University of Nairobi

School of Mathematics

ISSN: 2410-1397

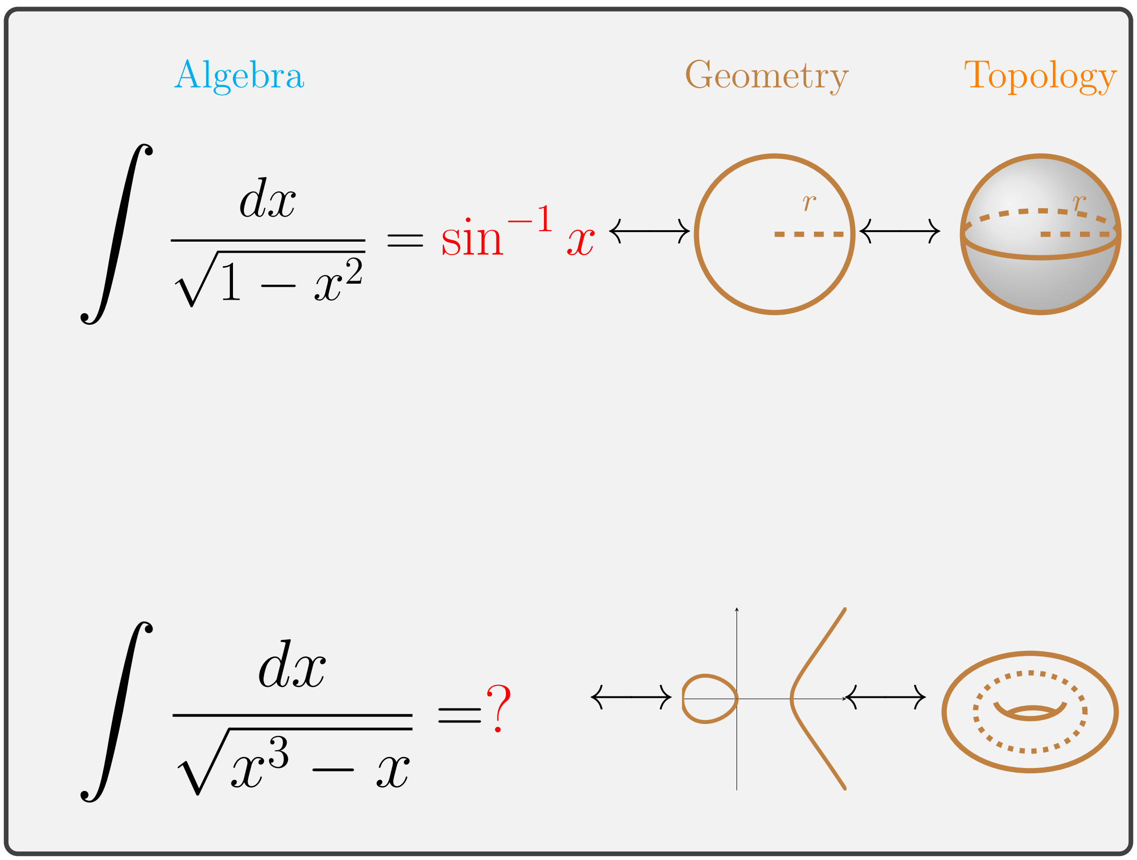
Dissertation in Pure Mathematics

Geometry of Elliptic Curves and Elliptic Integrals

Research Report in Mathematics, Number 03, 2019

Lawrence Muthama Paul

June 2019



Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

Dissertation in Pure Mathematics **University of Nairobi**
June 2019

Geometry of Elliptic Curves and Elliptic Integrals

Research Report in Mathematics, Number 03, 2019

Lawrence Muthama Paul

School of Mathematics
College of Biological and Physical sciences
Chiromo, off Riverside Drive
30197-00100 Nairobi, Kenya

Master Thesis

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

In this project, we compute the period $\pi(\lambda) = \oint_{\gamma_\lambda} \omega_\lambda$ of the Landau-Ginzburg [LG] model $(E_\lambda, \omega_\lambda, \gamma_\lambda)$ consisting of one parameter family of nonsingular cubics E_λ, algebraic $n-$forms ω_λ on E_λ and cycles $\gamma_\lambda \in H_n(E_\lambda, \mathbb{Z})$. This is a mirror symmetry toy model extension in [CG19] for computing mirror pairs of elliptic curves E_λ.

Master Thesis in Mathematics at the University of Nairobi, Kenya.
ISSN 2410-1397: Research Report in Mathematics
©Lawrence Muthama Paul, 2019
DISTRIBUTOR: School of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

LAWRENCE MUTHAMA PAUL
Reg No. I56/8480/2017

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

Dr Jared ONGARO
School of Mathematics,
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: ongaro@uonbi.ac.ke

Signature Date

Dr Benjamin KIKWAI
School of Mathematics
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.

Dedication

This project is dedicated to my dear lovely parents and siblings.

Contents

Acknowledgments

I would like to recognise each person whose support and contribution was crucial in putting up this dissertation. I express my gratitude to my supervisor Jared Ongaro for the useful comments, remarks and engagement throughout the learning process of this master thesis. His infectious enthusiasm for clarity in communicating Mathematics has been extremely inspirational.

Many thanks to the Director University of Nairobi School of Mathematics, Patrick Weke. To my lecturers, I will ever be indebted to all of you for moulding me into the Mathematician I am today. I am especially grateful to Mwalimu Achola; he planted in me the attitude of discovering Mathematics on my own . To James Katende, I can't say enough thank you for the many kind advises.

Am thankful to the University of Nairobi for awarding me the Masters Scholarship to pursue this degree. Many thanks to EAUMP for giving me opportunity to attend the ICTP-EAUMP Summer Schools in 2018 and 2019. The Annual University of Nairobi Algebraic Geometry Workshop is the fountain where I drank my first drop of Algebraic Geometry and is the source of this project. Thanks to the organisers Damian Maingi and my supervisor Jared Ongaro.

To Alessio Corti, your correspondence was extremely useful, thank you. To my classmates Stephen Mathenge, Lydia Atuya, Joseph Matheka, Lewis Kivuva, Julius Muigai and Flora Mwikali; our many discussions and your kind and healthy suggestions improved this dissertation. I must express my very profound gratitude to Geoffrey Mboya, you persistently read my work and advised me accordingly despite your busy schedule.

To my mum Catherine, dad Paul , brother Daniel, sisters Pascaline, Rose and Angeline; thank you for your prayers and support. You have been so understanding.

Lawrence Muthama Paul

Nairobi, 2019.

1 Introduction

1.1 Why Elliptic Curves?

The Integrals of the form

$$\int \frac{\text{polynomial}}{\sqrt{\text{cubic with three distinct roots}}} dx$$

began to show up in many different applications (like the true description of a pendulum's path, or a $3-$ dimensional random walk). Because one of these integrals arose from the ellipse arclength problem, they were dubbed the name elliptic integrals. And of course, these denominators motivated people to study the underlying algebraic curve in

$$y = \sqrt{\text{cubic with three distinct roots}}.$$

Namely, if we square both sides, and we get elliptic curves.

1.2 Outline of the thesis

The outline of the thesis is as follows:

Chapter 2:
We use this chapter to introduce a complex projective space which is the space we shall work on in this thesis. We shall also define a non singular cubic curve and wrap the chapter by studying at intersection multiplicity and Bezout theorem.
Chapter 3:
Here, we introduce Riemann surfaces and specifically show the construction of a torus as a Riemann surface. We also discuss non singular cubic curves and more specifically the group law on cubic curves and then show the correspondence between non singular cubic curves and the torus.
Chapter 4:
In this chapter, we shall demonstrate how understanding the geometry of cubic curves can be useful in solving elliptic integrals of the first kind. In particular we discuss the addition law on cubics. Finally, we reach the depth of discussion here by presenting a solution to the picard - Fuchs equation.

2 Preliminaries

This chapter is intended to provide a basic background to the content of this project as well as serve to fix notations. Throughout, we work over the field $\mathbb{C}$ of complex numbers. The content follows [Kir92] , [Mir95],[ST92] and [Rei13].

2.1 General Background

Elliptic integrals arise naturally in computation of arc lengths of ellipses. In this project we re-interpret them as functions on complex projective curves.

2.2 The Complex projective space

Definition 2.2.1. *Consider the vector space W over field $\mathbb{C}$, then the collection of all subspaces of W of one dimension is called complex projective space over W denoted by $\mathbb{P}(W)$. If $W = \mathbb{C}^{k+1}$ write $\mathbb{CP}^k := \mathbb{P}(\mathbb{C}^{k+1})$ or simply by $\mathbb{P}^k$. We call $\mathbb{P}^1$ the complex projective line or the Riemann sphere and $\mathbb{P}^2$ the complex projective plane.*

Choose the coordinates $y \in \mathbb{P}^k$ then $y \neq 0$ and $y = (y_0, \ldots, y_k) \in \mathbb{C}^{k+1}$ as it represents a line through the origin. Moreover, for any nonzero α, $(y_0, \ldots, y_k)$ and $(\alpha y_0, \ldots, \alpha y_k)$ represents the same line in $\mathbb{C}^{k+1}$. Observe that this implies $(y_0, \ldots, y_k) = (\alpha y_0, \ldots, \alpha y_k)$ are the same points in $\mathbb{P}^k$.

Definition 2.2.2. *$(y_0, \ldots, y_k)$ are called the homogenous coordinates of y and we write*

$$y = [y_0, \ldots, y_k]$$

Thus

$$\mathbb{P}^n := \left\{ [y_0, \ldots, y_k] \,\middle|\, (y_0, \ldots, y_k) \in \mathbb{C}^{k+1} \setminus \{0\} \right\} \Big/ \sim \, .$$

where $[x_0, \ldots, x_k] \sim [y_0, \ldots, y_k]$ only when for some $\alpha \in \mathbb{C} \setminus \{0\}$ we have $x_s = \alpha y_s$ for every $s = 0, 1, \ldots k$.

Define a surjection $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ by $\pi(y_0, \ldots, y_k) = [y_0, \ldots, y_k]$ and induce a quotient topology on $\mathbb{P}^k$ from topology on $\mathbb{C}^{k+1} \setminus \{0\}$ hence making $\mathbb{P}^k$ a topological space. Namely, $U \subset \mathbb{P}^k$ is open exactly when $\pi^{-1}(U) \subset \mathbb{C}^{k+1} \setminus \{0\}$ is open.

Proposition 2.2.3. *The complex projective space $\mathbb{P}^k$ of dimension k is compact.*

Proof. Let $\mathbb{S}^{2k+1} = \left\{(x_0,\ldots,x_k) \in \mathbb{C}^{k+1} : \sum_{i=0}^{n} |x_i|^2 = 1\right\}$.

Then $\mathbb{S}^{2k+1}$ is a $2k+1-$dimensional sphere. In particular, $\mathbb{S}^{2k+1} \subset \mathbb{C}^{k+1}$ is both closed and bounded thus compact. The map $\pi : \mathbb{S}^{2k+1} \to \mathbb{P}^k$ given by $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is continuous, and so its image is compact. Since $\mathbb{S}^{2k+1}$ is compact, its image must be compact. Now, if $[y_0,\ldots,y_k] \in \mathbb{P}^k$ then

$$\alpha = |y_0|^2 + \ldots + |y_k|^2 > 0.$$

So

$$[y_0,\ldots,y_k] = [\alpha^{-\frac{1}{2}}y_0,\ldots,\alpha^{-\frac{1}{2}}y_k].$$

But

$$|\alpha^{-\frac{1}{2}}y_0|^2 + \ldots + |\alpha^{-\frac{1}{2}}y_k|^2 = 1.$$

So

$$[y_0,\ldots,y_k] \in \pi(\mathbb{S}^{2k+1}).$$

Thus $\pi : \mathbb{S}^{2k+1} \to \mathbb{P}^k$ is onto.

□

Proposition 2.2.4. *The $k-$ dimensional space $\mathbb{P}^k$ is Hausdorff.*

Proof. Our argument is as follows: if l and m are unique points of $\mathbb{P}^k$, then there are l and m disjoint open neighborhoods.

Let L_0 be an open set of $\mathbb{P}^k$ then $\psi : L_o \to \mathbb{C}^k$ is a homeomorphism.

Assume that l and m are inside L_0 thus M, N are neighborhoods of $\psi_0(l)$, $\psi_0(m)$ respectively such that $M \cap N = \varnothing$ and $\psi_0^{-1}(M)$, $\psi_0^{-1}(N)$ such that $\psi_0^{-1}(M) \cap \psi_0^{-1}(N) = \varnothing$ are neighborhoods of l and m in $\mathbb{P}^k$ respectively.

Specifically this is true for $l = [1,0,\ldots,0]$, $m = [1,1,\ldots,1]$. Generally there are points $l_0,\ldots,l_k$ of $\mathbb{P}^k$ with $l_0 = l$ and no $k+1$ of the $k+2$ points $l_0,\ldots,l_k$ and m is inside a hyperplane. Moreover $g : \mathbb{P}^k \to \mathbb{P}^k$ is a transformation taking l to $[1,\ldots,0]$, m to $[1,1,\ldots,1]$.

Hence $[1,\ldots,0]$ and $[1,1,\ldots,1]$ have neighborhoods $\psi_0^{-1}(M)$ and $\psi_0^{-1}(N)$ in $\mathbb{P}^k$ such that $\psi_0^{-1}(M) \cap \psi_0^{-1}(N) = \varnothing$.

With g both continuous and bijection then $g^{-1}(\psi_0^{-1}(M))$, $g^{-1}(\psi_0^{-1}(N))$ are neighborhoods of l, m respectively in $\mathbb{P}^k$. □

2.3 Irreducibility and Singularity of algebraic curves in $\mathbb{P}^2$

Definition 2.3.1. *Let $l(m,n) \in \mathbb{C}[\mathbb{A}^2]$ be a degree d inhomogeneous polynomial. A **affine algebraic curve** D in $\mathbb{C}^2$ is the zero set of l*

$$D := \{(m,n) \in \mathbb{C}^2 : l(m,n) = 0\} = \mathbb{V}(l) \subset \mathbb{C}^2.$$

Definition 2.3.2. *Let $L(m,n,r) = \widetilde{l} \in \mathbb{C}[\mathbb{A}^3]$ be a degree d homogeneous polynomial. A **projective algebraic curve** D in $\mathbb{P}^2$ is the zero set of L*

$$\widetilde{D} := \{[m,n,r] \in \mathbb{P}^2 : L([m,n,r]) = 0\} = \mathbb{V}(\widetilde{l}) \subset \mathbb{P}^2.$$

We then say **affine algebraic curve** D in $\mathbb{C}^2$ has been **compactified** into a **projective algebraic curve** $\widetilde{D}$ in $\mathbb{P}^2$.

We can always look at affine pieces $\widetilde{D}_i = \widetilde{D} \cap U_i$ of $\widetilde{D} \subset \mathbb{P}^2$ by dehomogenizing L at $x_i = 1$. The Zariski topology on $\mathbb{P}^2$ can then be used to say all about $\widetilde{D}$.

Theorem 2.3.3 (Hilberts Nullstellensatz, NSS)**.** *Take $k = \mathbb{C}$ or any algebraically closed field and $P, Q \in \mathbb{C}[\mathbb{A}^k]$ homogeneous polynomials, not necessarily of the same degree. Then*

$$D := \mathbb{V}(P) = \mathbb{V}(Q) =: D' \subset \mathbb{P}^k$$

precisely when $P|Q^m$ and $Q|P^n$ for some $m,n > 0$. So in $\mathbb{P}^k$, algebraic curves $D = D'$ exactly when their defining polynomials have equal factors which are also irreducible, but maybe multiplicities not the same.

Definition 2.3.4. *An algebraic curve $\widetilde{D} := \{[m,n,r] \in \mathbb{P}^2 : L([m,n,r]) = 0\} \subset \mathbb{P}^2$ is said to be **irreducible** if L has factors which are not repeated. We then say, $\deg(\widetilde{D}) = \deg(L) = d$ and write $\widetilde{D}_d$. By NSS, for irreducible algebraic curve we have $\mathbb{V}(L) = \mathbb{V}(\mu\alpha L)$ for $\alpha \in \mathbb{C} \setminus \{0\}$ and $\mu \in \{\pm 1, \pm i\}$ or units of $k = \overline{k}$. Further, every algebraic curve can be decomposed into finitely many irreducible component, $\widetilde{D} = E_1 \cup \ldots \cup E_k$, where $E_i = \mathbb{V}(L_i)$ are irreducible algebraic curves and $L = \mu\alpha \prod_{i=1}^k L_i^{m_i}$.*

Definition 2.3.5. *Let $D := \{(m,n) \in \mathbb{C}^2 : l(m,n) = 0\} \subset \mathbb{C}^2$ an affine algebraic curve. Then $(r,s) \in \mathbb{C}^2$ is said to be **singular** if $r \in \mathrm{Sing}(D) := \mathbb{V}\left(l, \frac{\partial l}{\partial m}, \frac{\partial l}{\partial n}\right)$. Whenever $\varnothing = \mathrm{Sing}(D) \subset \mathbb{C}^2$ we say D is **nonsingular affine curve**.*

*For $\widetilde{D} := \{[m,n,r] \in \mathbb{CP}^2 : L([m,n,r]) = 0\} \subset \mathbb{P}^2$ projective algebraic curve. A point $[t,v,w] \in \widetilde{D} \in \mathbb{P}^2$ is said to be **singular** if*

$$t\frac{\partial L}{\partial m}([t,v,w]) + v\frac{\partial L}{\partial n}([t,v,w]) + w\frac{\partial L}{\partial r}([t,v,w]) = L([a,b,c]) = 0.$$

*Whenever $\varnothing = Sing(\widetilde{D}) := \mathbb{V}\left(m\frac{\partial L}{\partial m}([m,n,r]) + n\frac{\partial L}{\partial n}([m,n,r]) + r\frac{\partial L}{\partial r}([m,n,r])\right) \subset \mathbb{P}^2$ we say $\widetilde{D}$ is **nonsingular projective curve**.*

Lemma 2.3.6. *A projective curve $C = \{[m,n,r] \in \mathbb{P}^2 : P(m,n,r) = 0\} \subset \mathbb{P}^2$ is compact and Hausdorff.*

Proof. To show the compactness of C is enough to show that C is a closed subset of $\mathbb{P}^2$. We employ the fact that $\mathbb{P}^2$ is compact and a closed subset of a compact space is compact. Moreover a subset A of $\mathbb{P}^k$ is closed only when $\pi^{-1}(A) \subset \mathbb{C}^{k+1} \setminus \{0\}$ is closed and $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is given by

$$\pi(y_0, \dots, y_k) = [y_0, \dots, y_k].$$

Hence $\pi^{-1}(C) = \{(m,n,r) \in \mathbb{C}^3 \setminus \{0\} : P(m,n,r) = 0\}$ is closed by the argument that polynomials are continuous. Hence C is compact.

Since $\mathbb{P}^k$ is Hausdorff and C is a subset of $\mathbb{P}^k$ we have that C is also Hausdorff by the fact that any subset of a Hausdorff space is Hausdorff. $\square$

Definition 2.3.7. ***Resultant*** *$R_{l,s} \in \mathbb{C}$ is the determinant of an $(l+s) \times (l+s)$ matrix obtained from the coefficients of polynomials*

$$\begin{aligned} l(x) =& c_m x^m + \ldots + c_1 x + c_0 \\ s(x) =& d_n x^n + \ldots + d_1 x + d_0. \end{aligned}$$

As follows

$$\begin{bmatrix} c_0 & c_1 & c_2 & \ldots & c_m & 0 & 0 & 0 & \ldots & 0 \\ 0 & c_0 & c_1 & c_2 & \ldots & c_m & 0 & 0 & \ldots & 0 \\ 0 & 0 & c_0 & c_1 & c_2 & \ldots & c_m & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & c_0 & c_1 & c_2 & \ldots & c_m \\ d_0 & d_1 & d_2 & \ldots & d_n & 0 & 0 & 0 & \ldots & 0 \\ 0 & b_0 & d_1 & d_2 & \ldots & d_n & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & d_0 & d_1 & d_2 & \ldots & d_n \end{bmatrix}.$$

And

$$R_{l,s} = \alpha^n \beta^m \prod_{(1,1)\le(i,j)\le(m,n)} (\mu_i - \alpha_j) \equiv 0$$

precisely when

$$\begin{aligned} l(x) &= \alpha(x-\alpha)\ldots(x-\alpha_m) \\ s(x) &= \beta(x-\mu_1)\ldots(x-\mu_n) \end{aligned}$$

have at least one common factor. You can also factorise resultant by factoring one of the polynomials as $R_{p,qr} = R_{p,q} \times R_{p,r}$ which is useful in computing intersection multiplicities of curves.

Remark 2.3.8. *For homogeneous $L,S \in \mathbb{C}[A^3]$ we can find resultant $R_{L,S}(y,z) \in \mathbb{C}[A^2]$ with respect to x by treating both of them as $L,S \in \mathbb{C}[y,z][x]$. Here $deg\big(a_i(y,z)\big) = m-i$ and $deg\big(b_j(y,z)\big) = n-j$.*

2.4 Intersection Multiplicity and Bëzouts Theorem

Definition 2.4.1. *Let $M := \mathbb{V}(P), N := \mathbb{V}(Q) \subset \mathbb{P}^2$ be two projective algebraic curves without a common component and $[t,v,w] = l \in M \cap N$ a nonsingular point. Then **intersection multiplicity** $I_{l=[t,v,w]}(M,N) = s$ of the two curves at l is the number of times they meet at $[t,v,w] = l$ and (calculated by brute force) sdefined as the highest s for which*

$$(cy-bz)^s | R_{P,Q}(y,z).$$

Or using 6,7 and 8 from Proposition below.

Proposition 2.4.2. *Let $M := \mathbb{V}(P), N := \mathbb{V}(Q) \subset \mathbb{CP}^2$ and $\mathfrak{l}_1, \mathfrak{l}_2 \subset \mathbb{CP}^2$ distinct lines, then*

1. $I_l(M,N) = I_l(N,M)$.
2. $I_l(M,N) = \infty$ *if M and N have a common component.*
3. $I_l(M,N) = 0$ *if $l \notin M \cap N$.*
4. $I_l(\mathfrak{l}_1, \mathfrak{l}_2) = 1$ *if $\mathfrak{l}_1 \cap \mathfrak{l}_2 = \{l\}$.*
5. $I_l(M, T_lM) > 1$ *if $l \in \mathrm{Sing}(M)$ where*

   $$T_lM := \left\{[x,y,z] \in M : (x-t)\frac{\partial P}{\partial x}(t,v,w) + (y-v)\frac{\partial P}{\partial y}(t,v,w) + (z-w)\frac{\partial P}{\partial z}(t,v,w) = 0\right\}$$

   is the tangent space to M at $l = [t,v,w]$.
6. $I_l((x-\alpha)^k, M) = k \times I_l(x-\alpha, M)$ *for $m \geq 1$.*
7. $I_l(M,N) = \sum_{i=1}^{n} I_l(M,N_i)$ *where $N_i := \mathbb{V}(Q_i)$ are the irreducible components of N.*
8. *Let $E := \mathbb{V}(PR+Q) \subset \mathbb{P}^2$ for some $R \in \mathbb{C}[A^3]$, then $I_l(M,N) = I_l(M,E)$.*

Definition 2.4.3. *Let $M_d := \mathbb{V}(K) \subset \mathbb{CP}^2$ be a nonsingular projective algebraic curve. The **Hessian** is defined as*

$$\mathscr{H}_l(x,y,z) = \det \begin{bmatrix} K_{rr} & P_{rs} & K_{rt} \\ K_{st} & K_{ss} & K_{st} \\ K_{tr} & K_{ts} & K_{tt} \end{bmatrix},$$

where $K_r = \frac{\partial K}{\partial r},\ K_{rs} = \frac{\partial^2 K}{\partial r \partial s}, etc.$*Also using Euler's relation*

$$rK_r + sK_s + tK_t = dK$$

and simple row and column operations on $\mathscr{H}_K(r,s,t)$, *we have that*

$$t\mathscr{H}_K(r,s,t) = (d-1)\det\begin{bmatrix} K_{rr} & K_{rs} & K_r \\ K_{rs} & K_{ss} & K_s \\ K_{tr} & K_{ts} & K_t \end{bmatrix}$$

and further that,

$$t^2\mathscr{H}_K(r,s,t) = (d-1)^2\det\begin{bmatrix} K_{rr} & K_{rs} & K_r \\ K_{sr} & K_{ss} & K_s \\ K_r & K_s & \frac{d}{d-1}K \end{bmatrix}.$$

Definition 2.4.4. *A nonsingular point* $l \in M_d := \mathbb{V}(P) \subset \mathbb{CP}^2$ *on a projective algebraic curve is called an* ***inflection point*** *or a shorthand* ***flex*** *if* $I_l(M, T_lM) \geq 3$. *A nonsingular point* $l = [t, v, w]$ *is a flex precise when* $\mathscr{H}_P(t, v, w) = 0$.

Theorem 2.4.5 (Bëzout Theorem, Weak form)**.** *Let* $M_m := \mathbb{V}(T), N_n := \mathbb{V}(U) \subset \mathbb{P}^2$ *be algebraic curves(projective) of degrees* m,n *respectively and without common factors. Then*

$$\sharp M \cap N \leq mn.$$

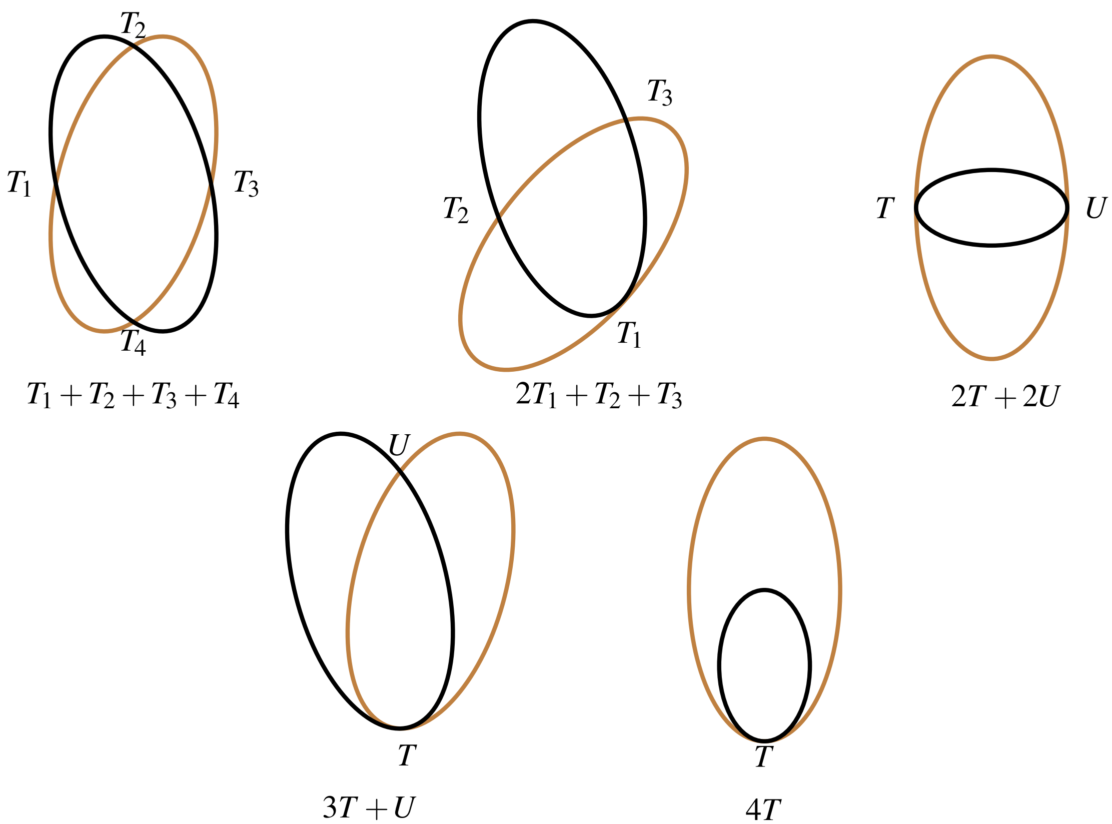


Figure 1. Five possible ways of intersection of two conics.

Proof. Proceed by contradiction. Idea: We pick $mn+1$ points $T_1, \ldots, T_{mn+1} \in M_m \cap N_n$ and proceed to show that if polynomials(homogeneous) $T(x,y,z), U(x,y,z)$ of degree m,n respectively defining two curves respectively have equal factor then M_m and N_n have a common component.

Choose $u \in \mathbb{CP}^2 \setminus M_m \cup N_n \cup_{1 \leq i < j \leq mn+1} \mathfrak{l}_{ij}$ where $\mathfrak{l}_{ij}$ is the line joining t_i and t_j for all i,j. Apply projective transformation so that $u = [1,0,0]$, hence $t_i = [a_i, b_i, c_i]$ are such that b_i and c_i are NOT both zeros by choice of u. We then have that $t([1,0,0]) \neq 0$ and $U([1,0,0]) \neq 0$. In particular, $a_m(y,z) \neq 0$ and $b_n(y,z) \neq 0$.

Now, $x - a_i | T([x,b_i,c_i]) \times U([x,b_i,c_i])$ for all $t_i \in C_m \cap D_n$. Hence from definition of resultant, we have that $R_{T,U}(b_i,c_i) = 0$ which implies that $c_i y - b_i z | R_{T,U}(y,z)$. Finally, since b_i and c_i are not both zeros and $t_i's$ distinct, $\left(c_i y - b_i z, c_j y - b_j z\right) = 1$ for all i,j. Therefore,

$$\prod_{i=1}^{mn+1} \left(c_i y - b_i z\right) \big| R_{T,U}(y,z).$$

But $\deg R_{T,U}(y,z) = mn$ and that of the left hand side is $mn+1$ showing that $R_{T,U}(y,z) \equiv 0$ showing that $T(x,y,z)$ and $U(x,y,z)$ have a common factor $U(x,y,z)$ hence a common component $E := V(U)$. And because the hypothesis doesn't want this, it must be that $\sharp M_m \cap N_n \leq mn$. $\square$

Theorem 2.4.6 (Bëzout Theorem, Strong form). *Let $M_m, N_n \subset \mathbb{P}^2$ be degree m and n projective algebraic curves respectively and with no common component. Then*

$$\sum_{p \in M \cap N} I_p(M,N) = mn.$$

Proof. From weak form of Bëzout Theorem, we have that $\sharp M_m \cap N_n = \{p_1, \ldots, p_k\}$ is such that $k \leq mn$ and that $\left(c_i y - b_i z, c_j y - b_j z\right) = 1$ since $q = [1,0,0], p_i = [a_i, b_i, c_i]$ and $p_j = [a_j, b_j, c_j]$ are NOT collinear. We then have, from the definition of intersection multiplicity $I_{p_i}(M_m, N_n)$ and the fact that $p_i's$ are distinct, that

$$R_{P,Q}(y,z) = \alpha \prod_{i=1}^{k} \left(c_i y - b_i z\right)^{I_{p_i}(M_m,N_n)}$$

for $0 \neq \alpha \in \mathbb{C}$. Now equating degree on both sides we have that

$$\sum_{i=1}^{k} I_{p_i}(M_m, D_n) = mn.$$

□

Example 2.4.7. *Find the intersections of the following pairs of curves in $\mathbb{P}^2$:*

$$x(y^2 - xz)^2 - y^5 = 0$$
$$y^4 + y^3 z - x^2 z^2 = 0$$

Solution:

$$Case1 : y = 1, then, (x : 1 : z)$$

$$Case2 : y = 0, then, (x : 0 : z)$$

Then we get

$$\begin{cases} x(1 - xz)^2 - 1 = 0 \\ 1 + z - x^2 z^2 = 0 \end{cases}$$

Equivalently the system gives,

$$x^3 z^2 - 2x^2 z + x - 1 = 0 \quad (1)$$
$$-x^2 z^2 + z + 1 = 0 \quad (2)$$

We choose to see (1) *and* (2) *as polynomials in* $\mathbb{C}[x][z]$

$$\begin{vmatrix} x^3 & -2x^2 & x-1 & 0 \\ 0 & x^3 & -2x^2 & x-1 \\ -x^2 & 1 & 1 & 0 \\ 0 & -x^2 & 1 & 1 \end{vmatrix}$$

$$= x^3 \begin{vmatrix} x^3 & -2x^2 & x-1 \\ 1 & 1 & 0 \\ -x^2 & 1 & 1 \end{vmatrix} - x^2 \begin{vmatrix} -2x^2 & x-1 & 0 \\ x^3 & -2x^2 & x-1 \\ -x^2 & 1 & 1 \end{vmatrix}.$$

$$\begin{aligned}
=& x^3(x^3+x-1+x^2(x-1)+2x^2) - x^2(x^4-x^2(x-1)^2+2x^2(x-1)-x^3(x-1)) \\
=& x^2\left[(x^4+x^2-x+x^3(x-1)+2x^3-4x^4+x^2(x-1)^2-2x^2(x-1)+x^3(x-1)\right] \\
=& x^2(x^4+x^2-x+x^4x^3+2x^3-4x^4+x^4-2x^3+x^2-2x^3+x^2-2x^3+2x^2+x^4-x^3) \\
=& x^2\left[-4x^3+4x^2-x\right] \\
=& -x^3(4x^2-4x+1) = -x^3(2x-1)^2.
\end{aligned}$$

The roots of the resultant are:

$$x = 0, \quad x = \frac{1}{2}$$

For $x = 0$, *this is the point* $(0:1:z)$
and for $x = \frac{1}{2}$, *this is the point* $(\frac{1}{2}:1:z)$

Case 2: If $y = 0$, *the system becomes,*

$$\begin{cases} x^3z^2 = 0 \\ -x^2z^2 = 0 \end{cases}$$

Either $x = 0$ *or* $z = 0$
$(0:0:z) \qquad (x:0:0)$

$(0:0:1) \qquad (1:0:0)$

But $(0:1:z)$ *is not "there" since* $x(y^2-xz)^2-y^5=0$ *becomes* $-1=0$

$(\frac{1}{2}:1:z) \in C_1 \cap C_2$. *Now solving for* $z^2-4z-4=0$ *we have* $z_{1,2} = 2 \pm 2\sqrt{2}$

The points in $\mathbb{P}^2$ *are* $(\frac{1}{2}:1:2+2\sqrt{2})$ *and* $(\frac{1}{2}:1:2-\sqrt{2})$.

3 Riemann surfaces

Definition 3.0.1. *A $1-$dimensional complex manifold S (i.e locally $\mathbb{R}^2$) equipped with complex atlas of charts $\mathscr{A} = \{(\varphi_i, U_i, V_i) \mid i \in I, \varphi_i : U_i \to V_i \subset \mathbb{C}$ homeomorphism $\}$. The pair $(S, \mathscr{A})$ is called a* ***Riemann surface*** *if*

(i). $S = \cup_{i \in I} U_i$.

(ii). *The transition function $\varphi_j \circ \phi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is biholomorphic for every $i, j \in I$.*

We call $\mathscr{A}$ a complex structure on S.

Example 3.0.2. *Let $S = \mathbb{P}^1$. By taking the charts*

$$\begin{aligned} U_0 =& \{[1,y] : y \in \mathbb{C}\}, V_0 = \mathbb{C}, \varphi_0 : [1,y] \mapsto y \\ U_1 =& \{[x,1] : x \in \mathbb{C}\}, V_1 = \mathbb{C}, \varphi_1 : [x,1] \mapsto x. \end{aligned}$$

Define the transition functions, $\varphi_1 \circ \varphi_0^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ by $z \mapsto \frac{1}{z}$. Then $\varphi_1 \circ \varphi_0^{-1}$ is biholomorphic. The Riemann surface $\mathbb{P}^1$ is also called ***Riemann sphere****.*

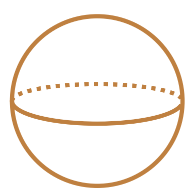


Figure 2. Riemann Sphere

Proposition 3.0.3. *A nonsingular curve D in $\mathbb{CP}^2$ is a Riemann surface complete with holomorphic atlas.*

From the classification theorem, for every integer $g \geq 0$ there exists exactly one Riemann surface. We usually draw spheres with $g-$handles to represent such surfaces. And that the Euler characteristic of C is given by $\chi(C) = 2 - 2g$.

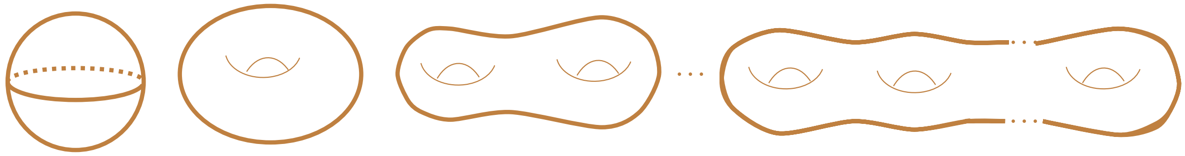


Figure 3. Cartoons representing smooth connected compact Riemann surfaces.

Definition 3.0.4. *Let $C=(S,\mathscr{A})$ be a Riemann surface. A **holomorphic function** $f: C\to\mathbb{C}$ is a function for which every chart $(\varphi_i,U_i,V_i)\in\mathscr{A}$, $f\circ\varphi_i^{-1}:V_i\to\mathbb{C}$ is holomorphic (i. e has a Taylor Series at every point in its domain). Further, we say f is **meromorphic function** if there is a finite set of points $\Sigma=\{p_i,\dots,p_n\}\subset C$ for which $f:C\setminus\Sigma\to\mathbb{C}$ is holomorphic such that*

$$\lim_{x\to p_i}|f(x)|=\infty$$

for every $p_i\in\Sigma$.

Definition 3.0.5. *A **covering** is a map $\pi: C_{d_1}\to C_{d_2}$ between Riemann Surfaces. A point $p\in C_{d_1}$ is a **ramification point** of π if there is a neighbourhood $C_{d_1}\supset V_p\ni p$ such that $\pi|_{V_p}$ is injective. A point $q\in C_{d_2}$ for which $\pi^{-1}(x)$ contains a ramification point is called a **branch point**. We say π is **unramified** if it has no branch points.*

Theorem 3.0.6 (The degree-genus Formula)**.** *Let $C_k:=\mathbb{V}(P)\subset\mathbb{CP}^2$ be a nonsingular degree k algebraic curve. Then the genus $g:=g(C_k)$ of C_k viewed as Riemann surface is given by*

$$g=\frac{1}{2}(k-1)(k-2)=\binom{k-1}{2}.$$

Proof. For $k=1$, every point of C_1 is a flex and $C_1\cong\mathbb{CP}^1$ with $g=0$. For $k\geq 2$, we have that C_k has $\leq 3k(k-2)$ flexes.By transformation, $\tau(C_k)=\mathscr{C}_k$, so that $[1,0,0]\notin\mathscr{C}_k\cup T_l\mathscr{C}_k$ where $l=[a,b,c]\in\mathscr{C}_k$ and

$$T_l\mathscr{C}_k:=\Big\{[x,y,z]\in C: xP_x(a,b,c)+yP_y(a,b,c)+zP_z(a,b,c)=0\Big\}.$$

Now, define a map $\pi:\mathscr{C}_k\to\mathbb{CP}^1$; $\pi([x,y,z])=[y,z]$, which is meromorphic as in proof of proposition above. Also, $l=[t,v,w]$ is a ramification point precisely when $T_l\mathscr{C}_k$ passes through $[1,0,0]$ or precisely when $l_x(a,b,c)=0$ and subsequently $I_l(\mathscr{C}_k,T_l\mathscr{C}_k)=2$ and so it is not a flex. Let $D_{k-1}:=\mathbb{V}(P_x)$, then $\mathscr{C}_k\cap D_{k-1}$ is the set of ramification points. $T_lD_{k-1}=\Big\{[x,y,z]\in D_{k-1}: xl_{xx}(a,b,c)+yl_{xy}(a,b,c)+zl_{xz}(a,b,c)=0: l_{xx}(a,b,c)\neq 0\Big\}$ does not pass through $[1,0,0]$ and so $T_l\mathscr{C}_k\neq T_lD_{D-1}$, hence $I_l(\mathscr{C}_k,D_{k-1})=1$. Now by strong form of Bëzout Theorem, we have

$$\sum_{l\in\mathscr{C}_k\cap D_{k-1}} I_l(\mathscr{C}_k,D_{k-1})=\sum_{l\in\mathscr{C}_k\cap D_{k-1}}1=k(k-1).$$

Hence the set of ramification points $\mathscr{C}_k \cap D_{k-1} = \{l_1, \ldots, l_{k(k-1)}\}$ and corresponding branch points $q_i = \pi(l_i) \in \mathbb{CP}^1$.

We can fine tune π so that p_i are distinct and $\sharp\pi^{-1}([b,c]) = \begin{cases} k-1 & \text{if } [b,c] = q_i, \text{ some } i \\ k & \text{if } [b,c] \neq q_i, \text{ for all } i \end{cases}$

Choosing a triangulation of $\mathbb{CP}^1$ with $k(k-1) \geq 2($ for $k \geq 2)$ V vertices E edges and F faces. We then have that

$$\chi(\mathbb{CP}^1) = k(k-1) - E + F = 2. \tag{3}$$

With $\sharp\pi^{-1}(q_i) = k-1$ and for $q \neq q_i$, $\sharp\pi^{-1}(q) = k$, this triangulation on $\mathbb{CP}^1$ lifts to a triangulation on $\mathscr{C}_k$ with vertices $\widetilde{V} = (k-1)k(k-1)$, edges $\widetilde{E} = kE$ and faces $\widetilde{F} = kF$. We then have from 3 that

$$\begin{aligned}
\chi(\mathscr{C}_k) = \chi(C_k) =& \widetilde{V} - \widetilde{E} + \widetilde{F} \\
=& (k-1)k(k-1) - kE + kF \\
=& k\big[k(k-1) - E + F\big] - (k(k-1) \\
=& 2k - k(k-1) \\
=& 2 - 2\Big(\frac{1}{2}(k-1)(k-2)\Big) \\
=& 2 - 2g(C_k).
\end{aligned}$$

Hence $g(C_k) = \frac{1}{2}(k-1)(k-2)$. □

Corollary 3.0.7. *There is no nonsingular projective algebraic curve in $\mathbb{CP}^2$ of genus $g = 2, 4, 5, 7, 8, 9, 11$, etc. So most Riemann surfaces whose cartoons are drawn above are NOT isomorphic to a nonsingular curve $C_m \subset \mathbb{CP}^2$ for any m. However, they can be embedded in some $\mathbb{CP}^N$ for some large N but with self intersections.*

3.1 The Weirstrass $\wp$-function

Let $\omega_1, \omega_2 \in \mathbb{C}^*$ which are linearly independent over $\mathbb{R}$. We denote by Λ the **lattice** generated by ω_1, ω_2 and defined by

$$\Lambda := \omega_1\mathbb{Z} + \omega_2\mathbb{Z} = \Big\{ n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z} \Big\} \cong \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2.$$

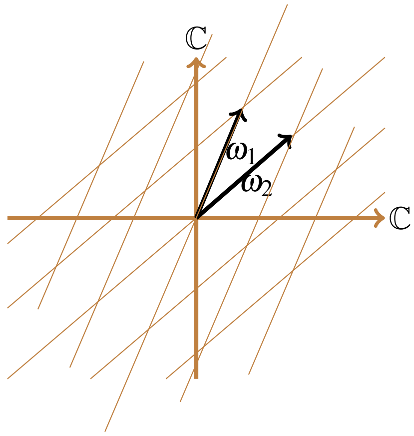


Figure 4. Translations of the Latttice Λ by nonzero $z \in \mathbb{C}$

Proposition 3.1.1. *$\wp(z)$ is a meromorphic function given by*

$\wp(z) = \frac{1}{z^2} \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$, *with the derivative*

$$\wp'(z) = \sum_{\omega \in \Lambda} -2\frac{1}{(z-\omega)^3}.$$

Lemma 3.1.2. *There is some $\delta > 0$ such that $\mid x\omega_1 + y\omega_2 \mid \geq \delta\sqrt{x^2+y^2} \qquad \forall x, y \in \mathbb{R}$*

Proof. Let $f : [0, 2\pi] \to \mathbb{R}$ defined by $f(\theta) = \mid \cos(\theta)\omega_1 + sin(\theta)\omega_2 \mid$
be a continuous function. Now $[0, 2\pi]$ is compact hence f is bounded and attains its bounds.
Also $f(\theta) > 0$ for all $\theta \in [0, 2\pi]$ from the fact that ω_1 and ω_2 are linearly independent over $\mathbb{R}$. Therefore there is some $\delta > 0$ such that $f(\theta) > \delta$ for all $\theta \in [0, 2\pi]$. It follows that

$$\mid x\omega_1 + y\omega_2 \mid \geq \delta\sqrt{x^2+y^2} \qquad \forall x, y \in R \times R.$$

□

Definition 3.1.3. *The function $\wp(z)$ is known as the Weierstrass $\wp-$ function associated to Λ.*

Lemma 3.1.4. $-\wp(z) = \wp(z) = \wp(z+\zeta)$ *for all $z \in \mathbb{C}$ and $\zeta \in \Lambda$.*

Proof. We first note, for $\zeta \in \Lambda$ then $\wp'(z+\zeta) = -2\sum_{\omega \in \Lambda}(z+\zeta-\omega)^{-3}$ since the tail end of this series converges absolutely and also since $\omega - \zeta$ runs over Λ as ω runs over Λ, we rearrange the series and substitute ω for $\omega - \zeta$ to obtain $\wp'(z+\zeta) = \wp'(z)$ for every $z \in \mathbb{C}$

$\Rightarrow \wp(z+\zeta) = \wp(z) + c(\zeta)$, $c(\zeta)$ depends on ζ and not on z. Substituting $z = -\frac{1}{2}\zeta$ we get $c(\zeta) = \wp(\frac{1}{2}\zeta) - \wp(-\frac{1}{2}\zeta)$. Now we have that

$$\wp(-z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - 0} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right)$$

replacing ω by $-\omega$ we get $\wp(-z) = \wp(z) \qquad \forall z \in \mathbb{C}$ In particular

$$c(\zeta) = \wp(\frac{1}{2}\zeta) - \wp(-\frac{1}{2}\zeta) = 0.$$

Hence, $\wp(z+\zeta) = \wp(\zeta)$. $\square$

Definition 3.1.5. *The function g on $\mathbb{C}$ with $g(z+\zeta) = g(z) \forall z \in \mathbb{C}, \forall \zeta \in \Lambda$, or equivalently $g(z+\omega_1) = g(z) = g(z+\omega_2) \forall z \in \mathbb{C}$, are referred to as doubly periodic lattice Λ (with periods ω_1 and ω_2).Hence Weirstrass $\wp-$ function on $\mathbb{C}$ is doubly periodic meromorphic function.*

Lemma 3.1.6. *A doubly periodic holomorphic function f on $\mathbb{C}$ is constant.*

Theorem 3.1.7. *Any bounded function(holomorphic) on $\mathbb{C}$ is constant.*

Lemma 3.1.8. $\wp'(z)^2 = 4\wp(z)^3 + k_2 \wp(z) + k_3$ *where*

$$k_2 = k_2(\Lambda) = 60 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^4},$$

$$k_3 = k_3(\Lambda) = 140 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^6}.$$

Proposition 3.1.9. *The weirstrass $\wp-$ function is surjective.*

Definition 3.1.10. *Let $C_\Lambda \subset \mathbb{P}^2$ be the curve given by*
$Q_\Lambda(m,n,r) = n^2 r - 4m^3 - k_2 m r^2 - k_3 r^3$ where $k_2 = k_2(\Lambda)$ and $k_3 = k_3(\Lambda)$ as in lemma 3.1.8

Lemma 3.1.11. *The curve C_Λ is non singular.*

Proof. Let $\alpha = \wp(\frac{1}{2}\rho_1), \qquad \beta = \wp(\frac{1}{2}\rho_2), \qquad \gamma = \wp\Big(\frac{1}{2}(\rho_1+\rho_2)\Big).$
To show that C_Λ is nonsingular it is enough to show that α, β, γ are distinct complex numbers and that
$Q_\Lambda(m,n,r) = n^2 r - 4(m-\alpha r)(m-\beta r)(m-\gamma r)$. The fact that α, β, γ are distinct follows from prop 3.1.9. Since $\wp$ is an even doubly periodic function its derivative is an odd doubly periodic function with the same periods ρ_1 and ρ_2. Thus

$$\wp'(\frac{1}{2}\rho_1) = \wp'(\frac{1}{2}\rho_1 - \rho_1) = \wp'(-\frac{1}{2}\rho_1) = -\wp'(\frac{1}{2}\rho_1).$$

and so $\wp'(\frac{1}{2}\rho_1) = 0$. By lemma 3.1.8 we have $4\alpha^3 + g_2\alpha + g_3 = \wp'(\frac{1}{2}\rho_1)^2 = 0$ and so α, and similarly β and γ, are the roots of the polynomial $4\alpha^3 - g_2\alpha - g_3$. Thus

$$Q_\Lambda(m,n,r) = n^2r - 4(m-\alpha r)(m-\beta r)(m-\gamma r)$$

with α, β, γ distinct, and hence the curve C_Λ defined by $Q_\Lambda(m,n,r)$ is non singular. $\square$

Remark 3.1.12. *If we regard the lattice Λ as an additive subgroup of $\mathbb{C}$ then we can construct the group(quotient). $\mathbb{C}/\Lambda = \{\Lambda + a : a \in \mathbb{C}\}$*
This quotient group is furnished with the quotient topology which is inherited from the standard topology on $\mathbb{C}$ as follows.
Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the surjective map defined by $\pi(a) = \Lambda + a$. Then a subset U of $\mathbb{C}/\Lambda$ is open in the quotient topology on $\mathbb{C}/\Lambda$ only when its inverse image $\pi^{-1}(U)$ is open in $\mathbb{C}$.

$\mathbb{C}/\Lambda$ is compact. Topologically $\mathbb{C}/\Lambda$ is a torus. This is due to the fact that we can identify $\mathbb{C}/\Lambda$ topologically with the parallelogram P by gluing its two pairs of opposite sides together. Glued one pair of the sides together gives a ***cylinder*** *and glueing the ends of the cylinder together gives a* ***torus****. Here we shall refer $\mathbb{C}/\Lambda$ as a* ***complex torus****.*

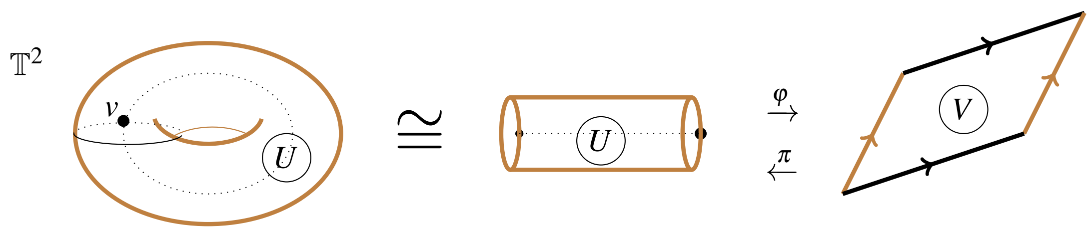


Figure 5. Torus as a Riemann Surface.

Now, define a function

$$\Phi : \mathbb{C}/\Lambda \to C_\Lambda$$

by

$$\Phi(z+\Lambda) = \begin{cases} [\wp(z) : \wp'(z) : 1] & \text{if } z \in \mathbb{C} \setminus \Lambda \\ [0:1:0] & \text{if } z \in \Lambda. \end{cases}$$

where

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(z-\omega)^2} + \frac{1}{\omega^2}\right)$$

is the Weierstrass $\wp-$ function. It is uniformly convergent on any compact subset U of $\mathbb{C}/\Lambda$ to a holomorphic function. $\wp$ has a double pole at $\omega \in \Lambda$. So the Weierstrass $\wp-$ function $\wp : \mathbb{C} \to \mathbb{C} \sqcup \{\infty\} = \mathbb{CP}^1$ is meromorphic with $\wp(\omega) = \infty$, $\omega \in \Lambda$ and is translation invariant for all $z \in \mathbb{C}$ and $\omega \in \Lambda$ i.e $\wp(z) = \wp(z+\omega)$. This periodicity of $\wp$

helps us conclude that it has no poles at $\omega \in \Lambda$ so that in fact, it has no poles at all. Now, by Maximum Principle Theorem of complex analysis (i.e. holomorphic functions on a Riemann surface without poles is a constant), it is easy to check that the Weierstrass $\wp-$ function satisfies the ODE

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 - k_2\wp(z) - k_3 \tag{4}$$

for all $z \in \mathbb{C}$ where $k_2, k_3 \in \mathbb{C}$ with $\wp'(z)$ the ordinary first derivative of $\wp$ with respect to z. Finally, if $\Phi(u+\Lambda) = [m:n:r]$ then from equation 4 i.e $\left(\wp'(z)\right)^2 = 4\wp(z)^3 - k_2\wp(z) - k_3$ we have $\Phi\big(\mathbb{C}/\Lambda\big)$ is a cubic $Q([m:n:r]) = n^2r - 4m^3 + k_2mr^2 + k_3r^3 = 0$ which can be re-written as

$$\Phi\big(\mathbb{C}/\Lambda\big) = \mathbb{V}(Q): \text{ with } Q: n^2r = 4m^3 - k_2mr^2 - k_3r^3. \tag{5}$$

The right hand side of Q has distinct root since the cubic who curve we started with was nonsingular and so can be written as a product of 3 distinct factors. Hence equation 4 and 5 have the same normal form. So that non-singular cubics like $y^2z = x(x-z)(x+z)$ above are equivalent to a torus with a unique pair $(k_2, k_3) \in \mathbb{C}^2$.

3.2 The group law on a cubic curve

Take two points K and L on a non singular cubic curve D defined on a complex projective plane $\mathbb{P}^2$ and draw the line KL. This line intersects the cubic curve on a third point M. We take another point on D say $\mathscr{O}$. Then from M we draw a line through $\mathscr{O}$ to intersect D on N. In the same fashion we can extend to special cases: If $K = L$, we construct tangent to the curve at the point $K = L$; If the x coordinate of K, L are the same , then the equation of line KL is $x = c$, we take $M = \infty$. Hence we define an addition $\oplus$ on points of $D \cup \{\infty\}$ by taking

$$K \oplus L \oplus M = \mathscr{O}$$

whenever M is the point associated to K, L.

Theorem 3.2.1. *The points of a non singular curve D of degree 3 defined over a field $R = \mathbb{C}$ forms a commutative group under the operation $\oplus$. The identity is given by $\mathscr{O} = \infty$ and the inverse of $K \in D$ is $-K$.*

Before we show that $k + l$ is a commutative group law on the points of D, with identity $\mathscr{O} = \infty$ and the inverse of k given by $-k$,we need the following technical lemma.

Lemma 3.2.2. *Let $P_1, \cdots, P_8$ be such that none of the 4 points lie on a line and none of the 7 points lie on a conic. Then there exists a unique point P_9 which is a 9th point of intersection of any two cubics passing through $P_1, \cdots, P_8$.*

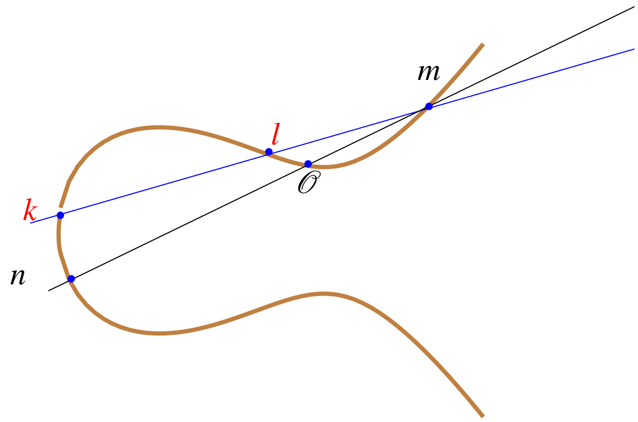


Figure 6. The construction of the group law

Proof of theorem 3.2.1. We need to show that associativity holds for the group law of cubic curves.

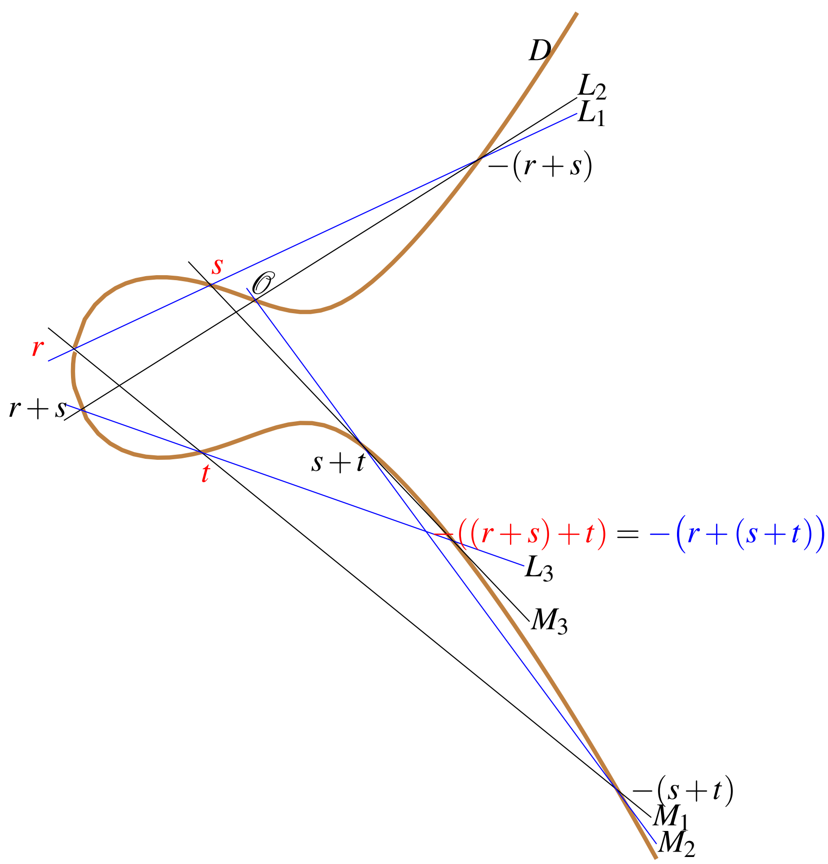


Figure 7. Associativity of the group Law on Cubic Curve

Consider the points r, s, t which are arbitrary points of D. Let L_1 be the line in $\mathbb{P}^2$ which meets D in the points $r+s, -(r+s)$ counted with multiplicity. Similarly let L_2, L_3, M_1, M_2 and M_3 be the lines in $\mathbb{P}^2$ which meet D in the points $\mathcal{O}, r+s, -(r+s)$;
$t, r+s, -((r+s)+t)$; $s, t, -(s+t)$; $\mathcal{O}, s+t, -(s+t)$ and $r, s+t$,
$-\big(r+(s+t)\big)$ respectively.

Let $E = L_1 \cup M_2 \cup L_3$ (Blue lines) and $F = M_1 \cup L_2 \cup M_3$ (Black lines) be the reducible cubic curves. From fig 7 above we can see that E meets D in the points
$\mathcal{O}, r, s, t, r+s, s+t, -(r+s), -(s+t), -((r+s)+t)$
whereas F meets D meets in the points $\mathcal{O}, r, s, t, r+s, s+t, -(r+s), -(s+t)$,
$-\big(r+(s+t)\big)$. $\square$

Remark 3.2.3. *1. Always take $\mathcal{O} = [0,1,0]$ unless stated otherwise.*

2. *For a point $k = (m,n,r)$, if we take the line through k and $k_0 = (0,1,0)$ then the other meeting point is $(m,-n,r)$ which is the point $-r$. Hence the inverse of a point, say r , under the group law is its reflection in the $x-$axis.*

3. *For the elliptic curve $n^2r = m^3 + Amr^2 + Br^3$ with the point $\mathcal{O}$ and the points (m,n,r) we have; let $m = (m_3, n_3, r_3)$ as defined in 7. Then $r+s = (m_3, -n_3, r_3)$, which is m reflected on the $x-$axis.*

From the fact that every non singular cubic curve C_Λ is isomorphic to a torus $\mathbb{C}/\Lambda$, under this isomorphism, the group operation $\oplus$ corresponds to the obvious abelian group operation on the set of points of $\mathbb{C}/\Lambda$, coming from the addition of complex numbers.

3.3 Holomorphic differentials on Riemann surfaces

We recall that for the lattice Λ in $\mathbb{C}$ we have associated a nonsingular cubic curve D_Λ in $\mathbb{P}^2$ defined by

$$y^2z = 4x^3 - g_2xz^2 - g_3z^3.$$

Our main work here is to see if given the curve C_Λ whether we can recover the lattice Λ. To achieve this we need the concept of the integral of a holomorphic differential along a piecewise smooth path in a Riemann surface.

Definition 3.3.1. *Let K be a Riemann surface. A piecewise smooth path in K is a continuous map β from a closed interval $[p,q]$ in $\mathbb{R}$ to K such that if $\psi : M \to N$ is a holomorphic chart on an open subset M of K and $[r,s] \subseteq \beta_{-1}(M)$ then $\psi \circ \gamma : [r,s] \to N$ is a piecewise-smooth path in the open subset N of $\mathbb{C}$.*

Remark 3.3.2. *For a compact Riemann surface K(e.g non-singular projective curves) the meromorphic functions on K are of much interest than holomorphic functions $f : K \to \mathbb{C}$ due to the fact holomorphic function on a compact Riemann surface is a constant but there are lots of meromorphic functions $f : K \to \mathbb{P}^1$ e.g the Weirstrass $\mathfrak{p}-$ function on a complex torus.*

Definition 3.3.3. *Let K be a Riemann surface and $g,h \in K$ meromorphic functions. Then gdh is called a* ***meromorphic differential*** *on K. If $\tilde{g},\tilde{h} \in K$ are some meromorphic functions then we have that $gdh = \tilde{g}d\tilde{h} \Leftrightarrow$ for every holomorphic chart $\psi : M \to N$ on $M \subseteq K$, open we have that*

$$(g \circ \psi^{-1})(h \circ \psi^{-1})' = (\tilde{g} \circ \psi^{-1})(\tilde{h} \circ \psi^{-1})'.$$

Definition 3.3.4. *Let $\left\{\psi_\sigma M_\sigma \to N_\sigma : \sigma \in A\right\}$ be a holomorphic atlas on Riemann surface K. A meromorphic differential η on K is given by a collection*

$$\left\{\eta_\sigma : N_\sigma \to \mathbb{P}^1 : \sigma \in A\right\}$$

of meromorphic functions on the open subsets N_σ of $\mathbb{C}$ such that $\sigma,\beta \in A$ and $m \in M_\sigma \cap M_\beta$ then

$$\eta_\sigma\Big(\psi_\sigma(m)\Big) = \eta_\beta\Big(\psi_\beta(m)\Big)\Big(\psi_\beta \circ \psi_\sigma^{-1}\Big)'\Big(\psi_\sigma(m)\Big).$$

Given two meromorphic functions g and h on K we are able to define a meromorphic differential gdh on K in this sense by $gdh = \eta$ where

$$\eta_\sigma = (g \circ \psi_\sigma^{-1})(h \circ \psi_\sigma^{-1})'.$$

Definition 3.3.5. *The meromorphic differential has a pole at n in K if the function(meromorphic) $(g \circ \psi_\sigma^{-1})(h \circ \psi_\sigma^{-1})'$ has a pole at $\psi(n)$ where $\psi : M \to N$ is a holomorphic chart on an open neighbourhood M of n in K. gdh is called a holomorphic differential if it has no poles.*

Definition 3.3.6. *If gdh is a holomorphic differential on K then the integral of gdh along a piecewise-smooth path $\xi : [p,q] \to K$ is*

$$\int_\xi gdh = \int_p^q (g \circ \xi)(t)(h \circ \xi)'(t)dt.$$

Remark 3.3.7. *If $\psi : [r,s] \to [p,q]$ is a piecewise-smooth map between the intervals $[r,s]$ and $[p,q]$ then $\eta \circ \psi : [r,s] \to K$ is a piecewise-smooth path in K and on substituting $t = \psi(k)$ we define*

$$\int_\xi gdh = \int_r^s g \circ \xi \circ \psi(k)(g \circ \xi)'(\psi(k))\psi'(k)dk = \int_{\xi\circ\psi} gdh$$

Example 3.3.8. *(1) If $K = \mathbb{C}$ then*

$$\int_\sigma gdh = \int_\sigma g(z)h'(z)dz$$

is the integral of $g(z)h'(z)$ along σ in the usual sense of complex analysis.

(2) If $g : S \to \mathbb{C}$ is a complex valued holomorphic mapping on any Riemann surface K then

$$\int_\sigma dg = g(\sigma(b)) - g(\sigma(a)).$$

Definition 3.3.9. *If $\psi : K \to R$ is a holomorphic mapping between Riemann surfaces K and R and if gdh is a holomorphic differential on R then we define a holomorphic differential $\psi^*(gdh)$ on K by*

$$\psi^*(gdh) = (g \circ \psi)d(h \circ \psi).$$

Then if $\sigma : [p,q] \to K$ is a piecewise-smooth path in K we have

$$\int_\sigma \psi^*(gdh) = \int_p^q g \circ \psi \circ \sigma(t)(h \circ \psi \circ \sigma)'(t)dt = \int_{\psi \circ \sigma} gdh.$$

Given lattice Λ in $\mathbb{C}$ we defined a biholomorphism

$$u : \mathbb{C}/\Lambda \to D_\Lambda$$

where $D_\Lambda \subset \mathbb{P}^2$ is non singular cubic curve given by $y^2z = x(x-z)(x+z)$.

There is a meromorphic differential on D_Λ given inhomogenous coordinates $[x,y,1]$ by $y^{-1}dx$. Let

$$\eta = u^*(y^{-1}dx).$$

Then η is a meromorphic differential on $\mathbb{C}/\Lambda$. Moreover if $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is defined as

$$\pi(z) = \Lambda + z$$

then

$$\begin{aligned}
\pi^*\eta =& \pi^*u^*(y^{-1}dx) \\
=& (u \circ \pi)^*(y^{-1}dx) \\
=& (\mathfrak{p}')^{-1}d\mathfrak{p} \\
=& (\mathfrak{p}')^{-1}\mathfrak{p}'dz \\
=& dz.
\end{aligned}$$

Proposition 3.3.10. $\Lambda = \left\{ \int_\sigma \eta : \sigma \text{ is a closed piecewise-smooth path in } \mathbb{C}/\Lambda \right\}$

Since $u : \mathbb{C}/\Lambda \to D_\Lambda$ is a bijection with a holomorphic inverse and $\eta = u^*(y^{-1}dx)$

Corollary 3.3.11. $\Lambda = \left\{ \int_\sigma y^{-1}dx : \sigma \text{ is a closed piecewise-smooth path in } D_\Lambda \right\}$

This means that we can recover the lattice Λ from the curve D_Λ in $\mathbb{P}^2$.

Also the function $u^{-1} : D_\Lambda \to \mathbb{C}/\Lambda$ in terms of integrals of the differential $y^{-1}dx$ on C_Λ.

Proposition 3.3.12. *The inverse of the holomorphic bijection*

$$u : \mathbb{C}/\Lambda \to D_\Lambda$$

is given by

$$u^{-1}(p) = \Lambda + \int_{[0,1,0]}^{p} y^{-1}dx$$

where the integral is over any piecewise-smooth path γ in C_Λ from $[0,1,0]$ to p.

Remark 3.3.13. *If γ_1 and γ_2 are both piecewise-smooth paths in C_Λ from $[0,1,0]$ to p then $\int_{\gamma_1} y^{-1}dx - \int_{\gamma} y^{-1}dx$ is the integral of $y^{-1}dx$ along a closed piecewise-smooth path in C_Λ and hence belongs to Λ.*

3.4 Abel's theorem

Let $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. Then the biholomorphic bijection $u : \mathbb{C}/\Lambda \to D_\Lambda$ is a group isomorphism with respect to the group structure(with p_0=[0,1,0]) on D_Λ and the quotient group structure on $\mathbb{C}/\Lambda$.

Remark 3.4.1. *Abel's theorem provides an alternative means of proof of the existence of an additive group structure on curve C with required properties on cubic curves of the form C_Λ. Every non singular projective cubic curve in $\mathbb{P}^2$ is equivalent under a projective transformation to one of the form D_Λ for some lattice Λ on $\mathbb{C}$.*

Recall that a line in $\mathbb{P}^2$ meets a non singular projective curve D in $\mathbb{P}^2$ either in:

1. 3 *distinct points l, m, n each with multiplicity* 1.
2. 2 *distinct points; l of multiplicity* 1 *and m of multiplicity* 2.
3. 1 *point l with multiplicity* 3.

The group structure on D_Λ is such that if l, m, n are unique points on D_Lambda then $l+m+n = 0$ only when l, m, n are all on a line in $\mathbb{P}^2$, $l+l+m = 0$ if and only if the tangent on D_Λ at l passes through m and $l+l+l = 0$ only when l is a point of inflection on D_Λ.

In particular, the points of inflection on D_Λ are the points of order 1 or 3.
Under group isomorphism these correspond to the points of order 1 or 3 in D/Λ. There are 9 precisely such points in D/Λ.

Theorem 3.4.2. *(Abel's theorem for tori) If $r, s, t \in \mathbb{C}$ then $r+s+t \in \Lambda$ only when there is a line $L \subset \mathbb{P}^2$ whose intersection with D_Λ has the points $u(\Lambda + r), u(\Lambda + s)$ and $u(\Lambda + t)$.*

Equivalently if $l, m, n \in C_\Lambda$ then

$$\Lambda + \int_{[0,1,0]}^{l} y^{-1}dx + \int_{[0,1,0]}^{m} y^{-1}dx + \int_{[0,1,0]}^{n} y^{-1}dx = \Lambda + 0$$

only when l, m, n are the meeting points of D_Λ with a line in $\mathbb{P}^2$.

Remark 3.4.3. *Abel's theorem can be interpreted as an addition formula modulo Λ for elliptic integrals of the form*

$$\int_{[0,1,0]}^{p} y^{-1}dx$$

on C_Λ.

3.5 The Riemann-Roch theorem

The Riemann Roch theorem relates dimensions of vector spaces of meromorphic functions with prescribed poles and zeros on a non singular projective curve C in $\mathbb{P}^2$. Consequence of the Riemann Roch theorem include, a proof of the law of associativity for the additive group structure on a non-singular cubic, a proof that every meromorphic function on a non singular projective curve is rational and the important fact that the genus g of a curve C can be described using the zeroes and poles of any non-zero meromorphic differential on C from $2g-2$.

Definition 3.5.1. *A divisor D on a non- singular projective curve C is a formal sum $D = \sum_{p \in C} n_p.p$ such that $n_p \in \mathbb{Z}$ for every $p \in C$ and $n_p = 0$ for all but finitely many $p \in C$.*

The degree of D is then $\deg(D) = \sum_{p \in C} n_p$.

Remark 3.5.2. *The set of all divisors on C is an abelian group, denoted $Div(C)$ and the degree defines homomorphism from $Div(C)$ to $\mathbb{Z}$.*

If $n_p \geq 0 \forall p \in C$ we write $D \geq 0$ and say that D is effective or positive.

Principal divisor is a divisor which is the divisor of some meromorphic function.

Two divisors D and D' are said to be linearly equivalent i.e $D \sim D'$, if $D - D'$ is a principal divisor.

The divisor of a meromorphic differential is called a ***canonical divisor*** *and is often written as κ.*

If η is another meromorphic differential on C which is not identically zero then there is function(meromorphic) g on C such that $\eta = g\lambda$, and hence

$$(\eta) = (g) + (\lambda) \sim (\lambda)$$

Thus any two canonical divisors are linearly independent.

Proposition 3.5.3. *A principal divisor on C has degree zero, i.e a function(meromorphic) on C which is not identically zero has equal number of zeroes and poles, counted with multiplicities.*

Corollary 3.5.4. *Two divisors(linearly equivalent) on C have equal degree. In short, canonical divisors on C all have equal degree.*

Proposition 3.5.5. *If κ is a canonical divisor on C then*

$$\deg \kappa = 2g - 2.$$

Definition 3.5.6. *Let $D = \sum_{p \in C} n_p . p$ be a divisor on C; then $L(D)$ is the set of all meromorphic functions on C satisfying*

$$(f) + D \geq 0$$

together with the zero function.

We define

$$l(D) = \dim L(D)$$

Corollary 3.5.7. *If $\deg D < 0$ then $l(D) = 0$*

Lemma 3.5.8. *If $D \sim D'$ then $l(D) = l(D')$*

Theorem 3.5.9. *(Riemann-Roch) If D is any divisor on a non singular projective curve C of genus g in $\mathbb{P}^2$ and κ is a canonical divisor on C, then*

$$l(D) - l(\kappa - D) = \deg(D) + 1 - g$$

Corollary 3.5.10. *The genus of a non singular projective curve C in $\mathbb{P}^2$ equals the dimension $l(\kappa)$.*

4 Elliptic integrals and cubic curves.

4.1 Addition law for Integrals.

In this section we shall demonstrate how understanding the geometry of cubic curves can help in solving the integral in 6 below.

$$\int \frac{dx}{\sqrt{x^3 - x}}. \tag{6}$$

We start by recalling the lattice

$$\Lambda := \mathbb{Z} + \omega\mathbb{Z} = \Big\{ n + m\omega \mid n, m \in \mathbb{Z} \Big\} \cong \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2 \subset \mathbb{C}, \qquad \mathrm{im}\,\omega > 0.$$

The integral 6 above is an integral on a cubic curve. Since non singular cubic curves are isomorphic to a 2-torus $\mathbb{C}/\Lambda$ we shall do some integration on the 2-torus.

The differential dz on $\mathbb{C}$ turns out to be the differential on $\mathbb{C}/\Lambda$. Now, $\mathbb{C}/\Lambda$ is a quotient group from the fact that $\mathbb{C}$ is a group and Λ is a subgroup. Thus the extra group structure on $\mathbb{C}/\Lambda$ will give us the addition law.

By translating the differential dz on the torus we find that it is an invariant differential. i.e If we have a change of variables by using $c \in \mathbb{C}$ or $\mathbb{C}/\Lambda$ to translate dz, then we have that

$$y = z + c,$$
$$\Rightarrow dy = dz.$$

Hence dz is invariant .

Now suppose that we want to integrate dz along some path γ on the 2-torus which goes form a to b, i.e $\int_\gamma dz$. Shifting the curve γ with some constant c we get a new curve $\gamma + c$. i.e $\int_{\gamma+c} dz$

Again by a simple change of variables we observe that

$$\int_{\gamma+c} dz = \int_\gamma dz \tag{7}$$

4.2 Addition law

Now, using the results in 7;

$$\int_0^r dz + \int_0^s dz = \int_0^r dz + \int_r^{r+s} dz = \int_0^{r+s} dz. \tag{8}$$

Since for integral on a manifold you state the path of integration, changing the path leads to a different answer. So the integrals on 8 are defined only upto some $\lambda \in \Lambda$

It is now evident that for the addition law to work, the group structure on the 2-torus and an invariant differential are needed.Since the non singular cubic curves and the 2-torus are isomorphic then cubic curves also have these two properties.

Theorem 4.2.1. *Let $l(m,n,r) \in \mathbb{C}[m,n,r]$ be given by $l(m,n,r) = n^2r - m^3 - \alpha mr^2 - \beta r^3$ define a cubic curve(non singular) D i.e $D = \{[m,n,r] : l(m,n,r) = 0\} \subset \mathbb{P}^2, \alpha, \beta \in \mathbb{C}$. Then D has:*

1. *an abelian group structure, say $\oplus$ and zero at ∞.*
2. *invariant differential. i.e*
$$\frac{dm}{n} = \frac{dm}{\sqrt{m^3 + \alpha m + \beta}}.$$

Now we have the addition law which we can summarise it as : Let $(m_1, n_1)(m_2, n_2)$ be points on the curve D then;

$$\int_\infty^{(m_1,n_1)} \frac{dm}{n} + \int_\infty^{(m_2,n_2)} \frac{dm}{n} = \int_\infty^{(m_1,n_1)\oplus(m_2,n_2)} \frac{dm}{n}.$$

4.3 The operator $D = \lambda \frac{d}{d\lambda}$ and the group of cubic integrals

For all $k, n \in \mathbb{N},\ \alpha \in \mathbb{C}$ we have that

1. $D\lambda^n = n\lambda^n$.
2. $D^k\lambda^n = n^k\lambda^n$.
3. $(D+\alpha)\lambda^n = (n+\alpha)\lambda^n$.
4. $(D+\alpha)^k\lambda^n = (n+\alpha)^k\lambda^n$.

Definition 4.3.1. *Let $f, g : X \to Y$ be continuous functions between topological spaces and $I = [0,1] \subset \mathbb{R}$ the unit interval. We say X is homotopic to Y if there exists a family $(h_t(x) := H(x,t) | \, t \in I)$ of continuous functions from X to Y indexed by I where $H : X \times I \to Y$, called the* ***homotopy*** *of f to g and denoted $f \sim g$, is defined for all $x \in X$ by*

$$H(x,t) = \begin{cases} h_0(x) = f(x) & \text{if } t = 0 \\ h_1(x) = g(x) & \text{if } t = 1. \end{cases}$$

Proposition 4.3.2. *Homotopy is an equivalence relation.*

Definition 4.3.3. *Let X be a topological spaces and $I = [0,1] \subset \mathbb{R}$ the unit interval. A loops based at $\lambda_0 \in X$ is a collection of continuous functions $L_0 = \{ f : I \to X : f(0) = f(1) = \lambda_0 \}$ which starts and ends at λ_0. The* ***Fundamental group of loops on X based at*** *λ_0, denoted as $\pi_1(X, \lambda_0)$ is given by*

$$\pi_1(X, \lambda_0) = \left(L_0/h, * : [f] * [g] = [f * g], \; [f]^{-1} = [f^{-1}] \right)$$

where

$$(f * g)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ f(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}, \qquad f^{-1}(t) = f(1-t)$$

and h is the homotopy relation on L_0.

Now, we can have another alternative view of Picard-Fuchs equation $\Pi(\lambda)$ using representation theoretic tools on the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \lambda_0)$ of loops based at $\lambda_0 \in X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and which avoids (circles) $0, 1$ and ∞ on the Riemann sphere.

Let $\gamma_1, \gamma_2 \in L_0$ be loops based at λ_0 and circling 0 and ∞ respectively, then$\gamma_1 * \gamma_2$ is the loop circling 1 with "twice the speed" and goes behind the Riemann sphere by first following γ_1 followed by γ_2. We then have that

$$\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \lambda_0) = \mathbb{Z}_{\gamma_1} * \mathbb{Z}_{\gamma_2}.$$

Lemma 4.3.4. *Let G be the group associated to cubic integrals. We have that*

$$G \simeq \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \lambda_0) = \mathbb{Z}_{\gamma_1} * \mathbb{Z}_{\gamma_2}.$$

Now consider the degree 2 representation $\rho : G \to Gl(2, \mathbb{C})$ and two elements of G

$$[\gamma_1] = 2 \int_0^1 \frac{dx}{y} = \int_{\gamma_1} \frac{dx}{y}$$

and

$$[\gamma_2] = 2\int_1^\lambda \frac{dx}{y} = \int_{\gamma_2} \frac{dx}{y}.$$

Also If $\rho([\gamma_1]) = T_0, \rho([\gamma_2]) = T_\infty, \rho([\gamma_1 * \gamma_2]) = T_1 \in GL(2,\mathbb{C})$ when λ loops around $0,\infty$ and 1 respectively; we can completely understand ρ. We have that $T_0 : \begin{bmatrix}[\gamma_2]\\ [\gamma_1]\end{bmatrix} \mapsto \begin{bmatrix}1.[\gamma_2]+2.[\gamma_1]\\ 0.[\gamma_2]+1.[\gamma_1]\end{bmatrix}$ hence $T_0 = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$. Similarly $T_\infty = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$ so that, as free generators,

$$\langle T_0, T_1\rangle = \Gamma(2) = \left\{ \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in Sl(2,\mathbb{Z}) \big| b,c \equiv 0 \mod 2 \right\} \subset Sl(2,\mathbb{Z}) \xleftarrow{\rho} \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, \lambda_0) = \mathbb{Z}_{\gamma_0} * \mathbb{Z}_{\gamma_\infty}$$

where $\rho : \begin{bmatrix}\gamma_0\\ \gamma_\infty\end{bmatrix} \mapsto \begin{bmatrix}T_0\\ T_\infty\end{bmatrix}$. It worth noting that under the Möbius action on the upper half plane

$$\mathbb{C} \supset \mathbb{H} := \{z : \operatorname{Im} z > 0\} \curvearrowleft \Gamma(2) \text{ defined by } \begin{bmatrix}a & b\\ c & d\end{bmatrix} \mapsto \frac{az+b}{cz+d},$$

we have that

$$\mathbb{H}/\Gamma(2) = \mathbb{P}^1 \setminus \{0,1,\infty\}.$$

4.4 Solving Picard - Fuchs equation

The elements $[\gamma_1]$ and $[\gamma_2]$ above are called **periods** of

$$E_\lambda = \mathbb{V}\left(y^2 = x(x-1)(x-\lambda)\right) \subset \mathbb{CP}^2$$

and depends on λ. We then recognise that the Picard-Fuchs equation 13 can be written as

$$\Pi(\lambda) = [\gamma_1] + [\gamma_2] = 2\int_0^\lambda \frac{dx}{y}. \tag{9}$$

Hence, $[\gamma_i]$ satisfies the Picard-Fuchs equation $\Pi(\lambda)$, finding them explicitly is tantamount to having solved the equation. The above fact 9 reveals that using residue theorem and integrating over the path γ as $\lambda \to 0$, equation 6 becomes

$$\int_\gamma \frac{dx}{x(x-1)^{1/2}} = Res_0 \left[\frac{1}{x(x-1)^{1/2}} = 2\pi i(-i)\right] = 2\pi.$$

This explains why the power series of $\Pi(\lambda)$ in in theorem 4.4.1 has a factor $\frac{1}{2\pi}$.

It is worth noting at this pointy that

$$\frac{\partial}{\partial\lambda}\left[\sqrt{x(x-1)(x-\lambda)}\right]=\frac{1}{2}\frac{\sqrt{x(x-1)(x-\lambda)}}{(x-\lambda)}$$

and

$$\frac{\partial^2}{\partial\lambda^2}\left[\sqrt{x(x-1)(x-\lambda)}\right]=\frac{3}{4}\frac{\sqrt{x(x-1)(x-\lambda)}}{(x-\lambda)^2} \tag{10}$$

so that some linear combination of $[\gamma_i], \frac{d[\gamma_i]}{d\lambda}$ and $\frac{d^2[\gamma_i]}{d\lambda^2}$ must be zero [0]; this hints at a posible direction in solving our equation $\Pi(\lambda)$.

Back to our big question; to solve 16 ,therefore, we reformulate it as follows: with $\lambda=\lambda_0$ fixed as before and $\omega=\frac{dx}{y}=\frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$, we notice that by differentiating the RHS of 10 [ignore the factor 3/4] with respect to x, we get

$$\frac{d}{dx}\left[\frac{\sqrt{x(x-1)(x-\lambda)}}{(x-\lambda)^2}\right]=-\frac{1}{2}\omega-(4\lambda-2)\frac{\partial\omega}{\partial\lambda}-2\lambda(\lambda-1)\frac{\partial^2\omega}{\partial\lambda^2}.$$

Now with the path γ as above [as $\lambda\to 0$] we have that

$$\int_\gamma d\frac{\sqrt{x(x-1)(x-\lambda)}}{(x-\lambda)^2}=\int_\gamma\left[-\frac{1}{2}\omega-(4\lambda-2)\frac{\partial\omega}{\partial\lambda}-2\lambda(\lambda-1)\frac{\partial^2\omega}{\partial\lambda^2}\right]dx.$$

Which reduces to

$$\lambda(\lambda-1)\frac{d^2[\gamma_i]}{d\lambda^2}+(2\lambda-1)\frac{d[\gamma_i]}{d\lambda}+\frac{1}{4}[\gamma_i]=0.$$

Our desirable Picard-Fuchs equation in $[\gamma]=[\gamma_i]$ is therefore given by

$$\frac{d^2[\gamma]}{d\lambda^2}+\frac{(2\lambda-1)}{\lambda(\lambda-1)}\frac{d[\gamma]}{d\lambda}+\frac{1}{4\lambda(\lambda-1)}[\gamma]=0 \tag{11}$$

which we can solve by quite elementary methods. Notice that the equation 11 is second-order of the form

$$\frac{d^2[\gamma]}{d\lambda^2}+\frac{P(\lambda)}{\lambda}\frac{d[\gamma]}{d\lambda}+\frac{Q(\lambda)}{\lambda^2}[\gamma]$$

for functions

$$P=\frac{2\lambda-1}{\lambda-1}\text{ and }Q=\frac{\lambda}{4(\lambda-1)}.$$

Since both P and Q are holomorphic at $\lambda=0$, it would mean that 11 has a **regular singular point at** $\lambda=0$. We then compute the indicial equation

$$r(r-1)+rP(0)+Q(0)=r^2;$$

which has a zero of multiplicity 2 at $r=0$. This would mean [check general theory of indicial equations] that the 2−dimensional solution space is generated by

$$[\gamma_1](\lambda) \text{ and } [\gamma_2](\lambda) = [\gamma_1](\lambda).\log(\lambda) + \phi(\lambda)$$

where ϕ is some holomorphic function and $[\gamma_1]$ is holomorphic at $\lambda = 0$. It is worth noting that $[\gamma_2]$ is multi-valued due to the logarithmic term.

Theorem 4.4.1.

$$\Pi(\lambda) = \oint \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \tag{12}$$

satisfies an algebraic ordinary differential equation on $\mathbb{P}^1 \setminus \{0,1,\infty\}$.

The idea here is to compute $\Pi(\lambda)$ as a power series in λ, then we extract the differential equation 11 from the power series.

By simple calculation we see that

$$\Pi(\lambda) := \int \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = \frac{1}{2\pi}\frac{1}{i}\int \frac{1}{x}(1-x)^{-\frac{1}{2}}(1-\frac{\lambda}{x})^{-\frac{1}{2}}dx$$

Moreover, we have that $\forall k \geq 0$ and $\forall m \geq 0$

$$\begin{aligned}
\Pi(\lambda) =& \frac{1}{2\pi i}\int \frac{1}{x}\sum_{k\geq 0}\begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} x^k \times \sum_{m\geq 0}\begin{pmatrix} -\frac{1}{2} \\ m \end{pmatrix}\left(\frac{\lambda}{x}\right)^m dx \\
=& \frac{1}{2\pi i}\oint x^{-1}\sum_{k\geq 0}\begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} x^k \times \sum_{m\geq 0}\begin{pmatrix} -\frac{1}{2} \\ m \end{pmatrix} x^{-m}\lambda^m dx \\
=& \frac{1}{2\pi i}\sum_{n\geq 0}\left[\sum_{k-m=n}\begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ m \end{pmatrix}\lambda^m\right]\oint_\gamma x^{n-1}dx \\
=& \frac{1}{2\pi i}\sum_{n\geq 0}\left[\sum_{n\leq k}\begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ k-n \end{pmatrix}\lambda^{k-n}\right]\int_{|z|\leq 1} f(z)dz \\
=& \frac{1}{\cancel{2\pi i}}\sum_{n\geq 0}\left[\sum_{n\leq k}\begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ k-n \end{pmatrix}\right]\lambda^n \times \cancel{2\pi i} \\
\stackrel{\text{below}}{=}& \sum_{n\geq 0}\begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix}^2 \lambda^n
\end{aligned}$$

This is "the" solution to the Picard-Fuchs equation which stays bounded near the singular point $\lambda = 0$. It is not an suprise us that the coefficients in this power series are rational, we shall soon see in the next section that they indeed are!

We now use the properties of the operator $D = \lambda \frac{d}{d\lambda}$ to show that the Picard Fuch's equation

$$\Pi(\lambda) = \sum_{n\geq 0} a_n \lambda^n$$

is as given above and it satisfies the ODE

$$\left[D^2 - \lambda\left(D + \frac{1}{2}\right)^2\right]\Pi(\lambda) = 0. \tag{13}$$

Indeed, with $D^2\Pi(\lambda) = \sum_{n\geq 0} a_n D^2 \lambda^n$, we get

$$D^2\Pi(\lambda) = \sum_{n\geq 0} n^2 a_n \lambda^n \tag{14}$$

which becomes to

$$(D+1)^2\Pi(\lambda) = \sum_{n\geq 0}(n+1)^2 a_{n+1}\lambda^n$$

by making a single step rightshift in 14. Further

$$\lambda\left(D + \frac{1}{2}\right)^2\Pi(\lambda) = \sum_{n\geq 0}\left(n + \frac{1}{2}\right)^2 a_n \lambda^n.$$

So that

$$\sum_{n\geq 0}\left[\left(n + \frac{1}{2}\right)^2 a_n - (n+1)^2 a_{n+1}\right]\lambda^n = 0. \tag{15}$$

$a_0 = 1$ and

$$a_{n+1} = \left[\frac{n+\frac{1}{2}}{n+1}\right]^2 a_n \implies a_n = \begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix}^2.$$

Alternatively, the coefficient a_n can be obtained from **Chu-Vandermonde Identity** on the sum of products of binomial coefficients

$$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}.$$

It is then easy to show equality of the coefficients so that $\Pi(\lambda)$ satisfies the differential equation 13 . From 13, since $\Pi(\lambda) \neq 0$, it remains to fix $\lambda_0 \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ then find the solution $\varphi(\lambda)$ of the new boundary value Picard-Fuch's ODE

$$D^2 - \lambda\left(D + \frac{1}{2}\right)^2 = 0, \text{ subject to } \varphi(\lambda_0) = A,\ \frac{d\varphi}{d\lambda}(\lambda_0) = B \in \mathbb{C}. \tag{16}$$

Theorem 4.4.2.

$$\Pi(\lambda) = [\gamma_1](\lambda) = \sum_{n\geq 0} \begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix}^2 \lambda^n = \begin{cases} \sum_{n\geq 0} \frac{(3n)!}{3^{3n}(n!)^3}\lambda^{3n} & \textit{if } \lambda \neq 0 \\ 1 & \textit{if } \lambda = 0. \end{cases}.$$

Proof. The holomorphic solution can be found as a power series

$$[\gamma_1](\lambda) = \sum_{n\geq 0} a_n \lambda^n$$

from which we have that

$$\frac{d[\gamma]}{d\lambda}(\lambda) = \sum_{n\geq 0}(n+1)a_{n+1}\lambda^n$$

$$\frac{d^2[\gamma]}{d\lambda^2}(\lambda) = \sum_{n\geq 0}(n+2)(n+1)a_{n+2}\lambda^n$$

giving a recursive relation $(n+2)(n+1)a_n = (n+3)^2 a_{n+3}$ with $a_0 = 1$ on coefficients. From this, we solve for

$$a_n = \begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix}^2 = \frac{(3n)!}{3^{3n}(n!)^3}$$

so that

$$\Pi(\lambda) = [\gamma_1](\lambda) = \begin{cases} \sum_{n\geq 0} \frac{(3n)!}{3^{3n}(n!)^3}\lambda^{3n} & \text{if } \lambda \neq 0 \\ 1 & \text{if } \lambda = 0. \end{cases}.$$

$\square$

Bibliography

[CG19] A. Corti and G. Gugiatti *Hyperelliptic Integrals and Mirrors of the Johnson-Kollár del Pezzo Surfaces*,arXiv:1901.09026v1 [math.AG] 2019.

[Sze] B. Szendroi *Cubic curves: a short survey*, ICAMMP 2005.

[Kir92] F. KIRWAN *Complex Algebraic Curves*, Cambridge LMSST 1992.

[ST92] J. H. Silverman and J. Tate *Rational points on elliptic Curves*, Springer-Verlag 1992.

[Rei13] M. REID *Undergraduate Algebraic Geometry*, Cambridge LMSST 2013.

[PS97] V. Prasdov and Y. Solovyev *Elliptic Functions and Elliptic Integrals*, AMS 1997.

[Mir95] R. MIRANDA *Algebraic Curves and Riemann Surfaces*, AMS 1995.