UNIVERSITY OF NAIROBI

CONSTRUCTING A STRING THEORY BACKGROUND USING BRANE-SIMPLEX DUALITY

By:

KINYANJUI CARRINGTONE

Reg No. I56/9841/2018

A Research Project Submitted for Partial Fulfillment of Requirements for the Degree of Master of Science in Physics of the University of Nairobi.

May, 2019

Declaration

I declare that this project is my original work and has not been submitted elsewhere for research. Where other people's work or my own work has been used, this has properly been acknowledged and referenced in accordance with the University of Nairobi's requirements.

Signature:.. Date 23/05/2019

Kinyanjui Carringtone

I56/9841/2018

Department of Physics

School of Physical Sciences

University of Nairobi

This project has been submitted for examination with our approval as research supervisors:

	Signature	Date
Prof. Collins O. Mito	....	21/05/2019

Department of Physics
University of Nairobi
P.O Box 30197-00100
Nairobi Kenya
collins@uonbi.ac.ke

Dr. Dismas S. Wamalwa	....	20/05/2019

Department of Physical Sciences
Meru University of Science and Technology
P.O. Box 972-60200
Meru, Kenya
dwamalwa@must.ac.ke

Dedication

I dedicate this work to my mother, Joyce Wanjiru Njoroge

Acknowledgment

I'd like to begin by acknowledging my family: Joyce Njoroge my mother for instilling curiosity and the mental discipline necessary for academic work in me, my brother Evans Njoroge for supporting me in cash and kind throughout my higher education studies, my brother Ezekiel Chege for being of tremendous support. I also would like to thank my uncle and aunt Haron Njoroge and Mary Njoroge for supporting me throughout my studies and being a second family away from home. I take this opportunity to thank all my extended family for the help and support they've given me through the testing times over the past year.
I thank Professor Lydia Njenga, Mrs. Catherine Njue and Madam Lucy Njeri of the Graduate School,and Professor Julius Mwabora of the Department of Physics. Without the administrative decision they took to support me through a scholarship, I would not have been in school. I thank Professor Peter Wilkinson of the University of Manchester for helpful discussions and criticism which helped me further improve my conceptual understanding of the work.
I acknowledge my supervisors: Professor Collins Mito of the Department of Physics, and Dr. Dismas Wamalwa of the Department of Physical Sciences at the Meru University of Science and Technology. They have been of great assistance ever since I approached them with a rough idea of what I want to do. Through refinements in thinking and helpful discussions and debates, this work has emerged.
The following friends have been of intellectual, moral and financial support: Lorna Njanja, Samson Mulandi Mutunga, Moses Langa Otieno, Brian Kipkirui, Eric Kamau, James Chege, Chelmis Muthoni, Emmanuel Wangila, Lameck Wekesa Muibanda, Bonface Osoro, Dennis Kulvinder Gharial, Paul Akumu, Samyukta Manikumar, Maurice Wafula, Andre Bezou, Othlamane Thibelang, Manoa Andriamanado, Sange Maxaku and Eric Meli.
I finally thank all scientists, philosophers and thinkers, who in their work, contributed directly and indirectly to this research project.

Abstract

String theory is a candidate theory for a unified description of physics. Unification of the micro and macrocosm and also of the four fundamental forces. String theory is however plagued by conceptual potholes, chief of which is the presence of extra dimensions, large or compactified. It also lacks a consistent description of spacetime. This is the problem of quantum gravity. Causal dynamical triangulation is a framework which seeks to quantise gravity by quantising spacetime. This can be done because it accepts the geometrodynamic definition of gravitation from general relativity. The quantum of spacetime is called a simplex. Spacetime is then constructed by "gluing" these simplices together. Causal dynamical triangulation is background independent. We investigate the possibility of generating spacetime using the modes of vibrations of strings in string theory. This is done by adding a version of the Regge action, generated in causal dynamical triangulation, to the Polyakov action of bosonic string theory. After this, the equations of motion are derived from application of the Euler-Lagrange equations on the Polyakov action. The resulting differential equations are solved to generate mode expansions. From these mode expansions, we get the Virasoro operators which can be used to generate the generators of the Lorentz algebra of the theory. The Lorentz algebra of the theory then defines the dimensionality of the model. A similar process is performed for superstring theory. We thust formulate a "chimeric" string model using the Ramond-Neveu-Schwarz formalism in superstring theory. The model exhibits spacetime simplices as a type of mode expansion of the string. It is thus be background independent. There is no effect on the dimensionality on string theory by the new model.

Contents

List of Figures

List of Tables

List of Symbols

- R Ricci scalar
- $R_{\mu\nu}$ Ricci tensor
- $T_{\mu\nu}$ Stress energy tensor
- G Newton's gravitational constant
- c Speed of light
- $g_{\mu\nu}$ Covariant metric tensor
- π Pi
- ψ Bispinor field
- $\bar{\psi}$ Dirac adjoint of the bispinor Field
- γ^{μ} Dirac matrices
- A_{μ} Current four vector
- $F_{\mu\nu}$ Electromagnetic stress-energy tensor
- μ_0 Magnetic permeability of free space
- X^{μ} Ambient spacetime worldsheet coordinates
- l_s Length of the vibrating string
- $\alpha_k^{\mu}, d_k^{\mu}, b_k^{\mu}$ Expansion modes of the vibrating string
- $\eta_{\mu\nu}$ Minkowski metric
- $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$ Four-derivatives
- $\partial_{\pm} = \partial_{\mu} \pm \partial_{\nu}$ Derivative in light cone coordinates
- x^{μ} Ambient spacetime coordinates
- p^{μ} Conjugate momentum coordinates
- $X(\tau, \sigma)$ Worldsheet coordinates
- $\dot{X}, X'$ Derivatives of the worldsheet coordinates with respect to τ and σ respectively
- T String tension
- $h^{\alpha\beta}$ Auxiliary field("Gauge")
- k Wave number
- $\delta_{\mu\nu}$ Kronecker delta
- L_m Virasoro operator

- M Mass of the expansion modes
- N Number of expansion modes
- D Number of dimensions
- $F(g_1, g_2; t)$ Path integral over all spacetimes over a time t
- g_1, g_2 Initial and final spacetime configurations respectively
- S_{EH} Einstein-Hilbert action

1 Chapter One: Introduction

1.1 General Introduction

Unification has been a particularly challenging problem in theoretical physics for the last half century. Unification proposes a coherent, single, and simple description of the four fundamental forces using one mathematical framework, replacing the current disparate descriptions of reality: The standard model of particle physics, and the general theory of relativity (McMahon, 2009).

Several mathematical frameworks have already been proposed.The frameworks can be placed under two general approaches: The quantum gravity approach, and the geometrodynamical approach. Because of the mathematical sophistication of quantum field theory, quantum gravity, which seeks a quantum field theory of gravity, is preferred to the geometrodynamical approach. String theory (McMahon, 2009), loop quantum gravity (Perez, 2009), and causal dynamical triangulation (Forcier, 2011) are examples of quantum gravity theories.

1.2 The Problem of Quantum Gravity

There are four fundamental forces in the universe:

i)The strong force

ii)The weak force

iii)The electromagnetic force

iv)The gravitational force

Because of the work of Feynman (Feynman,1948) and Schwinger (Schwinger, 1948), there exists a quantum theory of electromagnetism. In the theory, electromagnetic attraction and repulsion are modeled by the exchange of virtual particles. In the case of quantum electrodynamics, these particles are photons. This is known as the theory of quantum electrodynamics.

The lagrangian of quantum electrodynamics can be written as (McMahon,2008):

$$L_{QED} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\psi\bar{\psi} - q\bar{\psi}\gamma^{\mu}\psi A_{\mu}. \tag{1.1}$$

where:

L_{QED} is the Lagrangian of quantum electrodynamics,

μ_0 is the magnetic permeability of space,

$F_{\mu\nu}$ and $F^{\mu\nu}$ are the electromagnetic stress-energy tensors in covariant and contravariant forms respectively,

$i = \sqrt{-1}$ is the imaginary number,

m is the mass of the electron, ψ and $\bar{\psi}$ are the Dirac field and its complex conjugate respectively,

q is the charge coupling to the free fields,

γ^{μ} are the Dirac matrices,

A^{μ} is the four-current.

From the action, the equations of motion can be derived. In the interaction picture, the equations of motion can be difficult to keep track of. The coupling of these fields represents a challenge. This informs our use of the Feynman diagram shown below, in the case of an electron-electron scattering event(McMahon, 2009):

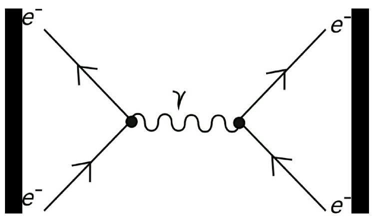

Figure 1.1: Feynman diagram for the scattering of electrons

The Feynman diagrams help us track of interactions and hence, calculate their probability in quantum electrodynamics.

Similar theories exist for the strong and weak nuclear force, where the exchange particles are gluons and W/Z bosons. The theories that describe these forces are known as quantum chromodynamics, and quantum flavordynamics respectively (Buchmuller, Ludelling 2006).

It has been shown that these theories derive their mathematical structure from the theory of groups. Under these symmetry groups, it is possible to unify these theories. For example, the symmetry group governing quantum electrodynamics is the $U(1)$ symmetry group [6]. Quantum flavordynamics on the other hand is governed by $SU(2)$ symmetry group. Therefore, by considering $U(1) \times SU(2)$ we unify the electromagnetic and the weak nuclear force. The resulting theory is called the electroweak theory, developed by Sheldon Glashow (1959) and Abdus Salam (Salam and Ward, 1959).

A similar class of theories exist that seek to unify electroweak theory and quantum chromodynamics. These class of theories are known as grand unified theories. The quantum field theory that describes these interactions is known as the standard model (Buchmuller and Ludeling, 1959) .

Gravitation remains particularly challenging. The main reason is that gravitation is not described by a quantum field theory. Gravitation is modeled by the general theory of relativity, proposed by Einstein (1915). In general relativity, gravity is understood to be the result of the curvature of spacetime, a four dimensional Hausdorff manifold. This curvature is caused by coupling to the stress energy tensor $T_{\mu\nu}$, a relativistic generalisation of mass. This is codified in the Einstein field equations (McMahon, 2006):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}. \tag{1.2}$$

where:

$R_{\mu\nu}$ is the Ricci tensor,

$g_{\mu\nu}$ is the metric which solves the Einstein Field Equations,

R is the Ricci scalar,

$\kappa = \frac{8\pi G}{c^4}$ is the coupling constant, $T_{\mu\nu}$ is the stress-energy tensor.

The equations are stated without the cosmological constant. The solution of these equations is the metric $g_{\mu\nu}$, of which the Minkowski $\eta_{\mu\nu}$ is the simplest case:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.3}$$

The metric describes a continuous, infinitely differentiable manifold. This is inconsistent with the general research program of quantisation.

For gravity to be unified with the other forces, it needs to be rewritten in the language of quantum field theory: quantum gravity. There exist quantum fields that can describe gravitation (Feynman, 1995). However, as will be expounded in the next section, the quantisation of the fields presents a challenge because of their non-renormalizability. Because of this, "naive" quantisation of the gravitational fields fails to be predictive (McMahon, 2009). This is the problem of quantum gravity.

1.3 Background Independence

Background independence is the ability of a model to produce the space in which it "lives", using internal mechanisms(Smolin, 2009). Background independence has the mathematical benefit of making the theories that possess it coordinate free. Coordinate-free descriptions have the virtue of allowing the laws of physics to be invariant with respect to coordinate transformations. This is in general desirable for a physical theory.

An example of a theory which suffers background dependence is classical mechanics. In the model, Euclidean vector space is clearly preferred. It assumes the axioms of Euclidean space to hold true generally. While Newtonian laws hold some form of invariance with respect to coordinate transformations, the model does not interact consistently with classical electrodynamics. This necessitates the introduction of special relativity (McMahon, 2006). The new model has a restricted notion of invariance with respect to Lorentz transformations. General relativity removes these restrictions by introducing the notion of general covariance (McMahon, 2006). Thus, general relativity restructures its own spacetime by associating it with a variable metric tensor. By definition, tensors are generally covariant. Thus general relativity can be said to have manifest background independence.

The solution of the Einstein field equations is a metric which then decribes the topology of the ambient spacetime. This is another way of defining background independence: the ability of a model to have its ambient spacetime as solutions of equations generated within the theory.

Quantum mechanics is also not background independent. Time is not defined within the theory. Time is thought to be observer-defined. The role played by space in the theory is also not clear. Quantum field theory is a relativistic generalisation of quantum mechanics. The theory seeks to understand the behaviour of forces using the exchange of virtual particles. The fields on which the particles are based are known as relativistic quantum fields. This poses a conceptual challenge: relativity requires that space and time are put on the same footing: all backgrounds are created equal. In standard non-relativistic quantum mechanics, an operator exists that describes a particle's position in space, but none exists that can do the same for time.

Quantum mechanics is thus inconsistent with relativity. Quantum field theory resolves this impasse by demoting the position operator to a parameter (McMahon, 2006). The relation of space and time is parametrised by the Minkowski metric. Physically this means that quantum field theory is defined on an external ambient spacetime. Therefore, it is not background independent.

The status of background independence in "theory of everything" type of theories is an active area of research. Our position is that theories of everything ought to be background independent: they are after all theories of everything (Smolin, 2009).

This informs our current interest in string theory. We have earlier introduced string theory as a generalisation of quantum field theory. It therefore occurs in an ambient relativistic spacetime. By this association, string theory is able to preserve Lorentz invariance. However, with this "virtue" comes a "vice". The theory is background dependent. There exist schemes that can introduce background independence into string theory. An example is to use AdS/CFT correspondence into the theory. AdS here refers to an Anti-De Sitter spacetime. It is a spacetime with global negative curvature. It is a specific solution of the Einstein field equation. CFT refers in full to conformal field theory. Such theories maintain invariance under conformal transformations. Part of conformal transformations is the change of scale. Thus conformal theories are scale invariant. They can be thought of as generalizations of quantum field theories which allow string theories to be expressed in a simpler and more useful form.

AdS/CFT correspondence postulates that a conformal field theory on the surface of an AdS hyperplane conforms to a gravitational type theory on the volume bounded by the surface.

However, it is not clear how AdS/CFT can be introduced for any general spacetime topology.The correspondence has not been established for any general topological transformations. Proving this to be the case will introduce some form of invariance into string theory. It is not agreed that string theory has background independence. What is needed is a method of generating a background internally within string theory. This will be the main thrust of our work.

The causal dynamical triangulation model is manifestly background independent. The theory seeks to construct spacetime from 4-simplices. The possible configurations of these 4-simplices are analogous to the possible solutions of the Einstein field equations.The arrow that is affixed onto the 4-simplice is a primordial notion of time. It imposes directionality on the fourth dimension. While it can be argued that causality has been imposed from external considerations to the theory, causality is a philosophical predisposition, it is not a background. Thus it is evident that causal dynamical triangulation is a background independent model.

1.4 Statement of the Problem

String theory has successfully merged the four fundamental forces. It has also shown all the matter particles to be the result of supersymmetric transformations on bosonic states. However, the mechanism of the generation of spacetime in the theory is not well explored. In fact, in standard string theory, the existence of a continuous spacetime, defined by an external theory, special relativity, is assumed. For this reason, we hold that string theory is not background independent.

1.5 Main Objective

To establish a string-simplex duality which will enable simplex-string transformations, thus introducing a spacetime background to string theory.

1.5.1 Specific Objectives

I. To Establish string-simplex duality for bosonic string theory.

II. To investigate the effect of string-simplex duality on the dimensionality of bosonic string theory.

III. To generalise string-simplex duality in superstring theory.

1.6 Justification

String theory is a promising "theory of everything". However, it is unable to describe spacetime internally. As such, this has crippled its potential of being a self-contained theory. The preference given to a specific spacetime may limit the development of the theory. An example of such is the fact that limiting mechanics to a Euclidean vector space inherently limits any possible relativistic generalisation. As such, we hold the position that string theory needs a "richer" space. We propose one way of building such a space. The ability of string theory to describe spacetime within its own internal mechanisms may help us improve our understanding of the theory. This will further the aim of unification. Unification of all forces might propel humanity far beyond what our futurists can imagine.

2 Chapter Two: Literature Review

2.1 Quantum Field Theory

Quantum field theory began with the Dirac equation (Dirac, 1928), which applied to electrons.The attempt at describing the Dirac equation as governing single particles led to contradictions. It was eventually reinterpreted as a quantum field theory. Dirac then considered a quantum mechanical theory of the electromagnetic field (Dirac, 1927). In the theory Dirac quantised the electromagnetic field in the case of the electron and the photon. He also showed the transition between energy levels as the result of change in photon numbers. Later, Born and Jordan developed the second quantization formalism in which "creation" and annihilation operators were developed(Heisenberg *et al.*, 1925). It was a full formalism developed without regard to the conservation of particle numbers (Jordan, 1927). Thus, a search for a fundamental theory of particles was needed. The expression of the electromagnetic field using harmonic oscillators was also done by Born and Jordan. However, a problem of infinities persisted. It had to do with the self energy of the electron, or more clearly: what is the strength of an electromagnetic field as one arbitrarily approaches an electromagnetic source?

This was resolved by Schwinger(1948) and Feynman(1948) by a process that came to be referred to as renormalisation. A full, covariant description of the electromagnetic field was finally completed. It came to be known as quantum electrodynamics. Later the mathematical structure of quantum electrodynamics came to be understood as the $U(1)$ symmetry group. Glashow(1959) and Salam(1959) later found a symmetry group $U(1) \times SU(2)$ which unified the electromagnetic and weak interaction. Weinberg showed that under the action of a Higgs-like field, some vector bosons of the electro-weak model acquire mass by symmetry breaking:the W/Z bosons acquire mass while the photon remains massless (Weinberg, 1967). At this point, the search for a quantum field theory of the strong interaction thus began.

2.2 String Theory

Heisenberg had earlier thought that the notion of spacetime broke down on a sub-nuclear level. Following his typically positivistic intuitions, Heisenberg considered only what goes into a sub-nuclear scattering event, and what gets out (Heisenberg, 1943). The matrix which describes this event is called an "S-matrix". It was realised that the S-matrix did not contain enough information to determine the probability of scattering events in the theory(Heisenberg, 1943) .

Gell-Mann developed a dispersion relation (Gell-Mann, 1956) which was thought to add information to the "S-Matrix" theory developed by Heisenberg, in addition to the principle of Unitarity developed by himself. Later, Veneziano showed that a specific function known as the Euler beta function, was able to describe four particle scattering events in the sub-nuclear region(Veneziano, 1968). This was later generalised to the scattering of N particles by Koba and Nielsen(1969).

Later, Nielsen (Koba *et al.*,1969) and Susskind(1969) interpreted this function to be the physical description of the scattering of vibrating strings. This marked the birth of string theory as a physical theory. However, string theory was supplanted by quantum chromodynamics as the favourite model for strong interactions (Yukawa, 1935). Revival of interest in string theory came about when Yoneya showed that string theory predicted the existence of the graviton (Yoneya, 1974). It was also noted that the spectrum of the Hamiltonian could describe forces in the standard model. This model came to be known as bosonic string theory. However, the model faced some challenges. Lovelace showed that it only worked consistently in 26 dimensions and had some anomalies (Lovelace, 1971). Ramond (1971), Neveu and Schwarz(1971) showed the elimination of anomalies by the implementation of a duality known as supersymmetry. Gervais and Sakita implemented the spacetime supersymmetry established by Green and Schwarz to generate fermionic states (Gervais *et al.*, 1971). By 1980, there were five consistent string theories, as listed in the introduction. Since it was thought that string theory was a potential theory of everything, this was a bit strange.

Witten showed that the five versions of superstring theory were actually asymptotic, lowenergy versions of single theory (Witten, 1995). It was later demonstrated by Polchinski that for $N = 1$ supergravity to be consistent in eleven dimensional superstring theory, the theory had to contain n-dimensional analogues of strings (Polchinski, 1995). He called these objects Dirichlet Branes: D-Branes. The new theory formed from the Witten-like dualities and these D-Branes came to be referred to as M-Theory.

Later, Douglas showed that flux compactifications led to different string vacua. The different string vacua had different coupling constants (Douglas, 2003). This implies that the vacua correspond to different cosmologies. As such, it became important to understand the number of possible string vacua. A number was suggested by Ashok and Douglas to be 10^{500} (Ashok *et al.,* 2004). Maldacena proposed that a conformal field theory defined on a surface is similar to string theory defined supergravity defined on an anti-De Sitter bulk (Maldacena, 1998). This was referred to as AdS/CFT correspondence. This principle was helpful in deriving the Hawking Black Hole Entropy formula from first principles. Using this notion, Horowitz showed that it is possible, at least in principle, to construct spacetime on the CFT defined in the bulk6 of AdS geometry (Horowitz, 2005).

Recently, it has been hypothesised that string vacua with a positive cosmological constant are unstable in string theory, they are colloquially referred to as the "swampland of solutions" (Obied, 2018). This is interesting, since our cosmology requires a positive cosmological constant(McMahon, 2006).

2.3 Causal Dynamical Triangulation

Whereas the background independence of string theory cannot yet be established, other approaches to quantum gravity build spacetime from first principles and as such are manifestly background independent. An early approach that attempted to do this was Euclidean quantum gravity(Hawking, 1977). The model however had some failures. An improvement on the theory came from imposing causality on the simplicial structure of spacetime in Euclidean Quantum Gravity. This was done by Ambjorn et. al.(Ambjorn *et al.*, 2002) and gave rise to the theory of causal dynamical triangulations. The model was able to reproduce the global structure of four dimensional spacetime, and was compatible with the notion of a positive cosmological constant.

In this work we shall put together the ideas of symmetry developed from quantum field theory to combine string theory and causal dynamical triangulation into a single model.

3 Chapter Three: Theoretical Background

3.1 String Theory

String theory refers to a general class of theories that hope to solve the quantum gravity problem by replacing the point particles of quantum field theory by minute vibrating strands of energy (McMahon, 2009).

In the theory, the gravitational field is modeled by the action of a spin-2 quantum field, whose behaviour is the equivalent of curved spacetime in general relativity. This is because the Einstein-Hilbert action can be obtained by considering the behaviour of a nonlinear, massless, spin-2 quantum field (Feynman, 1995). This is is formalised in a type of string theory known as bosonic string theory. Bosonic string theory is formulated by deriving the equation of a relativistic vibrating string: the Nambu-Goto Action (McMahon, 2009):

$$S = -T \int_{\tau_2}^{\tau_1} d\tau \int_0^l d\sigma \sqrt{\dot{X} X'^2 - (\dot{X} \cdot X')^2}. \tag{3.1}$$

where:

T is the string tension,

τ is the worldsheet time coordinate,

σ is the worldsheet spatial coordinate,

X and X' are the worldsheet and its derivatives.

This is later reformulated into the Polyakov action using an auxiliary gauge field. This is to make the quantisation of the action easier:5

$$S_P = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{3.2}$$

where:

T is the string tension,

τ is the worldsheet time coordinate,

σ is the worldsheet spatial coordinate,

X^μ and $\partial_\alpha X^\mu$ are the worldsheet and its derivatives,

$h^{\alpha\beta}$ is the metric of the auxiliary field,

$\sqrt{-h}$ is the determinant of the metric,

$\eta_{\mu\nu}$ is the Minkowski metric.

From this equation, we can derive the equations of motions for bosonic string theory:

$$\partial_{-}\partial_{+}X^{\mu} = 0 \tag{3.3}$$

where:

X^{μ} is the worldsheet,

$\partial_{-} = (\partial_{\tau} - \partial_{\sigma})$ and $\partial_{+} = (\partial_{\tau} + \partial_{\sigma})$ are derivatives in light-cone coordinates.

From the equations of motions, we can solve for the spacetime coordinates X^{μ} in the left and right moving case:

$$X^{\mu}_{R/L} = \frac{x^{\mu}}{2} + \frac{l_s^2}{2}p^{\mu}(\tau \pm \sigma) + i\frac{l_s}{\sqrt{2}}\sum_{k\neq 0}\frac{\alpha^{\mu}_{k}}{k}e^{-ik(\tau\pm\sigma)}. \tag{3.4}$$

where:

$X^{\mu}_{R/L}$ are the left-moving and right-moving solutions,

x^{μ} is the four-position,

l_s is the length of the string,

p^{μ} is the four-momentum,

τ is the worldsheet time coordinate,

σ is the worldsheet spatial coordinate,

α^{μ}_{k} are the mode expansions, interpretable as the modes of vibration of the string.

After this, appropriate boundary conditions can be explored. We can also quantise the string action by imposing commutation relations on the mode expansions α^{μ}_{k}:

$$[\alpha^{\mu}_{k}, \alpha^{\mu}_{l}] = m\eta^{\mu\nu}\delta_{m+n,0}. \tag{3.5}$$

$\eta^{\mu\nu}$ is the Minkowski metric,

$\delta_{m+n,0}$ is the Kronecker delta.

The mode expansions are actually operators. We can use them to define new operators called Virasoro operators:

$$L_m = \frac{1}{2}\sum_m : \alpha_{m-n}.\alpha_n : . \tag{3.6}$$

where :: denotes a normal ordered product.

String theory actually works by replacing the zero dimensional particles in the Feynman diagram by extensible strings. Instead of worldlines, we have worldtubes for closed strings, and worldsheets for open-ended strings. We can therefore easily construct feynman diagrams using these notions(McMahon, 2009):

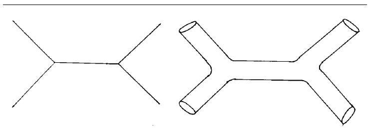

Figure 3.1: Feynman diagram for point particle adjacent to a Feynman diagram in string theory

Because of the extensibility of the strings, it is possible to calculate the scattering amplitude of gravitons resulting in finite results. Thus in string theory, it is possible to quantise the gravitational field.

As discussed above, the earlier models of spin-2 quantum fields were plagued by non-renormalizability. This means that unresolvable divergences existed when one considered the scattering matrix of the theory. By extending the formulation of the zero dimensional graviton in quantum field theory to one dimensional vibrating strings, bosonic string theory resolves this problem.

However, bosonic string theory is plagued by anomalies. The ground state of bosonic string theory is a tachyon field. This can be established by using the Virasoro operator to generate a mass-shell condition:

$$M^2 = -\frac{1}{\alpha'}, \tag{3.7}$$

where:

$$\alpha' = \frac{1}{2\pi T} \tag{3.8}$$

, M^2 is the mass-squared operator,

T is the string tension,

This is the mass shell condition got from the ground state. Tachyon fields present a serious challenge to the theory because it is formulated relativistically. The theory of relativity disallows the formation of tachyonic fields. Also, the presence of tachyonic fields mean that the vacuum of bosonic string theory is unstable. Moreover, the theory fails to consistently describe fermionic states. The theory also suffers from extra dimensions, having a consistent description at twenty-six dimensions. This can also be established using the mass shell condition:

$$M^2 = \frac{1}{\alpha'}\left(N - \frac{D-2}{24}\right). \tag{3.9}$$

where:

N is the number of string states,

D is the number of dimensions,

α' is the inverse of the string tension as earlier defined.

Requiring the second term to be 1 forces $D = 26$.

String theory attempts to solve these problems by invoking supersymmetry(McMahon, 2009). It is an extension of quantum field theory that establishes a duality between fermionic and bosonic states. This is done by introducing a supersymmetric charge operator, which implements the duality. Fermions are particles of half integer spin. Bosons are particles of integer spin. Mathematically, bosons are associated with symmetric wavefunctions, whereas fermions are associated with antisymmetric wavefunctions. Thus, a supersymmetric charge operator transforms anti-symmetric wavefunctions to symmetric wavefunctions and vice versa.

Supersymmetry can be implemented by adding the Dirac field action to the Polyakov action (McMahon, 2009):

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\left(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right). \tag{3.10}$$

where:

T is the string tension,

τ is the worldsheet time coordinate,

σ is the worldsheet spatial coordinate,

X^μ and $\partial_\alpha X^\mu$ are the worldsheet and its derivatives,

$h^{\alpha\beta}$ is the metric of the auxiliary field,

$\sqrt{-h}$ is the determinant of the metric,

$\eta_{\mu\nu}$ is the Minkowski metric.

$i = \sqrt{-1}$ is the imaginary number,

m is the mass of the electron, ψ and $\bar{\psi}$ are the Dirac field and its complex conjugate respectively,

ρ^α are the Dirac matrices,

From the Lagrangian we can build new solutions for fermions and bosons respectively as:

$$\psi^\mu_\pm(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_n d^\mu_n e^{in(\tau \mp \sigma)} \tag{3.11}$$

$$\phi^\mu_\pm(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_n b^\mu_n e^{in(\tau \pm \sigma)}. \tag{3.12}$$

where:

$psi^\mu_\pm$ is the fermionic solution.

d^μ_n are the new mode expansions.

The result is string theory with supersymmetry: superstring theory. It is able to describe bosonic and fermionic states.The number of dimensions needed to describe the new model consistently is ten. Superstring theory is actually a set of five self-consistent theories with different features(McMahon, 2009):

i) Type I string theory;

ii) Type IIA string theory;

iii) Type IIB string theory;

iv) $SO(32)$ theory;

v) $E_8 \times E_8$ theory.

By duality transformations, it can be shown that one class of string theory can be transformed into another (Witten, 1995). The dualities are known as conifold , T, and S dualities. These dualities can be argued to be evidence that these dualities are in fact sub-classes of a more fundamental theory: M-Theory.

3.2 Causal Dynamical Triangulation

Causal dynamical triangulation theory is an improvement on Euclidean quantum gravity (Forcier, 2011), (Ambjorn et al., 2002). Euclidean Quantum Gravity was proposed by Hawking (1977). In the model, the techniques of quantum field theory are applied to spacetime. It is taken to be a quantum field. A quantum field usually has excitations. In the case of the Higgs field, it is the Higgs boson.

The excitation of the four dimensional spacetime quantum field is the 4-simplex. It is often helpful to think of them as the four dimensional analogues of triangles. The hope of euclidean gravity is to construct spacetime from these four simplices. The standard approach of analysing the behaviour of quantum fields is using path integrals. In the case of spacetime this can be formalised as (Forcier, 2011):

$$F(g1, g2; t) = \sum e^{iS_{EH}}. \tag{3.13}$$

where,

$F(g1, g2; t)$ is the propagator from a certain simplice configuration $g1$ to a second simplice configuration $g2$ over a time period t,

S_{EH} is the Einstein-Hilbert action.

However, when the construction is done, the model fails (Ambjorn et al, 2002).

It results in an infinite dimensional spacetime. Each point in the spacetime is close to all other points. This spacetime pathology is not descriptive of our universe. Large scale description of the spacetime fails. The reason for this failure is the assumptions implicit in Euclidean quantum gravity. One of the main tools of quantum gravity is Wick rotations. A Wick rotation makes the replacement: $t \to it$. The effect on the metric will be such that the length of line element becomes:

$$ds^2 = -c^2dt^2 - (dx)^2 - (dy)^2 - (dz)^2. \tag{3.14}$$

where:

ds^2 is a length of line element squared,

c is the speed of light,

dx,dy and dz are infinitesimal elements along the x , y and z axes respectively,

dt is an element of time.

The result is to place space and time on an equal footing. The implication of this is that we have the possibility of moving both forward and backwards in time. This violation of causality also allows the formation of wormholes. These wormholes result in the multiply connected, infinite dimensional, we have discussed above.

Causal dynamical triangulation deals with the above problem by invoking causality(Forcier, 2011). Colloquially, causality is the notion that effect comes after cause. The principle fixes directionality to time.

Operationally, an arrow is assigned to all simplices. In gluing the simplices together, care is taken to make sure that the simplices have their "time arrows" alligned. This method of implementing causality disfavours the formation of wormholes. The problem of a pathological global spacetime is thus avoided.

Also, since time is included in the model, the spacetime geometry is dynamic and can allow for topological evolution of the spacetime geometry. The model develops four dimensional spacetime from first principles. It also has within it a mechanism for the inclusion of dark energy. This is favoured by the $\lambda - CDM$ model of cosmology (Forcier, 2011).

4 Chapter Four: Methodology

4.1 Introduction

The improvement of theories to accommodate new phenomena is usually done by extending the Lagrangian of the theory. In this section we introduce a spacetime background to string theory by doing this. We first add the Regge Lagrangian to the Polyakov Lagrangian. We then investigate duality transformations which leave the resulting action invariant, this is done by investigating for conserved currents on the worldsheet. We then investigate for conserved stress-energy tensor.

Equations of motion are then found by applying the principle of least action on the new action. Solving these equations yields mode expansions with appropriate boundary conditions. We investigate the properties of these mode expansions with respect to the distance operator of causal dynamical triangulations.This then allows us to quantise the theory. This is done by applying Virasoro algebra on the mode expansions, in essence specifying a commutator for the modes of vibration.

After this, the dimensionality of the theory is investigated. This is done by specifying a commutator of the generators of the Lorentz algebra. By specifying a normal ordering constant, we investigate the dimensionality of the theory.

We repeat above the same process, in the new case replacing Virasoro algebra by Super-Virasoro algebra.

4.2 Extension of Bosonic String Theory

4.2.1 Introduction

As discussed, we invoke the Ramond-Neveu-Schwarz(RNS) formalism to extend bosonic string theory. In the formalism, the Lagrangian of bosonic string theory, the Polyakov action, is extended by adding it to the Dirac Lagrangian. Symmetry principles are imposed between the Dirac Spinors ψ and the Bosonic worldsheet currents X^{μ}. These symmetries are then used to formulate the theory. This is done by defining invariants of transformation, specifying a stress-energy tensor and deriving the equations of motions of the string.

We follow this formalism in extending bosonic string theory to account for spacetime, using causal dynamical triangulations. In our case, the spinors are not fermions, but are simplices, whose properties are specified by causal dynamical triangulation. There are advantages and disadvantages of using the RNS scheme. The scheme implements duality transformations using supersymmetry, a physical hypothesis with no observational or experimental evidence. It is also not clear whether the RNS formalism can be generalised to other uses, apart from implementing boson-fermion duality. Nonetheless, we use it because it is well defined. It has a well formulated algebraic structure whose implementation in string theory has been extensively studied and documented.

We extend the Lagrangian of bosonic string theory by adding it to a reformulated Regge action. This is followed by an exploration of the symmetries of the action, beginning with spatial translation. We then take variations of the Lagrangian, investigating for conserved currents. This result will be used to derive a simplicial stress-energy tensor. We establish boundary conditions for the action. Finally, the equations of motion got from the zero divergence of the stress energy tensor will be solved to derive the mode expansions for the simplices. We now begin by extending the Lagrangian.

4.2.2 Extending The Lagrangian

We wish to add the Lagrangian of causal dynamical triangulation (CDT) to the Polyakov Lagrangian:

$$L_C = L_P + L_{CDT} \tag{4.1}$$

where:

L_C is the resulting Lagrangian, the “Chimeric” Lagrangian,

L_P is the Polyakov Lagrangian,

L_{CDT} is the CDT Lagrangian. We can rewrite (4.1) in terms of the action, which is the time integral of the Lagrangian, i.e;

$$S_C = S_P + S_{CDT} \tag{4.2}$$

where:

S_C is the resulting action,

L_P is the Polyakov action,

L_{CDT} is the CDT action.

We can rewrite (1) in terms of the action, which is the time integral of the Lagrangian. Now;

$$S_P = -\frac{T}{2}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu \tag{4.3}$$

and,

$$S_{CDT} = \frac{2\Lambda}{16\pi G}\sum_n V_n \tag{4.4}$$

where:

Λ is the cosmological constant,

V_n is the volume of the simplices,

T is the string tension,

σ is the worldsheet spatio-temporal coordinate,

G is the Gravitational constant,

X^μ and $\partial_\alpha X^\mu$ are the worldsheet and its derivatives.

Putting equations (4.3) and (4.4) in equation (4.2),we obtain:

$$S_C = -\frac{T}{2}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu + \frac{2\Lambda}{16\pi G}\sum_n V_n \tag{4.5}$$

We want to write the second term on the right hand side of equation (4.5) in a way that is amenable to the RNS formalism. We consider V_n by writing it as the integral of a certain density function $n(T)$ such that:

$$\sum_n V_n = \int d^2\sigma n(T) \tag{4.6}$$

We now rewrite equation (4.4) as:

$$S_{CDT} = \frac{2\Lambda}{16\pi G}\int d^2\sigma n(T) \tag{4.7}$$

Adding equation (4.7) to equation (4.3), we get:

$$\begin{aligned} S_C &= -\frac{T}{2}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu + \frac{2\Lambda}{16\pi G}\int d^2\sigma n(T) \\ &= -\frac{T}{2}\int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - \frac{2}{T}\frac{2\Lambda}{16\pi G} n(T)\right) \\ &= -\frac{T}{2}\int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - \beta n(T)\right) \end{aligned} \tag{4.8}$$

where: $\beta = \dfrac{4\Lambda}{16\pi TG}$.

We now introduce a postulate of the model. The assumption is that the simplices which contribute to the density function $n(T)$ are spinors. We shall however not rigorously prove this statement. Intuitively, simplices can point forwards or backwards in time. This means that they are describable by two numbers. This is analogous to the bispinor fields ϕ that are used to describe fermions. We implement this postulate mathematically:

$$n(T) = \bar{\phi}\phi. \tag{4.9}$$

Where:

ϕ is the spinor,

$\bar{\phi}$ is its complex conjugate. We now rewrite equation (4.8)

$$S_C = -\frac{T}{2}\int d^2\sigma\,(\partial_\alpha X^\mu \partial^\alpha X_\mu - \beta\phi^*\phi) \tag{4.10}$$

Consider the behaviour of the derivative of spinors:

$$\begin{aligned} \partial_\alpha \phi &= \kappa\phi \\ \therefore \frac{1}{\kappa}\partial_\alpha\phi &= \phi \end{aligned} \tag{4.11}$$

Where κ is an eigenvalue of the derivative operator. We insert equation (4.11) in equation (4.10) and absorb κ in β giving,

$$S_C = -\frac{T}{2}\int d^2\sigma\left(\partial_\alpha X^\mu \partial^\alpha X_\mu - \bar{\phi}\beta\partial_\alpha\phi\right). \tag{4.12}$$

We can thus read off the definition of the Chimeric Lagrangian:

$$L_C = -\frac{T}{2}\left(\partial_\alpha X^\mu \partial^\alpha X_\mu - \bar{\phi}\beta\partial_\alpha\phi\right) \tag{4.13}$$

and can thus define the causal dynamical triangulation Lagrangian L_{CDT}:

$$L_{CDT} = \frac{T}{2}\bar{\phi}\beta\partial_\alpha\phi.$$

4.2.3 Symmetries

We now investigate the symmetries of the system. It is expected that the Lagrangian and correspondingly, the action, will be invariant under certain transformations of the system. This will result in the fact that the action-derived equations of motion will maintain the same invariance with respect to these transformations. Physically, this means that the laws of physics derived from this action are invariant with respect to these specified transformations.

Before we investigate the symmetries of the system we have set up, it is important that we introduce duality transformations between X^μ and ϕ, i.e;

$$\delta X^\mu = \bar{\epsilon}\phi^\mu \tag{4.14}$$

$$\delta\phi^\mu = \partial_\alpha X^\mu \epsilon \tag{4.15}$$

where:

δ are variations,

$\epsilon, \bar{\epsilon}$ are Grassman numbers.

X^μ are worldsheet coordinates,

ϕ^μ are the spinors.

Time should be taken to interpret equations (4.14) and (4.15), since they form the crux of our work. These are the brane-simplex duality transformations. The implication of the analogues of these equations in the standard RNS formalism is that there is a non-zero probability that bosons can transform into fermions and vice versa. This is supersymmetry, and hence superstring theory. In our case, the equations are to be interpreted to mean that simplices can transform into bosons, and vice versa. A duality exists between the building blocks of matter, forces and those of spacetime. These building blocks are essentially the same entity. We posit that at the order of the Planck length, a distinction between these two building blocks cannot be made.

We now investigate invariance under spatial translations. We can displace the worldsheet coordinates by a value b^μ:

$$X^\mu = X^\mu + b^\mu \tag{4.16}$$

The simplices are not functions of the worldsheet coordinates so they are not subjected to the displacement. Putting equation (4.16) in equation (4.13):

$$\begin{aligned} L'_C &= -\frac{T}{2}\left[\partial_\alpha(X^\mu + b^\mu)\partial^\alpha(X_\mu + b_\mu) - \bar{\phi}\beta\partial_\alpha\phi\right] \\ &= -\frac{T}{2}\left[\partial_\alpha(X^\mu + b^\mu)\partial^\alpha(X_\mu + b_\mu)\right] + L_{CDT} \end{aligned}$$

$$= -\frac{T}{2}\left[(\partial_\alpha X^\mu + \partial_\alpha b^\mu)(\partial^\alpha X_\mu + \partial^\alpha b_\mu)\right] + L_{CDT}$$
$$= -\frac{T}{2}\left[\partial_\alpha X^\mu \partial^\alpha X_\mu + \partial_\alpha X^\mu \partial^\alpha b_\mu + \partial^\alpha b_\mu \partial^\alpha X_\mu + \partial_\alpha b^\mu \partial^\alpha b_\mu\right] + L_{CDT}$$
$$= -\frac{T}{2}\left[\partial_\alpha X^\mu \partial^\alpha X_\mu\right] + L_{CDT} - \frac{T}{2}\left[\partial_\alpha X^\mu \partial_\alpha b^\mu + \partial_\alpha b^\mu \partial_\alpha X^\mu\right]$$

where we have dropped the second order term in the derivative of the perturbation: $\partial_\alpha b^\mu \partial^\alpha b_\mu$.

The first two terms are our Lagrangian:

$$\begin{aligned} L'_C &= L_C - \frac{T}{2}\left[\partial_\alpha X^\mu \partial_\alpha b^\mu + \partial_\alpha b^\mu \partial_\alpha X^\mu\right] \\ -\delta L &= -\frac{T}{2}\left[\partial_\alpha X^\mu \partial^\alpha b_\mu + \partial_\alpha b^\mu \partial^\alpha X_\mu\right] \\ \delta L &= \frac{T}{2}\left[\partial_\alpha X^\mu \partial^\alpha b_\mu + \partial_\alpha b^\mu \partial^\alpha X_\mu\right] \end{aligned} \tag{4.17}$$

We wish to rearrange equation (4.17) in using index permutation and the Minkowski metric. First, we reorder the product considering commutativity.

$$\begin{aligned} \partial_\alpha b^\mu \partial^\alpha X_\mu &= \partial^\alpha X_\mu \partial_\alpha b^\mu \\ &= \partial^\alpha(\eta_{\mu\nu} X^\nu)\partial_\alpha(\eta^{\mu\sigma} b_\sigma) \\ &= \eta_{\mu\nu}\eta^{\mu\sigma}\partial^\alpha X^\nu \partial_\alpha b_\sigma \\ &= \delta^\sigma_\nu \partial^\alpha X^\nu \partial_\alpha b_\sigma \\ \partial^\alpha X^\nu \partial_\alpha b_\nu &= \partial^\alpha X^\mu \partial_\alpha b_\mu \end{aligned} \tag{4.18}$$

Where we have used the property of dummy variables. We can similarly write equation (4.18) as:

$$\begin{aligned} \partial^\alpha X^\mu \partial_\alpha b_\mu &= h^{\alpha\beta}\partial_\beta X^\mu h_{\alpha\gamma}\partial^\gamma b_\mu \\ &= h^{\alpha\beta} h_{\alpha\gamma}\partial_\beta X^\mu \partial^\gamma b_\mu \\ &= \delta^\beta_\gamma \partial_\beta X^\mu \partial^\gamma b_\mu \end{aligned}$$

$$= \partial_\beta X^\mu \partial^\beta b_\mu$$
$$= \partial_\alpha X^\mu \partial^\alpha b_\mu \tag{4.19}$$

application of equations (4.18) and (4.19) in equation (4.17) yields:

$$\delta L = T \partial_\alpha X^\mu \partial^\alpha b_\mu \tag{4.20}$$

We can therefore define a momentum field conjugate to position. This is consistent with the Noether theorem in which translations in coordinate space imply the conservation of momentum. We can thus read off momentum as the term which multiplies the derivative of the displacement b^μ as:

$$P^\mu_\alpha = T \partial_\alpha X^\mu \tag{4.21}$$

where;
P^μ_α are the four momenta,
T is the string tension,
X^μ is the worldsheet We have thus defined an invariance with respect to spatial translations. Finding a momentum field consistent with Noether's theorem.
We now want to take variations of the entire Lagrangian. We shall invoke equations (4.14) and (4.15) to establish symmetries which we use to find conserved currents. We also use these equations to derive a stress-energy tensor for the Lagrangian of causal dynamical triangulation.

4.2.4 Variations of the Lagrangian

We take a variation of equation (4.13):

$$\delta L_C = -\frac{T}{2} \delta \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - \bar{\phi} \beta \partial_\alpha \phi \right) \tag{4.22}$$

where:
δL_C is the chimeric Lagrangian.

We expand equation (4.22) fully as:

$$\delta L_C = -\frac{T}{2}\left(\partial_\alpha(\delta X^\mu)\partial^\alpha X_\mu + \partial_\alpha X^\mu \partial^\alpha(\delta X_\mu) - (\delta\bar{\phi^\mu})\beta\partial_\alpha\phi - \bar{\phi^\mu}\beta\partial_\alpha(\delta\phi)\right)$$

Now, the order of the variation of the worldsheet coordinates does not matter, it is commutative. This equation can be rewritten as:

$$\delta L_C = -\frac{T}{2}\left(2\partial_\alpha(\delta X^\mu)\partial^\alpha X_\mu - (\delta\bar{\phi^\mu})\beta\partial_\alpha\phi - \bar{\phi^\mu}\beta\partial_\alpha(\delta\phi)\right) \tag{4.23}$$

We now invoke the brane-simplex duality transformations. Putting equations (4.14) and (4.15) in equation (4.23) yields:

$$\delta L_C = -\frac{T}{2}\left(2\partial_\alpha(\bar{\epsilon}\phi^\mu)\partial^\alpha X_\mu - (\delta\bar{\phi^\mu})\beta\partial_\alpha\phi - \bar{\phi^\mu}\beta\partial_\alpha(\epsilon\partial_\beta X^\mu)\right) \tag{4.24}$$

Application of the properties of the Grassman numbers on the third term of equation (4.24) gives this equation as:

$$\begin{aligned}\delta L_C = -\frac{T}{2}&\left(2\partial_\alpha(\bar{\epsilon}\phi^\mu)\partial^\alpha X_\mu - (\bar{\epsilon}\partial_\beta X^\mu)\beta\partial_\alpha\phi - \bar{\phi^\mu}\beta\partial_\alpha(\epsilon\partial_\beta X^\mu)\right)\\ &= -\frac{T}{2}\left(2\partial_\alpha(\bar{\epsilon}\phi^\mu)\partial^\alpha X_\mu - 2(\bar{\epsilon}\partial_\beta X^\mu)\beta\partial_\alpha\phi\right)\\ &= -T\left[\partial_\alpha(\bar{\epsilon}\phi^\mu)\partial^\alpha X_\mu - (\bar{\epsilon}\partial_\beta X^\mu)\beta\partial_\alpha\phi^\mu\right]\end{aligned} \tag{4.25}$$

We use the product rule to expand the last term of equation (4.25) to yield:

$$\delta L_C = -T\left[\partial_\alpha(\bar{\epsilon}\phi^\mu)\partial^\alpha X_\mu - \partial_\beta((\bar{\epsilon}X^\mu)\beta\partial_\alpha\phi^\mu) + (\bar{\epsilon}X^\mu)\beta\partial_\alpha\partial_\beta\phi^\mu\right] \tag{4.26}$$

We expand term 2 of equaton (4.26) by taking the derivative:

$$\delta L_C = -T\left[\partial_\alpha(\bar{\epsilon}\phi^\mu\partial^\alpha X_\mu)) - \cancel{\bar{\epsilon}\phi^\mu\partial_\alpha\partial^\alpha X_\mu\beta} - \beta(\partial_\beta\bar{\epsilon})X^\mu\phi^\mu + \cancel{\beta(\partial_\beta\epsilon)\partial_\beta X^\mu\phi^\mu}\right]$$

We write the variation of the Lagrangian as:

$$\delta L_C = -T\left[\partial_\alpha(\bar{\epsilon}\phi^\mu\partial^\alpha X_\mu)) - \beta(\partial_\beta\bar{\epsilon})\partial_\beta X^\mu\phi^\mu\right] \tag{4.27}$$

We now want to look for the conserved current. The first term is a total derivative and thus has no effect on the variation. The second term is relevant to our investigation. It has coefficients multiplying the derivative $\partial_\beta\bar{\epsilon}$ which we take as our independent variable. We can thus read off the conserved current as:

$$J^\mu_\beta = \beta\partial_\beta X^\mu\phi^\mu \tag{4.28}$$

4.2.5 The Simplicial Stress-Energy Tensor

Following the RNS formalism, we investigate for the stress energy tensor associated with the worldsheet currents X^μ and the spinors ϕ^μ. The stress-energy tensor associated with the worldsheet currents is well documented as the bosonic stress energy tensor. We shall thus focus our efforts on the stress energy of the spinors, which are associated with simplices. We will then interpret the result.

We begin by introducing a spatial translation on the worldsheet coordinates:

$$\sigma^\alpha = \sigma^\alpha + \vartheta^\alpha \tag{4.29}$$

where:

σ^α are four-coordinates,

ϑ^α is the translation.

This will introduce a perturbation on X^μ and ϕ^μ:

$$X^\mu = X^\mu + \vartheta^\alpha\partial_\alpha X^\mu \tag{4.30}$$

$$\phi^\mu = \phi^\mu + \vartheta^\alpha\partial_\alpha\phi^\mu \tag{4.31}$$

We restate the definition of the simplicial Lagrangian:

$$L_{CDT} = -\frac{1}{2}\bar{\phi^{\mu}}\beta\partial_{\alpha}\phi^{\mu} \tag{4.32}$$

where:

L_{CDT} is the Lagrangian of causal dynamical triangulation,

ϕ^{μ} is the spinor,

$\bar{\phi^{\mu}}$ is the complex conjugate of the field,

where we have dropped the string tension term: T. We now introduce a variation of the Lagrangian in equation (4.32):

$$\delta L_{CDT} = -\frac{1}{2}(\delta\bar{\phi^{\mu}})\beta\partial_{\alpha}\phi^{\mu} - \frac{1}{2}\bar{\phi^{\mu}}\beta\partial_{\alpha}(\delta\phi^{\mu}) \tag{4.33}$$

where:

δL_{CDT} is a variation of the causal dynamical triangulation.

Now, consider a variation of ϕ^{μ}:

$$\begin{aligned} \phi^{\mu} &= \phi^{\mu} + \delta\phi^{\mu} \\ \Rightarrow \delta\phi^{\mu} &= \vartheta^{\alpha}\partial_{\alpha}\phi^{\mu} \end{aligned} \tag{4.34}$$

where $\delta\phi^{\mu}$ is the variation.

Using equation (4.34) in equation (4.33):

$$\delta L_{CDT} = -\frac{1}{2}(\vartheta^{\alpha}\partial_{\alpha}\bar{\phi^{\mu}})\beta\partial^{\alpha}\phi^{\mu} - \frac{1}{2}\bar{\phi^{\mu}}\beta\partial_{\alpha}(\vartheta^{\alpha}\partial_{\alpha}\phi^{\mu}) \tag{4.35}$$

Opening up the second term on the right hand side of equation (4.35);

$$L_{CDT} = -\frac{1}{2}(\vartheta^{\alpha}\partial_{\alpha}\bar{\phi^{\mu}})\beta\partial^{\alpha}\phi^{\mu} - \frac{1}{2}\bar{\phi^{\mu}}\beta\partial_{\alpha}\vartheta^{\alpha}\partial_{\alpha}\phi^{\mu} - \frac{1}{2}\bar{\phi^{\mu}}\beta\epsilon^{\alpha}\partial_{\beta}\partial_{\alpha}\phi^{\mu} \tag{4.36}$$

We now introduce integrals to reduce the equation (4.36).

$$L_{CDT} = -\frac{1}{2}\int(\vartheta^\alpha\partial_\alpha\bar{\phi}^\mu)\beta\partial^\alpha\phi^\mu - \frac{1}{2}\int\bar{\phi}^\mu\beta\partial_\alpha\vartheta^\alpha\partial_\alpha\phi^\mu - \frac{1}{2}\int\bar{\phi}^\mu\beta\vartheta^\alpha\partial_\beta\partial_\alpha\phi^\mu \tag{4.37}$$

We do it term by term. We begin with the third term on the right hand side of equation (4.36):

$$\int UdV = -\frac{1}{2}\int\bar{\phi}^\mu\beta\epsilon\partial_\alpha\partial_\beta\phi_\mu \tag{4.38}$$

We remove the constant negative fraction term to ease our writing:

$$\begin{aligned}\int UdV &= UV - \int VdU\\ U &= \bar{\phi}^\mu\beta\epsilon\\ dV &= \partial_\alpha\partial_\beta\phi_\mu\\ V &= \partial_\alpha\phi_\mu\\ dU &= \partial_\beta(\bar{\phi}^\mu\beta\epsilon)\end{aligned}$$

Now, the boundary term vanishes, using Dirichlet conditions:

$$\int UdV = \cancelto{0}{\int UV} - \int VdU \tag{4.39}$$

Using equations (4.37) and (4.38):

$$\int UdV = \frac{1}{2}\int\partial_\alpha\phi_\mu\partial_\beta(\bar{\phi}^\mu\beta\epsilon) \tag{4.40}$$

Removing the integrals, and putting equation (4.39) in equation (4.36):

$$\delta L_{CDT} = -\frac{1}{2}(\epsilon^\alpha\partial_\alpha\bar{\phi}^\mu)\beta\phi^\mu - \frac{1}{2}\bar{\phi}^\mu\beta\partial_\alpha\epsilon^\alpha\partial_\alpha\phi^\mu + \frac{1}{2}\partial_\alpha\phi_\mu\partial_\beta(\bar{\phi}^\mu\beta\epsilon) \tag{4.41}$$

Expanding the third term again and canceling like terms, we obtain:

$$\delta L_{CDT} = -\frac{1}{2}(\epsilon^{\alpha}\partial_{\alpha}\bar{\phi^{\mu}})\beta\phi^{\mu} - \frac{1}{2}\bar{\phi^{\mu}}\beta\partial_{\alpha}\epsilon^{\alpha}\partial_{\alpha}\phi^{\mu} + \frac{1}{2}\partial_{\alpha}\phi_{\mu}\partial_{\beta}\bar{\phi^{\mu}}\beta\epsilon + \frac{1}{2}\partial_{\alpha}\phi_{\mu}\bar{\phi^{\mu}}\beta\partial_{\beta}\epsilon \tag{4.42}$$

$$\delta L_{CDT} = \cancel{-\frac{1}{2}(\epsilon^{\alpha}\partial_{\alpha}\bar{\phi^{\mu}})\beta\phi^{\mu}} - \frac{1}{2}\bar{\phi^{\mu}}\beta\partial_{\alpha}\epsilon^{\alpha}\partial_{\alpha}\phi^{\mu} + \cancel{\frac{1}{2}\partial_{\alpha}\phi_{\mu}\partial_{\beta}\bar{\phi^{\mu}}\beta\epsilon} + \frac{1}{2}\partial_{\alpha}\phi_{\mu}\bar{\phi^{\mu}}\beta\partial_{\beta}\epsilon$$

$$\delta L_{CDT} = -\frac{1}{2}\bar{\phi^{\mu}}\beta\partial_{\alpha}\epsilon^{\alpha}\partial_{\alpha}\phi^{\mu} + \frac{1}{2}\partial_{\alpha}\phi_{\mu}\bar{\phi^{\mu}}\beta\partial_{\beta}\epsilon$$

The perturbation is constant:

$\partial_{\beta}\epsilon$ vanishes such that:

$$\delta L_{CDT} = -\frac{1}{2}\bar{\phi^{\mu}}\beta\partial_{\alpha}\epsilon^{\alpha}\partial_{\alpha}\phi^{\mu} + \cancelto{0}{\frac{1}{2}\partial_{\alpha}\phi_{\mu}\bar{\phi^{\mu}}\beta\partial_{\beta}\epsilon}$$

$$\delta L_{CDT} = -\frac{1}{2}\bar{\phi^{\mu}}\beta\partial_{\alpha}\epsilon^{\alpha}\partial_{\alpha}\phi^{\mu} \tag{4.43}$$

Rewriting equation (4.42):

$$\delta L_{CDT} = \partial_{\alpha}\epsilon^{\alpha}\left(-\frac{1}{2}\bar{\phi^{\mu}}\beta\partial_{\alpha}\phi^{\mu}\right)$$

From this we can read off the stress energy tensor associated with the simplices as:

$$T^{(CDT)}_{\mu\nu} = -\frac{1}{2}\bar{\phi^{\mu}}\beta\partial_{\alpha}\phi^{\mu}$$

where $T^{(CDT)}_{\mu\nu}$ is the stress-energy tensor of causal dynamical triangultion.

We symmetrize the terms because we wish to have a symmetrical stress energy tensor.

We make the relabeling: $\alpha \to \nu$.

$$T^{(CDT)}_{\mu\nu} = -\frac{1}{4}\bar{\phi^{\mu}}\beta\partial_{\nu}\phi_{\mu} - \frac{1}{4}\bar{\phi^{\mu}}\beta\partial_{\nu}\phi_{\mu} \tag{4.44}$$

Equation (4.43) has an interesting consequence: the simplices have a stress-energy tensor. This means that the simplices have energy. Now we recognise that these simplices are to be used to construct spacetime. Therefore, the spacetime constructed from these simplices will have an associated intrinsic energy. This could be interpreted as dark energy. Vafa et. al(2018) have proposed that dark energy is inconsistent with a stable universe in the landscape of solutions in string theory. Universes with dark energy occupy the "swampland" solutions.

Equation (4.43) represents a possible way in which string theory could be formulated in such a way that it not only is consistent with dark energy, but requires it. This could provide an explanatory basis for dark energy, subject to further investigations.

We can write the full stress energy tensor, including the contribution from bosonic terms The bosonic stress energy tensor is (McMahon, 2006):

$$T_{\alpha\beta}^{(Bosonic)} = \partial_\alpha X^\mu \partial_\beta X^\mu$$

$$T_{\alpha\beta} = T_{\alpha\beta}^{(CDT)} + T_{\alpha\beta}^{(Bosonic)}$$

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}\rho_\beta \bar{\phi^\mu} \beta \partial_\alpha \rho_\beta \phi_\mu - \frac{1}{4}\bar{\phi^\mu} \beta \partial_\alpha \rho_\beta \phi_\mu \tag{4.45}$$

We now invoke lightcone coordinates: this will later aid in quantisation. In this case it will help us arrive to the equations of motion with relative ease. We can decompose the tensor and the simplicial fields into "positive-positive" and "negative-negative" components.

$$T_{\alpha\beta} = T_{++} + T_{--} \tag{4.46}$$

$$\bar{\phi^\mu} = \bar{\phi^\mu_+} + \bar{\phi^\mu_-} \tag{4.47}$$

$$\phi^\mu = \phi_+ + \phi_- \tag{4.48}$$

$$\partial_\alpha = \partial_+ + \partial_- \tag{4.49}$$

where:

T_{++} and T_{--} are the respective stress-energy tensor components,

ϕ_+ and ∂_- are the bispinor field components in lightcone coordinate.

We consider the bosonic stress-energy tensor $T_{\alpha\beta}^{(Bosonic)}$ together with equations (4.45) through (4.48) to get:

$$T_{\alpha\beta}^{(Bosonic)} = (\partial_+ + \partial_-)\, X^\mu\, (\partial_+ + \partial_-)\, X_\mu \tag{4.50}$$

We expand equation (4.49):

$$T_{\alpha\beta}^{(Bosonic)} = (\partial_+ X^\mu + \partial_- X^\mu)\,(\partial_+ X_\mu + \partial_- X_\mu) \tag{4.51}$$

$$T_{\alpha\beta}^{(Bosonic)} = \partial_+ X^\mu \partial_+ X_\mu + \partial_- X^\mu \partial_+ X_\mu + \partial_- X^\mu \partial_+ X_\mu + \partial_- X^\mu \partial_- X_\mu \tag{4.52}$$

We have imposed the condition of symmetry on the stress-energy tensor: we do not envision cross terms:

$$T_{+-} = T_{-+} = 0 \tag{4.53}$$

where:

$$T_{++}^{(Bosonic)} = \partial_+ X^\mu \partial_+ X_\mu \tag{4.54}$$

$$T_{--}^{(Bosonic)} = \partial_- X^\mu \partial_- X_\mu \tag{4.55}$$

We now work out the simplicial stress energy tensor in terms of light cone coordinates. We first decompose the simplicial stress energy tensor into its lightcone components.

$$T_{\alpha\beta}^{(CDT)} = T_{++}^{(CDT)} + T_{--}^{(CDT)} \tag{4.56}$$

We will now expand equation (4.43) in terms of equations (4.46) to (4.48). We work with only one term because of the similarity of the two terms:

We drop the tensor notation in our following computations

$$T_{\alpha\beta}^{(CDT)} = \frac{1}{4}\left(\bar{\phi_+} + \bar{\phi_-}\right)(\partial_+ + \partial_-)\,(\phi_+ + \phi_-) \tag{4.57}$$

Expanding equation (4.57) while at the same time dropping cross-terms:

$$\begin{aligned} T^{(CDT)}_{\alpha\beta} &= \frac{1}{4}\left(\bar{\phi_+} + \bar{\phi_-}\right)\left(\partial_+\phi_+ + \partial_+\phi_- + \partial_-\phi_+ + \partial_-\phi_-\right) \\ &= \frac{1}{4}\left(\bar{\phi_+} + \bar{\phi_-}\right)\left(\partial_+\phi_+ + \partial_-\phi_-\right) \\ &= \frac{1}{4}\left(\bar{\phi_+}\partial_+\phi_+ + \bar{\phi_+}\partial_-\phi_- + \bar{\phi_-}\partial_+\phi_+ + \bar{\phi_-}\partial_-\phi_-\right) \end{aligned} \tag{4.58}$$

We open the brackets:

$$T^{(CDT)}_{\alpha\beta} = \frac{1}{4}\bar{\phi_+}\partial_+\phi_+ + \frac{1}{4}\bar{\phi_+}\partial_-\phi_- + \frac{1}{4}\bar{\phi_-}\partial_+\phi_+ + \frac{1}{4}\bar{\phi_-}\partial_-\phi_-$$

We again drop cross terms:

$$T^{(CDT)}_{\alpha\beta} = \frac{1}{4}\bar{\phi_+}\partial_+\phi_+ + \frac{1}{4}\bar{\phi_-}\partial_-\phi_- \tag{4.59}$$

Recalling that we had left out a term from equation (4.53), we will now double equation (4.59):

$$T^{(CDT)}_{\alpha\beta} = \frac{1}{2}\bar{\phi_+}\partial_+\phi_+ + \frac{1}{2}\bar{\phi_-}\partial_-\phi_- \tag{4.60}$$

By comparison of equation (4.54) and equation (4.58), we can establish that:

$$T^{(CDT)}_{++} = \frac{1}{2}\bar{\phi_+}\partial_+\phi_+ \tag{4.61}$$

$$T^{(CDT)}_{--} = \frac{1}{2}\bar{\phi_-}\partial_-\phi_- \tag{4.62}$$

We finally write the stress energy tensor components in full:

$$\begin{aligned} T_{++} &= T^{(CDT)}_{++} + T^{(Bosonic)}_{++} \\ &= \partial_+X^\mu\partial_+X^\mu + \frac{1}{2}\bar{\phi_+}\partial_+\phi_+ \end{aligned} \tag{4.63}$$

and:

$$T_{--} = T^{(CDT)}_{--} + T^{(Bosonic)}_{--}$$

$$= \partial_- X^\mu \partial_- X^\mu + \frac{1}{2}\bar{\phi}_- \partial_+ \phi_- \tag{4.64}$$

From these equations, we can get the equations of motion of the simplices and worldsheet currents of the bosonic string as:

$$\partial_+ \phi_\mu^- = \partial_- \phi_+^\mu = 0 \tag{4.65}$$

and:

$$\partial_- \partial_+ X_\mu = 0 \tag{4.66}$$

4.2.6 Establishing Boundary Conditions

We wish to rewrite the simplicial action in lightcone coordinates. We introduce the Dirac matrices to assist us. We restate the simplicial action with the Dirac matrices:

$$S_{CDT} = \bar{\phi}^\mu \rho^\alpha \phi_\mu \tag{4.67}$$

where:

ϕ^μ is the spinor,

$\bar{\phi}^\mu$ is the complex conjugate of the field,

ρ^α are the Dirac matrices.

We sum over repeated indices: We consider a two dimensional case, with one of space and one of time.

$$S_{CDT} = \bar{\phi}^\mu \left(\rho^0 \phi_0 + \rho^1 \phi_1\right) \phi_\mu \tag{4.68}$$

We expand equation (4.68) using the full form of the Dirac matrices and complete the computation:

$$\rho^0 \phi_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \partial_\tau$$

$$\rho^1\phi_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \partial_\sigma$$

Thus the term inside the brackets of equation (4.68) becomes:

$$\begin{aligned}
(\rho^0\phi_0 + \rho^1\phi_1) &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \partial_\tau + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \partial_\sigma \\
&= \begin{pmatrix} 0 & -i\partial_\tau \\ i\partial_\tau & 0 \end{pmatrix} + \begin{pmatrix} 0 & i\partial_\sigma \\ i\partial_\sigma & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -i(\partial_\tau - \partial_\sigma) \\ i(\partial_\tau + \partial_\sigma) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2i\partial_- \\ 2i\partial_+ & 0 \end{pmatrix}
\end{aligned} \tag{4.69}$$

We put equation (4.69) in equation (4.68):

$$\begin{aligned}
S_{CDT} &= \bar{\phi}^\mu \begin{pmatrix} 0 & -2i\partial_- \\ 2i\partial_+ & 0 \end{pmatrix} \phi^\mu \\
&= \begin{pmatrix} \phi_- & \phi_+ \end{pmatrix} \begin{pmatrix} 0 & -2i\partial_- \\ 2i\partial_+ & 0 \end{pmatrix} \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix} \\
&= \begin{pmatrix} \phi_- & \phi_+ \end{pmatrix} \begin{pmatrix} -2i\partial_-\phi_+ \\ 2i\partial_+\phi_- \end{pmatrix} \\
&= \begin{pmatrix} \phi_- & \phi_+ \end{pmatrix} 2i \begin{pmatrix} -\partial_-\phi_+ \\ \partial_+\phi_- \end{pmatrix} \\
&= \begin{pmatrix} \phi_- & \phi_+ \end{pmatrix} 2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -\partial_-\phi_+ \\ \partial_+\phi_- \end{pmatrix} \\
&= \begin{pmatrix} \phi_- & \phi_+ \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} -\partial_-\phi_+ \\ \partial_+\phi_- \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix}\phi_- & \phi_+\end{pmatrix}\begin{pmatrix}2\partial_+\phi_- \\ 2\partial_-\phi_+\end{pmatrix}$$

$$\begin{aligned}&\begin{pmatrix}\phi_- & \phi_+\end{pmatrix}2\begin{pmatrix}\partial_+\phi_- \\ \partial_-\phi_+\end{pmatrix}\\ &= 2\left(\phi_-\partial_+\phi_- + \phi_+\partial_-\phi_+\right)\end{aligned} \tag{4.70}$$

where we have used the fact that $\bar{\phi^{\mu}} = \rho^0\phi^*$ since ϕ is a Dirac field.

Equation (4.70) is our action without constants. We use equation (4.70) without the multiplicative factor 2. Consider the resulting simplicial action :

$$S_{CDT} = \int d^2\sigma(\phi_-\partial_+\phi_- + \phi_-\partial_+\phi_-) \tag{4.71}$$

We break equation (4.71) into parts to ease the integration process:

$$\begin{aligned} S_{CDT} = \int d^2\sigma(\phi_-\partial_+\phi_-) + \int d^2\sigma(\phi_-\partial_+\phi_-)\\ S_{CDT_{\frac{1}{2}}} = \int d^2\sigma(\phi_-\partial_+\phi_-)\end{aligned} \tag{4.72}$$

We take variations of equation (4.72):

$$\begin{aligned}\delta S_{CDT_{\frac{1}{2}}} &= \delta\int d^2\sigma(\phi_-\partial_+\phi_-)\\ &= \int d^2\sigma[\delta\phi_-\partial_+\phi_- + \phi_-\partial_+\delta(\phi_-)]\\ &= \int d^2\sigma[\delta\phi_-\partial_+\phi_-] + \int d^2\sigma[\phi_-\partial_+\delta(\phi_-)]\end{aligned} \tag{4.73}$$

From the equations (4.65) of motion , we notice that the first term of equation (4.73) vanishes. We thus consider the second term of equation (4.73). We use integration by parts to reduce it:

$$\begin{aligned}\int d^2\sigma[\phi_-\partial_+\delta(\phi_-)] = \int UdV = UV - \int VdU\\ V = \delta\phi_-\end{aligned}$$

$$dU = \partial_+\phi_-$$

$$U = \phi_-$$

$$dV = \partial_+(\delta\phi_-)$$

$$\therefore \int d^2\sigma[\phi_-\partial_+\delta(\phi_-)] = \phi_-\delta\phi_-\Big|_{\sigma=0}^{\sigma=\pi} - \int d^2\sigma\delta\phi_-\cancelto{0}{\partial_+\phi_-} \tag{4.74}$$

$$\Rightarrow \int d^2\sigma[\phi_-\partial_+\delta(\phi_-)] = \phi_-\delta\phi_-\Big|_{\sigma=0}^{\sigma=\pi} \tag{4.75}$$

Where we have again used equation (4.65) to conclude that the second term of equation (4.74) vanishes. We do the same for the first term of equation (4.73):

$$\int d^2\sigma[\phi_+\partial_-\delta(\phi_+)] = \int UdV = UV - \int Vdu$$

$$V = \delta\phi_+$$

$$dU = \partial_-\phi_+$$

$$U = \phi_+$$

$$dV = \partial_-(\delta\phi_+)$$

$$\therefore \int d^2\sigma[\phi_+\partial_-\delta(\phi_+)] = \phi_+\delta\phi_+\Big|_{\sigma=0}^{\sigma=\pi} - \int d^2\sigma\delta\phi_+\cancelto{0}{\partial_-\phi_+} \tag{4.76}$$

$$\Rightarrow \int d^2\sigma[\phi_+\partial_-\delta(\phi_+)] = \phi_+\delta\phi_+\Big|_{\sigma=0}^{\sigma=\pi} = 0 \tag{4.77}$$

We now put the equations together to come up with:

$$\int d\tau(\phi_+\delta\phi_+ - \phi_-\delta\phi_-)\Big|_{\sigma=\pi} - (\phi_+\delta\phi_+ - \phi_-\delta\phi_-)\Big|_{\sigma=0} \tag{4.78}$$

The result in equation (4.78) should vanish: it is a variation of the Lagrangian. We thus select appropriate boundary conditions such that this requirement is satisfied. We want the simplices to be distinct, that is, the simplices should have different "quantum numbers". We therefore use fermionic boundary conditions:

$$\phi_-^\mu(0,\tau) = \phi_+^\mu(0,\tau) \tag{4.79}$$

$$\phi_-^\mu(\pi,\tau) = \phi_+^\mu(\pi,\tau) \tag{4.80}$$

4.2.7 Solving the Equation of Motion

We begin with the fact that:

$$\partial_- \partial_+ \phi^\mu_- = 0 \tag{4.81}$$

where:

$\partial_- = (\partial_\tau - \partial_\sigma)$ and $\partial_+ = (\partial_\tau + \partial_\sigma)$ are derivatives in light-cone coordinates,

ϕ^μ_- are the bispinor fields.

$$\partial_- = \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} \tag{4.82}$$

$$\partial_+ = \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma} \tag{4.83}$$

Putting equations (4.82) and (4.83) in equation (4.81):

$$\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma}\right)\left(\frac{\partial}{\partial_\tau} + \frac{\partial}{\partial \sigma}\right)\phi^\mu_- = 0$$

$$\left(\frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\cancel{\partial} \tau}\frac{\cancel{\partial}}{\partial \sigma} - \frac{\partial}{\cancel{\partial} \sigma}\frac{\cancel{\partial}}{\partial \tau} - \frac{\partial^2}{\partial \sigma^2}\right)\phi^\mu_- = 0$$

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2}\right)\phi^\mu_- = 0 \tag{4.84}$$

We now open equation (4.84):

$$\left(\frac{\partial^2}{\partial_{\tau^2}}\right)\phi^\mu_- - \left(\frac{\partial^2}{\partial_{\sigma^2}}\right)\phi^\mu_- = 0 \tag{4.85}$$

Now, ϕ^μ_- is a multivariate function of σ and τ. We thus invoke separation of variables. We can express ϕ^μ_- as the product of two functions:

$$\phi^\mu_- = S(\tau)M(\sigma) \tag{4.86}$$

where:

S is a function of a single variable τ,

M is a function of a single variable σ. Putting equation (4.86) in equation (4.85):

$$\begin{aligned}
\frac{\partial^2}{\partial_{\tau^2}} S(\tau)M(\sigma) - \frac{\partial^2}{\partial_{\sigma^2}} S(\tau)M(\sigma) &= 0 \\
M(\sigma)\frac{d^2}{d\tau^2} S(\tau) - S(\tau)\frac{d^2}{\partial\sigma^2} M(\sigma) &= 0 \\
\frac{1}{S(\tau)M(\sigma)} M(\sigma)\frac{d^2}{d\tau^2} S(\tau) - \frac{1}{S(\tau)M(\sigma)} S(\tau)\frac{d^2}{\partial\sigma^2} M(\sigma) &= 0 \\
\frac{1}{S(\tau)}\frac{d^2}{d\tau^2} S(\tau) - \frac{1}{M(\sigma)}\frac{d^2}{\partial\sigma^2} M(\sigma) &= 0 \\
\frac{1}{S(\tau)}\frac{d^2}{d\tau^2} S(\tau) &= \frac{1}{M(\sigma)}\frac{d^2}{\partial\sigma^2} M(\sigma) \qquad (4.87)
\end{aligned}$$

Since the left hand side and right hand side of equation (4.87) are each a function of a distinct variable, application of the theory of ordinary differential equations demands that equation (4.87) is equal to a constant, say: $-\alpha^2$, so that:

$$\begin{aligned}
\frac{1}{S(\tau)}\frac{d^2}{d\tau^2} S(\tau) &= -\alpha^2 \\
\frac{1}{M(\sigma)}\frac{d^2}{\partial\sigma^2} M(\sigma) &= -\alpha^2 \\
\frac{d^2}{d\tau^2} S(\tau) &= -\alpha^2 S(\tau) \\
\frac{d^2}{\partial\sigma^2} M(\sigma) &= -\alpha^2 M(\sigma) \\
\frac{d^2}{d\tau^2} S(\tau) + \alpha^2 S(\tau) &= 0 \qquad (4.88) \\
\frac{d^2}{\partial\sigma^2} M(\sigma) + \alpha^2 M(\sigma) &= 0 \qquad (4.89)
\end{aligned}$$

We can now proceed to solve equations (4.88) and (4.89).

Solving equation (4.89) using the D operator where $D = \frac{d}{d\sigma}$ and dropping the functional dependence of M and N we obtain:

$$\begin{aligned}
D^2 S + \alpha^2 M &= 0 \\
(D^2 + \alpha^2)M &= 0 \\
(D - i\alpha)(D + i\alpha)M &= 0 \\
(D - i\alpha)M = 0 \ ; \ (D + i\alpha)M &= 0
\end{aligned}$$

$$\begin{aligned}
(D \pm i\alpha)M &= 0 \\
\left(\frac{d}{d\sigma} \pm i\alpha\right) M &= 0 \\
\frac{d}{d\sigma}M \pm i\alpha M &= 0 \\
\frac{d}{d\sigma}M &= \mp i\alpha M \\
\frac{dM}{M} &= \mp i\alpha d\sigma \\
\int \frac{dM}{M} &= \int \mp i\alpha d\sigma \\
\ln M &= \mp i\alpha\sigma + C \qquad (4.90) \\
M &= Ae^{\mp i\alpha\sigma} \qquad (4.91)
\end{aligned}$$

where we have taken exponentials about equation (4.90) to generate equation (4.91), and:

$$A = e^{c}$$

We can carry out a similar process for equation (4.88):

$$\begin{aligned}
D^2 S + \alpha^2 S &= 0 \\
(D^2 + \alpha^2)S &= 0 \\
(D - i\alpha)(D + i\alpha)S &= 0 \\
(D - i\alpha)S = 0 \ ; \ (D + i\alpha)S &= 0 \\
(D \pm i\alpha)S &= 0 \\
\left(\frac{d}{d\tau} \pm i\alpha\right) S &= 0 \\
\frac{d}{d\tau}S \pm i\alpha S &= 0 \\
\frac{d}{d\tau}S &= \mp i\alpha S \\
\frac{dS}{S} &= \mp i\alpha d\tau \\
\int \frac{dS}{S} &= \int \mp i\alpha d\tau \\
\ln S &= \mp i\alpha\tau + C_1 \qquad (4.92) \\
S &= Be^{\mp i\alpha\tau} \qquad (4.93)
\end{aligned}$$

We put equations (4.93) and (4.91) in equation (4.86):

$$\phi^{\mu}_{-} = ABe^{\mp i\alpha\tau}e^{\mp i\alpha\sigma}$$

$$\phi^{\mu}_{-} = Le^{\mp i\alpha\tau}e^{\mp i\alpha\sigma}$$

$$\phi^{\mu}_{-} = Le^{\mp i\alpha\tau\mp i\alpha\sigma}$$

$$\phi^{\mu}_{-} = Le^{i\alpha(\mp\tau\mp\sigma)} \tag{4.94}$$

We impose causality on the proper time coordinate. We reject $-\tau$. We express L in tensor notation D^{μ}

$$\phi^{\mu}_{-} = D^{\mu}e^{i\alpha(\tau\mp\sigma)} \tag{4.95}$$

We can carry out a Fourier decomposition equation (4.95) and express it as the sum of different functions with different coefficient weights.

$$\phi^{\mu}_{-} = \sum_{n} D^{\mu}_{n}e^{i\alpha(\tau\mp\sigma)} \tag{4.96}$$

The weights D^{μ}_{n} are the expansion modes of the simplices. In the next section we proceed to quantise the theory by expressing a commutation relation for these expansion modes

4.3 Calculating the Number of Critical Dimensions

4.3.1 Introduction

Since we have defined the theory in terms of the extended Polyakov action and understood the corresponding symmetries, we are now in a position to quantise the theory. We inherit the standard quantisation procedures from the Ramond-Neveu-Schwarz formalism. In this case, this is done by promoting the mode expansions into operators by defining a commutation relation for the same. Since Virasoro modes are defined using expansion modes, this amounts to defining a commutation relation for the Virasoro modes, in effect promoting them into operators. The associated algebra is known as Virasoro algebra. In the case of superstring theory, this is known as super-virasoro algebra.

We're specifically concerned with generators of the Poincare algebra which can be defined in terms of the (now quantised) expansion modes. It has a physical consequence that is particularly interesting to us. String theory is formulated on a manifold with properties consistent with general relativity, and quantum field theory. The theory is expected to preserve Lorentz invariance. It can be demonstrated that the case of the classical string maintains this invariance: generators of the Poincare algebra are commutative. Thus Lorentz invariance is one of the symmetries of the Polyakov action. It should be noted that this symmetry is continuous.

The challenge with quantisation is that it breaks the continuous symmetry, and hence the Lorentz invariance. Thus quantised string theory is not consistent with relativity in general. This is an inconsistency that must be cured, since it implies an inconsistency within the theory itself. There exists a procedure of doing this in the theory. One begins by imposing that some of the commutators of the generators vanish and finding out what variables in the theory must be fixed. It turns out that for the theory to preserve Lorentz invariance, the variable that has to be fixed is the number of dimensions. This fixed number is referred to as the critical dimension of the theory. While involving, this calculation has been referred to as the most important calculation in string theory, since it gives the theory its distinctive feature of fixed number of dimensions. In the section, we calculate the critical dimension of the theory.

4.3.2 Defining the Generators

The generators of the Poincare algebra were derived in the earlier section in the classical case. We consider the quantisation relations imposed on the mode expansions, and writing the generators in terms of both light cone and transverse coordinates:

$$J^{\mu\nu} = \frac{1}{2}\left(x^{\mu}p^{\nu} - p^{\mu}x^{\nu}\right) - x^{\mu}p^{\nu} - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^{\mu}_{-n}\varpi^{\nu} - \varpi^{\nu}_{-n}\varpi^{\mu}_{n}\right) - \frac{i}{2}\sum_{n=0}^{\infty}\left[\phi^{\mu}_{-n}, \phi^{\nu}_{\nu}\right] \tag{4.97}$$

where:

$J^{\mu\nu}$ is the generator of the Lorentz algebra,

x^{μ} is the four-position vector,

p^{μ} is the four momentum,

ϖ^{μ}_{-n} are expansion modes,

ϕ^{μ}_{-n} are the new simplicial fields.

For simplicity, we split this equation into three separate terms;

$$\Theta^{\mu\nu} = \frac{1}{2}\left(x^{\mu}p^{\nu} - p^{\mu}x^{\nu}\right) - x^{\mu}p^{\nu} \tag{4.98}$$

$$\Xi^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^{\mu}_{-n}\varpi^{\nu} - \varpi^{\nu}_{-n}\varpi^{\mu}_{n}\right)$$

$$\Phi^{\mu\nu} = -\frac{i}{2}\sum_{n=0}^{\infty}\left[\phi^{\mu}_{-n}, \phi^{\nu}_{n}\right]$$

Such that equations (4.97) becomes:

$$J^{\mu\nu} = \Theta^{\mu\nu} + \Xi^{\mu\nu} + \Phi^{\mu\nu} \tag{4.99}$$

We can define another generator $J^{\alpha\beta}$ such that:

$$J^{\alpha\beta} = \Theta^{\alpha\beta} + \Xi^{\alpha\beta} + \Phi^{\alpha\beta} \tag{4.100}$$

We now calculate the commutator $\left[J^{\mu\nu}, J^{\alpha\beta}\right]$:

$$\begin{aligned}\left[J^{\mu\nu}, J^{\alpha\beta}\right] &= \left[\Theta^{\mu\nu} + \Xi^{\mu\nu} + \Phi^{\mu\nu}, \Theta^{\alpha\beta} + \Xi^{\alpha\beta} + \Phi^{\alpha\beta}\right] \\ &= \Theta^{\mu\nu}\Theta^{\alpha\beta} + \Theta^{\mu\nu}\Xi^{\alpha\beta} + \Theta^{\mu\nu}\Phi^{\alpha\beta} + \Xi^{\mu\nu}\Theta^{\alpha\beta} + \Xi^{\mu\nu}\Xi^{\alpha\beta} + \\ &\Xi^{\mu\nu}\Phi^{\alpha\beta} + \Phi^{\mu\nu}\Theta^{\alpha\beta} + \Phi^{\mu\nu}\Xi^{\alpha\beta} + \Phi^{\mu\nu}\Phi^{\alpha\beta} - \\ &\Theta^{\alpha\beta}\Theta^{\mu\nu} - \Theta^{\alpha\beta}\Theta^{\mu\nu} - \Theta^{\alpha\beta}\Phi^{\mu\nu} - \Xi^{\alpha\beta}\Theta^{\mu\nu} - \Xi^{\alpha\beta}\Xi^{\mu\nu} - \\ &\Xi^{\alpha\beta}\Phi^{\mu\nu} - \Phi^{\alpha\beta}\Theta^{\mu\nu} - \Phi^{\alpha\beta}\Xi^{\mu\nu} - \Phi^{\alpha\beta}\Phi^{\mu\nu} \\ &= \Theta^{\mu\nu}\Theta^{\alpha\beta} - \Theta^{\alpha\beta}\Theta^{\mu\nu} + \Theta^{\mu\nu}\Xi^{\alpha\beta} - \Xi^{\alpha\beta}\Theta^{\mu\nu} + \Theta^{\mu\nu}\Phi^{\alpha\beta} - \Phi^{\alpha\beta}\Theta^{\mu\nu} \\ &+ \Xi^{\mu\nu}\Theta^{\alpha\beta} - \Theta^{\alpha\beta}\Xi^{\mu\nu} + \Xi^{\mu\nu}\Xi^{\alpha\beta} - \Xi^{\alpha\beta}\Xi^{\mu\nu} + \Xi^{\mu\nu}\Phi^{\alpha\beta} - \Phi^{\alpha\beta}\Xi^{\mu\nu} \\ &+ \Phi^{\mu\nu}\Theta^{\alpha\beta} - \Theta^{\alpha\beta}\Phi^{\mu\nu} + \Phi^{\mu\nu}\Xi^{\alpha\beta} - \Xi^{\alpha\beta}\Phi^{\mu\nu} + \Phi^{\mu\nu}\Phi^{\alpha\beta} - \Phi^{\alpha\beta}\Phi^{\mu\nu}\end{aligned}$$

where we have expanded the commutator and rearranged the terms to get the following sum of independent commutators:

$$\begin{aligned}\left[J^{\mu\nu}, J^{\alpha\beta}\right] &= \left[\Theta^{\mu\nu}, \Theta^{\alpha\beta}\right] + \left[\Theta^{\mu\nu}, \Xi^{\alpha\beta}\right] + \left[\Theta^{\mu\nu}, \Phi^{\alpha\beta}\right] \\ &+ \left[\Xi^{\mu\nu}, \Theta^{\alpha\beta}\right] + \left[\Xi^{\mu\nu}, \Xi^{\alpha\beta}\right] + \left[\Xi^{\mu\nu}, \Phi^{\alpha\beta}\right] \\ &+ \left[\Phi^{\mu\nu}, \Theta^{\alpha\beta}\right] + \left[\Phi^{\mu\nu}, \Xi^{\alpha\beta}\right] + \left[\Phi^{\mu\nu}, \Phi^{\alpha\beta}\right]\end{aligned} \tag{4.101}$$

We want to work in both transverse and light cone coordinates. The first two coordinates will be labeled by "+" and "−". That is, the coordinates in configuration space will be labeled by x^+ and x^-. The remaining coordinates will be labelled by i and j. It is these coordinates that are referred to as transverse coordinates. We use this system because pertinent commutators are well defined in it. We thus make the replacement:

$$\begin{aligned}\mu &\to i \\ \alpha &\to j \\ \nu, \beta &\to -\end{aligned}$$

Such that equation (4.101) now becomes:

$$
\begin{aligned}
\left[J^{i-}, J^{j-}\right] = & \left[\Theta^{i-}, \Theta^{j-}\right] + \left[\Theta^{i-}, \mathit{\Xi}^{j-}\right] + \left[\Theta^{i-}, \Phi^{j-}\right] \\
& + \left[\mathit{\Xi}^{i-}, \Theta^{j-}\right] + \left[\mathit{\Xi}^{i-}, \mathit{\Xi}^{j-}\right] + \left[\mathit{\Xi}^{i-}, \Phi^{j-}\right] \\
& + \left[\Phi^{i-}, \Theta^{j-}\right] + \left[\Phi^{i-}, \mathit{\Xi}^{j-}\right] + \left[\Phi^{i-}, \Phi^{j-}\right]
\end{aligned}
\tag{4.102}
$$

To compute the nine commutators in equation (4.102), we will make use of a standard commutation table in string theory:

Table 1: Commutators in Quantized String Theory

[,]	p^+	p^-	p^j	x^j	x^-	ϖ^j_m	ϖ^j_m
p^+	0	0	0	0	i	0	0
p^-	0	0	0	$-i\frac{p^i}{p^+}$	$-i\frac{p^-}{p^+}$	$-\frac{2\pi T}{p^+}m\varpi^j_m$	$-\frac{2\pi T}{p^+}\varpi^-_m$
p^i	0	0	0	$i\delta_{ij}$	0	0	0
x^i	0	$-i\frac{p^i}{p^+}$	$i\delta_{ij}$	0	0	$i\frac{\delta^{ij}\delta_m}{4\pi T}$	$i\frac{\varpi^i_m}{p^+}$
x^-	$-i$	$i\frac{p^-}{p^+}$	0	0	0	0	$i\frac{\varpi^-_m}{p^+}$
ϖ^i_n	0	$\frac{2\pi T}{p^+}n\varpi^i_n$	0	$-i\frac{\delta^{ij}\delta_n}{4\pi T}$	0	$n\delta^{ij}\delta_{m+n}$	$\frac{\sqrt{4\pi T}}{p^+}n\varpi^i_{n+m}$
ϖ^-_n	0	$\frac{2\pi T}{p^+}n\varpi^-_n$	0	$-i\frac{\varpi^i_n}{p^+}$	$-i\frac{\varpi^i}{p^+}$	-$\frac{4\pi T}{p^+}m\varpi^i_{n+m}$	$[\varpi^-_n, \varpi^j_m]$

where:

$p^+ = p^\tau + p^\sigma$ up to a factor,

$p^+ = p^\sigma - p^\tau$ up to a factor,

p^i where $(i = 1, 2, 3)$ is the momentum,

x^i where $(i = 1, 2, 3)$ is the four vector,

n and m define the number of string states, and are also an index of states,

T is the string tension,

δ_{ij} is the Kronecker delta,

δ_{m-n} is the Dirac delta.

The table shows the commutation relations between x, y and the expansion modes of the string in various components. The table is computed using Poisson brackets, and we reproduce the table here for the reader's benefit:

Our tools are now in place, we now begin the computation:

4.3.3 Calculating commutator $[\Theta^{i-}, \Theta^{j-}]$

The first term in equation (4.97) is usually referred to as the angular momentum generator. We calculate their commutators in this section. We begin by expressing the commutator in full:

$$\left[\Theta^{i-}, \Theta^{j-}\right] = \left[\frac{1}{2}\left(x^i p^- - p^- x^i\right) - x^- p^i, \frac{1}{2}\left(x^j p^- - p^- x^j\right) - x^- p^j\right] \tag{4.103}$$

Opening up brackets:

$$\left[\Theta^{i-}, \Theta^{j-}\right] = \left[\frac{1}{2}x^i p^- - \frac{1}{2}p^- x^i - x^- p^i, \frac{1}{2}x^j p^- - \frac{1}{2}p^- x^j - x^- p^j\right]$$

We now create a dictionary to ease our computation:

$$\begin{aligned} A &= \frac{1}{2}x^i p^- \\ B &= -\frac{1}{2}p^- x^i \\ C &= -x^- p^i \\ D &= \frac{1}{2}x^j p^- \\ E &= -\frac{1}{2}p^- x^j \\ F &= -x^- p^j \end{aligned} \tag{4.104}$$

Putting equations (4.104) in equation (4.103) we get:

$$\begin{aligned} [A+B+C, D+E+F] &= AD + AE + AF + BD + BE + BF + CD + CE + CF \\ &\quad - DA - DB - DC - EA - EB - EC - FA - FB - FC \\ &= (AD - DA) + (AE - EA) + (AF - FA) + (BD - DB) + (BE - EB) \\ &\quad + (BF - FB) + (CD - DC) + (CE - EC) + (CF - FC) \\ &= [A, D] + [A, E] + [A, F] + [B, D] + [B, E] + [B, F] + [C, D] + [C, E] + [C, F] \end{aligned} \tag{4.105}$$

Putting equation (4.104) in equation (4.105) we get:

$$
\begin{aligned}
\left[\Theta^{i-},\Theta^{j-}\right] = &\left[\frac{1}{2}x^ip^-,\frac{1}{2}x^jp^-\right]+\left[\frac{1}{2}x^ip^-,-\frac{1}{2}p^-x^j\right]+\left[\frac{1}{2}x^ip^-,-x^-p^j\right]\\
&+\left[-\frac{1}{2}p^-x^i,\frac{1}{2}x^jp^-\right]+\left[-\frac{1}{2}p^-x^i,-\frac{1}{2}p^-x^j\right]+\left[-\frac{1}{2}p^-x^i,-x^-p^j\right]\\
&+\left[-x^-p^i,\frac{1}{2}x^jp^-\right]+\left[-x^-p^i,-\frac{1}{2}p^-x^j\right]+\left[-x^-p^i,-x^-p^j\right]\\
&=\frac{1}{4}\left[x^ip^-,x^jp^-\right]-\frac{1}{4}\left[x^ip^-,p^-x^j\right]-\frac{1}{2}\left[x^ip^-,x^-p^j\right]\\
&-\frac{1}{4}\left[p^-x^i,x^jp^-\right]+\frac{1}{4}\left[p^-x^i,p^-x^j\right]+\frac{1}{2}\left[p^-x^i,x^-p^j\right]\\
&-\frac{1}{2}\left[x^-p^i,x^jp^-\right]+\frac{1}{2}\left[x^-p^i,p^-x^j\right]+\left[x^-p^i,x^-p^j\right]
\end{aligned}
\tag{4.106}
$$

We need to calculate the nine commutators in terms of equation (4.106) individually. We use the identities in table 1 above to calculate these commutators. We start with term 1.

$$
\begin{aligned}
\frac{1}{4}\left[x^ip^-,x^jp^-\right] &= \frac{1}{4}\left(x^ip^-x^jp^- - x^jp^-x^ip^-\right)\\
&=\frac{1}{4}\left(x^ip^-x^jp^- - x^jp^-x^ip^- - p^-x^ix^jp^- + p^-x^ix^jp^-\right)\\
&=\frac{1}{4}\left(x^ip^-x^jp^- - p^-x^ix^jp^- - x^jp^-x^ip^- + p^-x^jx^ip^-\right)\\
&=\frac{1}{4}\left(x^j(x^ip^- - p^-x^i)p^- - (x^jp^- - p^-x^j)x^ip^-\right)\\
&=\frac{1}{4}\left(x^j[x^i,p^-]p^- - [x^j,p^-]x^ip^-\right)
\end{aligned}
\tag{4.107}
$$

From equation (4.107), we can read off the commutators from table 1:

$$
\begin{aligned}
&=\frac{1}{4}\left(x^ji\frac{p^i}{p^+}p^- - i\frac{p^j}{p^+}x^ip^-\right)\\
&=i\frac{1}{4}\left(x^j\frac{p^i}{p^+}p^- - \frac{p^j}{p^+}x^ip^-\right)\\
&=\frac{i}{4}\left(x^jp^i - x^ip^j\right)\frac{p^-}{p^+}
\end{aligned}
\tag{4.108}
$$

We proceed to term 2 of equation (4.106):

$$
\begin{aligned}
-\frac{1}{4}[p^-x^i,p^-x^j] &= \frac{1}{4}(p^-x^ip^-x^j - p^-x^jp^-x^i)\\
&=\frac{1}{4}(p^-x^ip^-x^j - p^-x^jp^-x^i - p^ix^ix^jp^- + p^ix^ix^jp^-)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}(p^-x^ix^jp^- - p^-x^jp^-x^i + p^-x^ip^-x^j - p^-x^ix^jp^-) \\
&= \frac{1}{4}(p^-x^j(x^ip^-p^-x^i) - (x^jp^- - p^-x^j)p^-x^i) \\
&= \frac{1}{4}(p^-x^j[x^i,p^-] - x^i[x^j,p^-] \\
&= \frac{1}{4}\left(x^ji\frac{p^i}{p+}p^-i\frac{p^j}{p^+}x^ip^-\right) \\
&= \frac{i}{4}\left(p^ix^j - p^jx^i\right)\frac{p^-}{p+}
\end{aligned}
\tag{4.109}
$$

Term 3:

$$
\begin{aligned}
-\frac{1}{2}[x^ip^-, x^-p^j] &= -\frac{1}{2}(x^ip^-x^-p^j - x^-p^jx^ip^-) \\
&= -\frac{1}{2}(x^ip^-x^-p^j - x^-p^jx^ip^- + p^-x^-x^ip^- - p^-x^-x^ip^-) \\
&= -\frac{1}{2}(x^ip^-x^-p^j - x^-p^jx^ip^- + p^-x^-x^ip^- - p^-x^ix^-p^-) \\
&= -\frac{1}{2}((x^ip^j - p^jx^i)x^-p^- - (x^-p^- - p^-x^-)x^jp^-) \\
&= -\frac{1}{2}([x^i,p^j]x^-p^- - [x^-,p^-]x^jp^-) \\
&= -\frac{1}{2}\left(i\delta^{ij}x^-p^- - i\frac{p^j}{p+}x^ip^-\right) \\
&= -\frac{i}{2}\left(x^-p^-\delta^{ij} - \frac{p^-}{p+}x^ip^j\right) \\
&= -\frac{i}{2}\left(x^-p^-\delta^{ij} - x^ip^j\frac{p^-}{p+}\right)
\end{aligned}
\tag{4.110}
$$

Term 4:

$$
\begin{aligned}
-\frac{1}{4}[p^-x^j, p^-x^i] &= \frac{1}{4}(p^-x^jp^-x^i - p^-x^ip^-x^j) \\
&= \frac{1}{4}(p^-x^jp^-x^i - p^-x^ip^-x^j - p^jx^ix^jp^- + p^jx^ix^jp^-) \\
&= \frac{1}{4}(+p^jx^jx^ip^- - p^-x^ip^-x^j + p^-x^jp^-x^i - p^jx^jx^ip^-) \\
&= \frac{1}{4}(p^-x^i(x^jp^- - p^-x^j) - x^i(x^ip^- - p^-x^i)p^-) \\
&= \frac{1}{4}(p^-x^i[x^j,p^-] - x^i[x^i,p^-]) \\
&= \frac{1}{4}\left(x^ii\frac{p^j}{p+}p^-i\frac{p^i}{p^+}x^jp^-\right) \\
&= \frac{i}{4}\left(p^jx^i - p^ix^j\right)\frac{p^-}{p+}
\end{aligned}
\tag{4.111}
$$

Term 5:

$$
\begin{aligned}
\frac{1}{4}\left[p^- x^i, p^- x^j\right] &= \frac{1}{4}\left(p^- x^i p^- x^j - p^- x^j p^- x^i\right) \\
= \frac{1}{4}\left(p^- x^i p^- x^j - p^- x^j p^- x^i - p^- x^i x^j p^- + p^- x^i x^j p^-\right) \\
= \frac{1}{4}\left(x^i p^- x^j p^- - x^j p^- x^i p^- - p^- x^i x^j p^- + p^- x^i x^j p^-\right) \\
= \frac{1}{4}\left(x^j (x^i p^- - p^- x^i) p^- - (x^j p^- - p^- x^j) x^i p^-\right) \\
= \frac{1}{4}\left(x^j [x^i, p^-] p^- - [x^j, p^-] x^i p^-\right) \\
= \frac{1}{4}\left(x^j i \frac{p^i}{p^+} p^- - i \frac{p^j}{p^+} x^i p^-\right) \\
= i\frac{1}{4}\left(x^j \frac{p^i}{p^+} p^- - \frac{p^j}{p^+} x^i p^-\right) \\
= \frac{i}{4}\left(x^i p^j - p^j x^i\right) \frac{p^-}{p^+}
\end{aligned} \tag{4.112}
$$

Term 6:

$$
\begin{aligned}
-\frac{1}{2}[p^- x^i, x^- p^j] &= -\frac{1}{2}(p^- x^i x^- p^j - x^- p^j p^- x^i) \\
= -\frac{1}{2}(p^- x^i x^- p^j - x^- p^j x^i p^- + p^- x^- x^i p^- - p^- x^- x^i p^-) \\
= -\frac{1}{2}(x^i p^- x^- p^j - x^- p^j x^i p^- + p^- x^- x^i p^- - p^- x^i x^- p^-) \\
= -\frac{1}{2}((x^i p^j - p^j x^i) x^- p^- - (x^- p^- - p^- x^-) x^j p^-) \\
= -\frac{1}{2}([x^i, p^j] x^- p^- - [x^-, -p^-] x^j p^-) \\
= -\frac{1}{2}\left(i\delta^{ij} x^- p^- - i\frac{p^j}{p+} x^i p^-\right) \\
= -\frac{i}{2}\left(x^- p^- \delta^{ij} - \frac{p^-}{p+} x^i p^j\right) \\
= -\frac{i}{2}\left(p^j x^i \frac{p^-}{p+} - p^- x^- \delta^{ij}\right)
\end{aligned} \tag{4.113}
$$

Term 7:

$$
\begin{aligned}
-\frac{1}{2}[x^- p^i, x^j p^-] &= -\frac{1}{2}(x^- p^i x^j p^- - x^j p^- x^- p^i) \\
= -\frac{1}{2}(x^i p^- x^- p^j - x^- p^j x^i p^- + p^- x^- x^i p^- - p^- x^- x^i p^-)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}(x^i p^- x^- p^j - x^- p^j x^i p^- + p^- x^- x^i p^- - p^- x^i x^- p^-) \\
&= -\frac{1}{2}((x^i p^j - p^j x^i) x^- p^- - (x^- p^- - p^- x^-) x^j p^-) \\
&= -\frac{1}{2}([x^i, p^j] x^- p^- - [x^-, -p^-] x^j p^-) \\
&= -\frac{1}{2}\left(i\delta^{ij} x^- p^- - i\frac{p^j}{p+} x^i p^-\right) \\
&= -\frac{i}{2}\left(x^- p^- \delta^{ij} - \frac{p^-}{p+} x^i p^j\right) \\
&= -\frac{i}{2}\left(x^j p^i \frac{p^-}{p+} - x^- p^- \delta^{ij} -\right)
\end{aligned}
\tag{4.114}
$$

Term 8:

$$
\begin{aligned}
-\frac{1}{2}[x^- p^i, p^- x^j] &= -\frac{1}{2}(x^- p^i p^- x^j - p^- x^j x^- p^i) \\
&= -\frac{1}{2}(x^i p^- x^- p^j - x^- p^j x^i p^- + p^- x^- x^i p^- - p^- x^- x^i p^-) \\
&= -\frac{1}{2}(x^i p^- x^- p^j - x^- p^j x^i p^- + p^- x^- x^i p^- - p^- x^i x^- p^-) \\
&= -\frac{1}{2}((x^i p^j - p^j x^i) x^- p^- - (x^- p^- - p^- x^-) x^j p^-) \\
&= -\frac{1}{2}([x^i, p^j] x^- p^- - [x^-, p^-] x^j p^-) \\
&= -\frac{1}{2}\left(i\delta^{ij} x^- p^- - i\frac{p^j}{p+} x^i p^-\right) \\
&= -\frac{i}{2}\left(x^- p^- \delta^{ij} - \frac{p^-}{p+} x^i p^j\right) \\
&= \frac{i}{2}\left(x^j p^i \frac{p^-}{p+} - x^- p^- \delta^{ij} -\right)
\end{aligned}
\tag{4.115}
$$

And finally, term 9:

$$
\begin{aligned}
[x^- p^i, p^- x^j] &= (x^- p^i p^- x^j - p^- x^j x^- p^i) \\
&= (x^i p^- x^- p^j - x^- p^j x^i p^- + p^- x^- x^i p^- - p^- x^- x^i p^-) \\
&= (x^i p^- x^- p^j - x^- p^j x^i p^- + p^- x^- x^i p^- - p^- x^i x^- p^-) \\
&= ((x^i p^j - p^j x^i) x^- p^- - (x^- p^- - p^- x^-) x^j p^-) \\
&= ([x^i, p^j] x^- p^- - [x^-, -p^-] x^j p^-) \\
&= \left(i\delta^{ij} x^- p^- - i\frac{p^j}{p+} x^i p^-\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(x^- p^- \delta^{ij} - \frac{p^-}{p+} x^i p^j\right) \\
&= \left(x^j p^i \frac{p^-}{p+} - x^- p^- \delta^{ij} -\right)
\end{aligned} \tag{4.116}
$$

Putting equation (4.107) through to equation (4.116) together, we get:

$$
\begin{aligned}
\left[\Theta^{i-}, \Theta^{j-}\right] &= \frac{1}{4}(x^j p^i - x^i p^j)\frac{p^-}{p^+} + \frac{i}{4}(p^i x^j - x^i p^j)\frac{p^-}{p+} - \frac{i}{2}\left(x^- p^- \delta^{ij} - x^i p^j \frac{p^-}{p^+}\right) \\
&\quad -\frac{i}{4}(p^i x^j - x^j p^i)\frac{p^-}{p+} - \frac{i}{4}(p^j x^i - p^i x^j)\frac{p+}{p^-} + \frac{i}{2}\left(\frac{p^-}{p^+} p^j x^i - p^- x^- \delta^{ij}\right) \\
+\frac{i}{2}\left(\frac{p^-}{p+} p^j x^i - p^- x^- \delta^{ij}\right) &- \frac{i}{2}\left(x^j p^i \frac{p^-}{p^+} - x^- p^- \delta^{ij}\right) - \frac{1}{2}\left(\frac{p^-}{p+} x^j p^i - x^- p^- \delta^{ij}\right)
\end{aligned}
$$

Opening up brackets:

$$
\begin{aligned}
\left[\Theta^{i-}, \Theta^{j-}\right] &= \frac{i}{4} x^j p^i \frac{p^-}{p^+} - \frac{i}{4} x^i p^j \frac{p^-}{p^+} + \frac{i}{4} p^i x^j \frac{p^-}{p+} - \frac{i}{4} x^i p^j \frac{p^-}{p+} - \frac{i}{2} x^- p^- \delta^{ij} + \frac{i}{2} x^i p^j \frac{p^-}{p^+} \\
&\quad -\frac{i}{4} p^i x^j \frac{p^-}{p+} + \frac{i}{4} x^j p^i \frac{p^-}{p+} - \frac{i}{4} p^j x^i \frac{p+}{p^-} + \frac{i}{4} p^i x^j \frac{p+}{p^-} + \frac{i}{2}\frac{p^-}{p^+} p^j x^i - \frac{i}{2} p^- x^- \delta^{ij} \\
&\quad +\frac{i}{2}\frac{p^-}{p+} p^j x^i - p^- x^- \frac{i}{2}\delta^{ij} - \frac{i}{2} x^j p^i \frac{p^-}{p^+} + \frac{i}{2} x^- p^- \delta^{ij} - \frac{1}{2}\frac{p^-}{p+} x^j p^i - x^- p \frac{1}{2}\delta^{ij}
\end{aligned} \tag{4.117}
$$

To simplify equation (4.117), we split it into blocks of terms: terms pre-multiplied by $\frac{p^-}{p^+}$, post-multiplied by $\frac{p^-}{p^+}$ and the δ^{ij} terms. We evaluate the blocks of terms separately:
Pre- multiplied block:

$$
\begin{aligned}
&= -\frac{i}{4}\frac{p^-}{p^+} p^j x^i + \frac{i}{4}\frac{p^-}{p^+} x^j p^i - \frac{i}{4}\frac{p^-}{p^+} x^j p^i - \frac{i}{4}\frac{p^-}{p^+} p^j x^i + \frac{i}{4}\frac{p^-}{p^+} p^i x^j - \frac{i}{2}\frac{p^-}{p^+} p^j x^i - \frac{i}{2}\frac{p^-}{p^+} x^j p^i \\
&= -\frac{i}{4}\frac{p^-}{p^+}(-p^j x^i + x^j p^i - x^j p^i - p^j x^i + p^i x^j - p^j x^i - x^j p^i) \\
&= -\frac{i}{4}\frac{p^-}{p^+}(-2p^j x^i + 2p^j x^i + x^j p^i - 2x^j p^i + p^i x^j) \\
&= \frac{i}{4}\frac{p^-}{p^+}(-x^j p^i + p^i x^j) \\
&= \frac{i}{4}\frac{p^-}{p^+}(p^i x^j - x^j p^i) \\
&= \frac{i}{4}\frac{p^-}{p^+}[p^i, x^j]
\end{aligned} \tag{4.118}
$$

The post-multiplied block

$$
\begin{aligned}
&= \frac{i}{4}x^j p^i \frac{p^-}{p^+} - \frac{i}{4}x^i p^j \frac{p^-}{p^+} + \frac{i}{4}p^i x^j \frac{p^-}{p^+} + \frac{i}{4}p^i x^j \frac{p^-}{p^+} - \frac{i}{4}x^i p^j \frac{p^-}{p^+} - \frac{i}{2}x^j p^i \frac{p^-}{p^+} + \frac{i}{2}x^i p^j \frac{p^-}{p^+} \\
&= \frac{i}{4}(x^j p^i - x^i p^j + p^i x^j - x^i p^j - 2x^i p^j + 2x^i p^j)\frac{p^-}{p^+} \\
&= \frac{i}{4}(x^j p^i + p^i x^j - 2x^j p^i)\frac{p^-}{p^+} \\
&= \frac{i}{4}(-x^j p^i + p^i x^j)\frac{p^-}{p^+} \\
&= \frac{i}{4}[p^i, x^j]\frac{p^-}{p^+}
\end{aligned}
\tag{4.119}
$$

The Kronecker delta block:

$$
\begin{aligned}
&= \frac{i}{2}x^- p^- \delta^{ij} + \frac{i}{2}x^- p^- \delta^{ij} - \frac{i}{2}p^- x^- \delta^{ij} - \frac{i}{2}x^- p^- \delta^{ij} \\
&= \frac{i}{2}(x^- p^- + x^- p^- - p^- x^- - x^- p^-)\delta^{ij} \\
&= \frac{i}{2}(x^- p^- - p^- x^-)\delta^{ij} \\
&= \frac{i}{2}[x^-, p^-]\delta^{ij}
\end{aligned}
\tag{4.120}
$$

Putting equations (4.118), (4.119) and (4.120) together:

$$
\begin{aligned}
\left[\Theta^{i-}, \Theta^{j-}\right] &= i[x^-, p^-]\delta^{ij} + \frac{i}{4}[p^i, x^j]\frac{p^-}{p^+} + \frac{i}{4}\frac{p^-}{p^+}[p^i, x^j] \\
&= \frac{i}{4}(-i\delta^{ij})\frac{p^-}{p^+} + \frac{i}{4}(-i\delta^{ij})\frac{p^-}{p^+} + \frac{i}{2}i\frac{p^+}{p^-}\delta^{ij} \\
&= \frac{i}{4}(-i\delta^{ij})\frac{p^-}{p^+} + \frac{i}{4}(-i\delta^{ij})\frac{p^-}{p^+} + \frac{i}{2}i\frac{p^+}{p^-}\delta^{ij} \\
&= \frac{1}{4}\delta^{ij}\frac{p^-}{p^+} + \frac{1}{4}\delta^{ij}\frac{p^-}{p^+} - \frac{1}{2}\frac{p^+}{p^-}\delta^{ij} \\
&= \frac{1}{2}\delta^{ij}\frac{p^-}{p^+} - \frac{1}{2}\frac{p^+}{p^-}\delta^{ij} \\
&= \frac{1}{2}\delta^{ij}\frac{p^-}{p^+} - \frac{1}{2}\delta^{ij}\frac{p^+}{p^-} \\
&= 0
\end{aligned}
\tag{4.121}
$$

Therefore,

$$\left[\Theta^{i-}, \Theta^{j-}\right] = 0 \tag{4.122}$$

Thus the generators of angular momentum commute.

We now proceed to calculate terms 2 and 4 of equation (4.97):

4.3.4 Calculating commutator $[\Theta^{i-}, \varXi^{j-}] + [\varXi^{i-}, \Theta^{j-}]$

We first take cognisance of the following definitions:

$$\begin{aligned}
\Theta^{i-} &= \frac{1}{2}\left(x^i p^- - p^- x^i\right) - x^i p^- \\
\varXi^{i-} &= -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^i_{-n}\varpi^- - \varpi^-_{-n}\varpi^i_n\right) \\
\Theta^{j-} &= \frac{1}{2}\left(x^j p^- - p^- x^i\right) - x^j p^- \\
\varXi^{j-} &= -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^j_{-n}\varpi^- - \varpi^-_{-n}\varpi^j_n\right)
\end{aligned} \tag{4.123}$$

Such that,

$$\begin{aligned}
\left[\Theta^{i-}, \varXi^{j-}\right] + \left[\varXi^{i-}, \Theta^{j-}\right] = &\left[\frac{1}{2}\left(x^i p^- - p^- x^i\right) - x^i p^-, -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^j_{-n}\varpi^- - \varpi^-_{-n}\varpi^j_n\right)\right] \\
&+ \left[-i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^i_{-n}\varpi^- - \varpi^-_{-n}\varpi^i_n\right), \frac{1}{2}\left(x^j p^- - p^- x^i\right) - x^j p^-\right]
\end{aligned} \tag{4.124}$$

We will work this equation out term by term: We start with term one,

$$\begin{aligned}
\left[\Theta^{i-}, \varXi^{j-}\right] &= \left[\frac{1}{2}\left(x^i p^- - p^- x^i\right) - x^i p^-, -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^j_{-n}\varpi^- - \varpi^-_{-n}\varpi^j_n\right)\right] \\
&= \left[\frac{1}{2}x^i p^- - \frac{1}{2}p^- x^i - x^i p^-, -i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^j_{-n}\varpi^- + i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^-_{-n}\varpi^j_n\right]
\end{aligned} \tag{4.125}$$

We invoke a dictionary for simplification:

$$
\begin{aligned}
A &= \frac{1}{2} x^i p^- \\
B &= -\frac{1}{2} p^- x^i \\
C &= -x^i p^- \\
D &= -i \sum_{n=1}^{\infty} \frac{1}{n} \varpi^j_{-n} \varpi^- \\
E &= i \sum_{n=1}^{\infty} \frac{1}{n} \varpi^-_{-n} \varpi^j_n
\end{aligned} \tag{4.126}
$$

So equation (4.124) becomes:

$$
\begin{aligned}
\left[\Theta^{i-}, \varXi^{j-}\right] &= [A + B + C, D + E] \\
&= (AD + AE + BD + BE + CD + CE - DA - DB - DC - EA - EB - EC) \\
&= AD - DA + AE - EA + BD - DB + BE - EB + CD - DC + CE - EC \\
&= [A, D] + [A, E] + [B, D] \\
&\quad + [B, E] + [C, D] + [C, E]
\end{aligned} \tag{4.127}
$$

Writing equation (4.129) in terms of equation (4.126):

$$
\begin{aligned}
\left[\Theta^{i-}, \varXi^{j-}\right] = {} & [\frac{1}{2} x^i p^-, -i \sum_{n=1}^{\infty} \frac{1}{n} \varpi^j_{-n} \varpi^-] + [\frac{1}{2} x^i p^-, -i \sum_{n=1}^{\infty} \frac{1}{n} \varpi^j_{-n} \varpi^-] + \\
& [-\frac{1}{2} p^- x^i, -i \sum_{n=1}^{\infty} \frac{1}{n} \varpi^j_{-n} \varpi^-] + [-\frac{1}{2} p^- x^i, i \sum_{n=1}^{\infty} \frac{1}{n} \varpi^-_{-n} \varpi^j_n] + \\
& [-x^i p^-, -i \sum_{n=1}^{\infty} \frac{1}{n} \varpi^j_{-n} \varpi^-] + [-x^i p^-, i \sum_{n=1}^{\infty} \frac{1}{n} \varpi^-_{-n} \varpi^j_n] \\
= {} & -\frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n} [x^i p^-, \varpi^j_{-n} \varpi^-] + \frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n} [x^i p^-, \varpi^j_{-n} \varpi^-] + \\
& \frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n} [p^- x^i, \varpi^j_{-n} \varpi^-] - \frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n} [p^- x^i, \varpi^-_{-n} \varpi^j_n] + \\
& i \sum_{n=1}^{\infty} \frac{1}{n} [x^i p^-, \varpi^j_{-n} \varpi^-] - i \sum_{n=1}^{\infty} \frac{1}{n} [x^i p^-, \varpi^-_{-n} \varpi^j_n]
\end{aligned} \tag{4.128}
$$

We carry out a similar process for $[\Xi^{i-}, \Theta^{j-}]$:

$$\begin{aligned}
\left[\Xi^{i-}, \Theta^{j-}\right] &= \left[-i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^{i}_{-n}\varpi^{-} - \varpi^{-}_{-n}\varpi^{i}_{n}\right), \frac{1}{2}\left(x^{j}p^{-} - p^{-}x^{i}\right) - x^{j}p^{-}\right] \\
&= \left[-i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{i}_{-n}\varpi^{-} + i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{-}_{-n}\varpi^{i}_{n}, \frac{1}{2}x^{j}p^{-} - \frac{1}{2}p^{-}x^{i} - x^{j}p^{-}\right] \\
F &= -i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{i}_{-n}\varpi^{-} \\
G &= i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{-}_{-n}\varpi^{i}_{n} \\
H &= \frac{1}{2}x^{j}p^{-} \\
I &= -\frac{1}{2}p^{-}x^{i} \\
J &= -x^{j}p^{-} \\
&= [F+G, H+I+J] \\
&= (FH + FI + FJ + GH + GI + GJ - HF - HG - IF - IG - JF - JG) \\
&= FH - HF + FI - IF + FJ - JF + GH - HG + GI - IG + GJ - JG \\
&= [F,H] + [F,I] + [F,J] + [G,H] + [G,I] + [G,J]
\end{aligned}$$

$$\begin{aligned}
&= [-i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{i}_{-n}\varpi^{-}, \frac{1}{2}x^{j}p^{-}] + [-i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{i}_{-n}\varpi^{-}, -\frac{1}{2}p^{-}x^{i}] + [-i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{i}_{-n}\varpi^{-}, -x^{j}p^{-}] + \\
&\qquad [i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{-}_{-n}\varpi^{i}_{n}, \frac{1}{2}x^{j}p^{-}] + [i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{-}_{-n}\varpi^{i}_{n}, -\frac{1}{2}p^{-}x^{i}] + [i\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{-}_{-n}\varpi^{i}_{n}, -x^{j}p^{-}] \\
&= -\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[\varpi^{i}_{-n}\varpi^{-}, x^{j}p^{-}] + \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[\varpi^{i}_{-n}\varpi^{-}, p^{-}x^{i}] + i\sum_{n=1}^{\infty}\frac{1}{n}[\varpi^{i}_{-n}\varpi^{-}, x^{j}p^{-}] + \\
&\qquad \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[\varpi^{-}_{-n}\varpi^{i}_{n}, x^{j}p^{-}] - \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[\varpi^{-}_{-n}\varpi^{i}_{n}, p^{-}x^{i}] - i\sum_{n=1}^{\infty}\frac{1}{n}[\varpi^{-}_{-n}\varpi^{i}_{n}, x^{j}p^{-}]
\end{aligned} \tag{4.129}$$

We now want to simplify the equations further: We will factor out terms and manipulate them to simplify the equations. We start with equation (4.129). Combining terms (1), (2), (3), (4):

$$= -\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[x^i p^-, \varpi^j_{-n}\varpi^-_n] + \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[x^i p^-, \varpi^-_{-n}\varpi^j_n] +$$

$$\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[p^- x^i, \varpi^j_{-n}\varpi^-_n] - \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[p^- x^i, \varpi^-_{-n}\varpi^j_n]$$

$$= \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}(-[x^i p^-, \varpi^j_{-n}\varpi^-_n] + [x^i p^-, \varpi^-_{-n}\varpi^j_n] +$$

$$[p^- x^i, \varpi^j_{-n}\varpi^-_n] - [p^- x^i, \varpi^-_{-n}\varpi^j_n])$$

$$= \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}(-(x^i p^- \varpi^j_{-n}\varpi^-_n - \varpi^j_{-n}\varpi^-_n x^i p^-) + (x^i p^- \varpi^-_{-n}\varpi^j_n - \varpi^-_{-n}\varpi^j_n x^i p^-) +$$

$$(p^- x^i \varpi^j_{-n}\varpi^-_n - \varpi^j_{-n}\varpi^-_n p^- x^i) - (p^- x^i \varpi^-_{-n}\varpi^j_n - \varpi^-_{-n}\varpi^j_n p^- x^i))$$

$$= \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}(-x^i p^- \varpi^j_{-n}\varpi^-_n + \varpi^j_{-n}\varpi^-_n x^i p^- + x^i p^- \varpi^-_{-n}\varpi^j_n - \varpi^-_{-n}\varpi^j_n x^i p^- +$$

$$p^- x^i \varpi^j_{-n}\varpi^-_n - \varpi^j_{-n}\varpi^-_n p^- x^i - p^- x^i \varpi^-_{-n}\varpi^j_n + \varpi^-_{-n}\varpi^j_n p^- x^i)$$

Collecting like terms in the coordinate factors:

$$\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}(-x^i p^- \varpi^j_{-n}\varpi^-_n + \varpi^j_{-n}\varpi^-_n x^i p^- + x^i p^- \varpi^-_{-n}\varpi^j_n - \varpi^-_{-n}\varpi^j_n x^i p^- +$$

$$p^- x^i \varpi^j_{-n}\varpi^-_n - \varpi^j_{-n}\varpi^-_n p^- x^i - p^- x^i \varpi^-_{-n}\varpi^j_n + \varpi^-_{-n}\varpi^j_n p^- x^i)$$

$$= \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}(x^i p^-(\varpi^-_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^-_n) - p^- x^i(\varpi^-_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^-_n)$$

$$+(\varpi^-_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^-_n)x^i p^- - (\varpi^-_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^-_n)p^- x^i)$$

$$= \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}([x^i, p^-](\varpi^-_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^-_n) + (\varpi^-_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^-_n)[x^i, p^-]$$

$$= \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}([x^i, p^-]2(\varpi^-_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^-_n))$$

$$= i\sum_{n=1}^{\infty}\frac{1}{n}([x^i, p^-](\varpi^-_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^-_n))$$

$$= i\sum_{n=1}^{\infty}\frac{1}{n}(i\frac{p^j}{p+}(\varpi^-_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^-_n))$$

$$= -\frac{1}{p+}\sum_{n=1}^{\infty}\frac{1}{n}(p^j(\varpi^-_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^-_n))$$

$$= -\frac{1}{p+}\sum_{n=1}^{\infty}\frac{1}{n}((\varpi^-_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^-_n)p^i) \qquad (4.130)$$

We perform a similar computation for the remaining terms of equation (4.129):

$$= -\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[\varpi^i_{-n}\varpi^-_n, x^jp^-] + \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[\varpi^i_{-n}\varpi^-_n, p^-x^j]$$

$$+\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[\varpi^-_{-n}\varpi^i_n, x^jp^-] - \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}[\varpi^-_{-n}\varpi^i_n, p^-x^j]$$

$$= -\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}([\varpi^i_{-n}\varpi^-_n, x^jp^-] + [\varpi^i_{-n}\varpi^-_n, p^-x^j]$$

$$+[\varpi^-_{-n}\varpi^i_n, x^jp^-] - [\varpi^-_{-n}\varpi^i_n, p^-x^j])$$

$$= \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}(-(\varpi^i_{-n}\varpi^-_n x^jp^- - x^jp^-\varpi^i_{-n}\varpi^-_n) + (\varpi^i_{-n}\varpi^-_n p^-x^j - p^-x^j\varpi^i_{-n}\varpi^-_n)$$

$$+(\varpi^-_{-n}\varpi^i_n x^jp^- - x^jp^-\varpi^-_{-n}\varpi^i_n x^jp^-) - (\varpi^-_{-n}\varpi^i_n p^-x^j - \varpi^-_{-n}\varpi^i_n p^-x^j))$$

$$= \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}(-\varpi^i_{-n}\varpi^-_n x^jp^- + x^jp^-\varpi^i_{-n}\varpi^-_n + \varpi^i_{-n}\varpi^-_n p^-x^j - p^-x^j\varpi^i_{-n}\varpi^-_n$$

$$+\varpi^-_{-n}\varpi^i_n x^jp^- - x^jp^-\varpi^-_{-n}\varpi^i_n x^jp^- - \varpi^-_{-n}\varpi^i_n p^-x^j + \varpi^-_{-n}\varpi^i_n p^-x^j)$$

$$= \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}(x^jp^-(\varpi^-_{-n}\varpi^i_n - \varpi^i_{-n}\varpi^-_n) - p^-x^j(\varpi^-_{-n}\varpi^i_n - \varpi^i_{-n}\varpi^-_n)$$

$$+(\varpi^-_{-n}\varpi^i_n - \varpi^i_{-n}\varpi^-_n)x^jp^- - (\varpi^-_{-n}\varpi^i_n - \varpi^i_{-n}\varpi^-_n)p^-x^j)$$

$$\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}([x^j, p^-](\varpi^-_{-n}\varpi^i_n - \varpi^i_{-n}\varpi^-_n) + (\varpi^-_{-n}\varpi^i_n - \varpi^i_{-n}\varpi^-_n)[x^j, p^-]$$

$$= \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}([x^j, p^-]2(\varpi^-_{-n}\varpi^i_n - \varpi^i_{-n}\varpi^-_n))$$

$$= -i\sum_{n=1}^{\infty}\frac{1}{n}([x^j, p^-](\varpi^-_{-n}\varpi^i_n - \varpi^i_{-n}\varpi^-_n))$$

$$= -i\sum_{n=1}^{\infty}\frac{1}{n}(i\frac{p^i}{p+}(\varpi^-_{-n}\varpi^i_n - \varpi^i_{-n}\varpi^-_n))$$

$$= \frac{1}{p+}\sum_{n=1}^{\infty}\frac{1}{n}(p^i(\varpi^-_{-n}\varpi^i_n - \varpi^i_{-n}\varpi^-_n))$$

$$= \frac{1}{p+}\sum_{n=1}^{\infty}\frac{1}{n}((\varpi^-_{-n}\varpi^i_n - \varpi^i_{-n}\varpi^-_n)p^j) \quad (4.131)$$

Putting equations (4.131) and (4.130) together with the remainder terms, we get:

$$\left[\Theta^{i-},\Xi^{j-}\right]+\left[\Xi^{i-},\Theta^{j-}\right]=2\frac{p^-}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^j_n-\varpi^j_{-n}\varpi^i_n)+\frac{1}{p+}\sum_{n=1}^{\infty}\frac{1}{n}((\varpi^-_{-n}\varpi^i_n-\varpi^i_{-n}\varpi^-_n)p^j)$$
$$-\frac{1}{p+}\sum_{n=1}^{\infty}\frac{1}{n}((\varpi^-_{-n}\varpi^j_n-\varpi^j_{-n}\varpi^-_n)p^i) \tag{4.132}$$

4.3.5 Calculating commutator $[\Xi^{i-},\Xi^{j-}]$

We now proceed to calculate the fifth term of equation (4.97). We start by studying the commutator of Ξ^{i-} with individual expansion modes. This will help us simplify our computation of the commutator. We begin by calculating $[\Xi^{i-},\varpi^-_m]$.

Writing in full:

$$[\Xi^{i-},\varpi^-_m]=\left[-i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^i_{-n}\varpi^- -\varpi^-_{-n}\varpi^i_n\right),\varpi^-_m\right] \tag{4.133}$$

We manipulate the commutator further:

$$[\Xi^{i-},\varpi^-_m]=-i\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\varpi^i_{-n}\varpi^-_n-\varpi^-_{-n}\varpi^i_n\right),\varpi^-_m\right]$$
$$=-i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^i_{-n}\varpi^-_n\varpi^-_m-\varpi^-_{-n}\varpi^i_n\varpi^-_m-\varpi^-_m\varpi^i_{-n}\varpi^-_n+\varpi^-_m\varpi^-_{-n}\varpi^i_n\right)$$
$$=-i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_n\varpi^-_m-\varpi^-_{-n}\varpi^i_n\varpi^-_m-\varpi^-_m\varpi^i_{-n}\varpi^-_n+\varpi^-_m\varpi^-_{-n}\varpi^i_n$$
$$+\varpi^i_n\varpi^-_m\varpi^-_n-\varpi^i_n\varpi^-_m\varpi^-_n+\varpi^-_{-n}\varpi^-_m\varpi^i_n-\varpi^-_{-n}\varpi^-_m\varpi^i_n)$$
$$=-i\sum_{n=1}^{\infty}\frac{1}{n}((\varpi^i_{-n}\varpi^-_m-\varpi^-_m\varpi^i_{-n})\varpi^-_n+\varpi^i_{-n}(\varpi^-_n\varpi^-_m-\varpi^-_m\varpi^-_n)$$
$$-(\varpi^-_{-n}\varpi^-_m-\varpi^-_m\varpi^-_{-n})\varpi^i_n+\varpi^-_{-n}(\varpi^-_m\varpi^i_n-\varpi^i_n\varpi^-_m))$$
$$=-i\sum_{n=1}^{\infty}\frac{1}{n}([\varpi^i_{-n},\varpi^-_m]\varpi^-_n+\varpi^i_{-n}[\varpi^-_n,\varpi^-_m]-[\varpi^-_{-n},\varpi^-_m]\varpi^i_n+\varpi^-_{-n}[\varpi^-_m,\varpi^i_n]) \tag{4.134}$$

From here, we can read off the commutators from table 1:

$$[\varpi^i_{-n},\varpi^-_m]=\frac{\sqrt{4\pi T}}{p^+}\varpi^i_{m-n} \tag{4.135}$$

$$\left[\varpi^-_n, \varpi^-_m\right] = \sqrt{4\pi T}\frac{m-n}{p^+}\varpi^-_{m+n} + \sqrt{4\pi T}\left(\frac{D-2}{12}m(m^2-1)+2am\right)\frac{\delta_{m+n}}{(p^+)^2} \quad (4.136)$$

$$\left[\varpi^-_{-n}, \varpi^-_m\right] = \sqrt{4\pi T}\frac{m+n}{p^+}\varpi^-_{m+n} + \sqrt{4\pi T}\left(\frac{D-2}{12}m(m^2-1)+2am\right)\frac{\delta_{m-n}}{(p^+)^2} \quad (4.137)$$

Putting (4.135), (4.136) and (4.137) in (4.134) we get:

$$\begin{aligned}[\varXi^{i-}, \varpi^-_m] = -i\sum_{n=1}^{\infty}\frac{1}{n}(&-\frac{\sqrt{4\pi T}}{p^+}n\varpi^i_{m-n} + \varpi^i_{-n}\left(\frac{\sqrt{4\pi T}}{p^+}(m-n)\varpi^-_{m+n}\right) \\ &+\left(\frac{\sqrt{4\pi T}}{p^+}(m+n)\varpi^-_{m-n}\right)\varpi^i_n - \frac{4\pi T}{p+}n\varpi^-_{-n}\varpi^i_{n+m}) \\ &+i\frac{\chi(m)}{m}\end{aligned} \quad (4.138)$$

where:

$$\begin{aligned}\chi(m) = &\sqrt{4\pi T}\frac{m(m-n)}{p^+}\varpi^-_{m+n} + \sqrt{4\pi T}\left(\frac{D-2}{12}m^2(m^2-1)+2am\right)\frac{\delta_{m+n}}{(p^+)^2} \\ &+\sqrt{4\pi T}\frac{m(m+n)}{p^+}\varpi^-_{m+n} + \sqrt{4\pi T}\left(\frac{D-2}{12}m^2(m^2-1)+2am\right)\frac{\delta_{m-n}}{(p^+)^2}\end{aligned} \quad (4.139)$$

D is the number of dimensions.

We now manipulate equation (4.138):

$$\begin{aligned}[\varXi^{i-}, \varpi^-_m] = -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(&-n\varpi^i_{m-n} + \varpi^i_{-n}(m-n)\varpi^-_{m+n} \\ &+(m+n)\varpi^-_{m-n})\varpi^i_n - n\varpi^-_{-n}\varpi^i_{n+m}) \\ &+i\frac{\chi(m)}{m}\end{aligned}$$

$$\begin{aligned}= -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(&-n\varpi^i_{m-n} + (m-n))\varpi^i_{-n}\varpi^-_{m+n} \\ &+(m+n)\varpi^-_{m-n})\varpi^i_n - n\varpi^-_{-n}\varpi^i_{n+m}) \\ &+i\frac{\chi(m)}{m}\end{aligned}$$

$$= -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(-n\varpi^i_{m-n} + m\varpi^i_{-n}\varpi^-_{m+n} - n\varpi^i_{-n}\varpi^-_{m+n}$$

$$+m\varpi^-_{m-n}\varpi^i_n + n\varpi^-_{m-n}\varpi^i_n - n\varpi^-_{-n}\varpi^i_{n+m})$$

$$+i\frac{\chi(m)}{m}$$

$$= -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(-n\varpi^i_{m-n} + m\varpi^i_{-n}\varpi^-_{m+n} - n\varpi^i_{-n}\varpi^-_{m+n}$$

$$+m\varpi^-_{m-n}\varpi^i_n + n\varpi^-_{m-n}\varpi^i_n - n\varpi^-_{-n}\varpi^i_{n+m})$$

$$+i\frac{\chi(m)}{m}$$

$$= -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(-n\varpi^i_{m-n} - n\varpi^i_{-n}\varpi^-_{m+n}$$

$$+n\varpi^-_{m-n}\varpi^i_n - n\varpi^-_{-n}\varpi^i_{n+m}) - i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(-m\varpi^i_{-n}\varpi^-_{m+n} + m\varpi^-_{m-n}\varpi^i_n)$$

$$+i\frac{\chi(m)}{m}$$

$$= -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}n(-\varpi^i_{m-n}\varpi^-_n - \varpi^i_{-n}\varpi^-_{m+n}$$

$$+\varpi^-_{m-n}\varpi^i_n - \varpi^-_{-n}\varpi^i_{n+m}) - i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}m(-\varpi^i_{-n}\varpi^-_{m+n} + \varpi^-_{m-n}\varpi^i_n)$$

$$+i\frac{\chi(m)}{m}$$

$$= -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}(-\varpi^i_{m-n}\varpi^-_n - \varpi^i_{-n}\varpi^-_{m+n}$$

$$+\varpi^-_{m-n}\varpi^i_n - \varpi^-_{-n}\varpi^i_{n+m}) - i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}m(-\varpi^i_{-n}\varpi^-_{m+n} + \varpi^-_{m-n}\varpi^i_n)$$

$$+i\frac{\chi(m)}{m}$$

$$= -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}(-\varpi^i_{m-n}\varpi^-_n - \varpi^i_{-n}\varpi^-_{m+n}$$

$$+\varpi^-_{m-n}\varpi^i_n - \varpi^-_{-n}\varpi^i_{n+m}) - i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{m}{n}(-\varpi^i_{-n}\varpi^-_{m+n} + \varpi^-_{m-n}\varpi^i_n) \tag{4.140}$$

$$+i\frac{\chi(m)}{m}$$

We are interested in terms (1) and (3) of equation (4.140) which we manipulate further: We begin by changing the summation limit from infinity to m:

$$= \sum_{n=1}^{m} \left(\varpi^i_{-n} \varpi^-_{m+n} - \varpi^-_{-n} \varpi^i_{n+m} \right)$$
$$= \left(\varpi^i_{-1} \varpi^-_{m+1} - \varpi^-_{m-1} \varpi^i_{1+m} \right) + ... \left(\varpi^i_{-m} \varpi^-_{2m} - \varpi^-_{-m} \varpi^i_{2m} \right) \approx 0 \tag{4.141}$$

Therefore these terms partially cancel. We are thus left with:

$$\begin{aligned}[\Xi^{i-}, \varpi^-_m] = -i \frac{\sqrt{4\pi T}}{p^+} \sum_{n=1}^{m} (-\varpi^i_{m-n} \varpi^-_n + \varpi^-_{m-n} \varpi^i_n) \\ -i \frac{\sqrt{4\pi T}}{p^+} \sum_{n=1}^{\infty} \frac{m}{n} (-\varpi^i_{-n} \varpi^-_{m+n} + \varpi^-_{m-n} \varpi^i_n) \\ +i \frac{\chi(m)}{m}\end{aligned} \tag{4.142}$$

We focus on the first two terms of equation (4.142) for further manipulation. We make the substitution: $k = m - 1$

$$\sum_{n=1}^{m} \varpi^-_n \varpi^i_{m-n} = \sum_{k=m-1}^{0} \varpi^-_{m-k} \varpi^i_k = \sum_{n=0}^{m-1} \varpi^-_n \varpi^i_{m-n} \tag{4.143}$$

Putting equation (4.143) back in the first two terms of equation (4.142)

$$= \sum_{n=1}^{m} (-\varpi^i_{m-n} \varpi^-_n + \varpi^-_{m-n} \varpi^i_n)$$
$$= -\sum_{n=1}^{m} \varpi^i_{m-n} \varpi^-_n + \sum_{n=1}^{m} \varpi^-_{m-n} \varpi^i_n$$
$$= -\sum_{n=1}^{m-1} \varpi^i_{m-n} \varpi^-_n - \varpi^i_0 \varpi^-_m + \sum_{n=1}^{m-1} \varpi^-_{m-n} \varpi^i_n + \varpi^i_0 \varpi^-_m$$

$$= \varpi^i_0 \varpi^-_m - \varpi^i_0 \varpi^-_m - \sum_{n=1}^{m-1} \varpi^i_{m-n} \varpi^-_n + \sum_{n=1}^{m-1} \varpi^-_{m-n} \varpi^i_n$$
$$= \varpi^i_0 \varpi^-_m - \varpi^i_0 \varpi^-_m - \sum_{n=1}^{m-1} \left(-\varpi^i_{m-n} \varpi^-_n + \varpi^-_{m-n} \varpi^i_n \right)$$

$$= \varpi_0^i \varpi_m^- + \varpi_0^i \varpi_m^- + \sum_{n=1}^{m-1} \left(\varpi_{m-n}^- \varpi_n^i - \varpi_{m-n}^i \varpi_n^- \right) \tag{4.144}$$

We now focus on the term in the summation:

$$\sum_{n=1}^{m-1} \left(\varpi_{m-n}^- \varpi_n^i - \varpi_{m-n}^i \varpi_n^- \right)$$
$$= (\varpi_{m-1}^- \varpi_1^i - \varpi_{m-1}^i \varpi_1^-) + \cdots + (\varpi_1^- \varpi_{m-1}^i - \varpi_1^i \varpi_{m-1}^-) \tag{4.145}$$

Where we have performed the full summation to get equation (4.145). We now rearrange it by coupling the first term to the last, the second to the second last, and so on and so forth:

$$\begin{aligned}
\sum_{n=1}^{m-1} \left(\varpi_{m-n}^- \varpi_n^i - \varpi_{m-n}^i \varpi_n^- \right) &= (\varpi_{m-1}^- \varpi_1^i - \varpi_1^i \varpi_{m-1}^-) + \cdots + (\varpi_{m-1}^i \varpi_1^- - \varpi_1^- \varpi_{m-1}^i) \\
&= \left[\varpi_{m-1}^-, \varpi_1^i\right] + \cdots + \left[\varpi_{m-1}^i, \varpi_1^-\right] \\
&= \sum_{n=1}^{m-1} [\varpi_n^-, \varpi_{m-n}^i] = -\sum_{n=1}^{m-1} [\varpi_{m-n}^i, \varpi_n^-] \\
&= \sum_{n=1}^{m-1} \frac{\sqrt{4\pi T}}{p^+} (m-n) \varpi_m^i \\
&= \frac{\sqrt{4\pi T}}{p^+} \varpi_m^i \sum_{n=1}^{m-1} (m-n) \\
&= \frac{\sqrt{4\pi T}}{p^+} \varpi_m^i \frac{m(m-1)}{2}
\end{aligned} \tag{4.146}$$

Putting result (4.146) into expression (4.144) and putting the resulting terms in equation (4.142) we get:

$$= \varpi_0^i \varpi_m^- + \varpi_0^i \varpi_m^- + \frac{\sqrt{4\pi T}}{p^+} \varpi_m^i \frac{m(m-1)}{2} \tag{4.147}$$

$$[\Xi^{i-}, \varpi_m^-] = i \frac{\sqrt{4\pi T}}{p^+} (\varpi_0^i \varpi_m^- + \varpi_0^i \varpi_m^-) - i \frac{\sqrt{4\pi T}}{p^+} \frac{m}{n} (-\varpi_{-n}^i \varpi_{m+n}^- + \varpi_{m-n}^- \varpi_n^i)$$
$$+ i \frac{4\pi T}{(p^+)^2} \left(\frac{m(m-1)}{2} - \frac{f(m)}{m} \right) \varpi_{-m}^i \tag{4.148}$$

We now proceed to calculate another commutator $[\Xi^{i-}, \varpi^j_m]$

$$[\Xi^{i-}, \varpi^j_m] = \left[-i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^i_{-n}\varpi^- - \varpi^-_{-n}\varpi^i_n\right), \varpi^j_m\right] \tag{4.149}$$

$$= -i\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\varpi^i_{-n}\varpi^- - \varpi^-_{-n}\varpi^i_n\right), \varpi^j_m\right]$$

$$= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_n\varpi^j_m - \varpi^-_{-n}\varpi^i_n\varpi^j_m - \varpi^j_m\varpi^i_{-n}\varpi^-_n + \varpi^j_m\varpi^-_{-n}\varpi^i_n) \tag{4.150}$$

We manipulate terms 2 and 3 of equation (4.150) further:

$$\begin{aligned}
[\Xi^{i-}, \varpi^j_m] &= -\varpi^-_{-n}\varpi^i_n\varpi^j_m - \varpi^j_m\varpi^i_{-n}\varpi^-_n - \varpi^-_{-n}\varpi^j_m\varpi^i_n + \varpi^-_{-n}\varpi^j_m\varpi^i_n \\
&= -\varpi^-_{-n}\varpi^i_n\varpi^j_m - \varpi^-_{-n}\varpi^j_m\varpi^i_n - \varpi^j_m\varpi^i_{-n}\varpi^-_n + \varpi^-_{-n}\varpi^j_m\varpi^i_n \\
&= -\varpi^-_{-n}\varpi^j_m\varpi^i_n - \varpi^j_m\varpi^i_{-n}\varpi^-_n + \varpi^-_{-n}\varpi^j_m\varpi^i_n - \varpi^-_{-n}\varpi^i_n\varpi^j_m \\
&= -\varpi^-_{-n}\varpi^j_m\varpi^i_n - \varpi^j_m\varpi^i_{-n}\varpi^-_n + \varpi^-_{-n}(\varpi^j_m\varpi^i_n - \varpi^i_n\varpi^j_m) \\
&= -\varpi^-_{-n}\varpi^j_m\varpi^i_n - \varpi^j_m\varpi^i_{-n}\varpi^-_n + \varpi^-_{-n}[\varpi^j_m, \varpi^i_n]
\end{aligned}$$

Reading off the commutator tables;

$$[\Xi^{i-}, \varpi^j_m] = -\varpi^-_{-n}\varpi^j_m\varpi^i_n - \varpi^j_m\varpi^i_{-n}\varpi^-_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm} \tag{4.151}$$

Putting equation (4.151) in equation (4.150)

$$[\Xi^{i-}, \varpi^j_m] = -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_n\varpi^j_m + \varpi^j_m\varpi^-_{-n}\varpi^i_n - \varpi^-_{-n}\varpi^j_m\varpi^i_n - \varpi^j_m\varpi^i_{-n}\varpi^-_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm}) \tag{4.152}$$

$$\begin{aligned}
&= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_n\varpi^j_m + \varpi^j_m\varpi^-_{-n}\varpi^i_n - \varpi^-_{-n}\varpi^j_m\varpi^i_n - \varpi^i_{-n}\varpi^j_m\varpi^-_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm}) \\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_n\varpi^j_m - \varpi^i_{-n}\varpi^j_m\varpi^-_n + \varpi^j_m\varpi^-_{-n}\varpi^i_n - \varpi^-_{-n}\varpi^j_m\varpi^i_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm}) \\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}(\varpi^-_n\varpi^j_m - \varpi^j_m\varpi^-_n) + (\varpi^j_m\varpi^-_{-n} - \varpi^-_{-n}\varpi^j_m)\varpi^i_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm}) \\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}[\varpi^-_n, \varpi^j_m] + [\varpi^j_m, \varpi^-_{-n}]\varpi^i_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm})
\end{aligned}$$

$$= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}[\varpi^-_n, \varpi^j_m] + [\varpi^j_m, \varpi^-_{-n}]\varpi^i_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm}) \tag{4.153}$$

$$\begin{aligned}
&= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}[\varpi^-_n, \varpi^j_m] + [\varpi^j_m, \varpi^-_{-n}]\varpi^i_n + \cancelto{0}{\varpi^-_{-n}n\delta^{ij}\delta_{nm})}\\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\sqrt{4\pi T}}{p^+}m\varpi^i_{-n}\varpi^i_{n+m} - \frac{\sqrt{4\pi T}}{p^+}m\alpha^i_{n+m}\varpi^i_n\right)\\
&= -i\sum_{n=1}^{\infty}\frac{m}{n}\left(\frac{\sqrt{4\pi T}}{p^+}\varpi^i_{-n}\varpi^i_{n+m} - \frac{\sqrt{4\pi T}}{p^+}\varpi^i_{n+m}\varpi^i_n\right)\\
&= -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{m}{n}\left(\varpi^i_{-n}\varpi^i_{n+m} - \varpi^i_{-n+m}\varpi^i_n\right)
\end{aligned} \tag{4.154}$$

We repeat the computation for $[\Xi^{i-}, \varpi^j_{-m}]$

$$[\Xi^{i-}, \varpi^j_{-m}] = \left[-i\sum_{n=1}^{\infty}\frac{1}{n}\left(\varpi^i_{-n}\varpi^-_n - \varpi^-_{-n}\varpi^i_n\right), \varpi^j_{-m}\right] \tag{4.155}$$

$$= -i\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\varpi^i_{-n}\varpi^- - \varpi^-_{-n}\varpi^i_n\right), \varpi^j_m\right]$$

$$= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_n\varpi^j_{-m} - \varpi^-_{-n}\varpi^i_n\varpi^j_{-m} - \varpi^j_{-m}\varpi^i_{-n}\varpi^-_n + \varpi^j_{-m}\varpi^-_{-n}\varpi^i_n) \tag{4.156}$$

$$= -\varpi^-_{-n}\varpi^i_n\varpi^j_{-m} - \varpi^j_{-m}\varpi^i_{-n}\varpi^-_n - \varpi^-_{-n}\varpi^j_{-m}\varpi^i_n + \varpi^-_{-n}\varpi^j_{-m}\varpi^i_n$$

$$\begin{aligned}
&= -\varpi^-_{-n}\varpi^i_n\varpi^j_{-m} - \varpi^-_{-n}\varpi^j_{-m}\varpi^i_n - \varpi^j_{-m}\varpi^i_{-n}\varpi^-_n + \varpi^-_{-n}\varpi^j_{-m}\varpi^i_n\\
&= -\varpi^-_{-n}\varpi^j_{-m}\varpi^i_n - \varpi^j_{-m}\varpi^i_{-n}\varpi^-_n + \varpi^-_{-n}\varpi^j_{-m}\varpi^i_n - \varpi^-_{-n}\varpi^i_n\varpi^j_{-m}\\
&= -\varpi^-_{-n}\varpi^j_{-m}\varpi^i_n - \varpi^j_{-m}\varpi^i_{-n}\varpi^-_n + \varpi^-_{-n}(\varpi^j_{-m}\varpi^i_n - \varpi^i_n\varpi^j_{-m})
\end{aligned}$$

$$\begin{aligned}
&= -\varpi^-_{-n}\varpi^j_{-m}\varpi^i_n - \varpi^j_{-m}\varpi^i_{-n}\varpi^-_n + \varpi^-_{-n}[\varpi^j_{-m}, \varpi^i_n]\\
&= -\varpi^-_{-n}\varpi^j_{-m}\varpi^i_n - \varpi^j_{-m}\varpi^i_{-n}\varpi^-_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm}
\end{aligned}$$

$$= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_n\varpi^j_{-m} + \varpi^j_{-m}\varpi^-_{-n}\varpi^i_n - \varpi^-_{-n}\varpi^j_{-m}\varpi^i_n - \varpi^i_{-n}\varpi^j_{-m}\varpi^-_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm})$$

$$\begin{aligned}
&= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_n\varpi^j_{-m} - \varpi^i_{-n}\varpi^j_{-m}\varpi^-_n + \varpi^j_{-m}\varpi^-_{-n}\varpi^i_n - \varpi^-_{-n}\varpi^j_{-m}\varpi^i_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm})\\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}(\varpi^-_n\varpi^j_{-m} - \varpi^j_{-m}\varpi^-_n) + (\varpi^j_{-m}\varpi^-_{-n} - \varpi^-_{-n}\varpi^j_{-m})\varpi^i_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm})\\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}[\varpi^-_n, \varpi^j_{-m}] + [\varpi^j_{-m}, \varpi^-_{-n}]\varpi^i_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm})\\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}[\varpi^-_n, \varpi^j_{-m}] + [\varpi^j_{-m}, \varpi^-_{-n}]\varpi^i_n + \varpi^-_{-n}n\delta^{ij}\delta_{nm})
\end{aligned}$$

$$\begin{aligned}
&= -i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}[\varpi^-_n, \varpi^j_{-m}] + [\varpi^j_{-m}, \varpi^-_{-n}]\varpi^i_n + \cancelto{0}{\varpi^-_{-n}n\delta^{ij}\delta_{nm}})\\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\sqrt{4\pi T}}{p^+}m\varpi^i_{-n}\varpi^i_{n-m} - \frac{\sqrt{4\pi T}}{p^+}m\alpha^i_{n-m}\varpi^i_n\right)\\
&= -i\sum_{n=1}^{\infty}\frac{-m}{n}\left(\frac{\sqrt{4\pi T}}{p^+}\varpi^i_{-n}\varpi^i_{n-m} - \frac{\sqrt{4\pi T}}{p^+}\varpi^i_{n-m}\varpi^i_n\right)\\
&= -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{-m}{n}\left(\varpi^i_{-n}\varpi^i_{n-m} - \varpi^i_{-n-m}\varpi^i_n\right)
\end{aligned} \tag{4.157}$$

Putting these results in the commutator $[\Xi^{i-}, \Xi^{j-}]$ we obtain:

$$\left[\Xi^{i-}, \Xi^{j-}\right] = \frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(-\varpi^j_{-n}\varpi^-_0\varpi^i_n + \varpi^j_{-n}\varpi^i_0\varpi^-_n + \varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^-_n\varpi^i_0\varpi^j_n)$$

$$\cdots - \sum_{n=1}^{\infty}\frac{4\pi T}{(p+)^2}\left(\frac{n-1}{2} - \frac{\chi(n)}{n^2}\right)(\varpi^i_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^i_n)$$

$$\cdots + (\varpi^i_{-n}\varpi^-_{n-m}\varpi^j_m - \varpi^j_{-m}\varpi^-_{m-n}\varpi^i_n)\varpi^j_m$$

$$\cdots - \varpi^-_{-m}(\varpi^i_{-n}\varpi^j_{m+n} - \varpi^j_{m-n}\varpi^i_n) \tag{4.158}$$

Indeed, we can demonstrate that this is the commutator. We do this by manipulating term two of equation (4.158).

$$
\begin{aligned}
&-\sum_{n=1}^{\infty}\frac{4\pi T}{(p+)^2}\left(\frac{n-1}{2}-\frac{\chi(n)}{n^2}\right)(\varpi^i_{-n}\varpi^j_n-\varpi^j_{-n}\varpi^i_n)\\
&=-\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\left(\frac{n-1}{2}-\frac{\chi(n)}{n^2}\right)(\varpi^i_{-n}\varpi^j_n-\varpi^j_{-n}\varpi^i_n)\\
&=\left(-\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{n-1}{2}+\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{\chi(n)}{n^2}\right)(\varpi^i_{-n}\varpi^j_n-\varpi^j_{-n}\varpi^i_n)\\
&=-\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{n-1}{2}\varpi^i_{-n}\varpi^j_n+\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{\chi(n)}{n^2}\varpi^i_{-n}\varpi^j_n\\
&\cdots+\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{n-1}{2}\varpi^j_n\varpi^j_n-\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{\chi(n)}{n^2}\varpi^j_{-n}\varpi^i_n
\end{aligned}
\tag{4.159}
$$

We consider terms (1) and (2) of equation (4.159):

$$
=-\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{n-1}{2}\varpi^i_{-n}\varpi^j_n+\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{\chi(n)}{n^2}\varpi^i_{-n}\varpi^j_n
$$

Term (1) can be rewritten as:

$$
\begin{aligned}
-\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{n-1}{2}\varpi^i_{-n}\varpi^j_n&=-\sum_{n=1}^{\infty}\varpi^i_{-n}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{n-1}{2}\varpi^j_n\\
&=-\frac{\sqrt{4\pi T}}{(p+)}\sum_{n=1}^{\infty}\varpi^i_{-n}\frac{\sqrt{4\pi T}}{(p+)}\frac{1}{n}\frac{n(n-1)}{2}\varpi^j_n
\end{aligned}
\tag{4.160}
$$

Now,

$$
\begin{aligned}
\frac{\sqrt{4\pi T}}{(p+)}\frac{n(n-1)}{2}\varpi^j_n&=\frac{\sqrt{4\pi T}}{(p+)}\sum_{k}^{n}(n-k)\varpi^j_n\\
&=\sum_{k}^{n}\frac{\sqrt{4\pi T}}{(p+)}(n-k)\varpi^j_n\\
&=\sum_{k=}^{n}[\varpi^-_k,\varpi^j_{n-k}]
\end{aligned}
\tag{4.161}
$$

We now reassemble the solution by putting equation (4.161) into equation (4.160).

$$= -\frac{\sqrt{4\pi T}}{(p+)}\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{i}_{-n}\sum_{k=1}^{n-1}[\varpi^{-}_{k},\varpi^{j}_{n-k}] + \sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{\chi(n)}{n^2}\varpi^{i}_{-n}\varpi^{j}_{n} \tag{4.162}$$

We are interested in elucidating the commutator in term (1) of equation (4.162) further:

$$\begin{aligned}\sum_{k=1}^{n-1}[\varpi^{-}_{k},\varpi^{j}_{n-k}] &= \sum_{k=1}^{n-1}[\varpi^{-}_{k},\varpi^{j}_{n-k}] + \varpi^{-}_{0}\varpi^{j}_{k} - \varpi^{j}_{0}\varpi^{-}_{n} - \varpi^{-}_{0}\varpi^{j}_{k} + \varpi^{j}_{0}\varpi^{-}_{n}\\ &= \varpi^{-}_{0}\varpi^{j}_{k} - \varpi^{j}_{0}\varpi^{-}_{n} - \varpi^{-}_{0}\varpi^{j}_{k} + \varpi^{j}_{0}\varpi^{-}_{n} + \sum_{k=1}^{n-1}[\varpi^{-}_{k},\varpi^{j}_{n-k}]\\ &= \varpi^{-}_{0}\varpi^{j}_{k} - \varpi^{j}_{0}\varpi^{-}_{n} - \varpi^{-}_{0}\varpi^{j}_{k} + \varpi^{j}_{0}\varpi^{-}_{n} + \sum_{k=1}^{n-1}(\varpi^{-}_{k}\varpi^{j}_{n-k} - \varpi^{j}_{n-k}\varpi^{-}_{k})\\ &= \varpi^{-}_{0}\varpi^{j}_{k} - \varpi^{j}_{0}\varpi^{-}_{n} - \varpi^{-}_{0}\varpi^{j}_{k} + \varpi^{j}_{0}\varpi^{-}_{n} + \sum_{k=1}^{n-1}\varpi^{-}_{k}\varpi^{j}_{n-k} - \sum_{k=1}^{n-1}\varpi^{j}_{n-k}\varpi^{-}_{k}\end{aligned} \tag{4.163}$$

$$\begin{aligned}&= -\varpi^{-}_{0}\varpi^{j}_{k} + \varpi^{j}_{0}\varpi^{-}_{n} + \sum_{k=0}^{n-1}\varpi^{-}_{k}\varpi^{j}_{n-k} - \sum_{k=0}^{n-1}\varpi^{j}_{n-k}\varpi^{-}_{k}\\ &= -\varpi^{-}_{0}\varpi^{j}_{k} + \varpi^{j}_{0}\varpi^{-}_{n} + \sum_{k=0}^{n-1}(\varpi^{-}_{k}\varpi^{j}_{n-k} - \varpi^{j}_{n-k}\varpi^{-}_{k})\end{aligned} \tag{4.164}$$

Putting expression (4.164) back into equation (4.162):

$$\begin{aligned}&= -\frac{\sqrt{4\pi T}}{(p+)}\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{i}_{-n}\left(-\varpi^{-}_{0}\varpi^{j}_{k} + \varpi^{j}_{0}\varpi^{-}_{n} + \sum_{k=0}^{n-1}(\varpi^{-}_{k}\varpi^{j}_{n-k} - \varpi^{j}_{n-k}\varpi^{-}_{k})\right)\\ &\qquad \cdots + \sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{\chi(n)}{n^2}\varpi^{i}_{-n}\varpi^{j}_{n}\end{aligned} \tag{4.165}$$

So we can now repeat the process for terms (3) and (4) of equation (4.159). The two terms are:

$$= \sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{n-1}{2}\varpi^{j}_{-n}\varpi^{i}_{n} - \sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{\chi(n)}{n^2}\varpi^{j}_{-n}\varpi^{i}_{n} \tag{4.166}$$

We focus on term one of expression (4.166)

$$
\begin{aligned}
-\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{n-1}{2}\varpi^{j}_{-n}\varpi^{i}_{n} &= -\varpi^{i}_{-n}\sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{n-1}{2}\varpi^{i}_{n} \\
&= -\frac{\sqrt{4\pi T}}{(p+)}\sum_{n=1}^{\infty}\varpi^{i}_{-n}\frac{\sqrt{4\pi T}}{(p+)}\frac{1}{n}\frac{n(n-1)}{2}\varpi^{i}_{n} \\
\frac{\sqrt{4\pi T}}{(p+)}\frac{n(n-1)}{2}\varpi^{i}_{n} &= \frac{\sqrt{4\pi T}}{(p+)}\sum_{k}^{n}(n-k)\varpi^{i}_{n} \\
&= \sum_{k}^{n}\frac{\sqrt{4\pi T}}{(p+)}(n-k)\varpi^{i}_{n} \\
&= \sum_{k=}^{n}[\varpi^{-}_{k},\varpi^{i}_{n-k}]
\end{aligned}
$$

$$
= -\frac{\sqrt{4\pi T}}{(p+)}\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{j}_{-n}\sum_{k=1}^{n-1}[\varpi^{-}_{k},\varpi^{i}_{n-k}] + \sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{\chi(n)}{n^2}\varpi^{j}_{-n}\varpi^{i}_{n}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n-1}[\varpi^{-}_{k},\varpi^{i}_{n-k}] + \varpi^{-}_{0}\varpi^{i}_{k} - \varpi^{i}_{0}\varpi^{-}_{n} - \varpi^{-}_{0}\varpi^{i}_{k} + \varpi^{i}_{0}\varpi^{-}_{n} \\
&= \varpi^{-}_{0}\varpi^{i}_{k} - \varpi^{i}_{0}\varpi^{-}_{n} - \varpi^{-}_{0}\varpi^{i}_{k} + \varpi^{i}_{0}\varpi^{-}_{n} + \sum_{k=1}^{n-1}[\varpi^{-}_{k},\varpi^{i}_{n-k}] \\
&= \varpi^{-}_{0}\varpi^{i}_{k} - \varpi^{i}_{0}\varpi^{-}_{n} - \varpi^{-}_{0}\varpi^{i}_{k} + \varpi^{i}_{0}\varpi^{-}_{n} + \sum_{k=1}^{n-1}(\varpi^{-}_{k}\varpi^{i}_{n-k} - \varpi^{i}_{n-k}\varpi^{-}_{k}) \\
&= \varpi^{-}_{0}\varpi^{i}_{k} - \varpi^{i}_{0}\varpi^{-}_{n} - \varpi^{-}_{0}\varpi^{i}_{k} + \varpi^{i}_{0}\varpi^{-}_{n} + \sum_{k=1}^{n-1}\varpi^{-}_{k}\varpi^{i}_{n-k} - \sum_{k=1}^{n-1}\varpi^{i}_{n-k}\varpi^{-}_{k} \\
&= -\frac{\sqrt{4\pi T}}{(p+)}\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{j}_{-n}\left(-\varpi^{-}_{0}\varpi^{i}_{k} + \varpi^{i}_{0}\varpi^{-}_{n} + \sum_{k=0}^{n-1}(\varpi^{-}_{k}\varpi^{i}_{n-k} - \varpi^{i}_{n-k}\varpi^{-}_{k})\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\sqrt{4\pi T}}{(p+)}\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{j}_{-n}\left(-\varpi^{-}_{0}\varpi^{i}_{k} + \varpi^{i}_{0}\varpi^{-}_{n} + \sum_{k=0}^{n-1}(\varpi^{-}_{k}\varpi^{i}_{n-k} - \varpi^{i}_{n-k}\varpi^{-}_{k})\right) \\
&\qquad \cdots + \sum_{n=1}^{\infty}\frac{\sqrt{4\pi T}}{(p+)}\frac{\sqrt{4\pi T}}{(p+)}\frac{\chi(n)}{n^2}\varpi^{j}_{-n}\varpi^{i}_{n} \qquad (4.167)
\end{aligned}
$$

We now put equations (4.167) and (4.165) together:

$$
= -\frac{\sqrt{4\pi T}}{(p+)}\sum_{n=1}^{\infty}\frac{1}{n}\varpi^{i}_{-n}\left(-\varpi^{-}_{0}\varpi^{j}_{k} + \varpi^{j}_{0}\varpi^{-}_{n} + \sum_{k=0}^{n-1}(\varpi^{-}_{k}\varpi^{j}_{n-k} - \varpi^{j}_{n-k}\varpi^{-}_{k})\right)
$$

$$\cdots + \sum_{n=1}^{\infty} \frac{\sqrt{4\pi T}}{(p+)} \frac{\sqrt{4\pi T}}{(p+)} \frac{\chi(n)}{n^2} \varpi_{-n}^{i} \varpi_{n}^{j}$$

$$\cdots - \frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \varpi_{-n}^{j} \left(-\varpi_0^- \varpi_k^i + \varpi_0^i \varpi_n^- + \sum_{k=0}^{n-1} (\varpi_k^- \varpi_{n-k}^i - \varpi_{n-k}^i \varpi_k^-) \right)$$

$$\cdots + \sum_{n=1}^{\infty} \frac{\sqrt{4\pi T}}{(p+)} \frac{\sqrt{4\pi T}}{(p+)} \frac{\chi(n)}{n^2} \varpi_{-n}^{j} \varpi_{n}^{i}$$

Opening brackets partially:

$$= -\frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \left(-\varpi_{-n}^{i} \varpi_0^- \varpi_k^j + \varpi_{-n}^{i} \varpi_0^j \varpi_n^- \right) - \frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \varpi_{-n}^{i} \left(\sum_{k=0}^{n-1} (\varpi_k^- \varpi_{n-k}^j - \varpi_{n-k}^j \varpi_k^-) \right)$$

$$\cdots + \sum_{n=1}^{\infty} \frac{\sqrt{4\pi T}}{(p+)} \frac{\sqrt{4\pi T}}{(p+)} \frac{\chi(n)}{n^2} \varpi_{-n}^{i} \varpi_{n}^{j}$$

$$\cdots - \frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \left(-\varpi_{-n}^{j} \varpi_0^- \varpi_k^i + \varpi_{-n}^{j} \varpi_0^i \varpi_n^- \right) - \frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \varpi_{-n}^{j} \left(\sum_{k=0}^{n-1} (\varpi_k^- \varpi_{n-k}^i - \varpi_{n-k}^i \varpi_k^-) \right)$$

$$\cdots + \sum_{n=1}^{\infty} \frac{\sqrt{4\pi T}}{(p+)} \frac{\sqrt{4\pi T}}{(p+)} \frac{\chi(n)}{n^2} \varpi_{-n}^{j} \varpi_{n}^{i}$$

We now collect terms with the ground state of the mode expansions:

$$= -\frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \left(-\varpi_{-n}^{i} \varpi_0^- \varpi_k^j + \varpi_{-n}^{i} \varpi_0^j \varpi_n^- \right) - \frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \left(-\varpi_{-n}^{j} \varpi_0^- \varpi_k^i + \varpi_{-n}^{j} \varpi_0^i \varpi_n^- \right)$$

$$\cdots - \frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \varpi_{-n}^{j} \left(\sum_{k=0}^{n-1} (\varpi_k^- \varpi_{n-k}^i - \varpi_{n-k}^i \varpi_k^-) \right)$$

$$-\frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \varpi_{-n}^{i} \left(\sum_{k=0}^{n-1} (\varpi_k^- \varpi_{n-k}^j - \varpi_{n-k}^j \varpi_k^-) \right)$$

$$\cdots + \sum_{n=1}^{\infty} \frac{\sqrt{4\pi T}}{(p+)} \frac{\sqrt{4\pi T}}{(p+)} \frac{\chi(n)}{n^2} \varpi_{-n}^{j} \varpi_{n}^{i} + \sum_{n=1}^{\infty} \frac{\sqrt{4\pi T}}{(p+)} \frac{\sqrt{4\pi T}}{(p+)} \frac{\chi(n)}{n^2} \varpi_{-n}^{i} \varpi_{n}^{j}$$

Factoring out the summation:

$$= -\frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \left(-\varpi_{-n}^{i} \varpi_0^- \varpi_k^j + \varpi_{-n}^{i} \varpi_0^j \varpi_n^- + \varpi_{-n}^{j} \varpi_0^- \varpi_k^i + \varpi_{-n}^{j} \varpi_0^i \varpi_n^- \right)$$

$$\cdots - \frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \varpi^j_{-n} \left(\sum_{k=0}^{n-1} (\varpi^-_k \varpi^i_{n-k} - \varpi^i_{n-k} \varpi^-_k) \right)$$

$$\cdots - \frac{\sqrt{4\pi T}}{(p+)} \sum_{n=1}^{\infty} \frac{1}{n} \varpi^i_{-n} \left(\sum_{k=0}^{n-1} (\varpi^-_k \varpi^j_{n-k} - \varpi^j_{n-k} \varpi^-_k) \right)$$

$$\cdots + \sum_{n=1}^{\infty} \frac{4\pi T}{(p+)^2} \frac{\chi(n)}{n^2} \left(\varpi^j_{-n} \varpi^i_n + \varpi^i_{-n} \varpi^j_n \right) \tag{4.168}$$

Thus, we have recovered term (1) of equation (4.158) in (4.168). The others can be recovered using similar processes. Thus the commutator can be calculated using the sub-commutators developed in equations (4.148), (4.154) and (4.157). We now focus on the last term of equation (4.158). The algebraic manipulation of this term will help us link equation (4.158) with equation(4.168) later in the work when finalising the calculation of the critical dimensions: We will refer to the last term of equation (4.158) as C^{ij}. The reader trained in the cosmological steady state theory should not conflate this with the C-fields present in the theory.

$$C^{ij} = -\frac{\sqrt{4\pi T}}{(p+)} \sum_{m,n=1}^{\infty} \frac{1}{n} (\varpi^j_{-m} (\varpi^i_{-n} \varpi^-_{m+n} - \varpi^-_{m-n} \varpi^i_n) - (\varpi^i_{-n} \varpi^j_{n+m} - \varpi^j_{-m-n} \varpi^i_n) \varpi^-_m$$

$$\cdots + (\varpi^i_{-n} \varpi^j_{n+m} - \varpi^j_{-m-n} \varpi^i_n) \varpi^j_{-m} - \varpi^-_m (\varpi^i_{-n} \varpi^-_{n-m} - \varpi^-_{m-n} \varpi^i_n)) \tag{4.169}$$

Opening brackets:

$$C^{ij} = -\frac{\sqrt{4\pi T}}{(p+)} \sum_{m,n=1}^{\infty} \frac{1}{n} (\varpi^j_{-m} \varpi^i_{-n} \varpi^-_{m+n} - \varpi^j_{-m} \varpi^-_{m-n} \varpi^i_n) - (\varpi^i_{-n} \varpi^j_{n+m} - \varpi^j_{-m-n} \varpi^i_n) \varpi^-_m$$

$$\cdots + (\varpi^i_{-n} \varpi^j_{n+m} \varpi^j_m - \varpi^j_{-m-n} \varpi^i_n \varpi^j_m) - \varpi^-_{-m} (\varpi^i_{-n} \varpi^-_{n-m} - \varpi^-_{m-n} \varpi^i_n)) \tag{4.170}$$

We are interested in term (1) and term (4) of equation (4.170):

$$= \sum_{m,n=0}^{\infty} (\varpi^j_{-m} \varpi^i_{-n} \varpi^-_{m+n} - \varpi^i_{-n} \varpi^j_{n-m} \varpi^-_m) \tag{4.171}$$

We want to manipulate equation (4.171) further by invoking a change of indices:

$$\begin{aligned}
&\sum_{m,n=0}^{\infty}\left(\varpi^j_{-m}\varpi^i_{-n}\varpi^-_{m+n}-\varpi^i_{-n}\varpi^j_{n-m}\varpi^-_m\right) &(4.172)\\
&=\varpi^j_0\varpi^i_0\varpi^-_0-\varpi^i_0\varpi^j_0\varpi^-_0+\sum_{m,n=1}^{\infty}\left(\varpi^j_{-m}\varpi^i_{-n}\varpi^-_{m+n}-\varpi^i_{-n}\varpi^j_{n-m}\varpi^-_m\right)\\
&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\varpi^j_{-m}\varpi^i_{-n}\varpi^-_{m+n}-\varpi^i_{-n}\varpi^j_{n-m}\varpi^-_m)\\
&=\sum_{m=1}^{\infty}\sum_{n=1}^{m}(\varpi^j_{-m}\varpi^i_{-n}\varpi^-_{m+n}-\varpi^i_{-n}\varpi^j_{n-m}\varpi^-_m)\\
&=\sum_{m=1}^{\infty}(\varpi^j_{-m}\varpi^i_{-1}\varpi^-_{m+1}-\varpi^i_{-1}\varpi^j_{1-m}\varpi^-_m)\\
&\quad+\cdots+(\varpi^j_{-m}\varpi^i_{-m}\varpi^-_{2m}-\varpi^i_{-m}\varpi^j_0\varpi^-_m)\\
&=\sum_{m=1}^{\infty}(\varpi^j_{-m}\varpi^i_{-1}\varpi^-_{m+1}-\varpi^i_{-1}\varpi^j_{1-m}\varpi^-_m)\\
&\quad+\cdots+(\varpi^j_{-m}\varpi^i_{-m}\varpi^-_{2m}-\varpi^i_{-m}\varpi^j_0\varpi^-_m)\\
&\approxeq 0 &(4.173)
\end{aligned}$$

Similarly:

$$\sum_{n=1}^{m}(\varpi^-_{-n}\varpi^i_n\varpi^j_m+\varpi^-_{-m}\varpi^i_{-n}\varpi^j_{m+n})=\sum_{m=1}^{\infty}\sum_{n=1}^{m}(\varpi^-_{-m}\varpi^i_{-n}\varpi^j_{m+n}-\varpi^-_{-n}\varpi^i_n\varpi^j_m)$$

$$\begin{aligned}
&=\sum_{m=1}^{\infty}\sum_{n=1}^{m}(\varpi^-_{-m}\varpi^i_{-1}\varpi^j_{m+1}-\varpi^-_{-1}\varpi^i_1\varpi^j_m)\\
&\quad+\cdots+(\varpi^-_{-m}\varpi^i_{-m}\varpi^j_{2m}-\varpi^-_{-m}\varpi^i_m\varpi^j_m)\\
&\approxeq 0 &(4.174)
\end{aligned}$$

So that we're left with:

$$C^{ij}=\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{m}\left(-\sum_{m=1}^{n}\frac{1}{n}\varpi^i_{-n}\varpi^j_{n-m}\varpi^-_m+\sum_{m=1}^{n}\varpi^-_{-m}\varpi^j_{m-n}\varpi^i_n\right)$$

$$+\frac{\sqrt{4\pi T}}{p^+}\sum_{m,n}^{\infty}(\varpi^j_{-m-n}\varpi^i_n\varpi^-_m + \varpi^i_{-n}\varpi^i_{-n}\varpi^-_{n-m}\varpi^j_m - \varpi^j_{-m}\varpi^-_{m-n} - \varpi^-_{-m}\varpi^i_{-n}\varpi^j_{m+n}) \quad (4.175)$$

The first term of the second line of equation (4.174) is not normal ordered, this presents a quantum field theoretic challenge. With respect to the conservation of energy states. We do not expect that an annihilation operator annihilates the vacuum. So we need to manipulate the term in such a way that it can be consistent with these notions. This is the normal ordering procedure. We perform it below:

$$\begin{aligned}
\sum_{m,n}^{\infty}\varpi^j_{-m-n}\varpi^i_n\varpi^-_m &= \sum_{m,n}^{\infty}(\varpi^j_{-m-n}\varpi^i_n\varpi^-_m + \varpi^j_{-m-n}\varpi^-_m\varpi^i_n - \varpi^j_{-m-n}\varpi^-_m\varpi^i_n) \\
&= \sum_{m,n}^{\infty}(\varpi^j_{-m-n}\varpi^i_n\varpi^-_m - \varpi^j_{-m-n}\varpi^-_m\varpi^i_n + \varpi^j_{-m-n}\varpi^-_m\varpi^i_n) \\
&= \sum_{m,n}^{\infty}\varpi^j_{-m-n}(\varpi^i_n\varpi^-_m - \varpi^-_m\varpi^i_n) + \varpi^j_{-m-n}\varpi^-_m\varpi^i_n \\
&= \sum_{m,n}^{\infty}\varpi^j_{-m-n}[\varpi^i_n, \varpi^-_m] + \varpi^j_{-m-n}\varpi^-_m\varpi^i_n \\
&= \sum_{m,n}^{\infty}\varpi^j_{-m-n}(\varpi^j_{-m-n}\varpi^-_m\varpi^i_n + [\varpi^i_n, \varpi^-_m]) \\
&= \sum_{m,n}^{\infty}\varpi^j_{-m-n}\varpi^-_m\varpi^i_n + \sum_{m,n}^{\infty}\varpi^j_{-m-n}[\varpi^i_n, \varpi^-_m])
\end{aligned}$$

Writing in full and reading the commutator table:

$$\sum_{m,n}^{\infty}\varpi^j_{-m-n}\varpi^i_n\varpi^-_m = \sum_{m,n}^{\infty}\frac{1}{n}\varpi^j_{-m-n}\varpi^-_m\varpi^i_n + \frac{\sqrt{4\pi T}}{p^+}\sum_{m,n}^{\infty}\frac{1}{n}\varpi^j_{-m-n}\varpi^i_{n+m} \quad (4.176)$$

$$m + n = k$$

$$\frac{\sqrt{4\pi T}}{p^+}\sum_{m,n}^{\infty}\frac{1}{n}\varpi^j_{-k}\varpi^i_k$$

$$\sum_{n,m=1}^{\infty}\frac{1}{n}\varpi^-_{-m}\varpi^i_{-n}\varpi^j_{m+n} = \sum_{n,m=1}^{\infty}\frac{1}{n}\varpi^i_{-n}\varpi^-_{-m}\varpi^j_{m+n} + \frac{\sqrt{4\pi T}}{p^+}\sum_{k=2}^{\infty}(k-1)\varpi^i_{-k}\varpi^j_k$$

We now get:

$$C^{ij} = \frac{\sqrt{4\pi T}}{p^+}\sum_{m,n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{1}{n}\varpi^i_{-n}\varpi^j_{n-m}\varpi^-_m + \sum_{m=1}^{n}\frac{1}{n}\varpi^-_{-m}\varpi^j_{m-n}\varpi^i_n\right)$$

$$\frac{\sqrt{4\pi T}}{p^+}\sum_{m,n=1}^{\infty}\frac{1}{n}(\varpi^j_{-m-n}\varpi^-_m\varpi^i_n-\varpi^i_{-n}\varpi^-_{-m}\varpi^j_{m+n}-\varpi^i_{-n}\varpi^-_{n-m}\varpi^j_m-\varpi^j_{-m}\varpi^-_{m-n}\varpi^i_n)$$

$$\frac{4\pi T}{(p^+)^2}\sum_{n=2}^{\infty}(n-1)(\varpi^j_{-n}\varpi^i_n-\varpi^i_{-n}\varpi^j_n) \tag{4.177}$$

Partial cancellation allows us to get rid of the second line of equation (4.176)

$$\frac{\sqrt{4\pi T}}{p^+}\sum_{m,n=1}^{\infty}\frac{1}{n}(\varpi^j_{-m-n}\varpi^-_m\varpi^i_n-\varpi^i_{-n}\varpi^-_{-m}\varpi^j_{m+n}-\varpi^i_{-n}\varpi^-_{n-m}\varpi^j_m-\varpi^j_{-m}\varpi^-_{m-n}\varpi^i_n)$$

$$=\sum_{m,n}^{\infty}\frac{1}{n}\varpi^j_{-m-n}\varpi^j_{-m-n}\varpi^-_m\varpi^i_n+\frac{\sqrt{4\pi T}}{p^+}\sum_{m,n}^{\infty}\frac{1}{n}\varpi^j_{-m-n}\varpi^i_{n+m} \tag{4.178}$$

$$m+n=k$$

$$\frac{\sqrt{4\pi T}}{p^+}\sum_{m,n}^{\infty}\frac{1}{n}\varpi^j_{-k}\varpi^i_k$$

$$\sum_{n,m=1}^{\infty}\frac{1}{n}\varpi^-_{-m}\varpi^i_{-n}\varpi^j_{m+n}=\sum_{n,m=1}^{\infty}\frac{1}{n}\varpi^i_{-n}\varpi^-_{-m}\varpi^j_{m+n}+\frac{\sqrt{4\pi T}}{p^+}\sum_{k=2}^{\infty}(k-1)\varpi^i_{-k}\varpi^j_k$$

We now get:

$$C^{ij}=\frac{\sqrt{4\pi T}}{p^+}\sum_{m,n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{1}{n}\varpi^i_{-n}\varpi^j_{n-m}\varpi^-_m+\sum_{m=1}^{n}\frac{1}{n}\varpi^-_{-m}\varpi^j_{m-n}\varpi^i_n\right)$$

$$+\frac{4\pi T}{(p^+)^2}\sum_{n=2}^{\infty}(n-1)(\varpi^j_{-n}\varpi^i_n-\varpi^i_{-n}\varpi^j_n) \tag{4.179}$$

Opening brackets in line 1:

$$C^{ij}=\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{1}{n}\varpi^i_{-n}\varpi^j_{n-m}\varpi^-_m+\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{1}{n}\varpi^-_{-m}\varpi^j_{m-n}\varpi^i_n$$

$$\frac{4\pi T}{(p^+)^2}\sum_{n=1}^{\infty}(n-1)(\varpi^j_{-n}\varpi^i_n-\varpi^i_{-n}\varpi^j_n) \tag{4.180}$$

Using methods similar to the ones used in obtaining equation (4.179) we can obtain terms with the ground state mode expansions:

$$= \frac{4\pi T}{(p^+)^2}\sum_{n=1}^{\infty} = \frac{1}{n}(\varpi^j_{-n}\varpi^i_0\varpi^-_n - \varpi^-_{-n}\varpi^i_0\varpi^j_n - \varpi^i_{-n}\varpi^j_0\varpi^-_n + \varpi^-_{-n}\varpi^i_0\varpi^j_n - \varpi^i_{-n}\varpi^j_0\varpi^-_n$$
$$+\cdots\varpi^-_{-n}\varpi^j_0\varpi^i_n - \varpi^j_{-n}\varpi^-_0\varpi^i_n + \varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^i_{-n}\varpi^-_0\varpi^j_n + \varpi^i_{-n}\varpi^-_0\varpi^j_n + \varpi^j_{-n}\varpi^-_0\varpi^i_n)$$
$$+\varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^j_{-n}\varpi^-_0\varpi^i_n$$

Separating:

$$= \frac{4\pi T}{(p^+)^2}\sum_{n=1}^{\infty} = \frac{1}{n}(\varpi^j_{-n}\varpi^i_0\varpi^-_n - \varpi^-_{-n}\varpi^i_0\varpi^j_n - \varpi^i_{-n}\varpi^j_0\varpi^-_n + \varpi^-_{-n}\varpi^i_0\varpi^j_n - \varpi^i_{-n}\varpi^j_0\varpi^-_n$$
$$+\cdots\varpi^-_{-n}\varpi^j_0\varpi^i_n - \varpi^j_{-n}\varpi^-_0\varpi^i_n + \varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^i_{-n}\varpi^-_0\varpi^j_n + \varpi^i_{-n}\varpi^-_0\varpi^j_n + \varpi^j_{-n}\varpi^-_0\varpi^i_n)$$
$$+\varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^j_{-n}\varpi^-_0\varpi^i_n$$

$=$

$$\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^j_{-n}\varpi^i_0\varpi^-_n - \varpi^-_{-n}\varpi^i_0\varpi^j_n - \varpi^i_{-n}\varpi^j_0\varpi^-_n + \varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^-_{-n}\varpi^i_0\varpi^j_n -$$
$$\varpi^i_{-n}\varpi^j_0\varpi^-_n + \varpi^-_{-n}\varpi^j_0\varpi^i_n - \varpi^j_{-n}\varpi^-_0\varpi^i_n + \varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^j_n\varpi^-_0\varpi^i_n) \quad (4.181)$$
$$+\frac{4\pi T}{(p^+)^2}\sum_{n=0}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^j_{-n}\varpi^-_0\varpi^i_n + \varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^j_{-n}\varpi^-_0\varpi^i_n)$$

We now consider lines (1) and (2) of equation (4.181)

$$= \frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}((\varpi^j_{-n}\varpi^i_0\varpi^-_n - \varpi^-_{-n}\varpi^i_0\varpi^j_n) - (\varpi^j_{-n}\varpi^j_0\varpi^-_n - (\varpi^-_{-n}\varpi^j_0\varpi^i_n))$$
$$= \frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}((\varpi^j_{-n}\varpi^-_n - \varpi^-_{-n}\varpi^j_n)\frac{p^i}{\sqrt{4\pi T}} - (\varpi^j_{-n}\varpi^-_n - \varpi^-_{-n}\varpi^i_n)\frac{p^j}{\sqrt{4\pi T}}$$
$$= \frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}((\varpi^j_{-n}\varpi^-_n - \varpi^-_{-n}\varpi^j_n)\frac{p^i}{\sqrt{4\pi T}} - (\varpi^j_{-n}\varpi^-_n - \varpi^-_{-n}\varpi^i_n)\frac{p^j}{\sqrt{4\pi T}} \quad (4.182)$$

Considering term (2) of equation (4.181)

$$\frac{\sqrt{4\pi T}}{p^+}\sum_{n=0}^{\infty}\frac{1}{n}(2\varpi^i_{-n}\varpi^-_0\varpi^j_n - 2\varpi^j_{-n}\varpi^-_0\varpi^i_n)$$

$$\cdots = \frac{2\sqrt{4\pi T}}{p^+}\sum_{n=0}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^j_{-n}\varpi^-_0\varpi^i_n)$$
$$= \varpi^-_0\frac{2\sqrt{4\pi T}}{p^+}\sum_{n=0}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^j_{-n}\varpi^-_0\varpi^i_n)$$
$$= \varpi^-_0\frac{4\sqrt{\pi T}}{p^+}\sum_{n=0}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^j_{-n}\varpi^-_0\varpi^i_n) \tag{4.183}$$

Thus, the commutator $[\varXi^{i-}, \varXi^{j-}]$ becomes:

$$\left[\varXi^{i-}, \varXi^{j-}\right] = \varpi^-_0\frac{4\sqrt{\pi T}}{p^+}\sum_{n=0}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_0\varpi^j_n - \varpi^j_{-n}\varpi^-_0\varpi^i_n)$$

$$+\frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}((\varpi^j_{-n}\varpi^-_n - \varpi^-_{-n}\varpi^j_n)p^i - (\varpi^j_{-n}\varpi^-_n - \varpi^-_{-n}\varpi^i_n)p^j$$
$$-\sum_{n=1}^{\infty}\frac{4\pi T}{(p^+)^2}\left((2n-1) - \frac{\chi(n)}{n^2}\right)(\varpi^i_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^i_n) \tag{4.184}$$

4.3.6 Calculating commutator $[\varXi^{i-}, \Phi^{j-}]$

We now calculate the commutator $[\varXi^{i-}, \Phi^{j-}]$. We write the term Φ^{j-} as:

$$\Phi^{j-} = -\frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{n}(\sigma^j_n\sigma^-_{-n} - \sigma^-_{-n}\sigma^j_n) \tag{4.185}$$

Such that the commutator becomes:

$$\left[\varXi^{i-}, \Phi^{j-}\right] = \left[-i\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_n - \varpi^-_{-n}\varpi^i_n), -\frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{n}(\sigma^j_n\sigma^-_{-n} - \sigma^-_{-n}\sigma^j_n)\right] \tag{4.186}$$

We begin the manipulation,

$$\left[\varXi^{i-}, \Phi^{j-}\right] = i\sum_{n=1}^{\infty}\frac{1}{n}\frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{n}\left[(\varpi^i_{-n}\varpi^-_n - \varpi^-_{-n}\varpi^i_n), (\sigma^j_n\sigma^-_{-n} - \sigma^-_{-n}\sigma^j_n)\right] \tag{4.187}$$

Following earlier arguments, we shall invoke a dictionary to assist with our computations. This will be done by replacing each term with a letter.

$$\begin{aligned} A &= \varpi^i_{-n}\varpi^-_n \\ B &= -\varpi^-_{-n}\varpi^i_n \\ C &= \sigma^j_n\sigma^-_{-n} \\ D &= -\sigma^-_{-n}\sigma^j_n \end{aligned} \tag{4.188}$$

$$[A+B, C+D] = (AC + AD + BC + BD - CA - CB - DA - DB)$$

$$[A+B, C+D] = [A, C] + [A, D] + [B, C] + [B, D] \tag{4.189}$$

Replacing the letters for values:

$$[A+B, C+D] = [\varpi^i_{-n}\varpi^-_n, \sigma^j_n\sigma^-_{-n}] + [\varpi^i_{-n}\varpi^-_n, -\sigma^-_{-n}\sigma^j_n] + [-\varpi^-_{-n}\varpi^i_n, \sigma^j_n\sigma^-_{-n}] + [-\varpi^-_{-n}\varpi^i_n, -\sigma^-_{-n}\sigma^j_n]$$

$$= [\varpi^i_{-n}\varpi^-_n, \sigma^j_n\sigma^-_{-n}] + -[\varpi^i_{-n}\varpi^-_n, \sigma^-_{-n}\sigma^j_n] + -[\varpi^-_{-n}\varpi^i_n, \sigma^j_n\sigma^-_{-n}] + [\varpi^-_{-n}\varpi^i_n, \sigma^-_{-n}\sigma^j_n] \tag{4.190}$$

$$\begin{aligned} &= (\varpi^i_{-n}\varpi^-_n\sigma^j_n\sigma^-_{-n} - \sigma^j_n\sigma^-_{-n}\varpi^i_{-n}\varpi^-_n - \varpi^i_{-n}\varpi^-_n\sigma^-_{-n}\sigma^j_n + \varpi^i_{-n}\varpi^-_n\sigma^-_{-n}\sigma^j_n - \\ &\quad \varpi^-_{-n}\varpi^i_n\sigma^j_n\sigma^-_{-n} + \sigma^j_n\sigma^-_{-n}\varpi^-_{-n}\varpi^i_n + \varpi^-_{-n}\varpi^i_n\sigma^-_{-n}\sigma^j_n - \sigma^-_{-n}\sigma^j_n\varpi^-_{-n}\varpi^i_n) \end{aligned} \tag{4.191}$$

Collecting like terms in the bosonic mode expansions:

$$[A+B, C+D] = \varpi^i_{-n}\varpi^-_n[\sigma^j_n, \sigma^-_{-n}] + [\sigma^j_n, \sigma^-_{-n}]\varpi^i_{-n}\varpi^-_n + \varpi^i_{-n}\varpi^-_n[\sigma^-_{-n}\sigma^j_n] + [\sigma^j_n, \sigma^-_{-n}]\varpi^i_{-n}\varpi^-_n$$

$$-\varpi^i_{-n}\varpi^-_n[\sigma^j_n, \sigma^-_{-n}] - [\sigma^-_{-n}, \sigma^j_n]\varpi^i_{-n}\varpi^-_n - \varpi^i_{-n}\varpi^-_n[\sigma^-_{-n}\sigma^j_n] + [\sigma^j_n, \sigma^-_{-n}]\varpi^i_{-n}\varpi^-_n$$

$$= 0 \tag{4.192}$$

We can use result (4.192) to calculate $[\Phi^{i-}, \Xi^{j-}]$ i.e,

$$\left[\Phi^{i-}, \Xi^{j-}\right] = -\left[\Xi^{i-}, \Phi^{j-}\right]\delta^{ij} = 0 \tag{4.193}$$

4.3.7 Calculating commutator $[\Phi^{i-}, \Phi^{j-}]$

We calculate the final commutator:

$$\Phi^{i-} = -\frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{n}(\sigma_n^i\sigma_{-n}^- - \sigma_{-n}^-\sigma_n^i) \tag{4.194}$$

$$\Phi^{j-} = -\frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{n}(\sigma_n^j\sigma_{-n}^- - \sigma_{-n}^-\sigma_n^j) \tag{4.195}$$

Thus:

$$\left[\Phi^{i-}, \Phi^{j-}\right] = \left[-\frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{n}(\sigma_n^i\sigma_{-n}^- - \sigma_{-n}^-\sigma_n^i), -\frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{n}(\sigma_n^j\sigma_{-n}^- - \sigma_{-n}^-\sigma_n^j)\right] \tag{4.196}$$

We manipulate it further

$$= \frac{i}{2}\frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{n}\sum_{n=0}^{\infty}\frac{1}{n}\left[(\sigma_n^i\sigma_{-n}^- - \sigma_{-n}^-\sigma_n^i), (\sigma_n^j\sigma_{-n}^- - \sigma_{-n}^-\sigma_n^j)\right] \tag{4.197}$$

We aqain introduce a dictionary for further manipulation:

$$\begin{aligned} A &= \sigma_n^i\sigma_{-n}^- \\ B &= -\sigma_n^i\sigma_{-n}^- \\ C &= \sigma_n^i\sigma_{-n}^- \\ D &= -\sigma_{-n}^-\sigma_n^j \end{aligned} \tag{4.198}$$

$$[A+B, C+D] = (AC + AD + BC + BD - CA - CB - DA - DB)$$

$$[A+B, C+D] = [A, C] + [A, D] + [B, C] + [B, D] \tag{4.199}$$

We similarly put dictionary (4.198) into equation (4.199)

$$[A+B,C+D]=[\sigma^i_n\sigma^-_{-n},\sigma^i_n\sigma^-_{-n}]+[\sigma^i_n\sigma^-_{-n},-\sigma^-_{-n}\sigma^j_n]+[\sigma^i_n\sigma^-_{-n},\sigma^i_n\sigma^-_{-n}]+[\sigma^i_n\sigma^-_{-n},-\sigma^-_{-n}\sigma^j_n] \tag{4.200}$$

$$=[\sigma^i_n\sigma^-_{-n},\sigma^i_n\sigma^-_{-n}]-[\sigma^i_n\sigma^-_{-n},\sigma^-_{-n}\sigma^j_n]-[\sigma^i_n\sigma^-_{-n},\sigma^i_n\sigma^-_{-n}]+[\sigma^i_n\sigma^-_{-n},-\sigma^-_{-n}\sigma^j_n]$$
$$(\sigma^i_n\sigma^-_{-n}\sigma^i_n\sigma^-_{-n}-\sigma^i_n\sigma^-_{-n}\sigma^i_n\sigma^-_{-n}-\sigma^i_n\sigma^-_{-n}\sigma^-_{-n}\sigma^j_n+\sigma^-_{-n}\sigma^j_n\sigma^i_n\sigma^-_{-n}$$
$$-\sigma^i_n\sigma^-_{-n}\sigma^i_n\sigma^-_{-n}-\sigma^i_n\sigma^-_{-n}\sigma^i_n\sigma^-_{-n}+\sigma^i_n\sigma^-_{-n}\sigma^-_{-n}\sigma^j_n--\sigma^-_{-n}\sigma^j_n\sigma^i_n\sigma^-_{-n})$$
$$=0 \tag{4.201}$$

Putting equations (4.123), (4.132), (4.184), (4.193) and (4.201) together we arrive at the result:

$$\left[J^{i-},J^{j-}\right]=\varpi^-_0\frac{4\sqrt{\pi T}}{p^+}\sum_{n=0}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^-_0\varpi^j_n-\varpi^j_{-n}\varpi^-_0\varpi^i_n)$$
$$-\sum_{n=1}^{\infty}\frac{4\pi T}{(p^+)^2}\left(2(n-1)-\frac{\chi(n)}{n^2}\right)(\varpi^i_{-n}\varpi^j_n-\varpi^j_{-n}\varpi^i_n)$$
$$+\frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}((\varpi^j_{-n}\varpi^-_n-\varpi^-_{-n}\varpi^j_n)p^i-(\varpi^j_{-n}\varpi^-_n-\varpi^-_{-n}\varpi^i_n)p^j$$
$$2\frac{p^-}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^j_n-\varpi^j_{-n}\varpi^i_n)+$$
$$-\frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}((\varpi^j_{-n}\varpi^-_n-\varpi^-_{-n}\varpi^j_n)p^i-(\varpi^j_{-n}\varpi^-_n-\varpi^-_{-n}\varpi^i_n)p^j \tag{4.202}$$

The terms in $\frac{1}{p^+}$ cancel out leaving:

$$\left[J^{i-},J^{j-}\right]=\varpi^-_0\frac{4\sqrt{\pi T}}{p^+}\sum_{n=0}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^j_n-\varpi^j_{-n}\varpi^i_n)$$
$$-\sum_{n=1}^{\infty}\frac{4\pi T}{(p^+)^2}\left(2(n-1)-\frac{\chi(n)}{n^2}\right)(\varpi^i_{-n}\varpi^j_n-\varpi^j_{-n}\varpi^i_n)$$
$$-2\frac{p^-}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(\varpi^i_{-n}\varpi^j_n-\varpi^j_{-n}\varpi^i_n) \tag{4.203}$$

Collecting like terms allows us to group terms (1) and (3) of equation (4.203)

$$= (\varpi_0^- \frac{4\sqrt{\pi T}}{p^+} - 2\frac{p^-}{p^+}) \sum_{n=0}^{\infty} \frac{1}{n} (\varpi^i_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^i_n)$$

$$= \frac{1}{p^+} (\varpi_0^- 4\sqrt{\pi T} - 2p^-) \sum_{n=0}^{\infty} \frac{1}{n} (\varpi^i_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^i_n)$$

$$= \frac{1}{p^+} (\varpi_0^- 4\sqrt{\pi T} - 2p^-) \sum_{n=0}^{\infty} \frac{1}{n} (\varpi^i_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^i_n)$$

$$= \frac{1}{p^+} (\varpi_0^- 4\sqrt{\pi T} - 4\sqrt{\pi T}\hat{\varpi}_0) \sum_{n=0}^{\infty} \frac{1}{n} (\varpi^i_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^i_n)$$

$$= \frac{4\sqrt{\pi T}}{p^+} (\varpi_0^- - \hat{\varpi}_0) \sum_{n=0}^{\infty} \frac{1}{n} (\varpi^i_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^i_n)$$

Now, because of level matching, $\varpi_0^- = \hat{\varpi}_0$ and therefore $\cancelto{0}{\varpi_0^- - \varpi_0}$

We are then left with term (2) of equation (4.203) which we now write in full considering equation (4.139):

$$\left[J^{i-}, J^{j-}\right] = -\frac{4\pi T}{(p^+)^2} \sum_{n=1}^{\infty} \left[\left(\frac{d-2}{12} - 2 \right) n + \frac{1}{n} \left(2a - \frac{d-2}{12} \right) \right] (\varpi^i_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^i_n) \tag{4.204}$$

Now, one should remember that we require the Lorentz generators to commute with each other if we are to maintain Lorentz invariance. Thus, we expect the term in equation (4.204) to vanish. The mode expansions in equation (4.204) are arbitrary, we do not expect them to vanish, thus the term inside our brackets will vanish. We set $a = 1$ consistent with renormalisation in bosonic string theory. We have done this because none of the chimeric terms we have introduced contribute to the commutator.

Thus;

$$\sum_{n=1}^{\infty} \left[\left(\frac{d-2}{12} - 2 \right) n + \frac{1}{n} \left(2a - \frac{d-2}{12} \right) \right] = 0$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} n\left(\frac{d-2}{12}-2\right) = -\sum_{n=1}^{\infty}\frac{1}{n}\left(2a-\frac{d-2}{12}\right) \\
\sum_{n=1}^{\infty} n \neq \sum_{n=1}^{\infty}\frac{1}{n} \\
\therefore \left(\frac{d-2}{12}-2\right) = 0 \\
\frac{d-2}{12} = 2 \\
d-2 = 24 \\
d = 26
\end{aligned}
\tag{4.205}
$$

Equation (4.205) implies that the symmetry we have introduced has no effect on the dimensionality of string theory, at least in the case of the bosonic theory. However, it also shows that the symmetry can be implemented in a mathematically consistent way, without "breaking" the theory.

4.4 Extending Superstring Theory

4.4.1 Introduction

We now want to extend superstring theory using the same method we employed in bosonic string theory. Supersting theory, as discussed in the introduction, is an extension of bosonic string theory. The theory only describes bosons. By introducing supersymmetry, we can introduce fermionic fields into the theory. A by-product of the theory is to reduce the dimensionality of the new theory to 10 dimensions.

The model we have used is the Ramond-Neveu-Schwarz model as discussed. We introduce a new duality between the fermions and bosons while maintaining an equality between fermions and simplices.Thus we do not expect a transition between fermions and simplices: simplices are fermions.

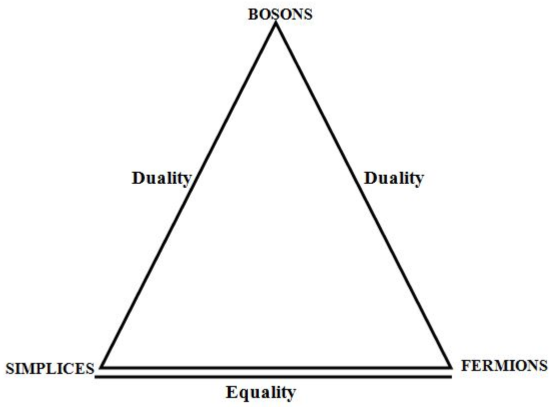

Figure 4.1: Illustration of the proposed dualities and equivalences.

We first write the superstring theory Lagrangian down, (McMahon, 2009) i.e:

$$S_{SST} = -\frac{T}{2}\int d^2\sigma(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\varphi}^\mu \gamma^\alpha \partial_\alpha \varphi_\mu) \tag{4.206}$$

where:

T is the string tension,

τ is the worldsheet time coordinate,

σ is the worldsheet spatial coordinate,

X^μ and $\partial_\alpha X^\mu$ are the worldsheet and its derivatives,

$i = \sqrt{-1}$ is the imaginary number,

m is the mass of the electron,

φ and $\bar{\varphi}$ are the Dirac field and its complex conjugate respectively,

γ^α are the Dirac matrices,

introducing the simplicial action and adding it to equation (4.206):

$$S_C = -\frac{T}{2}\int d^2\sigma(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\varphi}^\mu \gamma^\alpha \partial_\alpha \varphi_\mu - \bar{\phi}^\mu \beta \phi_\mu) \tag{4.207}$$

We now introduce the duality transformations between fermions, simplices and bosons. This means that, in addition to the duality transformations in Objective 1, we have supersymmetric transformations:

$$\begin{aligned} \delta X^\mu &= \bar{\epsilon}(\phi^\mu + \varphi^\mu) \\ \delta\phi^\mu &= \epsilon\partial_\alpha X^\mu \\ \delta\varphi^\mu &= \epsilon\partial_\alpha X^\mu \end{aligned} \tag{4.208}$$

These are the duality transformations that we will use in this section. From the action in equation (4.207) we can get the Lagrangian:

$$L_C = -\frac{T}{2}(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\varphi}^\mu \gamma^\alpha \partial_\alpha \varphi_\mu - \bar{\phi}^\mu \beta \phi_\mu) \tag{4.209}$$

We want to introduce a variation in equation (4.209)

$$\delta L_C = -\frac{T}{2}\delta\left(\partial_\alpha X^\mu \partial^\alpha X_\mu - -i\bar{\varphi}^\mu \gamma^\alpha \partial_\alpha \varphi_\mu - \bar{\phi}^\mu \beta \phi_\mu\right)$$

Expanding;

$$\delta L_C = -\frac{T}{2}\left(\partial_\alpha(\delta X^\mu)\partial^\alpha X_\mu + \partial_\alpha X^\mu \partial^\alpha(\delta X_\mu) - (\delta\bar{\phi^\mu})\beta\partial_\alpha\phi - \bar{\phi}^\mu\beta\partial_\alpha(\delta\phi) - i\delta\bar{\varphi}^\mu\gamma^\alpha\partial_\alpha\varphi^\mu - i\bar{\varphi}^\mu\gamma^\alpha\partial_\alpha\delta\varphi^\mu\right)$$

Collecting terms (1) and (2) together, reduces this equation to the form:

$$\delta L_C = -\frac{T}{2}\left(2\partial_\alpha(\delta X^\mu)\partial^\alpha X_\mu - (\delta\bar{\phi}^\mu)\beta\partial_\alpha\phi - \bar{\phi}^\mu\beta\partial_\alpha(\delta\phi) - i\delta\bar{\varphi}^\mu\rho^\alpha\partial_\alpha\varphi^\mu - i\bar{\varphi}^\mu\rho^\alpha\partial_\alpha\delta\varphi^\mu\right) \tag{4.210}$$

We invoke equations (4.208) in equation (4.210):

$$\begin{aligned}
\delta L_C &= -\frac{T}{2}\left(2\partial_\alpha(\bar\epsilon\phi^\mu)\partial^\alpha X_\mu - (\delta\bar{\phi}^\mu)\beta\partial_\alpha\phi - \bar{\phi}^\mu\beta\partial_\alpha(\partial_\beta X^\mu\epsilon) - i\bar{\varphi}^\mu\rho^\alpha\partial_\alpha(\bar\epsilon\rho^\beta\partial_\beta X^\mu) - i\bar{\varphi}^\mu\rho^\alpha\partial_\alpha\delta\varphi^\mu\right)\\
&= -\frac{T}{2}(2(\partial_\alpha(\bar\epsilon\phi^\mu\partial^\alpha X_\mu) + \partial_\alpha(\bar\epsilon\varphi^\mu)\partial^\alpha X_\mu) - (\delta\bar{\phi}^\mu)\beta\partial_\alpha\phi - \bar{\phi}^\mu\beta\partial_\alpha(\partial_\beta X^\mu\epsilon) - i\bar{\varphi}^\mu\rho^\alpha\partial_\alpha(\bar\epsilon\rho^\beta\partial_\beta X^\mu) -\\
&\qquad i\bar{\varphi}^\mu\rho^\alpha\partial_\alpha\delta\varphi^\mu))\\
&= -\frac{T}{2}(2(\partial_\alpha(\bar\epsilon\phi^\mu\partial^\alpha X_\mu) + \partial_\alpha(\bar\epsilon\varphi^\mu)\partial^\alpha X_\mu) - 2\bar\epsilon\partial_\beta X^\mu\beta\partial_\alpha\phi - 2\partial_\alpha(\bar\epsilon\rho^\beta\partial_\beta X^\mu\rho^\alpha\varphi_\mu)\\
&= -T((\partial_\alpha(\bar\epsilon\phi^\mu\partial^\alpha X_\mu) + \partial_\alpha(\bar\epsilon\varphi^\mu)\partial^\alpha X_\mu) - \bar\epsilon\partial_\beta X^\mu\beta\partial_\alpha\phi - \partial_\alpha(\bar\epsilon\rho^\beta\partial_\beta X^\mu\rho^\alpha\varphi_\mu)\\
&= -T((\partial_\alpha(\bar\epsilon\phi^\mu\partial^\alpha X_\mu) + \partial_\alpha(\bar\epsilon\varphi^\mu)\partial^\alpha X_\mu) - [\partial_\beta(\bar\epsilon X^\mu\beta\partial_\alpha\phi) -\\
&\qquad \bar\epsilon X^\mu\beta\partial_\alpha\partial_\beta\phi + \partial_\beta(\partial_\alpha\bar\epsilon\rho^\beta X^\mu\rho^\alpha\psi_\mu) - \partial_\alpha\partial_\beta\bar\epsilon\rho^\beta X^\mu\rho\varphi_\mu]\\
&= -T((\partial_\alpha(\bar\epsilon\phi^\mu\partial^\alpha X_\mu) + \partial_\alpha(\bar\epsilon\varphi^\mu)\partial^\alpha X_\mu) - \partial_\beta(\bar\epsilon X^\mu\beta\partial_\alpha\phi) +\\
&\qquad \bar\epsilon X^\mu\beta\partial_\alpha\partial_\beta\phi - \partial_\beta(\partial_\alpha\bar\epsilon\rho^\beta X^\mu\rho^\alpha\psi_\mu) + \partial_\alpha\partial_\beta\bar\epsilon\rho^\beta X^\mu\rho\varphi_\mu
\end{aligned} \tag{4.211}$$

$$= -T(\partial_\alpha(\bar\epsilon\phi^\mu)\partial^\alpha X^\mu - \partial_\alpha\bar\epsilon(\rho^\beta\partial_\beta X^\mu)\rho^\alpha\varphi_\mu - \bar\epsilon\rho^\beta\rho^\alpha(\partial_\alpha\partial_\beta X^\mu\varphi^\mu - \partial_\alpha(\bar\epsilon X^\mu\beta\partial_\alpha\phi) + \bar\epsilon X^\mu\beta\partial_\alpha\partial_\beta\phi) \tag{4.212}$$

We will introduce new terms using the product rule to cancel out like terms in equation (4.211) to obtain:

$$\begin{aligned}
\delta L_C = -T(&\partial_\alpha(\bar\epsilon\phi^\mu\partial^\alpha X^\mu) - \bar\epsilon\phi^\mu\partial^\alpha\partial_\alpha X^\mu + \partial_\alpha(\epsilon\varphi^\mu\partial^\alpha X^\mu) - \bar\epsilon\varphi^\mu\partial_\alpha\partial^\alpha X^\mu - \partial_\alpha\epsilon(\rho^\beta\partial_\beta X^\mu)\rho^\alpha\psi_\mu\\
&+\bar\epsilon\rho^\alpha\rho_\beta\rho_\alpha(\partial_\alpha\partial^\beta X^\mu)\varphi_\mu - \partial_\beta(\bar\epsilon X^\mu\beta\partial_\alpha\phi + \bar\epsilon X^\mu\beta\partial_\alpha\partial_\beta\phi)
\end{aligned}$$

$$\begin{aligned}
= -T(&\cancel{\partial^\alpha(\bar\epsilon\phi^\mu\partial^\alpha X^\mu)} - \cancel{\bar\epsilon\phi^\mu\partial^\alpha\partial_\alpha X^\mu} + \partial^\alpha(\epsilon\varphi^\mu\partial^\alpha X^\mu) - \cancel{\bar\epsilon\varphi^\mu\partial^\alpha\partial_\alpha X^\mu} - \partial_\alpha\epsilon(\rho^\beta\partial_\beta X^\mu)\rho^\alpha\varphi_\mu\\
&+\bar\epsilon\rho^\alpha\rho_\beta\rho_\alpha(\partial_\alpha\partial^\beta X^\mu)\varphi_\mu - \partial_\beta(\bar\epsilon X^\mu\beta\partial_\alpha\phi + \cancel{\bar\epsilon X^\mu\beta\partial_\alpha\partial_\beta\phi})
\end{aligned}$$

which leaves us with:

$$\delta L_C = -T(\partial^\alpha(\epsilon\varphi^\mu\partial^\alpha X^\mu) - \partial_\alpha\epsilon(\rho^\beta\partial_\beta X^\mu)\rho^\alpha\varphi_\mu + \bar{\epsilon}\rho^\alpha\rho_\beta\rho_\alpha(\partial_\alpha\partial^\beta X^\mu)\varphi_\mu - \partial_\beta(\bar{\epsilon}X^\mu\beta\partial_\alpha\phi)) \tag{4.213}$$

collecting like terms together, we get:

$$\delta L_C = -T(\partial^\alpha(\epsilon\varphi^\mu\partial^\alpha X^\mu) - \partial_\alpha\epsilon(\rho^\beta\partial_\beta X^\mu)\rho^\alpha\varphi_\mu + \bar{\epsilon}\rho^\alpha\rho_\beta\rho_\alpha(\partial_\alpha\partial^\beta X^\mu)\varphi_\mu - \partial_\beta(\bar{\epsilon}X^\mu\beta\partial_\alpha\phi)) \tag{4.214}$$

$$\begin{aligned}
\delta L_C &= -T(\partial^\alpha(\epsilon\phi^\mu\partial^\alpha X^\mu) + \partial^\alpha(\bar{\epsilon}\varphi^\mu\partial^\alpha X^\mu)) \\
&= -T\partial^\alpha((\epsilon\phi^\mu\partial^\alpha X^\mu) + (\epsilon\varphi^\mu\partial^\alpha X^\mu)) \\
&= -T\partial^\alpha\epsilon((\phi^\mu\partial^\alpha X^\mu) + (\varphi^\mu\partial^\alpha X^\mu)) \\
&= -T\partial^\alpha\epsilon(\phi^\mu + \varphi^\mu)\partial^\alpha X^\mu
\end{aligned} \tag{4.215}$$

We are not interested in the term in equation (4.214): it is a whole derivative. We are thus left with:

$$\delta L_C = -\partial_\beta\bar{\epsilon}(\rho^\beta\partial_\beta X^\mu)\rho^\alpha\varphi_\mu - \beta(\partial_\beta\bar{\epsilon})\partial_\beta X^\mu\phi^\mu$$

this leads to the conserved Noether current:

$$J^\mu_\beta = -\rho^\beta\partial_\beta X^\mu\rho^\alpha\varphi_\mu - \beta\partial_\beta X^\mu\phi^\mu \tag{4.216}$$

We now want to develop the stress energy tensor analogous to the procedure we used in objective 1. We will develop the fermionic part since we had already done the same for the simplicial part. We write the standard transformations:

$$\begin{aligned}
\sigma^\alpha &= \sigma^\alpha + \epsilon^\alpha \\
X^\mu &= X^\mu + \epsilon^\alpha\partial_\alpha X^\mu \\
\phi^\mu &= \phi^\mu + \epsilon^\alpha\partial_\alpha\phi^\mu
\end{aligned}$$

$$\varphi^{\mu} = \varphi^{\mu} + \epsilon\rho^{\alpha}\partial_{\alpha}\varphi^{\mu} \tag{4.217}$$

We recall the simplicial and fermionic lagrangian

$$\begin{aligned} L_{CDT} &= -\frac{1}{2}\bar{\phi}^{\mu}\beta\partial_{\alpha}\phi^{\mu} \\ L_F &= -\frac{i}{2}\varphi^{\mu}\rho^{\alpha}\partial_{\alpha}\varphi_{\mu} \end{aligned} \tag{4.218}$$

We add equations (4.217) together and take variations:

$$\delta L_{CDT} + \delta L_F = -\frac{1}{2}\delta\bar{\phi}^{\mu}\beta\partial_{\alpha}\phi^{\mu} - \frac{1}{2}\bar{\phi}^{\mu}\beta\partial_{\alpha}\delta\phi^{\mu} - \frac{i}{2}\delta\varphi^{\mu}\rho^{\alpha}\partial_{\alpha}\varphi_{\mu} - \frac{i}{2}\varphi^{\mu}\rho^{\alpha}\partial_{\alpha}\delta\varphi_{\mu} \tag{4.219}$$

Now we pay attention to the variations using these systems of equations:

$$\begin{aligned} \phi^{\mu} &= \phi^{\mu} + \delta\phi^{\mu} = \phi^{\mu} + \epsilon^{\alpha}\partial_{\alpha}\phi^{\mu} \\ \varphi^{\mu} &= \varphi^{\mu} + \delta\varphi^{\mu} = \phi^{\mu} + \epsilon\rho^{\alpha}\partial_{\alpha}\varphi^{\mu} \end{aligned} \tag{4.220}$$

Putting variations of equation (4.218) into equation (4.217), we get:

$$\delta L_{CDT} + \delta L_F = -\frac{1}{2}\beta\partial_{\alpha}\bar{\phi}^{\mu}\epsilon\partial_{\alpha}\phi^{\mu} - \frac{1}{2}\epsilon\partial_{\alpha}\bar{\phi}^{\mu}\beta\partial_{\alpha}\phi^{\mu} - \frac{i}{2}\epsilon\rho^{\alpha}\partial_{\alpha}\varphi^{\mu}\partial_{\alpha}\varphi_{\mu} - \frac{i}{2}\varphi^{\mu}\rho^{\alpha}\partial_{\alpha}\epsilon\rho^{\alpha}\partial_{\alpha}\varphi^{\mu} \tag{4.221}$$

We shall now concentrate with the fermionic part since the simplicial part has been developed.

$$\delta L_F = -\frac{i}{2}\epsilon^{\alpha}\rho^{\alpha}\partial_{\alpha}\varphi^{\mu}\rho^{\beta}\partial_{\beta}\varphi_{\mu} - \frac{i}{2}\varphi^{\mu}\rho^{\alpha}\partial_{\alpha}\epsilon\rho^{\alpha}\rho^{\beta}\partial_{\beta}\varphi^{\mu} \tag{4.222}$$

We further manipulate equation (4.218) in a way to make it further amenable to integration. We use the product rule on term 2 of equation (4.218):

$$\begin{aligned} \delta L_F &= -\frac{i}{2}\epsilon^{\alpha}\rho^{\alpha}\partial_{\alpha}\varphi^{\mu}\partial_{\alpha}\varphi_{\mu} - \frac{i}{2}\rho^{\alpha}\partial_{\alpha}(\varphi^{\mu}\epsilon^{\alpha})\rho^{\alpha}\partial_{\alpha}\varphi^{\mu} + \frac{i}{2}\rho^{\alpha}\varphi^{\mu}\epsilon^{\alpha}\rho^{\alpha}\partial^{\alpha}\partial_{\alpha}\varphi^{\mu} \\ \delta L_F &= -\frac{i}{2}\epsilon^{\alpha}\rho^{\alpha}\partial_{\alpha}\varphi^{\mu}\partial_{\alpha}\varphi_{\mu} - (\frac{i}{2}\rho^{\alpha}\partial_{\alpha}(\varphi^{\mu}\epsilon^{\alpha})\rho^{\alpha}\partial_{\alpha}\varphi^{\mu} - \frac{i}{2}\rho^{\alpha}\varphi^{\mu}\epsilon^{\alpha}\rho^{\alpha}\partial^{\alpha}\partial_{\beta}\varphi^{\mu}) \end{aligned} \tag{4.223}$$

To continue we must use integration by parts as we did in objective 1. We work on term

(3) of equation (4.222)

$$
\begin{aligned}
&= \frac{i}{2}\rho^{\alpha}\bar{\varphi}^{\mu}\epsilon^{\alpha}\rho^{\alpha}\partial^{\alpha}\partial_{\beta}\varphi^{\mu} \\
&= \frac{i}{2}\int \rho^{\alpha}\varphi^{\mu}\epsilon^{\alpha}\rho^{\alpha}\partial^{\alpha}\partial_{\beta}\varphi^{\mu} \qquad (4.224)\\
U &= \bar{\varphi}^{\mu} \\
dV &= \partial^{\alpha}\partial_{\beta}\varphi^{\mu} \\
dU &= \partial^{\alpha}(\bar{\varphi}^{\mu}\rho^{\alpha}\epsilon^{\alpha}) \\
V &= \partial_{\beta}\varphi^{\mu}
\end{aligned}
$$

So that equation (4.223) becomes:

$$\int U dV = UV - \int V dU$$

The integral thus becomes:

$$\int \rho^{\alpha}\varphi^{\mu}\epsilon^{\alpha}\rho^{\alpha}\partial^{\alpha}\partial_{\beta}\varphi^{\mu} = \bar{\varphi}^{\mu}\partial_{\beta}\varphi^{\mu} - \int \partial_{\beta}\varphi^{\mu}\partial^{\alpha}(\bar{\varphi}^{\mu}\rho^{\alpha}\epsilon^{\alpha})$$

$$\int \rho^{\alpha}\varphi^{\mu}\epsilon^{\alpha}\rho^{\alpha}\partial^{\alpha}\partial_{\beta}\varphi^{\mu} = \cancelto{0}{\bar{\varphi}^{\mu}\partial_{\beta}\varphi^{\mu}} - \int \partial_{\beta}\varphi^{\mu}\partial^{\alpha}(\bar{\varphi}^{\mu}\rho^{\alpha}\epsilon^{\alpha})$$

Such that the integral becomes:

$$\frac{i}{2}\int \rho^{\alpha}\varphi^{\mu}\epsilon^{\alpha}\rho^{\alpha}\partial^{\alpha}\partial_{\beta}\varphi^{\mu} = -\frac{i}{2}\int \partial_{\beta}\varphi^{\mu}\partial^{\alpha}(\bar{\varphi}^{\mu}\rho^{\alpha}\epsilon^{\alpha})$$

We open the third term up again:

$$\delta L_F = -\frac{1}{2}(\epsilon^{\alpha}\partial_{\alpha}\bar{\varphi}^{\mu})\rho^{\alpha}\psi^{\mu} - \frac{1}{2}\bar{\varphi}^{\mu}\rho^{\alpha}\partial_{\alpha}\epsilon^{\alpha}\partial_{\alpha}\varphi^{\mu} + \frac{1}{2}\partial_{\alpha}\varphi_{\mu}\partial_{\beta}\bar{\psi}^{\mu}\rho^{\alpha}\epsilon + \frac{1}{2}\partial_{\alpha}\varphi_{\mu}\bar{\varphi}^{\mu}\rho^{\alpha}\partial_{\beta}\epsilon \quad (4.225)$$

We can cancel out like terms and are left with:

$$\delta L_F = -\frac{1}{2}\bar{\varphi}^{\mu}\rho^{\alpha}\partial_{\alpha}\epsilon^{\alpha}\partial_{\alpha}\varphi^{\mu} + \frac{1}{2}\partial_{\alpha}\varphi_{\mu}\bar{\varphi}^{\mu}\beta\partial_{\beta}\epsilon$$

The perturbation is constant: $\partial_\beta \epsilon$ vanishes such that:

$$\delta L_F = -\frac{1}{2}\bar{\varphi}^\mu \beta \partial_\alpha \epsilon^\alpha \partial_\alpha \varphi^\mu + \cancelto{0}{\frac{1}{2}\partial_\alpha \psi_\mu \bar{\varphi}^\mu \beta \partial_\beta \epsilon}$$

$$\delta L_F = -\frac{1}{2}\bar{\varphi}^\mu \rho^\alpha \partial_\alpha \epsilon^\alpha \partial_\alpha \phi^\mu \tag{4.226}$$

Rewriting equation (4.225):

$$\delta L_F = \partial_\alpha \epsilon^\alpha \left(-\frac{1}{2}\bar{\varphi}^\mu \rho^\alpha \partial_\alpha \varphi^\mu\right)$$

From this we can read off the stress energy tensor associated with the fermions as:

$$T^{(F)}_{\mu\nu} = -\frac{1}{2}\bar{\varphi}^\mu \beta \partial_\alpha \varphi^\mu$$

We symmetrize the terms because we wish to have a symmetrical stress energy tensor:

$$T^{(F)}_{\mu\nu} = -\frac{i}{4}\bar{\varphi}^\mu \rho^\alpha \partial_\alpha \varphi_\mu - \frac{i}{4}\bar{\varphi}^\mu \beta \partial_\alpha \varphi_\mu \tag{4.227}$$

We now invoke lightcone coordinates: this will later aid in quantisation. We can decompose the tensor and the simplicial fields into “positive-positive” and “negative-negative” components.

$$T_{\alpha\beta} = T_{++} + T_{--} \tag{4.228}$$

$$\bar{\varphi}^\mu = \bar{\varphi}^\mu_+ + \bar{\varphi}^\mu_- \tag{4.229}$$

$$\varphi^\mu = \varphi^\mu_+ + \varphi^\mu_- \tag{4.230}$$

$$\partial_\alpha = \partial_+ + \partial_- \tag{4.231}$$

We now work out the fermionic stress energy tensor in terms of light cone coordinates.

We first decompose the fermionic stress energy tensor into its lightcone components as:

$$T_{\alpha\beta}^{(CDT)} = T_{++}^{(CDT)} + T_{--}^{(CDT)} \tag{4.232}$$

We now expand equation (4.226) in terms of equations (4.227) to (4.230). We will work with only one term because of the similarity of the two terms:
We drop the tensor notation in our following computations:

$$T_{\alpha\beta}^{(F)} = \frac{1}{4}\left(\bar{\varphi_+} + \bar{\varphi_-}\right)\left(\partial_+ + \partial_-\right)\left(\varphi_+ + \varphi_-\right) \tag{4.233}$$

Expanding equation (4.232) while at the same time dropping cross-terms:

$$\begin{aligned} T_{\alpha\beta}^{(F)} = \frac{1}{4}\left(\bar{\varphi_+} + \bar{\varphi_-}\right)\left(\partial_+\varphi_+ + \partial_+\varphi_- + \partial_-\varphi_+ + \partial_-\varphi_-\right) \\ = \frac{1}{4}\left(\bar{\varphi_+} + \bar{\varphi_-}\right)\left(\partial_+\varphi_+ +\right) + \partial_-\varphi_-) \\ = \frac{1}{4}\left(\bar{\varphi_+}\partial_+\varphi_+ + \bar{\psi_+}\partial_-\varphi_- + \bar{\varphi_-}\partial_+\varphi_+ + \bar{\varphi_-}\partial_-\varphi_-\right) \end{aligned} \tag{4.234}$$

We open the brackets:

$$T_{\alpha\beta}^{(F)} = \frac{1}{4}\bar{\varphi_+}\partial_+\varphi_+ + \frac{1}{4}\bar{\varphi_+}\partial_-\varphi_- + \frac{1}{4}\bar{\varphi_-}\partial_+\varphi_+ + \frac{1}{4}\bar{\varphi_-}\partial_-\varphi_-$$

We again drop cross terms:

$$T_{\alpha\beta}^{(F)} = \frac{1}{4}\bar{\varphi_+}\partial_+\varphi_+ + \frac{1}{4}\bar{\varphi_-}\partial_-\varphi_- \tag{4.235}$$

Consider that we had left out a term from equation (4.231) we will now double equation (4.234):

$$T_{\alpha\beta}^{(F)} = \frac{1}{2}\bar{\varphi_+}\partial_+\varphi_+ + \frac{1}{2}\bar{\varphi_-}\partial_-\varphi_- \tag{4.236}$$

From these equations, we can get the equations of motion of the simplices and worldsheet currents of the bosonic string.

$$\partial_+\varphi_\mu^- = \partial_-\varphi_+^\mu = 0 \tag{4.237}$$

and:

$$\partial_-\partial_+X_\mu = 0 \tag{4.238}$$

We use the standard RNS boundary conditions. We can invoke them after solving equation (4.237) in the same way we solved objective one as:

$$\partial_-\partial_+\varphi^\mu_- = 0 \tag{4.239}$$

$$\partial_- = \frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma} \tag{4.240}$$

$$\partial_+ = \frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma} \tag{4.241}$$

Putting equations (4.239) and (4.240) in equation (4.238):

$$\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma}\right)\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma}\right)\varphi^\mu_- = 0$$

$$\left(\frac{\partial^2}{\partial\tau^2} + \cancel{\frac{\partial}{\partial\tau}\frac{\partial}{\partial\sigma}} - \cancel{\frac{\partial}{\partial\tau}\frac{\partial}{\partial\sigma}} - \frac{\partial^2}{\partial\sigma^2}\right)\varphi^\mu_- = 0$$

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right)\varphi^\mu_- = 0 \tag{4.242}$$

We now open equation (4.241):

$$\left(\frac{\partial^2}{\partial\tau^2}\right)\varphi^\mu_- - \left(\frac{\partial^2}{\partial\sigma^2}\right)\varphi^\mu_- = 0 \tag{4.243}$$

Now, φ^μ_- is a multivariate function of σ and τ. We thus invoke separation of variables. We can express φ^μ_- as the product of two functions:

$$\varphi^\mu_- = L(\tau)N(\sigma) \tag{4.244}$$

Putting equation (4.243) in equation (4.242):

$$\frac{\partial^2}{\partial\tau^2}L(\tau)N(\sigma) - \frac{\partial^2}{\partial_{\sigma^2}}L(\tau)N(\sigma) = 0$$

$$
\begin{aligned}
N(\sigma)\frac{d^2}{d\tau^2}S(\tau) - L(\tau)\frac{d^2}{\partial\sigma^2}L(\sigma) &= 0 \\
\frac{1}{S(\tau)N(\sigma)}M(\sigma)\frac{d^2}{d\tau^2}L(\tau) - \frac{1}{N(\tau)N(\sigma)}L(\tau)\frac{d^2}{\partial\sigma^2}L(\sigma) &= 0 \\
\frac{1}{L(\tau)}\frac{d^2}{d\tau^2}L(\tau) - \frac{1}{N(\sigma)}\frac{d^2}{\partial\sigma^2}N(\sigma) &= 0 \\
\frac{1}{L(\tau)}\frac{d^2}{d\tau^2}L(\tau) = \frac{1}{N(\sigma)}\frac{d^2}{\partial\sigma^2}N(\sigma) & \qquad (4.245)
\end{aligned}
$$

We can now use separation of variables to separate equation (4.243). If the two terms are equal, then by the theorem of ordinary differential equations, they are equivalent to a third constant: we will use $-\alpha^2$.

$$
\begin{aligned}
\frac{1}{L(\tau)}\frac{d^2}{d\tau^2}L(\tau) &= -\alpha^2 \\
\therefore \frac{1}{N(\sigma)}\frac{d^2}{\partial\sigma^2}N(\sigma) &= -\alpha^2 \\
\frac{d^2}{d\tau^2}L(\tau) &= -\alpha^2 L(\tau) \\
\frac{d^2}{\partial\sigma^2}N(\sigma) &= -\alpha^2 N(\sigma) \\
\frac{d^2}{d\tau^2}S(\tau) + \alpha^2 S(\tau) &= 0 \qquad (4.246) \\
\frac{d^2}{\partial\sigma^2}N(\sigma) + \alpha^2 N(\sigma) &= 0 \qquad (4.247)
\end{aligned}
$$

We can now proceed to solve equations (4.245) and (3.246). We begin with equation (4.245). We will first begin by replacing the operator $\frac{d}{d\sigma}$ with the operator D We drop the function notation.

$$
\begin{aligned}
D^2 L + \alpha^2 N &= 0 \\
(D^2 + \alpha^2)N &= 0 \\
(D - i\alpha)(D + i\alpha)N &= 0 \\
(D - i\alpha)N = 0\ ;\ (D + i\alpha)N &= 0 \\
(D \pm i\alpha)N &= 0 \\
\left(\frac{d}{d\sigma} \pm i\alpha\right) N &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{d\sigma}N \pm i\alpha N &= 0 \\
\frac{d}{d\sigma}N &= \mp i\alpha N \\
\frac{dN}{N} &= \mp i\alpha d\sigma \\
\int \frac{dN}{N} &= \int \mp i\alpha d\sigma \\
\ln N &= \mp i\alpha\sigma + C \qquad (4.248)\\
N &= Be^{\mp i\alpha\sigma} \qquad (4.249)
\end{aligned}
$$

where we have taken exponentials on equation (4.247) to generate equation (4.248) and:

$$B = e^{c}$$

Similarly equation (4.245) yields:

$$LnL = \mp i\alpha\tau + C_1 \qquad (4.250)$$

$$L = Fe^{\mp i\alpha\tau} \qquad (4.251)$$

putting equations (4.250) and (4.248) in equation (4.243) we get:

$$
\begin{aligned}
\varphi_-^\mu &= DFe^{\mp i\alpha\tau}e^{\mp i\alpha\sigma} \\
\phi_-^\mu &= Le^{\mp i\alpha\tau}e^{\mp i\alpha\sigma} \\
\phi_-^\mu &= Le^{\mp i\alpha\tau \mp i\alpha\sigma} \\
\phi_-^\mu &= Le^{i\alpha(\mp\tau\mp\sigma)} \qquad (4.252)
\end{aligned}
$$

We impose causality on the proper time coordinate. We reject $-\tau$. We express L in tensor notation C_μ

$$\psi_-^\mu = C_\mu e^{i\alpha(\tau\mp\sigma)} \qquad (4.253)$$

$$\psi_-^\mu = \sum_n G_n^\mu e^{i\alpha(\tau\mp\sigma)} \qquad (4.254)$$

The weights G_n^μ are the expansion modes of the fermions. These solutions are equivalent to those obtained earlier in objective one for the simplices. This shows that there is an equivalence between fermionic and simplicial states, and second, that there may exist a duality between bosonic and fermionic, and simplicial states. We will not calculate the effect of quantisation on the generators of the Lorentz algebra, because as we have shown, we do not expect the simplicial states to contribute to the dimensionality of the theory. Thus,the dimensionality of the theory is still 10.

5 Chapter Five: Results and Discussions

5.1 Duality between Spacetime Simplices and Bosonic Strings

In objectives one and three of our work, we have established a duality transformation between bosons and simplices. These were introduced using RNS- like dualities on the bosonic fields modelled using four derivatives of the worldsheet. These bosonic fields were then transformed into simplices which in themselves were understood as Dirac fields. This was formalised using the duality transformations:

$$\delta X^\mu = \bar{\epsilon}\phi^\mu \tag{5.1}$$

$$\delta\phi^\mu = \partial_\alpha X^\mu \epsilon \tag{5.2}$$

where:
δ are variations,
$\epsilon, \bar{\epsilon}$ are Grassman numbers.
X^μ are worldsheet coordinates,
ϕ^μ are the spinors.
We have then investigated the duality further to establish its consistency. This has been done by studying the variations of the Lagrangian and then investigating the symmetries which conserve the principle of least action. This has allowed us to find conserved Noether currents:

$$P^\mu_\alpha = T\partial_\alpha X^\mu \tag{5.3}$$

Where:
P^μ_α is the four momentum,
T is the string tension,
$\partial_\alpha X^\mu$ are the derivatives of the worldsheet coordinates, the bosons.
By taking variations on the Lagrangian, we have obtained the stress energy tensor. The physical implications of this are discussed in the next section. Since we understand that four derivatives of the stress energy tensor vanish because of conservation of energy, we

have derived equations of motion and solve them to yield:

$$\phi_{-}^{\mu} = \sum_{n} D_{n}^{\mu} e^{i\alpha(\tau \mp \sigma)} \tag{5.4}$$

ϕ_{-}^{μ} is the solution,

D_{n}^{μ} are the simplicial mode expansions,

α is the separation constant,

τ is the time coordinate, σ is the spatial coordinate.

We have used this solution to establish that simplices can be modelled as mode expansions of a supersymmetric string. Thus the duality is established as mathematically consistent. However, is it physically correct, or even viable?

The question to be answered is whether we can envision a physical situation in which spacetime simplices can spontaneously transform into bosons. Is this merely a mathematical curiosity or does it carry physical implications? To answer this question we shall turn to general relativity.

In general relativity, the classical Newtonian field is replaced with spacetime curvature. The coupling of a body with matter and energy to spacetime curvature is modelled by the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{5.5}$$

where:

$R_{\mu\nu}$ is the Ricci tensor,

$g_{\mu\nu}$ is the metric which solves the Einstein Field Equations,

R is the Ricci scalar,

$\kappa = \frac{8\pi G}{c^4}$ is the coupling constant,

$T_{\mu\nu}$ is the stress-energy tensor.

Therefore, to interpret this equation in the case of a star system i.e a system consisting of a star with bodies several orders of magnitude lower than it in mass, we say that the star curves spacetime. The lighter bodies (planets, asteroids e.t.c) follow prescribed orbits in this curved spacetime. This class of orbits is known as a geodesic. This is illustrated in the figure below:

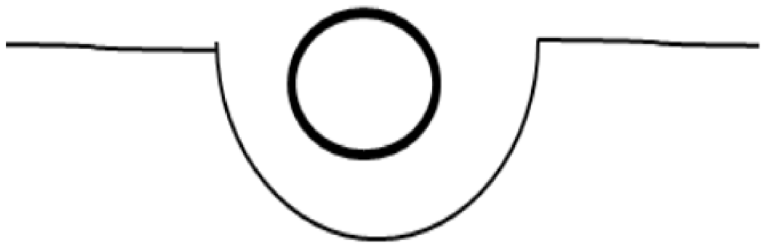

Figure 5.1: Sketch of a star curving spacetime, consistent with general relativity

Using approximation methods, Einstein was able to correctly model the orbit of Mercury and to explain the advance of its perihelion. Consider the formation of black hole. The mass of the star collapses beyond the event horizon. This collapse of the star marks the beginning of the formation of the black hole. The remainder of the process is not yet understood. However, in general relativity, black holes are modelled as a Vacuum field solution of the Einstein Field Equations. This means that the stress-energy tensor $T_{\mu\nu}$ in equation 5.5 vanishes:

$$T_{\mu\nu} = 0 \tag{5.6}$$

This is illustrated by the diagram below:

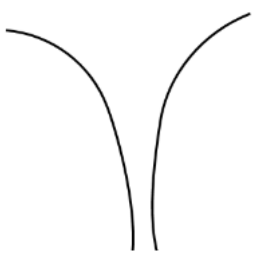

Figure 5.2: Sketch of a blackhole, showing a "hole" in spacetime, with matter absent. The hole exists formally as a manifestation of geodesic incompleteness

If we consider the collapse of a massive star forming a black hole, then a transition can be implied by studying the stress energy tensor:

$$T_{\mu\nu} \neq 0 \rightarrow T_{\mu\nu} = 0 \tag{5.7}$$

Strictly speaking, this is a violation of conservation of energy. The entire mass energy of a star is lost in the process of formation of a blackhole. This presents a challenge to our understanding of classical theories, of which general relativity is part of. To remedy this situation we try to understand the behaviour of spacetime in the process of collapse of the star. In the case of the star before collapse, spacetime has a well defined curvature, at least to an approximation. Furthermore, the curvature has a gentle transition. In the case of the black hole, the curvature diverges, at the centre. Thus, there is a change in the behaviour of spacetime from the collapse of the star to the blackhole. If we pay attention to both spacetime and the collapsing star, then our understanding of the process becomes clearer. We posit that a consistent description of the above process can be developed if the energy of the star is converted into spacetime. The increase of curvature of spacetime is accounted for by the loss of energy of the star. Thus, general relativity seems to require the existence of some form of spacetime-matter duality. Brane-simplex duality serves as a possible mathematical scheme of implementing this.

5.2 The Problem of Dark Energy in String Theory

From cosmological observations (Riess *et al.*, 1998), it is understood that the universe is undergoing an acceleration in its expansion. This is thought to be the result of a cosmological constant (Peebles *et al.*, 2003), or a quintessence field (Peebles *et al.*, 2003). The cosmological constant is thought to be energy intrinsic to space time modeled by an extra term in the Einstein Field Equations (McMahon, 2006) ;

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{5.8}$$

where:

Λ is the cosmological constant, coupled to the metric.

$R_{\mu\nu}$ is the Ricci tensor,

$g_{\mu\nu}$ is the metric which solves the Einstein Field Equations,

R is the Ricci scalar,

$\kappa = \frac{8\pi G}{c^4}$ is the coupling constant,

$T_{\mu\nu}$ is the stress-energy tensor.

This results in a uniformly accelerated expansion of the universe. While Λ is well understood mathematically, the physical mechanism underlying the extra term is not known. The possibility of using the quantum field vacua results in catastrophic disagreement with observation. The deviation between observational data and theoretically calculated value is 121 orders of magnitude (Carroll, 2001). This has led to the proposal of new models. A popular model is the quintessence field which is a scalar field modelled akin to the inflaton field causing a similar expansion of the primordial universe. The proposed model seems to be consistent with the cosmological constant. In the literature review, we have discussed the apparent inconsistency between string theory and dark energy. Dark energy is not acceptable as part of stable vacuum solutions. The vacuum solutions are obtained by flux compactifications. Superstring theory satisfies Lorentz invariance in 10 space time dimensions. To recover four dimensional space time, the extra dimensions are compactified to plank-scale lengths. The compactifications are described by topological spaces called Calabi-Yau manifolds. The mode expansions of the string are determined by those manifolds compactifications. This means that the physical properties of the universe are determined by these mode expansions, and ultimately by the compactification of the manifold. It has been estimated that the number of possible compactifications is 10^{500} which corresponds to 10^{500} possible universe states. This is known as the landscape of solutions. There exists a set of solutions that correspond to unstable vacuum.

These vacua can potentially decay because of tachyonic states which have negative energy states. Vafa et. al(2018) have shown that universe solutions with a cosmological constant have an unstable vacuum. This means that universes with dark energy live in the swampland. This is problematic for string theory, since it means that string theory is inconsistent with dark energy. In this work, we have posited that a duality exists between fermionic matter and simplices in causal dynamical triangulations. This gives the simplice Fermionic properties, including energy. This is determined by simplicial stress energy tensor as developed in subsections 4.1 and 4.3. The result we obtained was:

$$T_{\mu\nu}^{(CDT)} = -\frac{1}{4}\bar{\phi^{\mu}}\beta\partial_{\nu}\phi_{\mu} - \frac{1}{4}\bar{\phi^{\mu}}\beta\partial_{\nu}\phi_{\mu} \tag{5.9}$$

where:
$T_{\mu\nu}^{(CDT)}$ is the simplicial stress-energy tensor,
ϕ_{μ} and $\bar{\phi^{\mu}}$ are simplicial spinors and their Dirac adjoints respectively,
∂_{ν} are four derivatives.

The effect of this is to "thread" energy through space time, giving the effect of dark energy. Thus in the proposed model, RNS superstring models are not only consistent with, but also require dark energy .
As promising as this is, there is need for more work to understand the model and its phenomenological implications. It is not yet understood whether the simplicial stress energy tensor has all the properties that satisfy the cosmological constant, or quintessence. For example, is the simplicial stress energy tensor constant, does it lead to overall accelerated expansion of the universe, i.e negative pressure? The proposed model seems to be consistent with the cosmological constant. This will be subjected to more rigorous theoretical investigation.

5.2.1 Possible Mechanism for Generating the Cosmological Constant

If we accept that simplices are in fact fermions, then we can envision a mechanism for maintaining dark energy density as a constant. We know that the universe is expanding. Speaking in the language of causal dynamical triangulations, this implies a creation of more simplicies, which corresponds to more space and time. Now, if we accept that these simplicies have energy, then each time a simplice is created, energy is created. We must be careful with notions of density however: classically energy density is defined as the quotient of energy and the volume of space:

$$\rho_E = \frac{E}{V} \tag{5.10}$$

Where V is defined dimensionally as $(Length)^3$. Length and energy are usually taken as arbitrarily continuous, giving density as an arbitrarily continuous function. In the simplicial picture, this is replaced by discreteness. Thus the energy density of the simplices must be described as the energy per unit simplice. We use a two dimensional analogy in the figure below;

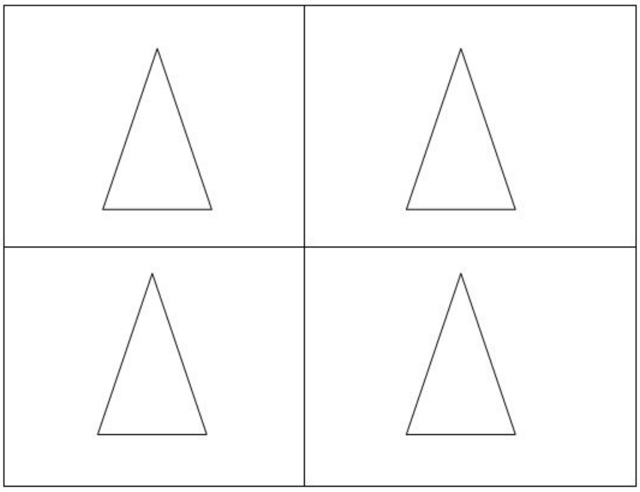

Figure 5.3: An area of spacetime containing four simplices

Classically we would calculate the area of the square by:

$$A = L \times L \tag{5.11}$$

Since we have lost these notions of continuity, we can only measure the area of the square as $4A$. If we assume that each simplice has an energy E then we can calculate the two dimensional classical energy density as;

$$\rho = \frac{4E}{A} = \frac{4E}{L^2} \tag{5.12}$$

.However, in the simplicial case, we should calculate it as:

$$\rho_E = \frac{4E}{4A} = \frac{E}{A} \tag{5.13}$$

Now, as we have discussed, the simplicial analogue of the expansion of the universe is the creation of new simplices. The new simplices created cannot share the same quantum state with the old simplices, they are "Fermionic" they thus have to go into new quantum states;

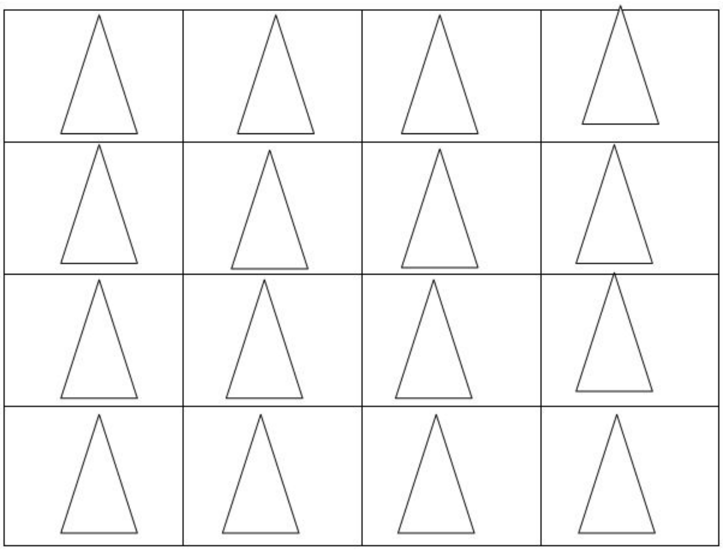

Figure 5.4: Expansion of spacetime, manifesting as the creation of new simplices

The new calculated energy density after expansion is:

$$\rho_E = \frac{16E}{16A} = \frac{E}{A} \tag{5.14}$$

Thus the energy density of the simplices is constant independent of the expansion of the universe. This "steady state" mode of dark energy is consistent with modern observations. This may satisfy the cosmological constant solutions.

5.3 The Dimensionality of String Theory and Flux Compactifications

As we discussed in our proposal, we expected that the duality would introduce a new "supersymmetry". The introduction of new Fermionic fields was expected to further cancel anomalies, leading to a reduction in the number of dimensions needed to describe RNS Superstrings consistently. This is the procedure which is invoked when making the transition from bosonic to superstring theory (McMahon,2009) theory. This expectation was however not fruitful. We established that the dimensionality of the theory is 26. There could be several reasons for this; including mathematical oversight. The most probable reason for this is that we did not implement supersymmetry in the simplicial states fully. In section 3.3, we defined the simplicio-fermionic field using the formula:

$$\Phi^{i-} = -\frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{n}(\sigma_n^i\sigma_{-n}^- - \sigma_{-n}^-\sigma_n^i) \tag{5.15}$$

$$\Phi^{j-} = -\frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{n}(\sigma_n^j\sigma_{-n}^- - \sigma_{-n}^-\sigma_n^j) \tag{5.16}$$

where:

Φ is defined by the equation above,

n is a counting integer denoting the nth expansion mode'

j and i are spacetime indices,

σ are the expansion modes of the dual simplice states.

This may not be the full definition of the field. It was established in cononance with the standard RNS formalism. Further, the commutation relations imposed between the fermionic and simplicial expansion modes ϖ^m and σ_i respectively may not have been correct. We imposed these on physical grounds, we do not envision a situation in which expansion modes do not commute with each other; the ideal measurement of expansion modes of a simplicial state should not affect the measurement of a fermionic state. However upon closer physical inspection, this argument may fail. After all, we are imposing dualities between separate states. Thus we may expect that a relationship between expansion modes exists in view of this duality. This relation may be properly expressed as a commutation relation. Thus the mode expansions may not commute. The argument against such a positon comes from RNS supersymmetry itself. The expansion modes of RNS superstrings commute with each other, thus, the formalism we have adopted requires that we accept that expansion modes for different fields mutually commute. If we accept that our implementation of the duality, and the corresponding commutation relations is correct then we are led into an interesting conundrum. If it is in fact true that there exist a symmetry between space time and matter to be implemented in a supersymmetry- like fashion then there are no more fields to be added to the theory. The perturbative form of the theory is background independent, yet it accounts for both fermions and bosons. With the addition of simplicial space time fields, there may be no more fields to add to the theory. If our implementation of the theory is correct, then it means that superstring theory really is 10-dimensional. This necessitates the introduction of flux compactifications to bring the theory into concordance with cosmological and astrophysical observations of a 4-dimensional universe. With this comes the ontological and epistemological measure problem: How do we do statistics on an infinite set of possibilities? Ultimately: How do we make testable predictions?

We had hoped to side-step this problem by reducing the number of dimensions in RNS superstrings, thus reducing the number of vacua states emergent from the flux compactifications. An optimum result would have been to obviate the need for flux compactifications. This was unsuccessful. We hold that flux compactifications are a worrying feature of RNS superstrings and string theories in general.

5.4 Is Spacetime Quantised?

Ever since the introduction of calculus to physics by Newton, space and time have been taken to be continuous. All the spaces associated with classical mechanics (vector space, parameter space, phase space) are held to be continuous and differentiable. With the introduction of special and general relativity, the introduction of differentiable manifolds further solidified the status of continuous spacetime in physics(McMahon, 2006). The introduction of quantum mechanics did not revolutionize our understanding of the parameter space. A particle is understood to have quantized energy but its wave function is a continuous function of position and time, which are themselves thought to be continuous. The momentum and energy of the particle are thought to be quantized, however, with the full understanding of Fourier transformation techniques, one can turn these quantized momenta into a continuous "Momentum Space" and do quantum mechanics consistently; quantum field theories traditionally live in the Minkowski Space time of Special Relativity which is continuous, and defined by the length of line interval:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{5.17}$$

where:
ds^2 is the length of line interval,
$c^2 \approx 9 \times 10^{16} m/s$ is the speed of light squared,
dt, dx, dy and dz are length of line intervals for time and the three spatial dimensions respectively.
Where dt^2, dx^2, dy^2, dz^2, and hence ds^2 can be made arbitrarily small. Thus, quantum field theory relies on some notion of arbitrary continuity. While there are forms of quantum field theory in a discrete space time such as lattice quantum chromodynamics, these are understood to be numerical approximations of a "true" continuum theory. With quantum gravity, we face challenges. It is understood that general relativity is defined on a continuous, differentiable manifold; the coupling of the stress energy tensor to the Einstein tensor gives rise to the Einstein field equations,

$$R_{\mu\nu}(x,t) - \frac{1}{2}g_{\mu\nu}(x,t) = G_{\mu\nu}(x,t) = \frac{8\pi G}{c^4}T_{\mu\nu}(x,t) \tag{5.18}$$

where the fact that the tensors are continuous functions of a continuous space time manifold is stressed. The Einstein tensor encodes the gravitational field: The gravitational field is geometrical nature. The problem of quantum gravity is thus stated: How do we express the gravitational field quantum mechanically? There are two general approaches to answering this question

5.4.1 Challenge Geometrodynamics; Accept Continuity.

In this picture, the notion that gravity is geometrical in nature is challenged. This is done by introducing a gravitation field whose vector is the graviton. The gravitational field is a spin 2 field which couples to the stress energy tensor in an equivalent way to the Einstein Tensor. The introduction of the graviton leaves space time no dynamics. The introduction of the graviton field however introduces new challenges. By power counting, the graviton scattering amplitudes are non- renormalisable, this is because the gravitational field is non-linear. String theory cures this problem by replacing point particles of quantum field theory with strings. This results in a renormalisable theory. The status of space time is less clear, at least in the case of perturbative string theory. String theory is deliberately formulated in such a sense that the theory satisfies Lorentz symmetry, a continuous symmetry which is the result of special relativity. In the classical formulation of the theory, it can be shown that the Lorentz symmetry is trivially satisfied. However, after quantization, the Lorentz symmetry is broken because of the non-zero commutation relations between mode expansions(McMahon, 2009). This break, including the presence of tachyonic ground state, has led workers in the field to impose Lorentz symmetry on string theory. This has led to an imposition of dimensionality of the theory because of the relation of commutation relations defined using the trace of the Mikowskian in the light-cone gauge. Thus, it has become accepted that, least in the perturbative formulation of the theory, a background which preserves Lorentz symmetry is required. This has the implication of making the background of theory continuous.

We followed similar arguments while calculating the dimensionality of the theory, where we used the "mixed index" components of the generators of the Lorentz algebra $J^{\mu\nu}$. Since the uniform components close under commutation, we required the mixed components to commute. The commutator we found was:

$$\left[J^{i-}, J^{j-}\right] = -\frac{4\pi T}{(p^+)^2}\sum_{n=1}^{\infty}\left[\left(\frac{d-2}{12}-2\right)n+\frac{1}{n}\left(2a-\frac{d-2}{12}\right)\right]\left(\varpi^i_{-n}\varpi^j_n - \varpi^j_{-n}\varpi^i_n\right) \quad (5.19)$$

where:

J^{i-} and J^{j-} are the mixed components of the generators of the Lorentz algebra,

$\pi = \frac{22}{7}$ is pi, the ratio of the diameter of a circle to its circumference,

T is the string tension,

p^+ is the momentum in lightcone coordinates,

$n \in 1, 2, 3... = N$ is a counting integer,

d is the number of dimensions,

a is the normal ordering constant, ϖ are the expansion modes.

We then fixed the normal ordering constant and forced $[J^{i-}, J^{j-}] = 0$. This led to the case:

$$d = 26 \tag{5.20}$$

In the case of the non- perturbative formulation of the theory, the understanding is a bit more nuanced and incomplete. A popular way of formulating string theory is using AdS/CFT duality discussed in the introduction. In this, two surprising results have emerged: The entanglement of states in the conformal field theory on the boundary corresponds to the definition of space time in the bulk. This interesting new development may shed light on the emergence of space time at high energy scales in string theory. However, there still exists a problem with the very definition of the AdS/CFT duality: The definition of the AdS boundary is a solution of the Einstein field equations. a continuous, differentiable manifold. Thus, we have spacetime emerging from structures dual to entangled states in continuous space time. A duality implies a transformation, or the existence of a more fundamental structure. For example, the duality of length contraction and time dilation measurements implies the existence of a more fundamental structure: spacetime. The existence of wave-particle duality implies the existence of a quantum field. The existence of excitations satisfy the Schrödinger equation, or its relativistic forms. A possible direction of research in perturbative string theory may be to understand the fundamental structure implied by AdS/CFT. The approach is popular among workers in quantum gravity because it preserves Lorentz invariance.

5.4.2 Accept Geometrodynamics, Challenge Continuity

In this case, we accept that gravity is inherently geometrical in nature. However, the notion of a continuous differentiable manifolds modelled by a metric, is rejected. Instead, the notion of quantization is introduced on space time. The theory works in an analogous scheme to Planck quantization of radiation. There are two major ways of implementing this: causal dynamical triangulations and loop quantum gravity. causal dynamical triangulations is a version of euclidean quantum gravity where simplices are the "quanta" of space time. The addition of all these simplices (integration over all possible configurations) yields our classical manifold. From this we can recover the usual classical notions of diffeomorphism invariance and the equivalence principle. As discussed in the introduction, causal dynamical triangulation improves Euclidean quantum gravity by imposing causality on the simplices. A similar implementation exists in loop quantum gravity, with spin networks being the analogous structures of simplices. Loop quantum gravity has become particularly useful in proposing "bounce" cosmologies and bounce astrophysics, in which there is a limit to to how much energy spacetime can hold. This places limits on the energy density of space time: any density higher than this will lead to a repulsive force analogous to the degeneracy pressure of neutron stars and brown dwarfs. This then leads to an expansion which is interpreted as a big bang (Ashtekar, 2009) in cosmology, or as a fast radio burst in stellar astrophysics. There is one challenge shared by both schemes: testability. String theory has the potential of providing a consistent, renormalisable quantum theory of gravity. It however provides us with a measure problem as earlier discussed. For us to quantize space time as scheme (2) proposes, we may need to break Lorentz invariance, at least in some scale. There is no evidence that this occurs in nature. Astrophysical measurements (Jacobson *et al.*,,2003) and more recently measurements in particle physics (Mestres, 1997) have shown that Lorentz invariance is not broken at the scales of the respective investigations. Further, some workers have proposed that the repulsive force arising from the reaching of the energy density bound levels to emissions by black holes in the form of fast radio bursts. however, there significant constraints on loop quantum gravity models with this picture.

In our scheme, we challenge both geometrodynamics and continuity. We hold that the introduction of a gravitation field with quantum field theoretical properties is the correct way of implementing quantum gravity. We make this decision on physical arguments; three of four fundamental forces are described by quantum fields, it would be strange if the fourth wasn't described by a quantum field. We also challenge continuity on the basis of physical arguments: imagine two points of space time that are arbitrarily close together. If we would wish to specify the positions of this points in configuration space, if the points are near infinitely close together (the definition of infinitesimals) then it would need more space than the entire universe to specify the positions of this space information. Theoretically, on the other hand, to resolve the two points, we would need more energy than that of the observable universe. It is on the basis that we reject the notion of continuity. However, there must be a mechanism of recovering some notion of effective continuity at the observable scale. This would allow us to use the continuous symmetries which preserve Lorentz invariance. Causal dynamical triangulation recovers the classical structure of space time (Forcier, 2011).

6 Chapter Six: Conclusions and Recommendations

6.1 Conclusions

We have postulated a new duality, brane-simplex duality which allows us to describe spacetime simplices in terms of fermionic functions. The bosonic string theory has been extended by linking the Polyakov action to the Regge action. We then studied the symmetries of the extended Polyakov action has then been studied leading to a stress energy tensor associated with the simplices. We have then discussed the possibility that this stress-energy tensor is dark energy. We have then solved for the fermionic simplices getting mode expansions for the extended bosonic string.

The generator of the Lorentz algebra was then invoked. The generators specified have to mutually commute if Lorentz invariance is to be maintained in the new extended theory. We have calculated the commutator considering the extra simplicial fermionic fields added to the traditional angular momentum generator because of the new duality. We have found no effect on the dimensionality of bosonic string theory. We have discussed the possible reasons for this.

The development of the full superstring theory has then been done. Akin to objective one, we have developed the symmetries of the action which gives us the stress-energy tensor in the standard fermionic case. We have not calculated the dimensionality of the new theory because it was demonstrated that the new duality does not affect dimensionality.

6.2 Recommendations

Work remains to be done to fully understand the duality. There is need for more research to be done to understand whether constructing spacetime out of fermionic fields is consistent.The stress energy tensor needs to predict the exact value of dark energy for the model to be phenomenologically and empirically relevant to cosmology. Another approach would be to tie the value of the cosmological constant to the theory, hopefully making it a "unique" string theory. We can then perform phenomenological computations using the theory. This may be thought analogically with the coupling constant renormalization in quantum field theory.

The fact that the new symmetry does not affect the dimensionality of the two respective string theories needs to be verified. If this is because of incomplete treatment and implementation of the new duality, then we recommend better implementations of the duality. This could possibly lead to the reduction of critical dimensions in the new "Chimera" theories.

The work could possibly be extended to non-perturbative string theory. We find the aspect of AdS/CFT duality particularly interesting. The duality provides for the generation of spacetime simplices in the bulk by linking it to entanglement of CFT states on the surface. The boundary acts like a set of boundary consitions of the model, whose existence is independent of the theory. By constructing spacetime using simplices, this boundary may be constructed within the theory, making non-perturbative formulations of string theory "fully" background independent.

References

A. Ashtekar, (2009). *Loop Quantum Cosmology: An Overview*, Gen. Rel. Grav. 41 (4): 707–741. arXiv:0812.0177. **Bibcode:2009GReGr..41..707A. doi:10.1007/s10714-009-0763-4.**

A. Einstein,*The Field Equations of Gravitation,Proceedings of the Prussian Academy of Sciences*, Part 2(1915)(884-887)

A. Neveu,J Schwarz, *Tachyon Free Dual Model with a Positive Intercept Trajectory*, (1971). **doi:10.1016/0370-2693(71)906609-1**

A. Salam, Ward J. C, *Weak and Electromagnetic Interaction*, Il Nuovo Cimento, (1959). **doi:10.1007/B02726525**

C. Lovelace, *Pomeron Factors from Dual Regge Cuts*, Physics Letters B.,(1971), **doi:10.1016/0370-2693(71)90665**

D. McMmahon, *Relativity Demystified*, McGraw-Hill, USA(2006).pp (37, 138)

D. McMmahon, *Quantum Field Theory Demystified*, McGraw-Hill, New York.(2008) pp. (139-160, 167-185, 209-231)

D. McMmahon, *String Theory Demystified*, McGraw-Hill Publishers, New York, (2009), pp (7-15, 32-87, 127-150, 188-193, 255-262)

G.T.Horowitz, *Spacetime in String Theory*, New Journal of Physics, (2005)

G. Veneziano,*Construction of a Crossing Symmetric Reggeon-behaved Amplitude for Linearly Rising Trajectories*, Il Nuovo Cimento, (1968). **doi:10.1007/BF02824451**

H. Nielsen, *An Almost Physical Interpretation of the Dual N-Point Function*, Nordita Preprint, (1969)

H. Yukawa, *On the Interaction of Elementary Particles*, Proceedings of The Mathematical Society of Japan, (1935)

J. Ambjorn, A Dasgupta, J Jurkiewicz and R. Loll, *A Lorentzian Cure for Euclidean Troubles*, arXiv:hep-th/0201104, 2002

J.L Gervais, B. Sakita, *Field Theory Interpretation of Supergauges in Dual Models*, Nuclear Physics B. (1971). **doi:101016/00550-3213(71)903351-8**

J. Maldacena, *The large N limit of Superconformal Field Theories and Supergravity*, Advances in Theoretical and Mathematical Physics, arXiv:hep-th/9711200(1998),**doi:101023/A:1026654312961**

J. Polchinski, *Dirichlet Branes and Ramond-Ramond Charges*, Physical Review D.,arXiv:hep-th/9510017.(1995) **doi:10.101103/PhysRevLett 75.724**

J. Schwinger, *On Quantum Electrodynamics and the Magnetic Moment of the Electron*, Physical Review Letters,(1948)

L. G. Mestres (1997-05-26). *Absence of Greisen-Zatsepin-Kuzmin Cutoff and Stability of Unstable Particles at Very High Energy, as a Consequence of Lorentz Symmetry Violation.* Proceedings of the 25th International Cosmic Ray Conference (held 30 July - 6 August. 6: 113. arXiv:physics/9705031. **Bibcode:1997ICRC6113G**.

L. Peebles, P. Ratra, E. Bharat (2003). *The cosmological constant and dark en-*

ergy. Reviews of Modern Physics. 75 (2): 559–606. arXiv:astro-ph/0207347. **Bibcode:2003RvMP75559P. doi:10.1103/RevModPhys.75.559.**

L. Susskind, *Harmonic Oscillator Analogy for the Veneziano Amplitude*, Physical Review Letters D.(1969)**doi:10.1103/PhysRevD1.1182**

M. Born, W. Heisenberg, P. Jordan, *On Quantum Mechanics II*, Magazine for Physics,(1925)

M. Douglas, *The Statistics of String/M-Theory Vacua*, Journal of High Energy Physics,**030546(2003)**, arXiv:hep-th/0301194

M. Gell-Mann, *Dispersion Relations in Pion-Pion and Photon-Nucleon Scattering*, High Energy Nuclear Physics: Proceedings of the Rochester Conference,(1956)

P. A. M. Dirac, *The Quantum Theory of the Emission and Absorption of Radiation*, Proceedings of the Royal Society of London, (1927)

P. A. M. Dirac, *The Quantum Theory of The Electron*, Proceedings of the Royal Society A:Mathematical Physical and Engineering Sciences, (1928)

P. Jordan, *On a New Foundation for Quantum Mechanics II*, Magazine for Physics(1-25), (1927)

P Ratra, L. Peebles, (1988). *Cosmological consequences of a rolling homogeneous scalar field* Physical Review D. 37 (12): 3406. **Bibcode:1988PhRvD..37.3406R. doi:10.1103/PhysRevD.37.3406**.

P. Ramond, *Dual Theory for Free Fermions*, Physical Review Letters D, (1971). **doi:10.1103PhysRevD.32415**

Riess, Adam G.; Filippenko; Challis; Clocchiatti; Diercks; Garnavich; Gilliland; Hogan; Jha; Kirshner; Leibundgut; Phillips; Reiss; Schmidt; Schommer; Smith; Spyromilio; Stubbs; Suntzeff; Tonry (1998). *Observational evidence from supernovae for an accelerating universe and a cosmological constant.* Astronomical Journal. 116 (3): 1009–38. arXiv:astro-ph/9805201. **Bibcode:1998AJ....116.1009R. doi:10.1086/300499**.

R. P. Feynman, *The Feynman Lectures on Gravitation*, Addison-Wesley Publishing Company, (1995) pp.(1-80)

R. Feynman, *A Relativistic Cutoff for Quantum Electrodynamics*, Physical Review (1430-1438), 1948, **doi:10.1103/PhysRev 74.939**

S. Carroll , (2001). *The cosmological constant.* Living Reviews in Relativity. 4 (1): 1. arXiv:astro-ph/0004075. **Bibcode:2001LR41C. doi:10.12942/lrr-2001-1**.

S. Glashow, *On the Renormalizability of Vector Meson Interactions*, Nuclear Physics,(1959)

S. W. Hawking, *A Path Integral Approach to Quantum Gravity in General Relativity*, Einstein Centenary Survey.(1977)

S.Ashok, M. Douglas, *Counting Flux Vacua* , Journal of High Energy Physics,(2004)

S. Weinberg, *A Model of Leptons*, Physical Review Letters, (1967)

T. Jacobson ,S. Liberati ,D. Mattingly (August 2003). *A strong astrophysical constraint on the violation of special relativity by quantum gravity.*

T. Yoneya, *Connection of Dual Models to Electrodynamics and Gravidynamics*, Progress in Theoretical Physics, (1974).

W. Buchmüller, C. Lüdeling, *Field Theory and The Standard Model*, arXiv preprint:http://arXiv.org/abs/hep-ph/0609174v1

W. Heisenberg, *The "observable quantities" in the theory of elementary particles*, Magazine for Physics, (1943)

Z. Koba, H. Nielsen, *Reaction Amplitude for N Mesons:A Generalisation of the Veneziano-Badarski-Ruegg-Virasoro Model* Nuclear Physics 13. 633-635,(1969). **doi:101016/050-3213(61)903319**

A. Forcier, *Introductory Causal Dynamical Triangulation*, ArXiv Preprint:https://arxiv.org/pdf/1109.3879.pdf,(2011)

A. Perez, *Introduction to Loop Quantum Gravity and Spin Foams*, Arxiv Preprint:http://arxiv.org/abs/gr-qc/0409061v3,(2009)

E. Witten, *String Theory Dynamics in Various Dimensions*, arXiv:hep-th/9503124, (1995)00158-0 **doi:10.1016100550-3213(95)**

G. Obied, H. Ooguri, L.Spodyneiko, C. Vafa, *De Sitter Space and the Swampland*, Arxiv Preprint:http://arxiv.org/abs/1806.08362v3, (2018)

L. Smolin, *The Case for Background Independence*, arXiv Preprint:http://arxiv.org/abs/hep-th/0507235v1,
(2009)

Turnitin Originality Report

Processed on: 03-May-2019 07:45 EAT
ID: 1124049752
Word Count: 25107
Submitted: 1

CONSTRUCTING A STRING THEORY BACKGROUND USING BRANE-SIMPLEX DUALITY By I56/9841/2018 Kinyanjui Carringtone

Similarity Index	Similarity by Source	
11%	Internet Sources:	4%
	Publications:	7%
	Student Papers:	8%

< 1% match (student papers from 22-Sep-2010)
Submitted to King's College on 2010-09-22

< 1% match (Internet from 11-Nov-2018)
https://particlephysicist.files.wordpress.com/2013/04/248.pdf

< 1% match (publications)
Lecture Notes in Computer Science, 2002.

< 1% match (publications)
"Quantum Field Theory II: Quantum Electrodynamics", Springer Nature, 2009

< 1% match (student papers from 30-May-2017)
Submitted to University of Nairobi on 2017-05-30

< 1% match (publications)
"Springer Handbook of Spacetime", Springer Nature, 2014

< 1% match (student papers from 02-May-2017)
Submitted to Imperial College of Science, Technology and Medicine on 2017-05-02

< 1% match (publications)
I. Kuruganti, S. Strickland, "Optimal importance sampling for Markovian systems with applications to tandem queues", Mathematics and Computers in Simulation, 1997

< 1% match (student papers from 02-May-2017)
Submitted to Imperial College of Science, Technology and Medicine on 2017-05-02

< 1% match (Internet from 23-Sep-2017)

DEPARTMENT OF PHYSICS
21 MAY 2019
UNIVERSITY OF NAIROBI
Approved
21/05/2019