Erlang Mixtures and Their Link With Exponential and Poisson Mixtures

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Master Thesis

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Abstract

Erlang mixtures have been constructed and expressed in three ways, which are, the Explicit form, the Bessel function of the third kind and the Confluent Hypergeometric functions. The direct method and the method of moments were used in the construction, and the two methods were equated to derive a Mathematical Identity.
The rth moment was also obtained and it was observed that it was expressed in terms of the rth moment of the reciprocal of the mixing distribution.
The link between Erlang mixtures and Exponential mixtures was shown, and it was deduced that the Erlang mixture with the shape parameter, n=1 becomes the Exponential mixture.
The link between Erlang mixtures and Poisson mixtures was also shown, and it was determined that the Poisson mixture is $\frac{t}{n}$ times the Erlang mixture. The basic difference-differential equations for a Poisson process were solved to obtain the Poisson distribution, and the first passage time distribution of the Poisson process was determined to be an Erlang mixture. The Probability Generating Functions of the Poisson mixtures were also obtaned.
The constuction of a four-parameter generalized Lindley distribution has also been introduced in this work which nests the one, two and three parameter Lindley distributions.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

BEATRICE MUGURE GATHONGO
Reg No. I56/8177/2017

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

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Dedication

This project is dedicated to my mother who was of great support throughout my studies and in all my other endeavors.

Abbreviations and Notations

Abbrevations and notations for specific chapters can be found in those chapters,and those that are generally used are listed below.

cdf Cumulative Density Function
pdf Probabilty Density Function
pmf Probability Mass Function
pgf Probability Generating Fuction
$f_n(t)$ Probability Density Function of a mixed Erlang distribution
$g(\lambda)$ Probability Density Function of a mixing distribution
G(s,t) Probability Generating Function of the Poisson distribution
$E(T^r)$ The rth moment of the mixture
Ψ(a,c;x) Tricomi Confluent Hypergeometric Function
${}_1F_1$(a,c;x) Kummer's Confluent Hypergeometric Function
$K_v(w)$ Modified Bessel Function of the third kind
$\gamma(a,x)$ Incomplete Gamma Function

Contents

Acknowledgments

First and foremost, I want to thank the Almighty God for the wisdom he bestowed upon me,the strength and good health He gave me that enabled me finish this project.
I would also like to express my special gratitude and thanks to my supervisor Prof. J.A.M Ottieno for imparting his knowledge and expertise in this project.
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Finally, my thanks and appreciations go to my classmates and the school of Mathematics fraternity for their support during the project and throughout my time at the University of Nairobi.

Beatrice Mugure Gathongo

Nairobi, 2019.

1 GENERAL INTRODUCTION

1.1 Background Information

One way of constructing probability distributions is through mixture, which is a way of combining two or more distributions to come up with a new one.
There are three types of mixtures,namely, finite, discrete and continuous mixtures.
In this work we look at continuous Erlang mixtures where continuous mixing distributions are mixed with the Erlang distribution.
Continuous mixtures was originated by Yule and Greenwood in 1920 who combined a Poisson distribution with a Gamma distribution to obtain a Negative Binomial distribution.
The Erlang distribution was developed by A.K. Erlang in 1986 at the Ericsson Computer Science Lab to examine the number of telephone calls that were made at a given time to the operators of the switching stations, and to generally provide a better way of programming telephony applications.
This work on telephone traffic engineering would later be used in modeling waiting times in queueing systems, and in stochastic processes.

1.2 Definitions and Terminologies

Let $f_n(t)$ be a function of a random variable t.
If

$$0 \leq f_n(t) \leq 1 \quad and \quad \sum_{t=-\infty}^{\infty} f_n(t) = 1$$

then t is a discrete random variable and $f_n(t)$ is known as a probability mass function of t.

If

$$f_n(t) > 0 \quad and \quad \int_{-\infty}^{\infty} f_n(t)dt = 1$$

then t is a continuous random variable and $f_n(t)$ is known as a probability density function of t.

So a continuous mixture will be given by;

$$f_n(t) = \int_0^\infty f(t/\lambda)g(\lambda)d\lambda$$

where

f(t/λ) is the conditional distribution

and

g(λ) is the pdf of a continuous random variable
λ and is known as the mixing distribution.

1.2.1 Erlang mixtures

The probability density function of the Erlang distribution is given by;

$$f(t/\lambda) = \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1}, \quad t > 0; \lambda > 0, n = 1, 2, 3, ..$$

where the parameter n is known as the shape parameter and the parameterr λ as the rate parameter.
And so the Erlang mixture is given by;

$$f_n(t) = \int_0^\infty \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} g(\lambda)d\lambda \tag{1.1}$$

An alternative definition of the Erlang distribution is;

$$f(t/\lambda) = \frac{e^{-\frac{t}{\mu}} t^{n-1}}{\mu^n \Gamma n}$$

where λ is replaced by $\frac{1}{\mu}$.

Other properties of the Erlang distribution include;

1. If X$\sim$ Erlang(n,λ) then a.X$\sim$ Erlang(n,$\frac{\lambda}{a}$) with a$\in \mathbb{R}$
2. If X$\sim$ Erlang(n_1,λ) and Y$\sim$ Erlang(n_2,λ) then X+Y$\sim$ Erlang ($n_1 + n_2$,λ)

1.2.2 Exponential mixtures

The Erlang distribution with the shape parameter,n=1 becomes the Exponential distribution.
The Erlang distribution is the distribution of a sum of n independent and identically distributed Exponential random variables,each with parameter λ and a mean $\frac{1}{\lambda}$.
That is,if $X_i \sim$Exponential(λ), then $\sum_{i=1}^{n} X_i \sim$Erlang(n,λ)
So the Exponential mixture becomes;

$$f(t) = \int_0^\infty \lambda e^{-\lambda t} g(\lambda) d(\lambda)$$

1.2.3 Poisson mixtures

The Erlang distribution is related to the Poisson distribution through the Poisson process as shown in chapter two.
The Poisson mixture is $\frac{t}{n}$ times the Erlang mixture.
i.e,

$$\begin{aligned} P_n(t) &= \frac{t}{n} f_n(t) \\ &= \frac{t}{n} \int_0^\infty \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} g(\lambda) d\lambda \\ &= \int_0^\infty \frac{\lambda^n}{n!} e^{-\lambda t} t^n g(\lambda) d\lambda \\ &= \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^n}{n!} g(\lambda) d\lambda \end{aligned}$$

1.3 Research Problem

As mentioned above there are three types of mixtures,namely; Finite,Discrete and Continuous mixtures.
In the literature more work on finite mixtures of Erlang distribution is the focus; for example;Willmot and Woo (2007); Fung et al (2000); Yin and Sheldon (2016); Antonio et al (2014); Verbenel (2014); Cosette et al (2016)
Very little seems to have been done on Continuous Erlang mixtures.
McNolty (1964) used the following five mixing distributions in the continuous Erlang mixture; Scaled Beta, Gamma, Rayleigh, Maxwell-Boltzman and the Bessel variate.

- McNolty (1964) did not come up with a general model. He considered every case independently.
- More mixing distributions need to be considered.
- Wakoli (2016) constructed Exponential mixtures and Sarguta (2017) constructed Poisson mixtures.
- Erlang mixture is related to Poisson mixture and Exponential mixture. This link however has not been shown.

1.4 Objectives

1.4.1 General Objectives

The main objective is to constuct Erlang mixtures and to show the link between Erlang mixtures and Poisson and Exponential mixtures.

1.4.2 Specific Objectives

1. To constuct and express Erlang mixtures in the following forms:

   I. In Explicit form.

   II. In terms of the Modified Bessel function of the third kind.

   III. In terms of the Confluent Hypergeometric functions.(Kummer's and Tricomi)

2. To Introduce the 4-parameter generalized Lindley mixing distribution and other forms of the 3-parameter generalized Lindley mixing distibutions.

3. To obtain the rth moments of the mixtures in all the forms listed above in (1).

4. To express the Erlang mixtures in two ways: The direct method and the method of moments, and equating the two methods to obtain a Mathematical Identity.

5. To obtain the Exponential mixture from the Erlang mixture after showing the link between them.

6. To obtain the Poisson mixture from the Erlang mixture after showing the link between them.

1.5 Methodology

Several methods have been used in the construction of the Erlang mixtures and they include;
I. Direct Integration and Substitution.
II.Special functions; Beta function, Gamma function, Modified Bessel function of the third kind and the Confluent Hypergeometric functions (Kummer's and Tricomi).
III.Transforms: Probability Generating Function.

1.6 Literature review

Although a lot has been covered on finite Erlang mixtures, very little seems to have been done on the continuous Erlang mixtures. Also,Exponential mixtures and Poisson mixtures have been studied but the link between Erlang mixtures and both Exponential and Poisson mixtures has not been shown.

Sarguta (2017) constructed continuous Poisson mixtures and expressed them in four ways: In explicit form where the direct integration was used and moments of the mixtures were obtained which included the moments about the origin(raw moments),moments about the mean(central moments) and the posterior rth moments; In terms of special functions which are the Modified Besel function of the third kind and the Confluent Hypergeometric functions(Kummer's and Tricomi); In recursive form where integration by parts was used and Wang's recursive approach was applied in determining the differential equations for the recursive models; and In expectation forms where the Laplace and Mellin transforms were used and the Probability Generating Function was used in obtaining moments which included the rth factorial moment, the raw moments and central moments. Mathematical Identities were also obtained by equating results derived using explicit forms and those expressed in terms of special functions to their corresponding method of moments.

Wakoli (2016) constructed type I and type II Exponential mixtures in explicit form and in terms of the special functions which are the Bessel function of the third kind and the Confluent Hypergeometric functions(Kummer's and Tricomi), obtained moments for these mixtures using Mellin transforms and conditional expectation, expressed mixed Poisson distributions in terms of hazard function of type I Exponential mixture and in terms of the Laplace transform, derived compound Poisson distribution in terms of Probability Generating Functions and recursive form, showed that a sum of hazard functions of Exponential mixtures results to a convolution of compound Poisson distributions, and obtained hazard functions using Laplace transforms of sums of independent continuous random variables.

McNolty (1964) derived probability density functions (which are Erlang mixtures) for the time to the (n+1)st failure, where the failure rate has a random distribution(which is the mixing distribution) and was not time-dependent.

1.7 Significance of the study

Continuous Erlang mixtures are applied;

a) In unifying Exponential mixtures and Poisson mixtures as shown in this work.

b) As waiting time distributions for a mixed Poisson process which are a non homogeneous birth process as shown in chapter two.

2 DISTRIBUTIONS ARISING FROM A POISSON PROCESS AND MIXED POISSON PROCESSES

2.1 Introduction

The objective of this chapter is to show that an Erlang distribution is a waiting time distribution in a Poisson process and an Erlang mixture is a waiting time distribution in a mixed Poisson process.

There are two approaches of deriving distributions arising from mixed Poisson processes.

The first one is based on a Poisson process with a randomized rate.

The other approach is based on a pure birth process.

The Poisson process is a special case of a pure birth process.

Solving the basic difference-differential equations for a Poisson process we obtain a Poisson distribution.

The waiting time for an nth event to occur in a Poisson process is shown to be an Erlang distribution.

We shall also express the first passage time distributions based on randomization in two forms.

Mathematical identities based on these two forms will be determined.

The first passage time distribution of the Poisson process is an Erlang mixture whose links with the Exponential mixture and the Poisson mixture are derived.

2.2 Solving the basic Difference-Differential Equation for a Poisson process

Let

$$X(t) = the \quad population \quad size \quad at \quad time \quad t$$

and

$$P_n(t) = Prob[X(t) = n]$$

The basic difference-differential equations for a pure birth process are given by;

$$P_0'(t) = -\lambda_o P_0(t)$$
$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), \quad n = 1, 2, 3, \ldots$$

where

$$P_n'(t) = \frac{d}{dt} P_n(t)$$

and

λ_n= birth rate when the population size is n.

For a Poisson process, λ_n=λ for all n.
Thus we have;

$$P_0'(t) = -\lambda P_0(t) \tag{2.1}$$
$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \quad n = 1, 2, 3, .. \tag{2.2}$$

Multiplying (2.2) by S^n and then summing the result over n;

$$\begin{aligned} \therefore \sum_{n=1}^{\infty} P_n'(t) S^n &= -\lambda \sum_{n=1}^{\infty} P_n(t) S^n + \lambda \sum_{n=1}^{\infty} P_{n-1}(t) S^n \\ &= -\lambda \sum_{n=1}^{\infty} P_n(t) S^n + \lambda S \sum_{n=1}^{\infty} P_{n-1}(t) S^n \end{aligned} \tag{2.3}$$

Define

$$
\begin{aligned}
G(s,t) &= \sum_{n=0}^{\infty} P_n(t)S^n \\
&= P_0(t) + \sum_{n=1}^{\infty} P_n(t)S^n \\
\implies \frac{\delta G}{\delta t} G(s,t) &= P_0^{'}(t) + \sum_{n=1}^{\infty} P_n^{'}(t)S^n
\end{aligned}
$$

G(s,t) can also be defined by;

$$
G(s,t) = \sum_{n=1}^{\infty} P_{n-1}(t)S^{n-1}
$$

Therefore equation (2.3) becomes

$$
\begin{aligned}
\frac{\delta G}{\delta t} - P_0^{'}(t) &= -\lambda[G(s,t) - P_0(t)] + \lambda SG(s,t) \\
&= -\lambda G(s,t) + \lambda P_0(t) + \lambda SG(s,t)
\end{aligned}
$$

Using equation (2.1), we have

$$
\begin{aligned}
\frac{\delta G}{\delta t} + \lambda P_0(t) &= -\lambda G(s,t) + \lambda P_0(t) + \lambda SG(s,t) \\
\therefore \frac{\delta G}{\delta t} &= -\lambda(1-S)G(s,t) \\
\therefore \frac{1}{G(s,t)} \frac{\delta G(s,t)}{\delta t} &= -\lambda(1-S) \\
\therefore \frac{\delta}{\delta t} lnG(s,t) &= -\lambda(1-S) \\
\therefore lnG(s,t) &= -\lambda(1-S)t + C
\end{aligned}
$$

$$\implies G(s,t) = e^{-(1-S)\lambda t + C}$$

Given the initial condition as

$$X(0) = 0 \implies P_0(0) = 1 \quad and \quad P_n(0) = 0 \quad for \quad n \neq 0$$

When t=0,

$$G(s,0) = e^C$$

But by definition,

$$\begin{aligned}
G(s,t) &= \sum_{n=0}^{\infty} P_n(t) S^n \\
&= P_0(t) + \sum_{n=1}^{\infty} P_n(t) S^n \\
\therefore G(s,0) &= P_0(0) + \sum_{n=1}^{\infty} P_n(0) S^n \\
&= 1 + 0 \\
&= 1 \\
\therefore G(s,t) &= e^{-(1-S)\lambda t + C} = e^C e^{-(1-S)\lambda t}
\end{aligned}$$

and

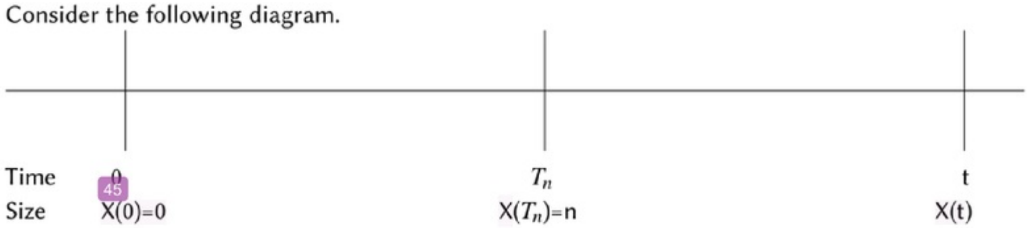
$$\begin{aligned}
G(s,0) = 1 &\implies e^C = 1 \implies c = 0 \\
\therefore G(s,t) &= e^{-\lambda t(1-s)} \qquad (2.4)
\end{aligned}$$

which is the pgf of a Poisson distribution with parameter λt; i.e

$$P_n(t) = \frac{e^{\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, 3, \ldots \qquad (2.5)$$

2.3 Waiting time distribution

Consider the following diagram.



$$
\begin{aligned}
t > T_n &\implies X(t) \geq X(T_n) \\
\therefore T_n < t &\implies X(t) \geq X(T_n) \\
T_n = t &\implies X(t) = X(T_n) \\
\therefore T_n \leq t &\implies X(t) \geq X(T_n) \\
\therefore Prob[T_n \leq t] &= Prob[X(t) \geq X(T_n)] \\
&= Prob[X(t) \geq n]
\end{aligned}
$$

Let

$$F_n(t) = Prob[T_n \leq t]$$

and since

$$P_n(t) = Prob[X(t) = n]$$

then

$$
\begin{aligned}
F_n(t) &= Prob[X(t) \geq n] \\
&= 1 - Prob[X(t) < n] \\
&= 1 - Prob[X(t) \leq n-1] \\
\therefore F_n(t) &= 1 - \sum_{j=0}^{n-1} P_j(t) \\
\therefore f_n(t) &= \frac{d}{dt} F_n(t) \\
&= -\sum_{j=0}^{n-1} \frac{d}{dt} P_j(t)
\end{aligned}
$$

For a Poisson process;

$$
\begin{aligned}
F_n(t) &= 1 - \sum_{j=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \\
f_n(t) &= -\sum_{j=0}^{n-1} \frac{1}{j!}\frac{d}{dt}e^{-\lambda t}(\lambda t)^j \\
&= -\sum_{j=0}^{n-1} \frac{1}{j!}[e^{-\lambda t} j(\lambda t)^{j-1}\lambda - \lambda e^{-\lambda t}(\lambda t)^j] \\
&= \sum_{j=0}^{n-1} \frac{1}{j!}[\lambda e^{-\lambda t}(\lambda t)^j - \lambda e^{-\lambda t} j(\lambda t)^{j-1}] \\
\therefore f_n(t) &= \lambda e^{-\lambda t}[\sum_{j=0}^{n-1} \frac{(\lambda t)^j}{j!} - \sum_{j=0}^{n-1} \frac{(\lambda t)^{j-1}}{(j-1)!}] \\
&= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \\
&= \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1}, \quad n = 1, 2, 3, .. \qquad \textbf{(2.6)}
\end{aligned}
$$

which is an Erlang distribution.

2.4 The first passage time distribution of a mixed Poisson Process

So,for a mixed Poisson process where n is fixed and λ is varying,the first passage time distribution becomes;

$$f_n(t) = \int_0^\infty \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} g(\lambda) d\lambda \qquad \textbf{(2.7)}$$

where g(λ) is a continuous mixing distribution.
This is an Erlang mixture which can be expressed in two ways,namely;
Method 1: **Direct method**

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} \int_0^\infty \lambda^n e^{-\lambda t} g(\lambda) d\lambda \\
&= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \qquad \textbf{(2.8)}
\end{aligned}
$$

We shall name this approach as the direct method.
Method 2: **Method of moments**

From method 1,

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n \sum_{k=0}^{\infty} \frac{(-\wedge t)^k}{k!}] \\ &= \frac{t^{n-1}}{\Gamma n} \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} E[\wedge^{n+k}] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{n+k-1}}{k!\Gamma n} E[\wedge^{n+k}] \end{aligned}$$

$$Let \quad n+k=j \quad \Longrightarrow k=j-n$$

$$f_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \tag{2.9}$$

Equating (2.8) and (2.9) we have the mathematical identity;

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} E(\wedge^j) = E[\wedge^n e^{-t\wedge}] \tag{2.10}$$

which has been proven below.

$$let \quad j-n=k \quad \Longrightarrow \quad j=n+k$$

$$\begin{aligned} \sum_{j-n=0}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} E(\wedge^j) &= \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} E(\wedge^{n+k}) \\ &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} E(\wedge^{n+k}) \\ &= E[\sum_{k=0}^{\infty} \frac{(-\wedge t)^k}{k!} \wedge^n] \\ &= E[\wedge^n e^{-t\wedge}] \end{aligned}$$

The rth moment of the Erlang mixture is given by;

$$\begin{aligned} E[T^r] &= EE[T^r/\wedge = \lambda] \\ &= E\int_0^\infty t^r f_n(t/\lambda)dt \\ &= E\int_0^\infty t^r \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} dt \\ &= E[\frac{\lambda^n}{\Gamma n}\int_0^\infty t^{n+r-1} e^{-\lambda t} dt] \\ &= E[\frac{\lambda^n}{\Gamma n}\frac{\Gamma(n+r)}{\lambda^{n+r}}] \\ E[T^r] &= \frac{\Gamma(n+r)}{\Gamma n} E(\frac{1}{\wedge})^r \end{aligned} \tag{2.11}$$

Thus, the rth moment of the Erlang mixture is expressed in terms of the rth moment of the reciprocal of the mixing distribution.
Therefore,

$$E(T) = nE(\frac{1}{\wedge}) \tag{2.12}$$

2.5 The Link Between an Erlang Mixture and an Exponential Mixture

$$\begin{aligned} f_n(t) &= \int_0^\infty \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} g(\lambda) d\lambda \\ \therefore f_1(t) &= \int_0^\infty \lambda e^{-\lambda t} g(\lambda) d\lambda \end{aligned} \tag{2.13}$$

which is the Exponential mixture and can be expressed as;

$$f_1(t) = E[\wedge e^{-t\wedge}] \tag{2.14}$$

and

$$f_1(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \tag{2.15}$$

The Identity is

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} E(\wedge^j) = E[\wedge e^{-t\wedge}] \tag{2.16}$$

The rth moment is

$$E(T^r) = r!E(\frac{1}{\wedge})^r \tag{2.17}$$

$$\therefore E(T) = E(\frac{1}{\wedge}) \tag{2.18}$$

2.6 The Link Between Erlang Mixture and Poisson Mixture

$$\begin{aligned} f_n(t) &= \int_0^{\infty} \frac{\lambda^n e^{-\lambda t} t^{n-1}}{\Gamma n} g(\lambda) d\lambda \\ &= \frac{n}{t} \int_0^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{\Gamma(n+1)} g(\lambda) d\lambda \\ &= \frac{n}{t} \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} g(\lambda) d\lambda \\ &= \frac{n}{t} P_n(t) \end{aligned} \tag{2.19}$$

where

$$P_n(t) = \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} g(\lambda) d\lambda \tag{2.20}$$

is a continuous Poisson mixture.

$$\therefore P_n(t) = \frac{t}{n} f_n(t), \quad n = 1, 2, 3, \ldots \tag{2.21}$$

Thus a Poisson mixture is $\frac{t}{n}$ times an Erlang mixture.
The factor $\frac{t}{n}$ transforms a continuous distribution to a discrete distribution.

The Poisson mixture $P_n(t)$ can be expressed as;

$$\begin{aligned} P_n(t) &= \frac{t}{n}\frac{t^{n-1}}{\Gamma n}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} P_n(t) &= \frac{t}{n}\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}E(\wedge^j) \\ &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j}}{n!(j-n)!}E(\wedge^j) \end{aligned} \quad (2.23)$$

The Identity is;

$$\begin{aligned} \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j}}{n!(j-n)!}E(\wedge^j) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\ \therefore \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}E(\wedge^j) &= E[\wedge^n e^{-t\wedge}] \end{aligned} \quad (2.24)$$

which is the same as (2.10) .

The Probability Generating Function (PGF) of the Poisson mixture is;

$$
\begin{aligned}
G(s,t) &= \sum_{n=0}^{\infty} P_n(t)S^n \\
&= \sum_{n=0}^{\infty} [\frac{t}{n} f_n(t)]S^n \\
&= \sum_{n=0}^{\infty} [\frac{t}{n} \int_0^{\infty} \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} g(\lambda) d\lambda] S^n \\
&= \sum_{n=0}^{\infty} \int_0^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} S^n g(\lambda) d\lambda \\
&= \int_0^{\infty} [e^{-\lambda t} (\sum_{n=0}^{\infty} \frac{(\lambda t S)^n}{n!}) g(\lambda) d\lambda] \\
&= \int_0^{\infty} e^{-\lambda t} e^{\lambda t S} g(\lambda) d\lambda \\
&= \int_0^{\infty} e^{-(1-s)\lambda t} g(\lambda) d\lambda \\
&= E[e^{-(1-s)t\wedge}] \\
G(s,t) &= L_{\wedge}[(1-s)t] && \textbf{(2.25)} \\
\frac{\sigma G}{\sigma S} &= \frac{\delta}{\delta S} E[e^{-t\wedge} e^{t\wedge S}] \\
&= E[t \wedge e^{-t\wedge} e^{t\wedge S}] && \textbf{(2.26)} \\
\frac{\sigma^2 G}{\sigma S^2} &= E[(t\wedge)^2 e^{-t\wedge} e^{t\wedge S}] && \textbf{(2.27)} \\
\frac{\sigma^r G}{\sigma S^r} &= E[(t\wedge)^r e^{-t\wedge} e^{t\wedge S}] && \textbf{(2.28)} \\
\frac{\sigma^r G(s,t)}{\sigma S^r} /_{s=1} &= E[t^r \wedge^r] && \textbf{(2.29)}
\end{aligned}
$$

i.e The rth factorial moment

We notice that the key unifying fuction in this work is

$$E[\wedge^n e^{-t\wedge}]$$

from which we can obtain

$$
\begin{aligned}
E[\wedge^j] &\quad when \quad n=j \quad and \quad t=0 \\
E[\wedge^r] &\quad when \quad n=r \quad and \quad t=0 \\
E[\wedge^{-r}] &\quad when \quad n=-r \quad and \quad t=0 \\
E[\wedge e^{-t\wedge}] &\quad when \quad n=1 \\
E[e^{-(1-s)t\wedge}] &\quad when \quad n=0 \quad and \quad t=(1-s)t
\end{aligned}
$$

So it is better to obtain E[∧n e-t∧] for a given mixing distribution g(λ), then obtain the other functions which are special cases.

3 ERLANG MIXTURES IN EXPLICIT FORM

3.1 Introduction

In this chapter Erlang Mixtures are expressed in Explicit form. They are obtained through direct integration.
Raw moments of the Erlang mixtures have been derived and specifically the first moment has been obtained.
The Exponential mixtures and Poisson mixtures have also been obtained and the PGFs determined in the Poisson mixtures.
Several mixing distributions have been used, they include, the Exponential, Gamma, Half-logistic, Lindley and Transmuted ditributions.

3.2 Erlang-Exponential Distribution

3.2.1 Erlang-Exponential Mixture

The Exponential mixing distribution is

$$g(\lambda) = \mu e^{-\mu\lambda}, \quad \lambda > 0; \quad \mu > 0 \tag{3.1}$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda} \mu e^{-\mu\lambda} \quad d\lambda \\
&= \mu \int_0^\infty \lambda^n e^{-\lambda(t+\mu)} d\lambda \\
&= \mu \frac{\Gamma(n+1)}{(t+\mu)^{n+1}} \\
&= \frac{\mu n \Gamma n}{(t+\mu)^{n+1}}
\end{aligned} \tag{3.2}$$

Construction by the direct method gives

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} \frac{\mu n \Gamma n}{(t+\mu)^{n+1}} \\
&= \frac{\mu n t^{n-1}}{(t+\mu)^{n+1}}, \quad t>0; \mu>0, n=1,2,3,\ldots
\end{aligned} \tag{3.3}
$$

By the method of moments we have

$$
\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\mu j \Gamma j}{\mu^{j+1}} \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{j!}{\mu^j}
\end{aligned} \tag{3.4}
$$

Identity 3.1

Equating the above two methods we get

$$
\begin{aligned}
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{j!}{\mu^j} &= \frac{n \mu t^{n-1}}{(t+\mu)^{n+1}} \\
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{j!}{\mu^j} &= \frac{\mu n \Gamma n}{(t+\mu)^{n+1}}
\end{aligned} \tag{3.5}
$$

The rth moment of the Erlang mixture is

$$
\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(1-r)}{\mu^{-r}} \\
&= \mu^r \frac{\Gamma(n+r)}{\Gamma n} \Gamma(1-r)
\end{aligned} \tag{3.6}
$$

$$
E(T) = \mu \frac{\Gamma(n+1)}{\Gamma n} \Gamma 0 = \infty \tag{3.7}
$$

3.2.2 Exponential-Exponential Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= \frac{\mu\Gamma 1}{(t+\mu)^2} \\ &= \frac{\mu}{(t+\mu)^2}, \quad t > 0; \mu > 0 \end{aligned} \tag{3.8}$$

which is a Pareto Distribution with parameters 1 and μ .
By the method of moments we have

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{j!}{\mu^j} \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1} j}{\mu^j} \end{aligned} \tag{3.9}$$

Identity 3.2
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1} j}{\mu^j} = \frac{\mu}{(t+\mu)^2} \tag{3.10}$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r! E(\wedge^{-r}) \\ &= r! \frac{\Gamma(1-r)}{\mu^{-r}} \\ &= r! \mu^r \Gamma(1-r) \end{aligned} \tag{3.11}$$

$$E(T) = \mu\Gamma 0 = \infty \tag{3.12}$$

3.2.3 Poisson-Exponential Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!} E(\wedge^n e^{-t\wedge}) \\ &= \frac{t^n}{n!} \frac{\mu n!}{(t+\mu)^{n+1}} \\ &= \frac{\mu t^n}{(t+\mu)^{n+1}} \\ &= (\frac{\mu}{t+\mu})(\frac{t}{t+\mu})^n, \quad t > 0; \mu > 0 \end{aligned} \tag{3.13}$$

which is a Geometric Distribution with parameter μ .

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \frac{j!}{\mu^j} \\ &= \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} (\frac{t}{\mu})^j \end{aligned} \tag{3.14}$$

Identity 3.3
Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} (\frac{t}{\mu})^j &= (\frac{\mu}{t+\mu})(\frac{t}{t+\mu})^n \\ \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n} j!}{(j-n)! \mu^j} &= \frac{\mu n!}{(t+\mu)^{n+1}} \end{aligned} \tag{3.15}$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= \frac{\mu}{(1-s)t+\mu} \end{aligned} \tag{3.16}$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r \frac{r!}{\mu^r} \\ &= r!(\frac{t}{\mu})^r \end{aligned} \tag{3.17}$$

$$E(T) = \frac{t}{\mu} \tag{3.18}$$

3.3 Erlang-One Parameter Gamma Distribution

3.3.1 Erlang-One Parameter Gamma Mixture

The One Parameter Gamma mixing distribution is

$$g(\lambda) = \frac{e^{-\lambda}\lambda^{\alpha-1}}{\Gamma\alpha} \quad ,\lambda > 0; \alpha > 0 \tag{3.19}$$

$$\begin{aligned} \therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda} \frac{e^{-\lambda}\lambda^{\alpha-1}}{\Gamma\alpha} d\lambda \\ &= \frac{1}{\Gamma\alpha} \int_0^\infty \lambda^{n+\alpha-1} e^{-\lambda(t+1)} d\lambda \\ &= \frac{1}{\Gamma\alpha} \frac{\Gamma(n+\alpha)}{(t+1)^{n+\alpha}} \\ &= \frac{\Gamma(n+\alpha)}{\Gamma\alpha} \frac{1}{(t+1)^{n+\alpha}} \end{aligned} \tag{3.20}$$

Construction by the direct method gives

$$\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} \frac{\Gamma(n+\alpha)}{\Gamma\alpha} \frac{1}{(t+1)^{n+\alpha}} \\
&= \frac{t^{n-1}}{(t+1)^{n+\alpha}} \frac{(n+\alpha-1)!}{(n-1)!(\alpha-1)!} \\
&= \frac{n}{t} \binom{n+\alpha-1}{n} (\frac{t}{t+1})^n (\frac{1}{t+1})^\alpha, \quad t > 0; \alpha > 0, n = 1, 2, 3, \ldots
\end{aligned} \tag{3.21}$$

By the method of moments we have

$$\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\Gamma(j+n)}{\Gamma\alpha}
\end{aligned} \tag{3.22}$$

Identity 3.4

Equating the above two methods we get

$$\begin{aligned}
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma\alpha} &= \frac{n}{t} \binom{\alpha+n-1}{n} (\frac{t}{t+1})^n (\frac{1}{t+1})^\alpha \\
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma\alpha} &= \frac{\Gamma(n+\alpha)}{\Gamma\alpha} \frac{1}{(t+1)^{n+\alpha}}
\end{aligned} \tag{3.23}$$

The rth moment of the Erlang mixture is

$$\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(\alpha-r)}{\Gamma\alpha}
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
E(T) &= \frac{\Gamma(n+1)}{\Gamma n} \frac{\Gamma(\alpha-1)}{\Gamma\alpha} \\
&= \frac{n}{\alpha-1}
\end{aligned} \tag{3.25}$$

3.3.2 Exponential-One Parameter Gamma Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= \frac{\Gamma(\alpha+1)}{\Gamma\alpha}\frac{1}{(t+1)^{\alpha+1}} \\ &= \frac{\alpha}{(t+1)^{\alpha+1}}, \quad t>0;\alpha>0 \end{aligned} \tag{3.26}$$

which is the Pareto Distribution with parameters α and 1.

By the method of moments we have

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\frac{\Gamma(j+\alpha)}{\Gamma\alpha} \\ f_1(t) &= \sum_{j=1}^{\infty}(-1)^{j-1}t^{j-1}j\binom{\alpha+j-1}{j} \end{aligned} \tag{3.27}$$

Identity 3.5
Equating the above two methods we get

$$\sum_{j=1}^{\infty}(-1)^{j-1}t^{j-1}j\binom{\alpha+j-1}{j} = \frac{\alpha}{(t+1)^{\alpha+1}} \tag{3.28}$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= r!\frac{\Gamma(\alpha-r)}{\Gamma\alpha} \end{aligned} \tag{3.29}$$

$$\begin{aligned} E(T) &= \frac{\Gamma(\alpha-1)}{\Gamma\alpha} \\ &= \frac{\Gamma(\alpha-1)}{(\alpha-1)\Gamma(\alpha-1)} \\ &= \frac{1}{\alpha-1} \end{aligned} \tag{3.30}$$

3.3.3 Poisson-One Parameter Gamma Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$\begin{aligned}
Pn(t) &= \frac{t^n}{n!} E(\wedge^n e^{-t\wedge}) \\
&= \frac{t^n}{n!} \frac{\Gamma(n+\alpha)}{\Gamma\alpha} \frac{1}{(t+1)^{n+\alpha}} \\
&= \frac{t^n}{(t+1)^{n+\alpha}} \binom{\alpha+n-1}{n} \\
&= \binom{\alpha+n-1}{n} (\frac{t}{t+1})^n (\frac{1}{t+1})^{\alpha}, \quad t>0; \alpha>0
\end{aligned} \tag{3.31}$$

which is the Negative Binomial Distribution with parameters α and 1.

By the method of moments we have

$$\begin{aligned}
P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j) \\
P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma\alpha}
\end{aligned} \tag{3.32}$$

Identity 3.6
Equating the above two methods we get

$$\begin{aligned}
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j \Gamma(j+\alpha)}{n!(j-n)!\Gamma\alpha} &= \binom{\alpha+n-1}{n} (\frac{t}{t+1})^n (\frac{1}{t+1})^{\alpha} \\
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n} \Gamma(j+\alpha)}{(j-n)!\Gamma\alpha} &= \frac{\Gamma(n+\alpha)}{\Gamma\alpha} (\frac{1}{t+1})^{n+\alpha}
\end{aligned} \tag{3.33}$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= \left(\frac{1}{(1-s)t+1}\right)^{\alpha} \end{aligned} \tag{3.34}$$

The rth moment of the Erlang mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r \frac{\Gamma(\alpha+r)}{\Gamma\alpha} \end{aligned} \tag{3.35}$$

$$\begin{aligned} E(T) &= t\frac{\Gamma(\alpha+1)}{\Gamma\alpha} \\ &= \alpha t \end{aligned} \tag{3.36}$$

3.4 Erlang-Type I Gamma Distribution and its Links

3.4.1 Erlang-Type I Gamma mixture

The Type I Gamma mixing distribution is

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma\alpha} e^{-\beta\lambda} \lambda^{\alpha-1} \quad ,\lambda > 0; \beta > 0, \alpha > 0 \tag{3.37}$$

$$\begin{aligned} \therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^{\infty} \lambda^n e^{-t\lambda} \frac{\beta^{\alpha}}{\Gamma\alpha} e^{-\beta\lambda} \lambda^{\alpha-1} d\lambda \\ &= \frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^{\infty} \lambda^{n+\alpha-1} e^{-\lambda(t+\beta)} d\lambda \\ &= \frac{\beta^{\alpha}}{\Gamma\alpha} \frac{\Gamma(n+\alpha)}{(t+\beta)^{n+\alpha}} \\ &= \frac{\Gamma(n+\alpha)}{\Gamma\alpha} \left(\frac{\beta}{t+\beta}\right)^{\alpha} \left(\frac{1}{t+\beta}\right)^{n} \end{aligned} \tag{3.38}$$

Construction by the direct method gives

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n}E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n}\frac{\Gamma(n+\alpha)}{\Gamma\alpha}(\frac{\beta}{t+\beta})^{\alpha}(\frac{1}{t+\beta})^{n} \\
&= \frac{n}{t}\binom{\alpha+n-1}{n}(\frac{\beta}{t+\beta})^{\alpha}(\frac{t}{t+\beta})^{n}, \quad t>0;\alpha>0,\beta>0,n=1,2,3,\ldots \qquad \textbf{(3.39)}
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}E(\wedge^j) \\
&= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}\frac{\Gamma(\alpha+j)}{\Gamma\alpha}(\frac{1}{\beta})^{j} \qquad \textbf{(3.40)}
\end{aligned}
$$

By Mcnolty's Approach
The Erlang mixture is given by

$$
f_n(t) = \int_0^{\infty}\frac{\lambda^{n+1}}{\Gamma(n+1)}e^{-\lambda t}t^n g(\lambda)d\lambda
$$

implying that

$$
\begin{aligned}
f_{n+1}(t) &= \int_0^{\infty}\frac{\lambda^{n+1}}{\Gamma(n+1)}e^{-\lambda t}t^n\frac{\alpha^{\beta}}{\Gamma\beta}e^{-\alpha\lambda}\lambda^{\beta-1}d\lambda \\
&= \frac{t^n}{\Gamma(n+1)}\frac{\alpha^{\beta}}{\Gamma\beta}\int_0^{\infty}\lambda^{(n+\beta+1)-1}e^{-\lambda(\alpha+t)}d\lambda \\
&= \frac{t^n}{\Gamma(n+1)}\frac{\alpha^{\beta}}{\Gamma\beta}\frac{\Gamma(n+\beta+1)}{(t+\alpha)^{n+\beta+1}} \\
&= \frac{t^n\alpha^{\beta}}{B(n+1,\beta)(t+\alpha)^{n+\beta+1}}, \quad t>0;\alpha>0,\beta>0
\end{aligned}
$$

This can be achieved by applying the method of moments as follows:-

From formula (3.40), we have

$$f_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\Gamma(\alpha+j)}{\Gamma\alpha} (\frac{1}{\beta})^j$$

$$Let \quad k = j-n \quad \implies \quad j = n+k$$

$$\therefore \quad f_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{n+k-1}}{k! \Gamma n} \frac{\Gamma(\alpha+n+k)}{\Gamma\alpha} (\frac{1}{\beta})^{n+k}$$

$$= \frac{t^{n-1}}{\Gamma n \beta^n \Gamma\alpha} \sum_{k=0}^{\infty} \frac{(-t)^k \Gamma(n+\alpha+k)}{k! \beta^k}$$

$$= \frac{t^{n-1}}{\Gamma n \beta^n \Gamma\alpha} \sum_{k=0}^{\infty} \frac{(\frac{-t}{\beta})^k \Gamma(n+\alpha+k)}{k!}$$

$$= \frac{t^{n-1} \Gamma(n+\alpha)}{\Gamma n \beta^n \Gamma\alpha} \sum_{k=0}^{\infty} \frac{(\frac{-t}{\beta})^k \Gamma(n+\alpha+k)}{k! \Gamma(n+\alpha)}$$

$$= \frac{t^{n-1}}{B(n,\alpha)\beta^n} \sum_{k=0}^{\infty} \binom{n+\alpha+k-1}{k} (\frac{-t}{\beta})^k$$

$$= \frac{t^{n-1}}{\beta^n B(n,\alpha)} \sum_{k=0}^{\infty} (-1)^k \binom{-(n+\alpha)}{k} (\frac{-t}{\beta})^k$$

$$= \frac{t^{n-1}}{\beta^n B(n,\alpha)} \sum_{k=0}^{\infty} \binom{-(n+\alpha)}{k} (\frac{t}{\beta})^k$$

$$= \frac{t^{n-1}}{\beta^n B(n,\alpha)} (1+\frac{t}{\beta})^{-(n+\alpha)}$$

$$= \frac{t^{n-1}}{\beta^n B(n,\alpha)} (\frac{\beta}{t+\beta})^{n+\alpha}$$

$$\therefore \quad f_{n+1}(t) = \frac{t^n}{\beta^{n+1} B(n+1,\alpha)} (\frac{\beta}{t+\beta})^{n+1+\alpha}$$

$$\therefore \quad f_{n+1}(t) = \frac{t^n}{\beta^{n+1}} \frac{\beta^{n+1+\alpha}}{(t+\beta)^{n+1+\alpha}} \frac{1}{B(n+1,\alpha)}$$

$$= \frac{t^n \beta^\alpha}{B(n+1,\alpha)(t+\beta)^{n+1+\alpha}}$$

Interchange α and β

$$\therefore \quad f_{n+1}(t) = \frac{t^n \alpha^\beta}{B(n+1,\beta)(t+\alpha)^{n+\beta+1}}$$

Identity 3.7

Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}\frac{\Gamma(\alpha+j)}{\Gamma\alpha}(\frac{1}{\beta})^{j}=\frac{n}{t}\binom{\alpha+n-1}{n}(\frac{\beta}{t+\beta})^{\alpha}(\frac{t}{t+\beta})^{n}$$
$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{\Gamma(\alpha+j)}{\Gamma\alpha}(\frac{1}{\beta})^{j}=\frac{\Gamma(n+\alpha)}{\Gamma\alpha}(\frac{\beta}{t+\beta})^{\alpha}(\frac{1}{t+\beta})^{n} \tag{3.41}$$

The rth moment of the Erlang mixture is

$$E(T^{r})=\frac{\Gamma(n+r)}{\Gamma n}E(\wedge^{-r})$$
$$=\frac{\Gamma(n+r)}{\Gamma n}\frac{\Gamma(\alpha-r)}{\Gamma\alpha}\beta^{r} \tag{3.42}$$
$$E(T)=\frac{\Gamma(n+1)}{\Gamma n}\frac{\Gamma(\alpha-1)}{\Gamma\alpha}\beta$$
$$=\frac{n\beta}{\alpha-1} \tag{3.43}$$

3.4.2 Exponential-Type I Gamma Mixture

Construction by the direct method gives

$$f_1(t)=E[\wedge e^{-t\wedge}]$$
$$=\frac{\Gamma(\alpha+1)}{\Gamma\alpha}(\frac{\beta}{t+\beta})^{\alpha}(\frac{1}{t+\beta})$$
$$=\alpha(\frac{\beta}{t+\beta})^{\alpha}(\frac{1}{t+\beta})$$
$$=\frac{\alpha\beta^{\alpha}}{(t+\beta)^{\alpha+1}},\quad t>0;\alpha>0,\beta>0 \tag{3.44}$$

which is Pareto II (Lomax) distribution with parameters (α,β).

By the method of moments we have

$$f_1(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} E(\wedge^j)$$
$$= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} \frac{\Gamma(\alpha+j)}{\beta^j \Gamma\alpha} \tag{3.45}$$

Identity 3.8
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} \frac{\Gamma(\alpha+j)}{\beta^j \Gamma\alpha} = \frac{\alpha\beta^{\alpha}}{(t+\beta)^{\alpha+1}} \tag{3.46}$$

The rth moment of the Exponential mixture is

$$E(T^r) = r!E(\wedge^{-r})$$
$$= r!\beta^r \frac{\Gamma(\alpha-r)}{\Gamma\alpha} \tag{3.47}$$
$$E(T) = \beta \frac{\Gamma(\alpha-1)}{\Gamma\alpha}$$
$$= \frac{\beta}{\alpha-1} \tag{3.48}$$

3.4.3 Poisson-Type I Gamma Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$P_n(t) = \frac{t^n}{n!} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^n}{n!} \frac{\Gamma(n+\alpha)}{\Gamma\alpha} (\frac{\beta}{t+\beta})^{\alpha} (\frac{1}{t+\beta})^n$$
$$= \binom{\alpha+n-1}{n} (\frac{\beta}{t+\beta})^{\alpha} (\frac{t^n}{t+\beta})^n, \quad t>0; \alpha>0, \beta>0 \tag{3.49}$$

which is the negative binomial distribution with parameters α and β.
By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{n!(j-n)!} E(\wedge^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{n!(j-n)!} \frac{\Gamma(\alpha+j)}{\beta^j \Gamma\alpha} \end{aligned} \tag{3.50}$$

Identity 3.9
Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{n!(j-n)!} \frac{\Gamma(\alpha+j)}{\beta^j \Gamma\alpha} &= t^n \binom{\alpha+n-1}{n} (\frac{\beta}{t+\beta})^{\alpha} (\frac{1}{t+\beta})^n \\ \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{\Gamma(\alpha+j)}{\beta^j \Gamma\alpha} &= \frac{\Gamma(n+\alpha)}{\alpha} (\frac{\beta}{t+\beta})^{\alpha} (\frac{1}{t+\beta})^n \end{aligned} \tag{3.51}$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= [\frac{\beta}{\beta+(1-s)t}]^{\alpha} \end{aligned} \tag{3.52}$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r \frac{\Gamma(\alpha+r)}{\beta^r \Gamma\alpha} \\ &= (\frac{t}{\beta})^r \frac{\Gamma(\alpha+r)}{\Gamma\alpha} \end{aligned} \tag{3.53}$$

$$\begin{aligned} E(T) &= (\frac{t}{\beta}) \frac{\Gamma(\alpha+1)}{\Gamma\alpha} \\ &= \frac{\alpha t}{\beta} \end{aligned} \tag{3.54}$$

3.5 Erlang-Type II Gamma Distribution and its Links

3.5.1 Erlang-Type II Gamma Mixture

The Type II Gamma mixing distribution is

$$g(\lambda) = \frac{1}{\beta^{\alpha}\Gamma\alpha} e^{\frac{-\lambda}{\beta}} \lambda^{\alpha-1}, \quad \lambda > 0; \beta > 0, \alpha > 0 \tag{3.55}$$

$$\begin{aligned}
\therefore \quad E[\wedge^{n} e^{-t\wedge}] &= \int_{0}^{\infty} \lambda^{n} e^{-t\lambda} \frac{e^{\frac{-\lambda}{\beta}} \lambda^{\alpha-1}}{\beta^{\alpha}\Gamma\alpha} d\lambda \\
&= \frac{1}{\beta^{\alpha}\Gamma\alpha} \int_{0}^{\infty} \lambda^{n+\alpha-1} e^{-\lambda(t+\frac{1}{\beta})} d\lambda \\
&= \frac{1}{\beta^{\alpha}\Gamma\alpha} \frac{\Gamma(n+\alpha)}{(t+\frac{1}{\beta})^{n+\alpha}} \\
&= \frac{\Gamma(n+\alpha)}{\Gamma\alpha} \frac{(\frac{1}{\beta})^{\alpha}}{(t+\frac{1}{\beta})^{n+\alpha}} \\
&= \frac{\Gamma(n+\alpha)}{\Gamma\alpha} \left(\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}}\right)^{\alpha} \left(\frac{1}{t+\frac{1}{\beta}}\right)^{n}
\end{aligned} \tag{3.56}$$

Constuction by the direct method gives

$$\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^{n} e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} \frac{\Gamma(n+\alpha)}{\Gamma\alpha} \left(\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}}\right)^{\alpha} \left(\frac{1}{t+\frac{1}{\beta}}\right)^{n} \\
&= \frac{n}{t} \binom{\alpha+n-1}{n} \left(\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}}\right)^{\alpha} \left(\frac{t}{t+\frac{1}{\beta}}\right)^{n}, \quad t > 0; \beta > 0, \alpha > 0, n = 1, 2, 3, \ldots
\end{aligned} \tag{3.57}$$

By the method of moments we have

$$\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^{j}) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\Gamma(\alpha+j)}{\Gamma\alpha} \beta^{j}
\end{aligned} \tag{3.58}$$

Identity 3.10
Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\Gamma(\alpha+j)}{\Gamma\alpha} \beta^j = \frac{n}{t} \binom{\alpha+n-1}{n} (\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}})^{\alpha} (\frac{t}{t+\frac{1}{\beta}})^n$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{\Gamma(\alpha+j)}{\Gamma\alpha} \beta^j = \frac{\Gamma(n+\alpha)}{\Gamma\alpha} (\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}})^{\alpha} (\frac{1}{t+\frac{1}{\beta}})^n \quad (3.59)$$

The rth moment of the Erlang mixture is

$$\begin{aligned} E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\ &= \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(\alpha-r)}{\Gamma\alpha} \frac{1}{\beta^r} \quad (3.60) \\ E(T) &= \frac{\Gamma(n+1)}{\Gamma n} \frac{\Gamma(\alpha-1)}{\Gamma\alpha} \frac{1}{\beta} \\ &= \frac{n}{\beta(\alpha-1)} \quad (3.61) \end{aligned}$$

3.5.2 Exponential-Type II Gamma Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= \frac{\Gamma(\alpha+1)}{\Gamma\alpha} (\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}})^{\alpha} (\frac{1}{t+\frac{1}{\beta}}) \\ &= \frac{\alpha(\frac{1}{\beta})^{\alpha}}{(t+\frac{1}{\beta})^{\alpha+1}}, \quad t>0; \alpha>0, \beta>0 \quad (3.62) \end{aligned}$$

which is the Pareto II (Lomax) distribution with parameters $(\alpha, \frac{1}{\beta})$

By the method of moments we have

$$f_1(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j)$$
$$= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{\Gamma(\alpha+j)}{\Gamma\alpha} \beta^j \quad (3.63)$$

Identity 3.11
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{\Gamma(\alpha+j)}{\Gamma\alpha} \beta^j = \frac{\alpha(\frac{1}{\beta})^{\alpha}}{(t+\frac{1}{\beta})^{\alpha+1}} \quad (3.64)$$

The rth moment of the Exponential mixture is

$$E(T^r) = r! E(\wedge^{-r})$$
$$= \frac{r!}{\beta^r} \frac{\Gamma(\alpha-r)}{\Gamma\alpha} \quad (3.65)$$
$$E(T) = \frac{1}{\beta} \frac{\Gamma(\alpha-1)}{\Gamma\alpha}$$
$$= \frac{1}{\beta(\alpha-1)} \quad (3.66)$$

3.5.3 Poisson-Type II Gamma Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$P_n(t) = \frac{t^n}{n!} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^n}{n!} \frac{\Gamma(n+\alpha)}{\Gamma\alpha} (\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}})^{\alpha} (\frac{1}{t+\frac{1}{\beta}})^n$$
$$= \binom{\alpha+n-1}{n} (\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}})^{\alpha} (\frac{t^n}{t+\frac{1}{\beta}})^n, \quad t>0; \alpha>0, \beta>0 \tag{3.67}$$

which is the negative binomial distribution with parameters α and $\frac{1}{\beta}$
By the method of moments we have

$$P_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \beta^j \frac{\Gamma(\alpha+j)}{\Gamma\alpha} \tag{3.68}$$

Identity 3.12
Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \beta^j \frac{\Gamma(\alpha+j)}{\Gamma\alpha} = \binom{\alpha+n-1}{n} (\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}})^{\alpha} (\frac{t^n}{t+\frac{1}{\beta}})^n$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \beta^j \frac{\Gamma(\alpha+j)}{\Gamma\alpha} = \frac{\Gamma(n+\alpha)}{\Gamma\alpha} (\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}})^{\alpha} (\frac{1}{t+\frac{1}{\beta}})^n \tag{3.69}$$

The PGF of the Poisson mixture is

$$G(s,t) = E[e^{-(1-s)t\wedge}]$$
$$= (\frac{\frac{1}{\beta}}{\frac{1}{\beta}+(1-s)t})^{\alpha} \tag{3.70}$$

The rth moment of the Poisson mixture is

$$E(T^r) = t^r E(\wedge^r)$$
$$= (t\beta)^r \frac{\Gamma(\alpha+r)}{\Gamma\alpha} \tag{3.71}$$
$$E(T) = (t\beta)\frac{\Gamma(\alpha+1)}{\Gamma\alpha}$$
$$= \alpha t\beta \tag{3.72}$$

3.6 Erlang-Shifted Gamma Distribution and Its Links

3.6.1 Erlang-Shifted Gamma Mixture

A two parameter Gamma distribution is given by;

$$f(x) = \frac{\beta^\alpha}{\Gamma\alpha} e^{-\beta x} x^{\alpha-1}, \quad x > 0; \alpha > 0, \beta > 0$$
$$let \quad x = y - \mu \quad \Longrightarrow y = x + \mu \quad \Longrightarrow dy = dx$$

Using Jacobian transformation;

$$g(y) = f(x)|\frac{dy}{dx}|$$
$$= \frac{\beta^\alpha}{\Gamma\alpha} e^{-\beta x} x^{\alpha-1}|1|$$
$$= \frac{\beta^\alpha}{\Gamma\alpha} e^{-\beta(y-\mu)}(y-\mu)^{\alpha-1}, \quad y > \mu; \alpha > 0, \beta > 0, \mu > 0$$

Replacing y with λ, we have the Shifted Gamma mixing distribution which is;

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma\alpha} e^{-\beta(\lambda-\mu)}(\lambda-\mu)^{\alpha-1}, \quad \lambda > \mu; \alpha > 0, \beta > 0, \mu > 0 \tag{3.73}$$

$$
\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_{\mu}^{\infty} \lambda^n e^{-t\lambda} \frac{\beta^{\alpha}}{\Gamma\alpha} e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1} d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma\alpha} \int_{\mu}^{\infty} \lambda^n (\lambda-\mu)^{\alpha-1} e^{-\lambda(t+\beta)+\beta\mu} d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma\alpha} e^{\beta\mu} \int_{\mu}^{\infty} \lambda^n (\lambda-\mu)^{\alpha-1} e^{-(\lambda-\mu+\mu)(t+\beta)} d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma\alpha} e^{-(\mu t)} \int_{\mu}^{\infty} \lambda^n (\lambda-\mu)^{\alpha-1} e^{-(\lambda-\mu)(t+\beta)} d\lambda
\end{aligned}
$$

$$
\begin{aligned}
let \quad & x = (\lambda-\mu)(t+\beta) \implies (\lambda-\mu) = \frac{x}{t+\beta} \implies \lambda = \frac{x}{t+\beta} + \mu \\
& dx = (t+\beta) d\lambda \implies d\lambda = \frac{dx}{t+\beta}
\end{aligned}
$$

$$
\begin{aligned}
E[\wedge^n e^{-t\wedge}] &= \frac{\beta^{\alpha}}{\Gamma\alpha} e^{-\mu t} \int_0^{\infty} \left(\frac{x+\mu(t+\beta)}{t+\beta}\right)^n \left(\frac{x}{t+\beta}\right)^{\alpha-1} e^{-x} \frac{dx}{t+\beta} \\
&= \frac{\beta^{\alpha} e^{-\mu t}}{\Gamma\alpha (t+\beta)^{n+\alpha}} \int_0^{\infty} x^{\alpha-1} [\mu(t+\beta)(1+\frac{x}{\mu(t+\beta)})]^n e^{-x} dx \\
&= \frac{\beta^{\alpha} e^{-\mu t} \mu^n (t+\beta)^n}{\Gamma\alpha (t+\beta)^{n+\alpha}} \int_0^{\infty} e^{-x} x^{\alpha-1} [1+\frac{x}{\mu(t+\beta)}]^n dx \\
&= \frac{\beta^{\alpha} e^{-\mu t} \mu^n}{\Gamma\alpha (t+\beta)^{\alpha}} \int_0^{\infty} e^{-x} x^{\alpha-1} [\sum_{k=0}^{n} \binom{n}{k} (\frac{x}{\mu(t+\beta)})^k] dx \\
&= \frac{\beta^{\alpha} e^{-\mu t} \mu^n}{\Gamma\alpha (t+\beta)^{\alpha}} \sum_{k=0}^{n} [\binom{n}{k} (\frac{1}{\mu(t+\beta)})^k \int_0^{\infty} e^{-x} x^{k+\alpha-1} dx] \\
&= \frac{\beta^{\alpha} e^{-\mu t} \mu^n}{\Gamma\alpha (t+\beta)^{\alpha}} \sum_{k=0}^{n} \binom{n}{k} (\frac{\Gamma(\alpha+k)}{\mu^k (t+\beta)^k}) \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} e^{-\mu t} \mu^{n-k} (\frac{\beta}{t+\beta})^{\alpha} (\frac{1}{t+\beta})^k \qquad \textbf{(3.74)}
\end{aligned}
$$

Construction by the direct method gives

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^{n-1}}{\Gamma n} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} e^{-\mu t} \mu^{n-k} (\frac{\beta}{t+\beta})^{\alpha} (\frac{1}{t+\beta})^{k} \\ &= \sum_{k=0}^{n} \binom{\alpha+k-1}{k} \frac{ne^{-\mu t}}{(n-k)!} \frac{(t\mu)^n}{t\mu^k} (\frac{\beta}{t+\beta})^{\alpha} (\frac{1}{t+\beta})^{k} \\ &= \frac{n}{t} \sum_{k=0}^{n} (\frac{e^{-\mu t}(\mu t)^{n-k}}{(n-k)!}) \binom{\alpha+k-1}{k} (\frac{\beta}{t+\beta})^{\alpha} (\frac{t}{t+\beta})^{k}, \quad t>0; \alpha>0, \beta>0, \mu>0, n=1,2,3,... \end{aligned} \tag{3.75}$$

By the method of moments we have

$$\begin{aligned} f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \sum_{k=o}^{j} \binom{j}{k} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} \frac{\mu^j}{(\mu\beta)^k} \end{aligned} \tag{3.76}$$

Identity 3.13
Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \sum_{k=o}^{j} \binom{j}{k} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} \frac{\mu^j}{(\mu\beta)^k} &= \frac{n}{t} \sum_{k=0}^{n} (\frac{e^{-\mu t}(\mu t)^{n-k}}{(n-k)!}) \binom{\alpha+k-1}{k} (\frac{\beta}{t+\beta})^{\alpha} (\frac{t}{t+\beta})^{k} \\ \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \sum_{k=o}^{j} \binom{j}{k} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} \frac{\mu^j}{(\mu\beta)^k} &= \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} e^{-\mu t} \mu^{n-k} (\frac{\beta}{t+\beta})^{\alpha} (\frac{1}{t+\beta})^{k} \end{aligned} \tag{3.77}$$

The rth moment of the Erlang mixture is

$$E(T^r) = \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r})$$
$$= \frac{\Gamma(n+r)}{\Gamma n} \sum_{k=0}^{\infty} \binom{-r}{k} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} \frac{\mu^{-r}}{(\mu\beta)^k}$$
$$= \sum_{k=0}^{\infty} \binom{-r}{k} \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} \frac{1}{(\mu\beta)^k \mu^r} \quad (3.78)$$
$$E(T) = \sum_{k=0}^{\infty} \binom{-1}{k} \frac{\Gamma(n+1)}{\Gamma n} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} \frac{1}{(\mu\beta)^k \mu}$$
$$= \sum_{k=0}^{\infty} \binom{-1}{k} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} \frac{n}{\mu} \frac{1}{(\mu\beta)^k} \quad (3.79)$$

3.6.2 Exponential-Shifted Gamma Mixture

Construction by the direct method gives

$$f_1(t) = E[\wedge e^{-t\wedge}]$$
$$= \sum_{k=0}^{1} \binom{1}{k} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} e^{-\mu t} \mu^{1-k} (\frac{\beta}{t+\beta})^{\alpha} (\frac{1}{t+\beta})^k$$
$$= e^{-\mu t} \mu (\frac{\beta}{t+\beta})^{\alpha} + e^{-\mu t} \frac{\Gamma(\alpha+1)}{\Gamma\alpha} (\frac{\beta}{t+\beta})^{\alpha} (\frac{1}{t+\beta})$$
$$= e^{-\mu t} (\frac{\beta}{t+\beta})^{\alpha} [\mu + \frac{\alpha}{t+\beta}], \quad t > 0; \alpha > 0, \beta > 0, \mu > 0 \quad (3.80)$$

By the method of moments we have

$$f_1(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j)$$
$$= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \sum_{k=o}^{j} \binom{j}{k} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} \frac{\mu^j}{(\mu\beta)^k} \quad (3.81)$$

Identity 3.15
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \sum_{k=o}^{j} \binom{j}{k} \frac{\Gamma(\alpha+k)}{\Gamma\alpha} \frac{\mu^j}{(\mu\beta)^k} = e^{-\mu t} (\frac{\beta}{t+\beta})^{\alpha} [\mu + \frac{\alpha}{t+\beta}] \quad (3.82)$$

The rth moment of the Exponential mixture is

$$E(T^r) = r!E(\wedge^{-r})$$
$$= r!\sum_{k=o}^{\infty}\binom{-r}{k}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}\frac{\mu^{-r}}{(\mu\beta)^k}$$
$$= \sum_{k=o}^{\infty}\binom{-r}{k} \quad r!\frac{\Gamma(\alpha+k)}{\Gamma\alpha}\frac{1}{(\mu\beta)^k\mu^r} \tag{3.83}$$
$$E(T) = \sum_{k=o}^{\infty}\binom{-1}{k}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}\frac{1}{(\mu\beta)^k\mu} \tag{3.84}$$

3.6.3 Poisson-Shifted Gamma Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$P_n(t) = \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}e^{-\mu t}\mu^{n-k}(\frac{\beta}{t+\beta})^{\alpha}(\frac{1}{t+\beta})^k$$
$$= \sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}\frac{e^{-\mu t}(\mu t)^n}{n!}(\frac{\beta}{t+\beta})^{\alpha}(\frac{1}{\mu(t+\beta)})^k$$
$$= \sum_{k=0}^{n}\frac{e^{-\mu t}(\mu t)^{n-k}}{(n-k)!}\binom{\alpha+k-1}{k}(\frac{\beta}{t+\beta})^{\alpha}(\frac{t}{\mu(t+\beta)})^k, \quad t>0;\alpha>0,\beta>0,\mu>0 \tag{3.85}$$

which is a Delaporte distribution. By the method of moments we have

$$P_n(t) = \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j)$$
$$= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\sum_{k=o}^{j}\binom{j}{k}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}\frac{\mu^j}{(\mu\beta)^k} \tag{3.86}$$

Identity 3.15
Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\sum_{k=o}^{j}\binom{j}{k}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}\frac{\mu^j}{(\mu\beta)^k}=\sum_{k=0}^{n}\frac{e^{-\mu t}(\mu t)^{n-k}}{(n-k)!}\binom{\alpha+k-1}{k}(\frac{\beta}{t+\beta})^{\alpha}(\frac{t}{\mu(t+\beta)})^k$$
$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\sum_{k=o}^{j}\binom{j}{k}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}\frac{\mu^j}{(\mu\beta)^k}=\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}e^{-\mu t}\mu^{n-k}(\frac{\beta}{t+\beta})^{\alpha}(\frac{1}{t+\beta})^k \quad (3.87)$$

The PGF of the Poisson mixture is

$$\begin{aligned}G(s,t)&=E[e^{-(1-s)t\wedge}]\\&=\sum_{k=0}^{0}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}e^{-\mu t(1-s)}\mu^{-k}(\frac{\beta}{\beta+(1-s)t})^{\alpha}(\frac{1}{\beta+(1-s)t})^k\\&=e^{-\mu(1-s)t}(\frac{\beta}{\beta+(1-s)t})^{\alpha}\end{aligned} \quad (3.88)$$

The rth moment of the Poisson mixture is

$$\begin{aligned}E(T^r)&=t^rE(\wedge^r)\\&=t^r\sum_{k=o}^{r}\binom{r}{k}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}\frac{\mu^r}{(\mu\beta)^k}\\&=\sum_{k=o}^{r}\binom{r}{k}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}\frac{(\mu t)^r}{(\mu\beta)^k}\end{aligned} \quad (3.89)$$

$$\begin{aligned}E(T)&=\sum_{k=o}^{1}\binom{1}{k}\frac{\Gamma(\alpha+k)}{\Gamma\alpha}\frac{(\mu t)}{(\mu\beta)^k}\\&=\mu t+\frac{\Gamma(\alpha+1)}{\Gamma\alpha}\frac{\mu t}{\mu\beta}\\&=\mu t(1+\frac{\alpha}{\mu\beta})\end{aligned} \quad (3.90)$$

3.7 Erlang-Half logistic Distribution and Its Links

3.7.1 Erlang-Half logistic mixture

The Half logistic mixing distribution is

$$g(\lambda) = \frac{2\mu e^{-\mu\lambda}}{(1+e^{-\mu\lambda})^2}, \quad \lambda > 0; \mu > 0 \tag{3.91}$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda} \frac{2\mu e^{-\mu\lambda}}{(1+e^{-\mu\lambda})^2} d\lambda \\
&= 2\mu \int_0^\infty \lambda^n e^{-\lambda(t+\mu)} (1+e^{-\mu\lambda})^{-2} d\lambda \\
&= 2\mu \int_0^\infty \lambda^n e^{-\lambda(t+\mu)} [\sum_{k=0}^\infty \binom{-2}{k} (e^{-\mu\lambda})^k] d\lambda \\
&= 2\mu \sum_{k=0}^\infty \binom{-2}{k} [\int_0^\infty \lambda^n e^{-\lambda(t+\mu+\mu k)} d\lambda] \\
&= 2\mu \sum_{k=0}^\infty \binom{-2}{k} \frac{\Gamma(n+1)}{((t+\mu+\mu k)^{n+1}} \\
&= 2\mu n! \sum_{k=0}^\infty \binom{-2}{k} [\frac{1}{t+\mu(1+k)}]^{n+1}
\end{aligned} \tag{3.92}$$

Construction by the direct method gives

$$\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} 2\mu n! \sum_{k=0}^\infty \binom{-2}{k} [\frac{1}{t+\mu(1+k)}]^{n+1} \\
&= \sum_{k=0}^\infty \binom{-2}{k} \frac{2\mu n t^{n-1}}{[t+\mu(1+k)]^{n+1}}, \quad t > 0; \mu > 0, n = 1,2,3,..
\end{aligned} \tag{3.93}$$

By the method of moments we have

$$\begin{aligned}
f_n(t) &= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} 2\mu j! \sum_{k=0}^\infty \binom{-2}{k} \frac{1}{[\mu(1+k)]^{j+1}}
\end{aligned} \tag{3.94}$$

Identity 3.16
Equating the above two methods we get

$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n(j-n)!} 2\mu j! \sum_{k=0}^{\infty} \binom{-2}{k} \frac{1}{[\mu(1+k)]^{j+1}} \quad = 2\mu n \sum_{k=0}^{\infty} \binom{-2}{k} \frac{t^{n-1}}{[t+\mu(1+k)]^{n+1}}$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} 2\mu j! \sum_{k=0}^{\infty} \binom{-2}{k} \frac{1}{[\mu(1+k)]^{j+1}} \quad = 2\mu n! \sum_{k=0}^{\infty} \binom{-2}{k} [\frac{1}{t+\mu(1+k)}]^{n+1} \tag{3.95}$$

The rth moment of the Erlang mixture is

$$\begin{aligned} E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\ &= \frac{\Gamma(n+r)}{\Gamma n} 2\mu\Gamma(1-r) \sum_{k=0}^{\infty} \binom{-2}{k} \frac{1}{[\mu(1+k)]^{1-r}} \\ &= \sum_{k=0}^{\infty} \binom{-2}{k} 2\mu \frac{\Gamma(n+r)\Gamma(1-r)}{\Gamma n[\mu(1+k)]^{1-r}} \end{aligned} \tag{3.96}$$

$$\begin{aligned} E(T) &= \sum_{k=0}^{\infty} \binom{-2}{k} 2\mu \frac{\Gamma(n+1)\Gamma(0)}{\Gamma n} \\ &= \infty \end{aligned} \tag{3.97}$$

3.7.2 Exponential-Half logistic Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= \sum_{k=0}^{\infty} \binom{-2}{k} \frac{2\mu}{[t+\mu(1+k)]^2}, \quad t>0; \mu>0 \end{aligned} \tag{3.98}$$

By the method of moments we have

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} 2\mu j! \sum_{k=0}^{\infty} \binom{-2}{k} \frac{1}{[\mu(1+k)]^{j+1}} \end{aligned} \tag{3.99}$$

Identity 3.17

Equating the above two methods we get

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}2\mu j!\sum_{k=0}^{\infty}\binom{-2}{k}\frac{1}{[\mu(1+k)]^{j+1}} = 2\mu\sum_{k=0}^{\infty}\binom{-2}{k}[\frac{1}{t+\mu(1+k)}]^2 \quad (3.100)$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= r!\sum_{k=0}^{\infty}\binom{-2}{k}\frac{2\mu\Gamma(1-r)}{[\mu(1+k)]^{1-r}} \quad (3.101) \end{aligned}$$

$$E(T) = \sum_{k=0}^{\infty}\binom{-2}{k}2\mu\Gamma 0 = \infty \quad (3.102)$$

3.7.3 Poisson-Half logistic mixture

$$P_n(t) = \frac{t}{n}fn(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!}2\mu n!\sum_{k=0}^{\infty}\binom{-2}{k}[\frac{1}{t+\mu(1+k)}]^{n+1} \\ &= \sum_{k=0}^{\infty}\binom{-2}{k}\frac{2\mu t^n}{[t+\mu(1+k)]^{n+1}}, \quad t>0;\mu>0 \quad (3.103) \end{aligned}$$

By the method of moments we have

$$Pn(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \sum_{k=0}^{\infty} \binom{-2}{k} \frac{2\mu j!}{[\mu(1+k)]^{j+1}}$$
$$= \sum_{j=n}^{\infty} (-1)^{j-n} t^j \binom{j}{n} \sum_{k=0}^{\infty} \binom{-2}{k} \frac{2\mu}{[\mu(1+k)]^{j+1}} \quad \textbf{(3.104)}$$

Identity 3.18

Equating the above two methods we get

$$\sum_{j=n}^{\infty} (-1)^{j-n} t^j \binom{j}{n} \sum_{k=0}^{\infty} \binom{-2}{k} \frac{2\mu}{[\mu(1+k)]^{j+1}} = \sum_{k=0}^{\infty} \binom{-2}{k} \frac{2\mu t^n}{[t+\mu(1+k)]^{n+1}}$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \sum_{k=0}^{\infty} \binom{-2}{k} \frac{2\mu j!}{[\mu(1+k)]^{j+1}} = \sum_{k=0}^{\infty} \binom{-2}{k} \frac{2\mu n!}{[t+\mu(1+k)]^{n+1}} \quad \textbf{(3.105)}$$

The PGF of the Poisson mixture is

$$G(s,t) = E[e^{-(1-s)t\wedge}]$$
$$= \sum_{k=0}^{\infty} \binom{-2}{k} \frac{2\mu}{(1-s)t+(1+k)\mu} \quad \textbf{(3.106)}$$

The rth moment of the Poisson mixture is

$$E(T^r) = t^r E(\wedge^r)$$
$$= t^r \sum_{k=0}^{\infty} \binom{-2}{k} \frac{2\mu r!}{[\mu(1+k)]^{r+1}}$$
$$= \sum_{k=0}^{\infty} \binom{-2}{k} (\frac{t}{\mu})^r \frac{2r!}{[(1+k)]^{r+1}} \quad \textbf{(3.107)}$$
$$E(T) = \sum_{k=0}^{\infty} \binom{-2}{k} (\frac{t}{\mu}) \frac{2}{[(1+k)]^2} \quad \textbf{(3.108)}$$

3.8 Four Parameter Generalized Lindley (G4L) Distribution and Its Special Cases

3.8.1 Construction Based on a finite mixture of two Gamma distributions and a parameterization

A finite mixture is defined as

$$g(\lambda) = \sum_{i=1}^{k} \omega_i g_i(\lambda), \quad where \quad \sum \omega_i = 1, \omega_i > 0 \tag{3.109}$$

Let

$$g(\lambda) = \omega g_1(\lambda) + (1-\omega) g_2(\lambda) \tag{3.110}$$

be a finite mixture, where ω + (1-ω) = 1, ω>0, (1-ω) >0.
Let

$$\begin{aligned} g_1(\lambda) &\sim Gamma(\alpha, \theta) \\ g_2(\lambda) &\sim Gamma(\alpha+1, \theta) \end{aligned}$$

$$\therefore \quad g(\lambda) = \omega \frac{\theta^{\alpha} e^{-\theta\lambda} \lambda^{\alpha-1}}{\Gamma\alpha} + (1-\omega) \frac{\theta^{\alpha+1} e^{-\theta\lambda} \lambda^{\alpha}}{\Gamma(\alpha+1)}$$

Let

$$\omega = \frac{\theta}{\theta+r} \quad \Longrightarrow \quad (1-\omega) = \frac{r}{\theta+r}$$

$$\begin{aligned} g(\lambda) &= \frac{\theta}{\theta+r} \frac{\theta^{\alpha} e^{-\theta\lambda} \lambda^{\alpha-1}}{\Gamma\alpha} + \frac{r}{\theta+r} \frac{\theta^{\alpha+1} e^{-\theta\lambda} \lambda^{\alpha}}{\Gamma(\alpha+1)} \\ &= \frac{\theta^{\alpha+1}}{\theta+r} e^{-\theta\lambda} [\frac{\lambda^{\alpha-1}}{\Gamma\alpha} + \frac{r\lambda^{\alpha}}{\Gamma(\alpha+1)}] \\ &= \frac{\theta^{\alpha+1}}{\theta+r} e^{-\theta\lambda} \frac{\lambda^{\alpha-1}}{\Gamma(\alpha+1)} [\alpha + r\lambda] \end{aligned}$$

Let r=$\frac{\delta}{\beta}$

$$
\begin{aligned}
g(\lambda) &= \frac{\theta^{\alpha+1}}{\theta+\frac{\delta}{\beta}} e^{-\theta\lambda} \frac{\lambda^{\alpha-1}}{\Gamma(\alpha+1)}[\alpha+\frac{\delta}{\beta}\lambda] \\
&= \frac{\theta^{\alpha+1}}{\beta\theta+\delta}[\beta\alpha+\delta\lambda] e^{-\theta\lambda} \frac{\lambda^{\alpha-1}}{\Gamma(\alpha+1)} \\
&= \frac{\theta^{\alpha+1}[\beta\alpha+\delta\lambda]}{\beta\theta+\delta} \frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)}, \quad \lambda>0;\theta>0,\alpha>0,\beta>0,\delta>0 \qquad (3.111)
\end{aligned}
$$

which is a 4-Parameter generalized Lindley distribution, with the following special cases.

3.8.2 Special cases

(i) δ= 1, we have Type I 3-parameter Lindley distribution.

$$
g(\lambda) = \frac{\theta^{\alpha+1}[\beta\alpha+\lambda]}{\beta\theta} \frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)}, \quad \lambda>0;\theta>0,\alpha>0,\beta>0, \qquad (3.112)
$$

(ii) β=1, we have Type II 3-parameter Lindley distribution.

$$
g(\lambda) = \frac{\theta^{\alpha+1}[\alpha+\delta\lambda]}{\theta+\delta} \frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)}, \quad \lambda>0;\theta>0,\alpha>0,\delta>0 \qquad (3.113)
$$

(iii) α=1, we have Type III 3-parameter Lindley distribution.

$$
g(\lambda) = \frac{\theta^{2}[\beta+\delta\lambda]}{\beta\theta+\delta} e^{-\theta\lambda}, \quad \lambda>0;\theta>0,\beta>0,\delta>0 \qquad (3.114)
$$

(iv) β=δ=1, we have Type I 2-parameter Lindley distribution.

$$
g(\lambda) = \frac{\theta^{\alpha+1}[\alpha+\lambda]}{\theta+1} \frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)}, \quad \lambda>0;\theta>0,\alpha>0, \qquad (3.115)
$$

(v) α=δ=1, we have Type II 2-parameter Lindley distribution.

$$
g(\lambda) = \frac{\theta^{2}[\beta+\lambda]}{\beta\theta+1} e^{-\theta\lambda}, \quad \lambda>0;\theta>0,\beta>0, \qquad (3.116)
$$

(vi) α=β=1, we have Type III 2-parameter Lindley distribution.

$$g(\lambda) = \frac{\theta^2[1+\delta\lambda]}{\theta+\delta}e^{-\theta\lambda}, \quad \lambda > 0; \theta > 0, \delta > 0 \tag{3.117}$$

(vii) α=β=δ=1, we have 1-parameter Lindley distribution.

$$g(\lambda) = \frac{\theta^2[1+\lambda]}{\theta+1}e^{-\theta\lambda}, \quad \lambda > 0; \theta > 0, \tag{3.118}$$

3.9 Erlang-One Parameter Lindley distribution and Its Links

3.9.1 Erlang-One Parameter Lindley mixture

The One Parameter Lindley mixing distribution is

$$g(\lambda) = \frac{\theta^2[1+\lambda]}{\theta+1}e^{-\theta\lambda}, \quad \lambda > 0; \theta > 0, \tag{3.119}$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda} \frac{\theta^2[1+\lambda]}{\theta+1} e^{-\theta\lambda} d\lambda \\
&= \frac{\theta^2}{\theta+1} \int_0^\infty \lambda^n [1+\lambda] e^{-\lambda(t+\theta)} d\lambda \\
&= \frac{\theta^2}{\theta+1} \int_0^\infty [\lambda^{n+1} e^{-\lambda(t+\theta)} + \lambda^n e^{-\lambda(t+\theta)}] d\lambda \\
&= \frac{\theta^2}{\theta+1} [\frac{\Gamma(n+2)}{(t+\theta)^{n+2}} + \frac{\Gamma(n+1)}{(t+\theta)^{n+1}}] \\
&= \frac{\theta^2}{\theta+1} \frac{\Gamma(n+1)}{(t+\theta)^{n+1}} [\frac{n+1}{t+\theta} + 1] \\
&= \frac{\theta^2}{\theta+1} \frac{n!}{(t+\theta)^{n+2}} [n+1+t+\theta]
\end{aligned} \tag{3.120}$$

Construction by the direct method gives

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma n}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^{n-1}}{\Gamma n}\frac{\theta^2}{\theta+1}\frac{n!}{(t+\theta)^{n+2}}[n+1+t+\theta] \\ &= \frac{\theta^2}{\theta+1}\frac{nt^{n-1}}{(t+\theta)^{n+2}}[n+1+t+\theta], \quad t>0;\theta>0, n=1,2,3,.. \end{aligned} \tag{3.121}$$

By the method of moments we have

$$\begin{aligned} f_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}E[\wedge^j] \\ &\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}\frac{\theta^2}{\theta+1}\frac{j!}{\theta^{j+2}}[j+1+\theta] \\ &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}\frac{j!}{\theta^j}\frac{j+1+\theta}{\theta+1} \end{aligned} \tag{3.122}$$

Identity 3.19
Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}\frac{j!}{\theta^j}\frac{j+1+\theta}{\theta+1} &= \frac{\theta^2}{\theta+1}\frac{nt^{n-1}}{(t+\theta)^{n+2}}[n+1+t+\theta] \\ \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{j!}{\theta^j}\frac{j+1+\theta}{\theta+1} &= \frac{\theta^2}{\theta+1}\frac{n!}{(t+\theta)^{n+2}}[n+1+t+\theta] \end{aligned} \tag{3.123}$$

The rth moment of the Erlang mixture is

$$\begin{aligned} E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n}E[\wedge^{-r}] \\ &= \frac{\Gamma(n+r)}{\Gamma n}\frac{\Gamma(1-r)}{\theta^{-r}}\frac{\theta-r+1}{\theta+1} \\ &= \frac{\Gamma(n+r)}{\Gamma n}\Gamma(1-r)\theta^r\frac{\theta-r+1}{\theta+1} \end{aligned} \tag{3.124}$$

$$E(T) = \frac{\Gamma(n+1)}{\Gamma n}\Gamma(0)\theta\frac{\theta}{\theta+1} = \infty \tag{3.125}$$

3.9.2 Exponential-One Parameter Lindley Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= \frac{\theta^2}{\theta+1}\frac{1}{(t+\theta)^3}[t+\theta+2] \\ &= \frac{\theta^2}{\theta+1}\frac{t+\theta+2}{(t+\theta)^3}, \quad t>0;\theta>0 \end{aligned} \tag{3.126}$$

By the method of moments we have

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}E[\wedge^j] \\ &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\frac{j!}{\theta^j}\frac{j+1+\theta}{\theta+1} \\ &= \sum_{j=1}^{\infty}(-1)^{j-1}t^{j-1}\frac{j}{\theta^j}\frac{j+1+\theta}{\theta+1} \end{aligned} \tag{3.127}$$

Identity 3.20
Equating the above two methods we get

$$\sum_{j=1}^{\infty}(-1)^{j-1}t^{j-1}\frac{j}{\theta^j}\frac{j+1+\theta}{\theta+1} = \frac{\theta^2}{\theta+1}\frac{t+\theta+2}{(t+\theta)^3} \tag{3.128}$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E[\wedge^{-r}] \\ &= r!\frac{\Gamma(1-r)}{\theta^{-r}}\frac{\theta-r+1}{\theta+1} \\ &= r!\Gamma(1-r)\theta^r\frac{\theta-r+1}{\theta+1} \end{aligned} \tag{3.129}$$

$$E(T) = \Gamma(0)\theta\frac{\theta}{\theta+1} = \infty \tag{3.130}$$

3.9.3 Poisson-One Parameter Lindley Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!} E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!} \frac{\theta^2}{\theta+1} \frac{n!}{(t+\theta)^{n+2}} [n+1+t+\theta] \\ &= \frac{\theta^2 t^n}{\theta+1} \frac{t+\theta+n+1}{(t+\theta)^{n+2}}, \quad t>0; \theta>0 \end{aligned} \tag{3.131}$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E[\wedge^j] \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \frac{j!}{\theta^j} \frac{j+1+\theta}{\theta+1} \\ &= \sum_{j=n}^{\infty} (-1)^{j-n} \frac{t^j}{\theta^j} \binom{j}{n} \frac{j+1+\theta}{\theta+1} \end{aligned} \tag{3.132}$$

Identity 3.21
Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty} (-1)^{j-n} \left(\frac{t}{\theta}\right)^j \binom{j}{n} \frac{\theta+j+1}{\theta+1} &= t^n \frac{\theta^2}{\theta+1} \frac{t+\theta+n+1}{(t+\theta)^{n+2}} \\ \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{j!}{\theta^j} \frac{\theta+j+1}{\theta+1} &= n! \frac{\theta^2}{\theta+1} \frac{t+\theta+n+1}{(t+\theta)^{n+2}} \end{aligned} \tag{3.133}$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= \frac{\theta^2}{\theta+1} \frac{(1-s)t+\theta+1}{[(1-s)t+\theta]^2} \end{aligned} \tag{3.134}$$

The rth moment of the Poisson mixture is

$$E(T^r) = t^r E(\wedge^r)$$
$$= t^r \frac{r!}{\theta^r}\frac{\theta+r+1}{\theta+1}$$
$$= r!(\frac{t}{\theta})^r \frac{\theta+r+1}{\theta+1} \tag{3.135}$$
$$E(T) = \frac{t}{\theta}\frac{\theta+2}{\theta+1} \tag{3.136}$$

3.10 Erlang-Type I Two-Parameter Lindley Distribution and its Links

3.10.1 Erlang-Type I Two-Parameter Lindley Mixture

The Type I Two Parameter Lindley mixing distribution is

$$g(\lambda) = \frac{\theta^{\alpha+1}[\alpha+\lambda]}{\theta+1}\frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)}, \quad \lambda > 0; \theta > 0, \alpha > 0, \tag{3.137}$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_0^\infty \lambda^n e^{-t\lambda}\frac{\theta^{\alpha+1}[\alpha+\lambda]}{\theta+1}\frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)}d\lambda$$
$$= \frac{\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}\int_0^\infty \lambda^{n+\alpha-1}e^{-\lambda(t+\theta)}[\alpha+\lambda]d\lambda$$
$$= \frac{\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}[\alpha\int_0^\infty e^{-\lambda(t+\theta)}\lambda^{n+\alpha-1}d\lambda + \int_0^\infty e^{-\lambda(t+\theta)}\lambda^{n+\alpha}d\lambda]$$
$$= \frac{\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}[\frac{\alpha\Gamma(n+\alpha)}{(t+\theta)^{n+\alpha}} + \frac{\Gamma(n+\alpha+1)}{(t+\theta)^{n+\alpha+1}}]$$
$$= \frac{\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}\frac{\Gamma(n+\alpha)}{(t+\theta)^{n+\alpha}}[\alpha+\frac{n+\alpha}{t+\theta}]$$
$$= \frac{\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}\frac{\Gamma(n+\alpha)}{(t+\theta)^{n+\alpha+1}}[\alpha(t+\theta)+n+\alpha]$$
$$= \frac{\theta^{\alpha+1}}{(\theta+1)}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)}\frac{[\alpha(t+\theta+1)+n]}{(t+\theta)^{n+\alpha+1}} \tag{3.138}$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^{n-1}}{\Gamma n} \frac{\theta^{\alpha+1}}{(\theta+1)} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{[\alpha(t+\theta+1)+n]}{(t+\theta)^{n+\alpha+1}}$$
$$= \binom{n+\alpha-1}{n-1} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\alpha(t+\theta+1)+n}{t(\theta+1)}], \quad t>0; \theta>0, \alpha>0, n=1,2,3,.. \tag{3.139}$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\theta^{\alpha+1}}{\theta^{j+\alpha+1}} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{[\alpha(\theta+1)+j]}{\theta+1}$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{1}{\theta^j} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{[\alpha(\theta+1)+j]}{\theta+1} \tag{3.140}$$

Identity 3.22

Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{1}{\theta^j} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{\alpha(\theta+1)+j}{\theta+1} = \binom{n+\alpha-1}{n-1} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\alpha(t+\theta+1)+n}{t(\theta+1)}]$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{1}{\theta^j} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{\alpha(\theta+1)+j}{\theta+1} = \frac{\theta^{\alpha+1}}{(\theta+1)} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\alpha(t+\theta+1)+n}{(t+\theta)^{n+\alpha+1}} \tag{3.141}$$

The rth moment of the Erlang mixture is

$$E(T^r) = \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r})$$
$$= \frac{\Gamma(n+r)}{\Gamma n} \frac{1}{\theta^{-r}} \frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)} [\frac{\alpha(\theta+1)-r}{\theta+1}]$$
$$= \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)} \theta^r [\frac{\alpha(\theta+1)-r}{\theta+1}] \tag{3.142}$$
$$E(T) = \frac{\Gamma(n+1)}{\Gamma n} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)} \theta [\frac{\alpha(\theta+1)-1}{\theta+1}]$$
$$= \frac{n}{\alpha(\alpha-1)} \frac{\theta}{\theta+1} [\alpha(\theta+1)-1] \tag{3.143}$$

3.10.2 Exponential-Type I-Two Parameter Lindley Mixture

Construction by the direct method gives

$$\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= \frac{\theta^{\alpha+1}}{(\theta+1)}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\frac{\alpha(t+\theta+1)+1}{(t+\theta)^{\alpha+2}} \\
&= (\frac{\theta}{(\theta+t)})^{\alpha+1}\frac{\alpha(t+\theta+1)+1}{(t+\theta)(\theta+1)}, \quad t>0;\theta>0,\alpha>0
\end{aligned} \tag{3.144}$$

By the method of moments we have

$$\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}E(\wedge^j) \\
&= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\frac{\theta^{\alpha+1}}{\theta^{j+\alpha+1}}\frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)}\frac{\alpha(\theta+1)+j}{\theta+1} \\
&= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\frac{1}{\theta^j}\frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)}\frac{\alpha(\theta+1)+j}{\theta+1}
\end{aligned} \tag{3.145}$$

Identity 3.23

Equating the above two methods we get

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\frac{1}{\theta^j}\frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)}\frac{\alpha(\theta+1)+j}{\theta+1} = (\frac{\theta}{(\theta+t)})^{\alpha+1}\frac{\alpha(t+\theta+1)+1}{(t+\theta)(\theta+1)} \tag{3.146}$$

The rth moment of the Exponential mixture is

$$\begin{aligned}
E(T^r) &= r!E[\wedge^{-r}] \\
&= r!\frac{1}{\theta^{-r}}\frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)}\frac{\alpha(\theta+1)-r}{\theta+1} \\
&= r!\theta^r\frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)}\frac{\alpha(\theta+1)-r}{\theta+1}
\end{aligned} \tag{3.147}$$

$$\begin{aligned}
E(T) &= \theta\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}\frac{\alpha(\theta+1)-1}{\theta+1} \\
&= \frac{\theta}{\theta+1}\frac{\alpha(\theta+1)-1}{\alpha(\alpha-1)}
\end{aligned} \tag{3.148}$$

3.10.3 Poisson-Type I-Two Parameter Lindley mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$\begin{aligned}
P_n(t) &= \frac{t^n}{n!} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^n}{n!} \frac{\theta^{\alpha+1}}{(\theta+1)} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\alpha(t+\theta+1)+n}{(t+\theta)^{n+\alpha+1}} \\
&= \frac{1}{n!} (\frac{t}{\theta+t})^n (\frac{\theta}{\theta+t})^{\alpha+1} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} [\frac{\alpha(t+\theta+1)+n}{\theta+1}] \\
&= \binom{n+\alpha-1}{n-1} (\frac{t}{\theta+t})^n (\frac{\theta}{\theta+t})^{\alpha+1} [\frac{\alpha(t+\theta+1)+n}{n(\theta+1)}], \quad t>0; \theta>0, \alpha>0
\end{aligned} \tag{3.149}$$

By the method of moments we have

$$\begin{aligned}
P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \frac{1}{\theta^j} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{\alpha(\theta+1)+j}{\theta+1} \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n}}{n!(j-n)!} (\frac{t}{\theta})^j \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{\alpha(\theta+1)+j}{\theta+1}
\end{aligned} \tag{3.150}$$

Identity 3.24
Equating the above two methods we get

$$\begin{aligned}
\sum_{j=n}^{\infty} \frac{(-1)^{j-n}}{n!(j-n)!} (\frac{t}{\theta})^j \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{\alpha(\theta+1)+j}{\theta+1} &= \binom{n+\alpha-1}{n-1} (\frac{t}{\theta+t})^n (\frac{\theta}{\theta+t})^{\alpha+1} [\frac{\alpha(t+\theta+1)+n}{n(\theta+1)}] \\
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{1}{\theta^j} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{\alpha(\theta+1)+j}{\theta+1} &= \frac{\theta^{\alpha+1}}{(\theta+1)} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\alpha(t+\theta+1)+n}{(t+\theta)^{n+\alpha+1}}
\end{aligned} \tag{3.151}$$

The PGF of the Poisson mixture is

$$
\begin{aligned}
G(s,t) &= E[e^{-(1-s)t\wedge}] \\
&= \frac{\theta^{\alpha+1}}{\theta+1}\frac{\Gamma\alpha}{\Gamma(\alpha+1)}\frac{\alpha[(1-s)t+\theta+1]}{[\theta+(1-s)t]^{\alpha+1}} \\
&= (\frac{\theta}{\theta+(1-s)t})^{\alpha+1}(\frac{(1-s)t+\theta+1}{\theta+1})
\end{aligned} \tag{3.152}
$$

The rth moment of the Poisson mixture is

$$
\begin{aligned}
E(T^r) &= t^r E[\wedge^r] \\
&= t^r \frac{1}{\theta^r}\frac{\Gamma(r+\alpha)}{\Gamma(\alpha+1)}\frac{\alpha(\theta+1)+r}{\theta+1} \\
&= (\frac{t}{\theta})^r \frac{\Gamma(r+\alpha)}{\Gamma(\alpha+1)}\frac{\alpha(\theta+1)+r}{\theta+1}
\end{aligned} \tag{3.153}
$$

$$
\begin{aligned}
E(T) &= (\frac{t}{\theta})\frac{\Gamma(1+\alpha)}{\Gamma(\alpha+1)}\frac{\alpha(\theta+1)+1}{\theta+1} \\
&= \frac{t}{\theta}\frac{\alpha(\theta+1)+1}{\theta+1}
\end{aligned} \tag{3.154}
$$

3.11 Erlang-Type II-Two Parameter Lindley distribution

3.11.1 Erlang-Type II-Two Parameter Lindley Mixture

The Type II-Two Parameter Lindley mixing distribution is

$$
g(\lambda) = \frac{\theta^2[\beta+\lambda]}{\beta\theta+1}e^{-\theta\lambda}, \quad \lambda > 0; \theta > 0, \beta > 0, \tag{3.155}
$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_0^\infty \lambda^n e^{-t\lambda} \frac{\theta^2[\beta+\lambda]}{\beta\theta+1} e^{-\theta\lambda} d\lambda$$
$$= \frac{\theta^2}{\beta\theta+1} \int_0^\infty \lambda^n e^{-\lambda(t+\theta)} [\beta+\lambda] d\lambda$$
$$= \frac{\theta^2}{\beta\theta+1} [\frac{\beta\Gamma(n+1)}{(t+\theta)^{n+1}} + \frac{\Gamma(n+2)}{(t+\theta)^{n+2}}]$$
$$= \frac{\theta^2}{\beta\theta+1} \frac{\Gamma(n+1)}{(t+\theta)^{n+1}} [\beta + \frac{n+1}{t+\theta}]$$
$$= \frac{\theta^2}{\beta\theta+1} \frac{\Gamma(n+1)}{(t+\theta)^{n+2}} [\beta(t+\theta)+n+1]$$
$$= \Gamma(n+1) \frac{\theta^2}{(t+\theta)^{n+2}} [\frac{\beta(t+\theta)+n+1}{\beta\theta+1}] \quad (3.156)$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^{n-1}}{\Gamma n} \frac{\theta^2}{(t+\theta)^{n+2}} \Gamma(n+1) [\frac{\beta(t+\theta)+n+1}{\beta\theta+1}]$$
$$= \frac{n}{t} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^2 [\frac{\beta(t+\theta)+n+1}{\beta\theta+1}], \quad t>0; \theta>0, \beta>0, n=1,2,3,.. \quad (3.157)$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{1}{\theta^j} \Gamma(j+1) [\frac{\beta\theta+j+1}{\beta\theta+1}]$$
$$= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\theta^j} \binom{j}{n} [\frac{\beta\theta+j+1}{\beta\theta+1}] \quad (3.158)$$

Identity 3.25

Equating the above two methods we get

$$= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\theta^j} \binom{j}{n} [\frac{\beta\theta+j+1}{\beta\theta+1}] = \frac{n}{t} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^2 [\frac{\beta(t+\theta)+n+1}{\beta\theta+1}]$$
$$\sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{j!}{\theta^j} [\frac{\beta\theta+j+1}{\beta\theta+1}] = \Gamma(n+1) \frac{\theta^2}{(t+\theta)^{n+2}} [\frac{\beta(t+\theta)+n+1}{\beta\theta+1}] \quad (3.159)$$

The rth moment of the Erlang mixture is

$$\begin{aligned} E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\ &= \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(1-r)}{\theta^{-r}} [\frac{\beta\theta - r + 1}{\beta\theta + 1}] \\ &= \frac{\Gamma(n+r)}{\Gamma n} \Gamma(1-r)\theta^r [\frac{\beta\theta - r + 1}{\beta\theta + 1}] \end{aligned} \tag{3.160}$$

$$E(T) = \frac{\Gamma(n+1)}{\Gamma n} \Gamma(0)\theta [\frac{\beta\theta}{\beta\theta + 1}] = \infty \tag{3.161}$$

3.11.2 Exponential-Type II-Two Parameter Lindley Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= \frac{\theta^2}{(t+\theta)^3} [\frac{\beta(t+\theta)+2}{\beta\theta+1}] \\ &= (\frac{\theta}{t+\theta})^2 [\frac{\beta(t+\theta)+2}{(t+\theta)(\beta\theta+1)}], \quad t > 0; \theta > 0, \beta > 0 \end{aligned} \tag{3.162}$$

By the method of moments we have

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{j!}{\theta^j} [\frac{\beta\theta + j + 1}{\beta\theta + 1}] \\ &= \sum_{j=1}^{\infty} (-1)^{j-1} t^{j-1} \frac{j}{\theta^j} [\frac{\beta\theta + j + 1}{\beta\theta + 1}] \end{aligned} \tag{3.163}$$

Identity 3.26
Equating the above two methods we get

$$\sum_{j=1}^{\infty} (-1)^{j-1} t^{j-1} \frac{j}{\theta^j} [\frac{\beta\theta + j + 1}{\beta\theta + 1}] = (\frac{\theta}{t+\theta})^2 [\frac{\beta(t+\theta)+2}{(t+\theta)(\beta\theta+1)}] \tag{3.164}$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= r!\Gamma(1-r)\theta^r[\frac{\beta\theta-r+1}{\beta\theta+1}] \end{aligned} \quad (3.165)$$

$$E(T) = \Gamma(0)\theta[\frac{\beta\theta}{\beta\theta+1}] \quad (3.166)$$

3.11.3 Poisson-Type II-Two Parameter Lindley Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!}\frac{\theta^2}{(t+\theta)^{n+2}}\Gamma(n+1)[\frac{\beta(t+\theta)+n+1}{\beta\theta+1}] \\ &= (\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^2[\frac{\beta(t+\theta)+n+1}{\beta\theta+1}], \quad t>0;\theta>0,\beta>0 \end{aligned} \quad (3.167)$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\ &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{j!}{\theta^j}[\frac{\beta\theta+j+1}{\beta\theta+1}] \\ &= \sum_{j=n}^{\infty}(-1)^{j-n}(\frac{t}{\theta})^j\binom{j}{n}[\frac{\beta\theta+j+1}{\beta\theta+1}] \end{aligned} \quad (3.168)$$

Identity 3.27
Equating the above two methods we get

$$\sum_{j=n}^{\infty}(-1)^{j-n}(\frac{t}{\theta})^j\binom{j}{n}[\frac{\beta\theta+j+1}{\beta\theta+1}]=(\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^2[\frac{\beta(t+\theta)+n+1}{\beta\theta+1}]$$

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{j!}{\theta^j}[\frac{\beta\theta+j+1}{\beta\theta+1}]=\Gamma(n+1)\frac{\theta^2}{(t+\theta)^{n+2}}[\frac{\beta(t+\theta)+n+1}{\beta\theta+1}] \tag{3.169}$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= [\frac{\theta}{(1-s)t+\theta}]^2[\frac{\beta[(1-s)t+\theta]+1}{\beta\theta+1}] \end{aligned} \tag{3.170}$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r\frac{r!}{\theta^r}[\frac{\beta\theta+r+1}{\beta\theta+1}] \\ &= r!(\frac{t}{\theta})^r[\frac{\beta\theta+r+1}{\beta\theta+1}] \end{aligned} \tag{3.171}$$

$$E(T)=(\frac{t}{\theta})[\frac{\beta\theta+2}{\beta\theta+1}] \tag{3.172}$$

3.12 Erlang-Type III-Two Parameter Lindley Distribution and Its Links

3.12.1 Erlang-Type III-Two Parameter Lindley Mixture

The Type III-Two Parameter Lindley mixing distribution is

$$g(\lambda)=\frac{\theta^2[1+\delta\lambda]}{\theta+\delta}e^{-\theta\lambda}, \quad \lambda>0;\theta>0,\delta>0 \tag{3.173}$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_0^\infty \lambda^n e^{-t\lambda} \frac{\theta^2[1+\delta\lambda]}{\theta+\delta} e^{-\theta\lambda} d\lambda$$
$$= \frac{\theta^2}{\theta+\delta}\int_0^\infty \lambda^n e^{-\lambda(t+\theta)}[1+\delta\lambda]d\lambda$$
$$= \frac{\theta^2}{\theta+\delta}[\frac{\Gamma(n+1)}{(t+\theta)^{n+1}} + \frac{\delta\Gamma(n+2)}{(t+\theta)^{n+2}}]$$
$$= \frac{\theta^2}{\theta+\delta}\frac{\Gamma(n+1)}{(t+\theta)^{n+1}}[1+\frac{\delta(n+1)}{t+\theta}]$$
$$= \frac{\theta^2}{\theta+\delta}\frac{\Gamma(n+1)}{(t+\theta)^{n+2}}[t+\theta+\delta(n+1)]$$
$$= \frac{\theta^2}{(t+\theta)^{n+2}}\Gamma(n+1)[\frac{t+\theta+\delta(n+1)}{\theta+\delta}] \quad (3.174)$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n}E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^{n-1}}{\Gamma n}\frac{\theta^2}{(t+\theta)^{n+2}}\Gamma(n+1)[\frac{t+\theta+\delta(n+1)}{\theta+\delta}]$$
$$= \frac{n}{t}(\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^2[\frac{t+\theta+\delta(n+1)}{\theta+\delta}], \quad t>0;\theta>0,\delta>0,n=1,2,3,.. \quad (3.175)$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}E(\wedge^j)$$
$$= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}\frac{1}{\theta^j}\Gamma(j+1)[\frac{\theta+\delta(j+1)}{\theta+\delta}]$$
$$= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\theta^j}n\binom{j}{n}[\frac{\theta+\delta(j+1)}{\theta+\delta}] \quad (3.176)$$

Identity 3.28

Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\theta^j}n\binom{j}{n}[\frac{\theta+\delta(j+1)}{\theta+\delta}] = \frac{n}{t}(\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^2[\frac{t+\theta+\delta(n+1)}{\theta+\delta}]$$
$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{j!}{\theta^j}[\frac{\theta+\delta(j+1)}{\theta+\delta}] = \Gamma(n+1)\frac{\theta^2}{(t+\theta)^{n+2}}[\frac{t+\theta+\delta(n+1)}{\theta+\delta}] \quad (3.177)$$

The rth moment of the Erlang mixture is

$$\begin{aligned} E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\ &= \frac{\Gamma(n+r)}{\Gamma n} \frac{1}{\theta^{-r}} \Gamma(1-r)[\frac{\theta+\delta(1-r)}{\theta+\delta}] \\ &= \frac{\Gamma(n+r)}{\Gamma n} \theta^r \Gamma(1-r)[\frac{\theta+\delta(1-r)}{\theta+\delta}] \end{aligned} \tag{3.178}$$

$$E(T) = \frac{\Gamma(n+1)}{\Gamma n} \theta \Gamma(0)[\frac{\theta+\delta(0)}{\theta+\delta}] = \infty \tag{3.179}$$

3.12.2 Exponential-Type III-Two Parameter Lindley Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= \frac{\theta^2}{(t+\theta)^3}[\frac{t+\theta+2\delta}{\theta+\delta}] \\ &= (\frac{\theta}{t+\theta})^2[\frac{t+\theta+2\delta}{(t+\theta)(\theta+\delta)}], \quad t>0; \theta>0, \delta>0 \end{aligned} \tag{3.180}$$

By the method of moments we have

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{j!}{\theta^j}[\frac{\theta+\delta(j+1)}{\theta+\delta}] \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{\theta^j} j[\frac{\theta+\delta(j+1)}{\theta+\delta}] \end{aligned} \tag{3.181}$$

Identity 3.29
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{\theta^j} j[\frac{\theta+\delta(j+1)}{\theta+\delta}] = (\frac{\theta}{t+\theta})^2[\frac{t+\theta+2\delta}{(t+\theta)(\theta+\delta)}] \tag{3.182}$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= r!\frac{1}{\theta^{-r}}\Gamma(1-r)[\frac{\theta+\delta(1-r)}{\theta+\delta}] \\ &= r!\theta^r\Gamma(1-r)[\frac{\theta+\delta(1-r)}{\theta+\delta}] \end{aligned} \quad (3.183)$$

$$E(T) = \theta\Gamma(0)[\frac{\theta+\delta(0)}{\theta+\delta}] = \infty \quad (3.184)$$

3.12.3 Poisson-Type III-Two Parameter Lindley Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!}\frac{\theta^2}{(t+\theta)^{n+2}}\Gamma(n+1)[\frac{t+\theta+\delta(n+1)}{\theta+\delta}] \\ &= (\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^2[\frac{t+\theta+\delta(n+1)}{\theta+\delta}], \quad t>0; \theta>0, \delta>0 \end{aligned} \quad (3.185)$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\ &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{j!}{\theta^j}[\frac{\theta+\delta(j+1)}{\theta+\delta}] \\ &= \sum_{j=n}^{\infty}(-1)^{j-n}(\frac{t}{\theta})^j\binom{j}{n}[\frac{\theta+\delta(j+1)}{\theta+\delta}] \end{aligned} \quad (3.186)$$

Identity 3.30
Equating the above two methods we get

$$\sum_{j=n}^{\infty}(-1)^{j-n}(\frac{t}{\theta})^j\binom{j}{n}[\frac{\theta+\delta(j+1)}{\theta+\delta}] = (\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^2[\frac{t+\theta+\delta(n+1)}{\theta+\delta}]$$

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{j!}{\theta^j}[\frac{\theta+\delta(j+1)}{\theta+\delta}] = \Gamma(n+1)\frac{\theta^2}{(t+\theta)^{n+2}}[\frac{t+\theta+\delta(n+1)}{\theta+\delta}] \quad (3.187)$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= [\frac{\theta}{(1-s)t+\theta}]^2[\frac{\theta+\delta+(1-s)t}{\theta+\delta}] \end{aligned} \quad (3.188)$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r\frac{r!}{\theta^r}[\frac{\theta+\delta(r+1)}{\theta+\delta}] \\ &= r!(\frac{t}{\theta})^r[\frac{\theta+\delta(r+1)}{\theta+\delta}] \end{aligned} \quad (3.189)$$

$$E(T) = (\frac{t}{\theta})[\frac{\theta+2\delta}{\theta+\delta}] \quad (3.190)$$

3.13 Erlang-Type I-3 Parameter Generalized Lindley (G3L) Distribution and its Links

3.13.1 Erlang-Type I-3 Parameter Generalized Lindley (G3L) Mixture

The Type I-3 Parameter Generalized Lindley mixing distribution is

$$g(\lambda) = \frac{\theta^{\alpha+1}[\beta\alpha+\lambda]}{\beta\theta}\frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)}, \quad \lambda>0;\theta>0,\alpha>0,\beta>0, \quad (3.191)$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_0^\infty \lambda^n e^{-t\lambda} \frac{\theta^{\alpha+1}[\beta\alpha+\lambda]}{\beta\theta} \frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)} d\lambda$$
$$= \frac{\theta^{\alpha+1}}{\beta\theta\Gamma(\alpha+1)} \int_0^\infty \lambda^{n+\alpha-1} e^{-\lambda(t+\theta)}[\beta\alpha+\lambda] d\lambda$$
$$= \frac{\theta^{\alpha+1}}{\beta\theta\Gamma(\alpha+1)} [\beta\alpha \frac{\Gamma(n+\alpha)}{(t+\theta)^{n+\alpha}} + \frac{\Gamma(n+\alpha+1)}{(t+\theta)^{n+\alpha+1}}]$$
$$= \frac{\theta^{\alpha+1}}{\beta\theta\Gamma(\alpha+1)} \frac{\Gamma(n+\alpha)}{(t+\theta)^{n+\alpha+1}} [\beta\alpha(t+\theta)+n+\alpha]$$
$$= \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}} [\frac{\beta\alpha(t+\theta)+n+\alpha}{\beta\theta}] \tag{3.192}$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^{n-1}}{\Gamma n} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}} [\frac{\beta\alpha(t+\theta)+n+\alpha}{\beta\theta}]$$
$$= \binom{n+\alpha-1}{n-1} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\beta\alpha(t+\theta)+n+\alpha}{t\beta\theta}], \quad t>0; \theta>0, \alpha>0, \beta>0, n=1,2,3,.. \tag{3.193}$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\beta\alpha\theta+j+\alpha}{\beta\theta}] \tag{3.194}$$

Identity 3.31

Equating the above two methods we get

$$\sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\beta\alpha\theta+j+\alpha}{\beta\theta}] = \binom{n+\alpha-1}{n-1} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\beta\alpha(t+\theta)+n+\alpha}{t\beta\theta}]$$
$$\sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\beta\alpha\theta+j+\alpha}{\beta\theta}] = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\beta\alpha(t+\theta)+n+\alpha}{\beta\theta(t+\theta)^n}] \tag{3.195}$$

The rth moment of the Erlang mixture is

$$
\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)} \frac{1}{\theta^{-r}} [\frac{\beta\alpha\theta - r + \alpha}{\beta\theta}] \\
&= \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)} \theta^r [\frac{\beta\alpha\theta - r + \alpha}{\beta\theta}] \qquad (3.196) \\
E(T) &= \frac{\Gamma(n+1)}{\Gamma n} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)} \theta [\frac{\beta\alpha\theta - 1 + \alpha}{\beta\theta}] \\
&= \frac{n\theta}{\alpha(\alpha-1)} [\frac{\beta\alpha\theta - 1 + \alpha}{\beta\theta}] \qquad (3.197)
\end{aligned}
$$

3.13.2 Exponential-Type I-3 Parameter Generalized Lindley(G3L) Mixture

Construction by the direct method gives

$$
\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= \frac{\theta^{\alpha+1}}{(t+\theta)^{\alpha+2}} [\frac{\beta\alpha(t+\theta) + 1 + \alpha}{\beta\theta}] \\
&= (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\beta\alpha(t+\theta) + 1 + \alpha}{(t+\theta)\beta\theta}], \quad t > 0; \theta > 0, \alpha > 0, \beta > 0 \qquad (3.198)
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\
&= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\beta\alpha\theta + j + \alpha}{\beta\theta}] \qquad (3.199)
\end{aligned}
$$

Identity 3.32

$$
\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\beta\alpha\theta + j + \alpha}{\beta\theta}] = (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\beta\alpha(t+\theta) + 1 + \alpha}{(t+\theta)\beta\theta}] \qquad (3.200)
$$

The rth moment of the Exponential mixture is

$$
\begin{aligned}
E(T^r) &= r!E(\wedge^{-r}) \\
&= r!\frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)}\frac{1}{\theta^{-r}}[\frac{\beta\alpha\theta-r+\alpha}{\beta\theta}] \\
&= r!\frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)}\theta^r[\frac{\beta\alpha\theta-r+\alpha}{\beta\theta}] \qquad (3.201)\\
E(T) &= \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}\theta[\frac{\beta\alpha\theta-1+\alpha}{\beta\theta}] \\
&= \frac{\theta}{\alpha(\alpha-1)}[\frac{\beta\alpha\theta-1+\alpha}{\beta\theta}] \qquad (3.202)
\end{aligned}
$$

3.13.3 Poisson-Type I-3 Parameter Generalized Lindley (G3L) Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$
\begin{aligned}
P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^n}{n!}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)}\frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}}[\frac{\beta\alpha(t+\theta)+n+\alpha}{\beta\theta}] \\
&= \frac{1}{n!}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)}(\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^{\alpha+1}[\frac{\beta\alpha(t+\theta)+n+\alpha}{\beta\theta}] \\
&= \binom{n+\alpha-1}{n-1}(\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^{\alpha+1}[\frac{\beta\alpha(t+\theta)+n+\alpha}{n\beta\theta}], \quad t>0;\theta>0,\alpha>0,\beta>0
\end{aligned}
\qquad (3.203)
$$

By the method of moments we have

$$
\begin{aligned}
P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\
&= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)}\frac{1}{\theta^j}[\frac{\beta\alpha\theta+j+\alpha}{\beta\theta}] \\
&= \sum_{j=n}^{\infty}(-1)^{j-n}\frac{1}{j}\binom{\alpha+j-1}{j-1}\binom{j}{n}(\frac{t}{\theta})^j[\frac{\beta\alpha\theta+j+\alpha}{\beta\theta}] \qquad (3.204)
\end{aligned}
$$

Identity 3.33

Equating the above two methods we get

$$\sum_{j=n}^{\infty}(-1)^{j-n}\frac{1}{j}\binom{\alpha+j-1}{j-1}\binom{j}{n}(\frac{t}{\theta})^j[\frac{\beta\alpha\theta+j+\alpha}{\beta\theta}]=\binom{n+\alpha-1}{n-1}(\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^{\alpha+1}[\frac{\beta\alpha(t+\theta)+n+}{n\beta\theta}$$

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)}\frac{1}{\theta^j}[\frac{\beta\alpha\theta+j+\alpha}{\beta\theta}]=\frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)}\frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}}[\frac{\beta\alpha(t+\theta)+n+\alpha}{\beta\theta}] \quad (3.205)$$

The PGF of the Poisson mixture is

$$\begin{aligned}G(s,t)&=E[e^{-(1-s)t\Lambda}]\\&=\frac{\Gamma\alpha}{\Gamma(\alpha+1)}[\frac{\theta}{(1-s)t+\theta}]^{\alpha+1}[\frac{\beta\alpha[(1-s)t+\theta]+\alpha}{\beta\theta}]\\&=[\frac{\theta}{(1-s)t+\theta}]^{\alpha+1}[\frac{\beta[(1-s)t+\theta]+1}{\beta\theta}]\end{aligned} \quad (3.206)$$

The rth moment of the Poisson mixture is

$$\begin{aligned}E(T^r)&=t^rE(\Lambda^r)\\&=t^r\frac{\Gamma(r+\alpha)}{\Gamma(\alpha+1)}\frac{1}{\theta^r}[\frac{\beta\alpha\theta+r+\alpha}{\beta\theta}]\\&=\frac{\Gamma(r+\alpha)}{\Gamma(\alpha+1)}(\frac{t}{\theta})^r[\frac{\beta\alpha\theta+r+\alpha}{\beta\theta}]\end{aligned} \quad (3.207)$$

$$E(T)=(\frac{t}{\theta})[\frac{\beta\alpha\theta+1+\alpha}{\beta\theta}] \quad (3.208)$$

3.14 Erlang-Type II-3 Parameter Generalized Lindley (G3L) Distribution and its Links

3.14.1 Erlang-Type II-3 Parameter Generalized Lindley (G3L) Mixture

The Type II-3 Parameter Generalized Lindley mixing distribution is

$$g(\lambda)=\frac{\theta^{\alpha+1}[\alpha+\delta\lambda]}{\theta+\delta}\frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)},\quad \lambda>0;\theta>0,\alpha>0,\delta>0 \quad (3.209)$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_0^\infty \lambda^n e^{-t\lambda} \frac{\theta^{\alpha+1}[\alpha+\delta\lambda]}{\theta+\delta} \frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)} d\lambda$$

$$= \frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma(\alpha+1)} \int_0^\infty \lambda^{n+\alpha-1} e^{-\lambda(t+\theta)} [\alpha+\delta\lambda] d\lambda$$

$$= \frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma(\alpha+1)} [\frac{\alpha\Gamma(n+\alpha)}{(t+\theta)^{n+\alpha}} + \frac{\delta\Gamma(n+\alpha+1)}{(t+\theta)^{n+\alpha+1}}]$$

$$= \frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma(\alpha+1)} \frac{\Gamma(n+\alpha)}{(t+\theta)^{n+\alpha+1}} [\alpha(t+\theta)+\delta(n+\alpha)]$$

$$= \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}} [\frac{\alpha(t+\theta+\delta)+\delta n}{(\theta+\delta)}] \tag{3.210}$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$

$$= \frac{t^{n-1}}{\Gamma n} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}} [\frac{\alpha(t+\theta+\delta)+\delta n}{(\theta+\delta)}]$$

$$= \binom{n+\alpha-1}{n-1} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\alpha(t+\theta+\delta)+\delta n}{t(\theta+\delta)}], \quad t>0; \theta>0, \alpha>0, \delta>0, n=1,2,3,\ldots \tag{3.211}$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$

$$= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\alpha(\theta+\delta)+\delta j}{(\theta+\delta)}] \tag{3.212}$$

Identity 3.34
Equating the above two methods we get

$$\sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\alpha(\theta+\delta)+\delta j}{(\theta+\delta)}] = \binom{n+\alpha-1}{n-1} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\alpha(t+\theta+\delta)+\delta n}{t(\theta+\delta)}]$$

$$\sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\alpha(\theta+\delta)+\delta j}{(\theta+\delta)}] = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}} [\frac{\alpha(t+\theta+\delta)+\delta n}{(\theta+\delta)}] \tag{3.213}$$

The rth moment of the Erlang mixture is

$$
\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)} \theta^r [\frac{\alpha(\theta+\delta)-\delta r}{(\theta+\delta)}] \qquad (3.214) \\
E(T) &= \frac{\Gamma(n+1)}{\Gamma n} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)} \theta [\frac{\alpha(\theta+\delta)-\delta}{(\theta+\delta)}] \\
&= \frac{n}{\alpha(\alpha-1)} \theta [\frac{\alpha(\theta+\delta)-\delta}{(\theta+\delta)}] \qquad (3.215)
\end{aligned}
$$

3.14.2 Exponential-Type II-3 Parameter Generalized Lindley (G3L) Mixture

Construction by the direct method gives

$$
\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\alpha(t+\theta+\delta)+\delta}{(t+\theta)(\theta+\delta)}], \quad t>0; \theta>0, \alpha>0, \delta>0 \qquad (3.216)
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\
&= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\alpha(\theta+\delta)+\delta j}{(\theta+\delta)}] \qquad (3.217)
\end{aligned}
$$

Identity 3.35
Equating the above two methods we get

$$
\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\alpha(\theta+\delta)+\delta j}{(\theta+\delta)}] = (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\alpha(t+\theta+\delta)+\delta}{(t+\theta)(\theta+\delta)}] \qquad (3.218)
$$

The rth moment of the Exponential mixture is

$$E(T^r) = r!E(\wedge^{-r}) = r!\frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)}\theta^r[\frac{\alpha(\theta+\delta)-\delta r}{(\theta+\delta)}] \tag{3.219}$$

$$E(T) = \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}\theta[\frac{\alpha(\theta+\delta)-\delta}{(\theta+\delta)}] = \frac{\theta}{\alpha(\alpha-1)}[\frac{\alpha(\theta+\delta)-\delta}{(\theta+\delta)}] \tag{3.220}$$

3.14.3 Poisson-Type II-3 Parameter Generalized Lindley (G3L) Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{t\wedge}] \\ &= \frac{t^n}{n!}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)}\frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}}[\frac{\alpha(t+\theta+\delta)+\delta n}{(\theta+\delta)}] \\ &= \binom{n+\alpha-1}{n-1}(\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^{\alpha+1}[\frac{\alpha(t+\theta+\delta)+\delta n}{n(\theta+\delta)}], \quad t>0;\theta>0,\alpha>0,\delta>0 \end{aligned} \tag{3.221}$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\ &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)}\frac{1}{\theta^j}[\frac{\alpha(\theta+\delta)+\delta j}{(\theta+\delta)}] \\ &= \sum_{j=n}^{\infty}(-1)^{j-n}\binom{j}{n}\binom{\alpha+j-1}{j-1}\frac{1}{j}(\frac{t}{\theta})^j[\frac{\alpha(\theta+\delta)+\delta j}{(\theta+\delta)}] \end{aligned} \tag{3.222}$$

Identity 3.36
Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n}}{j} \binom{j}{n} \binom{\alpha+j-1}{j-1} (\frac{t}{\theta})^j [\frac{\alpha(\theta+\delta)+\delta j}{(\theta+\delta)}] = \binom{n+\alpha-1}{n-1} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\alpha(t+\theta+\delta)+}{n(\theta+\delta)}$$

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\alpha(\theta+\delta)+\delta j}{(\theta+\delta)}] = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}} [\frac{\alpha(t+\theta+\delta)+\delta n}{(\theta+\delta)}] \quad (3.223)$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= [\frac{\theta}{(1-s)t+\theta}]^{\alpha+1} \frac{\Gamma\alpha}{\alpha\Gamma\alpha} [\frac{\alpha[(1-s)t+\theta+\delta]}{\theta+\delta}] \\ &= [\frac{\theta}{(1-s)t+\theta}]^{\alpha+1} [\frac{(1-s)t+\theta+\delta}{\theta+\delta}] \end{aligned} \quad (3.224)$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r \frac{\Gamma(r+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^r} [\frac{\alpha(\theta+\delta)+\delta r}{(\theta+\delta)}] \\ &= \frac{\Gamma(r+\alpha)}{\Gamma(\alpha+1)} (\frac{t}{\theta})^r [\frac{\alpha(\theta+\delta)+\delta r}{(\theta+\delta)}] \end{aligned} \quad (3.225)$$

$$E(T) = (\frac{t}{\theta}) [\frac{\alpha(\theta+\delta)+\delta}{(\theta+\delta)}] \quad (3.226)$$

3.15 Erlang-Type III-3 Parameter Generalized Lindley (G3L) Distribution and Its Links

3.15.1 Erlang-Type III-3 Parameter Generalized Lindley (G3L) Mixture

The Type III-3 Parameter Generalized Lindley mixing distribution is

$$g(\lambda) = \frac{\theta^2[\beta+\delta\lambda]}{\beta\theta+\delta} e^{-\theta\lambda}, \quad \lambda > 0; \theta > 0, \beta > 0, \delta > 0 \quad (3.227)$$

$$
\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda} \frac{\theta^2[\beta+\delta\lambda]}{\beta\theta+\delta} e^{-\theta\lambda} d\lambda \\
&= \frac{\theta^2}{\beta\theta+\delta} \int_0^\infty \lambda^n e^{-\lambda(t+\theta)} [\beta+\delta\lambda] d\lambda \\
&= \frac{\theta^2}{\beta\theta+\delta} [\beta \frac{\Gamma(n+1)}{(t+\theta)^{n+1}} + \delta \frac{\Gamma(n+2)}{(t+\theta)^{n+2}}] \\
&= \frac{\theta^2}{\beta\theta+\delta} \frac{\Gamma(n+1)}{(t+\theta)^{n+2}} [\beta(t+\theta)+\delta(n+1)] \\
&= \Gamma(n+1) \frac{\theta^2}{(t+\theta)^{n+2}} [\frac{\beta(t+\theta)+\delta(n+1)}{\beta\theta+\delta}] \qquad (3.228)
\end{aligned}
$$

Construction by the direct method gives

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} \Gamma(n+1) \frac{\theta^2}{(t+\theta)^{n+2}} [\frac{\beta(t+\theta)+\delta(n+1)}{\beta\theta+\delta}] \\
&= \frac{n}{t} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^2 [\frac{\beta(t+\theta)+\delta(n+1)}{\beta\theta+\delta}], \quad t>0; \theta>0, \beta>0, \delta>0, n=1,2,3,\ldots \qquad (3.229)
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{j!}{\theta^j} [\frac{\beta\theta+\delta(j+1)}{\beta\theta+\delta}] \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\theta^j} n \binom{j}{n} [\frac{\beta\theta+\delta(j+1)}{\beta\theta+\delta}] \qquad (3.230)
\end{aligned}
$$

Identity 3.37

Equating the above two methods we get

$$
\begin{aligned}
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\theta^j} n \binom{j}{n} [\frac{\beta\theta+\delta(j+1)}{\beta\theta+\delta}] &= \frac{n}{t} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^2 [\frac{\beta(t+\theta)+\delta(n+1)}{\beta\theta+\delta}] \\
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{j!}{\theta^j} [\frac{\beta\theta+\delta(j+1)}{\beta\theta+\delta}] &= \Gamma(n+1) \frac{\theta^2}{(t+\theta)^{n+2}} [\frac{\beta(t+\theta)+\delta(n+1)}{\beta\theta+\delta}] \qquad (3.231)
\end{aligned}
$$

The rth moment of the Erlang mixture is

$$E(T^r) = \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r})$$
$$= \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(1-r)}{\theta^{-r}} [\frac{\beta\theta + \delta(1-r)}{\beta\theta + \delta}]$$
$$= \frac{\Gamma(n+r)}{\Gamma n} \Gamma(1-r)\theta^r [\frac{\beta\theta + \delta(1-r)}{\beta\theta + \delta}] \quad (3.232)$$
$$E(T) = \frac{\Gamma(n+1)}{\Gamma n} \Gamma(0)\theta [\frac{\beta\theta}{\beta\theta + \delta}] = \infty \quad (3.233)$$

3.15.2 Exponential-Type III-3 Parameter Generalized Lindley (G3L) Mixture

Construction by the direct method gives

$$f_1(t) = E[\wedge e^{-t\wedge}]$$
$$= \frac{\theta^2}{(t+\theta)^3} [\frac{\beta(t+\theta) + 2\delta}{\beta\theta + \delta}]$$
$$= (\frac{\theta}{t+\theta})^2 [\frac{\beta(t+\theta) + 2\delta}{(t+\theta)(\beta\theta + \delta)}], \quad t > 0; \theta > 0, \beta > 0, \delta > 0 \quad (3.234)$$

By the method of moments we have

$$f_1(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j)$$
$$= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{j!}{\theta^j} [\frac{\beta\theta + \delta(j+1)}{\beta\theta + \delta}]$$
$$= \sum_{j=1}^{\infty} (-1)^{j-1} t^{j-1} \frac{j}{\theta^j} [\frac{\beta\theta + \delta(j+1)}{\beta\theta + \delta}] \quad (3.235)$$

Identity 3.38
Equating the above two methods we get

$$\sum_{j=1}^{\infty} (-1)^{j-1} t^{j-1} \frac{j}{\theta^j} [\frac{\beta\theta + \delta(j+1)}{\beta\theta + \delta}] = (\frac{\theta}{t+\theta})^2 [\frac{\beta(t+\theta) + 2\delta}{(t+\theta)(\beta\theta + \delta)}] \quad (3.236)$$

The rth moment of the Exponential mixture is

$$E(T^r) = r!E(\wedge^{-r}) = r!\Gamma(1-r)\theta^r[\frac{\beta\theta+\delta(1-r)}{\beta\theta+\delta}] \quad (3.237)$$

$$E(T) = \Gamma(0)\theta[\frac{\beta\theta}{\beta\theta+\delta}] = \infty \quad (3.238)$$

3.15.3 Poisson-Type III-3 Parameter Generalized Lindley (G3L) Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!}\Gamma(n+1)\frac{\theta^2}{(t+\theta)^{n+2}}[\frac{\beta(t+\theta)+\delta(n+1)}{\beta\theta+\delta}] \\ &= (\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^2[\frac{\beta(t+\theta)+\delta(n+1)}{\beta\theta+\delta}], \quad t>0;\theta>0,\beta>0,\delta>0 \end{aligned} \quad (3.239)$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\ &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{j!}{\theta^j}[\frac{\beta\theta+\delta(j+1)}{\beta\theta+\delta}] \\ &= \sum_{j=n}^{\infty}(-1)^{j-n}\binom{j}{n}(\frac{t}{\theta})^j[\frac{\beta\theta+\delta(j+1)}{\beta\theta+\delta}] \end{aligned} \quad (3.240)$$

Identity 3.39
Equating the above two methods we get

$$\sum_{j=n}^{\infty}(-1)^{j-n}\binom{j}{n}(\frac{t}{\theta})^j[\frac{\beta\theta+\delta(j+1)}{\beta\theta+\delta}]=(\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^2[\frac{\beta(t+\theta)+\delta(n+1)}{\beta\theta+\delta}]$$
$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{j!}{\theta^j}[\frac{\beta\theta+\delta(j+1)}{\beta\theta+\delta}]=\Gamma(n+1)\frac{\theta^2}{(t+\theta)^{n+2}}[\frac{\beta(t+\theta)+\delta(n+1)}{\beta\theta+\delta}] \tag{3.241}$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= [\frac{\theta}{(1-s)t+\theta}]^2[\frac{\beta[(1-s)t+\theta]+\delta}{\beta\theta+\delta}] \end{aligned} \tag{3.242}$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^rE(\wedge^r) \\ &= t^r\frac{r!}{\theta^r}[\frac{\beta\theta+\delta(r+1)}{\beta\theta+\delta}] \\ &= r!(\frac{t}{\theta})^r[\frac{\beta\theta+\delta(r+1)}{\beta\theta+\delta}] \end{aligned} \tag{3.243}$$

$$E(T)=(\frac{t}{\theta})[\frac{\beta\theta+2\delta}{\beta\theta+\delta}] \tag{3.244}$$

3.16 Erlang-4 Parameter Generalized Lindley (G4L) Distribution and Its Links

3.16.1 Erlang-4 Parameter Generalized Lindley (G4L) Mixture

The 4 Parameter Generalized Lindley mixing distribution is

$$g(\lambda)=\frac{\theta^{\alpha+1}[\beta\alpha+\delta\lambda]}{\beta\theta+\delta}\frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)},\quad \lambda>0;\theta>0,\alpha>0,\beta>0,\delta>0 \tag{3.245}$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_0^\infty \lambda^n e^{-t\lambda} \frac{\theta^{\alpha+1}[\beta\alpha+\delta\lambda]}{\beta\theta+\delta} \frac{e^{-\theta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha+1)} d\lambda$$
$$= \frac{\theta^{\alpha+1}}{(\beta\theta+\delta)\Gamma(\alpha+1)} \int_0^\infty \lambda^{n+\alpha-1} e^{-\lambda(t+\theta)}[\beta\alpha+\delta\lambda] d\lambda$$
$$= \frac{\theta^{\alpha+1}}{(\beta\theta+\delta)\Gamma(\alpha+1)} [\beta\alpha \frac{\Gamma(n+\alpha)}{(t+\theta)^{n+\alpha}} + \delta \frac{\Gamma(n+\alpha+1)}{(t+\theta)^{n+\alpha+1}}]$$
$$= \frac{\theta^{\alpha+1}}{(\beta\theta+\delta)\Gamma(\alpha+1)} \frac{\Gamma(n+\alpha)}{(t+\theta)^{n+\alpha+1}} [\beta\alpha(t+\theta)+\delta(n+\alpha)]$$
$$= \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}} [\frac{\beta\alpha(t+\theta)+\delta(n+\alpha)}{\beta\theta+\delta}] \tag{3.246}$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^{n-1}}{\Gamma n} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}} [\frac{\beta\alpha(t+\theta)+\delta(n+\alpha)}{\beta\theta+\delta}]$$
$$= \binom{n+\alpha-1}{n-1} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\beta\alpha(t+\theta)+\delta(n+\alpha)}{t(\beta\theta+\delta)}], \quad t>0; \theta>0, \alpha>0, \beta>0, \delta>0, n=1,2,3, \tag{3.247}$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\beta\alpha\theta+\delta(j+\alpha)}{\beta\theta+\delta}] \tag{3.248}$$

Identity 3.40

Equating the above two methods we get

$$\sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\beta\alpha\theta+\delta(j+\alpha)}{\beta\theta+\delta}] = \frac{1}{t} \binom{n+\alpha-1}{n-1} (\frac{t}{t+\theta})^n (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\beta\alpha(t+\theta)+\delta(n+\alpha)}{\beta\theta+\delta}]$$
$$\sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\beta\alpha\theta+\delta(j+\alpha)}{\beta\theta+\delta}] = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}} [\frac{\beta\alpha(t+\theta)+\delta(n+\alpha)}{\beta\theta+\delta}] \tag{3.249}$$

The rth moment of the Erlang mixture is

$$E(T^r) = \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r})$$

$$= \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(\alpha - r)}{\Gamma(\alpha+1)} \theta^r [\frac{\beta\alpha\theta + \delta(\alpha - r)}{\beta\theta + \delta}] \quad (3.250)$$

$$E(T) = \frac{\Gamma(n+1)}{\Gamma n} \frac{\Gamma(\alpha - 1)}{\Gamma(\alpha+1)} \theta [\frac{\beta\alpha\theta + \delta(\alpha - 1)}{\beta\theta + \delta}]$$

$$= \frac{n\theta}{\alpha(\alpha-1)} [\frac{\beta\alpha\theta + \delta(\alpha - 1)}{\beta\theta + \delta}] \quad (3.251)$$

3.16.2 Exponential-4 Parameter Generalized Lindley (G4L) Mixture

Construction by the direct method gives

$$f_1(t) = E[\wedge e^{-t\wedge}]$$

$$= \frac{\theta^{\alpha+1}}{(t+\theta)^{\alpha+2}} [\frac{\beta\alpha(t+\theta) + \delta(\alpha+1)}{\beta\theta + \delta}]$$

$$= (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\beta\alpha(t+\theta) + \delta(\alpha+1)}{(t+\theta)(\beta\theta+\delta)}], \quad t > 0; \theta > 0, \alpha > 0, \beta > 0, \delta > 0 \quad (3.252)$$

By the method of moments we have

$$f_1(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j)$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\beta\alpha\theta + \delta(j+\alpha)}{\beta\theta + \delta}] \quad (3.253)$$

Identity 3.41
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)} \frac{1}{\theta^j} [\frac{\beta\alpha\theta + \delta(j+\alpha)}{\beta\theta + \delta}] = (\frac{\theta}{t+\theta})^{\alpha+1} [\frac{\beta\alpha(t+\theta) + \delta(\alpha+1)}{(t+\theta)(\beta\theta+\delta)}] \quad (3.254)$$

The rth moment of the Exponential mixture is

$$E(T^r) = r!E(\wedge^{-r})$$
$$= r!\frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)}\theta^r[\frac{\beta\alpha\theta+\delta(\alpha-r)}{\beta\theta+\delta}] \quad (3.255)$$
$$E(T) = \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}\theta[\frac{\beta\alpha\theta+\delta(\alpha-1)}{\beta\theta+\delta}]$$
$$= \frac{\theta}{\alpha(\alpha-1)}[\frac{\beta\alpha\theta+\delta(\alpha-1)}{\beta\theta+\delta}] \quad (3.256)$$

3.16.3 Poisson-4 Parameter Generalized Lindley (G4L) Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$P_n(t) = \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^n}{n!}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)}\frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}}[\frac{\beta\alpha(t+\theta)+\delta(n+\alpha)}{\beta\theta+\delta}]$$
$$= \binom{n+\alpha-1}{n-1}(\frac{t}{t+\theta})^n(\frac{\theta}{t+\theta})^{\alpha+1}[\frac{\beta\alpha(t+\theta)+\delta(n+\alpha)}{n(\beta\theta+\delta)}], \quad t>0;\theta>0,\alpha>0,\beta>0,\delta>0 \quad (3.257)$$

Bt the method of moments we have

$$P_n(t) = \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j)$$
$$= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)}\frac{1}{\theta^j}[\frac{\beta\alpha\theta+\delta(j+\alpha)}{\beta\theta+\delta}]$$
$$= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}}{j}\binom{j}{n}\binom{\alpha+j-1}{j-1}(\frac{t}{\theta})^j[\frac{\beta\alpha\theta+\delta(j+\alpha)}{\beta\theta+\delta}] \quad (3.258)$$

Identity 3.42

Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n}}{j}\binom{j}{n}\binom{\alpha+j-1}{j-1}(\frac{t}{\theta})^{j}[\frac{\beta\alpha\theta+\delta(j+\alpha)}{\beta\theta+\delta}] = \binom{n+\alpha-1}{n-1}(\frac{t}{t+\theta})^{n}(\frac{\theta}{t+\theta})^{\alpha+1}[\frac{\beta\alpha(t+\theta)+\delta}{n(\beta\theta+}$$

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{\Gamma(j+\alpha)}{\Gamma(\alpha+1)}\frac{1}{\theta^{j}}[\frac{\beta\alpha\theta+\delta(j+\alpha)}{\beta\theta+\delta}] = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)}\frac{\theta^{\alpha+1}}{(t+\theta)^{n+\alpha+1}}[\frac{\beta\alpha(t+\theta)+\delta(n+\alpha)}{\beta\theta+\delta}] \tag{3.259}$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= \frac{\Gamma\alpha}{\Gamma(\alpha+1)}\frac{\theta^{\alpha+1}}{[(1-s)t+\theta]^{\alpha+1}}[\frac{\beta\alpha[(1-s)t+\theta]+\delta\alpha}{\beta\theta+\delta}] \\ &= [\frac{\theta}{[(1-s)t+\theta]}]^{\alpha+1}[\frac{\beta[(1-s)t+\theta]+\delta}{\beta\theta+\delta}] \end{aligned} \tag{3.260}$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^{r}) &= t^{r}E(\wedge^{r}) \\ &= t^{r}\frac{\Gamma(\alpha+r)}{\Gamma(\alpha+1)}\frac{1}{\theta^{r}}[\frac{\beta\alpha\theta+\delta(\alpha+r)}{\beta\theta+\delta}] \\ &= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha+1)}(\frac{t}{\theta})^{r}[\frac{\beta\alpha\theta+\delta(\alpha+r)}{\beta\theta+\delta}] \end{aligned} \tag{3.261}$$

$$E(T) = (\frac{t}{\theta})[\frac{\beta\alpha\theta+\delta(\alpha+1)}{\beta\theta+\delta}] \tag{3.262}$$

3.17 Erlang-Transmuted Exponential Distribution and Its Links

Given the Transmuted probability distribution as

$$G(t) = (1+y)F(t) - y[F(t)]^{2}; \quad -1 \leq y \leq 1$$

where F(t) and f(t) are the old cdf and pdf respectively and G(t) and g(t) are the new cdf and pdf respectively.

The derivative is

$$g(t) = (1+y)f(t) - 2yF(t)f(t)$$

Let f(t) be an Exponential pdf;

$$f(t) = \theta e^{-\theta t}; \quad t > 0; \theta > 0$$

then

$$\begin{aligned}
G(t) &= (1+y)[1-e^{-\theta t}] - y[1-e^{-\theta t}]^2 \\
g(t) &= (1+y)\theta e^{-\theta t} - 2y[1-e^{-\theta t}]\theta e^{-\theta t} \\
&= (1+y)\theta e^{-\theta t} - 2y\theta e^{-\theta t} + 2y\theta e^{-2\theta t} \\
&= (1-y)\theta e^{-\theta t} + 2y\theta e^{-2\theta t} \\
&= (1-y)\theta e^{-\theta t} + y(2\theta e^{-2\theta t}), \quad t > 0; \theta > 0 \\
let \quad y &= \alpha \\
g(t) &= (1-\alpha)\theta e^{-\theta t} + 2\alpha\theta e^{-2\theta t}
\end{aligned}$$

which is the Transmuted Exponential distribution.

3.17.1 Erlang-Transmuted Exponential Mixture

The Transmuted Exponential mixing distribution is

$$g(\lambda) = (1-\alpha)\theta e^{-\theta\lambda} + 2\alpha\theta e^{-2\theta\lambda}, \quad \lambda > 0; \theta > 0, \alpha > 0 \tag{3.263}$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda}[(1-\alpha)\theta e^{-\theta\lambda} + 2\alpha\theta e^{-2\theta\lambda}]d\lambda \\
&= (1-\alpha)\theta \int_0^\infty \lambda^n e^{-\lambda(t+\theta)} d\lambda + 2\alpha\theta \int_0^\infty \lambda^n e^{-\lambda(t+2\theta)} d\lambda \\
&= (1-\alpha)\theta \frac{\Gamma(n+1)}{(t+\theta)^{n+1}} + 2\alpha\theta \frac{\Gamma(n+1)}{(t+2\theta)^{n+1}} \\
&= \theta\Gamma(n+1)[\frac{(1-\alpha)}{(t+\theta)^{n+1}} + \frac{2\alpha}{(t+2\theta)^{n+1}}]
\end{aligned} \tag{3.264}$$

Construction by the direct method gives

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^{n-1}}{\Gamma n} \theta\Gamma(n+1)[\frac{(1-\alpha)}{(t+\theta)^{n+1}} + \frac{2\alpha}{(t+2\theta)^{n+1}}] \\ &= \theta n t^{n-1}[\frac{(1-\alpha)}{(t+\theta)^{n+1}} + \frac{2\alpha}{(t+2\theta)^{n+1}}] \\ &= \frac{n}{t}(1-\alpha)(\frac{\theta}{t+\theta})(\frac{t}{t+\theta})^n + \frac{n\alpha}{t}(\frac{2\theta}{t+2\theta})(\frac{t}{t+2\theta})^n, \quad t>0;\theta>0,\alpha>0,n=1,2,3,... \end{aligned} \tag{3.265}$$

By the method of moments we have

$$\begin{aligned} f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\theta j!}{(2\theta)^{j+1}} [2^{j+1}(1-\alpha)+2\alpha] \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{j!}{(2\theta)^{j}} [2^{j}(1-\alpha)+\alpha] \end{aligned} \tag{3.266}$$

Identity 3.43

Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{j!}{(2\theta)^{j}} [2^{j}(1-\alpha)+\alpha] &= \frac{n}{t}(1-\alpha)(\frac{\theta}{t+\theta})(\frac{t}{t+\theta})^n + \frac{n\alpha}{t}(\frac{2\theta}{t+2\theta})(\frac{t}{t+2\theta})^n \\ \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{j!}{(2\theta)^{j}} [2^{j}(1-\alpha)+\alpha] &= \theta\Gamma(n+1)[\frac{(1-\alpha)}{(t+\theta)^{n+1}} + \frac{2\alpha}{(t+2\theta)^{n+1}}] \end{aligned} \tag{3.267}$$

The rth moment of the Erlang mixture is

$$\begin{aligned} E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\ &= \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(1-r)}{(2\theta)^{-r}} [2^{-r}(1-\alpha)+\alpha] \\ &= \frac{\Gamma(n+r)}{\Gamma n} \Gamma(1-r)(2\theta)^r[\frac{(1-\alpha)}{2^r}+\alpha] \end{aligned} \tag{3.268}$$

$$E(T) = \frac{\Gamma(n+1)}{\Gamma n} \Gamma(0)(2\theta)[\frac{(1-\alpha)}{2}+\alpha] = \infty \tag{3.269}$$

3.17.2 Exponential-Transmuted Exponential Mixture

Construction by the direct method gives

$$
\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= \theta[\frac{(1-\alpha)}{(t+\theta)^2} + \frac{2\alpha}{(t+2\theta)^2}], \quad t>0; \theta>0, \alpha>0
\end{aligned} \tag{3.270}
$$

By the method of moments we have

$$
\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} E(\wedge^j) \\
&= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} \frac{j!}{(2\theta)^j}[2^j(1-\alpha)+\alpha] \\
&= \sum_{j=1}^{\infty} (-1)^{j-1}t^{j-1} \frac{j}{(2\theta)^j}[2^j(1-\alpha)+\alpha]
\end{aligned} \tag{3.271}
$$

Identity 3.44
Equating the above two methods we get

$$
\sum_{j=1}^{\infty} (-1)^{j-1}t^{j-1} \frac{j}{(2\theta)^j}[2^j(1-\alpha)+\alpha] = \theta[\frac{(1-\alpha)}{(t+\theta)^2} + \frac{2\alpha}{(t+2\theta)^2}] \tag{3.272}
$$

The rth moment of the Exponential mixture is

$$
\begin{aligned}
E(T^r) &= r!E(\wedge^{-r}) \\
&= r!\frac{\Gamma(1-r)}{(2\theta)^{-r}}[2^{-r}(1-\alpha)+\alpha] \\
&= r!\Gamma(1-r)(2\theta)^r[\frac{(1-\alpha)}{2^r}+\alpha]
\end{aligned} \tag{3.273}
$$

$$
E(T) = \Gamma(0)(2\theta)[\frac{(1-\alpha)}{2}+\alpha] = \infty \tag{3.274}
$$

3.17.3 Poisson-Transmuted Exponential Mixture

$$
P_n(t) = \frac{t}{n} f_n(t)
$$

Construction by the direct method gives

$$\begin{aligned}
P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^n}{n!}\theta\Gamma(n+1)[\frac{(1-\alpha)}{(t+\theta)^{n+1}} + \frac{2\alpha}{(t+2\theta)^{n+1}}] \\
&= t^n\theta[\frac{(1-\alpha)}{(t+\theta)^{n+1}} + \frac{2\alpha}{(t+2\theta)^{n+1}}] \\
&= (1-\alpha)(\frac{\theta}{t+\theta})(\frac{t}{t+\theta})^n + \alpha(\frac{2\theta}{t+2\theta})(\frac{t}{t+2\theta})^n, \quad t>0;\theta>0,\alpha>0
\end{aligned} \tag{3.275}$$

By the method of moments we have

$$\begin{aligned}
P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\
&= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{j!}{(2\theta)^j}[2^j(1-\alpha)+\alpha] \\
&= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{(2\theta)^j}\binom{j}{n}[2^j(1-\alpha)+\alpha]
\end{aligned} \tag{3.276}$$

Identity 3.45
Equating the above two methods we get

$$\begin{aligned}
\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{(2\theta)^j}\binom{j}{n}[2^j(1-\alpha)+\alpha] &= (1-\alpha)(\frac{\theta}{t+\theta})(\frac{t}{t+\theta})^n + \alpha(\frac{2\theta}{t+2\theta})(\frac{t}{t+2\theta})^n \\
\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{j!}{(2\theta)^j}[2^j(1-\alpha)+\alpha] &= \theta\Gamma(n+1)[\frac{(1-\alpha)}{(t+\theta)^{n+1}} + \frac{2\alpha}{(t+2\theta)^{n+1}}]
\end{aligned} \tag{3.277}$$

The PGF of the Poisson mixture is

$$\begin{aligned}
G(s,t) &= E[e^{-(1-s)t\wedge}] \\
&= \theta[\frac{(1-s)}{(1-s)t+\theta} + \frac{2\alpha}{(1-s)t+2\theta}]
\end{aligned} \tag{3.278}$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r \frac{r!}{(2\theta)^r}[2^r(1-\alpha)+\alpha] \\ &= r!(\frac{t}{2\theta})^r[2^r(1-\alpha)+\alpha] \end{aligned} \tag{3.279}$$

$$\begin{aligned} E(T) &= (\frac{t}{2\theta})[2(1-\alpha)+\alpha] \\ &= (\frac{t}{2\theta})(2-\alpha) \end{aligned} \tag{3.280}$$

4 ERLANG MIXTURES BASED ON MODIFIED BESSEL FUNCTION OF THE THIRD KIND

4.1 Introduction

In this chapter Erlang mixtures are expressed based on Modified Bessel function of the third kind.
The modified Bessel function of the third kind has been defined and its properties given.
Moments about the origin (raw moments) of the Erlang mixtures have been derived and specifically the first moment has been obtained.
Special cases of the Generalized Inverse Gaussian Distribution have also been derived.
The Exponential mixtures and Poisson mixtures have also been obtained and the PGFs determined in the Poisson mixtures.

4.1.1 Definition

The modified Bessel function of the third kind is defined as;

$$K_v(w) = \frac{1}{2}\int_0^\infty x^{v-1} e^{-\frac{w}{2}(x+\frac{1}{x})} dx \tag{4.1}$$

which is a fuction of w with index v.

4.1.2 Properties of the Bessel function of the third kind

They include;

$$1. K_v(w) = K_{-v}(w) \tag{4.2}$$

$$2. K_{v+1}(w) = \frac{2v}{w} K_v(w) + K_{v-1}(w) \tag{4.3}$$

$$3. K_v'(w) = \frac{d}{dw} K_v(w) = -\frac{1}{2}[K_{v-1}(w) + K_{v+1}(w)] \tag{4.4}$$

$$4. K_{v+\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w}[1 + \sum_{i=1}^{v} \frac{(v+1)!(2w)^{-i}}{(v-1)!i!}] \tag{4.5}$$

$$5. K_{\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w} \tag{4.6}$$

4.2 Erlang-Inverse Gamma Distribution and Its Links

4.2.1 Erlang-Inverse Gamma Mixture

The Inverse Gamma mixing distribution is

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma\alpha} e^{\frac{-\beta}{\lambda}} \lambda^{-\alpha-1}, \quad \lambda > 0; \alpha > 0, \beta > 0 \tag{4.7}$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda} \frac{\beta^{\alpha}}{\Gamma\alpha} e^{\frac{-\beta}{\lambda}} \lambda^{-\alpha-1} d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^\infty \lambda^{n-\alpha-1} e^{-t\lambda-\frac{\beta}{\lambda}} d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^\infty \lambda^{(n-\alpha)-1} e^{-t(\lambda+\frac{\beta}{t}\frac{1}{\lambda})} d\lambda
\end{aligned}$$

$$let \quad \lambda = \sqrt{\frac{\beta}{t}}x \quad \Longrightarrow \quad d\lambda = \sqrt{\frac{\beta}{t}}dx$$

$$\begin{aligned}
E[\wedge^n e^{-t\wedge}] &= \frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^\infty (\sqrt{\frac{\beta}{t}}x)^{n-\alpha-1} e^{-t(\sqrt{\frac{\beta}{t}}x+\frac{\beta}{t}\frac{1}{\sqrt{\frac{\beta}{t}}x})} \sqrt{\frac{\beta}{t}}dx \\
&= \frac{\beta^{\alpha}}{\Gamma\alpha} (\sqrt{\frac{\beta}{t}})^{n-\alpha} \int_0^\infty x^{n-\alpha-1} e^{-t\sqrt{\frac{\beta}{t}}(x+\frac{1}{x})} dx \\
&= \frac{\beta^{\alpha}}{\Gamma\alpha} (\sqrt{\frac{\beta}{t}})^{n-\alpha} \int_0^\infty x^{(n-\alpha)-1} e^{\frac{-2\sqrt{\beta t}}{2}(x+\frac{1}{x})} dx \\
&= \frac{\beta^{\alpha}}{\Gamma\alpha} (\sqrt{\frac{\beta}{t}})^{n-\alpha} 2K_{n-\alpha}(2\sqrt{\beta t}) \qquad (4.8) \\
E(\wedge^j) &= \int_0^\infty \lambda^j \frac{\beta^{\alpha}}{\Gamma\alpha} e^{\frac{-\beta}{\lambda}} \lambda^{-\alpha-1} d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^\infty \lambda^{j-\alpha-1} e^{\frac{-\beta}{\lambda}} d\lambda
\end{aligned}$$

$$let \quad x = \frac{\beta}{\lambda} \quad \Longrightarrow \lambda = \frac{\beta}{x} \quad \Longrightarrow d\lambda = \frac{-\beta}{x^2}dx$$

$$\begin{aligned}
E(\wedge^j) &= \frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^\infty (\frac{\beta}{x})^{j-\alpha-1} e^{-x} (\frac{-\beta}{x^2}) dx \\
&= \frac{\beta^{\alpha}\beta^{j-\alpha}}{\Gamma\alpha} \int_0^\infty x^{\alpha-j-1} e^{-x} dx \\
&= \frac{\beta^j}{\Gamma\alpha}\Gamma(\alpha-j) \\
&= \beta^j \frac{\Gamma(\alpha-j)}{\Gamma\alpha} \qquad (4.9)
\end{aligned}$$

Construction by the direct method gives

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} \frac{\beta^\alpha}{\Gamma\alpha} (\sqrt{\frac{\beta}{t}})^{n-\alpha} 2K_{n-\alpha}(2\sqrt{\beta t}) \\
&= \frac{(\sqrt{\beta t})^{n+\alpha}}{t\Gamma n \Gamma\alpha} 2K_{n-\alpha}(2\sqrt{\beta t}) \qquad (4.10)
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \beta^j \frac{\Gamma(\alpha - j)}{\Gamma\alpha} \qquad (4.11)
\end{aligned}
$$

Identity 4.1

Equating the above two methods we get

$$
\begin{aligned}
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \beta^j \frac{\Gamma(\alpha - j)}{\Gamma\alpha} &= \frac{(\sqrt{\beta t})^{n+\alpha}}{t\Gamma n \Gamma\alpha} 2K_{n-\alpha}(2\sqrt{\beta t}) \\
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \beta^j \frac{\Gamma(\alpha - j)}{\Gamma\alpha} &= \frac{\beta^\alpha}{\Gamma\alpha} (\sqrt{\frac{\beta}{t}})^{n-\alpha} 2K_{n-\alpha}(2\sqrt{\beta t}) \qquad (4.12)
\end{aligned}
$$

The rth moment of the Erlang mixture is

$$
\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \beta^{-r} \frac{\Gamma(\alpha + r)}{\Gamma\alpha} \\
&= \frac{1}{\beta^r} \frac{\Gamma(n+r)}{\Gamma n} \frac{\Gamma(\alpha + r)}{\Gamma\alpha} \qquad (4.13) \\
E(T) &= \frac{1}{\beta} \frac{\Gamma(n+1)}{\Gamma n} \frac{\Gamma(\alpha + 1)}{\Gamma\alpha} \\
&= \frac{n\alpha}{\beta} \qquad (4.14)
\end{aligned}
$$

4.2.2 Exponential-Inverse Gamma Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= \frac{\beta^\alpha}{\Gamma\alpha}\left(\sqrt{\frac{\beta}{t}}\right)^{1-\alpha} 2K_{1-\alpha}(2\sqrt{\beta t}) \end{aligned} \tag{4.15}$$

By the method of moments we have

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} E(\wedge^j) \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} \beta^j \frac{\Gamma(\alpha-j)}{\Gamma\alpha} \end{aligned} \tag{4.16}$$

Identity 4.2
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} \beta^j \frac{\Gamma(\alpha-j)}{\Gamma\alpha} = \frac{\beta^\alpha}{\Gamma\alpha}\left(\sqrt{\frac{\beta}{t}}\right)^{1-\alpha} 2K_{1-\alpha}(2\sqrt{\beta t}) \tag{4.17}$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= \frac{r!}{\beta^r}\frac{\Gamma(\alpha+r)}{\Gamma\alpha} \end{aligned} \tag{4.18}$$

$$\begin{aligned} E(T) &= \frac{1}{\beta}\frac{\Gamma(\alpha+1)}{\Gamma\alpha} \\ &= \frac{\alpha}{\beta} \end{aligned} \tag{4.19}$$

4.2.3 Poisson-Inverse Gamma Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!}\frac{\beta^\alpha}{\Gamma\alpha}(\sqrt{\frac{\beta}{t}})^{n-\alpha}2K_{n-\alpha}(2\sqrt{\beta t}) \\ &= \frac{(\sqrt{\beta t})^{n+\alpha}}{n!\Gamma\alpha}2K_{n-\alpha}(2\sqrt{\beta t}) \end{aligned} \tag{4.20}$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\ &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\beta^j\frac{\Gamma(\alpha-j)}{\Gamma\alpha} \end{aligned} \tag{4.21}$$

Identity 4.3

Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\beta^j\frac{\Gamma(\alpha-j)}{\Gamma\alpha} &= \frac{(\sqrt{\beta t})^{n+\alpha}}{n!\Gamma\alpha}2K_{n-\alpha}(2\sqrt{\beta t}) \\ \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\beta^j\frac{\Gamma(\alpha-j)}{\Gamma\alpha} &= \frac{\beta^\alpha}{\Gamma\alpha}(\sqrt{\frac{\beta}{t}})^{n-\alpha}2K_{n-\alpha}(2\sqrt{\beta t}) \end{aligned} \tag{4.22}$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= \frac{\beta^\alpha}{\Gamma\alpha}(\sqrt{\frac{\beta}{(1-s)t}})^{1-\alpha}2K_{1-\alpha}(2\sqrt{\beta t(1-s)}) \end{aligned} \tag{4.23}$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r\beta^r\frac{\Gamma(\alpha-r)}{\Gamma\alpha} \\ &= (\beta t)^r\frac{\Gamma(\alpha-r)}{\Gamma\alpha} \end{aligned} \tag{4.24}$$

$$E(T) = (\beta t)\frac{\Gamma(\alpha-1)}{\Gamma\alpha}$$
$$= \frac{\beta t}{(\alpha-1)} \tag{4.25}$$

4.3 Erlang-Pearson Type V Distribution and Its Links

4.3.1 Erlang-Pearson Type V Mixture

The Pearson Type V mixing distribution is

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma\alpha}e^{-\frac{\beta}{\lambda-c}}(\lambda-c)^{-(\alpha+1)}, \quad \lambda > c; \alpha > 0, \beta > 0 \tag{4.26}$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_c^\infty \lambda^n e^{-t\lambda}\frac{\beta^\alpha}{\Gamma\alpha}e^{-\frac{\beta}{\lambda-c}}(\lambda-c)^{-(\alpha+1)}d\lambda$$
$$= \frac{\beta^\alpha}{\Gamma\alpha}\int_c^\infty \lambda^n(\lambda-c)^{-(\alpha+1)}e^{-t\lambda-\frac{\beta}{\lambda-c}}d\lambda$$

$$let \quad x = \lambda - c \quad \Longrightarrow dx = d\lambda$$

$$E[\wedge^n e^{-t\wedge}] = \frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty (x+c)^n(x)^{-\alpha-1}e^{-t(x+c)-\frac{\beta}{x}}dx$$
$$= \frac{\beta^\alpha}{\Gamma\alpha}e^{-tc}\int_0^\infty (x+c)^n(x)^{-\alpha-1}e^{-tx-\frac{\beta}{x}}dx$$
$$= \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha}[\sum_{k=0}^{n}\binom{n}{k}c^{n-k}\int_0^\infty (x)^{k-\alpha-1}e^{-t(x+\frac{\beta}{t}\frac{1}{x})}dx]$$

$$let \quad x = \sqrt{\frac{\beta}{t}}y \quad \Longrightarrow dx = \sqrt{\frac{\beta}{t}}dy$$

$$E[\wedge^n e^{-t\wedge}] = \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha}[\sum_{k=0}^{n}\binom{n}{k}c^{n-k}\int_0^\infty (\sqrt{\frac{\beta}{t}}y)^{k-\alpha-1}e^{-t(\sqrt{\frac{\beta}{t}}y+\frac{\beta}{t}\frac{1}{\sqrt{\frac{\beta}{t}}y})}\sqrt{\frac{\beta}{t}}dy]$$
$$= \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha}[\sum_{k=0}^{n}\binom{n}{k}c^{n-k}(\sqrt{\frac{\beta}{t}})^{k-\alpha}\int_0^\infty y^{k-\alpha-1}e^{-t\sqrt{\frac{\beta}{t}}(y+\frac{1}{y})}dy]$$
$$= \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha}[\sum_{k=0}^{n}\binom{n}{k}c^{n-k}(\sqrt{\frac{\beta}{t}})^{k-\alpha}\int_0^\infty y^{(k-\alpha)-1}e^{\frac{-2\sqrt{\beta t}}{2}(y+\frac{1}{y})}dy]$$
$$= \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha}\sum_{k=0}^{n}\binom{n}{k}c^{n-k}(\sqrt{\frac{\beta}{t}})^{k-\alpha}2K_{k-\alpha}(2\sqrt{\beta t}) \tag{4.27}$$

$$
\begin{aligned}
E(\wedge^j) &= \int_c^\infty \lambda^j \frac{\beta^\alpha}{\Gamma\alpha} e^{\frac{-\beta}{\lambda-c}} (\lambda-c)^{-(\alpha+1)} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma\alpha} \int_c^\infty \lambda^j (\lambda-c)^{-(\alpha+1)} e^{\frac{-\beta}{\lambda-c}} d\lambda
\end{aligned}
$$

$$
let \quad x = \lambda - c \quad \Longrightarrow dx = d\lambda
$$

$$
\begin{aligned}
E(\wedge^j) &= \frac{\beta^\alpha}{\Gamma\alpha} \int_0^\infty (x+c)^j x^{-\alpha-1} e^{\frac{-\beta}{x}} dx \\
&= \frac{\beta^\alpha}{\Gamma\alpha} [\sum_{k=0}^{j} \binom{j}{k} c^{j-k} \int_0^\infty x^{k-\alpha-1} e^{\frac{-\beta}{x}} dx]
\end{aligned}
$$

$$
let \quad y = \frac{\beta}{x} \quad \Longrightarrow x = \frac{\beta}{y} \quad \Longrightarrow dx = \frac{-\beta}{y^2} dy
$$

$$
\begin{aligned}
E(\wedge^j) &= \frac{\beta^\alpha}{\Gamma\alpha} [\sum_{k=0}^{j} \binom{j}{k} c^{j-k} \int_0^\infty (\frac{\beta}{y})^{k-\alpha-1} e^{-y} (\frac{-\beta}{y^2}) dy] \\
&= \frac{\beta^\alpha}{\Gamma\alpha} [\sum_{k=0}^{j} \binom{j}{k} c^{j-k} \beta^{k-\alpha} \int_0^\infty y^{k-\alpha-1} e^{-y} dy] \\
&= \frac{\beta^\alpha}{\Gamma\alpha} [\sum_{k=0}^{j} \binom{j}{k} c^{j-k} \beta^{k-\alpha} \Gamma(\alpha-k)] \\
&= \sum_{k=0}^{j} \binom{j}{k} c^{j-k} \beta^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \\
&= \sum_{k=0}^{j} \binom{j}{k} c^j (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \qquad \textbf{(4.28)}
\end{aligned}
$$

Construction by the direct method gives

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha} \sum_{k=0}^{n} \binom{n}{k} c^{n-k} (\sqrt{\frac{\beta}{t}})^{k-\alpha} 2K_{k-\alpha}(2\sqrt{\beta t}) \\
&= \frac{(tc)^n}{t\Gamma n} \frac{\beta^\alpha}{\Gamma\alpha} e^{-tc} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{c^k} (\sqrt{\frac{\beta}{t}})^{k-\alpha} 2K_{k-\alpha}(2\sqrt{\beta t}) \qquad \textbf{(4.29)}
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \sum_{k=0}^{j} \binom{j}{k} c^j (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \qquad (4.30)
\end{aligned}
$$

Identity 4.4
Equating the above two methods we get

$$
\begin{aligned}
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \sum_{k=0}^{j} \binom{j}{k} c^j (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} &= \frac{(tc)^n}{t\Gamma n} \frac{\beta^\alpha}{\Gamma\alpha} e^{-tc} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{c^k} (\sqrt{\frac{\beta}{t}})^{k-\alpha} 2K_{k-\alpha}(2\sqrt{\beta t}) \\
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \sum_{k=0}^{j} \binom{j}{k} c^j (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} &= \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha} \sum_{k=0}^{n} \binom{n}{k} c^{n-k} (\sqrt{\frac{\beta}{t}})^{k-\alpha} 2K_{k-\alpha}(2\sqrt{\beta t})
\end{aligned}
\qquad (4.31)
$$

The rth moment of the Erlang mixture is

$$
\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \sum_{k=0}^{\infty} \binom{-r}{k} c^{-r} (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \\
&= \frac{\Gamma(n+r)}{c^r \Gamma n} \sum_{k=0}^{\infty} \binom{-r}{k} (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \qquad (4.32) \\
E(T) &= \frac{\Gamma(n+1)}{c\Gamma n} \sum_{k=0}^{\infty} \binom{-1}{k} (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \\
&= \frac{n}{c} \sum_{k=0}^{\infty} \binom{-1}{k} (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \qquad (4.33)
\end{aligned}
$$

4.3.2 Exponential-Pearson Type V Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha} \sum_{k=0}^{1} \binom{1}{k} c^{1-k} (\sqrt{\frac{\beta}{t}})^{k-\alpha} 2K_{k-\alpha}(2\sqrt{\beta t}) \\ &= \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha} [c(\sqrt{\frac{\beta}{t}})^{-\alpha} 2K_{-\alpha}(2\sqrt{\beta t}) + (\sqrt{\frac{\beta}{t}})^{1-\alpha} 2K_{1-\alpha}(2\sqrt{\beta t})] \\ &= \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha} 2(\sqrt{\frac{\beta}{t}})^{-\alpha} [cK_\alpha(2\sqrt{\beta t}) + (\sqrt{\frac{\beta}{t}}) K_{\alpha-1}(2\sqrt{\beta t})] \end{aligned} \quad (4.34)$$

By the method of moments we have

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \sum_{k=0}^{j} \binom{j}{k} c^j (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \end{aligned} \quad (4.35)$$

Identity 4.5
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \sum_{k=0}^{j} \binom{j}{k} c^j (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} = \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha} 2(\sqrt{\frac{\beta}{t}})^{-\alpha} [cK_\alpha(2\sqrt{\beta t}) + (\sqrt{\frac{\beta}{t}}) K_{\alpha-1}(2\sqrt{\beta t})] \quad (4.36)$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r! E(\wedge^{-r}) \\ &= r! \sum_{k=0}^{\infty} \binom{-r}{k} c^{-r} (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \\ &= \frac{r!}{c^r} \sum_{k=0}^{\infty} \binom{-r}{k} (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \end{aligned} \quad (4.37)$$

$$E(T) = \frac{1}{c} \sum_{k=0}^{\infty} \binom{-1}{k} (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \quad (4.38)$$

4.3.3 Poisson-Pearson Type V Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$\begin{aligned}
P_n(t) &= \frac{t^n}{n!} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^n}{n!} \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha} \sum_{k=0}^{n} \binom{n}{k} c^{n-k} (\sqrt{\frac{\beta}{t}})^{k-\alpha} 2K_{k-\alpha}(2\sqrt{\beta t}) \\
&= \frac{2(ct)^n \beta^\alpha e^{-tc}}{n!\Gamma\alpha} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{c^k} (\sqrt{\frac{\beta}{t}})^{k-\alpha} K_{k-\alpha}(2\sqrt{\beta t})
\end{aligned} \tag{4.39}$$

By the method of moments we have

$$\begin{aligned}
P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \sum_{k=0}^{j} \binom{j}{k} c^j (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} (tc)^j}{n!(j-n)!} \sum_{k=0}^{j} \binom{j}{k} (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha}
\end{aligned} \tag{4.40}$$

Identity 4.6

Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} (tc)^j}{n!(j-n)!} \sum_{k=0}^{j} \binom{j}{k} (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} = \frac{2(ct)^n \beta^\alpha e^{-tc}}{n!\Gamma\alpha} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{c^k} (\sqrt{\frac{\beta}{t}})^{k-\alpha} K_{k-\alpha}(2\sqrt{\beta t})$$

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \sum_{k=0}^{j} \binom{j}{k} c^j (\frac{\beta}{c})^k \frac{\Gamma(\alpha-k)}{\Gamma\alpha} = \frac{\beta^\alpha e^{-tc}}{\Gamma\alpha} \sum_{k=0}^{n} \binom{n}{k} c^{n-k} (\sqrt{\frac{\beta}{t}})^{k-\alpha} 2K_{k-\alpha}(2\sqrt{\beta t}) \tag{4.41}$$

The PGF of the Poisson mixture is

$$\begin{aligned}
G(s,t) &= E[e^{-(1-s)t\wedge}] \\
&= \frac{\beta^{\alpha}e^{-ct(1-s)}}{\Gamma\alpha}\sum_{k=0}^{0}\binom{0}{k}c^{-k}(\sqrt{\frac{\beta}{(1-s)t}})^{k-\alpha}2K_{k-\alpha}(2\sqrt{\beta(1-s)t}) \\
&= \frac{\beta^{\alpha}e^{-ct(1-s)}}{\Gamma\alpha}(\sqrt{\frac{\beta}{(1-s)t}})^{-\alpha}2K_{-\alpha}(2\sqrt{\beta(1-s)t}) \\
&= \frac{2\beta^{\alpha}e^{-ct(1-s)}}{\Gamma\alpha}(\sqrt{\frac{(1-s)t}{\beta}})^{\alpha}K_{\alpha}(2\sqrt{\beta(1-s)t}) \qquad (4.42)
\end{aligned}$$

The rth moment of the Poisson mixture is

$$\begin{aligned}
E(T^r) &= t^r E(\wedge^r) \\
&= t^r\sum_{k=0}^{r}\binom{r}{k}c^r(\frac{\beta}{c})^k\frac{\Gamma(\alpha-k)}{\Gamma\alpha} \\
&= (tc)^r\sum_{k=0}^{r}\binom{r}{k}(\frac{\beta}{c})^k\frac{\Gamma(\alpha-k)}{\Gamma\alpha} \qquad (4.43) \\
E(T) &= (tc)\sum_{k=0}^{1}\binom{1}{k}(\frac{\beta}{c})^k\frac{\Gamma(\alpha-k)}{\Gamma\alpha} \\
&= (tc)[\frac{\Gamma\alpha}{\Gamma\alpha}+(\frac{\beta}{c})\frac{\Gamma(\alpha-1)}{\Gamma\alpha}] \\
&= (tc)[1+\frac{\beta}{c(\alpha-1)}] \qquad (4.44)
\end{aligned}$$

4.4 Erlang-Inverse Gaussian Distribution and Its Links

4.4.1 Erlang-Inverse Gaussian Mixture

The Inverse Gaussian mixing distribution is

$$g(\lambda) = (\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\frac{\phi}{\mu}}\lambda^{-\frac{3}{2}}exp[-\frac{\phi\lambda}{2\mu^2}-\frac{\phi}{2\lambda}], \quad \lambda > 0; -\infty < \mu < \infty \qquad (4.45)$$

$$let \quad \mu = \sqrt{\frac{\phi}{\rho}} \quad \Longrightarrow \mu^2 = \frac{\phi}{\rho}$$

$$g(\lambda) = (\frac{\phi}{2\pi})^{\frac{1}{2}} \lambda^{-\frac{3}{2}} e^{\sqrt{\rho\phi}} exp[-\frac{1}{2}(\rho\lambda + \frac{\phi}{\lambda})]$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda} (\frac{\phi}{2\pi})^{\frac{1}{2}} \lambda^{-\frac{3}{2}} e^{\sqrt{\rho\phi}} exp[-\frac{1}{2}(\rho\lambda + \frac{\phi}{\lambda})] d\lambda \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} \int_0^\infty \lambda^{n-\frac{3}{2}} exp[-t\lambda - \frac{1}{2}(\rho\lambda + \frac{\phi}{\lambda})] d\lambda \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} \int_0^\infty \lambda^{(n-\frac{1}{2})-1} exp[-(\frac{2t+\rho}{2})(\lambda + \frac{\phi}{(2t+\rho)} \frac{1}{\lambda})] d\lambda
\end{aligned}$$

$$let \quad \lambda = \sqrt{\frac{\phi}{2t+\rho}} x \quad \Longrightarrow d\lambda = \sqrt{\frac{\phi}{2t+\rho}} dx$$

$$\begin{aligned}
E[\wedge^n e^{-t\wedge}] &= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} \int_0^\infty (\sqrt{\frac{\phi}{2t+\rho}} x)^{(n-\frac{1}{2})-1} exp[-(\frac{2t+\rho}{2})(\sqrt{\frac{\phi}{2t+\rho}} x + \frac{\phi}{(2t+\rho)} \frac{1}{\sqrt{\frac{\phi}{2t+\rho}} x})] \sqrt{\frac{\phi}{2t+\rho}} dx \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{2t+\rho}})^{(n-\frac{1}{2})} \int_0^\infty x^{n-\frac{1}{2}-1} exp[-(\frac{2t+\rho}{2})(\sqrt{\frac{\phi}{2t+\rho}})(x + \frac{1}{x})] dx \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{2t+\rho}})^{(n-\frac{1}{2})} \int_0^\infty x^{(n-\frac{1}{2})-1} exp[-\frac{\sqrt{\phi(2t+\rho)}}{2}(x + \frac{1}{x})] dx \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{2t+\rho}})^{(n-\frac{1}{2})} 2K_{n-\frac{1}{2}}(\sqrt{\phi(2t+\rho)}) \qquad (4.46)
\end{aligned}$$

$$E(\wedge^j) = (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{\rho}})^{(j-\frac{1}{2})} 2K_{j-\frac{1}{2}}(\sqrt{\phi\rho}) \qquad (4.47)$$

Construction by the direct method gives

$$\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{2t+\rho}})^{(n-\frac{1}{2})} 2K_{n-\frac{1}{2}}(\sqrt{\phi(2t+\rho)}) \qquad (4.48)
\end{aligned}$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{\rho}})^{(j-\frac{1}{2})} 2K_{j-\frac{1}{2}}(\sqrt{\phi\rho}) \tag{4.49}$$

Identity 4.7
Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{\rho}})^{(j-\frac{1}{2})} 2K_{j-\frac{1}{2}}(\sqrt{\phi\rho}) = \frac{t^{n-1}}{\Gamma n} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{2t+\rho}})^{(n-\frac{1}{2})} 2K_{n-\frac{1}{2}}(\sqrt{\phi(2t}$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{\rho}})^{(j-\frac{1}{2})} 2K_{j-\frac{1}{2}}(\sqrt{\phi\rho}) = (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{2t+\rho}})^{(n-\frac{1}{2})} 2K_{n-\frac{1}{2}}(\sqrt{\phi(2t+\rho)}) \tag{4.50}$$

The rth moment of the Erlang mixture is

$$E(T^r) = \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r})$$
$$= \frac{\Gamma(n+r)}{\Gamma n} (\frac{\phi}{2\pi})^{\frac{1}{2}} 2e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{\rho}})^{(-r-\frac{1}{2})} K_{-r-\frac{1}{2}}(\sqrt{\phi\rho})$$
$$= \frac{\Gamma(n+r)}{\Gamma n} (\frac{\phi}{2\pi})^{\frac{1}{2}} 2e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{\rho}})^{-(r+\frac{1}{2})} K_{r+\frac{1}{2}}(\sqrt{\phi\rho}) \tag{4.51}$$
$$E(T) = \frac{\Gamma(n+1)}{\Gamma n} (\frac{\phi}{2\pi})^{\frac{1}{2}} 2e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{\rho}})^{-\frac{3}{2}} K_{\frac{3}{2}}(\sqrt{\phi\rho})$$
$$= 2ne^{\sqrt{\rho\phi}} (\frac{\phi}{2\pi})^{\frac{1}{2}} (\sqrt{\frac{\phi}{\rho}})^{-\frac{3}{2}} K_{\frac{3}{2}}(\sqrt{\phi\rho}) \tag{4.52}$$

4.4.2 Exponential-Inverse Gaussian Mixture

Construction by the direct method gives

$$
\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= 2e^{\sqrt{\rho\phi}}(\frac{\phi}{2\pi})^{\frac{1}{2}}(\sqrt{\frac{\phi}{2t+\rho}})^{\frac{1}{2}}K_{\frac{1}{2}}(\sqrt{\phi(2t+\rho)}) \\
&= 2e^{\sqrt{\rho\phi}}(\frac{\phi}{2\pi})^{\frac{1}{2}}(\sqrt{\frac{\phi}{2t+\rho}})^{\frac{1}{2}}\sqrt{\frac{\pi}{2\sqrt{\phi(2t+\rho)}}}e^{-\sqrt{\phi(2t+\rho)}} \\
&= 2e^{\sqrt{\rho\phi}-\sqrt{\phi(2t+\rho)}}(\frac{\phi}{2\pi})^{\frac{1}{2}}(\sqrt{\frac{\phi}{2t+\rho}\frac{1}{\phi(2t+\rho)}})^{\frac{1}{2}}(\sqrt{\frac{\pi}{2}})^{\frac{1}{2}} \\
&= 2e^{\sqrt{\phi}[\sqrt{\rho}-\sqrt{(2t+\rho)}]}\frac{\sqrt{\phi}}{2}(\frac{1}{2t+\rho})^{\frac{1}{2}} \\
&= e^{\sqrt{\phi}[\sqrt{\rho}-\sqrt{(2t+\rho)}]}\sqrt{\frac{\phi}{2t+\rho}} \qquad (4.53)
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}E(\wedge^j) \\
&= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}2e^{\sqrt{\rho\phi}}(\frac{\phi}{2\pi})^{\frac{1}{2}}(\sqrt{\frac{\phi}{\rho}})^{(j-\frac{1}{2})}K_{j-\frac{1}{2}}(\sqrt{\phi\rho}) \qquad (4.54)
\end{aligned}
$$

Identity 4.8

Equating the above two methods we get

$$
\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}2e^{\sqrt{\rho\phi}}(\frac{\phi}{2\pi})^{\frac{1}{2}}(\sqrt{\frac{\phi}{\rho}})^{(j-\frac{1}{2})}K_{j-\frac{1}{2}}(\sqrt{\phi\rho}) = e^{\sqrt{\phi}[\sqrt{\rho}-\sqrt{(2t+\rho)}]}\sqrt{\frac{\phi}{2t+\rho}} \qquad (4.55)
$$

The rth moment of the Exponential mixture is

$$
\begin{aligned}
E(T^r) &= r!E(\wedge^{-r}) \\
&= r!2e^{\sqrt{\rho\phi}}(\frac{\phi}{2\pi})^{\frac{1}{2}}(\sqrt{\frac{\phi}{\rho}})^{(-r-\frac{1}{2})}K_{-r-\frac{1}{2}}(\sqrt{\phi\rho}) \\
&= 2r!e^{\sqrt{\rho\phi}}(\frac{\phi}{2\pi})^{\frac{1}{2}}(\sqrt{\frac{\phi}{\rho}})^{-(r+\frac{1}{2})}K_{r+\frac{1}{2}}(\sqrt{\phi\rho}) \qquad (4.56)
\end{aligned}
$$

$$
\begin{aligned}
E(T) &= 2e^{\sqrt{\rho\phi}}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}}\left(\sqrt{\frac{\phi}{\rho}}\right)^{-\frac{3}{2}}K_{\frac{3}{2}}(\sqrt{\phi\rho}) \\
K_{1+\frac{1}{2}}(\sqrt{\phi\rho}) &= \sqrt{\frac{\pi}{2\sqrt{\phi\rho}}}e^{-\sqrt{\rho\phi}}\left[1+\sum_{i=1}^{1}\frac{(1+i)!(2\sqrt{\phi\rho})^{-i}}{(1-i)!i!}\right] \\
&= \sqrt{\frac{\pi}{2\sqrt{\phi\rho}}}e^{-\sqrt{\rho\phi}}\left[1+\frac{2}{2\sqrt{\phi\rho}}\right] \\
&= \sqrt{\frac{\pi}{2\sqrt{\phi\rho}}}e^{-\sqrt{\rho\phi}}\left[\frac{\sqrt{\phi\rho}-1}{2\sqrt{\phi\rho}}\right] \\
E(T) &= 2e^{\sqrt{\rho\phi}}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}}\left(\sqrt{\frac{\phi}{\rho}}\right)^{-\frac{3}{2}}\sqrt{\frac{\pi}{2\sqrt{\phi\rho}}}e^{-\sqrt{\rho\phi}}\left[\frac{\sqrt{\phi\rho}-1}{2\sqrt{\phi\rho}}\right] \\
&= 2\left(\frac{\phi}{2\pi}\frac{\pi}{2\sqrt{\phi\rho}}\right)^{\frac{1}{2}}\left(\sqrt{\frac{\phi}{\rho}}\right)^{-\frac{3}{2}}\left[\frac{\sqrt{\phi\rho}-1}{2\sqrt{\phi\rho}}\right] \\
&= \left(\sqrt{\frac{\phi}{\rho}}\right)^{\frac{1}{2}}\left(\sqrt{\frac{\phi}{\rho}}\right)^{-\frac{3}{2}}\left[\frac{\sqrt{\phi\rho}-1}{\sqrt{\phi\rho}}\right] \\
&= \left(\sqrt{\frac{\phi}{\rho}}\right)^{-1}\left[\frac{\sqrt{\phi\rho}-1}{\sqrt{\phi\rho}}\right] \\
&= \left(\sqrt{\frac{\rho}{\phi\phi\rho}}\right)\left[\sqrt{\phi\rho}-1\right] \\
&= \frac{1}{\phi}\left[\sqrt{\phi\rho}-1\right]
\end{aligned} \tag{4.57}
$$

4.4.3 Poisson-Inverse Gaussian Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$
\begin{aligned}
P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^n}{n!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}}e^{\sqrt{\rho\phi}}\left(\sqrt{\frac{\phi}{2t+\rho}}\right)^{(n-\frac{1}{2})}2K_{n-\frac{1}{2}}(\sqrt{\phi(2t+\rho)})
\end{aligned} \tag{4.58}
$$

By the method of moments we have

$$P_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} 2e^{\sqrt{\rho\phi}} (\frac{\phi}{2\pi})^{\frac{1}{2}} (\sqrt{\frac{\phi}{\rho}})^{(j-\frac{1}{2})} K_{j-\frac{1}{2}}(\sqrt{\phi\rho}) \quad (4.59)$$

Identity 4.9
Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} 2e^{\sqrt{\rho\phi}} (\frac{\phi}{2\pi})^{\frac{1}{2}} (\sqrt{\frac{\phi}{\rho}})^{(j-\frac{1}{2})} K_{j-\frac{1}{2}}(\sqrt{\phi\rho}) = \frac{t^n}{n!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{2t+\rho}})^{(n-\frac{1}{2})} 2K_{n-\frac{1}{2}}(\sqrt{\phi(2t+\rho)})$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} 2e^{\sqrt{\rho\phi}} (\frac{\phi}{2\pi})^{\frac{1}{2}} (\sqrt{\frac{\phi}{\rho}})^{(j-\frac{1}{2})} K_{j-\frac{1}{2}}(\sqrt{\phi\rho}) = (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\rho\phi}} (\sqrt{\frac{\phi}{2t+\rho}})^{(n-\frac{1}{2})} 2K_{n-\frac{1}{2}}(\sqrt{\phi(2t+\rho)}) \quad (4.60)$$

The PGF of the Poisson mixture is

$$G(s,t) = E[e^{-(1-s)t\wedge}]$$
$$= 2e^{\sqrt{\rho\phi}} (\frac{\phi}{2\pi})^{\frac{1}{2}} (\sqrt{\frac{\phi}{2t(1-s)+\rho}})^{-\frac{1}{2}} K_{-\frac{1}{2}}(\sqrt{\phi(2t(1-s)+\rho)})$$
$$= 2e^{\sqrt{\rho\phi}} (\frac{\phi}{2\pi})^{\frac{1}{2}} (\sqrt{\frac{2t(1-s)+\rho}{\phi}})^{\frac{1}{2}} \sqrt{\frac{\pi}{2\sqrt{\phi(2t(1-s)+\rho)}}} e^{-\sqrt{\phi(2t(1-s)+\rho)}}$$
$$= 2e^{\sqrt{\rho\phi}} e^{-\sqrt{\phi(2t(1-s)+\rho)}} (\frac{\phi}{2\pi} \frac{\pi}{2\sqrt{\phi(2t(1-s)+\rho)}})^{\frac{1}{2}} (\sqrt{\frac{2t(1-s)+\rho}{\phi}})^{\frac{1}{2}}$$
$$= 2e^{\sqrt{\rho\phi}-\sqrt{\phi(2t(1-s)+\rho)}} (\sqrt{\frac{\phi}{2t(1-s)+\rho}})^{\frac{1}{2}} (\sqrt{\frac{2t(1-s)+\rho}{\phi}})^{\frac{1}{2}}$$
$$= 2e^{\sqrt{\rho\phi}-\sqrt{\phi(2t(1-s)+\rho)}} \quad (4.61)$$

The rth moment of the Poisson mixture is

$$E(T^r) = t^r E(\wedge^r)$$
$$= t^r 2e^{\sqrt{\rho\phi}} (\frac{\phi}{2\pi})^{\frac{1}{2}} (\sqrt{\frac{\phi}{\rho}})^{(r-\frac{1}{2})} K_{r-\frac{1}{2}}(\sqrt{\phi\rho}) \quad (4.62)$$

$$
\begin{aligned}
E(T) &= 2te^{\sqrt{\rho\phi}}(\frac{\phi}{2\pi})^{\frac{1}{2}}(\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}}K_{\frac{1}{2}}(\sqrt{\phi\rho}) \\
&= 2te^{\sqrt{\rho\phi}}(\frac{\phi}{2\pi})^{\frac{1}{2}}(\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}}\sqrt{\frac{\pi}{2\sqrt{\phi\rho}}}e^{-\sqrt{\phi\rho}} \\
&= 2t(\frac{\phi}{2\pi}\frac{\pi}{2\sqrt{\phi\rho}})^{\frac{1}{2}}(\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}} \\
&= \frac{2t}{2}(\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}}(\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}} \\
&= t\sqrt{\frac{\phi}{\rho}}
\end{aligned} \tag{4.63}
$$

4.5 Erlang-Reciprocal Inverse Gaussian Distribution and Its Links

4.5.1 Erlang-Reciprocal Inverse Gaussian Mixture

The Reciprocal Inverse Gaussian mixing distribution is

$$
g(\lambda) = (\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}\lambda^{-\frac{1}{2}}exp[-\frac{\phi}{2}\lambda - \frac{\rho}{2}\frac{1}{\lambda}], \quad \lambda > 0 \tag{4.64}
$$

$$
\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda}(\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}\lambda^{-\frac{1}{2}}exp[-\frac{\phi}{2}\lambda - \frac{\rho}{2}\frac{1}{\lambda}]d\lambda \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}\int_0^\infty \lambda^{n-\frac{1}{2}}exp[-t\lambda - \frac{\phi}{2}\lambda - \frac{\rho}{2}\frac{1}{\lambda}]d\lambda \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}\int_0^\infty \lambda^{n-\frac{1}{2}}exp[-\frac{1}{2}[(2t+\phi)\lambda + \frac{\rho}{\lambda}]]d\lambda \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}\int_0^\infty \lambda^{n-\frac{1}{2}}exp[-\frac{(2t+\phi)}{2}[\lambda + \frac{\rho}{(2t+\phi)}\frac{1}{\lambda}]]d\lambda \\
let \quad \lambda &= \sqrt{\frac{\rho}{2t+\phi}}x \quad \Longrightarrow d\lambda = \sqrt{\frac{\rho}{2t+\phi}}dx \\
E[\wedge^n e^{-t\wedge}] &= (\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}\int_0^\infty (\sqrt{\frac{\rho}{2t+\phi}}x)^{n-\frac{1}{2}}exp[-\frac{(2t+\phi)}{2}[\sqrt{\frac{\rho}{2t+\phi}}x + \frac{\rho}{(2t+\phi)}\frac{1}{\sqrt{\frac{\rho}{2t+\phi}}x}]]\sqrt{\frac{\rho}{2t+\phi}} \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}(\sqrt{\frac{\rho}{2t+\phi}})^{n+\frac{1}{2}}\int_0^\infty x^{n+\frac{1}{2}-1}exp[-\frac{(2t+\phi)}{2}\sqrt{\frac{\rho}{2t+\phi}}(x+\frac{1}{x})]dx
\end{aligned}
$$

$$E[\wedge^n e^{-t\wedge}] = (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{n+\frac{1}{2}} \int_0^\infty x^{n+\frac{1}{2}-1} exp[-\frac{\sqrt{\rho(2t+\phi)}}{2}(x+\frac{1}{x})]dx$$
$$= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{n+\frac{1}{2}} 2K_{n+\frac{1}{2}}(\sqrt{\rho(2t+\phi)})$$
$$= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{n+\frac{1}{2}} 2\sqrt{\frac{\pi}{2\sqrt{\rho(2t+\phi)}}} e^{-\sqrt{\rho(2t+\phi)}} [1+\sum_{i=1}^{n} \frac{(n+1)!(2\sqrt{\rho(2t+\phi)})^{-i}}{(n-1)!i!}] \tag{4.65}$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^{n-1}}{\Gamma n} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{n+\frac{1}{2}} 2K_{n+\frac{1}{2}}(\sqrt{\rho(2t+\phi)})$$
$$= \frac{t^{n-1}}{\Gamma n} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{n+\frac{1}{2}} 2\sqrt{\frac{\pi}{2\sqrt{\rho(2t+\phi)}}} e^{-\sqrt{\rho(2t+\phi)}} [1+\sum_{i=1}^{n} \frac{(n+1)!(2\sqrt{\rho(2t+\phi)})^{-i}}{(n-1)!i!}] \tag{4.66}$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{\phi}})^{j+\frac{1}{2}} 2K_{j+\frac{1}{2}}(\sqrt{\rho\phi}) \tag{4.67}$$

Identity 4.10
Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{\phi}})^{j+\frac{1}{2}} 2K_{j+\frac{1}{2}}(\sqrt{\rho\phi}) = \frac{t^{n-1}}{\Gamma n} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{n+\frac{1}{2}} 2K_{n+\frac{1}{2}}(\sqrt{\rho(2t+\phi)})$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{\phi}})^{j+\frac{1}{2}} 2K_{j+\frac{1}{2}}(\sqrt{\rho\phi}) = (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{n+\frac{1}{2}} 2K_{n+\frac{1}{2}}(\sqrt{\rho(2t+\phi)}) \tag{4.68}$$

The rth moment of the Erlang mixture is

$$
\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{\phi}})^{-r+\frac{1}{2}} 2K_{-r+\frac{1}{2}}(\sqrt{\rho\phi}) \qquad \text{(4.69)} \\
E(T) &= \frac{\Gamma(n+1)}{\Gamma n} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{\phi}})^{-\frac{1}{2}} 2K_{-\frac{1}{2}}(\sqrt{\rho\phi}) \\
&= n(\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}} 2\sqrt{\frac{\pi}{2\sqrt{\rho\phi}}} e^{-\sqrt{\rho\phi}} \\
&= 2n(\frac{\phi}{2\pi}\frac{\pi}{2\sqrt{\rho\phi}})^{\frac{1}{2}} (\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}} \\
&= n(\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}} (\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}} \\
&= n\sqrt{\frac{\phi}{\rho}} \qquad \text{(4.70)}
\end{aligned}
$$

4.5.2 Exponential-Reciprocal Inverse Gaussian Mixture

Construction by the direct method gives

$$
\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{\frac{3}{2}} 2K_{1+\frac{1}{2}}(\sqrt{\rho(2t+\phi)}) \\
K_{1+\frac{1}{2}}(\sqrt{\rho(2t+\phi)}) &= \sqrt{\frac{\pi}{2\sqrt{\rho(2t+\phi)}}} e^{-\sqrt{\rho(2t+\phi)}} [1 + \sum_{i=1}^{1} \frac{(1+i)!(2\sqrt{\rho(2t+\phi)})^{-i}}{(1-i)!i!}] \\
&= \sqrt{\frac{\pi}{2\sqrt{\rho(2t+\phi)}}} e^{-\sqrt{\rho(2t+\phi)}} [1 + 2!(2\sqrt{\rho(2t+\phi)})^{-1}] \\
&= \sqrt{\frac{\pi}{2\sqrt{\rho(2t+\phi)}}} e^{-\sqrt{\rho(2t+\phi)}} [1 + \frac{1}{\sqrt{\rho(2t+\phi)}}] \\
f_1(t) &= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{\frac{3}{2}} 2\sqrt{\frac{\pi}{2\sqrt{\rho(2t+\phi)}}} e^{-\sqrt{\rho(2t+\phi)}} [\frac{\sqrt{\rho(2t+\phi)}+1}{\sqrt{\rho(2t+\phi)}}] \\
&= (\frac{\phi}{2\pi}\frac{\pi}{2\sqrt{\rho(2t+\phi)}})^{\frac{1}{2}} e^{\sqrt{\phi\rho}-\sqrt{\rho(2t+\phi)}} 2(\sqrt{\frac{\rho}{2t+\phi}})^{\frac{3}{2}} [\frac{\sqrt{\rho(2t+\phi)}+1}{\sqrt{\rho(2t+\phi)}}]
\end{aligned}
$$

$$
\begin{aligned}
f_1(t) &= (\sqrt{\frac{\phi^2}{\rho(2t+\phi)}})^{\frac{1}{2}}(\sqrt{\frac{\rho}{2t+\phi}})^{\frac{1}{2}}(\sqrt{\frac{\rho}{2t+\phi}})[\frac{\sqrt{\rho(2t+\phi)}+1}{\sqrt{\rho(2t+\phi)}}]e^{\sqrt{\phi\rho}[1-\sqrt{\frac{2t}{\phi}+1}]} \\
&= (\frac{\phi}{2t+\phi}\frac{\rho}{2t+\phi}\frac{1}{\rho(2t+\phi)})^{\frac{1}{2}}[\sqrt{\rho(2t+\phi)}+1]e^{\sqrt{\phi\rho}[1-\sqrt{\frac{2t}{\phi}+1}]} \\
&= (\frac{\phi}{2t+\phi})^{\frac{1}{2}}[\frac{\sqrt{\rho(2t+\phi)}+1}{2t+\phi}]e^{\sqrt{\phi\rho}[1-\sqrt{\frac{2t}{\phi}+1}]} \qquad \textbf{(4.71)}
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}E(\wedge^j) \\
&= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}(\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}(\sqrt{\frac{\rho}{\phi}})^{j+\frac{1}{2}}2K_{j+\frac{1}{2}}(\sqrt{\rho\phi}) \qquad \textbf{(4.72)}
\end{aligned}
$$

Identity 4.11
Equating the above two methods we get

$$
\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}(\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}(\sqrt{\frac{\rho}{\phi}})^{j+\frac{1}{2}}2K_{j+\frac{1}{2}}(\sqrt{\rho\phi}) = (\frac{\phi}{2t+\phi})^{\frac{1}{2}}[\frac{\sqrt{\rho(2t+\phi)}+1}{2t+\phi}]e^{\sqrt{\phi\rho}[1-\sqrt{\frac{2t}{\phi}+1}]} \qquad \textbf{(4.73)}
$$

The rth moment of the Exponential mixture is

$$
\begin{aligned}
E(T^r) &= r!E(\wedge^{-r}) \\
&= r!(\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}(\sqrt{\frac{\rho}{\phi}})^{-r+\frac{1}{2}}2K_{-r+\frac{1}{2}}(\sqrt{\rho\phi}) \\
&= r!(\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}(\sqrt{\frac{\phi}{\rho}})^{r-\frac{1}{2}}2K_{r-\frac{1}{2}}(\sqrt{\rho\phi}) \qquad \textbf{(4.74)} \\
E(T) &= (\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}(\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}}2K_{\frac{1}{2}}(\sqrt{\rho\phi}) \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\sqrt{\phi\rho}}(\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}}2\sqrt{\frac{\pi}{2\sqrt{\rho\phi}}}e^{-\sqrt{\rho\phi}} \\
&= 2(\frac{\phi}{2\pi}\frac{\pi}{2\sqrt{\rho\phi}})^{\frac{1}{2}}(\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}}
\end{aligned}
$$

$$E(T) = (\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}}(\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}}$$
$$= \sqrt{\frac{\phi}{\rho}} \quad (4.75)$$

4.5.3 Poisson-Reciprocal Inverse Gaussian Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$P_n(t) = \frac{t^n}{n!} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^n}{n!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{n+\frac{1}{2}} 2K_{n+\frac{1}{2}}(\sqrt{\rho(2t+\phi)}) \quad (4.76)$$

By the method of moments we have

$$P_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{\phi}})^{j+\frac{1}{2}} 2K_{j+\frac{1}{2}}(\sqrt{\rho\phi}) \quad (4.77)$$

Identity 4.12
Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{\phi}})^{j+\frac{1}{2}} 2K_{j+\frac{1}{2}}(\sqrt{\rho\phi}) = \frac{t^n}{n!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{n+\frac{1}{2}} 2K_{n+\frac{1}{2}}(\sqrt{\rho(2t+\phi)})$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{\phi}})^{j+\frac{1}{2}} 2K_{j+\frac{1}{2}}(\sqrt{\rho\phi}) = (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2t+\phi}})^{n+\frac{1}{2}} 2K_{n+\frac{1}{2}}(\sqrt{\rho(2t+\phi)})$$
$$(4.78)$$

The PGF of the Poisson mixture is

$$
\begin{aligned}
G(s,t) &= E[e^{-(1-s)t\wedge}] \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2(1-s)t+\phi}})^{\frac{1}{2}} 2K_{\frac{1}{2}}(\sqrt{\rho(2(1-s)t+\phi)}) \\
&= (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{2(1-s)t+\phi}})^{\frac{1}{2}} 2\sqrt{\frac{\pi}{2\sqrt{\rho(2(1-s)t+\phi)}}} e^{-\sqrt{\rho(2(1-s)t+\phi)}} \\
&= 2(\frac{\phi}{2\pi}\frac{\pi}{2\sqrt{\rho(2(1-s)t+\phi)}})^{\frac{1}{2}} e^{\sqrt{\phi\rho}-\sqrt{\rho(2(1-s)t+\phi)}} (\sqrt{\frac{\rho}{2(1-s)t+\phi}})^{\frac{1}{2}} \\
&= e^{\sqrt{\phi\rho}-\sqrt{\rho(2(1-s)t+\phi)}} (\sqrt{\frac{\phi^2}{\rho(2(1-s)t+\phi)}})^{\frac{1}{2}} (\sqrt{\frac{\rho}{2(1-s)t+\phi}})^{\frac{1}{2}} \\
&= e^{\sqrt{\phi\rho}-\sqrt{\rho(2(1-s)t+\phi)}} (\frac{\phi}{(2(1-s)t+\phi)})^{\frac{1}{2}} \qquad (4.79)
\end{aligned}
$$

The rth moment of the Poisson mixture is

$$
\begin{aligned}
E(T^r) &= t^r E(\wedge^r) \\
&= t^r (\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{\phi}})^{r+\frac{1}{2}} 2K_{r+\frac{1}{2}}(\sqrt{\rho\phi}) \qquad (4.80)
\end{aligned}
$$

$$
\begin{aligned}
E(T) &= t(\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{\phi}})^{\frac{3}{2}} 2K_{1+\frac{1}{2}}(\sqrt{\rho\phi}) \\
&= t(\frac{\phi}{2\pi})^{\frac{1}{2}} e^{\sqrt{\phi\rho}} (\sqrt{\frac{\rho}{\phi}})^{\frac{3}{2}} 2\sqrt{\frac{\pi}{2\sqrt{\rho\phi}}} e^{-\sqrt{\rho\phi}} [1+\frac{1}{\sqrt{\rho\phi}}] \\
&= 2t(\frac{\phi}{2\pi}\frac{\pi}{2\sqrt{\rho\phi}})^{\frac{1}{2}} (\sqrt{\frac{\rho}{\phi}})^{\frac{3}{2}} [\frac{\sqrt{\rho\phi}+1}{\sqrt{\rho\phi}}] \\
&= t(\sqrt{\frac{\phi}{\rho}})^{\frac{1}{2}} (\sqrt{\frac{\rho}{\phi}})^{\frac{1}{2}} (\sqrt{\frac{\rho}{\phi}}) [\frac{\sqrt{\rho\phi}+1}{\sqrt{\rho\phi}}] \\
&= \frac{t}{\phi}[\sqrt{\rho\phi+1}] \qquad (4.81)
\end{aligned}
$$

4.6 Erlang-Generalized Inverse Gaussian (GIG) Distribution and Its Links

4.6.1 Erlang-Generalized Inverse Gaussian (GIG) Mixture

The Generalized Inverse Gaussian mixing distribution is

$$g(\lambda) = \frac{(\frac{\rho}{\phi})^{\frac{v}{2}}}{2K_v(\sqrt{\rho\phi})}\lambda^{v-1}exp[-\frac{1}{2}(\rho\lambda + \frac{\phi}{\lambda})], \quad \lambda > 0 \tag{4.82}$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda} \frac{(\frac{\rho}{\phi})^{\frac{v}{2}}}{2K_v(\sqrt{\rho\phi})}\lambda^{v-1}exp[-\frac{1}{2}(\rho\lambda + \frac{\phi}{\lambda})]d\lambda \\
&= \frac{(\frac{\rho}{\phi})^{\frac{v}{2}}}{2K_v(\sqrt{\rho\phi})}\int_0^\infty \lambda^{n+v-1}exp[-t\lambda - \frac{1}{2}(\rho\lambda + \frac{\phi}{\lambda})]d\lambda \\
&= \frac{(\frac{\rho}{\phi})^{\frac{v}{2}}}{2K_v(\sqrt{\rho\phi})}\int_0^\infty \lambda^{n+v-1}exp[-\frac{(2t+\rho)}{2}(\lambda + \frac{\phi}{(2t+\rho)}\frac{1}{\lambda})]d\lambda \\
let \quad \lambda &= \sqrt{\frac{\phi}{2t+\rho}}x \quad \Longrightarrow d\lambda = \sqrt{\frac{\phi}{2t+\rho}}dx
\end{aligned}$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \frac{(\frac{\rho}{\phi})^{\frac{v}{2}}}{2K_v(\sqrt{\rho\phi})}\int_0^\infty (\sqrt{\frac{\phi}{2t+\rho}}x)^{n+v-1}exp[-\frac{(2t+\rho)}{2}(\sqrt{\frac{\phi}{2t+\rho}}x + \frac{\phi}{(2t+\rho)}\frac{1}{\sqrt{\frac{\phi}{2t+\rho}}x})]\sqrt{\frac{\phi}{2t\cdots}} \\
&= \frac{(\frac{\rho}{\phi})^{\frac{v}{2}}}{2K_v(\sqrt{\rho\phi})}(\sqrt{\frac{\phi}{2t+\rho}})^{n+v}\int_0^\infty x^{n+v-1}exp[-\frac{(2t+\rho)}{2}\sqrt{\frac{\phi}{2t+\rho}}(x + \frac{1}{x})]dx \\
&= \frac{(\frac{\rho}{\phi})^{\frac{v}{2}}}{2K_v(\sqrt{\rho\phi})}(\sqrt{\frac{\phi}{2t+\rho}})^{n+v}\int_0^\infty x^{n+v-1}exp[-\sqrt{\phi(2t+\rho)}(x + \frac{1}{x})]dx \\
&= (\frac{\rho}{\phi})^{\frac{v}{2}}(\sqrt{\frac{\phi}{2t+\rho}})^{n+v}\frac{K_{n+v}(\sqrt{\phi(2t+\rho)})}{K_v(\sqrt{\rho\phi})}
\end{aligned} \tag{4.83}$$

Construction by the direct method gives

$$\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n}E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n}(\sqrt{\frac{\rho}{\phi}})^v(\sqrt{\frac{\phi}{2t+\rho}})^{n+v}\frac{K_{n+v}(\sqrt{\phi(2t+\rho)})}{K_v(\sqrt{\rho\phi})}
\end{aligned} \tag{4.84}$$

By the method of moments we have

$$
\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \left(\sqrt{\frac{\rho}{\phi}}\right)^{v} \left(\sqrt{\frac{\phi}{\rho}}\right)^{j+v} \frac{K_{j+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \left(\sqrt{\frac{\phi}{\rho}}\right)^{j} \frac{K_{j+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})}
\end{aligned} \tag{4.85}
$$

Identity 4.13
Equating the above two methods we get

$$
\begin{aligned}
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \left(\sqrt{\frac{\phi}{\rho}}\right)^{j} \frac{K_{j+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} &= \frac{t^{n-1}}{\Gamma n} \left(\sqrt{\frac{\rho}{\phi}}\right)^{v} \left(\sqrt{\frac{\phi}{2t+\rho}}\right)^{n+v} \frac{K_{n+v}(\sqrt{\phi(2t+\rho)})}{K_v(\sqrt{\rho\phi})} \\
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \left(\sqrt{\frac{\phi}{\rho}}\right)^{j} \frac{K_{j+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} &= \left(\sqrt{\frac{\rho}{\phi}}\right)^{v} \left(\sqrt{\frac{\phi}{2t+\rho}}\right)^{n+v} \frac{K_{n+v}(\sqrt{\phi(2t+\rho)})}{K_v(\sqrt{\rho\phi})}
\end{aligned} \tag{4.86}
$$

The rth moment of the Erlang mixture is

$$
\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \left(\sqrt{\frac{\phi}{\rho}}\right)^{-r} \frac{K_{-r+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} \\
&= \frac{\Gamma(n+r)}{\Gamma n} \left(\sqrt{\frac{\rho}{\phi}}\right)^{r} \frac{K_{v-r}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})}
\end{aligned} \tag{4.87}
$$

$$
\begin{aligned}
E(T) &= \frac{\Gamma(n+1)}{\Gamma n} \left(\sqrt{\frac{\rho}{\phi}}\right) \frac{K_{v-1}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} \\
&= n\left(\sqrt{\frac{\rho}{\phi}}\right) \frac{K_{v-1}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})}
\end{aligned} \tag{4.88}
$$

4.6.2 Exponential-Generalized Inverse Gaussian (GIG) Mixture

Construction by the direct method gives

$$f_1(t) = E[\wedge e^{-t\wedge}]$$
$$= (\sqrt{\frac{\rho}{\phi}})^{v}(\sqrt{\frac{\phi}{2t+\rho}})^{1+v}\frac{K_{v+1}(\sqrt{\phi(2t+\rho)})}{K_v(\sqrt{\rho\phi})} \tag{4.89}$$

By the method of moments we have

$$f_1(t) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}E(\wedge^j)$$
$$= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}(\sqrt{\frac{\phi}{\rho}})^{j}\frac{K_{j+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} \tag{4.90}$$

Identity 4.14
Equating the above two methods we get

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}(\sqrt{\frac{\phi}{\rho}})^{j}\frac{K_{j+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} = (\sqrt{\frac{\rho}{\phi}})^{v}(\sqrt{\frac{\phi}{2t+\rho}})^{1+v}\frac{K_{v+1}(\sqrt{\phi(2t+\rho)})}{K_v(\sqrt{\rho\phi})} \tag{4.91}$$

The rth moment of the Exponential mixture is

$$E(T^r) = r!E(\wedge^{-r})$$
$$= r!(\sqrt{\frac{\phi}{\rho}})^{-r}\frac{K_{-r+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})}$$
$$= r!(\sqrt{\frac{\rho}{\phi}})^{r}\frac{K_{v-r}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} \tag{4.92}$$
$$E(T) = (\sqrt{\frac{\rho}{\phi}})\frac{K_{v-1}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} \tag{4.93}$$

4.6.3 Poisson-Generalized Inverse Gaussian (GIG) Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!}(\sqrt{\frac{\rho}{\phi}})^v(\sqrt{\frac{\phi}{2t+\rho}})^{n+v}\frac{K_{n+v}(\sqrt{\phi(2t+\rho)})}{K_v(\sqrt{\rho\phi})} \end{aligned} \quad (4.94)$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\ &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}(\sqrt{\frac{\phi}{\rho}})^j\frac{K_{j+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} \end{aligned} \quad (4.95)$$

Identity 4.15
Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}(\sqrt{\frac{\phi}{\rho}})^j\frac{K_{j+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} &= \frac{t^n}{n!}(\sqrt{\frac{\rho}{\phi}})^v(\sqrt{\frac{\phi}{2t+\rho}})^{n+v}\frac{K_{n+v}(\sqrt{\phi(2t+\rho)})}{K_v(\sqrt{\rho\phi})} \\ \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}(\sqrt{\frac{\phi}{\rho}})^j\frac{K_{j+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} &= (\sqrt{\frac{\rho}{\phi}})^v(\sqrt{\frac{\phi}{2t+\rho}})^{n+v}\frac{K_{n+v}(\sqrt{\phi(2t+\rho)})}{K_v(\sqrt{\rho\phi})} \end{aligned} \quad (4.96)$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= (\sqrt{\frac{\rho}{\phi}})^v(\sqrt{\frac{\phi}{2(1-s)t+\rho}})^v\frac{K_v(\sqrt{\phi(2(1-s)t+\rho)})}{K_v(\sqrt{\rho\phi})} \\ &= (\sqrt{\frac{\rho}{2(1-s)t+\rho}})^v\frac{K_v(\sqrt{\phi(2(1-s)t+\rho)})}{K_v(\sqrt{\rho\phi})} \end{aligned} \quad (4.97)$$

The rth moment of the Poisson mixture is

$$
\begin{aligned}
E(T^r) &= t^r E(\wedge^r) \\
&= t^r (\sqrt{\frac{\phi}{\rho}})^r \frac{K_{r+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} \\
&= (t\sqrt{\frac{\phi}{\rho}})^r \frac{K_{r+v}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} \qquad (4.98)
\end{aligned}
$$

$$
E(T) = t\sqrt{\frac{\phi}{\rho}} \frac{K_{v+1}(\sqrt{\phi\rho})}{K_v(\sqrt{\rho\phi})} \qquad (4.99)
$$

4.7 Special Cases of Erlang-Generalized Inverse Gaussian (GIG) Distribution

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} (\sqrt{\frac{\rho}{\phi}})^v (\sqrt{\frac{\phi}{2t+\rho}})^{n+v} \frac{\frac{1}{2}(\sqrt{\frac{2t+\rho}{\phi}})^{n+v} \int_0^\infty \lambda^{n+v-1} e^{-\frac{1}{2}[(2t+\rho)\lambda + \frac{\phi}{\lambda}]} d\lambda}{\frac{1}{2}(\sqrt{\frac{\rho}{\phi}})^v \int_0^\infty \lambda^{v-1} e^{-\frac{1}{2}(\rho\lambda + \frac{\phi}{\lambda})} d\lambda} \\
&= \frac{t^{n-1}}{\Gamma n} \frac{\int_0^\infty \lambda^{n+v-1} e^{-\frac{1}{2}[(2t+\rho)\lambda + \frac{\phi}{\lambda}]} d\lambda}{\int_0^\infty \lambda^{v-1} e^{-\frac{1}{2}(\rho\lambda + \frac{\phi}{\lambda})} d\lambda} \qquad (4.100)
\end{aligned}
$$

4.7.1 Erlang-Inverse Gaussian Distribution

When v=-$\frac{1}{2}$

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} (\sqrt{\frac{\rho}{\phi}})^{-\frac{1}{2}} (\sqrt{\frac{\phi}{2t+\rho}})^{n-\frac{1}{2}} \frac{K_{n-\frac{1}{2}}(\sqrt{\phi(2t+\rho)})}{K_{-\frac{1}{2}}(\sqrt{\rho\phi})} \\
&= \frac{t^{n-1}}{\Gamma n} (\sqrt{\frac{\rho}{\phi}})^{-\frac{1}{2}} (\sqrt{\frac{\phi}{2t+\rho}})^{n-\frac{1}{2}} \frac{K_{n-\frac{1}{2}}(\sqrt{\phi(2t+\rho)})}{\sqrt{\frac{\pi}{2\sqrt{\rho\phi}}} e^{-\sqrt{\rho\phi}}} \\
&= \frac{t^{n-1}}{\Gamma n} (\sqrt{\frac{\phi}{\rho}} \frac{2\sqrt{\rho\phi}}{\pi})^{\frac{1}{2}} (\sqrt{\frac{\phi}{2t+\rho}})^{n-\frac{1}{2}} e^{\sqrt{\rho\phi}} K_{n-\frac{1}{2}}(\sqrt{\phi(2t+\rho)}) \\
&= \frac{t^{n-1}}{\Gamma n} (\frac{2\phi}{\pi})^{\frac{1}{2}} (\sqrt{\frac{\phi}{2t+\rho}})^{n-\frac{1}{2}} e^{\sqrt{\rho\phi}} K_{n-\frac{1}{2}}(\sqrt{\phi(2t+\rho)}) \qquad (4.101)
\end{aligned}
$$

4.7.2 Erlang-Reciprocal Inverse Gaussian Distribution

When v=$\frac{1}{2}$

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n}(\sqrt{\frac{\rho}{\phi}})^{\frac{1}{2}}(\sqrt{\frac{\phi}{2t+\rho}})^{n+\frac{1}{2}}\frac{K_{n+\frac{1}{2}}(\sqrt{\phi(2t+\rho)})}{K_{\frac{1}{2}}(\sqrt{\rho\phi})} \\
&= \frac{t^{n-1}}{\Gamma n}(\sqrt{\frac{\rho}{\phi}})^{\frac{1}{2}}(\sqrt{\frac{\phi}{2t+\rho}})^{n+\frac{1}{2}}\frac{K_{n+\frac{1}{2}}(\sqrt{\phi(2t+\rho)})}{\sqrt{\frac{\pi}{2\sqrt{\rho\phi}}}e^{-\sqrt{\rho\phi}}} \\
&= \frac{t^{n-1}}{\Gamma n}(\sqrt{\frac{\rho}{\phi}}\frac{2\sqrt{\rho\phi}}{\pi})^{\frac{1}{2}}(\sqrt{\frac{\phi}{2t+\rho}})^{n+\frac{1}{2}}e^{\sqrt{\rho\phi}}K_{n+\frac{1}{2}}(\sqrt{\phi(2t+\rho)}) \\
&= \frac{t^{n-1}}{\Gamma n}(\frac{2\rho}{\pi})^{\frac{1}{2}}(\sqrt{\frac{\phi}{2t+\rho}})^{n+\frac{1}{2}}e^{\sqrt{\rho\phi}}K_{n+\frac{1}{2}}(\sqrt{\phi(2t+\rho)}) \qquad (4.102)
\end{aligned}
$$

4.7.3 Erlang-Gamma Distribution

When v>0, ϕ=0, ρ>0

$$
f_n(t) = \frac{t^{n-1}}{\Gamma n}\frac{\int_0^\infty \lambda^{n+v-1}e^{-\frac{1}{2}[(2t+\rho)\lambda]}d\lambda}{\int_0^\infty \lambda^{v-1}e^{-\frac{1}{2}(\rho\lambda)}d\lambda}
$$

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n}\frac{\frac{\Gamma(n+v)}{(\frac{2t+\rho}{2})^{n+v}}}{\frac{\Gamma n}{(\frac{\rho}{2})^v}} \\
&= \frac{t^{n-1}}{\Gamma n}\frac{\Gamma(n+v)}{(\frac{2t+\rho}{2})^{n+v}}\frac{(\frac{\rho}{2})^v}{\Gamma n} \\
&= \frac{1}{t}\frac{\Gamma(n+v)}{(\Gamma n)^2}(\frac{\rho}{2t+\rho})^v(\frac{2t}{2t+\rho})^n; \quad \rho > 0, v > 0 \qquad (4.103)
\end{aligned}
$$

4.7.4 Erlang-Exponential Distribution

When v=1, ϕ=0, ρ>0

$$\begin{aligned} f_n(t) &= \frac{1}{t}\frac{\Gamma(n+1)}{(\Gamma n)^2}(\frac{\rho}{2t+\rho})(\frac{2t}{2t+\rho})^n \\ &= \frac{n}{t\Gamma n}(\frac{\rho}{2t+\rho})(\frac{2t}{2t+\rho})^n; \quad \rho > 0 \end{aligned} \tag{4.104}$$

4.7.5 Erlang-Inverse Gamma Distribution

When v<0, ϕ>0, ρ=0

$$f_n(t) = \frac{t^{n-1}}{\Gamma n}\frac{\int_0^\infty \lambda^{n+v-1}e^{-\frac{1}{2}[2t\lambda+\frac{\phi}{\lambda}]}d\lambda}{\int_0^\infty \lambda^{v-1}e^{-\frac{1}{2}(\frac{\phi}{\lambda})}d\lambda}$$

$$let \quad \lambda = \sqrt{\frac{\phi}{2t}}x \quad \Longrightarrow d\lambda = \sqrt{\frac{\phi}{2t}}dx$$

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma n}\frac{\int_0^\infty(\sqrt{\frac{\phi}{2t}}x)^{n+v-1}e^{-\frac{1}{2}[2t\sqrt{\frac{\phi}{2t}}x+\frac{\phi}{\sqrt{\frac{\phi}{2t}}x}]}\sqrt{\frac{\phi}{2t}}dx}{\frac{\Gamma(-v)}{(\frac{\phi}{2})^{-v}}} \\ &= \frac{t^{n-1}}{\Gamma n}\frac{(\frac{\phi}{2})^{-v}}{\Gamma(-v)}(\sqrt{\frac{\phi}{2t}})^{n+v}\int_0^\infty x^{n+v-1}e^{-\frac{2t}{2}(\sqrt{\frac{\phi}{2t}})[x+\frac{1}{x}]}dx \\ &= \frac{t^{n-1}}{\Gamma n\Gamma(-v)}(\frac{\phi}{2})^{-v}(\sqrt{\frac{\phi}{2t}})^{n+v}K_{n+v}(\sqrt{(2t)\phi}) \\ &= \frac{t^{-(v+1)}}{\Gamma n\Gamma(-v)}(\sqrt{\frac{\phi}{2}})^{n-v}K_{n+v}(\sqrt{(2t)\phi}) \end{aligned} \tag{4.105}$$

4.7.6 Erlang-Levy Distribution

When v=-$\frac{1}{2}$, ϕ>0, ρ=0

$$\begin{aligned} f_n(t) &= \frac{t^{-(-\frac{1}{2}+1)}}{\Gamma n\Gamma(\frac{1}{2})}(\sqrt{\frac{\phi}{2}})^{n+\frac{1}{2}}K_{n-\frac{1}{2}}(\sqrt{(2t)\phi}) \\ &= \frac{1}{\Gamma n\sqrt{t\pi}}(\sqrt{\frac{\phi}{2}})^{n+\frac{1}{2}}K_{n-\frac{1}{2}}(\sqrt{(2t)\phi}) \end{aligned} \tag{4.106}$$

4.7.7 Erlang-Positive Hyperbolic Distribution

When v=1,ϕ>0, ρ>0

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma n}\frac{\int_0^\infty \lambda^{n+1-1}e^{-\frac{1}{2}[(2t+\rho)\lambda+\frac{\phi}{\lambda}]}d\lambda}{\int_0^\infty \lambda^{1-1}e^{-\frac{1}{2}(\rho\lambda+\frac{\phi}{\lambda})}d\lambda} \\ &= \frac{t^{n-1}}{\Gamma n}(\sqrt{\frac{\rho}{\phi}})(\sqrt{\frac{\phi}{2t+\rho}})^{n+1}\frac{K_{n+1}(\sqrt{\phi(2t+\rho)})}{K_1(\sqrt{\rho\phi})} \end{aligned} \quad (4.107)$$

4.7.8 Erlang-Harmonic Distribution

When v=0, ϕ=an, $\rho=\frac{a}{n}$

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma n}(\sqrt{\frac{an}{2t+\frac{a}{n}}})^n\frac{K_n(\sqrt{an(2t+\frac{a}{n})})}{K_0(\sqrt{an\frac{a}{n}})} \\ &= \frac{t^{n-1}}{\Gamma n}(\sqrt{\frac{an^2}{2tn+a}})^n\frac{K_n(\sqrt{a(2tn+a)})}{K_0(a)} \end{aligned} \quad (4.108)$$

5 ERLANG MIXTURES BASED ON CONFLUENT HYPERGEOMETRIC FUNCTIONS

5.1 Introduction

In this chapter Erlang mixtures are expressed in terms of the Confluent Hypergeometric Functions which are the Kummer's and Tricomi.
The Confluent Hypergeometric Functions have been defined and their properties given.
The Incomplete Gamma function has also been defined and its relation to the Confluent Hypergeometric Functions shown.
Moments about the origin (raw moments) of the Erlang mixtures have been derived and specifically the first moment has been obtained.
The Exponential mixtures and Poisson mixtures have also been obtained and the PGFs determined in the Poisson mixtures.

5.2 Confluent Hypergeometric Functions

5.2.1 Kummer's Confluent Hypergeometric Function

It is defined as;

$$\begin{aligned} {}_1F_1(f,g;t) &= 1+\frac{f}{g}\frac{t}{1!}+\frac{f(f+1)t^2}{g(g+1)2!}+\ldots.. \\ &= 1+\sum_{n=1}^{\infty}\frac{f(f+1)(f+2)\ldots..(f+n-1)}{g(g+1)(g+2)\ldots..(g+n-1)}\frac{t^n}{n!} \end{aligned} \quad (5.1)$$

$g \neq 0,-1,-2,-3,\ldots$

$$\begin{aligned} {}_1F_1(f,g;t) &= 1+\sum_{n=1}^{\infty}\frac{(f+n-1)(f+n-2)\ldots..(f+2)(f+1)f\Gamma f\Gamma g}{(g+n-1)(g+n-2)\ldots..(g+2)(g+1)g\Gamma g\Gamma f}\frac{t^n}{n!} \\ &= 1+\sum_{n=1}^{\infty}\frac{\Gamma(f+n)}{\Gamma(g+n)}\frac{\Gamma g}{\Gamma f}\frac{t^n}{n!} \\ &= 1+\sum_{n=1}^{\infty}\frac{\Gamma(f+n)\Gamma(g-f)}{\Gamma(g+n)}\frac{\Gamma g}{\Gamma f\Gamma(g-f)}\frac{t^n}{n!} \\ &= 1+\sum_{n=1}^{\infty}B(f+n,g-f)\frac{1}{B(f,g-f)}\frac{t^n}{n!} \end{aligned}$$

$$ {}_1F_1(f,g;t) = 1 + \frac{1}{B(f,g-f)} \sum_{n=1}^{\infty} B(f+n, g-f) \frac{t^n}{n!} $$

Expressing Kummer's Confluent Hypergeometric function in the integral form we have

$$ \begin{aligned} {}_1F_1(f,g;t) &= 1 + \frac{1}{B(f,g-f)} \sum_{n=1}^{\infty} \int_0^1 y^{f+n-1}(1-y)^{g-f-1} \frac{t^n}{n!} dy \\ &= 1 + \frac{1}{B(f,g-f)} \sum_{n=1}^{\infty} \int_0^1 y^{f-1}(1-y)^{g-f-1} \frac{(ty)^n}{n!} dy \\ &= 1 + \frac{1}{B(f,g-f)} \int_0^1 y^{f-1}(1-y)^{g-f-1} [\sum_{n=1}^{\infty} \frac{(ty)^n}{n!}] dy \\ &= 1 + \frac{1}{B(f,g-f)} \int_0^1 y^{f-1}(1-y)^{g-f-1} [e^{ty} - 1] dy \\ &= 1 + \frac{1}{B(f,g-f)} [\int_0^1 y^{f-1}(1-y)^{g-f-1} e^{ty} dy - \int_0^1 y^{f-1}(1-y)^{g-f-1} dy] \\ {}_1F_1(f,g;t) &= \frac{1}{B(f,g-f)} \int_0^1 y^{f-1}(1-y)^{g-f-1} e^{ty} dy \end{aligned} \tag{5.2} $$

This can be expressed in another form by letting x=(1-y) $\implies$ dx=dy

$$ \begin{aligned} {}_1F_1(f,g;t) &= \frac{1}{B(f,g-f)} \int_0^1 x^{g-f-1}(1-x)^{f-1} e^{t(1-x)} dx \\ &= \frac{e^t}{B(f,g-f)} \int_0^1 x^{g-f-1}(1-x)^{g-(g-f)-1} e^{-tx} dx \\ &= e^t {}_1F_1(g-f,g;-t) \end{aligned} \tag{5.3} $$

5.2.2 Tricomi Confluent Hypergeometric Function

It is defined as;

$$ \Psi(f,g;t) = \frac{1}{\Gamma f} \int_0^{\infty} y^{f-1}(1+y)^{g-f-1} e^{-ty} dy \tag{5.4} $$

Also

$$ \Psi(f,g;t) = t^{1-g} \Psi(f-g+1, 2-g;t) \tag{5.5} $$

The following relation between Tricomi and Kummer's Confluent Hypergeometric Functions holds

$$\Psi(f,g;t) = \frac{\Gamma(1-g)}{\Gamma(f-g+1)}\,{}_1F_1(f,g;t) + \frac{\Gamma(g-1)x^{1-g}}{\Gamma f}\,{}_1F_1(f-g+1,2-g;t) \quad \textbf{(5.6)}$$

$g \neq 0,-1,-2,...$

5.2.3 Incomplete Gamma Function

It is defined as

$$\gamma(f,t) = \int_0^t y^{f-1}e^{-y}dy \quad \textbf{(5.7)}$$

and its relation to the Confluent Hypergeometric Functions is as shown below.

$$\begin{aligned}
\gamma(f,t) &= \int_0^t y^{f-1}e^{-y}dy \\
&= \int_0^t y^{f-1}\sum_{n=0}^{\infty}\frac{(-y)^n}{n!}dy \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^t y^{f+n-1}dy \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{t^{f+n}}{f+n} \\
&= t^f\sum_{n=0}^{\infty}\frac{1}{f+n}\frac{(-t)^n}{n!} \\
&= t^f[\frac{1}{f} + \frac{1}{f+1}\frac{(-t)}{1!} + \frac{1}{f+2}\frac{(-t)^2}{2!} + \frac{1}{f+3}\frac{(-t)^3}{3!} + ...] \\
&= \frac{t^f}{f}[1 + \frac{f}{f+1}\frac{(-t)}{1!} + \frac{f}{f+2}\frac{(-t)^2}{2!} + \frac{f}{f+3}\frac{(-t)^3}{3!} + ...] \\
&= \frac{t^f}{f}[1 + \frac{f}{f+1}\frac{(-t)}{1!} + \frac{f(f+1)}{(f+1)(f+2)}\frac{(-t)^2}{2!} + \frac{f(f+1)(f+2)}{(f+1)(f+2)(f+3)}\frac{(-t)^3}{3!} + ...] \\
&= \frac{t^f}{f}[1 + \sum_{n=1}^{\infty}\frac{f(f+1)(f+2).....(f+n-1)}{(f+1)(f+2).....(f+n)}\frac{(-t)^n}{n!}]
\end{aligned}$$

$$\therefore \quad \gamma(f,t) = \frac{t^f}{f}\,{}_1F_1(f,f+1;-t) \quad \textbf{(5.8)}$$

Also, from (5.3) we get

$$\gamma(f,t) = \frac{t^f}{f} e_1^{-t} F_1(1, f+1; -t) \tag{5.9}$$

5.3 Erlang-Beta I Distribution and Its Links

5.3.1 Erlang-Beta I Mixture

The Beta I mixing distribution is

$$g(\lambda) = \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 < \lambda < 1; \alpha > 0, \beta > 0 \tag{5.10}$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^1 \lambda^n e^{-t\lambda} \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)} d\lambda \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 \lambda^{n+\alpha-1}(1-\lambda)^{\beta-1} e^{-t\lambda} d\lambda \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 \lambda^{n+\alpha-1}(1-\lambda)^{(n+\alpha)+\beta-(n+\alpha)-1} e^{-t\lambda} d\lambda \\
&= \frac{B(n+\alpha,\beta)}{B(\alpha,\beta)} {}_1F_1(n+\alpha, n+\alpha+\beta; -t)
\end{aligned} \tag{5.11}$$

Construction by the direct method gives

$$\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} \frac{B(n+\alpha,\beta)}{B(\alpha,\beta)} {}_1F_1(n+\alpha, n+\alpha+\beta; -t), \quad t > 0; \alpha > 0, \beta > 0, n = 1, 2, 3, ..
\end{aligned} \tag{5.12}$$

By the method of moments we have

$$
\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)} {}_1F_1(j+\alpha, j+\alpha+\beta; 0) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)} \qquad (5.13)
\end{aligned}
$$

Identity 5.1

Equating the above two methods we get

$$
\begin{aligned}
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)} &= \frac{t^{n-1}}{\Gamma n} \frac{B(n+\alpha,\beta)}{B(\alpha,\beta)} {}_1F_1(n+\alpha, n+\alpha+\beta; -t) \\
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)} &= \frac{B(n+\alpha,\beta)}{B(\alpha,\beta)} {}_1F_1(n+\alpha, n+\alpha+\beta; -t) \qquad (5.14)
\end{aligned}
$$

The rth moment of the Erlang mixture is

$$
\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \frac{B(\alpha-r,\beta)}{B(\alpha,\beta)} \qquad (5.15) \\
E(T) &= \frac{\Gamma(n+1)}{\Gamma n} \frac{B(\alpha-1,\beta)}{B(\alpha,\beta)} \\
&= n\frac{\alpha+\beta-1}{\alpha-1} \qquad (5.16)
\end{aligned}
$$

5.3.2 Exponential-Beta I Mixture

Construction by the direct method gives

$$
\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} {}_1F_1(\alpha+1, \alpha+\beta+1; -t) \\
&= \frac{\alpha}{\alpha+\beta} {}_1F_1(\alpha+1, \alpha+\beta+1; -t), \quad t>0; \alpha>0, \beta>0 \qquad (5.17)
\end{aligned}
$$

By the method of moments we have

$$f_1(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j)$$
$$= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{B(j+\alpha, \beta)}{B(\alpha, \beta)} \tag{5.18}$$

Identity 5.2
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{B(j+\alpha, \beta)}{B(\alpha, \beta)} = \frac{\alpha}{\alpha+\beta} {}_1F_1(\alpha+1, \alpha+\beta+1; -t) \tag{5.19}$$

The rth moment of the Exponential mixture is

$$E(T^r) = r! E(\wedge^{-r})$$
$$= r! \frac{B(\alpha - r, \beta)}{B(\alpha, \beta)} \tag{5.20}$$
$$E(T) = \frac{B(\alpha - r, \beta)}{B(\alpha, \beta)}$$
$$= \frac{\alpha+\beta-1}{\alpha-1} \tag{5.21}$$

5.3.3 Poisson-Beta I Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$P_n(t) = \frac{t^n}{n!} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^n}{n!} \frac{B(n+\alpha, \beta)}{B(\alpha, \beta)} {}_1F_1(n+\alpha, n+\alpha+\beta; -t), \quad t > 0; \alpha > 0, \beta > 0 \tag{5.22}$$

By the method of moments we have

$$P_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)} \tag{5.23}$$

Identity 5.3
Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)} = \frac{t^n}{n!} \frac{B(n+\alpha,\beta)}{B(\alpha,\beta)} {}_1F_1(n+\alpha, n+\alpha+\beta; -t)$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)} = \frac{B(n+\alpha,\beta)}{B(\alpha,\beta)} {}_1F_1(n+\alpha, n+\alpha+\beta; -t) \tag{5.24}$$

The PGF of the Poisson mixture is

$$G(s,t) = E[e^{-(1-s)t\wedge}]$$
$$= \frac{B(\alpha,\beta)}{B(\alpha,\beta)} {}_1F_1(\alpha, \alpha+\beta; -(1-s)t)$$
$$= {}_1F_1(\alpha, \alpha+\beta; -(1-s)t) \tag{5.25}$$

The rth moment of the Poisson mixture is

$$E(T^r) = t^r E(\wedge^r)$$
$$= t^r \frac{B(r+\alpha,\beta)}{B(\alpha,\beta)} \tag{5.26}$$
$$E(T) = t \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$
$$= \frac{t\alpha}{\alpha+\beta} \tag{5.27}$$

5.4 Erlang-Uniform Distribution and Its Links

5.4.1 Erlang-Uniform Mixture

The Uniform mixing distribution is

$$g(\lambda) = \frac{1}{b-a}, \quad a \leq \lambda \leq b, a > 0, b > 0 \tag{5.28}$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_a^b \lambda^n e^{-t\wedge} \frac{1}{b-a} d\lambda \\
&= \frac{1}{b-a} \int_a^b \lambda^n e^{-t\wedge} d\lambda \\
&= \frac{1}{b-a} [\int_0^b \lambda^n e^{-t\wedge} d\lambda - \int_0^a \lambda^n e^{-t\wedge} d\lambda]
\end{aligned}$$

$$let \quad x = \lambda t \quad \Longrightarrow dx = t d\lambda$$

$$\begin{aligned}
E[\wedge^n e^{-t\wedge}] &= \frac{1}{b-a} [\int_0^{bt} (\frac{x}{t})^n e^{-x} \frac{dx}{t} - \int_0^{at} (\frac{x}{t})^n e^{-x} \frac{dx}{t}] \\
&= \frac{1}{b-a} [\frac{\gamma(n+1, bt)}{t^{n+1}} - \frac{\gamma(n+1, at)}{t^{n+1}}] \\
&= \frac{1}{(b-a)t^{n+1}} [\frac{(bt)^{n+1}}{n+1} {}_1F_1(n+1, n+2; -bt)] - \frac{1}{(b-a)t^{n+1}} [\frac{(at)^{n+1}}{n+1} {}_1F_1(n+1, n+2; -at)] \\
&= \frac{1}{(b-a)(n+1)} [b^{n+1} {}_1F_1(n+1, n+2; -bt) - a^{n+1} {}_1F_1(n+1, n+2; -at)]
\end{aligned} \tag{5.29}$$

Construction by the direct method gives

$$\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} \frac{1}{(b-a)(n+1)} [b^{n+1} {}_1F_1(n+1, n+2; -bt) - a^{n+1} {}_1F_1(n+1, n+2; -at)], \quad t > 0; a > 0, b > 0, n = 1,
\end{aligned} \tag{5.30}$$

By the method of moments we have

$$\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{[b^{j+1} - a^{j+1}]}{(b-a)(j+1)}
\end{aligned} \tag{5.31}$$

Identity 5.4

Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}\frac{[b^{j+1}-a^{j+1}]}{(b-a)(j+1)}=\frac{t^{n-1}}{\Gamma n}\frac{1}{(b-a)(n+1)}[b^{n+1}{}_1F_1(n+1,n+2;-bt)-a^{n+1}{}_1F_1(n+1,n+2;-$$

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{[b^{j+1}-a^{j+1}]}{(b-a)(j+1)}=\frac{1}{(b-a)(n+1)}[b^{n+1}{}_1F_1(n+1,n+2;-bt)-a^{n+1}{}_1F_1(n+1,n+2;-at)] \quad (5.32)$$

The rth moment of the Erlang mixture is

$$\begin{aligned}E(T^r)&=\frac{\Gamma(n+r)}{\Gamma n}E(\wedge^{-r})\\&=\frac{\Gamma(n+r)}{\Gamma n}\frac{[b^{1-r}-a^{1-r}]}{(b-a)(1-r)}\end{aligned} \quad (5.33)$$

$$E(T)=\frac{\Gamma(n+1)}{\Gamma n}\frac{[1-1]}{(b-a)(0)}=\infty \quad (5.34)$$

5.4.2 Exponential-Uniform Mixture

Construction by the direct method gives

$$\begin{aligned}f_1(t)&=E[\wedge e^{-t\wedge}]\\&=\frac{1}{(b-a)(1+1)}[b^{1+1}{}_1F_1(1+1,1+2;-bt)-a^{1+1}{}_1F_1(1+1,1+2;-at)]\\&=\frac{1}{2(b-a)}[b^2{}_1F_1(2,3;-bt)-a^2{}_1F_1(2,3;-at)],\quad t>0;a>0,b>0\end{aligned} \quad (5.35)$$

By the method of moments we have

$$\begin{aligned}f_1(t)&=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}E(\wedge^j)\\&=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\frac{[b^{j+1}-a^{j+1}]}{(b-a)(j+1)}\end{aligned} \quad (5.36)$$

Identity 5.5
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} \frac{[b^{j+1}-a^{j+1}]}{(b-a)(j+1)} = \frac{1}{2(b-a)}[b^2{}_1F_1(2,3;-bt)-a^2{}_1F_1(2,3;-at)] \quad (5.37)$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= r!\frac{[b^{1-r}-a^{1-r}]}{(b-a)(1-r)} \\ &= \frac{r![b^{1-r}-a^{1-r}]}{(b-a)(1-r)} \quad (5.38) \\ E(T) &= \frac{[1-1]}{(b-a)(1-1)} = \infty \quad (5.39) \end{aligned}$$

5.4.3 Poisson-Uniform Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!}\frac{1}{(b-a)(n+1)}[b^{n+1}{}_1F_1(n+1,n+2;-bt)-a^{n+1}{}_1F_1(n+1,n+2;-at)], \quad t>0; a>0, b>0 \end{aligned} \quad (5.40)$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{[b^{j+1}-a^{j+1}]}{(b-a)(j+1)} \quad (5.41) \end{aligned}$$

Identity 5.6

Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{[b^{j+1}-a^{j+1}]}{(b-a)(j+1)}=\frac{t^n}{n!}\frac{1}{(b-a)(n+1)}[b^{n+1}{}_1F_1(n+1,n+2;-bt)-a^{n+1}{}_1F_1(n+1,n+2;-a$$

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{[b^{j+1}-a^{j+1}]}{(b-a)(j+1)}=\frac{1}{(b-a)(n+1)}[b^{n+1}{}_1F_1(n+1,n+2;-bt)-a^{n+1}{}_1F_1(n+1,n+2;-at)] \quad (5.42)$$

The PGF of the Poisson mixture is

$$\begin{aligned}G(s,t)&=E[e^{-(1-s)t\wedge}]\\&=\frac{1}{(b-a)}[b_1F_1(1,2;-b(1-s)t)-a_1F_1(1,2;-a(1-s)t)]\end{aligned} \quad (5.43)$$

The rth moment of the Poisson mixture is

$$\begin{aligned}E(T^r)&=t^rE(\wedge^r)\\&=t^r\frac{[b^{r+1}-a^{r+1}]}{(b-a)(r+1)}\\&=\frac{t^r[b^{r+1}-a^{r+1}]}{(b-a)(r+1)}\end{aligned} \quad (5.44)$$

$$E(T)=\frac{t[b^2-a^2]}{2(b-a)} \quad (5.45)$$

5.5 Erlang-Beta II Distribution and Its Links

5.5.1 Erlang-Beta II Mixture

The Beta II mixing distribution is

$$g(\lambda)=\frac{\lambda^{\alpha-1}}{B(\alpha,\beta)(1+\lambda)^{\alpha+\beta}},\quad \lambda>0;\alpha>0,\beta>0 \quad (5.46)$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_0^\infty \lambda^n e^{-t\lambda} \frac{\lambda^{\alpha-1}}{B(\alpha,\beta)(1+\lambda)^{\alpha+\beta}} d\lambda$$
$$= \frac{1}{B(\alpha,\beta)} \int_0^\infty \frac{\lambda^{n+\alpha-1} e^{-t\lambda}}{(1+\lambda)^{\alpha+\beta}} d\lambda$$
$$= \frac{1}{B(\alpha,\beta)} \int_0^\infty \lambda^{n+\alpha-1} (1+\lambda)^{n+1-\beta-(n+\alpha)-1} e^{-t\lambda} d\lambda$$
$$= \frac{\Gamma(n+\alpha)}{B(\alpha,\beta)} \Psi(n+\alpha, n-\beta+1; t) \tag{5.47}$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^{n-1}}{\Gamma n} \frac{\Gamma(n+\alpha)}{B(\alpha,\beta)} \Psi(n+\alpha, n-\beta+1; t), \quad t > 0; \alpha > 0, \beta > 0, n = 1, 2, 3, .. \tag{5.48}$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{B(j+\alpha, \beta-j)}{B(\alpha,\beta)} \tag{5.49}$$

Identity 5.7

Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{B(j+\alpha, \beta-j)}{B(\alpha,\beta)} = \frac{t^{n-1}}{\Gamma n} \frac{\Gamma(n+\alpha)}{B(\alpha,\beta)} \Psi(n+\alpha, n-\beta+1; t)$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{B(j+\alpha, \beta-j)}{B(\alpha,\beta)} = \frac{\Gamma(n+\alpha)}{B(\alpha,\beta)} \Psi(n+\alpha, n-\beta+1; t) \tag{5.50}$$

The rth moment of the Erlang mixture is

$$
\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \frac{B(\alpha - r, \beta + r)}{B(\alpha, \beta)} \qquad (5.51) \\
E(T) &= \frac{\Gamma(n+1)}{\Gamma n} \frac{B(\alpha - 1, \beta + 1)}{B(\alpha, \beta)} \\
&= \frac{n\beta}{\alpha - 1} \qquad (5.52)
\end{aligned}
$$

5.5.2 Exponential-Beta II Mixture

Construction by the direct method gives

$$
\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= \frac{\Gamma(\alpha+1)}{B(\alpha, \beta)} \Psi(\alpha + 1, 2 - \beta; t), \quad t > 0; \alpha > 0, \beta > 0 \qquad (5.53)
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\
&= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{B(j + \alpha, \beta - j)}{B(\alpha, \beta)} \qquad (5.54)
\end{aligned}
$$

Identity 5.8
Equating the above two methods we get

$$
\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{B(j + \alpha, \beta - j)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+1)}{B(\alpha, \beta)} \Psi(\alpha + 1, 2 - \beta; t) \qquad (5.55)
$$

The rth moment of the Exponential mixture is

$$
\begin{aligned}
E(T^r) &= r!E(\wedge^{-r}) \\
&= r!\frac{B(\alpha-r,\beta+r)}{B(\alpha,\beta)} && (5.56) \\
E(T) &= \frac{B(\alpha-1,\beta+1)}{B(\alpha,\beta)} \\
&= \frac{\beta}{\alpha-1} && (5.57)
\end{aligned}
$$

5.5.3 Poisson-Beta II Mixture

$$
P_n(t) = \frac{t}{n} f_n(t)
$$

Construction by the direct method gives

$$
\begin{aligned}
P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^n}{n!}\frac{\Gamma(n+\alpha)}{B(\alpha,\beta)}\Psi(n+\alpha, n-\beta+1; t), \quad t>0; \alpha>0, \beta>0 && (5.58)
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\
&= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{B(j+\alpha,\beta-j)}{B(\alpha,\beta)} && (5.59)
\end{aligned}
$$

Identity 5.9
Equating the above two methods we get

$$
\begin{aligned}
\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{B(j+\alpha,\beta-j)}{B(\alpha,\beta)} &= \frac{t^n}{n!}\frac{\Gamma(n+\alpha)}{B(\alpha,\beta)}\Psi(n+\alpha,n-\beta+1;t) \\
\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{B(j+\alpha,\beta-j)}{B(\alpha,\beta)} &= \frac{\Gamma(n+\alpha)}{B(\alpha,\beta)}\Psi(n+\alpha,n-\beta+1;t) && (5.60)
\end{aligned}
$$

The PGF of the Poisson mixture is

$$
\begin{aligned}
G(s,t) &= E[e^{-(1-s)t\wedge}] \\
&= \frac{\Gamma(\alpha)}{B(\alpha,\beta)}\Psi(\alpha,1-\beta;(1-s)t)
\end{aligned} \tag{5.61}
$$

The rth moment of the Poisson mixture is

$$
\begin{aligned}
E(T^r) &= t^r E(\wedge^r) \\
&= t^r \frac{B(r+\alpha,\beta-r)}{B(\alpha,\beta)} \\
&= \frac{t^r B(r+\alpha,\beta-r)}{B(\alpha,\beta)}
\end{aligned} \tag{5.62}
$$

$$
\begin{aligned}
E(T) &= \frac{tB(\alpha+1,\beta-1)}{B(\alpha,\beta)} \\
&= \frac{t\alpha}{\beta-1}
\end{aligned} \tag{5.63}
$$

5.6 Erlang-Scaled Beta Distribution and Its Links

5.6.1 Erlang-Scaled Beta Mixture

Given the Beta I distribution as;

$$
h(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 < x < 1; \alpha > 0, \beta > 0
$$

let;

$$
x = \frac{\lambda}{\mu} \quad \Longrightarrow \lambda = x\mu \quad \Longrightarrow \frac{dx}{d\lambda} = \frac{1}{\mu}
$$

The Scaled Beta mixing distribution is;

$$
\begin{aligned}
g(\lambda) &= h(x)|\frac{dx}{d\lambda}| = h(x)\frac{1}{\mu} \\
\therefore \quad g(\lambda) &= \frac{(\frac{\lambda}{\mu})^{\alpha-1}(1-\frac{\lambda}{\mu})^{\beta-1}\frac{1}{\mu}}{B(\alpha,\beta)}, \quad 0 < \lambda < \mu; \alpha > 0, \beta > 0, \mu > 0
\end{aligned} \tag{5.64}
$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_0^{\mu} \lambda^n e^{-t\lambda} \frac{(\frac{\lambda}{\mu})^{\alpha-1}(1-\frac{\lambda}{\mu})^{\beta-1}}{B(\alpha,\beta)} \frac{d\lambda}{\mu}$$
$$= \mu^{n-1} \int_0^{\mu} \frac{(\frac{\lambda}{\mu})^{n}(1-\frac{\lambda}{\mu})^{\beta-1}}{B(\alpha,\beta)} (\frac{\lambda}{\mu})^{\alpha-1} e^{-t\lambda} d\lambda$$
$$= \frac{\mu^{n-1}}{B(\alpha,\beta)} \int_0^{\mu} (\frac{\lambda}{\mu})^{n+\alpha-1}(1-\frac{\lambda}{\mu})^{\beta-1} e^{-t\lambda} d\lambda$$

$$let \quad \lambda = \mu x \quad \Longrightarrow d\lambda = \mu dx$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \frac{\mu^{n}}{B(\alpha,\beta)} \int_0^{1} x^{n+\alpha-1}(1-x)^{\beta-1} e^{-\mu t x} dx$$
$$= \frac{\mu^n B(n+\alpha, n+\alpha+\beta)}{B(\alpha,\beta)} \int_0^1 \frac{x^{n+\alpha-1}(1-x)^{\beta-1}}{B(n+\alpha, n+\alpha+\beta)} e^{-\mu t x} dx$$
$$= \frac{\mu^n B(n+\alpha, n+\alpha+\beta)}{B(\alpha,\beta)} \int_0^1 \frac{x^{n+\alpha-1}(1-x)^{n+\alpha+\beta-(n+\alpha)-1}}{B(n+\alpha, n+\alpha+\beta)} e^{-\mu t x} dx$$
$$= \frac{\mu^n B(n+\alpha, n+\alpha+\beta)}{B(\alpha,\beta)} {}_1F_1(n+\alpha, n+\alpha+\beta; -\mu t) \tag{5.65}$$

$$E(\wedge^j) = \frac{\mu^j}{B(\alpha,\beta)} \int_0^1 x^{j+\alpha-1}(1-x)^{\beta-1} dx$$
$$= \mu^j \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)} \tag{5.66}$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^{n-1}}{\Gamma n} \frac{\mu^n B(n+\alpha, n+\alpha+\beta)}{B(\alpha,\beta)} {}_1F_1(n+\alpha, n+\alpha+\beta; -\mu t)$$
$$= \frac{(\mu t)^n}{t\Gamma n} \frac{B(n+\alpha, n+\alpha+\beta)}{B(\alpha,\beta)} {}_1F_1(n+\alpha, n+\alpha+\beta; -\mu t), \quad t>0; \alpha>0, \beta>0, \mu>0, n=1,2,3,.. \tag{5.67}$$

By the method of moments we have

$$
\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\mu^j B(j+\alpha,\beta)}{B(\alpha,\beta)} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k t^{n+k-1} \mu^{n+k}}{k! \Gamma n} \frac{B(n+k+\alpha,\beta)}{B(\alpha,\beta)} \\
&= \frac{t^{n-1}\mu^n}{\Gamma n} \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t)^k}{k!} \frac{B(n+k+\alpha,\beta)}{B(\alpha,\beta)} \qquad (5.68)
\end{aligned}
$$

By McNolty's Approach

$$
\begin{aligned}
f_{n+1}(t) &= \int_0^a \frac{\lambda^{n+1}}{\Gamma(n+1)} e^{-\lambda t} t^n g(\lambda) d\lambda \\
&= \int_0^a \frac{\lambda^{n+1}}{\Gamma(n+1)} \frac{e^{-\lambda t} t^n \lambda^{p-1} (1-\frac{\lambda}{a})^{q-1}}{a^p B(p,q)} d\lambda \\
&= \int_0^a \frac{\lambda^{n+1} e^{-\lambda t} t^n}{\Gamma n B(p,q)} (\frac{\lambda}{a})^{p-1} (1-\frac{\lambda}{a})^{q-1} \frac{d\lambda}{a} \\
\therefore \quad f_{n+1}(t) &= \frac{a^n t^n}{\Gamma n B(p,q)} \int_0^a (\frac{\lambda}{a})^{n+1+p-1} (1-\frac{\lambda}{a})^{q-1} e^{-\lambda t} d\lambda \\
&= \frac{(at)^n}{\Gamma n B(p,q)} \int_0^a (\frac{\lambda}{a})^{n+1+p-1} (1-\frac{\lambda}{a})^{q-1} e^{-at\frac{\lambda}{a}} d\lambda
\end{aligned}
$$

$$
let \quad z = \frac{\lambda}{a} \quad \Longrightarrow \quad adz = d\lambda
$$

$$
\begin{aligned}
\therefore \quad f_{n+1}(t) &= \frac{(at)^n}{\Gamma n B(p,q)} \int_0^1 z^{n+1+p-1} (1-z)^{q-1} e^{-atz} adz \\
&= \frac{a^{n+1} t^n}{\Gamma n B(p,q)} \int_0^1 z^{n+1+p-1} (1-z)^{q-1} \sum_{k=0}^{\infty} \frac{(-at)^k z^k}{k!} dz \\
&= \frac{a^{n+1} t^n}{\Gamma n B(p,q)} \sum_{k=0}^{\infty} [\frac{(-at)^k}{k!} \int_0^1 z^{n+1+p+k-1} (1-z)^{q-1} dz] \\
&= \frac{a^{n+1} t^n}{\Gamma n B(p,q)} \sum_{k=0}^{\infty} \frac{(-at)^k}{k!} B(n+p+k+1,q) \qquad (5.69)
\end{aligned}
$$

This result can be achieved using the result given by the method of moments;i.e.,

$$f_n(t) = \frac{t^{n-1}\mu^n}{\Gamma n}\sum_{k=0}^{\infty}\frac{(-\mu t)^k}{k!}\frac{B(n+k+\alpha,\beta)}{B(\alpha,\beta)}$$

$$\therefore \quad f_{n+1}(t) = \frac{t^n\mu^{n+1}}{\Gamma(n+1)}\sum_{k=0}^{\infty}\frac{(-\mu t)^k}{k!}\frac{B(n+1+k+\alpha,\beta)}{B(\alpha,\beta)}$$

$let \quad \mu = a, \quad \alpha = p \quad and \quad \beta = q$

$$\therefore \quad f_{n+1}(t) = \frac{t^n a^{n+1}}{\Gamma(n+1)}\sum_{k=0}^{\infty}\frac{(-at)^k}{k!}\frac{B(n+1+k+p,q)}{B(p,q)} \tag{5.70}$$

Identity 5.10

Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}\frac{\mu^j B(j+\alpha,\beta)}{B(\alpha,\beta)} = \frac{(\mu t)^n}{t\Gamma n}\frac{B(n+\alpha,\beta)}{B(\alpha,\beta)}\,{}_1F_1(n+\alpha,n+\alpha+\beta;-\mu t)$$

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{\mu^j B(j+\alpha,\beta)}{B(\alpha,\beta)} = \frac{\mu^n B(n+\alpha,\beta)}{B(\alpha,\beta)}\,{}_1F_1(n+\alpha,n+\alpha+\beta;-\mu t) \tag{5.71}$$

The rth moment of the Erlang mixture is

$$E(T^r) = \frac{\Gamma(n+r)}{\Gamma n}E(\wedge^{-r})$$

$$= \frac{\Gamma(n+r)}{\Gamma n}\frac{\mu^{-r}B(\alpha-r,\beta)}{B(\alpha,\beta)} \tag{5.72}$$

$$E(T) = \frac{\Gamma(n+1)}{\Gamma n}\frac{\mu^{-1}B(\alpha-1,\beta)}{B(\alpha,\beta)}$$

$$= \frac{n}{\mu}\frac{(\alpha+\beta-1)}{\alpha-1} \tag{5.73}$$

5.6.2 Exponential-Scaled Beta Mixture

Construction by the direct method gives

$$f_1(t) = E[\wedge e^{-t\wedge}]$$

$$= \frac{\mu B(\alpha+1,\beta)}{B(\alpha,\beta)}\,{}_1F_1(\alpha+1,\alpha+\beta+1;-\mu t)$$

$$= \frac{\mu\alpha}{\alpha+\beta}\,{}_1F_1(\alpha+1,\alpha+\beta+1;-\mu t), \quad t>0;\alpha>0,\beta>0,\mu>0 \tag{5.74}$$

By the method of moments we have

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{\mu^j B(j+\alpha,\beta)}{B(\alpha,\beta)} \end{aligned} \tag{5.75}$$

Identity 5.11
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{\mu^j B(j+\alpha,\beta)}{B(\alpha,\beta)} = \frac{\mu\alpha}{\alpha+\beta} {}_1F_1(\alpha+1,\alpha+\beta+1;-\mu t) \tag{5.76}$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r! E(\wedge^{-r}) \\ &= r! \frac{\mu^{-r} B(\alpha-r,\beta)}{B(\alpha,\beta)} \\ &= \frac{r!}{\mu^r} \frac{B(\alpha-r,\beta)}{B(\alpha,\beta)} \end{aligned} \tag{5.77}$$

$$\begin{aligned} E(T) &= \frac{1}{\mu} \frac{B(\alpha-1,\beta)}{B(\alpha,\beta)} \\ &= \frac{\alpha+\beta-1}{\mu(\alpha-1)} \end{aligned} \tag{5.78}$$

5.6.3 Poisson-Scaled Beta Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$
\begin{aligned}
P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^n}{n!}\frac{\mu^n B(n+\alpha,\beta)}{B(\alpha,\beta)}{}_1F_1(n+\alpha,n+\alpha+\beta;-\mu t) \\
&= \frac{(\mu t)^n}{n!}\frac{B(n+\alpha,\beta)}{B(\alpha,\beta)}{}_1F_1(n+\alpha,n+\alpha+\beta;-\mu t), \quad t>0;\alpha>0,\beta>0,\mu>0
\end{aligned} \tag{5.79}
$$

By the method of moments we have

$$
\begin{aligned}
P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\
&= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{\mu^j B(j+\alpha,\beta)}{B(\alpha,\beta)}
\end{aligned} \tag{5.80}
$$

Identity 5.12
Equating the above two methods we get

$$
\begin{aligned}
\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{\mu^j B(j+\alpha,\beta)}{B(\alpha,\beta)} &= \frac{(\mu t)^n}{n!}\frac{B(n+\alpha,\beta)}{B(\alpha,\beta)}{}_1F_1(n+\alpha,n+\alpha+\beta;-\mu t) \\
\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{\mu^j B(j+\alpha,\beta)}{B(\alpha,\beta)} &= \frac{\mu^n B(n+\alpha,\beta)}{B(\alpha,\beta)}{}_1F_1(n+\alpha,n+\alpha+\beta;-\mu t)
\end{aligned} \tag{5.81}
$$

The PGF of the Poisson mixture is

$$
\begin{aligned}
G(s,t) &= E[e^{-(1-s)t\wedge}] \\
&= \frac{B(\alpha,\beta)}{B(\alpha,\beta)}{}_1F_1(\alpha,\alpha+\beta;-\mu(1-s)t) \\
&= {}_1F_1(\alpha,\alpha+\beta;-\mu(1-s)t)
\end{aligned} \tag{5.82}
$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r \frac{\mu^r B(r+\alpha,\beta)}{B(\alpha,\beta)} \\ &= (\mu t)^r \frac{B(r+\alpha,\beta)}{B(\alpha,\beta)} \qquad (5.83) \\ E(T) &= (\mu t) \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \\ &= \frac{\mu t \alpha}{\beta+\alpha} \qquad (5.84) \end{aligned}$$

5.7 Erlang-Full Beta Distribution and Its Links

5.7.1 Erlang-Full Beta Mixture

Given the Beta II distribution as

$$h(x) = \frac{x^{p-1}}{B(p,q)(1+x)^{p+q}}, \quad x > 0; p > 0, q > 0$$

let

$$x = b\lambda \quad \Longrightarrow dx = bd\lambda$$

The Full Beta mixing distribution is

$$g(\lambda) = \frac{b^p}{B(p,q)} \frac{\lambda^{p-1}}{(1+b\lambda)^{p+q}}, \quad \lambda > 0; b > 0, p > 0, q > 0 \qquad (5.85)$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_0^\infty \lambda^n e^{-t\lambda} \frac{b^p}{B(p,q)} \frac{\lambda^{p-1}}{(1+b\lambda)^{p+q}} d\lambda$$

$$= \frac{b^p}{B(p,q)} \int_0^\infty \frac{\lambda^{n+p-1} e^{-t\lambda}}{(1+b\lambda)^{p+q}} d\lambda$$

$$let \quad b\lambda = x \quad \Longrightarrow bd\lambda = dx$$

$$E[\wedge^n e^{-t\wedge}] = \frac{b^p}{B(p,q)} \int_0^\infty (\frac{x}{b})^{n+p-1} e^{-t\frac{x}{b}} (1+b\frac{x}{b})^{-(p+q)} \frac{dx}{b}$$

$$= \frac{b^p}{b^{n+p}B(p,q)} \int_0^\infty x^{n+p-1} e^{-\frac{t}{b}x} (1+x)^{n+1-q-(n+p)-1} dx$$

$$= \frac{\Gamma(n+p)}{b^n B(p,q)} \Psi(n+p, n+1-q; \frac{t}{b}) \tag{5.86}$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$

$$= \frac{t^{n-1}}{\Gamma n} \frac{\Gamma(n+p)}{b^n B(p,q)} \Psi(n+p, n+1-q; \frac{t}{b})$$

$$= \frac{1}{t} (\frac{t}{b})^n \frac{\Gamma(n+p)}{\Gamma n B(p,q)} \Psi(n+p, n+1-q; \frac{t}{b}), \quad t>0; b>0, p>0, q>0, n=1,2,3,.. \tag{5.87}$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$

$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{B(j+p, q-j)}{b^j B(p,q)} \tag{5.88}$$

Identity 5.13

Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{B(j+p, q-j)}{b^j B(p,q)} = \frac{1}{t} (\frac{t}{b})^n \frac{\Gamma(n+p)}{\Gamma n B(p,q)} \Psi(n+p, n+1-q; \frac{t}{b})$$

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{B(j+p, q-j)}{b^j B(p,q)} = \frac{\Gamma(n+p)}{b^n B(p,q)} \Psi(n+p, n+1-q; \frac{t}{b}) \tag{5.89}$$

The rth moment of the Erlang mixture is

$$\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \frac{B(p-r,q+r)}{b^{-r}B(p,q)} \\
&= \frac{\Gamma(n+r)}{\Gamma n} \frac{b^r B(p-r,q+r)}{B(p,q)} && (5.90) \\
E(T) &= \frac{\Gamma(n+1)}{\Gamma n} \frac{bB(p-1,q+1)}{B(p,q)} \\
&= \frac{nbq}{p-1} && (5.91)
\end{aligned}$$

5.7.2 Exponential-Full Beta Mixture

Construction by the direct method gives

$$\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= \frac{\Gamma(p+1)}{bB(p,q)} \Psi(p+1, 2-q; \frac{t}{b}), \quad t>0; b>0, p>0, q>0 && (5.92)
\end{aligned}$$

By the method of moments we have

$$\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\
&= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{B(j+p,q-j)}{b^j B(p,q)} && (5.93)
\end{aligned}$$

Identity 5.14
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{B(j+p,q-j)}{b^j B(p,q)} = \frac{\Gamma(p+1)}{bB(p,q)} \Psi(p+1, 2-q; \frac{t}{b}) \quad (5.94)$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= r!\frac{B(p-r,q+r)}{b^{-r}B(p,q)} \\ &= \frac{r!b^r B(p-r,q+r)}{B(p,q)} \qquad (5.95) \\ E(T) &= \frac{bB(p-1,q+1)}{B(p,q)} \\ &= \frac{bq}{p-1} \qquad (5.96) \end{aligned}$$

5.7.3 Poisson-Full Beta Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!b^n}\frac{\Gamma(n+p)}{B(p,q)}\Psi(n+p,n+1-q;\frac{t}{b}) \\ &= (\frac{t}{b})^n\frac{\Gamma(n+p)}{n!B(p,q)}\Psi(n+p,n+1-q;\frac{t}{b}), \quad t>0; b>0, p>0, q>0 \qquad (5.97) \end{aligned}$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\ &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{B(j+p,q-j)}{b^j B(p,q)} \qquad (5.98) \end{aligned}$$

Identity 5.15
Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{B(j+p,q-j)}{b^jB(p,q)}=(\frac{t}{b})^n\frac{\Gamma(n+p)}{n!B(p,q)}\Psi(n+p,n+1-q;\frac{t}{b})$$
$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{B(j+p,q-j)}{b^jB(p,q)}=\frac{\Gamma(n+p)}{b^nB(p,q)}\Psi(n+p,n+1-q;\frac{t}{b}) \tag{5.99}$$

The PGF of the Poisson mixture is

$$\begin{aligned}G(s,t)&=E[e^{-(1-s)t\wedge}]\\&=\frac{\Gamma(p)}{B(p,q)}\Psi(p,1-q;\frac{(1-s)t}{b})\end{aligned} \tag{5.100}$$

The rth moment of the Poisson mixture is

$$\begin{aligned}E(T^r)&=t^rE(\wedge^r)\\&=t^r\frac{B(r+p,q-r)}{b^rB(p,q)}\\&=(\frac{t}{b})^r\frac{B(r+p,q-r)}{B(p,q)}\end{aligned} \tag{5.101}$$

$$\begin{aligned}E(T)&=(\frac{t}{b})\frac{B(p+1,q-1)}{B(p,q)}\\&=\frac{tp}{b(q-1)}\end{aligned} \tag{5.102}$$

5.8 Erlang-Pearson Type I Distribution and Its Links

5.8.1 Erlang-Pearson Type I Mixture

The Pearson Type I mixing distribution is

$$g(\lambda)=\frac{1}{B(p,q)}\frac{(\lambda-a)^{p-1}}{(b-a)^{p-1}}\frac{(b-\lambda)^{q-1}}{(b-a)^{q-1}}\frac{1}{b-a},\quad a\le\lambda\le b, a>0, b>0, p>0, q>0 \tag{5.103}$$

$$
\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_a^b \lambda^n e^{-t\lambda} \frac{1}{B(p,q)} \frac{(\lambda-a)^{p-1}}{(b-a)^{p-1}} \frac{(b-\lambda)^{q-1}}{(b-a)^{q-1}} \frac{d\lambda}{b-a} \\
&= \frac{1}{B(p,q)} \int_a^b \lambda^n e^{-t\lambda} (\frac{\lambda-a}{b-a})^{p-1} (1-\frac{\lambda-a}{b-a})^{q-1} \frac{d\lambda}{b-a} \\
let \quad x &= \frac{\lambda-a}{b-a} \quad \Longrightarrow x(b-a)+a=\lambda \quad \Longrightarrow d\lambda=(b-a)dx \\
E[\wedge^n e^{-t\wedge}] &= \frac{1}{B(p,q)} \int_0^1 (x(b-a)+a)^n e^{-t(x(b-a)+a)} x^{p-1}(1-x)^{q-1} dx \\
&= \frac{e^{-ta}}{B(p,q)} \int_0^1 e^{-tx(b-a)} [\sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k (b-a)^k] x^{p-1}(1-x)^{q-1} dx \\
&= \frac{e^{-ta}}{B(p,q)} \sum_{k=0}^{n} [\binom{n}{k} a^{n-k}(b-a)^k \int_0^1 x^{k+p-1}(1-x)^{k+p+q-(k+p)-1} e^{-tx(b-a)} dx] \\
&= e^{-ta} \sum_{k=0}^{n} [\binom{n}{k} a^{n-k}(b-a)^k \frac{B(k+p,q)}{B(p,q)} {}_1F_1(k+p,k+p+q;-(b-a)t)]
\end{aligned}
\tag{5.104}
$$

Construction by the direct method gives

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} e^{-ta} \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(b-a)^k \frac{B(k+p,q)}{B(p,q)} {}_1F_1(k+p,k+p+q;-(b-a)t) \\
&= \frac{a^n t^{n-1} e^{-ta}}{\Gamma n} \sum_{k=0}^{n} \binom{n}{k} (\frac{b-a}{a})^k \frac{B(k+p,q)}{B(p,q)} {}_1F_1(k+p,k+p+q;-(b-a)t), \quad t>0; a>0, b>0, p>0, q>
\end{aligned}
\tag{5.105}
$$

By the method of moments we have

$$
\begin{aligned}
f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \sum_{k=0}^{j} \binom{j}{k} a^{j-k}(b-a)^k \frac{B(k+p,q)}{B(p,q)}
\end{aligned}
\tag{5.106}
$$

Identity 5.16
Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}\sum_{k=0}^{j}\binom{j}{k}a^{j-k}(b-a)^{k}\frac{B(k+p,q)}{B(p,q)}=\frac{a^{n}t^{n-1}e^{-ta}}{\Gamma n}\sum_{k=0}^{n}\binom{n}{k}(\frac{b-a}{a})^{k}\frac{B(k+p,q)}{B(p,q)}$$
$$_1F_1(k+p,k+p+q;-(b-a)t)$$
$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\sum_{k=0}^{j}\binom{j}{k}a^{j-k}(b-a)^{k}\frac{B(k+p,q)}{B(p,q)}=e^{-ta}\sum_{k=0}^{n}\binom{n}{k}a^{n-k}(b-a)^{k}\frac{B(k+p,q)}{B(p,q)}$$
$$_1F_1(k+p,k+p+q;-(b-a)t) \quad \textbf{(5.107)}$$

The rth moment of the Erlang mixture is

$$\begin{aligned}E(T^{r})&=\frac{\Gamma(n+r)}{\Gamma n}E(\wedge^{-r})\\&=\frac{\Gamma(n+r)}{\Gamma n}\sum_{k=0}^{\infty}\binom{-r}{k}a^{-r-k}(b-a)^{k}\frac{B(k+p,q)}{B(p,q)}\\&=\frac{\Gamma(n+r)}{a^{r}\Gamma n}\sum_{k=0}^{\infty}\binom{-r}{k}(\frac{b-a}{a})^{k}\frac{B(k+p,q)}{B(p,q)}\end{aligned} \quad \textbf{(5.108)}$$

$$\begin{aligned}E(T)&=\frac{\Gamma(n+1)}{a\Gamma n}\sum_{k=0}^{\infty}\binom{-1}{k}(\frac{b-a}{a})^{k}\frac{B(k+p,q)}{B(p,q)}\\&=\frac{n}{a}\sum_{k=0}^{\infty}\binom{-1}{k}(\frac{b-a}{a})^{k}\frac{B(k+p,q)}{B(p,q)}\end{aligned} \quad \textbf{(5.109)}$$

5.8.2 Exponential-Pearson Type I Mixture

Construction by the direct method gives

$$\begin{aligned}f_1(t)&=E[\wedge e^{-t\wedge}]\\&=e^{-ta}\sum_{k=0}^{1}\binom{1}{k}a^{1-k}(b-a)^{k}\frac{B(k+p,q)}{B(p,q)}{}_1F_1(k+p,k+p+q;-(b-a)t)\\&=e^{-ta}[a\frac{B(p,q)}{B(p,q)}{}_1F_1(p,p+q;-(b-a)t)+(b-a)\frac{B(p+1,q)}{B(p,q)}{}_1F_1(p+1,p+q+1;-(b-a)t)]\\&=e^{-ta}[a{}_1F_1(p,p+q;-(b-a)t)+(b-a)\frac{p}{p+q}{}_1F_1(p+1,p+q+1;-(b-a)t)],\quad t>0;a>0,b>\end{aligned}$$
(5.110)

By the method of moments we have

$$f_1(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j)$$
$$= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \sum_{k=0}^{j} \binom{j}{k} a^{j-k} (b-a)^k \frac{B(k+p,q)}{B(p,q)} \tag{5.111}$$

Identity 5.17
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \sum_{k=0}^{j} \binom{j}{k} a^{j-k} (b-a)^k \frac{B(k+p,q)}{B(p,q)} = e^{-ta}[{}_1F_1(p, p+q; -(b-a)t) + (b-a)\frac{p}{p+q}\,{}_1F_1(p+1, p+q+1; -(b-a)t)] \tag{5.112}$$

The rth moment of the Exponential mixture is

$$E(T^r) = r! E(\wedge^{-r})$$
$$= r! \sum_{k=0}^{\infty} \binom{-r}{k} a^{-r-k} (b-a)^k \frac{B(k+p,q)}{B(p,q)}$$
$$= \frac{r!}{a^r} \sum_{k=0}^{\infty} \binom{-r}{k} (\frac{b-a}{a})^k \frac{B(k+p,q)}{B(p,q)} \tag{5.113}$$
$$E(T) = \frac{1}{a} \sum_{k=0}^{\infty} \binom{-1}{k} (\frac{b-a}{a})^k \frac{B(k+p,q)}{B(p,q)} \tag{5.114}$$

5.8.3 Poisson-Pearson Type I Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$
\begin{aligned}
P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^n}{n!}e^{-ta}\sum_{k=0}^{n}\binom{n}{k}a^{n-k}(b-a)^k\frac{B(k+p,q)}{B(p,q)}{}_1F_1(k+p,k+p+q;-(b-a)t) \\
&= \frac{(at)^n}{n!}e^{-ta}\sum_{k=0}^{n}\binom{n}{k}(\frac{b-a}{a})^k\frac{B(k+p,q)}{B(p,q)}{}_1F_1(k+p,k+p+q;-(b-a)t) \\
&= \sum_{k=0}^{n}\binom{n}{k}(\frac{b-a}{a})^k\frac{B(k+p,q)}{B(p,q)}{}_1F_1(k+p,k+p+q;-(b-a)t), \quad t>0;a>0,b>0,p>0,q>0
\end{aligned}
\tag{5.115}
$$

By the method of moments we have

$$
\begin{aligned}
P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\
&= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\sum_{k=0}^{j}\binom{j}{k}a^{j-k}(b-a)^k\frac{B(k+p,q)}{B(p,q)}
\end{aligned}
\tag{5.116}
$$

Identity 5.18

Equating the above two methods we get

$$
\begin{aligned}
\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\sum_{k=0}^{j}\binom{j}{k}a^{j-k}(b-a)^k\frac{B(k+p,q)}{B(p,q)} &= \frac{(at)^n}{n!}e^{-ta}\sum_{k=0}^{n}\binom{n}{k}(\frac{b-a}{a})^k\frac{B(k+p,q)}{B(p,q)} \\
&\quad {}_1F_1(k+p,k+p+q;-(b-a)t) \\
\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\sum_{k=0}^{j}\binom{j}{k}a^{j-k}(b-a)^k\frac{B(k+p,q)}{B(p,q)} &= e^{-ta}\sum_{k=0}^{n}\binom{n}{k}a^{n-k}(b-a)^k\frac{B(k+p,q)}{B(p,q)} \\
&\quad {}_1F_1(k+p,k+p+q;-(b-a)t)
\end{aligned}
\tag{5.117}
$$

The PGF of the Poisson mixture is

$$
\begin{aligned}
G(s,t) &= E[e^{-(1-s)t\wedge}] \\
&= e^{-a(1-s)t}\sum_{k=0}^{0}\binom{0}{k}a^{-k}(b-a)^k\frac{B(k+p,q)}{B(p,q)}{}_1F_1(k+p,k+p+q;-(b-a)(1-s)t) \\
&= e^{-a(1-s)t}\frac{B(p,q)}{B(p,q)}{}_1F_1(p,p+q;-(b-a)(1-s)t) \\
&= e^{-a(1-s)t}{}_1F_1(p,p+q;-(b-a)(1-s)t)
\end{aligned}
\tag{5.118}
$$

The rth moment of the Poisson mixture is

$$
\begin{aligned}
E(T^r) &= t^r E(\wedge^r) \\
&= t^r \sum_{k=0}^{r} \binom{r}{k} a^{r-k}(b-a)^k \frac{B(k+p,q)}{B(p,q)} \\
&= (at)^r \sum_{k=0}^{r} \binom{r}{k} (\frac{b-a}{a})^k \frac{B(k+p,q)}{B(p,q)} \qquad (5.119) \\
E(T) &= (at) \sum_{k=0}^{1} \binom{1}{k} (\frac{b-a}{a})^k \frac{B(k+p,q)}{B(p,q)} \\
&= (at)[\frac{B(p,q)}{B(p,q)} + (\frac{b-a}{a})\frac{B(p+1,q)}{B(p,q)}] \\
&= (at)[1+(\frac{b-a}{a})(\frac{p}{p+q})] \qquad (5.120)
\end{aligned}
$$

5.9 Erlang-Pearson Type VI Distribution and Its Links

5.9.1 Erlang-Pearson Type VI Mixture

The Pearson Type VI mixing distribution is

$$
g(\lambda) = \frac{(\frac{\lambda-d}{d-c})^{b-a-1}\frac{1}{d-c}}{B(a,b-a)(1+\frac{\lambda-d}{d-c})^b}, \quad \lambda > d; a>0, b>0, c>0, d>0, n=1,2,3,.. \qquad (5.121)
$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_d^\infty \lambda^n e^{-t\lambda} \frac{(\frac{\lambda-d}{d-c})^{b-a-1}\frac{1}{d-c}}{B(a,b-a)(1+\frac{\lambda-d}{d-c})^b} d\lambda$$

$$let \quad x = \frac{\lambda-d}{d-c} \quad \Longrightarrow x(d-c)+d = \lambda \quad \Longrightarrow d\lambda = (d-c)dx$$

$$\begin{aligned} E[\wedge^n e^{-t\wedge}] &= \int_0^\infty [x(d-c)+d]^n \frac{e^{-t[x(d-c)+d]}x^{b-a-1}}{B(a,b-a)(1+x)^b} dx \\ &= \frac{1}{B(a,b-a)} \int_0^\infty \frac{[x(d-c)+d]^n e^{-t[x(d-c)+d]}x^{b-a-1}}{(1+x)^b} dx \\ &= \frac{e^{-td}}{B(a,b-a)} \int_0^\infty [x(d-c)+d]^n e^{-tx(d-c)} x^{b-a-1}(1+x)^{-b} dx \\ &= \frac{e^{-td}}{B(a,b-a)} \int_0^\infty [\sum_{k=0}^n \binom{n}{k} d^{n-k}(d-c)^k x^{k+b-a-1}](1+x)^{k-a+1-k-b+a-1} e^{-(d-c)tx} dx \\ &= \frac{e^{-td}d^n}{B(a,b-a)} \sum_{k=0}^n \binom{n}{k} (\frac{d-c}{d})^k \Gamma(k+b-a)\Psi(k+b-a,k-a+1;(d-c)t) \end{aligned} \tag{5.122}$$

Construction by the direct method gives

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^{n-1}}{\Gamma n} \frac{e^{-td}d^n}{B(a,b-a)} \sum_{k=0}^n \binom{n}{k} (\frac{d-c}{d})^k \Gamma(k+b-a)\Psi(k+b-a,k-a+1;(d-c)t) \\ &= \frac{(td)^n}{t\Gamma n} \frac{e^{-td}}{B(a,b-a)} \sum_{k=0}^n \binom{n}{k} (\frac{d-c}{d})^k \Gamma(k+b-a)\Psi(k+b-a,k-a+1;(d-c)t), \quad t>0; a,b,c,d > \end{aligned} \tag{5.123}$$

By the method of moments we have

$$\begin{aligned} f_n(t) &= \sum_{j=n}^\infty \frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!} E(\wedge^j) \\ &= \sum_{j=n}^\infty \frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!} d^j \sum_{k=0}^j \binom{j}{k} (\frac{d-c}{d})^k \frac{B(k+b-a,a-k)}{B(a,b-a)} \end{aligned} \tag{5.124}$$

Identity 5.19
Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}d^{j}\sum_{k=0}^{j}\binom{j}{k}(\frac{d-c}{d})^{k}\frac{B(k+b-a,a-k)}{B(a,b-a)}=\frac{(td)^{n}}{t\Gamma n}\frac{e^{-td}}{B(a,b-a)}\sum_{k=0}^{n}\binom{n}{k}(\frac{d-c}{d})^{k}$$
$$\Gamma(k+b-a)\Psi(k+b-a,k-a+1;(d-c)t)$$
$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}d^{j}\sum_{k=0}^{j}\binom{j}{k}(\frac{d-c}{d})^{k}\frac{B(k+b-a,a-k)}{B(a,b-a)}=\frac{e^{-td}d^{n}}{B(a,b-a)}\sum_{k=0}^{n}\binom{n}{k}(\frac{d-c}{d})^{k}$$
$$\Gamma(k+b-a)\Psi(k+b-a,k-a+1;(d-c)t) \tag{5.125}$$

The rth moment of the Erlang mixture is

$$\begin{aligned}E(T^{r})&=\frac{\Gamma(n+r)}{\Gamma n}E(\wedge^{-r})\\&=\frac{\Gamma(n+r)}{\Gamma n}d^{-r}\sum_{k=0}^{\infty}\binom{-r}{k}(\frac{d-c}{d})^{k}\frac{B(k+b-a,a-k)}{B(a,b-a)}\\&=\frac{\Gamma(n+r)}{d^{r}\Gamma n}\sum_{k=0}^{\infty}\binom{-r}{k}(\frac{d-c}{d})^{k}\frac{B(k+b-a,a-k)}{B(a,b-a)}\end{aligned} \tag{5.126}$$

$$\begin{aligned}E(T)&=\frac{\Gamma(n+1)}{d\Gamma n}\sum_{k=0}^{\infty}\binom{-1}{k}(\frac{d-c}{d})^{k}\frac{B(k+b-a,a-k)}{B(a,b-a)}\\&=\frac{n}{d}\sum_{k=0}^{\infty}\binom{-1}{k}(\frac{d-c}{d})^{k}\frac{B(k+b-a,a-k)}{B(a,b-a)}\end{aligned} \tag{5.127}$$

5.9.2 Exponential-Pearson Type VI Mixture

Construction by the direct method gives

$$\begin{aligned}f_{1}(t)&=E[\wedge e^{-t\wedge}]\\&=\frac{de^{-td}}{B(a,b-a)}\sum_{k=0}^{1}\binom{1}{k}(\frac{d-c}{d})^{k}\Gamma(k+b-a)\Psi(k+b-a,k-a+1;(d-c)t)\\&=\frac{de^{-td}}{B(a,b-a)}[\Gamma(b-a)\Psi(b-a,1-a;(d-c)t)+(\frac{d-c}{d})\Gamma(1+b-a)\Psi(1+b-a,2-a;(d-c)t)]\\&=\frac{de^{-td}\Gamma(b-a)}{B(a,b-a)}[\Psi(b-a,1-a;(d-c)t)+(\frac{d-c}{d})(b-a)\Psi(1+b-a,2-a;(d-c)t)],\quad t>0;a,b,c,d>\end{aligned} \tag{5.128}$$

By the method of moments we have

$$
\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\
&= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} d^j \sum_{k=0}^{j} \binom{j}{k} \left(\frac{d-c}{d}\right)^k \frac{B(k+b-a, a-k)}{B(a, b-a)}
\end{aligned} \tag{5.129}
$$

Identity 5.20
Equating the above two methods we get

$$
\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} d^j \sum_{k=0}^{j} \binom{j}{k} \left(\frac{d-c}{d}\right)^k \frac{B(k+b-a, a-k)}{B(a, b-a)} = \frac{d e^{-td} \Gamma(b-a)}{B(a, b-a)} [\Psi(b-a, 1-a; (d-c)t) + \left(\frac{d-c}{d}\right)(b-a) \Psi(1+b-a, 2-a; (d-c)t)] \tag{5.130}
$$

The rth moment of the Exponential mixture is

$$
\begin{aligned}
E(T^r) &= r! E(\wedge^{-r}) \\
&= r! d^{-r} \sum_{k=0}^{\infty} \binom{-r}{k} \left(\frac{d-c}{d}\right)^k \frac{B(k+b-a, a-k)}{B(a, b-a)} \\
&= \frac{r!}{d^r} \sum_{k=0}^{\infty} \binom{-r}{k} \left(\frac{d-c}{d}\right)^k \frac{B(k+b-a, a-k)}{B(a, b-a)}
\end{aligned} \tag{5.131}
$$

$$
E(T) = \frac{1}{d} \sum_{k=0}^{\infty} \binom{-1}{k} \left(\frac{d-c}{d}\right)^k \frac{B(k+b-a, a-k)}{B(a, b-a)} \tag{5.132}
$$

5.9.3 Poisson-Pearson Type VI Mixture

$$
P_n(t) = \frac{t}{n} f_n(t)
$$

Construction by the direct method gives

$$P_n(t) = \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^n}{n!}\frac{e^{-td}d^n}{B(a,b-a)}\sum_{k=0}^{n}\binom{n}{k}(\frac{d-c}{d})^k\Gamma(k+b-a)\Psi(k+b-a,k-a+1;(d-c)t)$$
$$= \frac{e^{-td}(td)^n}{n!B(a,b-a)}\sum_{k=0}^{n}\binom{n}{k}(\frac{d-c}{d})^k\Gamma(k+b-a)\Psi(k+b-a,k-a+1;(d-c)t), \quad t>0;a,b,c,d>0$$

(5.133)

By the method of moments we have

$$P_n(t) = \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j)$$
$$= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}d^j\sum_{k=0}^{j}\binom{j}{k}(\frac{d-c}{d})^k\frac{B(k+b-a,a-k)}{B(a,b-a)} \quad (5.134)$$

Identity 5.21
Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}d^j\sum_{k=0}^{j}\binom{j}{k}(\frac{d-c}{d})^k\frac{B(k+b-a,a-k)}{B(a,b-a)} = \frac{e^{-td}(td)^n}{n!B(a,b-a)}\sum_{k=0}^{n}\binom{n}{k}(\frac{d-c}{d})^k \Gamma(k+b-a)\Psi(k+b-a,k-a+1;(d-c)t)$$
$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}d^j\sum_{k=0}^{j}\binom{j}{k}(\frac{d-c}{d})^k\frac{B(k+b-a,a-k)}{B(a,b-a)} = \frac{e^{-td}d^n}{B(a,b-a)}\sum_{k=0}^{n}\binom{n}{k}(\frac{d-c}{d})^k \Gamma(k+b-a)\Psi(k+b-a,k-a+1;(d-c)t)$$

(5.135)

The PGF of the Poisson mixture is

$$G(s,t) = E[e^{-(1-s)t\wedge}]$$
$$= \frac{e^{-d(1-s)t}}{B(a,b-a)}\sum_{k=0}^{0}\binom{0}{k}(\frac{d-c}{d})^k\Gamma(k+b-a)\Psi(k+b-a,k-a+1;(d-c)(1-s)t)$$
$$= \frac{e^{-d(1-s)t}\Gamma(b-a)}{B(a,b-a)}\Psi(b-a,1-a;(d-c)(1-s)t) \quad (5.136)$$

The rth moment of the Poisson mixture is

$$E(T^r) = t^r E(\wedge^r)$$
$$= t^r d^r \sum_{k=0}^{r} \binom{r}{k} (\frac{d-c}{d})^k \frac{B(k+b-a, a-k)}{B(a, b-a)}$$
$$= (td)^r \sum_{k=0}^{r} \binom{r}{k} (\frac{d-c}{d})^k \frac{B(k+b-a, a-k)}{B(a, b-a)} \quad (5.137)$$
$$E(T) = (td) \sum_{k=0}^{1} \binom{r}{k} (\frac{d-c}{d})^k \frac{B(k+b-a, a-k)}{B(a, b-a)}$$
$$= (td)[\frac{B(b-a, a)}{B(a, b-a)} + (\frac{d-c}{d})\frac{B(1+b-a, a-1)}{B(a, b-a)}]$$
$$= (td)[1 + (\frac{d-c}{d})(\frac{b-a}{a-1})] \quad (5.138)$$

5.10 Erlang-Shifted Gamma (Pearson Type III) Distribution and Its Links

5.10.1 Erlang-Shifted Gamma (Pearson Type III) Mixture

The Shifted Gamma (Pearson Type III) mixing distribution is

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma\alpha} e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1}, \quad \lambda > \mu; \alpha > 0, \beta > 0, \mu > 0 \quad (5.139)$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_\mu^\infty \lambda^n e^{-t\lambda} \frac{\beta^\alpha}{\Gamma\alpha} e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1} d\lambda$$
$$= \frac{\beta^\alpha}{\Gamma\alpha} \int_\mu^\infty \lambda^n (\lambda-\mu)^{\alpha-1} e^{-t\lambda-\beta\lambda+\beta\mu} d\lambda$$
$$= \frac{\beta^\alpha}{\Gamma\alpha} e^{-\mu t} \int_\mu^\infty \lambda^n (\lambda-\mu)^{\alpha-1} e^{-(t+\beta)(\lambda-\mu)} d\lambda$$
$$let \quad x = \lambda - \mu \quad \Longrightarrow \lambda = x + \mu \quad \Longrightarrow d\lambda = dx$$
$$E[\wedge^n e^{-t\wedge}] = \frac{\beta^\alpha}{\Gamma\alpha} e^{-\mu t} \int_0^\infty (x+\mu)^n x^{\alpha-1} e^{-(t+\beta)x} dx$$
$$let \quad x = \mu y \quad \Longrightarrow dx = \mu dy$$

$$
\begin{aligned}
E[\wedge^n e^{-t\wedge}] &= \frac{\beta^\alpha}{\Gamma\alpha} e^{-\mu t} \int_0^\infty (\mu y + \mu)^n (\mu y)^{\alpha-1} e^{-(t+\beta)\mu y} \mu dy \\
&= \frac{\beta^\alpha}{\Gamma\alpha} e^{-\mu t} \mu^{n+\alpha} \int_0^\infty (y+1)^n y^{\alpha-1} e^{-(t+\beta)\mu y} dy \\
&= \frac{\mu^n (\beta\mu)^\alpha e^{-\mu t}}{\Gamma\alpha} \int_0^\infty (y+1)^{n+\alpha+1-\alpha-1} y^{\alpha-1} e^{-(t+\beta)\mu y} dy \\
&= \frac{\mu^n (\beta\mu)^\alpha e^{-\mu t}}{\Gamma\alpha} \Gamma\alpha \Psi(\alpha, \alpha+n+1; (t+\beta)\mu) \\
&= \mu^n (\beta\mu)^\alpha e^{-\mu t} \Psi(\alpha, \alpha+n+1; (t+\beta)\mu) \qquad (5.140)
\end{aligned}
$$

Construction by the direct method gives

$$
\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} \mu^n (\beta\mu)^\alpha e^{-\mu t} \Psi(\alpha, \alpha+n+1; (t+\beta)\mu) \\
&= \frac{(\mu t)^n (\beta\mu)^\alpha e^{-\mu t}}{t\Gamma n} \Psi(\alpha, \alpha+n+1; (t+\beta)\mu), \quad t>0; \alpha>0, \beta>0, \mu>0, n=1,2,3,..
\end{aligned}
\qquad (5.141)
$$

By the method of moments we have

$$
\begin{aligned}
f_n(t) &= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \mu^j (\beta\mu)^\alpha \Psi(\alpha, \alpha+j+1; \beta\mu) \qquad (5.142)
\end{aligned}
$$

Identity 5.22

Equating the above two methods we get

$$
\begin{aligned}
\sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \mu^j (\beta\mu)^\alpha \Psi(\alpha, \alpha+j+1; \beta\mu) &= \frac{(\mu t)^n (\beta\mu)^\alpha e^{-\mu t}}{t\Gamma n} \Psi(\alpha, \alpha+n+1; (t+\beta)\mu) \\
\sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \mu^j (\beta\mu)^\alpha \Psi(\alpha, \alpha+j+1; \beta\mu) &= \mu^n (\beta\mu)^\alpha e^{-\mu t} \Psi(\alpha, \alpha+n+1; (t+\beta)\mu)
\end{aligned}
\qquad (5.143)
$$

The rth moment of the Erlang mixture is

$$
\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \mu^{-r} (\beta\mu)^{\alpha} \Psi(\alpha, \alpha - r + 1; \beta\mu) \\
&= \frac{\Gamma(n+r)}{\mu^r \Gamma n} (\beta\mu)^{\alpha} \Psi(\alpha, \alpha - r + 1; \beta\mu) \qquad (5.144) \\
E(T) &= \frac{\Gamma(n+1)}{\mu \Gamma n} (\beta\mu)^{\alpha} \Psi(\alpha, \alpha - 1 + 1; \beta\mu) \\
&= \frac{n}{\mu} (\beta\mu)^{\alpha} \Psi(\alpha, \alpha; \beta\mu) \qquad (5.145)
\end{aligned}
$$

5.10.2 Exponential-Shifted Gamma (Pearson Type III) Mixture

Construction by the direct method gives

$$
\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= \mu(\beta\mu)^{\alpha} e^{-\mu t} \Psi(\alpha, \alpha + 2; (t+\beta)\mu), \quad t > 0; \alpha > 0, \beta > 0, \mu > 0 \qquad (5.146)
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\
&= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \mu^j (\beta\mu)^{\alpha} \Psi(\alpha, \alpha + j + 1; \beta\mu) \qquad (5.147)
\end{aligned}
$$

Identity 5.23
Equating the above two methods we get

$$
\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \mu^j (\beta\mu)^{\alpha} \Psi(\alpha, \alpha + j + 1; \beta\mu) = \mu(\beta\mu)^{\alpha} e^{-\mu t} \Psi(\alpha, \alpha + 2; (t+\beta)\mu) \qquad (5.148)
$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= r!\mu^{-r}(\beta\mu)^{\alpha}\Psi(\alpha,\alpha-r+1;\beta\mu) \\ &= \frac{r!}{\mu^r}(\beta\mu)^{\alpha}\Psi(\alpha,\alpha-r+1;\beta\mu) \end{aligned} \quad (5.149)$$

$$E(T) = \frac{(\beta\mu)^{\alpha}}{\mu}\Psi(\alpha,\alpha;\beta\mu) \quad (5.150)$$

5.10.3 Poisson-Shifted Gamma (Pearson Type III) Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!}\mu^n(\beta\mu)^{\alpha}e^{-\mu t}\Psi(\alpha,\alpha+n+1;(t+\beta)\mu) \\ &= \frac{(\mu t)^n(\beta\mu)^{\alpha}e^{-\mu t}}{n!}\Psi(\alpha,\alpha+n+1;(t+\beta)\mu), \quad t>0;\alpha>0,\beta>0,\mu>0 \end{aligned} \quad (5.151)$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\ &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\mu^j(\beta\mu)^{\alpha}\Psi(\alpha,\alpha+j+1;\beta\mu) \end{aligned} \quad (5.152)$$

Identity 5.24

Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\mu^j(\beta\mu)^{\alpha}\Psi(\alpha,\alpha+j+1;\beta\mu) = \frac{(\mu t)^n(\beta\mu)^{\alpha}e^{-\mu t}}{n!}\Psi(\alpha,\alpha+n+1;(t+\beta)\mu)$$

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\mu^j(\beta\mu)^{\alpha}\Psi(\alpha,\alpha+j+1;\beta\mu) = \mu^n(\beta\mu)^{\alpha}e^{-\mu t}\Psi(\alpha,\alpha+n+1;(t+\beta)\mu)$$

$$(5.153)$$

The PGF of the Poisson mixture is

$$
\begin{aligned}
G(s,t) &= E[e^{-(1-s)t\wedge}] \\
&= (\beta\mu)^{\alpha} e^{-\mu(1-s)t} \Psi(\alpha, \alpha+1; ([1-s]t+\beta)\mu)
\end{aligned} \tag{5.154}
$$

The rth moment of the Poisson mixture is

$$
\begin{aligned}
E(T^r) &= t^r E(\wedge^r) \\
&= t^r \mu^r (\beta\mu)^{\alpha} \Psi(\alpha, \alpha+r+1; \beta\mu) \\
&= (t\mu)^r (\beta\mu)^{\alpha} \Psi(\alpha, \alpha+r+1; \beta\mu)
\end{aligned} \tag{5.155}
$$

$$
E(T) = (t\mu)(\beta\mu)^{\alpha} \Psi(\alpha, \alpha+2; \beta\mu) \tag{5.156}
$$

5.11 Erlang-Right Truncated Distribution and Its Links

5.11.1 Erlang-Right Truncated Mixture

Given the two-parameter Gamma distribution as;

$$
h(x) = \frac{a^b}{\Gamma b} e^{-ax} x^{b-1}, \quad x > 0; a > 0, b > 0 \tag{5.157}
$$

In the integral

$$
I = \int_0^p e^{-ax} x^{b-1} dx; \quad p > 0
$$

Let;

$$
t = ax \quad \implies x = \frac{t}{a} \quad \implies dx = \frac{dt}{a}
$$

$$
\begin{aligned}
\therefore \quad I &= \int_0^{ap} e^{-t} \left(\frac{t}{a}\right)^{b-1} \frac{dt}{a} \\
&= \frac{1}{a^b} \int_0^{ap} e^{-t} t^{b-1} dt \\
&= \frac{1}{a^b} \gamma(b, ap)
\end{aligned}
$$

So

$$\frac{a^b}{\Gamma b}\int_0^p e^{-ax}x^{b-1}dx = \frac{a^b}{\Gamma b}\frac{1}{a^b}\gamma(b,ap) = \frac{\gamma(b,ap)}{\Gamma b}$$

where γ(b,ap) is an Incomplete Gamma function.

$$\therefore \quad \frac{a^b}{\Gamma b}\frac{\Gamma b}{\gamma(b,ap)}\int_0^p e^{-ax}x^{b-1}dx = 1$$
$$\int_0^p \frac{a^b}{\gamma(b,ap)}e^{-ax}x^{b-1}dx = 1$$
$$\int_0^p e^{-ax}x^{b-1}dx = \frac{\gamma(b,ap)}{a^b}$$

The Right Truncated distribution then becomes;

$$g(\lambda) = \frac{a^b}{\gamma(b,ap)}e^{-a\lambda}\lambda^{b-1}, \quad 0<\lambda<p; a>0, b>0, p>0 \tag{5.158}$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_0^p \lambda^n e^{-t\lambda}\frac{a^b}{\gamma(b,ap)}e^{-a\lambda}\lambda^{b-1}d\lambda = \frac{a^b}{\gamma(b,ap)}\int_0^p \lambda^{n+b-1}e^{-\lambda(t+a)}d\lambda$$

$$let \quad x = \lambda(t+a) \implies dx = (t+a)d\lambda$$

$$\begin{aligned} E[\wedge^n e^{-t\wedge}] &= \frac{a^b}{\gamma(b,ap)}\int_0^{p(t+a)}\left(\frac{x}{t+a}\right)^{n+b-1}e^{-x}\frac{dx}{t+a} \\ &= \frac{a^b}{\gamma(b,ap)}\frac{\gamma(n+b,p(t+a))}{(t+a)^{n+b}} \\ &= \frac{a^b}{(t+a)^{n+b}}\frac{\gamma(n+b,p(t+a))}{\gamma(b,ap)} \\ &= \frac{(ap)^b p^n}{[p(t+a)]^{n+b}}\frac{\frac{[p(t+a)]^{n+b}}{n+b}\,{}_1F_1(n+b,n+b+1;-p(t+a))}{\frac{(ap)^b}{b}\,{}_1F_1(b,b+1;-ap)} \\ &= \frac{bp^n}{n+b}\frac{{}_1F_1(n+b,n+b+1;-p(t+a))}{{}_1F_1(b,b+1;-ap)} \end{aligned} \tag{5.159}$$

Or

$$E[\wedge^n e^{-t\wedge}] = p^n \frac{b}{n+b} \frac{e^{-p(t+a)}}{e^{-ap}} \frac{{}_1F_1(1,n+b+1;p(t+a))}{{}_1F_1(1,b+1;ap)}$$
$$= \frac{bp^n}{n+b} e^{-pt} \frac{{}_1F_1(1,n+b+1;p(t+a))}{{}_1F_1(1,b+1;ap)} \tag{5.160}$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^{n-1}}{\Gamma n} \frac{bp^n}{n+b} \frac{{}_1F_1(n+b,n+b+1;-p(t+a))}{{}_1F_1(b,b+1;-ap)}$$
$$= \frac{b(pt)^n}{t(n+b)\Gamma n} \frac{{}_1F_1(n+b,n+b+1;-p(t+a))}{{}_1F_1(b,b+1;-ap)}, \quad t>0; a>0, b>0, p>0, n=1,2,3,.. \tag{5.161}$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{bp^j}{j+b} \frac{{}_1F_1(j+b,j+b+1;-ap)}{{}_1F_1(b,b+1;-ap)} \tag{5.162}$$

Identity 5.25

Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{bp^j}{j+b} \frac{{}_1F_1(j+b,j+b+1;-ap)}{{}_1F_1(b,b+1;-ap)} = \frac{b(pt)^n}{t(n+b)\Gamma n} \frac{{}_1F_1(n+b,n+b+1;-p(t+a))}{{}_1F_1(b,b+1;-ap)}$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{bp^j}{j+b} \frac{{}_1F_1(j+b,j+b+1;-ap)}{{}_1F_1(b,b+1;-ap)} = \frac{bp^n}{n+b} \frac{{}_1F_1(n+b,n+b+1;-p(t+a))}{{}_1F_1(b,b+1;-ap)} \tag{5.163}$$

The rth moment of the Erlang mixture is

$$
\begin{aligned}
E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\
&= \frac{\Gamma(n+r)}{\Gamma n} \frac{bp^{-r}}{b-r} \frac{{}_1F_1(b-r, b-r+1; -ap)}{{}_1F_1(b, b+1; -ap)} \\
&= \frac{\Gamma(n+r)}{p^r \Gamma n} \frac{b}{b-r} \frac{{}_1F_1(b-r, b-r+1; -ap)}{{}_1F_1(b, b+1; -ap)} \qquad (5.164) \\
E(T) &= \frac{\Gamma(n+1)}{p\Gamma n} \frac{b}{b-1} \frac{{}_1F_1(b-1, b; -ap)}{{}_1F_1(b, b+1; -ap)} \\
&= \frac{n}{p} \frac{b}{b-1} \frac{{}_1F_1(b-1, b; -ap)}{{}_1F_1(b, b+1; -ap)} \qquad (5.165)
\end{aligned}
$$

5.11.2 Exponential-Right Truncated Mixture

Construction by the direct method gives

$$
\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= \frac{bp}{b+1} \frac{{}_1F_1(b+1, b+2; -p(t+a))}{{}_1F_1(b, b+1; -ap)}, \quad t > 0; a > 0, b > 0, p > 0 \qquad (5.166)
\end{aligned}
$$

By the method of moments we have

$$
\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\
&= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{bp^j}{j+b} \frac{{}_1F_1(j+b, j+b+1; -ap)}{{}_1F_1(b, b+1; -ap)} \qquad (5.167)
\end{aligned}
$$

Identity 5.26

Equating the above two methods we get

$$
\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{bp^j}{j+b} \frac{{}_1F_1(j+b, j+b+1; -ap)}{{}_1F_1(b, b+1; -ap)} = \frac{bp}{b+1} \frac{{}_1F_1(b+1, b+2; -p(t+a))}{{}_1F_1(b, b+1; -ap)} \qquad (5.168)
$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= r!\frac{bp^{-r}}{b-r}\frac{{}_1F_1(b-r,b-r+1;-ap)}{{}_1F_1(b,b+1;-ap)} \\ &= \frac{r!}{p^r}\frac{b}{b-r}\frac{{}_1F_1(b-r,b-r+1;-ap)}{{}_1F_1(b,b+1;-ap)} \end{aligned} \tag{5.169}$$

$$E(T) = \frac{1}{p}\frac{b}{b-1}\frac{{}_1F_1(b-1,b;-ap)}{{}_1F_1(b,b+1;-ap)} \tag{5.170}$$

5.11.3 Poisson-Right Truncated Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!}\frac{bp^n}{n+b}\frac{{}_1F_1(n+b,n+b+1;-p(t+a))}{{}_1F_1(b,b+1;-ap)} \\ &= \frac{(pt)^n}{n!}\frac{b}{n+b}\frac{{}_1F_1(n+b,n+b+1;-p(t+a))}{{}_1F_1(b,b+1;-ap)}, \quad t>0; a>0, b>0, p>0 \end{aligned} \tag{5.171}$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{bp^j}{j+b}\frac{{}_1F_1(j+b,j+b+1;-ap)}{{}_1F_1(b,b+1;-ap)} \end{aligned} \tag{5.172}$$

Identity 5.27

Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{bp^j}{j+b}\frac{{}_1F_1(j+b,j+b+1;-ap)}{{}_1F_1(b,b+1;-ap)}=\frac{(pt)^n}{n!}\frac{b}{n+b}\frac{{}_1F_1(n+b,n+b+1;-p(t+a))}{{}_1F_1(b,b+1;-ap)}$$

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{bp^j}{j+b}\frac{{}_1F_1(j+b,j+b+1;-ap)}{{}_1F_1(b,b+1;-ap)}=\frac{bp^n}{n+b}\frac{{}_1F_1(n+b,n+b+1;-p(t+a))}{{}_1F_1(b,b+1;-ap)} \tag{5.173}$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= \frac{b}{b}\frac{{}_1F_1(b,b+1;-p[(1-s)t+a])}{{}_1F_1(b,b+1;-ap)} \\ &= \frac{{}_1F_1(b,b+1;-p[(1-s)t+a])}{{}_1F_1(b,b+1;-ap)} \end{aligned} \tag{5.174}$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r\frac{bp^r}{b+r}\frac{{}_1F_1(r+b,r+b+1;-ap)}{{}_1F_1(b,b+1;-ap)} \\ &= \frac{b(tp)^r}{b+r}\frac{{}_1F_1(r+b,r+b+1;-ap)}{{}_1F_1(b,b+1;-ap)} \end{aligned} \tag{5.175}$$

$$E(T) = \frac{btp}{b+1}\frac{{}_1F_1(b+1,b+2;-ap)}{{}_1F_1(b,b+1;-ap)} \tag{5.176}$$

5.12 Erlang-Left Truncated Distribution and Its Links

5.12.1 Erlang-Left Truncated Mixture

Given the Gamma distribution with two parameters as;

$$h(x)=\frac{\beta^{\alpha}}{\Gamma\alpha}e^{-\beta x}x^{\alpha-1},\quad x>0;\alpha>0,\beta>0 \tag{5.177}$$

Then

$$\frac{\beta^{\alpha}}{\Gamma\alpha}\int_{0}^{\infty} e^{-\beta x}x^{\alpha-1}dx = 1$$

$$\frac{\beta^{\alpha}}{\Gamma\alpha}[\int_{0}^{\lambda_o} e^{-\beta x}x^{\alpha-1}dx + \int_{\lambda_o}^{\infty} e^{-\beta x}x^{\alpha-1}dx] = 1$$

$$let \quad t = \beta x \quad \Longrightarrow dt = \beta dx$$

$$\begin{aligned}\int_{0}^{\lambda_o} e^{-\beta x}x^{\alpha-1}dx &= \int_{0}^{\beta\lambda_o} e^{-t}(\frac{t}{\beta})^{\alpha-1}\frac{dt}{\beta} \\ &= \frac{1}{\beta^{\alpha}}\int_{0}^{\beta\lambda_o} e^{-t}t^{\alpha-1}dt \\ &= \frac{\gamma(\alpha,\beta\lambda_o)}{\beta^{\alpha}}\end{aligned}$$

Therefore

$$\frac{\beta^{\alpha}}{\Gamma\alpha}[\int_{0}^{\lambda_o} e^{-\beta x}x^{\alpha-1}dx + \int_{\lambda_o}^{\infty} e^{-\beta x}x^{\alpha-1}dx] = 1$$

becomes

$$\frac{\gamma(\alpha,\beta\lambda_o)}{\Gamma\alpha} + \frac{\beta^{\alpha}}{\Gamma\alpha}\int_{\lambda_o}^{\infty} e^{-\beta x}x^{\alpha-1}dx = 1$$

And

$$\begin{aligned}\int_{\lambda_o}^{\infty} e^{-\beta x}x^{\alpha-1}dx &= \frac{\Gamma\alpha}{\beta^{\alpha}} - \frac{\gamma(\alpha,\beta\lambda_o)}{\beta^{\alpha}} \\ &= [\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)]\frac{1}{\beta^{\alpha}}\end{aligned}$$

$$\therefore \quad \int_{\lambda_o}^{\infty} \frac{\beta^{\alpha}e^{-\beta x}x^{\alpha-1}dx}{\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)} = 1$$

The Left Truncated mixing distribution then becomes

$$g(\lambda) = \frac{\beta^{\alpha}e^{-\beta\lambda}\lambda^{\alpha-1}}{\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)}, \quad \lambda > \lambda_o; \alpha > 0, \beta > 0, \lambda_o > 0 \tag{5.178}$$

$$where \quad \gamma(\alpha,\beta\lambda_o) = \int_{0}^{\beta\lambda_o} e^{-x}x^{\alpha-1}dx$$

$$\therefore \quad E[\wedge^n e^{-t\wedge}] = \int_{\lambda_o}^{\infty} \lambda^n e^{-t\lambda} \frac{\beta^\alpha e^{-\beta\lambda} \lambda^{\alpha-1}}{\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)} d\lambda$$

$$= \frac{\beta^\alpha}{\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)} \int_{\lambda_o}^{\infty} \lambda^{n+\alpha-1} e^{-\lambda(t+\beta)} d\lambda$$

$$let \quad x = \lambda(t+\beta) \quad \Longrightarrow dx = (t+\beta)d\lambda$$

$$E[\wedge^n e^{-t\wedge}] = \frac{\beta^\alpha}{\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)} \int_{\lambda_o(t+\beta)}^{\infty} (\frac{x}{t+\beta})^{n+\alpha-1} e^{-x} \frac{dx}{t+\beta}$$

$$= \frac{\beta^\alpha}{\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)} \frac{1}{(t+\beta)^{n+\alpha}} \int_{\lambda_o(t+\beta)}^{\infty} e^{-x} x^{n+\alpha-1} dx$$

$$= \frac{\beta^\alpha}{(t+\beta)^{n+\alpha}} \frac{\Gamma(\alpha+n) - \gamma(\alpha+n,(t+\beta)\lambda_o)}{\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)} \qquad \textbf{(5.179)}$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$

$$= \frac{t^{n-1}}{\Gamma n} \frac{\beta^\alpha}{(t+\beta)^{n+\alpha}} \frac{\Gamma(\alpha+n) - \gamma(\alpha+n,(t+\beta)\lambda_o)}{\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)}$$

$$= \frac{1}{t\Gamma n} (\frac{t}{t+\beta})^n (\frac{\beta}{t+\beta})^\alpha \frac{\Gamma(\alpha+n) - \gamma(\alpha+n,(t+\beta)\lambda_o)}{\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)}, \quad t>0; \alpha>0, \beta>0, \lambda_o>0, n=1,2,3,.. \qquad \textbf{(5.180)}$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$

$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\beta^\alpha}{\beta^{j+\alpha}} \frac{\Gamma(\alpha+j) - \gamma(\alpha+j,\beta\lambda_o)}{\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)}$$

$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{1}{\beta^j} \frac{\Gamma(\alpha+j) - \gamma(\alpha+j,\beta\lambda_o)}{\Gamma\alpha - \gamma(\alpha,\beta\lambda_o)} \qquad \textbf{(5.181)}$$

Identity 5.28
Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-1}}{\Gamma n(j-n)!}\frac{1}{\beta^j}\frac{\Gamma(\alpha+j)-\gamma(\alpha+j,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}=\frac{1}{t\Gamma n}(\frac{t}{t+\beta})^n(\frac{\beta}{t+\beta})^\alpha\frac{\Gamma(\alpha+n)-\gamma(\alpha+n,(t+\beta)\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}$$
$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{1}{\beta^j}\frac{\Gamma(\alpha+j)-\gamma(\alpha+j,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}=\frac{\beta^\alpha}{(t+\beta)^{n+\alpha}}\frac{\Gamma(\alpha+n)-\gamma(\alpha+n,(t+\beta)\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)} \tag{5.182}$$

The rth moment of the Erlang mixture is

$$\begin{aligned}E(T^r)&=\frac{\Gamma(n+r)}{\Gamma n}E(\wedge^{-r})\\&=\frac{\Gamma(n+r)}{\Gamma n}\beta^r\frac{\Gamma(\alpha-r)-\gamma(\alpha-r,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}\end{aligned} \tag{5.183}$$
$$\begin{aligned}E(T)&=\frac{\Gamma(n+1)}{\Gamma n}\beta\frac{\Gamma(\alpha-1)-\gamma(\alpha-1,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}\\&=n\beta\frac{\Gamma(\alpha-1)-\gamma(\alpha-1,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}\end{aligned} \tag{5.184}$$

5.12.2 Exponential-Left Truncated Mixture

Construction by the direct method gives

$$\begin{aligned}f_1(t)&=E[\wedge e^{-t\wedge}]\\&=\frac{\beta^\alpha}{(t+\beta)^{\alpha+1}}\frac{\Gamma(\alpha+1)-\gamma(\alpha+1,(t+\beta)\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}\\&=(\frac{1}{t+\beta})(\frac{\beta}{t+\beta})^\alpha\frac{\alpha\Gamma\alpha-\gamma(\alpha+1,(t+\beta)\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)},\quad t>0;\alpha>0,\beta>0,\lambda_o>0\end{aligned} \tag{5.185}$$

By the method of moments we have

$$\begin{aligned}f_1(t)&=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}E(\wedge^j)\\&=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\frac{1}{\beta^j}\frac{\Gamma(\alpha+j)-\gamma(\alpha+j,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}\end{aligned} \tag{5.186}$$

Identity 5.29
Equating the above two methods we get

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\frac{1}{\beta^{j}}\frac{\Gamma(\alpha+j)-\gamma(\alpha+j,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}=(\frac{1}{t+\beta})(\frac{\beta}{t+\beta})^{\alpha}\frac{\alpha\Gamma\alpha-\gamma(\alpha+1,(t+\beta)\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)} \tag{5.187}$$

The rth moment of the Exponential mixture is

$$E(T^r)=r!E(\wedge^{-r}) = r!\beta^{r}\frac{\Gamma(\alpha-r)-\gamma(\alpha-r,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)} \tag{5.188}$$

$$E(T)=\beta\frac{\Gamma(\alpha-1)-\gamma(\alpha-1,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)} \tag{5.189}$$

5.12.3 Poisson-Left Truncated Mixture

$$P_n(t)=\frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$\begin{aligned}P_n(t)&=\frac{t^n}{n!}E[\wedge^n e^{-t\wedge}]\\&=\frac{t^n}{n!}\frac{\beta^{\alpha}}{(t+\beta)^{n+\alpha}}\frac{\Gamma(\alpha+n)-\gamma(\alpha+n,(t+\beta)\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}\\&=\frac{1}{n!}(\frac{t}{t+\beta})^{n}(\frac{\beta}{t+\beta})^{\alpha}\frac{\Gamma(\alpha+n)-\gamma(\alpha+n,(t+\beta)\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)},\quad t>0;\alpha>0,\beta>0,\lambda_o>0\end{aligned} \tag{5.190}$$

By the method of moments we have

$$\begin{aligned}P_n(t)&=\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j}}{n!(j-n)!}E(\wedge^{j})\\&=\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j}}{n!(j-n)!}\frac{1}{\beta^{j}}\frac{\Gamma(\alpha+j)-\gamma(\alpha+j,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}\end{aligned} \tag{5.191}$$

Identity 3.30
Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\frac{1}{\beta^j}\frac{\Gamma(\alpha+j)-\gamma(\alpha+j,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}=\frac{1}{n!}(\frac{t}{t+\beta})^n(\frac{\beta}{t+\beta})^\alpha\frac{\Gamma(\alpha+n)-\gamma(\alpha+n,(t+\beta)\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}$$
$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\frac{1}{\beta^j}\frac{\Gamma(\alpha+j)-\gamma(\alpha+j,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}=\frac{\beta^\alpha}{(t+\beta)^{n+\alpha}}\frac{\Gamma(\alpha+n)-\gamma(\alpha+n,(t+\beta)\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)} \tag{5.192}$$

The PGF of the Poisson mixture is

$$\begin{aligned}G(s,t)&=E[e^{-(1-s)t\wedge}]\\&=\frac{\beta^\alpha}{[(1-s)t+\beta]^\alpha}\frac{\Gamma(\alpha)-\gamma(\alpha,[(1-s)t+\beta]\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}\\&=[\frac{\beta}{(1-s)t+\beta}]^\alpha\frac{\Gamma(\alpha)-\gamma(\alpha,[(1-s)t+\beta]\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}\end{aligned} \tag{5.193}$$

The rth moment of the Poisson mixture is

$$\begin{aligned}E(T^r)&=t^rE(\wedge^r)\\&=(\frac{t}{\beta})^r\frac{\Gamma(\alpha+r)-\gamma(\alpha+r,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)}\end{aligned} \tag{5.194}$$

$$E(T)=\frac{t}{\beta}\frac{\alpha\Gamma\alpha-\gamma(\alpha+1,\beta\lambda_o)}{\Gamma\alpha-\gamma(\alpha,\beta\lambda_o)} \tag{5.195}$$

5.13 Erlang-Truncated Gamma (from above and below) Distribution and Its Links

5.13.1 Erlang-Truncated Gamma (from above and below) Mixture

In the integral;

$$\int_a^b e^{-\beta x}x^{\alpha-1}dx = \int_0^b e^{-\beta x}x^{\alpha-1}dx - \int_0^a e^{-\beta x}x^{\alpha-1}dx$$

$$let \quad t = \beta x \quad \Longrightarrow dt = \beta dx$$

$$\begin{aligned}
\int_a^b e^{-\beta x}x^{\alpha-1}dx &= \int_0^{\beta b} e^{-t}(\frac{t}{\beta})^{\alpha-1}\frac{dt}{\beta} - \int_0^{\beta a} e^{-t}(\frac{t}{\beta})^{\alpha-1}\frac{dt}{\beta}\\
&= \frac{1}{\beta^\alpha}\int_0^{\beta b} e^{-t}t^{\alpha-1}dt - \frac{1}{\beta^\alpha}\int_0^{\beta a} e^{-t}t^{\alpha-1}dt\\
&= \frac{\gamma(\alpha,\beta b)}{\beta^\alpha} - \frac{\gamma(\alpha,\beta a)}{\beta^\alpha}
\end{aligned}$$

$$\int_a^b \beta^\alpha e^{-\beta x}x^{\alpha-1}dx = \gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)$$

$$\therefore \quad \int_a^b \frac{\beta^\alpha e^{-\beta x}x^{\alpha-1}dx}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)} = 1$$

and the Truncated Gamma (from above and below) mixing distribution is

$$g(\lambda) = \frac{\beta^\alpha e^{-\beta\lambda}\lambda^{\alpha-1}}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)}, \quad a < \lambda < b, \alpha > 0, \beta > 0, a > 0, b > 0 \tag{5.196}$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_a^b \lambda^n e^{-t\lambda}\frac{\beta^\alpha e^{-\beta\lambda}\lambda^{\alpha-1}}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)}d\lambda\\
&= \frac{\beta^\alpha}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)}\int_a^b \lambda^{n+\alpha-1}e^{-\lambda(t+\beta)}d\lambda
\end{aligned}$$

$$let \quad x = \lambda(t+\beta) \quad \Longrightarrow dx = (t+\beta)d\lambda$$

$$\begin{aligned}
E[\wedge^n e^{-t\wedge}] &= \frac{\beta^\alpha}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)}\int_{a(t+\beta)}^{b(t+\beta)}(\frac{x}{t+\beta})^{n+\alpha-1}e^{-x}\frac{dx}{t+\beta}\\
&= \frac{\beta^\alpha}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)}\frac{1}{(t+\beta)^{n+\alpha}}[\int_0^{b(t+\beta)} x^{n+\alpha-1}e^{-x}dx - \int_0^{a(t+\beta)} x^{n+\alpha-1}e^{-x}dx]\\
&= \frac{\beta^\alpha}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)}\frac{1}{(t+\beta)^{n+\alpha}}\gamma[n+\alpha,b(t+\beta)] - \gamma[n+\alpha,a(t+\beta)]\\
&= \frac{\beta^\alpha}{(t+\beta)^{n+\alpha}}\frac{\gamma[n+\alpha,b(t+\beta)] - \gamma[n+\alpha,a(t+\beta)]}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)}
\end{aligned} \tag{5.197}$$

Construction by the direct method gives

$$f_n(t) = \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}]$$
$$= \frac{t^{n-1}}{\Gamma n} \frac{\beta^\alpha}{(t+\beta)^{n+\alpha}} \frac{\gamma[n+\alpha, b(t+\beta)] - \gamma[n+\alpha, a(t+\beta)]}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)}$$
$$= \frac{1}{t\Gamma n} (\frac{t}{t+\beta})^n (\frac{\beta}{t+\beta})^\alpha \frac{\gamma[n+\alpha, b(t+\beta)] - \gamma[n+\alpha, a(t+\beta)]}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)}, \quad t > 0; \alpha > 0, \beta > 0, a > 0, b > 0, n \tag{5.198}$$

By the method of moments we have

$$f_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j)$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{\beta^\alpha}{\beta^{j+\alpha}} \frac{\gamma(j+\alpha, b\beta) - \gamma(j+\alpha, a\beta)}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)}$$
$$= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{1}{\beta^j} \frac{\gamma(j+\alpha, b\beta) - \gamma(j+\alpha, a\beta)}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)} \tag{5.199}$$

Identity 5.31
Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \frac{1}{\beta^j} \frac{\gamma(j+\alpha, b\beta) - \gamma(j+\alpha, a\beta)}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)} = \frac{1}{t\Gamma n} (\frac{t}{t+\beta})^n (\frac{\beta}{t+\beta})^\alpha \frac{\gamma[n+\alpha, b(t+\beta)] - \gamma[n+\alpha, a(t}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)}$$
$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{1}{\beta^j} \frac{\gamma(j+\alpha, b\beta) - \gamma(j+\alpha, a\beta)}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)} = \frac{\beta^\alpha}{(t+\beta)^{n+\alpha}} \frac{\gamma[n+\alpha, b(t+\beta)] - \gamma[n+\alpha, a(t+\beta)]}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)} \tag{5.200}$$

The rth moment of the Erlang mixture is

$$E(T^r) = \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r})$$
$$= \frac{\Gamma(n+r)}{\Gamma n} \frac{1}{\beta^{-r}} \frac{\gamma(\alpha - r, b\beta) - \gamma(\alpha - r, a\beta)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)}$$
$$= \frac{\Gamma(n+r)}{\Gamma n} \beta^r \frac{\gamma(\alpha - r, b\beta) - \gamma(\alpha - r, a\beta)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} \quad (5.201)$$
$$E(T) = \frac{\Gamma(n+1)}{\Gamma n} \beta \frac{\gamma(\alpha - 1, b\beta) - \gamma(\alpha - 1, a\beta)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)}$$
$$= n\beta \frac{\gamma(\alpha - 1, b\beta) - \gamma(\alpha - 1, a\beta)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} \quad (5.202)$$

5.13.2 Exponential-Truncated Gamma (from above and below) Mixture

Construction by the direct method gives

$$f_1(t) = E[\wedge e^{-t\wedge}]$$
$$= \frac{\beta^\alpha}{(t+\beta)^{\alpha+1}} \frac{\gamma[\alpha+1, b(t+\beta)] - \gamma[\alpha+1, a(t+\beta)]}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)}, \quad t > 0; \alpha > 0, \beta > 0, a > 0, b > 0 \quad (5.203)$$

By the method of moments we have

$$f_1(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j)$$
$$= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{1}{\beta^j} \frac{\gamma(j+\alpha, b\beta) - \gamma(j+\alpha, a\beta)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} \quad (5.204)$$

Identity 5.32
Equating the above two methods we get

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \frac{1}{\beta^j} \frac{\gamma(j+\alpha, b\beta) - \gamma(j+\alpha, a\beta)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} = \frac{\beta^\alpha}{(t+\beta)^{\alpha+1}} \frac{\gamma[\alpha+1, b(t+\beta)] - \gamma[\alpha+1, a(t+\beta)]}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} \quad (5.205)$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= r!\beta^r \frac{\gamma(\alpha-r,b\beta)-\gamma(\alpha-r,a\beta)}{\gamma(\alpha,\beta b)-\gamma(\alpha,\beta a)} \qquad (5.206) \end{aligned}$$

$$E(T) = \beta \frac{\gamma(\alpha-1,b\beta)-\gamma(\alpha-1,a\beta)}{\gamma(\alpha,\beta b)-\gamma(\alpha,\beta a)} \qquad (5.207)$$

5.13.3 Poisson-Truncated Gamma (from above and below) Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!} E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!} \frac{\beta^\alpha}{(t+\beta)^{n+\alpha}} \frac{\gamma[n+\alpha,b(t+\beta)]-\gamma[n+\alpha,a(t+\beta)]}{\gamma(\alpha,\beta b)-\gamma(\alpha,\beta a)} \\ &= \frac{1}{n!}\left(\frac{t}{t+\beta}\right)^n \left(\frac{\beta}{t+\beta}\right)^\alpha \frac{\gamma[n+\alpha,b(t+\beta)]-\gamma[n+\alpha,a(t+\beta)]}{\gamma(\alpha,\beta b)-\gamma(\alpha,\beta a)}, \quad t>0; \alpha>0, \beta>0, a>0, b>0 \end{aligned}$$

(5.208)

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \frac{1}{\beta^j} \frac{\gamma(j+\alpha,b\beta)-\gamma(j+\alpha,a\beta)}{\gamma(\alpha,\beta b)-\gamma(\alpha,\beta a)} \qquad (5.209) \end{aligned}$$

Identity 5.33

Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \frac{1}{\beta^j} \frac{\gamma(j+\alpha,b\beta)-\gamma(j+\alpha,a\beta)}{\gamma(\alpha,\beta b)-\gamma(\alpha,\beta a)} = \frac{1}{n!}\left(\frac{t}{t+\beta}\right)^n \left(\frac{\beta}{t+\beta}\right)^\alpha \frac{\gamma[n+\alpha,b(t+\beta)]-\gamma[n+\alpha,a(t+}{\gamma(\alpha,\beta b)-\gamma(\alpha,\beta a)}$$

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{1}{\beta^j} \frac{\gamma(j+\alpha,b\beta)-\gamma(j+\alpha,a\beta)}{\gamma(\alpha,\beta b)-\gamma(\alpha,\beta a)} = \frac{\beta^\alpha}{(t+\beta)^{n+\alpha}} \frac{\gamma[n+\alpha,b(t+\beta)]-\gamma[n+\alpha,a(t+\beta)]}{\gamma(\alpha,\beta b)-\gamma(\alpha,\beta a)}$$

(5.210)

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= \frac{\beta^{\alpha}}{[(1-s)t+\beta]^{\alpha}} \frac{\gamma[\alpha, b[(1-s)t+\beta]] - \gamma[\alpha, a[(1-s)t+\beta]]}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)} \\ &= [\frac{\beta}{(1-s)t+\beta}]^{\alpha} \frac{\gamma[\alpha, b[(1-s)t+\beta]] - \gamma[\alpha, a[(1-s)t+\beta]]}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)} \end{aligned} \tag{5.211}$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r \frac{1}{\beta^r} \frac{\gamma(r+\alpha, b\beta) - \gamma(r+\alpha, a\beta)}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)} \\ &= (\frac{t}{\beta})^r \frac{\gamma(r+\alpha, b\beta) - \gamma(r+\alpha, a\beta)}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)} \end{aligned} \tag{5.212}$$

$$E(T) = \frac{t}{\beta} \frac{\gamma(\alpha+1, b\beta) - \gamma(\alpha+1, a\beta)}{\gamma(\alpha,\beta b) - \gamma(\alpha,\beta a)} \tag{5.213}$$

5.14 Erlang-Truncated Pearson Type III Distribution and Its Links

5.14.1 Erlang-Truncated Pearson Type III Mixture

The Truncated Pearson Type III mixing distribution is

$$g(\lambda) = \frac{(1-\lambda)^{\beta-2} e^{\alpha\lambda}}{B(1,\beta-1)\,{}_1F_1(1,\beta;\alpha)}, \quad 0<\lambda<1; \beta>0, \alpha>0 \tag{5.214}$$

$$\begin{aligned} \therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^1 \lambda^n e^{-t\lambda} \frac{(1-\lambda)^{\beta-2} e^{\alpha\lambda}}{B(1,\beta-1)\,{}_1F_1(1,\beta;\alpha)} d\lambda \\ &= \frac{1}{B(1,\beta-1)\,{}_1F_1(1,\beta;\alpha)} \int_0^1 \lambda^n (1-\lambda)^{\beta-2} e^{-\lambda(t-\alpha)} d\lambda \\ &= \frac{1}{B(1,\beta-1)\,{}_1F_1(1,\beta;\alpha)} \int_0^1 \lambda^{n+1-1} (1-\lambda)^{\beta+n-(n+1)-1} e^{\lambda(\alpha-t)} d\lambda \\ &= \frac{B(n+1,\beta-1)\,{}_1F_1(n+1,n+\beta;\alpha-t)}{B(1,\beta-1)\,{}_1F_1(1,\beta;\alpha)} \\ &= nB(n,\beta) \frac{{}_1F_1(n+1,n+\beta;\alpha-t)}{{}_1F_1(1,\beta;\alpha)} \end{aligned} \tag{5.215}$$

Construction by the direct method gives

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^{n-1}}{\Gamma n} nB(n,\beta) \frac{{}_1F_1(n+1, n+\beta; \alpha - t)}{{}_1F_1(1,\beta;\alpha)} \\ &= \frac{nt^{n-1}\Gamma\beta}{\Gamma(n+\beta)} \frac{{}_1F_1(n+1,n+\beta;\alpha-t)}{{}_1F_1(1,\beta;\alpha)}, \quad t>0; \alpha>0, \beta>0, n=1,2,3,.. \end{aligned} \tag{5.216}$$

By the method of moments we have

$$\begin{aligned} f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} jB(j,\beta) \frac{{}_1F_1(j+1, j+\beta; \alpha)}{{}_1F_1(1,\beta;\alpha)} \end{aligned} \tag{5.217}$$

Identity 5.34

Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} jB(j,\beta) \frac{{}_1F_1(j+1, j+\beta; \alpha)}{{}_1F_1(1,\beta;\alpha)} &= \frac{nt^{n-1}\Gamma\beta}{\Gamma(n+\beta)} \frac{{}_1F_1(n+1,n+\beta;\alpha-t)}{{}_1F_1(1,\beta;\alpha)} \\ \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} jB(j,\beta) \frac{{}_1F_1(j+1, j+\beta; \alpha)}{{}_1F_1(1,\beta;\alpha)} &= nB(n,\beta) \frac{{}_1F_1(n+1,n+\beta;\alpha-t)}{{}_1F_1(1,\beta;\alpha)} \end{aligned} \tag{5.218}$$

The rth moment of the Erlang mixture is

$$\begin{aligned} E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\ &= \frac{\Gamma(n+r)}{\Gamma n} - rB(-r,\beta) \frac{{}_1F_1(1-r, \beta-r; \alpha)}{{}_1F_1(1,\beta;\alpha)} \\ &= \frac{-r\Gamma(n+r)}{\Gamma n} B(-r,\beta) \frac{{}_1F_1(1-r,\beta-r;\alpha)}{{}_1F_1(1,\beta;\alpha)} \end{aligned} \tag{5.219}$$

$$\begin{aligned} E(T) &= \frac{-1\Gamma(n+1)}{\Gamma n} B(-1,\beta) \frac{{}_1F_1(0,\beta-1;\alpha)}{{}_1F_1(1,\beta;\alpha)} \\ &= -nB(-1,\beta) \frac{{}_1F_1(0,\beta-1;\alpha)}{{}_1F_1(1,\beta;\alpha)} = \infty \end{aligned} \tag{5.220}$$

5.14.2 Exponential-Truncated Pearson Type III Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= B(1,\beta)\frac{{}_1F_1(2,\beta+1;\alpha-t)}{{}_1F_1(1,\beta;\alpha)}, \quad t>0;\alpha>0,\beta>0 \end{aligned} \tag{5.221}$$

By the method of moments we have

$$\begin{aligned} f_n(t) &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}E(\wedge^j) \\ &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}jB(j,\beta)\frac{{}_1F_1(j+1,j+\beta;\alpha)}{{}_1F_1(1,\beta;\alpha)} \end{aligned} \tag{5.222}$$

Identity 5.35
Equating the above two methods we get

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}jB(j,\beta)\frac{{}_1F_1(j+1,j+\beta;\alpha)}{{}_1F_1(1,\beta;\alpha)} = B(1,\beta)\frac{{}_1F_1(2,\beta+1;\alpha-t)}{{}_1F_1(1,\beta;\alpha)} \tag{5.223}$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= r!(-r)B(-r,\beta)\frac{{}_1F_1(1-r,\beta-r;\alpha)}{{}_1F_1(1,\beta;\alpha)} \end{aligned} \tag{5.224}$$

$$E(T) = (-1)B(-1,\beta)\frac{{}_1F_1(0,\beta-1;\alpha)}{{}_1F_1(1,\beta;\alpha)} = \infty \tag{5.225}$$

5.14.3 Poisson-Truncated Pearson Type III Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$\begin{aligned}
P_n(t) &= \frac{t^n}{n!} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^n}{n!} nB(n,\beta) \frac{{}_1F_1(n+1, n+\beta; \alpha - t)}{{}_1F_1(1,\beta;\alpha)} \\
&= \frac{t^n \Gamma\beta}{\Gamma(n+\beta)} \frac{{}_1F_1(n+1, n+\beta; \alpha - t)}{{}_1F_1(1,\beta;\alpha)}, \quad t > 0; \alpha > 0, \beta > 0
\end{aligned} \tag{5.226}$$

By the method of moments we have

$$\begin{aligned}
P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} jB(j,\beta) \frac{{}_1F_1(j+1, j+\beta; \alpha)}{{}_1F_1(1,\beta;\alpha)}
\end{aligned} \tag{5.227}$$

Identity 5.36

Equating the above two methods we get

$$\begin{aligned}
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} jB(j,\beta) \frac{{}_1F_1(j+1, j+\beta; \alpha)}{{}_1F_1(1,\beta;\alpha)} &= \frac{t^n \Gamma\beta}{\Gamma(n+\beta)} \frac{{}_1F_1(n+1, n+\beta; \alpha - t)}{{}_1F_1(1,\beta;\alpha)} \\
\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} jB(j,\beta) \frac{{}_1F_1(j+1, j+\beta; \alpha)}{{}_1F_1(1,\beta;\alpha)} &= nB(n,\beta) \frac{{}_1F_1(n+1, n+\beta; \alpha - t)}{{}_1F_1(1,\beta;\alpha)}
\end{aligned} \tag{5.228}$$

The PGF of the Poisson mixture is

$$\begin{aligned}
G(s,t) &= E[e^{-(1-s)t\wedge}] \\
&= \frac{\Gamma 1 \Gamma\beta}{\Gamma\beta} \frac{{}_1F_1(1,\beta;\alpha - (1-s)t)}{{}_1F_1(1,\beta;\alpha)} \\
&= \frac{{}_1F_1(1,\beta;\alpha - (1-s)t)}{{}_1F_1(1,\beta;\alpha)}
\end{aligned} \tag{5.229}$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r r B(r,\beta) \frac{{}_1F_1(r+1, r+\beta; \alpha)}{{}_1F_1(1,\beta;\alpha)} \\ &= r t^r B(r,\beta) \frac{{}_1F_1(r+1, r+\beta; \alpha)}{{}_1F_1(1,\beta;\alpha)} \end{aligned} \tag{5.230}$$

$$E(T) = tB(1,\beta)\frac{{}_1F_1(2,\beta+1;\alpha)}{{}_1F_1(1,\beta;\alpha)} \tag{5.231}$$

5.15 Erlang-Pareto I Distribution and Its Links

5.15.1 Erlang-Pareto I Mixture

The Pareto I mixing distribution is

$$g(\lambda) = \frac{\alpha\beta^{\alpha}}{\lambda^{\alpha+1}}, \quad \lambda > \beta; \beta > 0, \alpha > 0 \tag{5.232}$$

$$\begin{aligned} \therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_{\beta}^{\infty} \lambda^n e^{-t\lambda} \frac{\alpha\beta^{\alpha}}{\lambda^{\alpha+1}} d\lambda \\ &= \alpha\beta^{\alpha} \int_{\beta}^{\infty} e^{-t\lambda} \lambda^{n-\alpha-1} d\lambda \\ let \quad \lambda &= x+\beta \quad \Longrightarrow d\lambda = dx \\ E[\wedge^n e^{-t\wedge}] &= \alpha\beta^{\alpha} \int_0^{\infty} e^{-t(x+\beta)} (x+\beta)^{n-\alpha-1} dx \\ &= \alpha\beta^{\alpha} e^{-t\beta} \int_0^{\infty} e^{-tx} (x+\beta)^{n-\alpha-1} dx \\ let \quad x &= \beta y \quad \Longrightarrow dx = \beta dy \end{aligned}$$

$$\begin{aligned} E[\wedge^n e^{-t\wedge}] &= \alpha\beta^{\alpha} e^{-t\beta} \int_0^{\infty} e^{-t\beta y} (\beta y+\beta)^{n-\alpha-1} \beta dy \\ &= \alpha\beta^{\alpha} e^{-t\beta} \beta^{n-\alpha} \int_0^{\infty} e^{-t\beta y} (y+1)^{n-\alpha-1} dy \\ &= \alpha\beta^{n} e^{-t\beta} \int_0^{\infty} y^{1-1} (y+1)^{n-\alpha+1-1-1} e^{-t\beta y} dy \\ &= \alpha\beta^{n} e^{-t\beta} \Psi(1, n-\alpha+1; \beta t) \end{aligned} \tag{5.233}$$

Construction by the direct method gives

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^{n-1}}{\Gamma n} \alpha\beta^n e^{-t\beta} \Psi(1, n-\alpha+1; \beta t) \\ &= \frac{\alpha(t\beta)^n e^{-t\beta}}{t\Gamma n} \Psi(1, n-\alpha+1; \beta t), \quad t > 0; \alpha > 0, \beta > 0, n = 1, 2, 3, .. \end{aligned} \tag{5.234}$$

By the method of moments we have

$$\begin{aligned} f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \alpha\beta^j B(1, \alpha - j) \end{aligned} \tag{5.235}$$

Identity 5.37
Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \alpha\beta^j B(1, \alpha - j) &= \frac{\alpha(t\beta)^n e^{-t\beta}}{t\Gamma n} \Psi(1, n-\alpha+1; \beta t) \\ \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \alpha\beta^j B(1, \alpha - j) &= \alpha\beta^n e^{-t\beta} \Psi(1, n-\alpha+1; \beta t) \end{aligned} \tag{5.236}$$

The rth moment of the Erlang mixture is

$$\begin{aligned} E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\ &= \frac{\Gamma(n+r)}{\Gamma n} \alpha\beta^{-r} B(1, \alpha + r) \\ &= \frac{\Gamma(n+r)}{\Gamma n} \frac{\alpha}{\beta^r} B(1, \alpha + r) \end{aligned} \tag{5.237}$$

$$\begin{aligned} E(T) &= \frac{\Gamma(n+1)}{\Gamma n} \frac{\alpha}{\beta} B(1, \alpha + 1) \\ &= \frac{n\alpha}{\beta} B(1, \alpha + 1) \end{aligned} \tag{5.238}$$

5.15.2 Exponential-Pareto I Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= \alpha\beta e^{-t\beta}\Psi(1,2-\alpha;\beta t), \quad t>0;\alpha>0,\beta>0 \end{aligned} \tag{5.239}$$

By the method of moments we have

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}E(\wedge^j) \\ &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\alpha\beta^j B(1,\alpha-j) \end{aligned} \tag{5.240}$$

Identity 5.38
Equating the above two methods we get

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\alpha\beta^j B(1,\alpha-j) = \alpha\beta e^{-t\beta}\Psi(1,2-\alpha;\beta t) \tag{5.241}$$

The rth moment of the Exponential mixture is

$$\begin{aligned} E(T^r) &= r!E(\wedge^{-r}) \\ &= r!\frac{\alpha}{\beta^r}B(1,\alpha+r) \\ &= \frac{\alpha r!}{\beta^r}B(1,\alpha+r) \end{aligned} \tag{5.242}$$

$$\begin{aligned} E(T) &= \frac{\alpha}{\beta}B(1,\alpha+1) \\ &= \frac{\alpha}{\beta(\alpha+1)} \end{aligned} \tag{5.243}$$

5.15.3 Poisson-Pareto I Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$
\begin{aligned}
P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^n}{n!}\alpha\beta^n e^{-t\beta}\Psi(1, n-\alpha+1; \beta t) \\
&= \frac{\alpha(t\beta)^n e^{-t\beta}}{n!}\Psi(1, n-\alpha+1; \beta t), \quad t>0; \alpha>0, \beta>0
\end{aligned} \tag{5.244}
$$

By the method of moments we have

$$
\begin{aligned}
P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\
&= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\alpha\beta^j B(1, \alpha-j)
\end{aligned} \tag{5.245}
$$

Identity 5.39
Equating the above two methods we get

$$
\begin{aligned}
\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\alpha\beta^j B(1, \alpha-j) &= \frac{\alpha(t\beta)^n e^{-t\beta}}{n!}\Psi(1, n-\alpha+1; \beta t) \\
\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\alpha\beta^j B(1, \alpha-j) &= \alpha\beta^n e^{-t\beta}\Psi(1, n-\alpha+1; \beta t)
\end{aligned} \tag{5.246}
$$

The PGF of the Poisson mixture is

$$
\begin{aligned}
G(s,t) &= E[e^{-(1-s)t\wedge}] \\
&= \alpha e^{-\beta(1-s)t}\Psi(1, 1-\alpha; \beta(1-s)t)
\end{aligned} \tag{5.247}
$$

The rth moment of the Poisson mixture is

$$
\begin{aligned}
E(T^r) &= t^r E(\wedge^r) \\
&= t^r \alpha\beta^r B(1, \alpha-r) \\
&= \alpha(t\beta)^r B(1, \alpha-r)
\end{aligned} \tag{5.248}
$$

$$
\begin{aligned}
E(T) &= (\alpha t\beta)B(1, \alpha-1) \\
&= \frac{\alpha t\beta}{\alpha-1}
\end{aligned} \tag{5.249}
$$

5.16 Erlang-Pareto II (Lomax) Distribution and Its Links

5.16.1 Erlang-Pareto II (Lomax) Mixture

The Pareto II (Lomax) mixing distribution is

$$g(\lambda) = \frac{\alpha\beta^\alpha}{(\lambda+\beta)^{\alpha+1}}, \quad \lambda > 0; \alpha > 0, \beta > 0 \tag{5.250}$$

$$\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda} \frac{\alpha\beta^\alpha}{(\lambda+\beta)^{\alpha+1}} d\lambda \\
&= \alpha\beta^\alpha \int_0^\infty \lambda^n e^{-t\lambda} (\lambda+\beta)^{-\alpha-1} d\lambda \\
let \quad \lambda = \beta x \quad &\implies d\lambda = \beta dx \\
E[\wedge^n e^{-t\wedge}] &= \alpha\beta^\alpha \int_0^\infty (\beta x)^n e^{-\beta xt} (\beta x + \beta)^{-\alpha-1} \beta dx \\
&= \alpha\beta^\alpha \beta^{n-\alpha} \int_0^\infty x^n e^{-\beta tx} (1+x)^{-\alpha-1} dx \\
&= \alpha\beta^n \int_0^\infty x^{n+1-1} (1+x)^{n+1-\alpha-(n+1)-1} e^{-\beta tx} dx \\
&= \alpha\beta^n \Gamma(n+1) \Psi(n+1, n-\alpha+1; \beta t)
\end{aligned} \tag{5.251}$$

Construction by the direct method gives

$$\begin{aligned}
f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^{n-1}}{\Gamma n} \alpha\beta^n \Gamma(n+1) \Psi(n+1, n-\alpha+1; \beta t) \\
&= \frac{\alpha n (t\beta)^n}{t} \Psi(n+1, n-\alpha+1; \beta t), \quad t > 0; \alpha > 0, \beta > 0, n = 1, 2, 3, ..
\end{aligned} \tag{5.252}$$

By the method of moments we have

$$\begin{aligned}
f_n(t) &= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\
&= \sum_{j=n}^\infty \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \alpha\beta^j B(j+1, \alpha-j)
\end{aligned} \tag{5.253}$$

Identity 5.40
Equating the above two methods we get

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \alpha \beta^j B(j+1, \alpha - j) = \frac{\alpha n (t\beta)^n}{t} \Psi(n+1, n-\alpha+1; \beta t)$$

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \alpha \beta^j B(j+1, \alpha - j) = \alpha \beta^n \Gamma(n+1) \Psi(n+1, n-\alpha+1; \beta t) \tag{5.254}$$

The rth moment of the Erlang mixture is

$$\begin{aligned} E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\ &= \frac{\Gamma(n+r)}{\Gamma n} \alpha \beta^{-r} B(1-r, \alpha+r) \\ &= \frac{\Gamma(n+r)}{\Gamma n} \frac{\alpha}{\beta^r} B(1-r, \alpha+r) \end{aligned} \tag{5.255}$$

$$E(T) = \frac{\Gamma(n+1)}{\Gamma n} \frac{\alpha}{\beta} B(0, \alpha+1) = \infty \tag{5.256}$$

5.16.2 Exponential-Pareto II (Lomax) Mixture

Construction by the direct method gives

$$\begin{aligned} f_1(t) &= E[\wedge e^{-t\wedge}] \\ &= \alpha \beta \Psi(2, 2-\alpha; \beta t), \quad t > 0; \alpha > 0, \beta > 0 \end{aligned} \tag{5.257}$$

By the method of moments we have

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\wedge^j) \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \alpha \beta^j B(j+1, \alpha - j) \end{aligned} \tag{5.258}$$

Identity 5.41

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} \alpha \beta^j B(j+1, \alpha - j) = \alpha \beta \Psi(2, 2-\alpha; \beta t) \tag{5.259}$$

The rth moment of the Exponential mixture is

$$\begin{aligned}
E(T^r) &= r!E(\wedge^{-r}) \\
&= r!\frac{\alpha}{\beta^r}B(1-r,\alpha+r) \\
&= \frac{\alpha r!}{\beta^r}B(1-r,\alpha+r) \qquad (5.260) \\
E(T) &= \frac{\alpha}{\beta}B(0,\alpha+1) = \infty \qquad (5.261)
\end{aligned}$$

5.16.3 Poisson-Pareto II (Lomax) Mixture

$$P_n(t) = \frac{t}{n}f_n(t)$$

Construction by the direct method gives

$$\begin{aligned}
P_n(t) &= \frac{t^n}{n!}E[\wedge^n e^{-t\wedge}] \\
&= \frac{t^n}{n!}\alpha\beta^n\Gamma(n+1)\Psi(n+1,n-\alpha+1;\beta t) \\
&= \alpha(t\beta)^n\Psi(n+1,n-\alpha+1;\beta t), \quad t>0;\alpha>0,\beta>0 \qquad (5.262)
\end{aligned}$$

By the method of moments we have

$$\begin{aligned}
P_n(t) &= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}E(\wedge^j) \\
&= \sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\alpha\beta^jB(j+1,\alpha-j) \qquad (5.263)
\end{aligned}$$

Identity 5.42
Equating the above two methods we get

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^j}{n!(j-n)!}\alpha\beta^jB(j+1,\alpha-j) = \alpha(t\beta)^n\Psi(n+1,n-\alpha+1;\beta t)$$

$$\sum_{j=n}^{\infty}\frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\alpha\beta^jB(j+1,\alpha-j) = \alpha\beta^n\Gamma(n+1)\Psi(n+1,n-\alpha+1;\beta t) \qquad (5.264)$$

The PGF of the Poisson mixture is

$$
\begin{aligned}
G(s,t) &= E[e^{-(1-s)t\wedge}] \\
&= \alpha\Psi(1,1-\alpha;\beta(1-s)t) \qquad (5.265)
\end{aligned}
$$

The rth moment of the Poisson mixture is

$$
\begin{aligned}
E(T^r) &= t^r E(\wedge^r) \\
&= t^r \alpha\beta^r B(r+1,\alpha-r) \\
&= \alpha(t\beta)^r B(r+1,\alpha-r) \qquad (5.266) \\
E(T) &= \alpha t\beta B(2,\alpha-1) \qquad (5.267)
\end{aligned}
$$

5.17 Erlang-Generalized Pareto (Gamma-Gamma) Distribution and Its Links

5.17.1 Erlang-Generalized Pareto (Gamma-Gamma) Mixture

The Generalized Pareto (Gamma-Gamma) mixing distribution is

$$
g(\lambda) = \frac{\lambda^{\beta-1}\mu^{\alpha}}{B(\alpha,\beta)(\lambda+\mu)^{\alpha+\beta}}, \quad \lambda > 0; \alpha > 0, \beta > 0, \mu > 0 \qquad (5.268)
$$

$$
\begin{aligned}
\therefore \quad E[\wedge^n e^{-t\wedge}] &= \int_0^\infty \lambda^n e^{-t\lambda} \frac{\lambda^{\beta-1}\mu^{\alpha}}{B(\alpha,\beta)(\lambda+\mu)^{\alpha+\beta}} d\lambda \\
&= \frac{\mu^\alpha}{B(\alpha,\beta)} \int_0^\infty \lambda^{n+\beta-1} e^{-t\lambda} (\lambda+\mu)^{-\alpha-\beta} d\lambda \\
let \quad \lambda = \mu x \quad &\Longrightarrow d\lambda = \mu dx \\
E[\wedge^n e^{-t\wedge}] &= \frac{\mu^\alpha}{B(\alpha,\beta)} \int_0^\infty (\mu x)^{n+\beta-1} e^{-t\mu x} (\mu x+\mu)^{-\alpha-\beta} \mu dx \\
&= \frac{\mu^\alpha \mu^{n-\alpha}}{B(\alpha,\beta)} \int_0^\infty x^{n+\beta-1} (1+x)^{-\alpha-\beta} e^{-\mu t x} dx \\
&= \frac{\mu^n}{B(\alpha,\beta)} \int_0^\infty x^{n+\beta-1} (1+x)^{n+1-\alpha-(n+\beta)-1} e^{-\mu t x} dx \\
&= \frac{\mu^n}{B(\alpha,\beta)} \Gamma(n+\beta)\Psi(n+\beta, n-\alpha+1; \mu t) \qquad (5.269)
\end{aligned}
$$

Construction by the direct method gives

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma n} E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^{n-1}}{\Gamma n} \frac{\mu^n}{B(\alpha,\beta)} \Gamma(n+\beta)\Psi(n+\beta, n-\alpha+1; \mu t) \\ &= \frac{(\mu t)^n}{tB(\alpha,\beta)} \frac{\Gamma(n+\beta)}{\Gamma n} \Psi(n+\beta, n-\alpha+1; \mu t), \quad t>0; \alpha>0, \beta>0, \mu>0, n=1,2,3,.. \end{aligned} \tag{5.270}$$

By the method of moments we have

$$\begin{aligned} f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\wedge^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \mu^j \frac{B(j+\beta, \alpha-j)}{B(\alpha,\beta)} \end{aligned} \tag{5.271}$$

Identity 5.43
Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} \mu^j \frac{B(j+\beta, \alpha-j)}{B(\alpha,\beta)} &= \frac{(\mu t)^n}{tB(\alpha,\beta)} \frac{\Gamma(n+\beta)}{\Gamma n} \Psi(n+\beta, n-\alpha+1; \mu t) \\ \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \mu^j \frac{B(j+\beta, \alpha-j)}{B(\alpha,\beta)} &= \frac{\mu^n}{B(\alpha,\beta)} \Gamma(n+\beta)\Psi(n+\beta, n-\alpha+1; \mu t) \end{aligned} \tag{5.272}$$

The rth moment of the Erlang mixture is

$$\begin{aligned} E(T^r) &= \frac{\Gamma(n+r)}{\Gamma n} E(\wedge^{-r}) \\ &= \frac{\Gamma(n+r)}{\Gamma n} \mu^{-r} \frac{B(\beta-r, \alpha+r)}{B(\alpha,\beta)} \\ &= \frac{\Gamma(n+r)}{\mu^r \Gamma n} \frac{B(\beta-r, \alpha+r)}{B(\alpha,\beta)} \end{aligned} \tag{5.273}$$

$$\begin{aligned} E(T) &= \frac{\Gamma(n+1)}{\mu \Gamma n} \frac{B(\beta-1, \alpha+1)}{B(\alpha,\beta)} \\ &= \frac{n\alpha}{\mu(\beta-1)} \end{aligned} \tag{5.274}$$

5.17.2 Exponential-Generalized Pareto (Gamma-Gamma) Mixture

Construction by the direct method gives

$$
\begin{aligned}
f_1(t) &= E[\wedge e^{-t\wedge}] \\
&= \frac{\mu}{B(\alpha,\beta)}\Gamma(\beta+1)\Psi(\beta+1,2-\alpha;\mu t) \\
&= \frac{\mu\beta\Gamma(\alpha+\beta)}{\Gamma\alpha}\Psi(\beta+1,2-\alpha;\mu t), \quad t>0;\alpha>0,\beta>0,\mu>0
\end{aligned} \tag{5.275}
$$

By the method of moments we have

$$
\begin{aligned}
f_1(t) &= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}E(\wedge^j) \\
&= \sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\mu^j\frac{B(j+\beta,\alpha-j)}{B(\alpha,\beta)}
\end{aligned} \tag{5.276}
$$

Identity 5.44
Equating the above two methods we get

$$
\sum_{j=1}^{\infty}\frac{(-1)^{j-1}t^{j-1}}{(j-1)!}\mu^j\frac{B(j+\beta,\alpha-j)}{B(\alpha,\beta)} = \frac{\mu\beta\Gamma(\alpha+\beta)}{\Gamma\alpha}\Psi(\beta+1,2-\alpha;\mu t) \tag{5.277}
$$

The rth moment of the Exponential mixture is

$$
\begin{aligned}
E(T^r) &= r!E(\wedge^{-r}) \\
&= r!\mu^{-r}\frac{B(\beta-r,\alpha+r)}{B(\alpha,\beta)} \\
&= \frac{r!}{\mu^r}\frac{B(\beta-r,\alpha+r)}{B(\alpha,\beta)}
\end{aligned} \tag{5.278}
$$

$$
\begin{aligned}
E(T) &= \frac{1}{\mu}\frac{B(\beta-1,\alpha+1)}{B(\alpha,\beta)} \\
&= \frac{\alpha}{\mu(\beta-1)}
\end{aligned} \tag{5.279}
$$

5.17.3 Poisson-Generalized Pareto (Gamma-Gamma) Mixture

$$P_n(t) = \frac{t}{n} f_n(t)$$

Construction by the direct method gives

$$\begin{aligned} P_n(t) &= \frac{t^n}{n!} E[\wedge^n e^{-t\wedge}] \\ &= \frac{t^n}{n!} \frac{\mu^n}{B(\alpha,\beta)} \Gamma(n+\beta)\Psi(n+\beta, n-\alpha+1; \mu t) \\ &= \frac{(\mu t)^n}{n!} \frac{\Gamma(n+\beta)}{B(\alpha,\beta)} \Psi(n+\beta, n-\alpha+1; \mu t), \quad t>0; \alpha>0, \beta>0, \mu>0 \end{aligned} \tag{5.280}$$

By the method of moments we have

$$\begin{aligned} P_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\wedge^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \mu^j \frac{B(j+\beta, \alpha-j)}{B(\alpha,\beta)} \end{aligned} \tag{5.281}$$

Identity 5.45
Equating the above two methods we get

$$\begin{aligned} \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} \mu^j \frac{B(j+\beta, \alpha-j)}{B(\alpha,\beta)} &= \frac{(\mu t)^n}{n!} \frac{\Gamma(n+\beta)}{B(\alpha,\beta)} \Psi(n+\beta, n-\alpha+1; \mu t) \\ \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \mu^j \frac{B(j+\beta, \alpha-j)}{B(\alpha,\beta)} &= \frac{\mu^n}{B(\alpha,\beta)} \Gamma(n+\beta)\Psi(n+\beta, n-\alpha+1; \mu t) \end{aligned} \tag{5.282}$$

The PGF of the Poisson mixture is

$$\begin{aligned} G(s,t) &= E[e^{-(1-s)t\wedge}] \\ &= \frac{1}{B(\alpha,\beta)} \Gamma(\beta)\Psi(\beta, 1-\alpha; \mu(1-s)t) \\ &= \frac{\Gamma(\beta)}{B(\alpha,\beta)} \Psi(\beta, 1-\alpha; \mu(1-s)t) \end{aligned} \tag{5.283}$$

The rth moment of the Poisson mixture is

$$\begin{aligned} E(T^r) &= t^r E(\wedge^r) \\ &= t^r \mu^r \frac{B(r+\beta, \alpha - r)}{B(\alpha, \beta)} \\ &= (t\mu)^r \frac{B(r+\beta, \alpha - r)}{B(\alpha, \beta)} \end{aligned} \tag{5.284}$$

$$\begin{aligned} E(T) &= (t\mu) \frac{B(\beta+1, \alpha - 1)}{B(\alpha, \beta)} \\ &= \frac{\mu t \beta}{\alpha - 1} \end{aligned} \tag{5.285}$$

6 SUMMARY AND RECOMMENDATIONS

6.1 Summary

Continuous Erlang mixtures were constructed with various mixing distributions using the direct method and the method of moments, and the two methods were equated to deduce a Mathematical Identity. The rth moments were also obtained. The Erlang mixtures were expressed in three forms, namely, Explicit, the Bessel function of the third kind and the Confluent Hypergeometric funtions which are Kummer's and Tricomi.
Exponential mixtures and Poisson mixtures were obtained from the Erlang mixtures, and the direct method and the method of moments were used and equated to deduce a Mathematical Identity. The rth moments were also obtained and Probability Generating Functions obtained in the Poisson mixtures.

6.2 Recommendations and Future Research

- The 4-parameter generalized Lindley mixing distribution and other forms of the 3-parameter generalzed Lindley distribution have been introduced in this work. Other mixing distributions leading to Erlang mixtures could also be identified.
- The link between Erlang mixtures and both Exponential mixtures and Poisson mixtures has been shown. The link between these mixtures and other mixed distributions could also be established. For example the link between the Erlang distribution and the Pearson type III and the Chi-squared distribution could be worked on, where both distributions are special cases of the Erlang distribution. The link between Erlang mixtures and Exponential mixtures through the Pareto distribution could also be explored.
- The rth moments for the Erlang mixtures, Exponential mixtures and Poisson mixtures were deduced. Other moments such as the raw moment and the central moment could be obtained.

This research focuses on construction of mixed Erlang distributions and linking them to both Exponential mixtures and Poisson mixtures. Further work could be done on estimation and application of these three mixtures.

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