and we write $\mathcal{V} \overset{*}{<} \mathcal{U}$ if the collection $\{St(V, \mathcal{V}), V \in \mathcal{V}\}$ refines $\mathcal{U}$.

<u>Lemma 3.5</u>

If $\mathcal{V}_k$, k=1, 2, ....is a sequence of open Lebesque covers of a metric space $(X,\ell)$ s.t.

(i) $\mathcal{V}_{k+1} \overset{*}{<} \mathcal{V}_k \ \forall K \in \mathbb{N}$

(ii) The collection $\{St(x, \mathcal{V}_k)\ k=1, 2, ....\}$ is a neighbourhood base at x $\forall x \in X$ then $\exists$ a metric $\sigma$ equivalent to $\ell$ s.t. $\mathcal{V}_{k+1} < U(\sigma, 2^{-k}) < \mathcal{V}_k \ \forall k$.

For a proof of this lemma, see Isbell, theorem 4.

<u>Lemma 3.6</u> (Goto)

Let $(C_i, C'_i)$ $1 \leq i \leq r$ be r pairs of disjoint closed sets of a metric space $(X,\ell)$. Then $\exists$ a metric $\sigma$ on X and r continuous functions $f_i: X \longrightarrow I$ $1 \leq i \leq r$ s.t.

(i) $\sigma$ is equivalent to $\ell$ and $\ell > \sigma$ i.e. given $\delta > 0$, $\exists\ \varepsilon > 0$ s.t. $\ell(x,y) < \varepsilon \Rightarrow \sigma(x,y) < \delta \ \forall x,y \in X$.

(ii) $f_i(C_i) = \{0\}$ , $f_i(C'_i) = \{1\}$

(iii) For any $\varepsilon > 0$, $\exists$ an open set $U$ of $X$ s.t. $\sigma(U) < \varepsilon$ and $f_i|X-U$ is uniformly continuous w.r.t. $\sigma$ for each $i$.

Proof: Let $(X,\ell)$ and $(C_i, C'_i)$ $1 \leq i \leq r$ be as in the lemma. Let $\mathcal{U}_k = U(\ell, 2^{-k})$, $k = 2,3,4,\ldots$.

Then (a1) $\mathcal{U}_k$ is a uniform open cover of $(X,\ell)$.

(a2) $\mathcal{U}_{k+1} \overset{*}{<} \mathcal{U}_k$

(a3) mesh $\mathcal{U}_k < 1/k$ .

For $k = 2, 3, \ldots$ and $1 \leq i \leq r$, let

$A_{ki} = \{x \in X: \ell(x, C_i) \leq 1/k, \ell(x, C'_i) \leq 1/k\}$.

Let $A_k = \bigcup_{i=1}^{r} A_{ki}$.

Then (b1) $A_{k+1} \subset A_k$

(b2) For each $x \in X$, $\exists k_0$ s.t. $\ell(x, \bigcup_{k=k_0}^{\infty} A_k) \geq 1/k_0$.

(b1) is obvious. To see (b2), let $x \in X$. $\exists\, \delta > 0$ s.t. for each $i$, either $\ell(x, C_i) > \delta$ or $\ell(x, C'_i) > \delta$ .

Choose $k_0$ s.t. $1/k_0 < \frac{1}{2}\delta$ . Suppose $k \geq k_0$ and $y \in A_k$.

If $\ell(x, y) \leq 1/k_0$ then, since $\ell(y, C_i) \leq 1/k \leq 1/k_0$ and $\ell(y, C'_i) \leq 1/k \leq 1/k_0$ for some $i$, we have $\ell(x, C_i) \leq 2/k_0 < \delta$ and $\ell(x, C'_i) < \delta$ for some $i$ contradicting the choice of $\delta$ .

So $y \in A_k$, $k \geq k_0 \Rightarrow \ell(x, y) > 1/k_0$ which proves (b2).

If for some $k$ $A_k = \Phi$ then $\ell(C_i, C'_i) > 0$ for $1 \leq i \leq r$ and the metric $\ell$ and functions

$$\frac{\ell(x, C_i)}{\ell(x, C_i) + \ell(x, C'_i)}$$ would satisfy conditions (i) to (iii) of lemma 3.6. (iii) would be satisfied because

$\ell(x, Ci) + \ell(x, Ci') \geq \delta > 0 \ \forall\ x$ for some $\delta$ if $\ell(Ci, Ci') > 0$ and we would then proceed as in lemma 3.4.

So we assume $Ak \neq \Phi$ for all K.

Let $Gk = St(Ak, \mathcal{U}k)$ and let $\mathcal{V}k = \{Gk\} \cup \{U \in \mathcal{U}k : U \cap Ak = \phi\}$. Then $\mathcal{V}k$ is a Lebesgue open cover of X (since $\mathcal{U}k < \mathcal{V}k$ and $\mathcal{U}k$ is Lebesgue) and the sequence $\mathcal{V}k$, k = 2, 3,.....satisfies:-

(c1) $\mathcal{V}k+1 \overset{*}{<} \mathcal{V}k$

(c2) The collection $\{St(x, \mathcal{V}k)\ \ k = 1, 2, \ldots\}$ is a nbhd base at x for each $x \in X$.

To see (c1), suppose $V \in \mathcal{V}k+1$. We want to show $St(V, \mathcal{V}k+1) \subset V' \in \mathcal{V}k$ for some V'. Suppose V = Gk+1. Clearly $St(Gk+1, \mathcal{V}k+1) = St(Gk+1, \mathcal{U}k+1)$. Suppose $U \in \mathcal{U}k+1$ and $U \cap Gk+1 \neq \phi$. Then $St(U, \mathcal{U}k+1) \cap Ak+1 \neq \phi$. But $St(U, \mathcal{U}k+1) \subset U' \in \mathcal{U}k$ for some U'. $U' \cap Ak+1 \neq \phi$, so $U' \cap Ak \neq \phi$, so $U' \subset St(Ak, \mathcal{U}k) = Gk$. So $U \subset Gk$ (infact $St(U, \mathcal{U}k+1) \subset Gk$). It follows that $St(Gk+1, \mathcal{U}k+1) \subset Gk$ so $St(Gk+1, \mathcal{V}k+1) \subset Gk \in \mathcal{V}k$. Now suppose $V \neq Gk+1$. Then $V = U_0 \in \mathcal{U}k+1$. If $Gk+1 \cap U_0 = \phi$ then $St(U_0, \mathcal{V}k+1) = St(U_0, \mathcal{U}k+1) \subset U' \subset V'$ for some $U' \in \mathcal{U}k$ and $V' \in \mathcal{V}k$. If $Gk+1 \cap U_0 \neq \phi$, then, as above, $St(U_0, \mathcal{U}k+1) \subset Gk$. Now, clearly $St(U_0, \mathcal{V}_{k+1}) = St(U_0, \mathcal{U}_{k+1}) \cup G_{k+1}$. Since $A_{k+1} \subset A_k$ and $\mathcal{U}_{k+1} < \mathcal{U}_k$. $G_{k+1} \subset G_k$ so $St(U_0, \mathcal{V}_{k+1}) \subset G_k \subset \mathcal{V}_k$.

To show (c2), we only need show that for any $x \in X$ and $\varepsilon > 0, \exists\ k_0$ s.t. $\ell(St(x, \mathcal{V}k_0)) < \varepsilon$. Let $x \in X$. (b2) implies $\exists\ k_0$ s.t. $1/k_0 < \frac{1}{2}\varepsilon$ and $\ell(x, Ak_0) \geq 1/k_0$.

Since mesh $\upsilon k_0 < 1/k_0$, $x \notin Gk_0$. So if $x \varepsilon V \varepsilon \nabla k_0$ then $V \varepsilon \upsilon k_0$ and $\ell(V) < 1/k_0$. So $\ell(St(x, \nabla k_0)) \leq 2/k_0 < \varepsilon$.

Thus $\nabla k$, $k = 2, 3, \ldots$ satisfy the conditions of lemma 3.5 and $\exists$ a metric $\sigma$ on X s.t:- (d1) $\sigma$ is compatible with $\ell$ and $\nabla k+1 < U(\sigma, 2^{-k}) < \nabla k$ $\forall k$, $k = 2, 3, \ldots$.

Since $U(\ell, 2^{-k-1}) = \upsilon k+1 < \nabla k+1 < U(\sigma, 2^{-k})$, we have $\ell(x, y) < 2^{-k-1} \Rightarrow \sigma(x,y) < 2^{-k}$ and condition (i) of lemma 3.6 follows.

Claim: For each k, $k = 2, 3, 4, \ldots$ and $1 \leq i \leq r$ $St(Ci-Gk, \nabla k) \cap (C'_i - Gk) = \Phi$

For suppose $x \varepsilon St(Ci - Gk, \nabla k) \cap (C'i - Gk)$. Then $x \varepsilon V \varepsilon \nabla k$ for some V s.t. $V \cap (Ci-Gk) \neq \Phi$, $V \cap (Ci' - Gk) \neq \Phi$. Obviously $V \notin Gk$ so $V \varepsilon \upsilon k$ so $\ell(V) < 1/k$. But this implies $x \varepsilon Ak \subset Gk$ contrary to the fact that $x \varepsilon C'_i - Gk$.

Since $U(\sigma, 2^{-k}) < \nabla k$, $St(Ci-Gk, U(\sigma, 2^{-k})) \cap (C'_i-Gk) = \Phi$. Hence $\sigma(Ci-Gk, C'i-Gk) \geq 2^{-k}$.

Now let $Wk = \{x \varepsilon X: \sigma(x, Gk) < 2^{-k+1}\}$ $k = 2, 3, \ldots$. Since $Gk+1 \subset Gk$ it follows that $\sigma(Wk+1, X-Wk) \geq 2^{-k}$. Since $Gk \subset Wk$ we have $\sigma(Ci-Wk, C'i-Wk) \geq 2^{-k}$ $\forall$ i, $1 \leq i \leq r$. Now from lemma 3.4, $\exists$ for each i, a continuous function $fi: X \longrightarrow I$ s.t. $fi(Ci) = \{0\}$, $fi(C'i) = \{1\}$

and fi is uniformly continuous on X-Wk w.r.t. $\sigma$ for each k. To complete the proof, it is only necessary to show that $\lim_k \sigma(Wk) = 0$. Since $Gk \varepsilon \nabla k$ and $\nabla k < U(\sigma, 2^{-k+1})$ we have $\sigma(Gk) \leq 2^{-k+2}$. It then follows that $\sigma(Wk) \leq 2^{-k+3}$ and the proof is complete.

Lemma 3.7 (Goto)

Let $(X, \ell)$ be a metric space. Let $\tau$ be a metric on $I^r$, for some positive integer r, giving the usual topology of $I^r$. Let $f: X \to I^r$ be a continuous function s.t. for any $\varepsilon > 0$, $\exists$ an open set U of X s.t. $\ell(U) < \varepsilon$ and f is uniformly continuous on X-U w.r.t. $\ell, \tau$

If $\sigma: X \times X \to R$ is the function described by $\sigma(x, y) = \ell(x,y) + \tau(f(x), f(y))$, then $\sigma$ is an equivalent metric to $\ell$ and $d_2(X, \sigma) \leq \max\{d_2(X, \ell), r\}$.

Proof: We may assume $d_2(X, \ell) < \infty$ otherwise there is nothing to prove. It is clear that $\sigma$ is a metric equivalent to $\ell$. Let $m = \max\{d_2(X, \ell), r\}$.

Let $(Ci, C'i)$ $1 \leq i \leq m+1$ be m+1 pairs of closed sets of X s.t. $\sigma(Ci, C'i) > 0$ $1 \leq i \leq m+1$. Let $\delta = \min\{\sigma(Ci, C'i), 1 \leq i \leq m+1\}$. By hypothesis $\exists$ an open set U s.t. $\ell(U) < \frac{1}{4}\delta$ and f is uniformly continuous on X-U w.r.t. $\ell$, $\tau$. Let $V = \{x \in X: \ell(x, \overline{U}) < \frac{1}{8}\delta\}$.

Then $\ell(V) < \frac{1}{2}\delta$.

Claim: $\tau(f(Ci \cap V), f(C'i \cap V)) \geq \frac{1}{2}\delta$. $\forall i$, $1 \leq i \leq m+1$,

For, if $x \in Ci \cap V$, $y \in C'i \cap V$ and $\tau(f(x), f(y)) < \frac{1}{2}\delta$, we would have $\sigma(x,y) = \ell(x,y) + \tau(f(x), f(y)) < \frac{1}{2}\delta + \frac{1}{2}\delta = \delta$ contrary to the choice of $\delta$.

Let $Bi = \overline{f(Ci \cap V)}$, $B'_i = \overline{f(Ci' \cap V)}$, $1 \leq i \leq m+1$.

Then $Bi \cap B'_i = \phi$ $\forall i$, $1 \leq i \leq m+1$. Since $\dim I^r = r \leq m$ and using theorem 0.4 $\exists$ closed sets $Ei$, $E'i$ $1 \leq i \leq m+1$ of $I^r$ s.t.:-

(b1) $B_i \subset E_i$ and $B'_i \subset E'_i$

(b2) $E_i \cap E'_i = \phi$

(b3) $\bigcup_{i=1}^{m+1} (E_i \cup E'_i) = I^r$.

Since $I^r$ is compact, we have:-

(b4) $\tau(E_i, E'_i) > \delta 1 > 0 \ \forall\ i$ for some $\delta 1$.

We have for each i that $\ell(f^{-1}(E_i) - U, f^{-1}(E'_i) - U) > 0$.

To see this, let $\varepsilon > 0$ be s.t. $x, y \varepsilon X-U$ and $\ell(x, y) < \varepsilon \Rightarrow \tau(f(x), f(y)) < \delta 1$. Such an $\varepsilon$ exists because f is uniformly continuous on X-U. Let $x \varepsilon f^{-1}(E_i) - U$, $y \varepsilon f^{-1}(E'_i) - U$ for any i. Then $\ell(x,y) < \varepsilon \Rightarrow \tau(f(x), f(y)) < \delta 1$, contrary to (b4) since $f(x) \varepsilon E_i$, $f(y) \varepsilon E'_i$. So $\ell(x,y) \geq \varepsilon$. Thus $\ell(f^{-1}(E_i)-U, f^{-1}(E'_i)-U) \geq \varepsilon \ \forall i \ 1 \leq i \leq m+1$.

Now let $F_i = f^{-1}(E_i) \cap \overline{U}$, $F'_i = f^{-1}(E'_i) \cap \overline{U}$.

Then:-

(d1) $\ell(F_i - U, F'_i - U) > 0 \ \forall i$.

It is also clear that:-

(d2) $C_i \cap \overline{U} \subset F_i$, $C'_i \cap \overline{U} \subset F'_i$

(d3) $F_i \cap F'_i = \phi$

(d4) $\bigcup_{i=1}^{m+1} (F_i \cup F'_i) \supset U$.

d2 follows from (b1), (d3) from (b2) and (d4) from (b3).

Claim:-

(d5) $\ell(F_i - U, C'_i - U) > 0$, $\ell(F'_i - U, C_i - U) > 0 \ \forall i$.

To see this, let $\delta 2 = \min \{\frac{1}{8}\varepsilon, \ell(f^{-1}(E_i)-U, f^{-1}(E'_i)-U)$

Suppose $x \in C'i-U$ and $\ell(x, Fi-U) < \delta_2$.

Then $\ell(x, Fi-U) < \frac{1}{8}\varepsilon$ so $\ell(x, \overline{U}) < \frac{1}{8}\varepsilon$ since $Fi \subset \overline{U}$. So $x \in V$. So $x \in C'i \cap V \subset f^{-1}(E'i)$.

Since $x \notin U$, $x \in f^{-1}(E'i) - U$ and so $\ell(x, Fi-U) \geq \ell(x, f^{-1}(Ei) - U) \geq \ell(f^{-1}(Ei)-U, f^{-1}(E'i)-U) \geq \delta_2$, a contradiction. It follows that $\ell(Fi-U, C'i-U) \geq \delta_2 > 0$. Similarly $\ell(F'i - U, Ci - U) > 0$.

We also have:-

(d6) $\ell(Ci-U, C'i-U) > 0 \ \forall i, 1 \leq i \leq m+1$.

To see this, choose $\delta_3$ s.t.

$0 < \delta_3 < \min \{\sigma(Ci, C'i), 1 \leq i \leq m+1\}$. Since f is uniformly continuous on X-U, $\exists \ \varepsilon > 0$ s.t. $\varepsilon < \frac{1}{2}\delta$ and for $x, y \in X-U$ $\ell(x,y) < \varepsilon \Rightarrow \tau(f(x), f(y)) < \frac{1}{2}\delta_3$. Then if $x \in Ci-U$, $y \in C'i-U$ for any i, $\ell(x,y) < \varepsilon \Rightarrow \tau(f(x), f(y)) < \frac{1}{2}\delta_3$.

So $\sigma(x,y) = \ell(x,y) + \tau(f(x), f(y)) < \varepsilon + \frac{1}{2}\delta_3 < \delta_3$ contrary to the choice of $\delta_3$ and the fact that $x \in Ci$, $y \in C'i$. So for $x \in Ci-U$, $y \in C'i -U$ $\ell(x,y) \geq \varepsilon \ \forall \ i$ which implies $\ell(Ci-U, C'i-U) \geq \varepsilon > 0 \ \forall \ i$.

Let $Di = Ci \cup Fi$, $D'i = C'i \cup F'i$ $1 \leq i \leq m+1$.

Since Fi, F'i are disjoint closed sets of $\overline{U}$ s.t. $Ci \cap \overline{U} \subset Fi$ and $C'i \cap \overline{U} \subset F'i$, it follows that Di, D'i are disjoint closed sets of X (since Ci, C'i are disjoint). Furthermore, it follows from (d1), (d5) and (d6) that $\ell(Di-U, D'i-U) > 0 \ \forall i, 1 \leq i \leq m+1$.

Since $d_2(X, \ell) \leq m$, $\exists$ closed sets Ki, K'i of X s.t. $Ki \cap Ki' = \phi$, $Di - U \subset Ki$, $Di' - U \subset Ki'$ and $X = \bigcup_{i=1}^{m+1} (K_i \cup K'_i)$

see Fig 3.1

Let $W_i = (K_i - U) \cup D_i$, $W'_i = (K'_i - U) \cup D'_i$.

Since $(K_i - U) \supset (D_i - U)$, $(K'_i - U) \supset (D'_i - U)$ and $K_i - U$, $K'_i - U$ are disjoint, $W_i$, $W'_i$ are disjoint.

Clearly $C_i \subset D_i \subset W_i$, $C'_i \subset D'_i \subset W'_i$ and $\bigcup_{i=1}^{m+1} (W_i \cup W'_i)$

$$= \bigcup_{i=1}^{m+1} [(K_i - U) \cup (K'_i - U)] \cup [\bigcup_{i=1}^{m+1} (D_i \cup D'_i)] \supset$$

$$\{[\bigcup_{i=1}^{m+1} (K_i \cup K'_i)] - U\} \cup [\bigcup_{i=1}^{m+1} (F_i \cup F'_i)] \supset (X - U) \cup U = X$$

(using (d4) ).

So we have found disjoint closed sets $W_i$, $W'_i$ $1 \leq i \leq m+1$ s.t. $C_i \subset W_i$, $C'_i \subset W'_i$ and $\bigcup_{i=1}^{m+1} (W_i \cup W'_i) = X$. Thus

$d_2(X, \sigma) \leq m$ as required.

We are now ready to prove a result analogous to theorem 3.2 for the dimension function $d_2$.

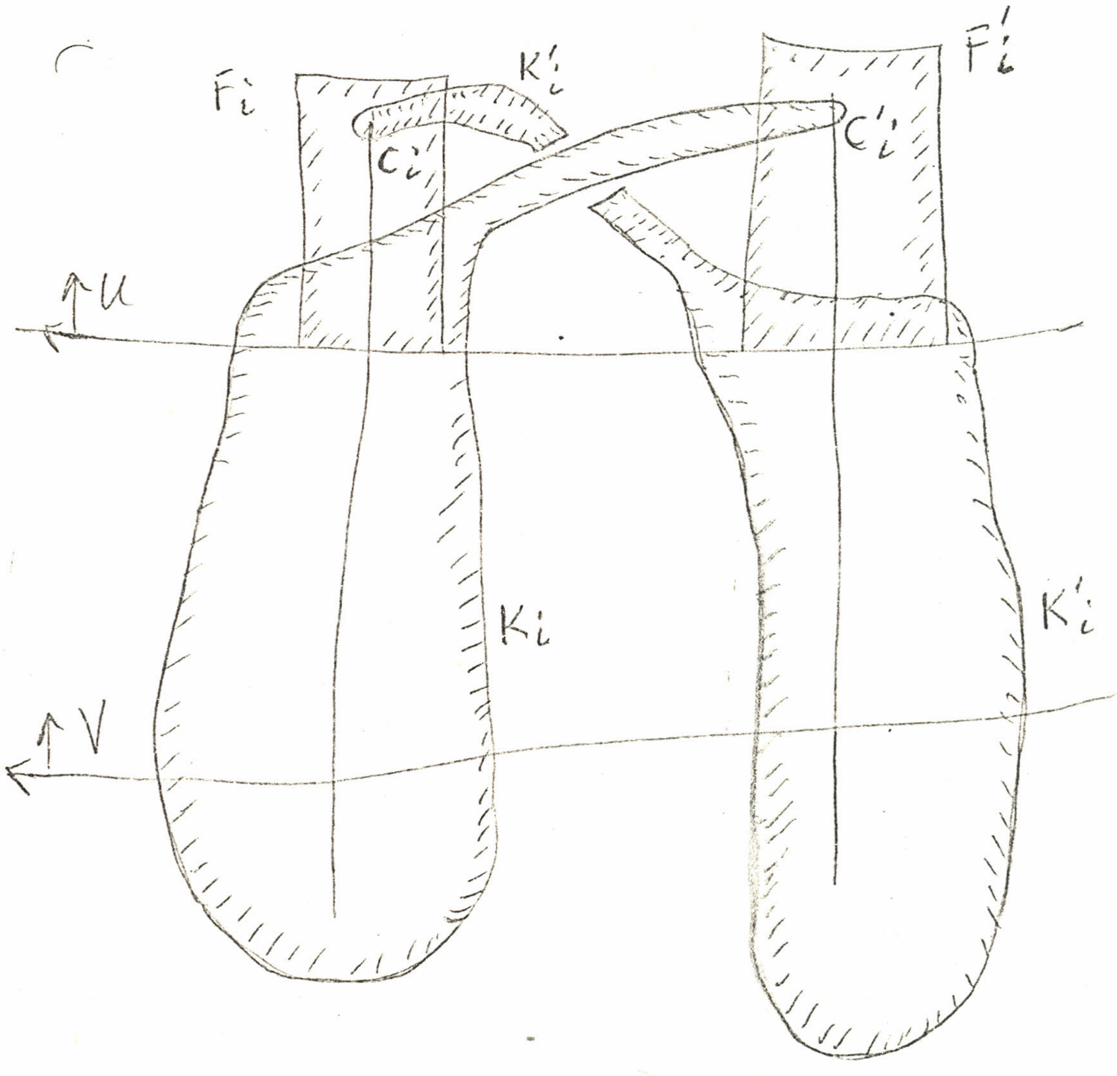
<u>Theorem 3.3.</u> (Goto)

Let $(X, \ell)$ be a metric space s.t. $d_2(X, \ell) = m < n = \dim X$. Then for each $i$, $m \leq i \leq n$, $\exists$ a metric $\ell_i$ on $X$ s.t. $\ell_i$ is equivalent to $\ell$ and $d_2(X, \ell_i) = i$.

Proof: Let $i > m$ (there is nothing to show if $i = m$). The conditions of the theorem imply $m \geq 0$ so $i \geq 1$. Since $\dim X > i - 1$, $\exists$ $i$ pairs of disjoint closed sets $(C_j, C'_j)$ $1 \leq j \leq i$ s.t. for any closed sets $Y_j$, $1 \leq j \leq i$, s.t. $Y_j$ separates $C_j$ and $C'_j$, we have $\bigcap_{j=1}^{i} Y_j \neq \phi$.

By lemma 3.6 $\exists$ a metric $\sigma$ on $X$ and continuous functions $f_j : X \to I$ $1 \leq j \leq i$ s.t.:-

Fig. 3.1



(i) $\sigma$ is equivalent to $\ell$ and $\ell > \sigma$.

(ii) $f_j(Cj) = 0$, $f_j(C'j) = 1$

(iii) Given $\varepsilon > 0$, $\exists$ an open set U of X s.t. $\sigma(U) < \varepsilon$ and $f_j \mid X-U$ is uniformly continuous w.r.t. $\sigma \forall j$, $1 \leq j \leq i$.

Let $f: X \longrightarrow I^i$ be given by $f(x) = (f_1(x), f_2(x), \ldots, f_i(x))$. Let $\tau$ be the metric on $I^i$ given by

$$\tau(x,y) = \sum_{j=1}^{i} |x_j - y_j| \text{ where } x = (x_1, x_2, \ldots x_i),$$

$y=(y_1, y_2, \ldots y_i)$. $\tau$ gives the usual topology of $I^i$. Condition (iii) above implies that f satisfies the uniformity condition of lemma 3.7; i.e. given $\varepsilon < 0$, $\exists$ an open set U of X s.t. $\sigma(U) < \varepsilon$ and $f|X-U$ is uniformly continuous w.r.t. $\sigma, \tau$. Let $\ell_i$ be given by $\ell i(x, y) = \ell(x, y) + \tau(f(x), f(y))$. Then from lemma 3.7, $d_2(X, \ell i) \leq \max\{d_2(X,\sigma), i\}$. Since $\ell > \sigma$, $d_2(X,\sigma) \leq d_2(X, \ell) < i$ so $d_2(X, \ell i) \leq i$. On the other hand, we have $\ell i(Cj, C'j) \geq 1$ $\forall j$, $1 \leq j \leq i$ and yet if $Yj$, $1 \leq j \leq i$ are closed sets separating $Cj$ and $C'j$ then $\bigcap_{j=1}^{i} Yj \neq \Phi$. This implies $d_2(X, \ell i) > i-1$, so $d_2(X, \ell i) = i$ as required.

We restate a special case of lemma 1.4.

<u>Lemma 3.8</u>

Let X be a topological space, C, C' be disjoint closed sets of X and $X = \bigcup_{i=1}^{k} Di$ where $Di$ $1 \leq i \leq k$ is open and $\overline{D}i \subset Di+1$. For each i, let $Fi$ be the closed set $\overline{D}i - Di-1$ $(D_0 = \Phi)$. Suppose $Bi$, $1 \leq i \leq k$ are closed sets of Fi s.t. Bi separates $C \cap Fi$ and $C' \cap Fi$ in Fi.

Then $\exists$ a closed set B of X s.t. B separates C and C' in X and $B \subset \bigcup_{i=1}^{k}$ (Bi $\cup$ bdry Di) This lemma is obtained from lemma 1.4 by putting Gj = X, $j \geq k$.

Theorem 3.4 (Nichols, 1969)

Let $(X, \ell)$ be a metric space and $f: X \to I$ a continuous function. Define $\sigma : X \times X \to R$ by $\sigma(x, y) = \ell(x, y) + |f(x) - f(y)|$ . Then $\sigma$ is an equivalent metric to $\ell$ and

(i) $d_5 (X, \sigma) \leq d_5 (X, \ell) + 1$

(ii) $d_3 (X, \sigma) \leq d_3 (X, \ell) + 1$.

Proof: Since the proofs of (i) and (ii) are similar, they are proved simultaneously. We have seen earlier that $\sigma$ and $\ell$ are equivalent.

Let $(X, \ell)$, f, $\sigma$ be as given with $d_5(X, \ell) \leq m$ (respectively $d_3(X, \ell) \leq m$).

Let A be a countable (resp. finite) set.

Let Cj, C'j $j \in A$ be pairs of disjoint closed sets of X s.t. $\sigma(Cj, C'j) \geq \varepsilon > 0 \ \forall \ j \in A$ for some $\varepsilon$.

Choose N s.t. $^1/N < \frac{1}{2} \varepsilon$.

Since A is countable, $\exists$ distinct members Mj, $j \in A$ of the interval $(0, {}^1/N)$.

For each j, let $M^0j = 0$, $M^1j = Mj$, $M^2j = Mj + {}^1/N$, $M^3j = Mj + {}^2/N$, ....$M^Nj = Mj + {}^{N-1}/N$, $Mj^{N+1} = 1$.

Put $k = N+1$.

Then (1) $M^ij < M^{i+1}j$ and $M^{i+1}j - M^ij \leq \frac{1}{2}\varepsilon \ \forall \ j \in A$,

$0 \leq i \leq k-1$.

(2) If $j \neq j'$ and $1 \leq i, i' \leq k-1$ then $M^i j \neq M^{i'} j'$.
Both claims are clear.
For $0 \leq i \leq k-1$, and $j \in A$, let $Dji = f^{-1}[0, M^i j)$ and let $Djk = X$. Then for fixed j, it is clear that the sets $Dji$ $0 \leq i \leq k$ satisfy the conditions of lemma 3.8. It is also clear that if $Fji$ is defined by $Fji = \overline{D}ji - Dji-1$ $1 \leq i \leq k$ (i.e. as in lemma 3.8), then:-

(3) $f(Fji) \subset [Mj^{i-1}, Mj^i]$.

Claim: (4). $\ell(Cj \cap Fji, C'j \cap Fji) \geq \frac{1}{2}\varepsilon$ $\forall$ $j \in A$, $1 \leq i \leq k$
For, if $x \in Cj \cap Fji$, $y \in C'j \cap Fji$, then, since $\sigma(Cj, C'j) > \varepsilon$. $\sigma(x, y) > \varepsilon$; but from (3) and (1), $|f(x) - f(y)| \leq \frac{1}{2}\varepsilon$ so $\ell(x,y) = \sigma(x,y) - |f(x)-f(y)| > \frac{1}{2}\varepsilon$ and the claim follows.

Thus the collection $(Cj \cap Fji, C'j \cap Fji)$ $j \in A$, $1 \leq i \leq k$ is a countable (resp. finite) collection satisfying (4). Since $d_5(X, \ell) \leq m$ (resp. $d_3(x, \ell) \leq m$), we can find closed sets $B'ji$, $j \in A$, $1 \leq i \leq k$ of X s.t. $B'ji$ separates $Cj \cap Fji$ and $C'j \cap Fji$ in X and ord $\{B'ji, j \in A, 1 \leq i \leq k\} \leq m-1$. Let $Bji = B'ji \cap Fji$. Then $Bji$ is a closed set of X (and $Fji$) separating $Cj \cap Fji$ and $C'j \cap Fji$ in $Fji$ and ord $\{Bji, j \in A.$ $1 \leq i \leq k\} \leq m-1$.

We want to construct closed sets $Bj$ of X s.t. $Bj$ separates $Cj$ and $C'j$ in X and ord $\{Bj, j \in A\} \leq$ ord $\{Bji$ $j \in A$, $1 \leq i \leq k\}$ +1.

From lemma 1, $\exists$ for each fixed j a closed set $Bj$ of

X s.t. Bj separates Cj and C'j and:-

(6) $Bj \subset \bigcup_{i=1}^{k} (Bji \cup \text{bdry } Dji)$

Claim: $\text{ord}\{Bj,\ j \in A\} \leq \text{ord}\{Bji,\ j \in A,\ 1 \leq i \leq k\} + 1.$

For, suppose $x \in \bigcap_{r=1}^{t} B_{j_r}$ where $j_r$, $1 \leq r \leq t$ are t distinct members of A(t>1). Then, from (6) we have:-

(7) $x \in \bigcup_{i=1}^{k} (Bj_r i \cup \text{bdry } Dj_r i)$ for each r, $1 \leq r \leq t$.

Now $\text{bdry } Dj_r i \subset Fj_r i \cap Fj_r i+1 \subset f^{-1}(Mj_r^i)$ (from (3))

Also $\text{bdry } Dj_r k = \text{bdry } X = \phi$

So $\bigcup_{i=1}^{k} \text{bdry } Dj_r i = \bigcup_{i=1}^{k-1} \text{bdry } Dj_r i$. We therefore have

$\bigcup_{i=1}^{k} \text{bdry } Dj_r i \subset \bigcup_{i=1}^{k-1} f^{-1}(Mj_r^i)$. Now from (2), $\left(\bigcup_{i=1}^{k-1} f^{-1}(Mj^i)\right) \cap \left(\bigcup_{i=1}^{k-1} f^{-1}(M_{j'}^i)\right) = \phi$ if $j \neq j'$.

Thus x can belong to the set $\bigcup_{i=1}^{k} \text{bdry } Dj_r i$ for at most one r, $1 \leq r \leq t$. Then from (7), x belongs to $\bigcup_{i=1}^{k} Bj_r i$ for at least t-1 indices r, say $1 \leq r \leq t-1$.

For each r, $1 \leq r \leq t-1, \exists\, i_r$, $1 \leq i_r \leq k$ s.t. $x \in Bj_r i_r$. Then $x \in Bj_r i_r$ for t-1 distinct pairs $i_r j_r$. It follows that

$\text{ord}\{B_j,\ j \in A\} \leq \text{ord}\{Bji,\ J \in A,\ 1 \leq i \leq k\} + 1$

Thur $\text{ord}\{Bj,\ j \in A\} \leq m$. This shows that $d_5(X, \sigma) \leq m+1$ (resp. $d_3(X, \sigma) \leq m+1$).

We finally prove a result analogous to theorem 3.2. for the dimension functions $d_3$ and $d_5$.

<u>Theorem 3.5</u> (Nichols, 1969).

Let $(X, \rho)$ be a metric space with $d_5(X, \rho) = m < n = \dim X$ (resp. $d_3(X, \rho) = m < n = \dim X$). Then for any integer s such that $m \leq s \leq n$ $\exists$ a metric $\rho_s$ equivalent to $\rho$ s.t. $d_5(X, \rho_s) = s$ (resp. $d_3(X, \rho_s) = s$).

Proof:

Since dim X > n-1, $\exists$ n pairs of disjoint closed sets ($C_j$, $C'_j$) $1 \leq j \leq n$ s.t. if $B_j$ $1 \leq j \leq n$ are closed sets of X s.t. $B_j$ separates $C_j$ and $C'_j$ for $1 \leq j \leq n$ then $\bigcap_{j=1}^{n} B_j \neq \phi$. By Urysohns lemma, $\exists$ for each j, $1 \leq j \leq n$, a continuous function $f_j : X \to I$ s.t. $f_j(C_j) = \{0\}$, $f_j(C'_j) = \{1\}$. For each i, $1 \leq i \leq n$ let $\rho_i(x, y) = \rho(x, y) + \sum_{j=1}^{i} |f_j(x) - f_j(y)|$.

Let $\rho_0 = \rho$. Then from theorem 3.4, $\rho_i$, $0 \leq i \leq n$ are equivalent and $d(X, \rho_i) \leq d(X, \rho_{i-1}) + 1$, $1 \leq i \leq n$, d=
As in lemma 3.1, $d_5(X, \rho_n) = n$ (resp. $d_3(X, \rho_n) = n$).
It follows from the above facts that for any s, $m \leq s \leq n$ $\exists$ a metric $\rho_s$ equivalent to $\rho$ s.t. $d_5(X, \rho_s) = s$ (resp. $d_3(X, \rho_s) = s$).

Historical notes:

The realization theorem for $d_3$ (theorem 3.5) was first proved for separable metric spaces only by Roberts (Roberts, 1968) in 1968. Nichols (Nichols 1969) generalized the result to all metric spaces in 1969. The same result for $d_2$ (theorem 3.3.) was first proved in a very special case (for the spaces ($Y_n$, $\rho_n$) in example 2.3 where $X = I^n$) by Nichols (Nichols, 1973) in 1973. Goto proved the result for all metric spaces in 1976.

## SECTION 4

In this section, we study some characterizations of the metric-dependent dimension functions μ-dim, $d_2$, $d_3$, $d_5$, $d_6$ and $d_7$ and prove a weak sum theorem for the dimension functions $d_2$, $d_3$, $d_6$, $d_7$ and μ-dim. In the proofs, we leave out trival cases where the dimension is -1.

Definition 4.1

A cover $\mathcal{U}$ of a metric space $(X, \rho)$ is said to be a Lebesgue Cover of $(X, \rho)$ if for some $\delta > 0$, every subset of X of diameter not exceeding $\delta$ is contained in some member of $\mathcal{U}$. Such a $\delta$ is called a Lebesgue number for $\mathcal{U}$.

Definition 4.2

A cover $\mathcal{U}$ of a metric space $(X, \rho)$ is said to be uniformly shrinkable if for some $\delta > 0$, $\exists$ a cover $\{F_U, U \in \mathcal{U}\}$ of X s.t. $\rho(F_U, X-U) \geq \delta \ \forall \ U \in \mathcal{U}$. (Recall that $\rho(x, \phi) = \infty \ \forall \ x \in X$ by convention). $\{F_U, U \in \mathcal{U}\}$ is called a uniform shrinking of $\mathcal{U}$.

Theorem 4.1

A cover $\mathcal{U}$ of a metric space $(X, \rho)$ is a Lebesgue cover of $(X, \rho)$ iff it is uniformly shrinkable.

Proof: Necessity: Let $\mathcal{U}$ be a Lebesgue cover of $(X, \rho)$ with Lebesgue number $\delta$. For each $U \in \mathcal{U}$ Let $F_U = \{x \in X: \rho(x, X-U) \geq \frac{1}{2}\delta\}$.

Then $\{F_U, U \in \mathcal{U}\}$ is a cover of X. For suppose $x \in X$. $B(x, \delta/2) \subset U_0$ for some $U_0 \in \mathcal{U}$. So $\rho(x, X-U_0) \geq \delta/2$ so $x \in F_{U_0}$. Obviously $\rho(F_U, X-U) \geq \delta/2 \;\forall\; U \in \mathcal{U}$. So $\mathcal{U}$ is uniformly shrinkable.

Sufficiency: Suppose a cover $\mathcal{U}$ of a metric space $(X, \rho)$ is uniformly shrinkable.

Let $\{F_U, U \in \mathcal{U}\}$ be a cover of X s.t. $\rho(F_U, X-U) \geq \delta > 0 \;\forall\; U \in \mathcal{U}$ for some $\delta$.

Let A be any subset of X with $\rho(A) \leq \frac{1}{2}\delta$.

Leaving out the trivial case $A = \phi$, $A \cap F_{U_0} \neq \phi$ for some $U_0 \in \mathcal{U}$. But this implies $A \subset U_0$ proving that $\mathcal{U}$ is Lebesgue.

Corollary 4.1

Every Lebesgue cover $\mathcal{U}$ of a metric space $(X, \rho)$ has an open Lebesgue refinement $\{G_U, U \in \mathcal{U}\}$ s.t. $G_U \subset U$.

Proof: If $\mathcal{U}$ is a Lebesgue cover of $(X, \rho)$, let $\{F_U, U \in \mathcal{U}\}$ be a uniform shrinking of $\mathcal{U}$ s.t. $(F_U, X-U) \geq \delta > 0 \;\forall\; U \in \mathcal{U}$ for some $\delta$.

Let $G_U = B(F_U, \delta)$. Then $\{G_U, U \in \mathcal{U}\}$ is an open Lebesgue cover of $(X, \rho)$ and $G_U \subset U$.

Defn. 4.3 Let $\{\mathcal{G}_\gamma, \gamma \in \Lambda\}$ be a collection of collections of subsets of a set X. Let $\Gamma$ be the set of functions $f: \Lambda \longrightarrow \bigcup_{\gamma \in \Lambda} \mathcal{G}_\gamma$ s.t. $f(\gamma) \in \mathcal{G}_\gamma \;\forall\; \gamma \in \Lambda$.

Then $\bigwedge_{\gamma \in \Lambda} \mathcal{G}_\gamma$ is defined to be the collection $\{\bigcap_{\gamma \in \Lambda} f(\gamma), f \in \Gamma\}$.

Defn 4.4. A collection $\mathcal{C} = \{ C_\alpha, \alpha \in \mathcal{A} \}$ of subsets of a set X is said to be m-point bounded if it is of order m-1. It is said to be point bounded if it is of finite order, and is said to be point finite if every point is contained in $C_\alpha$ for only finitely many $\alpha$ .

Lemma 4.1

Let $\{ \mathcal{G}_\gamma , \gamma \in \Lambda \}$ be a collection of collections of subsets of a metric space (X, $\rho$).

(i) If $\mathcal{G}_\gamma$ is a Lebesgue cover of (X, $\rho$) with Lebesgue number $\delta$ for each $\gamma \in \Lambda$ for some $\delta$ , then $\Lambda_{\gamma \in \Lambda} \mathcal{G}_\gamma$ is a Lebesque cover of (X, $\rho$) with Lebesgue number $\delta$ .

(ii) If $\mathcal{G}_\gamma = \{ G_\gamma , G'_\gamma \}$ and $\{ G'_\gamma , \gamma \in \Lambda \}$ is locally finite, then $\Lambda_{\gamma \in \Lambda} \mathcal{G}_\gamma$ is locally finite. It is countable if $\Lambda$ is countable.

(iii) If $\mathcal{G}_\gamma = \{ G_\gamma, G'_\gamma \}$ and $\{ G'_\gamma , \gamma \in \Lambda \}$ is point bounded, then $\Lambda_{\gamma \in \Lambda} \mathcal{G}_\gamma$ is point bounded. It is countable if $\Lambda$ is countable.

(iv) If $\mathcal{G}_\gamma = \{ G_\gamma, G'_\gamma \}$ and $\{ G'_\gamma , \gamma \in \Lambda \}$ is point finite, then $\Lambda_{\gamma \in \Lambda} \mathcal{G}_\gamma$ is point finite. It is countable if $\Lambda$ is countable

Proof: (i). Suppose each $\mathcal{G}_\gamma$ is Lebesgue with Lebesgue number $\delta$ . Let A be any set of diameter not exceeding $\delta$ . For each $\gamma \in \Lambda$ let $f(\gamma)$ be a member of $\mathcal{G}_\gamma$ containing A (Since $\mathcal{G}_\gamma$ is Lebesgue with number $\delta$ ). Then $A \subset \bigcap_{\gamma \in \Lambda} f(\gamma) \in \Lambda_{\gamma \in \Lambda} \mathcal{G}_\gamma$ .

This shows that $\wedge_{\gamma\in\Lambda}\mathcal{G}_\gamma$ is a Lebesgue cover of (X, $\mathcal{E}$) with Lebesque number $\delta$ .

(ii) Suppose $\mathcal{G}_\gamma = \{G_\gamma, G'_\gamma\}$ and $\{G'_\gamma, \gamma\in\Lambda\}$ is l.f. For any $x\in X$, $\exists$ a nbhd U of x s.t. U intersects only finitely many $G'_\gamma$'s, say $G'_{\gamma_1}, \ldots, G'_{\gamma_k}$. Put B = $\{\gamma_1, \ldots, \gamma_k\}$. Let $\Gamma$ be as in defn 4.3. If $f\in\Gamma$ is s.t. $U\cap(\bigcap_{\gamma\in\Lambda} f(\gamma)) \neq \phi$ we must have $f(\gamma) = G_\gamma$ for $\gamma\notin B$. There are only finitely many such f's in $\Gamma$ and so U intersects only finitely many members of $\wedge_{\gamma\in\Lambda}\mathcal{G}_\gamma$. So $\wedge_{\gamma\in\Lambda}\mathcal{G}_\gamma$ is l.f. To see that $\wedge_{\gamma\in\Lambda}\mathcal{G}_\gamma$ is countable, let $f\in\Gamma$ be s.t. $f(\gamma) = G'_\gamma$ for infinitely may $\gamma$'s in $\Lambda$. Then, since $\{G'_\gamma, \gamma\in\Lambda\}$ is l.f. $\bigcap_{\gamma\in\Lambda} f(\gamma) = \phi$. So if $\bigcap_{\gamma\in\Lambda} f(\gamma) \neq \phi$, we must have $f(\gamma) = G'_\gamma$

for only finitely many $\gamma$'s. There are only countably many such f's in $\Gamma$ and that implies $\wedge_{\gamma\in\Lambda}\mathcal{G}_\gamma$ is countable. The proof of (iii) and (iv) are similar to the proof of (ii).

<u>Lemma 4.2</u>

Let X be a normal space and $(F_\alpha, F'_\alpha), \alpha\in\mathcal{A}$ a collection of closed sets of X s.t. $F_\alpha \cap F'_\alpha = \phi\ \forall \alpha\in\mathcal{A}$ and $\{X-F'_\alpha, \alpha\in\mathcal{A}\}$ is point finite. If $\mathcal{G} = \wedge_{\alpha\in\mathcal{A}}\{X-F_\alpha, X-F'_\alpha\}$ (i.e. $\mathcal{G} = \wedge_{\alpha\in\mathcal{A}}\mathcal{G}_\alpha$ where $\mathcal{G}_\alpha = \{X-F_\alpha, X-F'_\alpha\}$ ) has an open refinement of order $\leq n \geq 0$ then $\exists$ closed sets $B_\alpha, \alpha\in\mathcal{A}$ s.t. $B_\alpha$ separates $F_\alpha$ and $F'_\alpha$ and ord $\{B_\alpha, \alpha\in\mathcal{A}\} \leq n-1$.

For a proof of this lemma, see Nagata "Modern dimension theory" II, 5, B pp 23-25.

In the rest of this section, Jn denotes the set $\{1, 2, 3, \ldots, n\}$ for a positive integer n.

Theorem 4.2 (Smith, 1968)

Let $(X, \ell)$ be a metric space. Then $d_2(X, \ell) \leq n$ iff for each collection $\{\mathcal{G}i, i \in Jn+1\}$ of n+1 binary Lebesgue covers of X (i.e. covers consisting of two members), the cover $\mathcal{G} = \bigwedge_{i \in Jn+1} \mathcal{G}i$ of X has an open refinement of order $\leq$ n.

Proof: Necessity: Suppose $d_2(X, \ell) \leq n \geq 0$. Let $\{\mathcal{G}i, i \in Jn+1\}$ be n+1 binary Lebesgue covers of X. From corollary 4.1, we may assume each $\mathcal{G}i$ to be an open cover. Clearly, each $\mathcal{G}i$ can be written as $\{Gi, X-Fi\}$ where $\ell(Fi, X-Gi) \geq \delta > 0$ $\forall i$ for some $\delta$. Then, since $d_2(X, \ell) \leq n$, $\exists$ open sets Ui, $i \in Jn+1$ s.t. $Fi \subset Ui \subset \overline{U}i \subset Gi$ and $\bigcap_{i=1}^{n+1} Bi = \phi$ where Bi = bdry Ui. For each non-empty subset I of Jn+1 let $\mathcal{C}I$ be the collection $\{C \cap (\bigcap_{i \notin I} Bi), C \in \bigwedge_{i \in I} \{Ui, X-\overline{U}i\}\}$.

(Take $\bigcap_{i \in \phi} Bi = X$). For $k \in Jn_{+1}$, let $\mathcal{F}k$ be the collection $\bigcup_{|I|=k} \mathcal{C}I$. $\bigcup_{k=1}^{n+1} \mathcal{F}k$ covers X (recall $\bigcap_{i=1}^{n+1} Bi = \phi$) and, clearly, refines $\bigwedge_{i \in Jn+1} \mathcal{G}i$. If F, F' are distinct members of $\mathcal{F}k$, then for some $i \in Jn_{+1}$ we must have a situation where $F \subset Ui$ and $F' \subset X-Ui$ or $F \subset X-\overline{U}i$ and $F' \subset \overline{U}i$ so that $F \cap \overline{F'} = \phi$. Thus $\mathcal{F}k$, being finite, is a disjoint collection of relatively open subsets of $\bigcup_{F \in \mathcal{F}k} F$.

Since X is completely normal and $\mathcal{F}k$ is finite, $\exists$ a disjoint collection $\mathcal{H}k$ of open subsets of X s.t. $\mathcal{H}k = \{H_F, F \in \mathcal{F}k\}$ and $F \subset H_F$. Let $H'_F = H_F \cap G$ where G is a member of $\Lambda_{i \in Jn+1} \mathcal{G}i$ containing F. Then $\mathcal{H}'k = \{H'_F, F \in \mathcal{F}_k\}$ is a disjoint collection of open sets of X and $\mathcal{F}k$ refines $\mathcal{H}'k$ (but they are not necessarily covers of X) which in turn refines $\Lambda_{i \in Jn+1} \mathcal{G}i$.

Since $\bigcup_{k \in Jn+1} \mathcal{F}k$ covers X, so does $\bigcup_{k \in Jn+1} \mathcal{H}'k = \mathcal{H}$, say .

Claim: ord $\mathcal{H} \leq n$. This is clear because each $\mathcal{H}k$ is a disjoint collection. It is also clear that $\mathcal{H}$ refines $\Lambda_{i \in Jn+1} \mathcal{G}i$ since each $\mathcal{H}k$ does. Thus we have found an open refinement of $\Lambda_{i \in Jn+1} \mathcal{G}i$ of order $\leq$ n.

Sufficiency: Assume that for each collection $\{\mathcal{G}i, i \in Jn+1\}$, of n+1 binary Lebesgue covers of $(X, \ell)$, $\Lambda_{i \in Jn+1} \mathcal{G}i$ has an open refinement of order $\leq$ n. Let (Ci, C'i) $i \in Jn+1$ be n+1 pairs of closed sets s.t. $\ell(Ci, C'i) > 0$. Then $\{X-Ci, X-C'i\}$, $i \in Jn+1$, are clearly n+1 binary Lebesgue covers of $(X, \ell)$. $\Lambda_{i \in J_{n+1}} \{X-Ci, X-C'i\}$ has an open refinement of order $\leq$ n by hypothesis. From lemma 4.2, $\exists$ closed sets Bi, $i \in Jn+1$ s.t. Bi separates Ci and C'i and ord $\{Bi, i \in Jn+1\} \leq n-1$. Thus $d_2(X, \ell) \leq n$.

<u>Theorem 4.3.</u> (Smith, 1968). Let $(X, \ell)$ be a metric space. Then $d_2(X, \ell) \leq n$ iff every Lebesgue cover $\mathcal{G} = \{G_1, G_2, \ldots, G_{n+2}\}$ of $(X, \ell)$, consisting of n+2 members has an open refinement of order $\leq$ n.

Proof: Necessity: Assume $d_2(X, \ell) \leq n$.

Let $\mathcal{G} = \{G_1 \ldots\ldots, G_{n+2}\}$ be a Lebesgue cover of $(X, \ell)$. From theorem 4.1 (and taking closures) $\exists$ a closed cover $\mathcal{F} = \{F_1, F_2, \ldots\ldots, F_{n+2}\}$ of X s.t. $\ell(Fi, X-Gi) > 0$ $\forall i$, $1 \leq i \leq n+2$. $\bigwedge_{i \in Jn+1} \{Gi, X-Fi\}$ refines $\mathcal{G}$. This is because if $H \in \bigwedge_{i \in Jn+1} \{Gi, X-Fi\}$, then either $H \subset Gi$ for some i, $i \in Jn+1$, or $H \subset X - \bigcup_{i \in Jn+1} Fi$ in which case $H \subset Fn+2 \subset Gn+2$ (since $\mathcal{F}$ covers X).

By theorem 4.2, $\bigwedge_{i \in Jn} \{Gi, X-Fi\}$ has an open refinement of order $\leq n$. Thus $\mathcal{G}$ has an open refinement of order $\leq n$.

Sufficiency: Suppose each Lebesgue cover $\mathcal{G} = \{G_1, \ldots\ldots, Gn+2\}$ of $(X, \ell)$ consisting of n+2 members has an open refinement of order $\leq n$. Let $(Ci, C'i)$ $i \in Jn+1$ be n+1 pairs of closed sets of X s.t. $\ell(Ci, C'i) > 0$. Let $\delta$ be s.t. $0 < \delta \leq \min \{\ell(Ci, C'i), i \in Jn+1\}$. For $i \in Jn+1$, let $Ui = B(Ci, \delta)$ and $Fi = B(Ci, \delta/4)$. Let $Un+2 = X - \bigcup_{i \in Jn+1} Fi$.

Then $\mathcal{U} = \{U_1, U_2, \ldots\ldots, U_{n+2}\}$ is uniformly shrinkable and so is a Lebesgue cover. To see that $\mathcal{U}$ is uniformly shrinkable, let $U'i = B(Ci, \delta/2)$ $i \in Jn+1$ and $U'n+2 = X - \bigcup_{i \in Jn+1} B(Ci, \delta/3)$. Then $\{U'_1, \ldots U'n+2\}$ is a uniform shrinking of $\mathcal{U}$. By hypothesis, $\mathcal{U}$ has an open refinement $\mathcal{W}$ of order $\leq n$. $\exists$ a function $f: \mathcal{W} \to \mathcal{U}$ s.t. for $W \in \mathcal{W}$, $W \subset f(W)$. For each $i \in Jn+2$, let $Hi = \bigcup_{\substack{W \in \mathcal{W} \\ f(W) = Ui}} W$. Then $\mathcal{H} = \{H_1, \ldots\ldots, H_{n+2}\}$ is an open refinement of $\mathcal{U}$ s.t. $H_i \subset U_i$ $i \in J_{n+2}$, and ord $\mathcal{H} \leq n$. Let $E_i = C_i - H_i$, $i \in J_{n+1}$. Let $Y_i = B(E_i, \delta/4)$, $i \in J_{n+1}$ (recall that $\rho(x, \phi) = \infty$ by convention), and let $V_i = H_i \cup Y_i$, $i \in J_{n+1}$, and $V_{n+2} = H_{n+2}$. $Y_i \cap H_{n+2} \subset Y_i \cap U_{n+2} = \phi$, $i \in J_{n+1}$ from the definition of $Y_i$ and $U_{n+2}$. This, together with ord $\mathcal{H} \leq n$ implies ord $\mathcal{V} \leq n$ where $\mathcal{V} = \{V_1, \ldots, V_{n+2}\}$. It is clear that $C_i \subset V_i \subset X - C'_i$ $i \in J_{n+1}$ and $\mathcal{V}$ covers X. Since X is normal, $\exists$ closed sets $D_i$, $i \in J_{n+2}$ s.t. $C_i \subset D_i \subset V_i$, $i \in J_{n+1}$ and $\mathcal{D} = \{D_1, \ldots D_{n+2}\}$ covers X. Again, since X is normal, $\exists$ open sets $K_i$, $i \in J_{n+1}$ s.t. $D_i \subset K_i \subset \bar{K}_i \subset V_i$, $i \in J_{n+1}$. Let $B_i = \text{bdry } K_i$, $i \in J_{n+1}$. Then clearly $B_i$ separates $C_i$ and $C'_i$.

Claim: ord $\{B_i, i \in J_{n+1}\} \leq n-1$.

Suppose $x \in \bigcap_{i \in J_{n+1}} B_i$. Then $x \notin D_i$ for $i \in J_n$. Thus $x \in D_{n+2}$ since $\mathcal{D}$ covers X. Thus $x \in V_{n+2}$. Also $x \in V_i$, $i \in J_{n+1}$ since $B_i \subset V_i$, $i \in J_{n+1}$, so $x \in \bigcap_{i \in J_{n+2}} V_i$ which is impossible since ord $\mathcal{V} \leq n$. This shows that $d_2(X, \rho) \leq n$.

<u>Defn 4.5</u> A collection $\mathcal{C}$ of subsets of a metric space $(X, \rho)$ is said to be <u>n-uniformly discrete</u>, $n \geq 1$, if $\mathcal{C} = \bigcup_{i=1}^{n} \mathcal{C}_i$ where $\mathcal{C}_i$, $1 \leq i \leq n$ satisfy the condition that $\exists\ \delta > 0$ s.t. $\forall i$, $1 \leq i \leq n$, and $C, C' \in \mathcal{C}_i$ with $C \neq C'$ we have $\rho(C, C') > \delta$.

Lemma 4.3 (Smith and Nichols)

If $\zeta$ is a Lebesgue cover of a metric space $(X, \ell)$ and $\zeta = \bigcup_{\alpha\varepsilon\Delta} \zeta\alpha$ where each $\zeta\alpha$ is m-point bounded, m a fixed positive integer, then $\zeta$ has a refinement $\Theta$ s.t. $\Theta = \bigcup_{\alpha\varepsilon\Delta} \Theta\alpha$ and each $\Theta\alpha$ is m-uniformly discrete.

Proof: The proof is by induction on m.
Suppose the result true for a positive integer m.
Let $\zeta$ be as in the lemma with m replaced by m+1. $\zeta$ has a Lebesque refinement $\Omega = \{F_C, C \varepsilon\zeta\}$ s.t. $\ell(F_C, X-C) > \delta > 0 \ \forall C\varepsilon\zeta$ for some $\delta$. For each $\alpha\varepsilon\Delta$ let $\dagger\alpha$ be the collection of all subcollections of $\zeta\alpha$ with m+1 members. For each $S \varepsilon \dagger\alpha$, let $Gs = \bigcap_{C\varepsilon S} F_C$.

Then for any $\alpha$, if $S, S'\varepsilon\dagger\alpha$, $S \neq S'$, $\exists\, C \varepsilon\zeta\alpha$ (W.L.G) s.t. $C\varepsilon S$, $C \notin S'$. Then, since $\zeta\alpha$ is m+1-point bounded, $Gs' \cap C = \Phi$. But $Gs \subset F_C$ so $\ell(Gs, Gs') > \delta$. Let $Y\alpha = \bigcup_{S\varepsilon\dagger\alpha} G_S$. Let $Y = \bigcup_{\alpha\varepsilon\Delta} Y\alpha$ and let $Z = X - Y$.

. For each $\alpha\varepsilon\Delta$, let $\Pi\alpha = \{ F_C \cap Z, C\varepsilon\zeta\alpha \}$.
Let $\Pi = \bigcup_{\alpha\varepsilon\Delta} \Pi\alpha$. Then $\Pi$ is a Lebesgue cover of $Z$.
For each $\alpha\varepsilon\Delta$, $\Pi\alpha$ is m-point bounded. By the induction hypothesis, $\Pi$ has a refinement $\Theta' = \bigcup_{\alpha\varepsilon\Delta} \Theta'\alpha$ where $\Theta'_\alpha$ is m-uniformly discrete with $\Theta'\alpha = \bigcup_{i=1}^{m} \Lambda\alpha i$

where $\Lambda\alpha i$, $1 \leq i \leq m$ satisfy the condition that for some $\delta' > 0$, $K, K' \varepsilon \Lambda\alpha i$, $K \neq K' \Rightarrow \ell(K, K') >$ for any i. For each $\alpha$, let $\Lambda\alpha\, m+1 = \{ Gs, S \varepsilon \dagger\alpha\}$.
Let $\Theta\alpha = \bigcup_{i=1}^{m+1} \Lambda\alpha i$. Then $\Theta = \bigcup_{\alpha\varepsilon\Delta} \Theta\alpha$ is the required refinement of $\zeta$. If m = 1 then

$\Omega = \{F_C, C \in \zeta\} = \bigcup_{\alpha \in \Delta} \Omega\alpha$ where $\Omega\alpha = \{F_C, C \in \zeta\alpha\}$

is the required refinement. This completes the induction.

<u>Lemma 4.4.</u> (Smith and Nichols)

Let $\zeta$ be a Lebesgue cover of a metric space $(X, \ell)$ s.t. $\zeta = \bigcup_{\alpha \in \Delta} \zeta\alpha$ where each $\zeta\alpha$ is m-point bounded.

Then $\zeta$ has an open Lebesgue refinement $\Theta$ s.t. $\Theta = \bigcup_{\alpha \in \Delta} \Theta\alpha$ where each $\Theta\alpha$ is m-uniformly discrete.

Proof: $\zeta$ has a Lebesgue refinement $\Omega = \{F_C, C \in \zeta\}$ s.t. $\ell(F_C, X-C) > \delta > 0 \ \forall\ C \in \zeta$ for some $\delta$. Let $\Omega\alpha = \{F_C, C \in \zeta\alpha\}$. Then $\Omega = \bigcup_{\alpha \in \Delta} \Omega\alpha$ and each $\Omega\alpha$ is m-point bounded.

By lemma 4.3, $\Omega$ has a refinement $\Pi = \bigcup_{\alpha \in \Delta} \Pi\alpha$ where each $\Pi\alpha$ is m-uniformly discrete. $\Pi\alpha = \bigcup_{i=1}^{m} \Lambda\alpha_i$ where for some $\delta'\alpha > 0$, $K, K' \in \Lambda\alpha_i$, $1 \leq i \leq m$, $K \neq K' \Rightarrow \ell(K, K') > \delta'\alpha$.

Let $\delta\alpha = \min\{\delta, \delta'\alpha\}$ for each $\alpha \in \Delta$.

Then if $\Theta\alpha = \{B(H, 1/4\ \delta\alpha), H \in \Pi\alpha\}$, $\Theta\alpha$ is an m-uniformly discrete open collection and $\Theta = \bigcup_{\alpha \in \Delta} \Theta\alpha$ is the required open Lebesgue refinement of $\zeta$.

<u>Corollary 4.2</u> (Smith and Nichols)

Let $\zeta$ be an n-point bounded Lebesgue cover of a metric space $(X, \ell)$. Then $\zeta$ has an n-uniformly discrete open Lebesgue refinement.

Proof: This is immediate from Lemma 4.4.

<u>Theorem 4.4.</u> (Smith and Nichols)

Let $(X, \ell)$ be a metric space. Then $d_2(X, \ell) \leq n$ iff every n+2 - point bounded Lebesgue cover of $(X, \ell)$ has an open refinement of order $\leq n$.

<u>Proof:</u> Necessity: Assume $d_2(X, \ell) \leq n$. Let $\zeta$ be an n+2 - point bounded Lebesgue cover of X. From Corollary 4.2, $\zeta$ has an open Lebesgue refinement $\Theta$ where $\Theta = \bigcup_{i=1}^{n+2} \Theta i$ and each $\Theta i$ is disjoint.

Let $Gi = \bigcup_{G \in \Theta i} G$. $\{Gi, 1 \leq i \leq n+2\}$ is a Lebesgue cover of X with n+2 members so, from Theorem 4.3, $\{Gi\}$ has an open refinement $\Pi$ of order $\leq n$. $\exists$ a function $f: \Pi \longrightarrow \{1, 2, \ldots, n+2\}$ s.t. $H \subset G_{f(H)}$ $\forall H \in \Pi$. Let $\Lambda = \{H \cap G, H \in \Pi, G \in \Theta_{f(H)}\}$. Then $\Lambda$ is an open refinement of $\zeta$ of order $\leq n$.

Sufficiency: This is clear from Theorem 4.3 since every collection consisting of n+2 members is n+2 point bounded.

<u>Theorem 4.5</u> (Smith, Smith and Nichols)

Let $(X, \ell)$ be a metric space. Then the following are equivalent:-

(i) $d_3(X, \ell) \leq n$

(ii) Every finite Lebesgue cover of $(X, \ell)$ has an open refinement of order $\leq n$.

(iii) Every point bounded Lebesgue cover of $(X, \ell)$ has an open refinement of order $\leq n$.

(iv) If $(C\alpha, C'\alpha)\ \alpha\varepsilon\Delta$ are pairs of closed sets of X s.t. $\ell(C\alpha, C'\alpha) > \delta > 0\ \forall\ \alpha\varepsilon\Delta$ for some $\delta$ and $\{X-C'\alpha, \alpha\varepsilon\Delta\}$ is point bounded then $\exists$ closed sets $B\alpha$, $\alpha\varepsilon\Delta$ s.t. $B\alpha$ separates $C\alpha$ and $C'\alpha$ and ord $\{B\alpha, \alpha\varepsilon\Delta\} \leq n-1$.

Proof: We prove (i) => (ii) => (iii) => (iv) => (i).

(i) => (ii).

Suppose $d_3(X, \ell) \leq n$. Let $\Theta = \{G_1, G_2, \ldots\ldots, G_k\}$ be a finite Lebesgue cover of $(X, \ell)$. From corollary 4.1, we may assume $\Theta$ to be an open cover. $\exists$, by theorem 4.1, a closed cover $\Omega = \{F_1, F_2, \ldots\ldots, F_k\}$ of X s.t. $\ell(F_i, X - G_i) > 0$ for $1 \leq i \leq k$.
Since $d_3(X, \ell) \leq n$, $\exists$ open sets $Ui$, $1 \leq i \leq k$ s.t. $Fi \subset Ui \subset \overline{U}i \subset Gi$, $1 \leq i \leq k$, and ord $\{\text{bdry } Ui, 1 \leq i \leq k\} \leq n-1$. By lemma 1.3, $\{Gi\}$ has an open refinement of order $\leq n$.

(ii) => (iii)

Assume (ii). Let $\zeta$ be a point bounded Lebesgue cover of $(X, \ell)$ so for some positive integer m $\zeta$ is m-point bounded. From corollary 4.2, $\zeta$ has an open Lebesgue refinement $\Theta$ where $\Theta = \bigcup_{i=1}^{m} \Theta i$ and each $\Theta i$ is disjoint. Let $Gi = \bigcup_{G\,\varepsilon\,\Theta i} G$. $\{Gi, 1 \leq i \leq m\}$ is a finite Lebesgue cover of $(X, \ell)$. From (ii), $\{Gi, 1 \leq i \leq m\}$ has an open refinement $\Pi$ of order $\leq n$.

$\exists$ a function $f: \Pi \longrightarrow \{1, 2, \ldots, m\}$ s.t. $H \subset G_{f(H)}$ $\forall H \in \Pi$. Let $\Lambda = \{H \cap G, H \in \Pi, G \in \Theta_{f(H)}\}$. Then $\Lambda$ is an open refinement of $\zeta$ of order $\leq n$.

(iii) $\Rightarrow$ (iv)

Assume (iii). Let $(C_\alpha, C'_\alpha)_{\alpha \in \Delta}$ and $\delta$ be as in (iv). $\{X - C_\alpha, X - C'_\alpha\}$ is a Lebesgue cover of $(X, \ell)$ $\forall \alpha \in \Delta$ and so, therefore, is $\Lambda_{\alpha \in \Delta} \{X - C_\alpha, X - C'_\alpha\}$ from Lemma 4.1. Since $\{X - C'_\alpha, \alpha \in \Delta\}$ is point bounded, $\Lambda_{\alpha \in \Delta} \{X - C_\alpha, X - C'_\alpha\}$ is point bounded by lemma 4.1. So from (iii), $\Lambda_{\alpha \in \Delta} \{X - C_\alpha, X - C'_\alpha\}$ has an open refinement of order $\leq n$. From lemma 4.2, $\exists$ closed sets $B_\alpha, \alpha \in \Delta$ s.t. $B_\alpha$ separates $C_\alpha$ and $C'_\alpha$ and ord $\{B_\alpha, \alpha \in \Delta\} \leq n-1$.

(iv) $\Rightarrow$ (i).

This is clear from the definition of $d_3$.

Lemma 4.5 (Smith, 1970)

Let $\Theta = \{G_\alpha, \alpha \in \Delta\}\}$ be a star-countable collection of subsets of a set X. Then $\exists$ a partition $\{\Delta_\beta, \beta \in \chi\}$ of $\Delta$ s.t. $\Delta_\beta$ is countable for each $\beta \in \chi$ and if we put $X_\beta = \bigcup_{\alpha \in \Delta_\beta} G_\alpha$ then if $\beta, \beta' \in \chi, \beta \neq \beta'$, then $X_\beta \cap X_{\beta'} = \phi$.

Proof: Define a relation $\sim$ on $\Delta$ as follows. $\alpha \sim \alpha'$ if $\exists$ a finite number of members $G_{\alpha_1}, G_{\alpha_2}, \ldots G_{\alpha_k}$ of $\Theta$ s.t. $G_\alpha \cap G_{\alpha_1} \neq \Phi$, $G_{\alpha_1} \cap G_{\alpha_2} \neq \Phi$, $\ldots G_{\alpha_k} \cap G_{\alpha'} \neq \Phi$. Clearly $\sim$ is an equivalence relation on $\Delta$. Let $\{\Delta_\beta, \beta \in \chi\}$ be the collection of equivalence classes of $\sim$ . Then the conditions of the lemma are satisfied.

Lemma 4.6 (Smith, 1968)

Every countable Lebesgue cover of a metric space $(X,\ell)$ has a countable, open, l.f. Lebesgue refinement.

Proof: Let $\Theta$ be a countable Lebesgue cover of a metric space $(X, \ell)$. From corollary 4.1, $\Theta$ has a countable open Lebesgue refinement $\Pi = \{H_1, H_2, \ldots\}$. Let $\Omega = \{F_1, F_2, \ldots\}$ be a uniform shrinking of $\Pi$ with $\ell(Fi, X-Hi) \geq \delta > 0 \; \forall i$, $i = 1, 2, \ldots$ for some $\delta$.

Let $Ui = Hi - \bigcup_{j<i} \overline{B(Fj, \delta/4)}$ $i = 2, 3, \ldots$ and $U_1 = H_1$. Clearly $\upsilon = \{U_1, U_2, \ldots\}$ is an open, countable, l.f. refinement of $\Theta$. Furthermore, if $Ei = B(Fi, \frac{1}{2}\delta) - \bigcup_{j<i} B(Fj, {}^1/3\,\delta)$ $i = 2, 3, \ldots$ and $E_1 = B(F_1, \frac{1}{2}\delta)$, then $\{E_1, E_2, \ldots\}$ is a uniform shrinking of $\upsilon$ so $\upsilon$ is Lebesgue.

Theorem 4.6 (Smith, Smith and Nichols)

Let $(X,\ell)$ be a metric space. Then the following conditions are equivalent

(i) $d_6(X,\ell) \leq n$

(ii) Every countable l.f. Lebesgue cover of $(X,\ell)$ has an open refinement of order $\leq n$.

(iii) Every countable Lebesgue cover of $(X,\ell)$ has an open refinement of order $\leq n$.

(iv) Every star-countable Lebesgue cover of $(X, \ell)$ has an open refinement of order $\leq n$.

(v) Every Lebesgue cover of $(X, \ell)$ representable as a union $\bigcup_{i=1}^{\infty} \mathcal{V}i$ with $\mathcal{V}i$ m-point bounded $i = 1, 2....$ for some positive integer m has an open refinement of order $\leq n$.

Proof: We prove (i) => (ii) => (iii) => (iv) => (v) => (ii) => (i).

(i) => (ii).

Let $\Theta = \{G_1, G_2, .....\}$ be a countable l.f. Lebesgue cover of $(X, \ell)$. From corollary 4.1., we may assume $\Theta$ is open. From theorem 4.1, $\Theta$ has a closed refinement $\Omega = \{F_1, F_2.....\}$ s.t. $\ell(Fi, X-Gi) \geq \delta > 0$ $\forall i$, $i = 1, 2, 3,....$ for some $\delta$. Since $d_6(X, \ell) \leq n$, $\exists$ open sets Ui, $i = 1, 2,....$ s.t. $Fi \subset Ui \subset \overline{U}i \subset Gi$ $i = 1, 2,....$ and ord {bdry Ui, $i = 1, 2,.....\} \leq n-1$. {Ui } is a cover of X and by lemma 1.3 $\Theta$ has an open refinement of order $\leq n$.

(ii) => (iii).

This is obvious from lemma 4.6.

(iii) => (iv).

Assume (iii). Let $\Theta = \{G\alpha, \alpha\varepsilon\Delta\}$ be a star countable Lebesgue cover of $(X, \ell)$. From lemma 4.5, $\exists$ a partition $\{\Delta\beta, \beta\varepsilon\chi\}$ of $\Delta$ s.t. each $\Delta\beta$ is countable and if $X\beta = \bigcup_{\alpha\varepsilon\Delta\beta} G\alpha$ then $X\beta \cap X\beta' = \Phi$ if $\beta \neq \beta', \beta, \beta' \varepsilon \chi$

Clearly, for each $\beta\varepsilon\chi, \upsilon\beta = \{G\alpha, \alpha\varepsilon\Delta\beta\} \cup \{X - X\beta\}$ $(= \{G\alpha, \alpha\varepsilon\Delta\beta\} \cup \{\bigcup_{\alpha\notin\Delta\beta} G\alpha\})$ is a countable Lebesgue cover of $(X, \ell)$. From (iii), $\upsilon\beta$ has an open refinement $\upsilon'\beta$ of order $\leq n$. If we let $\upsilon''\beta$ be the collection of those members of $\upsilon\beta$ which are contained in some $G\alpha, \alpha\varepsilon\Delta\beta$, then $\upsilon''\beta$ is an open cover of $X\beta$ of order $\leq n$ which refines $\{G\alpha, \alpha\varepsilon\Delta\beta\}$. Let $\upsilon = \bigcup_{\beta\varepsilon\chi} \upsilon''\beta$.

Then $\upsilon$ is an open refinement of $\Theta$ of order $\leq n$.

(iv) => (v)

Assume (iv). Let $\zeta$ be a Lebesgue cover of $(X, \ell)$ s.t. $\zeta = \bigcup_{i=1}^{\infty} \zeta i$ where $\exists$ an integer m s.t. $\zeta i$ is m-point bounded for each i. From lemma 4.4., $\zeta$ has an open Lebesgue refinement $\Theta$ s.t. $\Theta = \bigcup_{i=1}^{\infty} \Theta i$ and each $\Theta i = \bigcup_{j=1}^{m} \Pi ij$ where $\Pi ij$ are disjoint collections. For each i, i = 1, 2,.....and $1 \leq j \leq m$, let $Hij = \bigcup_{H\varepsilon \Pi ij} H$. Then $\Pi = \{Hij, i = 1, 2,....$ $1 \leq j \leq m\}$ is a countable Lebesgue cover of $(X, \ell)$. From (iv) (or even (iii)) $\Pi$ has an open refinement $\upsilon$ of order $\leq n$. $\exists$ a function $f: \upsilon \longrightarrow N \times N$ s.t. $U \subset Hf(U)$ $\forall$ $U \varepsilon \upsilon$.

Then $\{U \cap K, U\varepsilon\upsilon, K \varepsilon \Pi_{f(U)}\}$ is an open refinement of $\zeta$ of order $\leq n$.

(v) => (ii) is immediate.

(ii) => (i).

Assume (ii). Let (Ci, C'i) $i \in N$ be a collection of pairs of closed sets of X s.t. $\ell(Ci, C'i) \geq \delta > 0$ $\forall\, i \in N$ for some $\delta$ and $\{X-C'i,\ i \in N\}$ is l.f. From lemma 4.1, $\Theta = \wedge_{i \in N} \{X-Ci, X-C'i\}$ is a countable, l.f. Lebesgue cover of $(X, \ell)$. From (ii), $\Theta$ has an open refinement of order $\leq n$. From lemma 4.2, $\exists$ closed sets Bi, $i \in N$ s.t. Bi separates Ci and C'i and ord $\{Bi, i \in N\} \leq n-1$. This completes the proof.

<u>Lemma 4.7</u>

Every point finite Lebesgue cover of a metric space $(X, \ell)$ has a locally finite Lebesgue refinement.

Proof: Let $\upsilon$ be a point finite Lebesque cover of a metric space $(X, \ell)$. From Theorem 4.1 $\upsilon$ has a uniform shrinking $\{FU, U \in \upsilon\}$ s.t. $\ell(FU, X-U) > \delta > 0$ $\forall\, U \in \upsilon$ for some $\delta$. Let $GU = B(FU, \frac{1}{2}\delta)$.
Claim: $\Theta = \{GU, U \in \upsilon\}$ is a l.f. Lebesgue cover of $(X, \ell)$. $\Theta$ is Lebesgue because $\{FU, U \in \upsilon\}$ is a uniform shrinking of $\Theta$. To see that $\Theta$ is l.f., let $x \in X$. Then, since $\upsilon$ is point finite, x is contained in only finitely many members, say $U_1$ $U_2, \ldots, U_k$ of $\upsilon$. Now if $x \notin U \in \upsilon$, then $B(x, \frac{1}{2}\delta) \cap GU = \Phi$ so $B(x, \frac{1}{2}\delta)$ intersects at most a finite number of members $(GU_1, GU_2, \ldots GU_k)$ of $\Theta$.

<u>Theorem 4.7</u> (Smith, Smith and Nichols)

Let $(X, \ell)$ be a metric space. Then the following conditions are equivalent:-

(i) $d_7(X, \ell) \leq n$.

(ii) Every locally finite Lebesgue cover of $(X,\ell)$ has an open refinement of order $\leq n$.

(iii) Every point finite Lebesgue cover of $(X,\ell)$ has an open refinement of order $\leq n$.

(iv) If $(C\alpha, C'\alpha)\ \alpha\varepsilon\Delta$ is a collection of pairs of closed sets of $(X,\ell)$ s.t. $\ell(C\alpha, C'\alpha) > \delta > 0\ \forall\ \alpha\varepsilon\Delta$ for some $\delta$ and $\{X-C'\alpha, \alpha\varepsilon\Delta\}$ is point finite then $\exists$ closed sets $B\alpha, \alpha\varepsilon\Delta$ s.t. $B\alpha$ separates $C\alpha$ and $C'\alpha$ $\forall\ \alpha\varepsilon\Delta$ and ord $\{B\alpha, \alpha\varepsilon\Delta\} \leq n-1$

Proof: We prove (i) => (ii) => (iii) => (iv) => (i).

(i) => (ii).

Assume $d_7(X,\ell) \leq n$. Let $\Theta$ be a l.f. Lebesgue cover of $(X, \ell)$. From corollary 4.1, we may assume $\Theta$ to be open. From theorem 4.1, $\Theta$ has a closed refinement $\{F_G, G\ \varepsilon\Theta\}$ s.t. $\ell(FG, X-G) > \delta > 0\ \forall\ G\ \varepsilon\Theta$ for some $\delta$. Since $d_7(X, \ell) \leq n$, $\exists$ open sets $U_G$, $G\ \varepsilon\Theta$ s.t. $F_G \subset U_G \subset \overline{U}_G \subset G\ \forall\ G\ \varepsilon\Theta$ and ord $\{\text{bdry } U_G, G\ \varepsilon\Theta\} \leq n-1$. From lemma 1.3, $\Theta$ has an open refinement of order $\leq n$.

(ii) => (iii).

This is obvious from lemma 4.7

(iii) => (iv).

Let $(C\alpha, C'\alpha)_{\alpha\varepsilon\Delta}$ and $\delta$ be as in (iv).

$\{X-C\alpha, X-C'\alpha\}$ is a Lebesgue cover of $(X,\ell)$ with Lebesgue number $\delta$ $\forall$ $\alpha\varepsilon\Delta$. From lemma 4.1, $\Theta = \Lambda_{\alpha\varepsilon\Delta} \{X-C\alpha, X-C'\alpha\}$ is a point finite Lebesgue cover of $(X, \ell)$. From (iii) $\Theta$ has an open refinement of order $\leq n$. From lemma 4.2, $\exists$ closed sets $B\alpha$, $\alpha \varepsilon \Delta$ satisfying the condition in (iv).

(iv) => (i)

This is obvious.

<u>Theorem 4.8</u> (Smith, 1970)

Let $(X, \ell)$ be a metric space.

Then the following conditions are equivalent

(i) $d_5 (X, \ell) \leq n$

(ii) If $(C\alpha, C'\alpha)_{\alpha\varepsilon\Delta}$ is a collection of pairs of closed sets of $(X,\ell)$ s.t. $\ell(C\alpha, C'\alpha) > \delta > 0$ $\forall$ $\alpha\varepsilon\Delta$ for some $\delta$ and $\{X-C'\alpha, \alpha\varepsilon\Delta\}$ is star countable, then $\exists$ closed sets $B\alpha, \alpha\varepsilon\Delta$ s.t. $B\alpha$ separates $C\alpha$ and $C'\alpha$ $\forall\alpha\varepsilon\Delta$ and ord $\{B\alpha, \alpha\varepsilon\Delta\} \leq n-1$.

Proof:

(i) => (ii)

Assume $d_5(X,\ell) \leq n$. Let $(C\alpha, C'\alpha)_{\alpha\varepsilon\Delta}$ and $\delta$ be as in (ii). From lemma 4.5, $\exists$ a partition $\{\Delta\beta, \beta\varepsilon\chi\}$ of $\Delta$ s.t. each $\Delta\beta$ is countable and if $X\beta = \bigcup_{\alpha\varepsilon\Delta\beta} (X-C'\alpha)$ then $X\beta \cap X\beta' = \Phi$ if $\beta \neq \beta'$, $\beta, \beta' \varepsilon\chi$.

Since $\Delta\beta$ is countable and $d_5(X, \ell) \leq n$, $\exists$, for each $\beta\varepsilon\chi$, closed sets $B\alpha$, $\alpha\varepsilon\Delta\beta$ s.t. $B\alpha$ separates $C\alpha$ and $C'\alpha$ for each $\alpha\varepsilon\Delta\beta$ and ord $\{B\alpha, \alpha\varepsilon\Delta\beta\} \leq n-1$.

Then $\bigcup_{\beta\varepsilon\chi} \{B\alpha, \alpha\varepsilon\Delta\beta\} = \{B\alpha, \alpha\varepsilon\Delta\}$ is a collection of closed sets of $(X, \ell)$ s.t. $B\alpha$ separates $C\alpha$ and $C'\alpha$ $\forall\alpha\varepsilon\Delta$ and ord $\{B\alpha, \alpha\varepsilon\Delta\} \leq n-1$. To see that ord $\{B\alpha, \alpha\varepsilon\Delta\} \leq n-1$, we note that if $B\alpha$ separates $C\alpha$ and $C'\alpha$, then $B\alpha \subset X-C'\alpha \subset X\beta$ if $\alpha\varepsilon\Delta\beta, \beta\varepsilon\chi$.

So if $\beta\neq\beta'$, $\beta, \beta'\varepsilon\chi$, then for $\alpha\varepsilon\Delta\beta$ and $\lambda\varepsilon\Delta\beta'$, $B\alpha \cap B\lambda = \Phi$ (because $X\beta \cap X\beta' = \Phi$). This, together with the fact that ord $\{B\alpha, \alpha\varepsilon\Delta\beta\} \leq n-1$ implies that ord $\{B\alpha, \alpha\varepsilon\Delta\} \leq n-1$.

Thus (ii) holds.

(ii) => (i).

This is obvious.

Theorem 4.9 (Egorov)

For a metric space $(X, \ell)$, $\mu$-dim $(X, \ell) \leq n$ iff every Lebesgue cover of $(X, \ell)$ has an open refinement of order $\leq n$.

Proof: The proof is immediate.

From theorem 4.7, the following is now obvious:-

Theorem 4.10

For any metric space $(X, \ell)$, $d_7(X, \ell) \leq \mu$-dim $(X, \ell)$

This justifies the claim in remark 1.1.

We now use the Lebesgue cover characterizations derived so far to prove a weak sum theorem for $d_2$, $d_3$, $d_6$, $d_7$ and $\mu$-dim.

Defn 4.6

Let $\Theta$ be an open cover of a topological space X. Then $\Theta$-dim(X) is the smallest integer n s.t. $\exists$ an open refinement of $\Theta$ of order $\leq$ n. $\Theta$-dim (X) = $\infty$ if no such integer exists. If $Y \subset X$, then $\Theta$-dim Y = $\Theta|Y$-dim Y by definition.

Theorem 4.11 (Morita)

Let X be a normal topological space, $\{U\alpha, \alpha \in \Delta\}$ a l.f. open collection and $\{F\alpha, \alpha \in \Delta\}$ a closed collection s.t. $F\alpha \subset U\alpha \ \forall \alpha \in \Delta$. Let $\Theta$ be any l.f. open cover of X s.t. $\Theta$-dim $(F\alpha) \leq n \ \forall \alpha \in \Delta$. If dim $F\alpha \cap F\beta \leq n-1$ for $\alpha \neq \beta$, then $\Theta$-dim $\bigcup_{\alpha \in \Delta} F\alpha \leq n$.

The proof of this theorem can be found in Morita.

We generalize theorem 4.11 to the following:-

Theorem 4.12 (Smith, 1970).

Let X be a normal topological space, $\{U\alpha, \alpha \in \Delta\}$ a l.f. open collection and $\{F\alpha, \alpha \in \Delta\}$ a closed collection s.t. $F\alpha \subset U\alpha \ \forall \alpha \in \Delta$. Let $\Theta$ be any l.f. open cover of X s.t. $\Theta$-dim $(F\alpha) \leq n \ \forall \alpha \in \Delta$. If dim bdry $(F\alpha) \cap F\beta \leq n-1$ for $\alpha \neq \beta$, then $\Theta$-dim $\bigcup_{\alpha \in \Delta} F\alpha < \underline{n}$.

Proof: Let $\Theta$, $U\alpha$, $F\alpha$, $\alpha\varepsilon\Delta$ be as above. Let $S(\Delta)$ be the collection of all finite non-empty subsets of $\Delta$. For each $A \varepsilon S(\Delta)$, let $H_A = \bigcap_{\alpha\varepsilon A} F\alpha - \bigcup_{\substack{\alpha\varepsilon\Delta \\ \alpha\notin A}} \text{int } F\alpha$.

Let $V_A = \bigcap_{\alpha\varepsilon A} U\alpha$. Then $\Pi = \{H_A, A\varepsilon S(\Delta)\}$ and $\mathcal{V} = \{V_A, A \varepsilon S(\Delta)\}$ are locally finite, $\mathcal{V}$ is open, $\Pi$ is closed, and $H_A \subset V_A$ $\forall$ $A\varepsilon S(\Delta)$.

Claim (i) $\bigcup_{A\varepsilon S(\Delta)} H_A = \bigcup_{\alpha\varepsilon\Delta} F\alpha$.

(ii) If $A, A'\varepsilon S(\Delta)$ $A \neq A'$ then

$H_A \cap H_{A'} \subset \text{bdry}(F\alpha) \cap F\alpha'$ for some $\alpha, \alpha'$.

To see (i), suppose $x \varepsilon \bigcup_{\alpha\varepsilon\Delta} F\alpha$. Since $\{F\alpha, \alpha\varepsilon\Delta\}$ is l.f. $x$ is contained in only finitely many $F_\alpha$, so $\{\alpha\ \varepsilon\Delta: x\varepsilon F\alpha\} = A_o \varepsilon S(\Delta)$.

Then $x\varepsilon H_{A_o}$. To see (ii), suppose $A, A' \varepsilon S(\Delta)$ with $A\neq A'$. Either $A-A' \neq \Phi$ or $A'-A \neq \Phi$. W.L.G. assume $A-A' \neq \Phi$. Let $\alpha\varepsilon$ $A-A'$. Since $A' \neq \Phi$, $\exists$ $\beta$ $\varepsilon$ $A'$. Then $H_A \cap H_{A'} \subset (F\alpha - \text{int } F\beta) \cap F\beta \subset F\alpha \cap \text{bdry } F\beta$. Since $\Pi$ is a closed collection, we have $\dim H_A \cap H_{A'} \leq n-1$ if $A\neq A'$, $A, A' \varepsilon S(\Delta)$. Since $\Pi$ refines $\{F\alpha, \alpha\varepsilon\Delta\}$, $\Theta$-dim $(H_A) \leq n$ $\forall$ $A \varepsilon S(\Delta)$. We now apply theorem 4.11 to conclude that $\Theta$-dim $\bigcup_{\alpha\varepsilon\Delta} F\alpha = \Theta$-dim $\bigcup_{A\varepsilon S(\Delta)} H_A \leq n$.

<u>Theorem 4.13</u> (Smith, 1970)

Let $\{F\alpha, \alpha\ \varepsilon\Delta\}$ be a l.f. closed cover of a metric space

$(X,\mathcal{C})$ s.t. if $\alpha \neq \beta$ then dim bdry $(F_\alpha) \cap F_\beta \leq n-1$.
If $d(F_\alpha) \leq n$ $\forall \alpha \in \mathcal{A}$ where d is μ-dim, $d_7$, $d_6$, $d_3$ or $d_2$, then $d(X,\mathcal{C}) \leq n$.

Proof: Assume μ-dim $F_\alpha \leq n$ $\forall \alpha \in \mathcal{A}$ (respectively $d_7$, $d_6$, $d_3$ and $d_2$). Let $\mathcal{G}$ be a Lebesgue cover of $(X,\mathcal{C})$ (respectively l.f. Lebesgue cover, countable Lebesgue cover, finite Lebesgue cover or a Lebesgue cover with n+2 members). For each $\alpha \in \mathcal{A}$, $\mathcal{G}|F_\alpha$ is a Lebesgue cover of $F_\alpha$ (resp. l.f. Lebesgue cover, countable Lebesgue cover, finite Lebesgue cover or Lebesgue cover with n+2 members). Since μ-dim $F_\alpha \leq n$ (resp. $d_7(F_\alpha) \leq n$, $d_6(F_\alpha) \leq n$, $d_3(F_\alpha) \leq n$, $d_2(F_\alpha) \leq n$) $\mathcal{G}|F_\alpha$ has an open (in $F_\alpha$) l.f. (in $F_\alpha$) refinement $\mathcal{U}_\alpha$ s.t. ord $\mathcal{U}_\alpha \leq n$. Since $F_\alpha$ is normal, $\mathcal{U}_\alpha$ has a closed l.f. (in $F_\alpha$) refinement $\xi_\alpha$ s.t. ord $\xi_\alpha \leq n$. $\xi_\alpha$ is also l.f. in X. Furthermore, since $\{F_\alpha\}$ is l.f. and $E \in \xi_\alpha \Rightarrow E \subset F_\alpha$, $\xi = \bigcup_{\alpha \in \mathcal{A}} \xi_\alpha$ is l.f. in X. Also $\xi$ refines $\mathcal{G}$ . By lemma 1.2 $\exists$ an open l.f. refinement $\mathcal{V} = \{V_E, E \in \xi\}$ of $\mathcal{G}$ s.t. $E \subset V_E$ for each $E \in \xi$ and ord $\{V_E, E \in \xi_\alpha\} \leq n$. Clearly $\mathcal{V}$-dim $F_\alpha \leq n$ $\forall \alpha \in \mathcal{A}$. By theorem 4.12, $\mathcal{V}$-dim X $\leq n$. So $\mathcal{V}$ and therefore $\mathcal{G}$, has an open refinement of order $\leq n$. Thus μ-dim $(X,\mathcal{C}) \leq n$ (resp. $d_7(X,\mathcal{C}) \leq n$, $d_6(X,\mathcal{C}) \leq n$, $d_3(X,\mathcal{C}) \leq n$ and $d_2(X,\mathcal{C}) \leq n$).

Remark

It might be speculated that the various metric-dependent dimension functions satisfy other sum theorems e.g. the countable sum theorem (theorem 0.7), a monotone sum theorem (i.e. if $F_i$, $i \in N$ is an increasing sequence of closed sets s.t. $X = \bigcup_{i=1}^{\infty} F_i$ and $d(F_i) \leq n$ then $d(X) \leq n$), or a finite sum theorem.

J.C. Nichols and J.C. Smith have shown (Nichols and Smith) that none of the metric-dependent dimensions functions discussed above satisfy any of the sum theorems mentioned. They construct a metric space $(X,\ell)$ s.t. $X=A_1 \cup A_2$, $A_1$, $A_2$ closed in X with μ-dim $A_1 \leq 1$, μ-dim $A_2 \leq 1$ but $d_2(X,\ell) \geq 2$. This shows that none of the metric-dependent dimension functions satisfies the countable sum theorem or finite sum theorem. They also give an example of a metric space $(X,\ell)$ s.t. $X = \bigcup_{i \in N} A_i$, where each $A_i$ is closed, $A_i \subset A_{i+1}$, and μ-dim $A_i \leq 1$ for each i but $d_2(X,\ell) \geq 2$. This shows that none of the metric dependent dimension functions satisfies the monotone sum theorem.

## CONCLUDING REMARKS

Much of the current research in dimension theory involves the dimension theory of uniform spaces. A uniform space is a generalization of a metric space.

Of several possible definitions of a uniform space, we give only one.

Defn

Let X be a set. Let $\Delta$ denote the subset $\{(x, x), x \varepsilon X\}$ of XxX. If U, V are subsets of XxX, let $U \circ V$ denote the set

$\{(x,y) \varepsilon XxX$: for some $z \varepsilon X$, $(x, z) \varepsilon V$ and $(z,y) \varepsilon U\}$.

A diagonal uniformity on X is a collection $\Gamma(X)$ (or just $\Gamma$), of subsets of XxX, called surroundings, which satisfy:-

(a) $D \varepsilon \Gamma \Rightarrow \Delta \subset D$

(b) $D_1, D_2 \varepsilon \Gamma \Rightarrow D_1 \cap D_2 \varepsilon \Gamma$

(c) $D \varepsilon \Gamma \Rightarrow E \circ E \subset D$ for some $E \varepsilon \Gamma$

(d) $D \varepsilon \Gamma \Rightarrow E^{-1} \subset D$ for some $E \varepsilon \Gamma$ ($E^{-1}$ is the set $\{(y, x), (x, y) \varepsilon E\}$.)

(e) $D \varepsilon \Gamma$, $D \subset E \Rightarrow E \varepsilon \Gamma$

A uniform space $(X, \Gamma)$ is a set X together with a diagonal uniformity $\Gamma$ on X. A diagonal uniformity on X gives rise to a topology on X as follows. For $x \varepsilon X$ and $D \varepsilon \Gamma$, let $B(x,D) = \{y \varepsilon X: (x,y) \varepsilon D\}$

Then the collection $\{B(x, D), x\varepsilon X, D\varepsilon\Gamma\}$ is a base for a topology on X.

Any metric $\ell$ on X generates a diagonal uniformity $\{D_\varepsilon, \varepsilon > 0\}$ where $D\varepsilon = \{(x, y)\varepsilon X x X: \ell(x,y) < \varepsilon\}$. We therefore see that a uniform space is a generalization of a metric space. The condition that $\ell(x, y) < \varepsilon$ in a metric space is replaced by the condition that $(x,y) \varepsilon D$, $D \varepsilon \Gamma$ in a uniform space. Therefore the notion of two subsets being a positive distance apart or <u>distant</u> is meaningful in a uniform space. We say two subsets C,C' of a uniform space $(X, \Gamma)$ are <u>distant</u> if for some $D\varepsilon \Gamma$. $CxC' \cap D = \phi$. A collection $C\alpha$, $C'\alpha)$ $\alpha \; \varepsilon \upsilon$ of pairs of subsets of $(X, \Gamma)$ are <u>uniformly distant</u> if $\exists \; D\varepsilon \; \Gamma$ s.t.

$C\alpha \times C'\alpha \cap D = \phi \; \forall \; \alpha\varepsilon\upsilon$. We see therefore that all the metric-depedent dimension functions discussed above may be generalized to uniform spaces. For these generalizations, Soniat (Soniat) has obtained Lebesque - cover type characterizations for $\mu$-dim and $d_3$ while Smith (Smith ) has obtained Lebesque-cover type characterizations for $d_2$, $d_6$, and $d_7$. These dimension functions defined on uniform spaces fail to satisfy the equality $d_4$ = dim or the inequality dim $\leq 2d_2$ satisfied by metric-dependent dimension functions. Charalambous (Charalambous) has introduced dimension functions $\Gamma$-dim, $\Gamma$-Ind, $\Gamma$ $-d_1$, $\Gamma$ $-d_2$, $\Gamma$ $-d_3$, and $\Gamma$-$d_4$ for a uniform space $(X, \Gamma)$ which satisfy $\Gamma\text{-}d_1 \leq \Gamma\text{-}d_3 \leq \Gamma\text{-}d_4 = \Gamma\text{-dim} \leq 2\; \Gamma\text{-}d_2$ and $\Gamma\text{-}d_1 =$ $\Gamma$-Ind and $\Gamma$-dim further satisfies the countable sum theorem, a subset theorem, the Urysohn inequality and a product theorem. It agrees with dim on Lindelöf spaces and spaces with uniformity derivable from a metric.

There exist open problems in the theory of metric (uniformity) dependent dimension functions. Notably, is $d_3(X,\ell) = \mu\text{-dim}\,(X,\ell)$ for any (separable) metric space $(X,\ell)$?

More generally, which of the dimension functions $d_3$, $d_5$, $d_6$, $d_7$ and $\mu$-dim are equal and under what conditions?

Which subset theorems are satisfied by $d_6$ and $d_7$? Do $d_6$ and $d_7$ satisfy the realization theorem? (see theorem 3.2).

The notion of dimension is quite fundamental and of great intrinsic interest. Apart from that, dimension theory is a subject that could intersect with other areas of mathematics. Already, a strong relationship has been found between dimension and measure for metric spaces. (Hureuicz and Wallman, Chapter VII).

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