University of Nairobi

School of Mathematics

ISSN: 2410-1397

Master Project in Actuarial Science

Modelling Of Market Stock Using The Normal variance Models

Research Report in Mathematics, Number 28, 2019

Chege Leah Wanjiru

June 2019

Submitted to the School of Mathematics in partial fulfillment for a degree in Master of Science in Actuarial Science

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Chege Leah Wanjiru

School of Mathematics
College of Biological and Physical sciences
Chiromo, off Riverside Drive
30197-00100 Nairobi, Kenya

Master Thesis
Submitted to the School of Mathematics in partial fulfillment for a degree in Master of Science in Actuarial Science

Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

Great work has been done on modeling of financial instruments namely,shares,equities, stocks and many more.The focus of this thesis is mainly modeling of stocks based on normal mixtures.The essence of this work is to do a comparison between the Normal Variance Mean Model and Normal Variance model and determine which of the two is best for modeling stocks.

Normal mixtures is a combination of two distributions where the normal distribution is the conditional distribution and is mixed with another distribution as the mixing distribution.The two mixing distribution discussed in this thesis are both Gamma and Inverse Gaussian distributions ,out of which we get the Variance Gamma and Normal Inverse Gaussian distributions respectively.Data is fitted on the distributions, Normal variance model and the Normal Variance Mean Model and a comparison is done to ascertain which model gives the best goodness of fit and is the best model.

Construction of the two distributions based on Normal Variance is done using two approaches that is; Saralees Nadarajah and Ole Barndoff, and their respective properties are given. Their respective Maximum likelihood Estimators based on the two distributions is also determined. Estimation of the two models is effected using Method of Moments. Data from Standard and Poor's 500 index,January 1977- December has been used for the analysis of the given work,and the results have been discussed accordingly. With regard to AIC test done in the analysis Skewed Variance Gamma distribution gives the best goodness of fit as compared to the other distributions While Nadarajah's Approach Normal Inverse Gaussian distribution tends to be the model for fitting stock returns.

Master Thesis in Mathematics at the University of Nairobi, Kenya.
ISSN 2410-1397: Research Report in Mathematics

DISTRIBUTOR: School of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

LEAH WANJIRU CHEGE
Reg No. I56/7954/2017

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

Prof. J.A.M.M Ottieno
School of Mathematics,
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: joseph.otieno@uonbi.ac.ke

Signature Date

Dr Davis Bundi
School of Mathematics
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: dbundi@uonbi.ac.ke

Dedication

To my parents ,siblings Mary, Elijah and Dorcas,my sister in- law Lilian and nephew Leon Chege.

Contents

Acknowledgments

All glory to God Almighty for His grace,favour and strength throughout the process and completion of this thesis.

I will forever be grateful to my Supervisor Prof. J.A.M.M Ottieno for his guidance ,ideas and endless encouragement that has made this thesis work a success. To Dr. Bundi am grateful for the support,guidance and the follow up on the progress of my thesis. Dr. Ivivi Mwaniki words cannot express my gratitude for the words of encouragement , advice and insight in regards to the analysis part of my thesis. To Prof. Weke for your persistent follow up on the progress and encouragement during my study period. May God exceedingly bless and establish you all.

Last but not least, to my family and friends who have been there to support , encourage and pray for me throughout my studies.May God's favour be upon you and remain blessed.

Leah Wanjiru Chege

Nairobi, 2017.

0.1 Abbreviations and Notations

Abbreviation and notations for specific:

f(y)	Probability Density function of y
f(y/v)	Conditional Probability of y given v
g(v)	mixing distribution
$K_v(.)$	Hankel function index v
E(q)	Expectation of q
E(q/v)	Conditional Expectation of q given v
φ	$\{.\}$- set of variables
L(φ)	Likelihood function
Log L(φ)	log likelihood function
V.G	variance Gamma distribution
N.I.G	Normal Inverse Gaussian distribution
M.L.E	Maximum Likelihood Estimator
N.V.M	Normal Variance Mean Model
N.V	Normal Variance Model

1 Introduction

1.1 Background Information

In the financial world there has been broad work done based for the modeling of financial instruments such as shares,equities,stocks and many others. In this thesis great focus has been drawn to the modeling of stocks based on normal mixtures. Market stock is a type of security that signifies proportionate ownership in the issuing corporation. This entitles the stockholder to that proportion of the corporation's asset and earning. Market Stocks are bought and sold predominantly on stock exchange,though there can be private sales and are the foundation of nearly every portfolio.

The flow chart below demonstrates a vivid description of the stock market business. It sets out the relationship between a company with its investor and also elaborates the stock transactions. In order for a company to raise capital it issues shares for the first time(IPO) thereby securing finance at the first instance for its operations from the investor. The pricing of stocks at the initial stage is based on the estimated worth of the company and the number of shares being issued. This amount becomes the base of the company ,but also gains from future profits. The investor also acquires voting rights on account of the shares of the company as a stockholder. In order for the company to maintain its stockholders it pays dividends as a form of bonus. Dividends are just a percentage of the profit made by the company and are paid quaterly or annually according to the company's respective policy. In the stock market transaction we have both the buyer and seller. Buying and selling in the stock market is determined by the company percieved value.

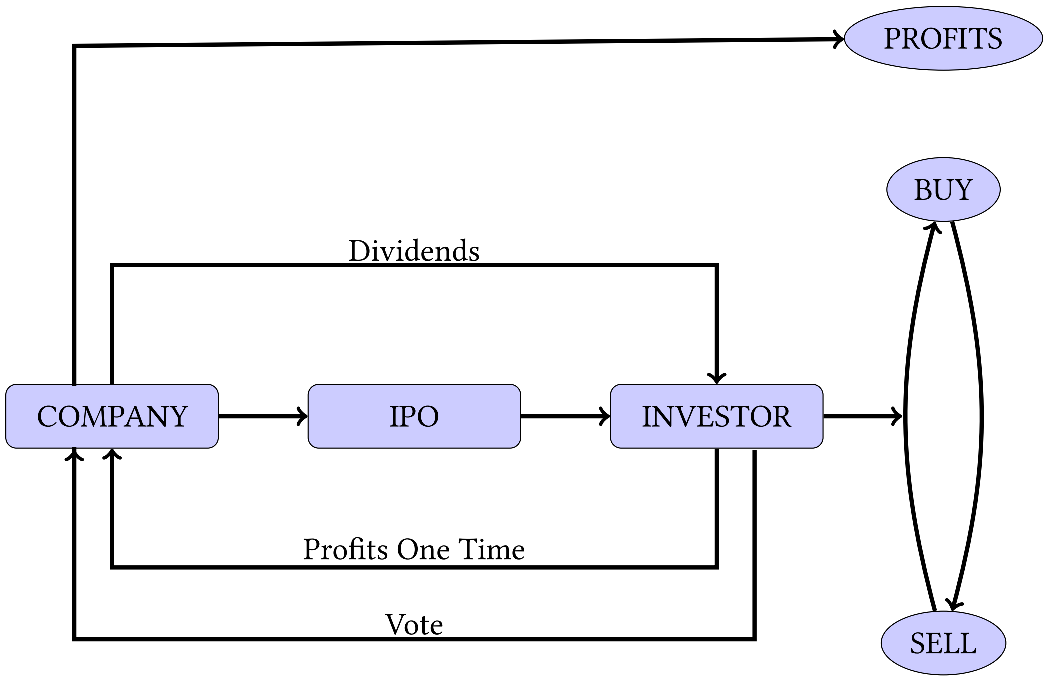

Figure 1. Illustration of the Stock Market

The figure below illustrates vividly the stock market movement. The bull market is when the stock rises,Investors who sell shares at this point are known as "bullish" investors. They that sell when prices go down that is in the bear market are known "bearish" investors. Share prices are described by the volume this is defined by the number of shares that change hands each day. High volume and Low volume depend on the differences between the selling and buying prices.

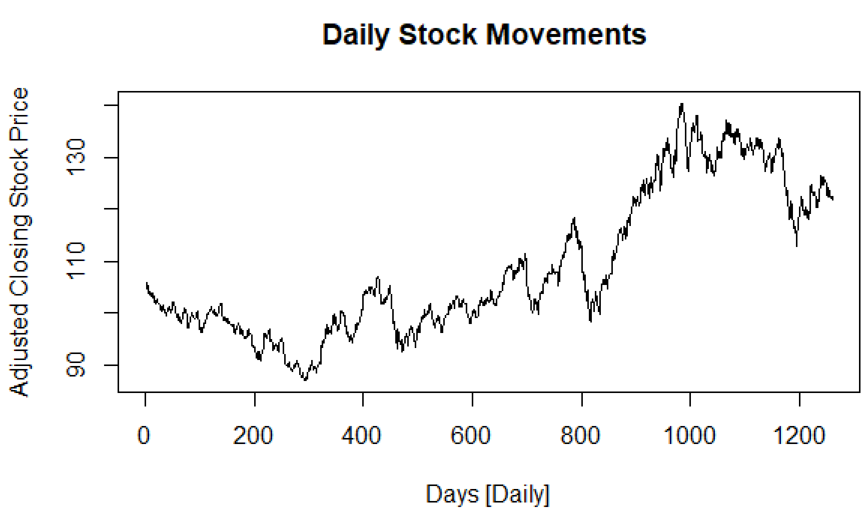

Figure 2. Stock Movement

In this thesis, the normal mixtures model is constructed where the normal distribution is the conditional distribution and both the Gamma and Inverse Gaussian distributions are the mixing distributions respectively. Based on two approaches normal mixtures distribution takes the form:-

1. Ole Barndoff(1977) Approach:Normal Mean variance.
   In this approach we consider the mean as a function of variance;
   where,$(x/v) \sim N(\mu+\beta v, v)$
   Therefore

$$f(x/v) = \frac{1}{\sqrt{2\pi v}} e^{\frac{-(x-\mu-\beta v)^2}{2v}}$$

$$\begin{aligned} f(x) &= \int_0^\infty f(x/v) g(v)\, dv \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi v}} e^{\frac{-(x-\mu-\beta v)^2}{2v}} g(v)\, dv \end{aligned}$$

   we take, $\mu = 0$ and $\beta = v$

$$f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi v}} e^{\frac{-x}{2v}} g(v)\, dv$$

2. Saralees Nadarajah (2012):Normal- variance. In this approach we consider;
   where,$(y/\lambda) \sim N(\mu, \lambda)$
   Therefore

$$f(y/\lambda) = \frac{1}{\sqrt{2\pi\lambda}} e^{\frac{-(y-\mu)^2}{2\lambda}}$$

   where, $-\infty < y > \infty$, $-\infty < \mu > \infty\ v > 0$

$$\begin{aligned} f(y) &= \int_0^\infty f(y/\lambda) g(\lambda) d\lambda \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi\lambda}} e^{\frac{-(y-\mu)^2}{2\lambda}} g(\lambda) d\lambda \end{aligned}$$

   we take, $\mu = 0$ and $\frac{y^2}{2} = x$

$$f(x) = \int_0^{\infty} \frac{1}{\sqrt{2\pi\lambda}} e^{\frac{-x}{\lambda}} g(\lambda) d\lambda$$

Chapter Five Maximum Likelihood Estimator has been discussed and derived accordingly based on the two distributions constructed. Chapter Six estimation is done based on the two models. Data analysis has been done based on the data obtained from Standard and Poor's 500 index from January 1977 to December 1981 .The S and P 500 or just the S and P was developed and continues to be maintained by S and P Dow Jones indices a joint venture majority owned by S and P global. It is an American stock market index based on the market capitalization of 500 large companies having common stock listed on the NYSE,NASDAQ or the Eboe BVX exchange.The results have been presented in Chapter Seven. Chapter Eight we have the conclusion and recommendations. Finally, Chapter Nine is the Appendix.

1.2 Statement Of The Problem

In the stock market, stocks returns are never normally distributed, this is because of the risk measures in the financial world. These risk measures include interest rates, Economic Outlook, Inflation, Deflation, Political Shocks, Terrorism, Investor Sentiments. Most research done in modeling stocks returns is based on the Normal variance Mean model.

The aim of this thesis is to model stocks returns using the Normal Variance Model based on two distributions Normal Inverse Gaussian and Variance Gamma distributions respectively. A comparison of the two distributions is done based on the N.V.M and N.V models respectively to identify the best model.

1.3 Study Objectives

The overall objective is to investigate the distribution that is best for modeling stock returns.

1.3.1 Specific objectives

i) To Construct the normal mixtures distributions, Variance Gamma and Normal Inverse Gaussian respectively, their respective properties and Maximum Likelihood Estimator.

ii) To Estimate the parameters of the respective distribution based on the Method of Moments.

iii) To Fit the Standard and Poor's 500 index from January 1977 to December 1981 data to the respective distributions.

1.4 Literature Review

1.4.1 Normal Mixtures

A discrete mixture of normal distributions is proposed to explain the observed significant kurtosis(fat tails)and significant positive skewness in the distribution of daily rates of returns for a sample of common stocks and indexes. The result comparison between the models is that the discrete mixture of normal distributions has substantially more descriptive validity than the student model.[KON84]

Mixtures of normal distribution is proposed to accomodate the normality and asymmetry characteristics of financial time series data as found in the distribution of monthly rates of returns for their indices.Maximum likelihood estimation via EM algorithm to fit the two component mixture of normal distribution using data sets on logarithmic stock returns of Bursa Malaysia indices.The mixture distribution was found to accommodate leptokurtic as well as skewed in the data .Mixture of normal distribution is proposed to accommodate the non-normality and asymetric characteristics of financial time series data as found in the distribution of monthly rates of returns for the indices.[KAM12]

The normal variance model has been used to describe stock return distribution.The model is based on taking the conditional stock return distribution to be normal with it's varaince.the estimation procedures are based on the method of moment and the method of maximum likelihood [NAD12]

The normal mixture distribution is proposed to accommodate the non-normality and asymmetry characteristics of financial time seried data as found in the distribution of returns.In support of determining the number of component in the mixture the information criterion for model selection is used.The goodness of fit measures provides supporting evidence in favour of the two component normal mixture model distribution at all frequency levels.Empirical results indicate that the normal mixture distribution offer a plausible description of the data, and is shown to be more superior compared the use of other distributions(Gumbel minimum distribution and Laplace distribution).[KAM15]

1.4.2 Normal Inverse Gaussian Distribution

The Normal Inverse Gaussian Distribution (NIG) distribution determines an homogenous levy process,and the process is representable through subbodination of brownian motion by the inverse gaussian process.The canonical, levy type,decomposition of the process is also determined.A brief review of the connection of the NIG distribution to the classes of generalived hyperbolic and inverse Gaussian distribution is also discussed ,[BARN97].In order to investigate if NIG levy process is a suitable model the uniform residuals by means of an algorithm which simulates random variables from the NIG distribution is calculated. The algorithm uses the characterivation of the NIG as a normal variance mean mixture.An approximation of the process that only relies on the fact that the process is a levy process with characteristic triplet is provided which will make it more tractable from a mathematical point of view [RYB97] In a comparison of various distribution to model the leptokurtic marginal distribution of asset is done.This is to identify a distribution the best fits empirical asset returns.Based on the results obtained after the analysis done for index returns,NIG distribution can be describe as a distribution which does not explain the tail properties acccurately enough,"underestimate the tail thickness" [HUR97]

The gap between the traditional ARCH modelling and recent advances on realived volatilities.The implied daily GARCH model with NIG errors estimated for the ECU returns results in very accurate out of sample predictions for the three years actual daily exchange rules.Empirical results coroborate the argument in favour the GARCH-NIG model as proving accurate and parsimonious representations of daily exchange rate dynamics.Hence,in the case of equities, skewness in the returns may necessitate the use of more general NIG mixture distribution,[FORS02]

Alternative approach for VaR calculation based on realized moments. The EMWA was used to compute the forecast realized moments which were then used to parameterive the NIG distribution in a method of moments,[LAU14]

1.4.3 Variance Gamma Distribution

A continuous time stochastic process, termed the VG model is introduced , for modeling the underlying uncertainity during stock market returns.Given that the GH distribution improves only marginally on the VG distribution and the t-distribution, if at all, we postulate that in most cases it is sufficient to consider only the latter two models. The authors concede that, although the VG distribution seems marginally superior here, modelling of rare events may be better captured by the t-distribution[MAD90].A three parameter stochastic process termed the VG prices that generalives brownian motion is developed as a model for the dynamics of log stock prices .These additional parameters provides control over the skewness and kurtosis of the return distribution.The additional parameters also correct for pricing biases of the blackscholes model,[MAD98]

Inside the class of processes which allow to build a model free from arbritrage,which we have seen corresponds to the class semi-martingales, the attention is focused on the variance gamma process.The Variance Gamma process can be obtained by replacing the two parameter compose with the black and scholes model, that allow to control the skeweness and kurtosis of the process followed by the underlying returns allowing to price options with different strikes without need to modify implied volatility or other parameter as the moneyness change,[FIO04].Fitting of the variance gamma distribution allows for skewness by moment method.The fitting procedure allows for possible dependence of increments in log returns, while retaining their stationarity.Standard estimation and hypothesis-testing theory depends on a large sample of observations which are independently as well as identically distributed and consequently may give inappropriate conclusions in the presence of dependence.[SEN04]

A review of the evolution of the theme of the pre-1990 papers is the estimation of parameters of log prices increment distributions that have real simple closed form characteristic function directly on simulated data and Sydney stock exchange data,[SEN06].Financial returns (log increments)data $Y_t, t = 1, 2....$, are treated as a stationery process, with the common distribution of each time point being not necessarily symmetric.Expository exploration of the applicability of the method of moments in both symmetric and skewed settings ,using moment estimates of $EY_t, VarY_t$,skewness and kurtosis to clarify the usefulness of this methods.[JET06]

A scaled self-decomposable stochastic process is used to model long term equity returns and options prices. This parsimonious model is compared to a number of other one dimensional continous time stochastic process that are commonly used in finance and the actuarial Sciences.A test is done to test if the models can reproduce a typical implied volatility surface seen in the market. Based on evidence from options prices the variance gamma scaled self- decomposable and the Regime switching lognormal process[MOL 10]

A method to process options using a multinomial method is proposed when the underlying price is modeled with a Variance Gamma process.The continuous time V.G process is approximated by a continuous time process with the same first four cumulants and then discretived in time and space. This approach is particulary convenient for pricing options which can be exercised before the expiration date.Numerical computations of the given options are presented and compared with results obtained with finite difference method and with the black scholes. It turns out that the multinomial method is easier to complement than the finite difference method . The algorithm does not involve any matrix multiplication or matrix inversion as in the case of implicit and explicit method PIDE's, which means the computational time is much smaller.[CAN18]

The outline of the thesis is as follows:

Chapter 2: Deriving the propeties and first derivative of Hankel function based in their respective intergration properties, Constuction of the Generalized Inverse Gaussian and its special cases .

Chapter 3:Construction of the Normal Inverse Gaussian distribution based on Nadarajah and Ole-Barndoff approach and deriving its properties
Chapter 4:Construction of the Variance Gamma distribution based on Nadarajah and Ole-Barndoff approach and deriving its properties
Chapter 5:Deriving of the Maximum likelihood estimators of the respective distributions based on the two approaches mentioned above
Chapter 6:Estimation based on Method of Moments of the two distributions respectively
Chapter 7:Data analysis done on the two distributions
Chapter 8:Conclusion and Recomendation
Chapter 9:Appendix

2 Generalised Inverse Gaussian Distribution

2.1 Introduction

In this chapter,Construction of GIG pdf and two of it's special cases is considered.Their construction is based on the Hankel function.

2.2 Hankel function

2.2.1 First Integral Representation and its properties

$$K_v(w) = \frac{1}{2}\int_0^\infty x^{v-1} e^{\frac{-w}{2}(x+\frac{1}{x})} dx \tag{1}$$

property one

$$K_v(w) = K_-v(w)(symmetry) \tag{2}$$

Proof.

$$K_v(w) = \frac{1}{2}\int_0^\infty x^{v-1} e^{\frac{-w}{2}(x+\frac{1}{x})} dx$$

$$let, \frac{w}{2s} = x, Therefore, d(x) = \frac{-w}{2s^2}\ ds$$

Thus:

$$K_v(w) = \frac{1}{2}\int_0^\infty \{\frac{w}{2s}^{v-1} e^{\frac{-w}{2}(\frac{w}{2s}+\frac{2s}{w})}(\frac{-w}{2s^2}\}\ ds$$

$$hencelet, s = \frac{1}{v}, thus, ds = \frac{-dv}{v^2}$$

$$\begin{aligned} K_v(w) =& \frac{1}{2}\int_\infty^0 (\frac{1}{v})^{v-1} e^{\frac{-w}{2}(\frac{1}{v}+v)}(\frac{-1}{v^2})dv \\ =& \frac{1}{2}\int_0^\infty \{(\frac{1}{v})^{-v-1} e^{\frac{w}{2}(\frac{1}{v}+v)}(\frac{-1}{v^2})\}\ dv \\ =& K_-v(w) \end{aligned}$$

□

2.2.2 Second Integral Representation and its properties

$$K_v(w) = (\frac{w}{2})^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})} \int_1^\infty (s^2-1)^{v-\frac{1}{2}} e^{-ws}\, ds \tag{3}$$

property two

$$K_{j+\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w} \{1 + \sum_{i=1}^{j} \frac{(j+1)!(2w)^{-i}}{(j-i)!i!}\} \tag{4}$$

Proof.

$$let, v = j + \frac{1}{2}, for, j = 0, 1, 2, ...$$

hence,

$$\begin{aligned} K_{j+\frac{1}{2}}(w) =& (\frac{w}{2})^{j+\frac{1}{2}} \frac{\Gamma(\frac{1}{2})}{\Gamma j+1} \int_1^\infty (s^2-1)^j e^{-ws}\, ds \\ =& \frac{\sqrt{\pi} w^{j+\frac{1}{2}}}{2^{j+\frac{1}{2}} j!} \int_1^\infty (s^2-1)^j e^{-ws}\, ds \\ =& \frac{\pi w^{j+\frac{1}{2}}}{2^j 2^{\frac{1}{2}} j!} \int_1^\infty (s^2-1)^j e^{-ws} ds \end{aligned}$$

Therefore,

$$\begin{aligned} =& \sqrt{\frac{w\pi}{2}} \frac{w^j}{2^j} \frac{1}{j!} \int_1^\infty (s^2-1)^j e^{-ws}\, ds \\ =& \sqrt{\frac{w\pi}{2}} \frac{w^j}{2^j} \frac{e^{-w}}{j!} \int_1^\infty (s^2-1)^j \frac{e^{-ws}}{e^{-w}}\, ds \\ =& \sqrt{\frac{w\pi}{2}} \frac{w^j}{2^j} \frac{e^{-w}}{j!} \int_1^\infty [(s-1)(s+1)]^j e^{-ws+w}\, ds \\ =& \sqrt{\frac{w\pi}{2}} \frac{w^j}{2^j} \frac{e^{-w}}{j!} \int_1^\infty [(s-1)(s+1)]^j e^{-w(s-1)}\, ds \end{aligned}$$

$$let, v = w(s-1), thus, \frac{v}{w} = s-1; s+1 = 2 + \frac{v}{w}, hence, ds = \frac{dv}{w}$$

$$
\begin{aligned}
K_{j+\frac{1}{2}}(w) =& \sqrt{\frac{w\pi}{2}} e^{-w} \frac{w^j}{2^j j!} \int_0^\infty \{\frac{v}{w}(2+\frac{v}{w})\}^j e^{-v} \frac{dv}{w} \\
=& \frac{1}{w} \sqrt{\frac{w\pi}{2}} e^{-w} \frac{w^j}{2^j j!} \int_0^\infty \{\frac{2v}{w}(1+\frac{v}{2w})\}^j e^{-v}\, dv \\
=& \sqrt{\frac{\pi}{2w}} \frac{e^{-w}}{j!} \frac{w^j}{2^j} \int_0^\infty \{(\frac{2v}{w})^j (1+\frac{v}{2w})\}^j e^{-v}\, dv \\
=& \sqrt{\frac{\pi}{2w}} \frac{e^{-w}}{j!} \int_0^\infty \{(v)^j (1+\frac{v}{2w})\}^j e^{-v}\, dv \\
=& \sqrt{\frac{\pi}{2w}} \frac{e^{-w}}{j!} \int_0^\infty (v)^j \sum_{i=0}^{n} \binom{j}{i} (\frac{v}{2w})^i e^{-v}\, dv \\
=& \sqrt{\frac{\pi}{2w}} \frac{e^{-w}}{j!} \sum_{i=0}^{j} \binom{j}{i} \frac{1}{(2w)^i} \int_0^\infty (v)^{(j+i+1)-1} e^{-v}\, dv \\
=& \sqrt{\frac{\pi}{2w}} \frac{e^{-w}}{j!} \sum_{i=0}^{j} \binom{j}{i} \frac{1}{(2w)^i} \Gamma(j+i+1) \\
=& \sqrt{\frac{\pi}{2w}} \frac{e^{-w}}{j!} \{\binom{j}{0} \frac{1}{(2w)^0} \Gamma(j+1) + \sum_{i=1}^{j} \binom{j}{i} \frac{1}{(2w)^i} \Gamma(j+i+1)\} \\
=& \sqrt{\frac{\pi}{2w}} \frac{e^{-w}}{j!} \{\Gamma(j+1) + \sum_{i=1}^{j} \binom{j}{i} \frac{1}{(2w)^i} (j+i)!\} \\
=& \sqrt{\frac{\pi}{2w}} \frac{e^{-w}}{j!} \{j! + \sum_{i=1}^{j} \binom{j}{i} (2w)^{-i} (j+i)!\}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
K_{j+\frac{1}{2}}(w) =& \sqrt{\frac{\pi}{2w}} e^{-w} \{1 + \sum_{i=1}^{j} \frac{1}{j!} \binom{j}{i} (2w)^{-i} (j+i)!\} \\
=& \sqrt{\frac{\pi}{2w}} e^{-w} \{1 + \sum_{i=1}^{j} \frac{j!(j+1)!}{j!(j-1)!i!} (2w)^{-i}\} \\
=& \sqrt{\frac{\pi}{2w}} e^{-w} \{1 + \sum_{i=1}^{j} \frac{(j+1)!(2w)^{-i}}{(j-1)!i!}\}
\end{aligned}
$$

□

First Corollary of property 2;

$$
K_{\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w} \tag{5}
$$

Proof.

$$K_{\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}}e^{-w}[1+0]$$
$$= \sqrt{\frac{\pi}{2w}}e^{-w}$$

□

Second Corollary of property 2;

$$K_{\frac{3}{2}}(w) = \sqrt{\frac{\pi}{2w}}e^{-w}\{1+\sum_{i=1}^{j}\frac{(j+1)!(2w)^{-i}}{(j-1)!i!}\} \tag{6}$$

Proof.

$$= \sqrt{\frac{\pi}{2w}}e^{-w}\{1+\frac{(1+1)!(2w)^{-1}}{(1-1)!1!}\}$$
$$= \sqrt{\frac{\pi}{2w}}e^{-w}\{1+\frac{2!(2w)^{-1}}{0!1!}\}$$
$$= \sqrt{\frac{\pi}{2w}}e^{-w}\{1+\frac{1}{w}\}$$

□

2.2.3 Derivatives of hankel function

First Derivatives

$$\frac{\partial}{\partial w}K_v(w) = \frac{-1}{2}\{K_{v+1}(w)+K_{v-1}(w)\} \tag{7}$$

Proof.

$$
\begin{aligned}
\frac{\partial}{\partial w}K_v(w) =&\frac{\partial}{\partial w}\frac{1}{2}\int_0^\infty p^{v-1}e^{\frac{-w}{2}(p+\frac{1}{p})}\,dp\\
=&\frac{1}{2}\int_0^\infty p^{v-1}\{\frac{-1}{2}(p+\frac{1}{p})\}e^{\frac{w}{2}(p+\frac{1}{p})}\,dp\\
=&\frac{-1}{2}\{\frac{1}{2}\int_0^\infty p^{v}e^{\frac{-w}{2}(p+\frac{1}{p})}\,dp+\frac{1}{2}\int_0^\infty p^{v-2}e^{\frac{-w}{2}(p+\frac{1}{p})}\,dp\}\\
=&\frac{-1}{2}\{K_{v+1}(w)+K_{v-1}(w)\}
\end{aligned}
$$

□

recall,

$$
K_v(w) = \frac{1}{2}(\frac{w}{2})^v\int_0^\infty p^{-v-1}e^{-p-\frac{w^2}{4p}}\,dp
$$

Proof.

$$
\begin{aligned}
\frac{\partial}{\partial w}K_v(w) =&\frac{\partial}{\partial w}\frac{1}{2}(\frac{w}{2})^v\int_0^\infty p^{-v-1}e^{-p-\frac{w^2}{4p}}\,dp\\
=&\frac{1}{2}\{\frac{v}{2}(\frac{w}{2})^{v-1}\int_0^\infty p^{-v-1}e^{-p-\frac{w^2}{4p}}\,dp+(\frac{w}{2})^v\int_0^\infty p^{-v-1}e^{-p-\frac{w^2}{4p}}(\frac{-2}{4p}w)\,dp\}\\
=&\frac{1}{2}\{\frac{v}{2}(\frac{w}{2})^{-1}(\frac{w}{2})^v\int_0^\infty p^{-v-1}e^{-p-\frac{w^2}{4p}}\,dp+(\frac{w}{2})^v\int_0^\infty p^{-v-1}e^{-p-\frac{w^2}{4p}}(\frac{-2}{4p}w)\,dp\}\\
=&\frac{1}{2}\{\frac{v}{2}(\frac{w}{2})^v\int_0^\infty p^{-v-1}e^{-p-\frac{w^2}{4p}}\,dp-(\frac{w}{2})^v(\frac{w}{2})\int_0^\infty p^{-v-1}e^{-p-\frac{w^2}{4p}}(\frac{-2}{4p}w)\,dp\}\\
=&\frac{v}{w}\{\frac{1}{2}(\frac{w}{2})^v\int_0^\infty p^{-v-1}e^{-p-\frac{w^2}{4p}}\,dp\}-\frac{1}{2}(\frac{w}{2})^{v+1}\int_0^\infty p^{-(v+1)-1}e^{-p-\frac{w^2}{4p}}\,dp\\
=&\frac{v}{w}K_v(w)-K_{v+1}(w)
\end{aligned}
$$

□

Equating the two derivatives we get,

$$
\begin{aligned}
\frac{-1}{2}K_{v+1}(w)-\frac{1}{2}K_{v-1}(w) &= \frac{v}{w}K_v(w)-K_{v+1}(w)\\
K_{v+1}(w) &= \frac{2v}{w}K_v(w)+\frac{1}{2}K_{v-1}(w) \qquad \textbf{(8)}
\end{aligned}
$$

2.3 GIG distribution with its special cases

2.3.1 Constuction of GIG

Generalised inverse Gaussian distribution has a pdf of the form;-

$$\frac{(\frac{\psi}{\chi})^{\frac{v}{2}}}{2k_v(\sqrt{\chi\psi})}x^{v-1}e^{\frac{-1}{2}(\psi x+\frac{\chi}{x})}$$

where $K_v(.)$ is the hankel function index v.
proof,
Barndoff - Nielsen et al (1977),proofs that GIG is infinetly divisible.
$K_v(w) = \frac{1}{2}\int_0^\infty x^{v-1}e^{\frac{-w}{2}(x+\frac{1}{x})}\,dx$
let,

$$w = \sqrt{\chi\psi}and, x = \sqrt{\frac{\psi}{\chi}}v, \Rightarrow dx = \sqrt{\frac{\psi}{\chi}}dv$$

maintaining the integration limits,

$$\begin{aligned}
K_v(\sqrt{\chi\psi}) =& \frac{1}{2}\int_0^\infty (\sqrt{\frac{\psi}{\chi}}v)^{v-1}exp\{\frac{-\sqrt{\chi}\psi}{2}(\sqrt{\frac{\psi}{\chi}}v+\sqrt{\frac{\chi}{\psi}}\frac{1}{v})\}\sqrt{\frac{\psi}{\chi}}\,dv \\
=& \frac{1}{2}\int_0^\infty (\sqrt{\frac{\psi}{\chi}})^v v^{v-1}exp\{\frac{-\sqrt{\chi}\psi}{2}(\sqrt{\frac{\psi}{\chi}}v+\sqrt{\frac{\chi}{\psi}}\frac{1}{v})\}\,dv \\
=& \frac{1}{2}\int_0^\infty (\sqrt{\frac{\psi}{\chi}})^v v^{v-1}exp\{\frac{-1}{2}(\sqrt{\frac{\psi\chi\psi}{\chi}}v+\sqrt{\frac{\chi\chi\psi}{\psi}}\frac{1}{v})\}\,dv \\
=& \frac{1}{2}(\sqrt{\frac{\psi}{\chi}})^v\int_0^\infty v^{v-1}exp\{\frac{-1}{2}(\frac{\psi}{\chi}v+\frac{chi}{v})\}\,dv
\end{aligned}$$

hence the pdf,

$$\begin{aligned}
1 =& \frac{1}{2}(\sqrt{\frac{\psi}{\chi}})^v\frac{1}{K_v(\sqrt{\chi\psi})}\int_0^\infty v^{v-1}exp\{\frac{-1}{2}(\frac{\psi}{\chi}v+\frac{chi}{v})\}\,dv \\
=& \int_0^\infty \frac{(\frac{\psi}{\chi})^{\frac{v}{2}}}{2K_v(\sqrt{\chi\psi})}v^{v-1}e^{\frac{-1}{2}(\psi v+\frac{\chi}{v})}\,dv
\end{aligned}$$

$\therefore$

$$f_{GIG}(x;v,\chi,\psi) = \frac{(\frac{\psi}{\chi})^{\frac{v}{2}}}{2K_v(\sqrt{\chi\psi})}v^{v-1}e^{\frac{-1}{2}(\psi v+\frac{\chi}{v})}$$

2.3.2 Special cases of GIG

Gamma distribution

Gamma$\sim$ $(\chi = 0, v > 0)$

$$f(x) = \frac{(\frac{\psi}{\chi})^{\frac{v}{2}} v^{v-1} e^{\frac{-1}{2}(\psi v + \frac{\chi}{v})}}{(\frac{\psi}{\chi})^{\frac{v}{2}} \int_0^\infty v^{v-1} e^{\frac{-1}{2}(\psi v + \frac{\chi}{v})}}$$
$$= \frac{v^{v-1} e^{\frac{-1}{2}(\psi v + \frac{\chi}{v})}}{\int_0^\infty v^{v-1} e^{\frac{-1}{2}(\psi v + \frac{\chi}{v})}}$$

let, $\chi = 0$,
substituting to the equation,

$$f(x) = \frac{v^{v-1} e^{\frac{-1}{2}}(\psi v)}{\frac{\Gamma(v)}{(\frac{\psi}{2})^v}}$$
$$= \frac{(\frac{\psi}{2})^v}{\Gamma(v)} v^{v-1} e^{\frac{-\psi}{2} v}$$

$$Gamma \sim (v, \frac{\psi}{2})$$

Inverse Gaussian Distribution

Inverse Gausssian is a special case of GIG,where v=$\frac{-1}{2}$,

$$f_{IG}(v; \chi, \psi) = f_{GIG}(v; \frac{-1}{2}, \chi, \psi)$$

$$f_{GIG}(x; \frac{-1}{2}, \chi, \psi) = \frac{(\frac{\psi}{\chi})^{\frac{\frac{-1}{2}}{2}}}{2K_{\frac{-1}{2}}(\sqrt{\chi\psi})} v^{\frac{-1}{2}-1} e^{\frac{-1}{2}(\psi v + \frac{\chi}{v})}$$
$$= \frac{(\frac{\psi}{\chi})^{\frac{-1}{4}}}{2K_{\frac{-1}{2}}(\sqrt{\chi\psi})} v^{\frac{-1}{2}-1} e^{\frac{-1}{2}(\psi v + \frac{\chi}{v})}$$

by symmetry,equation,2,

$$K_v(w) = K_{-v}(w)$$

$\therefore$

$$K_{\frac{1}{2}}(w) = K_{\frac{-1}{2}}(w)$$

Thus,equation,6,

$$K_{\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}}e^{-w}$$

let,$w = \sqrt{\chi\psi}$

$$\begin{aligned} K_{\frac{1}{2}}(\sqrt{\chi\psi}) =& \sqrt{\frac{\pi}{2\sqrt{\chi\psi}}}e^{-\sqrt{\chi\psi}} \\ =& \frac{\sqrt{\pi}}{\sqrt{2}(\chi\psi)^{\frac{1}{4}}}e^{-\sqrt{\chi\psi}} \end{aligned}$$

substituting back to the inverse gaussian pdf,

$$\begin{aligned} f_{IG} =& \frac{(\frac{\psi}{\chi})^{\frac{-1}{4}} v^{\frac{-1}{2}-1} e^{\frac{-1}{2}(\psi v + \frac{\chi}{v})}}{2\frac{\sqrt{\pi}}{\sqrt{2}(\chi\psi)^{\frac{1}{4}}}e^{-\sqrt{\chi\psi}}} \\ =& \frac{\sqrt{2}(\chi\psi)^{\frac{1}{4}}e^{\sqrt{\chi\psi}}}{2\sqrt{\pi}(\frac{\psi}{\chi})^{\frac{1}{4}}} v^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi v + \chi\frac{1}{v})} \\ =& \frac{1}{\sqrt{2}\sqrt{\pi}}(\frac{\chi\psi}{\frac{\psi}{\chi}})^{\frac{1}{4}} e^{\sqrt{\chi\psi}} v^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi v + \chi\frac{1}{v})} \\ =& \frac{\sqrt{\chi}}{\sqrt{2\pi}} v^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi v + \chi\frac{1}{v}) + \sqrt{\chi\psi}} \end{aligned}$$

2.3.3 Raw Moments for Normal Mixtures

let,$q/v \sim N(\delta + \theta v, \sigma^2 v)$

General formula

First Moments

$$\begin{aligned}E(q) =& E(E(q/v))\\ =& E(\delta+\theta v)\\ =& \delta+\theta E(v)\end{aligned}$$

Second Moments

$$\begin{aligned}E(q^2) =& E(E(q^2/v))\\ =& E(var(q/v)+(E(q/v))^2)\\ =& E(\sigma^2 v+(\delta+\theta v)^2)\\ =& \sigma^2 E(v)+E(\delta^2+2\theta\delta v+\theta^2 v^2)\\ =& \sigma^2+\delta^2+2\theta\delta+\theta^2 E(v^2)\end{aligned}$$

Third Moments

$$\begin{aligned}E(q^3) =& E(3\mu^* E(q^2/v)-2\mu^{*^3})\\ =& E(3(\delta+\theta v)(\sigma^2 v+(\delta+\theta v)^2)-2(\delta+\theta v)^3)\\ =& E(3(\delta+\theta v)\ \sigma^2 v+(\delta+\theta v)^3))\\ =& E(3\delta\sigma^2 v+3\theta\sigma^2 v^2+\delta^3+3\theta\delta^2 v+3\theta^2\delta v^2+\theta^3 v^3)\\ =& 3\delta\sigma^2+3\theta\sigma^2 E(v^2)+\delta^3+3\theta\delta^2+3\theta^2\delta E(v^2)+\theta^3 E(v^3)\end{aligned}$$

Fourth Moments

$$
\begin{aligned}
E(q^4/v) =& 3\sigma^{*^4} + 4\mu^* E(q^3/v) - \sigma^* \mu^2 E(q^2/v) + 3\mu^{*^4} \\
=& 3\sigma^{*^4} + 4\delta(E(q^3/v)) + 4\theta v E(q^3/v) + 3(\delta+\theta v)^4 - 6(\delta+\theta v)^2(\sigma^2 v + (\delta+\theta v)^2) \\
=& 3\sigma^{*^4} + 4\delta(3\delta\sigma^2 v + 3\theta\sigma^2 v^2 + \delta^3 + 3\theta\delta^2 v + 3\theta^2\delta v^2 + \theta^3 v^3) + 4\theta v(3\delta\sigma^2 v + 3\theta\sigma^2 v^2 + \delta^3 \\
& + 3\theta\delta^2 v + 3\theta^2\delta v^2 + \theta^3 v^3) - 6(\delta^2 + 2\theta\delta v + \theta^2 v^2)(\sigma^2 v + \delta^2 + 2\theta\delta v + \theta^2 v^2) + \\
& 3(\delta^4 + 4\theta\delta^3 v + 6\theta^3\delta^2 v^3 + \theta^4 v^4) \\
=& 3\sigma^{*^4} + 4(3\delta^2\sigma^2 v + 3\theta\delta\sigma^2 v^2 + \delta^4 + 3\theta\delta^2 v + 3\theta^2\delta^2 v^2 + \delta\theta^3 v^3) + 4(3\delta\theta\sigma^2 v^2 + 3\theta^2\sigma^2 v^3 \\
& + \delta^3\theta v + 3\delta^2\theta^2 v^2 + 3\delta\theta^3 v^3 + \theta^4 v^4) - 6(\delta^2\sigma^2 v + \delta^4 + 2\delta^3\theta v + \delta^2\theta^2 v^2) - 6(2\theta\delta\sigma^2 v^2 + \\
& 2\theta\delta^3 v + 4\theta^2\delta^2 v^2 + 2\theta^3\delta v^3) - 6(\theta^2\delta^2 v^3 + \delta^2\theta^2 v^2 + 2\delta\theta^3 v^3 + \theta^4 v^4) + 3(\delta^4 + 4\theta\delta^3 v + \\
& 6\theta^2\delta^2 v^2 + 4\theta^3\delta v^3 + \theta^4 v^4) \\
=& 3\sigma^4 v^2 + \delta^4 + 12\theta\delta\sigma^2 v^2 + 6\delta^2\sigma^2 v + 4\delta^3\theta v + 6\theta^2\delta^2 v^2 + 6\theta^2\sigma^2 v^3 + 4\theta^3\delta v^3 + \theta^4 v^4
\end{aligned}
$$

$$
\begin{aligned}
E(q^4) =& 3\sigma^4 E(v^2) + \delta^4 + 12\theta\delta\sigma^2 E(v^2) + 6\delta^2\sigma^2 + 4\delta^3\theta + 6\theta^2\delta^2 E(v^2) + 6\theta^2\sigma^2 E(v^3) + 4\theta^3\delta E(v^3) \\
& + \theta^4 E(v^4)
\end{aligned}
$$

3 Normal Inverse Gaussian Distribution

3.1 Introduction to Normal mixtures

Normal mixtures is a mixtures of the normal distribution as the conditional distribution and another distribution as a prior (mixing) distribution.In this chapter focus has been made on the construction of Normal Inverse Gaussian distribution,based on two approaches.

1. Ole Barndoff(1977) Approach:Normal Mean variance.
   In this approach we consider the mean as a function of variance;
   $(x/\sigma^2) = (x/z)$ where $(z = \sigma^2)$
   $(x/z) \sim N(\mu + \beta z, z)$
   Therefore,

$$f(x/z) = \frac{1}{\sqrt{2\pi z}} e^{\frac{-(x-\mu-\beta v)^2}{2v}}$$

$$\begin{aligned} f(x) &= \int_0^\infty f(x/v)g(z)dz \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi v}} e^{\frac{-(x-\mu-\beta v)^2}{2v}} g(z)dz \end{aligned}$$

   we take, $\mu = 0$ and $\beta = v$

$$f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi v}} e^{\frac{-x}{2z}} g(z)dz$$

2. Saralees Nadarajah (2012):Normal- variance. In this approach we consider;

$$(y/\sigma^2) = (y/\lambda), where(\lambda = \sigma^2), and, (y/\lambda) \sim N(\mu, \lambda)$$

   $\therefore$

$$f(y/\lambda) = \frac{1}{\sqrt{2\pi\lambda}} e^{\frac{-(y-\mu)^2}{2\lambda}}$$

where, $-\infty < y > \infty$, $-\infty < \mu > \infty \; v > 0$

$$\begin{aligned} f(y) &= \int_0^\infty f(y/\lambda) g(\lambda) d\lambda \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi\lambda}} e^{\frac{-(y-\mu)^2}{2\lambda}} g(\lambda) d\lambda \end{aligned}$$

we take, $\mu = 0$ and $\frac{y^2}{2} = x$

$$f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi\lambda}} e^{\frac{-x}{\lambda}} g(\lambda) d\lambda$$

3.2 Construction of Normal inverse Gaussian distribution

3.2.1 Barndoff Nielsen approach

$$f(x) = \int_z f(x/z) g(z) \, dz$$

where,

$$f(x/z) = \frac{1}{\sqrt{2\pi z}} e^{\frac{-(x)^2}{2z}}$$

and the mixing distribution is,

$$g(z) = \frac{\sqrt{\chi}}{\sqrt{2\pi}} z^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi z + \chi \frac{1}{z}) + \sqrt{\chi\psi}}$$

therefore,

$$
\begin{aligned}
f(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi z}} e^{\frac{-(x)^2}{2z}} \sqrt{\frac{\chi}{2\pi}} z^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi z+\chi \frac{1}{z})+\sqrt{\chi\psi}} dz \\
&= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\chi}}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{z}} e^{\frac{-(x)^2}{2z}} z^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi z+\chi \frac{1}{z})+\sqrt{\chi\psi}} dz \\
&= \frac{\sqrt{\chi}}{2\pi} \int_0^\infty z^{\frac{-1}{2}} e^{\frac{-(x)^2}{2z}} z^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi z+\chi \frac{1}{z})+\sqrt{\chi\psi}} dz \\
&= \frac{\sqrt{\chi}}{2\pi} \int_0^\infty z^{-2} e^{\frac{-(x)^2}{2z}} z^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi z+\chi \frac{1}{z})+\sqrt{\chi\psi}} dz \\
&= \frac{\sqrt{\chi} e^{\sqrt{\chi\psi}}}{2\pi} \int_0^\infty z^{-1-1} e^{\frac{-(x)^2}{2z}} z^{\frac{-3}{2}} e^{\frac{-1}{2}} (\psi z + (\chi + x^2) \frac{1}{z}) dz
\end{aligned}
$$

Introducing Barndoff Nielsen (1977);GIG pdf,

$$
\frac{(\frac{\psi}{\chi})^{\frac{v}{2}}}{2K_v(\sqrt{\chi\psi})} z^{v-1} e^{\frac{-1}{2}(\psi x + \frac{\chi}{x})}
$$

substituting in the function,

$$
f(x) = \frac{\sqrt{\chi}}{2\pi} e^{\sqrt{\psi\chi}} \frac{2K_{-1}(\sqrt{(\chi+x^2)\psi})}{(\frac{\psi}{\chi+x^2})^{\frac{-1}{2}}} \int_0^\infty \frac{(\frac{\psi}{\chi+x^2})^{\frac{-1}{2}}}{2K_{-1}(\sqrt{(\chi+x^2)\psi})} z^{-1-1} e^{\frac{-1}{2}(\psi z + \frac{\chi+x^2}{z})}
$$

Since,

$$
\int_0^\infty \frac{(\frac{\psi}{\chi+x^2})^{\frac{-1}{2}}}{2K_{-1}(\sqrt{(\chi+x^2)\psi})} z^{-1-1} e^{\frac{-1}{2}(\psi z + \frac{\chi+x^2}{z})} = 1
$$

therefore,

$$
\begin{aligned}
f(x) &= \frac{\sqrt{\chi}}{2\pi} e^{\sqrt{\psi\chi}} \frac{2K_{-1}(\sqrt{(\chi+x^2)\psi})}{(\frac{\psi}{\chi+x^2})^{\frac{-1}{2}}} \\
&= \frac{\sqrt{\chi}}{\pi} e^{\sqrt{\psi\chi}} K_{-1}(\sqrt{(\chi+x^2)\psi}) (\frac{\psi}{\chi+x^2})^{\frac{1}{2}} \\
&= \frac{\sqrt{\chi}}{\pi} e^{\sqrt{\psi\chi}} K_1(\sqrt{(\chi+x^2)\psi}) \sqrt{\frac{\psi}{\chi+x^2}}
\end{aligned}
$$

3.2.2 Moments of Inverse Gaussian Distribution

First moments

$$
\begin{aligned}
E(z) &= \int_0^\infty z\sqrt{\frac{\chi}{2\pi}}z^{\frac{-3}{2}}e^{\frac{-1}{2}(\psi z+\frac{\chi}{z})}e^{\sqrt{\chi\psi}}dz \\
&= \sqrt{\frac{\chi}{2\pi}}e^{\sqrt{\chi\psi}}\int_0^\infty z^{\frac{-3}{2}+1}e^{\frac{-1}{2}(\psi z+\frac{\chi}{z})}dz \\
&= \sqrt{\frac{\chi}{2\pi}}e^{\sqrt{\chi\psi}}\int_0^\infty z^{\frac{-1}{2}}e^{\frac{-1}{2}(\psi z+\frac{\chi}{z})}dz \\
&= \sqrt{\frac{\chi}{2\pi}}e^{\sqrt{\chi\psi}}\int_0^\infty z^{\frac{1}{2}-1}e^{\frac{-1}{2}(\psi z+\frac{\chi}{z})}dz
\end{aligned}
$$

Recall,

$$
\int_0^\infty x^{\nu-1}e^{\frac{-1}{2}(\psi x+\frac{\chi}{x})}\,dx = \frac{2K_\nu(\sqrt{\chi\psi})}{(\frac{\chi}{\psi})^{\frac{\nu}{2}}}
$$

hence,

$$
\int_0^\infty z^{\frac{1}{2}-1}e^{\frac{-1}{2}(\psi z+\frac{\chi}{z})} = \frac{2K_{\frac{1}{2}}(\sqrt{\chi\psi})}{(\frac{\chi}{\psi})^{\frac{1}{4}}}
$$

Hankel function ,equation 5

$$
K_{\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}}e^{-w}
$$

$$
K_{\frac{1}{2}}(\sqrt{\chi\psi}) = \sqrt{\frac{\pi}{2\sqrt{\chi\psi}}}e^{-\sqrt{\chi\psi}}
$$

Substitute in the function,

$$
\begin{aligned}
E(z) &= \sqrt{\frac{\chi}{2\pi}}e^{\sqrt{\chi\psi}}\frac{2\sqrt{\frac{\pi}{2\sqrt{\chi\psi}}}}{(\frac{\chi}{\psi})^{\frac{1}{4}}}e^{-\sqrt{\chi\psi}} \\
&= \frac{\sqrt{\chi}}{(\frac{\chi\psi}{\frac{\chi}{\psi}})^{\frac{1}{4}}} \\
&= \sqrt{\frac{\chi}{\psi}}
\end{aligned}
$$

Second Moments

$$
\begin{aligned}
E(z^2) &= \int_0^\infty z^2 \sqrt{\frac{\chi}{2\pi}} z^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi z + \frac{\chi}{z})} e^{\sqrt{\chi\psi}} dz \\
&= \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \int_0^\infty z^{\frac{-3}{2}+2} e^{\frac{-1}{2}(\psi z + \frac{\chi}{z})} dz \\
&= \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \int_0^\infty z^{\frac{1}{2}} e^{\frac{-1}{2}(\psi z + \frac{\chi}{z})} dz \\
&= \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \int_0^\infty z^{\frac{3}{2}-1} e^{\frac{-1}{2}(\psi z + \frac{\chi}{z})} dz
\end{aligned}
$$

hence,

$$
\int_0^\infty z^{\frac{3}{2}-1} e^{\frac{-1}{2}(\psi z + \frac{\chi}{z})} = \frac{2K_{\frac{3}{2}}(\sqrt{\chi\psi})}{(\frac{\chi}{\psi})^{\frac{3}{4}}}
$$

substutute in the function,

$$
E(z^2) = \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \frac{2K_{\frac{3}{2}}(\sqrt{\chi\psi})}{(\frac{\chi}{\psi})^{\frac{3}{4}}}
$$

Hankel function ,equation,6

$$
K_{\frac{3}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w}(1 + \frac{1}{w})
$$

$$
K_{\frac{3}{2}}(\sqrt{\chi\psi}) = \sqrt{\frac{\pi}{2\sqrt{\chi\psi}}} e^{-\sqrt{\chi\psi}}(1 + \frac{1}{\sqrt{\chi\psi}})
$$

substitute to the function,

$$
\begin{aligned}
E(z^2) &= \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \frac{2\sqrt{\frac{\pi}{2\sqrt{\chi\psi}}}}{(\frac{\chi}{\psi})^{\frac{3}{4}}} e^{-\sqrt{\chi\psi}}(1 + \frac{1}{\sqrt{\chi\psi}}) \\
&= \frac{\sqrt{\chi}}{\frac{(\chi\psi)^{\frac{1}{4}}}{\frac{\chi}{\psi}^{\frac{3}{4}}}}(1 + \frac{1}{\sqrt{\chi\psi}}) \\
&= \psi(1 + \frac{1}{\sqrt{\chi\psi}})
\end{aligned}
$$

Third moment

$$
\begin{aligned}
E(z^3) = &\int_0^\infty z^3 \sqrt{\frac{\chi}{2\pi}} z^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi z + \frac{\chi}{z})} e^{\sqrt{\chi\psi}} dz \\
= &\sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \int_0^\infty z^{\frac{-3}{2}+3} e^{\frac{-1}{2}(\psi z + \frac{\chi}{z})} dz \\
= &\sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \int_0^\infty z^{\frac{3}{2}} e^{\frac{-1}{2}(\psi z + \frac{\chi}{z})} dz \\
= &\sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \int_0^\infty z^{\frac{5}{2}-1} e^{\frac{-1}{2}(\psi z + \frac{\chi}{z})} dz
\end{aligned}
$$

hence,

$$
\int_0^\infty z^{\frac{5}{2}-1} e^{\frac{-1}{2}(\psi z + \frac{\chi}{z})} = \frac{2K_{\frac{5}{2}}(\sqrt{\chi\psi})}{(\frac{\chi}{\psi})^{\frac{5}{4}}}
$$

Substitute in the function,

$$
E(z^3) = \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \frac{2K_{\frac{5}{2}}(\sqrt{\chi\psi})}{(\frac{\chi}{\psi})^{\frac{5}{4}}}
$$

Hankel function,equation,4,

$$
K_{j+\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w} \{1 + \sum_{i=1}^{j} \frac{(j+1)!(2w)^{-i}}{(j-i)!i!}\}
$$

let,$j = 2$,

$$
\begin{aligned}
K_{2+\frac{1}{2}}(w) = &\sqrt{\frac{\pi}{2w}} e^{-w} \{1 + \sum_{i=1}^{2} \frac{(2+1)!(2w)^{-i}}{(2-i)!i!}\} \\
= &\sqrt{\frac{\pi}{2w}} e^{-w} [1 + \{\frac{3!}{1!1!(2w)} + \frac{4!}{0!2!(2w)^2}\}] \\
= &\sqrt{\frac{\pi}{2w}} e^{-w} [1 + \{\frac{6}{2w} + \frac{6}{(2w)^2}\}]
\end{aligned}
$$

therefore,

$$
K_{\frac{5}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w} (1 + \{\frac{3}{w} + \frac{3}{(w)^2}\})
$$

substituting $w = \sqrt{\chi\psi}$,

$$K_{\frac{5}{2}}(\sqrt{\chi\psi}) = \sqrt{\frac{\pi}{2\sqrt{\chi\psi}}}e^{-\sqrt{\chi\psi}}[1+\{\frac{3}{\sqrt{\chi\psi}}+\frac{3}{\chi\psi}\}]$$

substituting back to the equation:,

$$\begin{aligned} E(Z^3) =& \sqrt{\frac{\chi}{2\pi}}e^{\sqrt{\chi\psi}}2\frac{\sqrt{\frac{\pi}{2\sqrt{\chi\psi}}}}{(\frac{\chi}{\psi})^{\frac{5}{4}}}e^{-\sqrt{\chi\psi}}[1+\{\frac{3}{\sqrt{\chi\psi}}+\frac{3}{\chi\psi}\}] \\ =& \frac{\sqrt{\chi}}{\frac{(\chi\psi)^{\frac{1}{4}}\psi^{\frac{5}{4}}}{\chi^{\frac{5}{4}}}}[1+\{\frac{3}{\sqrt{\chi\psi}}+\frac{3}{\chi\psi}\}] \\ =& (\frac{\chi}{\psi})^{\frac{3}{2}}[1+\{\frac{3}{\sqrt{\chi\psi}}+\frac{3}{\chi\psi}\}] \end{aligned}$$

Fourth moment

$$\begin{aligned} E(z^4) =& \int_0^\infty z^4\sqrt{\frac{\chi}{2\pi}}z^{\frac{-3}{2}}e^{\frac{-1}{2}(\psi z+\frac{\chi}{z})}e^{\sqrt{\chi\psi}}dz \\ =& \sqrt{\frac{\chi}{2\pi}}e^{\sqrt{\chi\psi}}\int_0^\infty z^{\frac{-3}{2}+4}e^{\frac{-1}{2}(\psi z+\frac{\chi}{z})}dz \\ =& \sqrt{\frac{\chi}{2\pi}}e^{\sqrt{\chi\psi}}\int_0^\infty z^{\frac{5}{2}}e^{\frac{-1}{2}(\psi z+\frac{\chi}{z})}dz \\ =& \sqrt{\frac{\chi}{2\pi}}e^{\sqrt{\chi\psi}}\int_0^\infty z^{\frac{7}{2}-1}e^{\frac{-1}{2}(\psi z+\frac{\chi}{z})}dz \end{aligned}$$

hence,

$$\int_0^\infty z^{\frac{7}{2}-1}e^{\frac{-1}{2}(\psi z+\frac{\chi}{z})} = \frac{2K_{\frac{7}{2}}(\sqrt{\chi\psi})}{(\frac{\chi}{\psi})^{\frac{7}{4}}}$$

substutute in the function,

$$E(z^4) = \sqrt{\frac{\chi}{2\pi}}e^{\sqrt{\chi\psi}}\frac{2K_{\frac{7}{2}}(\sqrt{\chi\psi})}{(\frac{\chi}{\psi})^{\frac{7}{4}}}$$

Hankel function,equation ,4,

$$K_{j+\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}}e^{-w}(1+\sum_{i=1}^{j}\frac{(j+1)!(2w)^{-i}}{(j-i)!i!})$$

let,$j = 3$,

$$\begin{aligned}
K_{3+\frac{1}{2}}(w) =&\sqrt{\frac{\pi}{2w}}e^{-w}(1+\sum_{i=1}^{3}\frac{(3+1)!(2w)^{-i}}{(3-i)!i!}) \\
=&\sqrt{\frac{\pi}{2w}}e^{-w}(1+\{\frac{4!}{2!1!(2w)}+\frac{5!}{1!2!(2w)^2}+\frac{6!}{0!3!(2w)^3}\}) \\
=&\sqrt{\frac{\pi}{2w}}e^{-w}(1+\{\frac{6}{w}+\frac{15}{(w)^2}+\frac{15}{(w)^3}\})
\end{aligned}$$

substituting $w = \sqrt{\chi\psi}$,

$$K_{\frac{7}{2}}(\sqrt{\chi\psi}) = \sqrt{\frac{\pi}{2\sqrt{\chi\psi}}}e^{-\sqrt{\chi\psi}}[1+\{\frac{3}{\sqrt{\chi\psi}}+\frac{3}{\chi\psi}\}]$$

substituting back to the equation:,

$$\begin{aligned}
E(Z^4) =&\sqrt{\frac{\chi}{2\pi}}e^{\sqrt{\chi\psi}}2\frac{\sqrt{\frac{\pi}{2\sqrt{\chi\psi}}}}{(\frac{\chi}{\psi})^{\frac{7}{4}}}e^{-\sqrt{\chi\psi}}[1+\{\frac{6}{\sqrt{\chi\psi}}+\frac{15}{\chi\psi}+\frac{15}{(\sqrt{\chi\psi})^3}\}] \\
=&\frac{\sqrt{\chi}}{\frac{(\chi\psi)^{\frac{1}{4}}}{(\frac{\chi}{\psi})^{\frac{7}{4}}}}[1+\{\frac{6}{\sqrt{\chi\psi}}+\frac{15}{\chi\psi}+\frac{15}{(\sqrt{\chi\psi})^3}\}] \\
=&(\frac{\chi}{\psi})^2[1+\{\frac{6}{\sqrt{\chi\psi}}+\frac{15}{\chi\psi}+\frac{15}{(\sqrt{\chi\psi})^3}\}]
\end{aligned}$$

3.2.3 Convolution Property

Convolution property is one of the properties of NIG where it is said to be closed under convolution.
that is,

$$NIG(x_1,\phi_1,\mu) * NIG(x_2,\phi_2,\mu) = NIG(x_1+x_2,\phi_1+\phi_2,\mu)$$

let

$$Y = (x_1 + x_2)$$

based on MGF technique,

$$M_y(t) = M_{x_1}(t) * M_{x_2}(t)$$

Proof.

$$\begin{aligned}
M_x(t) =& E[e^{tx}] \\
=& EE[e^{\frac{tx}{\lambda}}] \\
=& E[e^{\frac{tx}{\lambda}}] \\
=& e^{\mu t + \frac{1}{2}\lambda t^2} \\
=& \int_0^\infty e^{\mu t + \frac{1}{2}\lambda t^2} g(\lambda) d\lambda
\end{aligned}$$

$let, \mu = 0$

$$\begin{aligned}
M_x(t) &= \int_0^\infty e^{\frac{1}{2}\lambda t^2} g(\lambda) d\lambda \\
&= \int_0^\infty e^{\frac{1}{2}\lambda t^2} \sqrt{\frac{\chi}{2\pi}} \lambda^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi\lambda + \frac{\chi}{\lambda})} e^{\sqrt{\chi\psi}} d\lambda \\
&= \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \int_0^\infty e^{\frac{1}{2}\lambda t^2} \lambda^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi\lambda + \frac{\chi}{\lambda})} d\lambda \\
&= \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \int_0^\infty \lambda^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi\lambda + \frac{\chi}{\lambda}) + \frac{1}{2}\lambda t^2} d\lambda \\
&= \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \int_0^\infty \lambda^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi\lambda + \frac{\chi}{\lambda} - \lambda t^2)} d\lambda \\
&= \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \int_0^\infty \lambda^{\frac{-3}{2}} e^{\frac{-1}{2}((\psi - t^2)\lambda + \frac{\chi}{\lambda})} d\lambda \\
&= \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} \int_0^\infty \lambda^{\frac{-3}{2}} e^{\frac{-(\psi - t^2)}{2}(\lambda + \frac{\chi}{(\psi - t^2)\lambda})} d\lambda
\end{aligned}$$

let,

$$\lambda = \sqrt{\frac{\chi}{(\psi - t^2)}} z, \Rightarrow d\lambda = \sqrt{\frac{\chi}{(\psi - t^2)}} dz$$

$$=\sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi}\psi} \int_0^\infty (\sqrt{\frac{\chi}{(\psi - t^2)}} z)^{\frac{-3}{2}} e^{\frac{-(\psi - t^2)}{2}(\sqrt{\frac{\chi}{(\psi - t^2)}} z + \frac{\chi}{(\psi - t^2)(\sqrt{\frac{\chi}{(\psi - t^2)}} z})} \sqrt{\frac{\chi}{(\psi - t^2)}} dz$$

$$=\sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi}\psi} \int_0^\infty (\sqrt{\frac{\chi}{(\psi - t^2)}})^{\frac{-1}{2}} z^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi - t^2)(\sqrt{\frac{\chi}{4 - t^2}})(z + \frac{1}{z})} dz$$

$$=\sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi}\psi} \int_0^\infty (\sqrt{\frac{\chi}{(\psi - t^2)}})^{\frac{-1}{2}} z^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi - t^2)(\sqrt{\frac{\chi}{4 - t^2}})(z + \frac{1}{z})} dz$$

$$=2\sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi}\psi} (\sqrt{\frac{(\psi - t^2)}{\chi}})^{\frac{1}{2}} \frac{1}{2} \int_0^\infty z^{\frac{-3}{2}} e^{\frac{-\sqrt{(\psi - t^2)\chi}}{2}(z + \frac{1}{z})} dz$$

$$=2\sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi}\psi} (\sqrt{\frac{(\psi - t^2)}{\chi}})^{\frac{1}{2}} \frac{1}{2} K_{\frac{-1}{2}}(\sqrt{\chi(\psi - t^2)})$$

$$=2\sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi}\psi} (\sqrt{\frac{(\psi - t^2)}{\chi}})^{\frac{1}{2}} \frac{1}{2} K_{\frac{1}{2}}(\sqrt{\chi(\psi - t^2)})$$

hankel property,equation5

$$K_{\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w}$$

$$K_{\frac{1}{2}}(\sqrt{\chi(\psi - t^2)}) = \sqrt{\frac{\pi}{2\sqrt{\chi(\psi - t^2)}}} e^{-\sqrt{\chi(\psi - t^2)}}$$

substitute in the equation,

$$=2\sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi}\psi} (\sqrt{\frac{(\psi - t^2)}{\chi}})^{\frac{1}{2}} \frac{1}{2} \sqrt{\frac{\pi}{2\sqrt{\chi(\psi - t^2)}}} e^{-\sqrt{\chi(\psi - t^2)}}$$

$$=e^{\sqrt{\chi\psi} - \sqrt{\chi(\psi - t^2)}}$$

$$=e^{\sqrt{\chi}(\sqrt{\psi} - \sqrt{\psi - t^2})}$$

proof for convolution property,

$$\begin{aligned}
M_y(t) =& E[e^{tx_1}] * E[e^{tx_2}] \\
=& M_{x_1}(t) * M_{x_1}(t) \\
=& e^{\sqrt{\chi_1}(\sqrt{\psi}-\sqrt{\psi - t^2})} * e^{\sqrt{\chi_2}(\sqrt{\psi}-\sqrt{\psi - t^2})} \\
=& e^{\sqrt{\chi_1 + \chi_2}(\sqrt{\psi}-\sqrt{\psi - t^2})}
\end{aligned}$$

□

3.3 Construction of NIG

3.3.1 Sareless Nadarajah's Approach

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi\lambda}} exp(\frac{-x}{\lambda})\sqrt{\frac{\mu\phi}{2\pi}}\lambda^{\frac{-3}{2}} exp\{(\frac{-\phi}{2})(\frac{\lambda}{\mu}+\frac{\mu}{\lambda})\}d\lambda \\
&= \frac{e^{\phi}\sqrt{\mu\phi}}{\sqrt{2\pi}\sqrt{2\pi}}\int_0^\infty \lambda^{\frac{-1}{2}} exp(\frac{-x}{\lambda})\lambda^{\frac{-3}{2}} exp\{(\frac{-\phi}{2})(\frac{\lambda}{\mu}+\frac{\mu}{\lambda})\}d\lambda \\
&= \frac{e^{\phi}\sqrt{\mu\phi}}{2\pi}\int_0^\infty \lambda^{-}2exp\{(\frac{-x}{\lambda})-(\frac{-\phi}{2})(\frac{\lambda}{\mu}+\frac{\mu}{\lambda})\}d\lambda \\
&= \frac{e^{\phi}\sqrt{\mu\phi}}{2\pi}\int_0^\infty \lambda^{-}2exp\{(\frac{-1}{2})(\frac{\phi}{\mu}\lambda+\frac{\phi}{\lambda}\mu+\frac{2x}{\lambda})\}d\lambda \\
&= \frac{e^{\phi}\sqrt{\mu\phi}}{2\pi}\int_0^\infty \lambda^{-}2exp\{(\frac{-1}{2})(\frac{\phi}{\mu}\lambda+\frac{\phi\mu+2x}{\lambda})\}d\lambda
\end{aligned}$$

Introducing Jϕrgansen(1982)concept to the integral part of the equation below,

$$f(x) = \frac{(\frac{\psi}{\chi})^{\frac{\lambda}{2}}}{2K_\lambda(\sqrt{\chi\psi})} x^{\lambda-1} exp\{(\frac{-1}{2})(\frac{\psi}{x}+x\psi)\}$$

$$Let, \frac{\phi}{\mu} = \psi, and, \phi\mu + 2x = \chi$$

Therefore,

$$f(x) = \frac{e^{\phi}\sqrt{\mu\phi}}{2\pi}\int_0^\infty \lambda^{-2} exp\{(\frac{-1}{2})(\psi\lambda+\frac{\chi}{\lambda})\}$$

Representing everything to take the form of modified bessel function ,

$$= \frac{e^{\phi}\sqrt{\mu\phi}}{2\pi}\frac{2k_{-}1(\sqrt{\psi\chi})}{(\frac{\psi}{\chi})^{\frac{-1}{2}}}\int_{0}^{\infty}\frac{(\psi\chi)^{\frac{-1}{2}}}{2K_{\frac{-1}{2}}(\sqrt{\chi\psi})}\lambda^{(-1-1)}exp\{(\frac{-1}{2})(\psi\lambda+\frac{\chi}{\lambda})\}d\lambda$$

Thus,

$$= \int_{0}^{\infty}\frac{(\psi\chi)^{\frac{-1}{2}}}{2K_{\frac{-1}{2}}(\sqrt{\chi\psi})}\lambda^{(-1-1)}exp\{(\frac{-1}{2})(\psi\lambda+\frac{\chi}{\lambda})\}d\lambda = 1$$

. since it is a pdf.

Hence,replacing back ψ and χ to their original variable representation we have

$$\begin{aligned} f(x) =& \frac{e^{\phi}\sqrt{\mu\phi}}{2\pi}\frac{2K_{-1}(\sqrt{\mu\phi+\frac{\phi}{\mu}2x})}{(\frac{\frac{\phi}{\mu}}{\mu\phi+2x})^{\frac{-1}{2}}} \\ =& \frac{\sqrt{\mu\phi}e^{\phi}}{\pi}\frac{\sqrt{\frac{\phi}{\mu}}}{\sqrt{\mu\phi+2x}}K_{-1}(\sqrt{(\mu\phi+2x)(\frac{\phi}{\mu})}) \\ =& \frac{e^{\phi}}{\pi}\sqrt{\frac{(\mu\phi)(\frac{\phi}{\mu})}{\mu\phi+2x}}K_{-1}(\sqrt{(\mu\phi+2x)(\frac{\phi}{\mu})}) \\ =& \frac{\phi e^{\phi}}{\pi\sqrt{\mu\phi+2x}}K_{-1}(\sqrt{\phi(\phi+\frac{2x}{\mu})}) \end{aligned}$$

3.3.2 Moments of IG

First moments

$$E[\lambda] = \int_0^\infty \lambda \sqrt{\frac{\mu\phi}{2\pi}} e^\phi \lambda^{\frac{-3}{2}} exp\{\frac{-\phi}{2}(\frac{\phi}{\mu}+\frac{\mu}{\lambda})\} d\lambda$$
$$= \sqrt{\frac{\mu\phi}{2\pi}} e^\phi \int_0^\infty \lambda^{\frac{-1}{2}} exp\{\frac{-1}{2}(\frac{\phi}{\mu}\lambda+\frac{\mu\phi}{\lambda})\} d\lambda$$

Introducing Jϕrgensen(1982)pdf,with intergral representation;

$$\int_0^\infty x^{v-1} e^{\frac{-1}{2}(\psi x+\chi x^{-1})} = 2K_v(\sqrt{\chi\psi})(\frac{\chi}{\psi})^{\frac{v}{2}}$$

let,$v = \frac{1}{2}, \frac{\phi}{\mu} = \psi$ and $\mu\phi = \chi$

therefore;

$$E[\lambda] = \sqrt{\frac{\mu\phi}{2\pi}} e^\phi \int_0^\infty \lambda^{\frac{1}{2}-1} exp\{\frac{-1}{2}(\psi\lambda+\chi\lambda^{-1}\} d\lambda$$
$$= \sqrt{\frac{\mu\phi}{2\pi}} e^\phi 2K_{\frac{1}{2}}(\sqrt{\chi\psi})(\frac{\chi}{\psi})^{\frac{\frac{1}{2}}{2}}$$

replacing back the original parameters;

$$= \sqrt{\frac{\mu\phi}{2\pi}} e^\phi 2K_{\frac{1}{2}}(\sqrt{(\mu\phi)(\frac{\phi}{\mu})})(\frac{\phi\mu}{\frac{\phi}{\mu}})^{\frac{1}{4}}$$
$$= \sqrt{\frac{\mu\phi}{2\pi}} e^\phi 2K_{\frac{1}{2}}(\phi)\mu^{\frac{1}{2}}$$

recall the equation(5)

$$K_{\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}}e^{-w}$$

thus;

$$K_{\frac{1}{2}}(\phi) = \sqrt{\frac{\pi}{2\phi}}e^{-\phi}$$

substitute to the equation;

$$\begin{aligned}
E(\lambda) =& \frac{\sqrt{\mu}\sqrt{\phi}}{\sqrt{2}\sqrt{\pi}}e^{\phi}\frac{2\frac{\sqrt{\pi}}{\sqrt{2}\sqrt{\pi}}}{\sqrt{\frac{1}{\mu}}}e^{-\phi} \\
=& \sqrt{\frac{\mu}{\frac{1}{\mu}}} = \sqrt{(\mu)^2} = \mu \\
=& \mu
\end{aligned}$$

Second Moments

$$\begin{aligned}
E(\lambda^2) =& \int_0^{\infty} \lambda^2 \sqrt{\frac{\mu\phi}{2\pi}}\lambda^{\frac{-3}{2}} e^{\frac{-\phi}{2}(\frac{\lambda}{\mu}+\frac{\mu}{\mu})} e^{\phi} d\lambda \\
=& \sqrt{\frac{\mu\phi}{2\pi}}e^{\phi}\int_0^{\infty} \lambda^{\frac{-3}{2}+2} e^{\frac{-1}{2}(\frac{\phi}{\mu}\lambda+\frac{\phi\mu}{\lambda})} d\lambda \\
=& \sqrt{\frac{\phi\mu}{2\pi}}e^{\phi}\int_0^{\infty} \lambda^{\frac{1}{2}} e^{\frac{-1}{2}(\frac{\phi}{\mu}\lambda+\frac{\phi\mu}{\lambda}} d\lambda \\
=& \sqrt{\frac{\mu\phi}{2\pi}}e^{\phi}\int_0^{\infty} \lambda^{\frac{3}{2}-1} e^{\frac{-1}{2}((\frac{\phi}{\mu}\lambda+\frac{\phi\mu}{\lambda}} d\lambda
\end{aligned}$$

hence,

$$\int_0^{\infty} \lambda^{\frac{3}{2}-1} e^{\frac{-1}{2}(\frac{\phi}{\mu}\lambda+\frac{\phi\mu}{\lambda})} d\lambda = \frac{2K_{\frac{3}{2}}(\sqrt{\frac{\phi}{\mu}*\phi\mu})}{(\frac{\frac{\phi}{\mu}}{\mu\phi})^{\frac{3}{4}}}$$

substutute in the function,

$$E(\lambda^2) = \sqrt{\frac{\mu\phi}{2\pi}} e^{\phi} \frac{2K_{\frac{3}{2}}(\phi)}{(\frac{\frac{\phi}{\mu}}{\mu\phi})^{\frac{3}{4}}}$$

Hankel function ;equation6.

$$K_{\frac{3}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w}(1 + \frac{1}{w})$$

$$K_{\frac{3}{2}}(\phi) = \sqrt{\frac{\pi}{2\phi}} e^{-\phi}(1 + \frac{1}{\phi})$$

substituting back to the equation,

$$\begin{aligned} E(\lambda^2) =& \frac{\sqrt{\mu}\sqrt{\phi}}{\sqrt{2}\sqrt{\pi}} e^{\phi} 2 \frac{\frac{\sqrt{\pi}}{\sqrt{2}\sqrt{\phi}}}{\frac{1}{\mu}^{\frac{3}{2}}} e^{\phi}(1 + \frac{1}{\phi}) \\ =& \mu^2(1 + \frac{1}{\phi}) \end{aligned}$$

Third moment

$$
\begin{aligned}
E(\lambda^3) &= \int_0^\infty \lambda^3 \sqrt{\frac{\mu\phi}{2\pi}} \lambda^{\frac{-3}{2}} e^{\frac{-\phi}{2}(\frac{\lambda}{\mu}+\frac{\mu}{\lambda})} d\lambda \\
&= \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \int_0^\infty \lambda^{\frac{-3}{2}+3} e^{\frac{-1}{2}(\frac{\lambda}{\mu}+\frac{\mu}{\lambda})} d\lambda \\
&= \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \int_0^\infty \lambda^{\frac{3}{2}} e^{\frac{-1}{2}(\frac{\phi\lambda}{\mu}+\frac{\phi\mu}{\lambda})} d\lambda \\
&= \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \int_0^\infty \lambda^{\frac{5}{2}-1} e^{\frac{-1}{2}(\frac{\phi\lambda}{\mu}+\frac{\phi\mu}{\lambda})} d\lambda
\end{aligned}
$$

thus,

$$
\int_0^\infty \lambda^{\frac{5}{2}-1} e^{\frac{-1}{2}(\frac{\phi\lambda}{\mu}+\frac{\phi\mu}{\lambda})} d\lambda = \frac{2K_{\frac{5}{2}}(\phi)}{(\frac{\frac{\phi}{\mu}}{\phi\mu})^{\frac{5}{4}}}
$$

substutute in the function,

$$
E(\lambda^3) = \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \frac{2K_{\frac{5}{2}}(\phi)}{(\frac{\frac{\phi}{\mu}}{\phi\mu})^{\frac{5}{4}}}
$$

Hankel function ,equation(4),

$$
K_{j+\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w} (1 + \sum_{i=1}^{j} \frac{(j+1)!(2w)^{-i}}{(j-i)!i!})
$$

let, $j = 2$,

$$K_{2+\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w}\{1 + \sum_{i=1}^{2} \frac{(2+1)!(2w)^{-i}}{(2-i)!i!}\}$$
$$= \sqrt{\frac{\pi}{2w}} e^{-w}[1 + \{\frac{3!}{1!1!(2w)} + \frac{4!}{0!2!(2w)^2}\}]$$
$$= \sqrt{\frac{\pi}{2w}} e^{-w}[1 + \{\frac{6}{2w} + \frac{6}{(2w)^2}\}]$$

therefore,

$$K_{\frac{5}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w}(1 + \{\frac{3}{w} + \frac{3}{(w)^2}\})$$

substituting $w = \phi$,

$$K_{\frac{5}{2}}(\phi) = \sqrt{\frac{\pi}{2\phi}} e^{-\phi}(1 + \{\frac{3}{\phi} + \frac{3}{\phi^2}\})$$

substituting back to the equation:,

$$E(\lambda^3) = \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} 2 \frac{\sqrt{\frac{\pi}{2\phi}}}{(\frac{1}{\phi})^{\frac{5}{2}}} e^{-\phi}(1 + \{\frac{3}{\phi} + \frac{3}{\phi^2}\})$$
$$= \frac{\sqrt{\mu}\sqrt{\phi}}{\sqrt{2}\sqrt{\pi}} e^{\phi} 2 \frac{\sqrt{\pi}}{\sqrt{2}\sqrt{\phi}} e^{-\phi}(1 + \{\frac{3}{\phi} + \frac{3}{\phi^2}\})\mu^{\frac{5}{2}}$$
$$= \mu^3(1 + \{\frac{3}{\phi} + \frac{3}{\phi^2}\})$$

Fourth moment

$$\begin{aligned}E(\lambda^4) =& \int_0^\infty \lambda^4 \sqrt{\frac{\phi\mu}{2\pi}} \lambda^{\frac{-3}{2}} e^{\frac{-1}{2}(\frac{\phi\lambda}{\mu}+\frac{\phi\mu}{\lambda})} e^{\phi} d\lambda \\ =& \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \int_0^\infty \lambda^{\frac{-3}{2}+4} e^{\frac{-1}{2}(\frac{\phi\lambda}{\mu}+\frac{\phi\mu}{\lambda})} d\lambda \\ =& \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \int_0^\infty \lambda^{\frac{5}{2}} e^{\frac{-1}{2}(\frac{\phi\lambda}{\mu}+\frac{\phi\mu}{\lambda})} d\lambda \\ =& \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \int_0^\infty \lambda^{\frac{7}{2}-1} e^{\frac{-1}{2}(\frac{\phi\lambda}{\mu}+\frac{\phi\mu}{\lambda})} d\lambda\end{aligned}$$

hence,

$$\int_0^\infty \lambda^{\frac{7}{2}-1} e^{\frac{-1}{2}(\frac{\phi\lambda}{\mu}+\frac{\phi\mu}{\lambda})} = \frac{2K_{\frac{7}{2}}(\phi)}{(\frac{\frac{\phi}{\mu}}{\phi\mu})^{\frac{7}{4}}}$$

substutute in the function,

$$E(\lambda^4) = \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \frac{2K_{\frac{7}{2}}(\phi)}{(\frac{\frac{\phi}{\mu}}{\phi\mu})^{\frac{7}{4}}}$$

Hankel function property 2:,

$$K_{j+\frac{1}{2}}(w) = \sqrt{\frac{\pi}{2w}} e^{-w} (1 + \sum_{i=1}^{j} \frac{(j+1)!(2w)^{-i}}{(j-i)!i!})$$

let, $j=3$,

$$
\begin{aligned}
K_{3+\frac{1}{2}}(w) =& \sqrt{\frac{\pi}{2w}}e^{-w}\{1+\sum_{i=1}^{3}\frac{(3+1)!(2w)^{-i}}{(3-i)!i!}\} \\
=& \sqrt{\frac{\pi}{2w}}e^{-w}(1+\{\frac{4!}{2!1!(2w)}+\frac{5!}{1!2!(2w)^2}+\frac{6!}{0!3!(2w)^3}\}) \\
=& \sqrt{\frac{\pi}{2w}}e^{-w}(1+\{\frac{6}{w}+\frac{15}{(w)^2}+\frac{15}{(w)^3}\})
\end{aligned}
$$

substituting $w=\phi$,

$$
K_{\frac{7}{2}}(\phi) = \sqrt{\frac{\pi}{2\phi}}e^{-\phi}(1+\{\frac{6}{\phi}+\frac{15}{\phi^2}+\frac{15}{(\phi)^3}\})
$$

substituting back to the equation:,

$$
\begin{aligned}
E(\lambda^4) =& \sqrt{\frac{\phi\mu}{2\pi}}e^{\phi}2\frac{\sqrt{\frac{\pi}{2\phi}}}{(\frac{\frac{\phi}{\mu}}{\phi\mu})^{\frac{7}{4}}}e^{-\phi}(1+\{\frac{6}{\phi}+\frac{15}{\phi^2}+\frac{15}{\phi^3}\}) \\
=& \frac{\sqrt{\mu}\sqrt{\phi}}{\sqrt{2}\sqrt{\pi}}e^{\phi}2\frac{\sqrt{\pi}}{\sqrt{2}\sqrt{\phi}}e^{\phi}(1+\{\frac{6}{\phi}+\frac{15}{\phi^2}+\frac{15}{\phi^3}\})\mu^{\frac{7}{2}} \\
=& \mu^4(1+\{\frac{6}{\phi}+\frac{15}{\phi^2}+\frac{15}{\phi^3}\})
\end{aligned}
$$

3.3.3 Convolution property

Convolution property is one of the properties of NIG where it is said to be closed under convolution.
that is,

$$
NIG(x_1,\phi_1,\mu)*NIG(x_2,\phi_2,\mu) = NIG(x_1+x_2,\phi_1+\phi_2,\mu)
$$

$$letY = (x_1 + x_2)$$

based on MGF technique,

$$M_y(t) = M_{x_1}(t) * M_{x_2}(t)$$

Proof.

$$\begin{aligned} M_x(t) =& E[e^{tx}] \\ =& EE[e^{\frac{tx}{\lambda}}] \\ =& E[e^{\frac{tx}{\lambda}}] \\ =& e^{\mu t + \frac{1}{2}\lambda t^2} \\ =& \int_0^\infty e^{\mu t + \frac{1}{2}\lambda t^2} g(\lambda) d\lambda \end{aligned}$$

$$let, \mu = 0$$

$$\begin{aligned} M_x(t) &= \int_0^\infty e^{\frac{1}{2}\lambda t^2} g(\lambda) d\lambda \\ &= \int_0^\infty e^{\frac{1}{2}\lambda t^2} \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \lambda^{\frac{-3}{2}} e^{\frac{-\phi}{2}(\frac{\lambda}{\mu} + \frac{\mu}{\lambda})} d\lambda \\ &= \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \int_0^\infty e^{\frac{1}{2}\lambda t^2} \lambda^{\frac{-3}{2}} e^{\frac{-1}{2}(\frac{\phi\lambda}{\mu} + \frac{\phi\mu}{\lambda})} d\lambda \\ &= \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \int_0^\infty \lambda^{\frac{-3}{2}} e^{\frac{-1}{2}(\frac{\phi\lambda}{\mu} + \frac{\phi\mu}{\lambda} + \frac{1}{2}\lambda t^2)} d\lambda \\ &= \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \int_0^\infty \lambda^{\frac{-3}{2}} e^{\frac{-1}{2}((\frac{\phi}{\mu} - t^2)\lambda + \frac{\phi\mu}{\lambda})} d\lambda \end{aligned}$$

Jϕgernsen ,IG pdf takes the form,

$$\int_0^\infty \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} x^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi x+\chi x^{-1})} = 1$$

$$x^{\frac{-3}{2}} e^{\frac{-1}{2}(\psi x+\chi x^{-1})} = \sqrt{\frac{2\pi}{\chi}} e^{-\sqrt{\chi\psi}}$$

$$let, \chi = \phi\mu, and, \sqrt{\chi\psi} = \sqrt{(\frac{\phi}{\mu} - t^2)\phi\mu}$$

Substitute back to the equation,

$$\begin{aligned}
&= \sqrt{\frac{\phi\mu}{2\pi}} e^{\phi} \sqrt{\frac{2\pi}{\phi\mu}} e^{-\sqrt{(\frac{\phi}{\mu}-t^2)\phi\mu}} \\
&= e^{\phi} e^{-\sqrt{(\frac{\phi}{\mu}-t^2)\phi\mu}} \\
&= e^{\phi-\sqrt{\phi(\phi-\mu t^2)}}
\end{aligned}$$

hence,

$$\begin{aligned}
M_y(t) &= M_{x_1}(t) * M_{x_2}(t) \\
&= e^{\phi_1-\sqrt{\phi_1(\phi_1-\mu t^2)}} * e^{\phi_2-\sqrt{\phi_2(\phi_2-\mu t^2)}} \\
&= e^{(\phi_1+\phi_2)-\sqrt{\phi_1(\phi_1-\mu t^2)}-\sqrt{\phi_2(\phi_2-\mu t^2)}}
\end{aligned}$$

This approach does not clearly show the concept of convolution property.

□

4 Variance Gamma

4.1 Introduction

In this chapter construction and properties of Gamma is considered.Variance Gamma is a mixture of normal distribution as the conditional distribution and Gamma distribution as the prior diatribution. The construction is based on two approaches;-

1. Ole Barndoff approach.

$$
\begin{aligned}
f(x) &= \int_{\theta} f(x/\theta)g(\theta)\, d\theta \\
&= \int_{0}^{\infty} f(x/z)g(z)dz
\end{aligned}
$$

where,

$$
f(x/z) = \frac{1}{\sqrt{2\pi z}} e^{\frac{-x^2}{2z}}
$$

$$
f(z) = \frac{(\frac{\psi}{2})^{\lambda}}{\Gamma(\lambda)} z^{\lambda-1} e^{-\frac{1}{2}\psi z}
$$

2. Nadarajah's approach.

$$
\begin{aligned}
f(x) &= \int_{\theta} f(x/\theta)g(\theta)\, d\theta \\
&= \int_{0}^{\infty} f(x/\lambda)g(\lambda)d\lambda
\end{aligned}
$$

where,

$$f(x/\lambda) = \frac{1}{\sqrt{2\pi\lambda}} e^{\frac{-x^2}{2\lambda}}$$

$$g(\lambda) = \frac{\lambda^{\beta-1} exp(\frac{-\lambda}{\mu})}{\mu^{\beta}\Gamma(\beta)}$$

4.2 Constuction of Variance Gamma

4.2.1 Based on Ole Barndoff approach

$$\begin{aligned}
f(x) &= \int_0^\infty f(x/z) g(z) dz \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi z}} e^{\frac{-x^2}{2z}} \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)} z^{\lambda-1} e^{-\frac{1}{2}\psi z} dz \\
&= \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)\sqrt{2\pi}} \int_0^\infty \frac{1}{z} e^{\frac{-x^2}{2z}} z^{\lambda-1} e^{-\frac{1}{2}\psi z} dz \\
&= \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)\sqrt{2\pi}} \int_0^\infty z^{\lambda-\frac{1}{2}-1} e^{-\frac{1}{2}(\psi z + \frac{-x^2}{z})} dz \\
&= \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)\sqrt{2\pi}} 2K_{\lambda-\frac{1}{2}}(\sqrt{\chi^2\psi})(\frac{\chi^2}{\psi})^{\frac{\lambda-\frac{1}{2}}{2}} \\
&= \frac{\psi^\lambda 2}{2^\lambda \Gamma(\lambda)\sqrt{\pi}\sqrt{2}} K_{\lambda-\frac{1}{2}}(\chi\sqrt{\psi})(\chi^2)^{\frac{1}{2}(\lambda-\frac{1}{2})} \psi^{\frac{-1}{2}(\lambda-\frac{1}{2})} \\
&= \frac{1}{2^{\lambda-\frac{1}{2}}\Gamma(\lambda)\sqrt{\pi}} K_{\lambda-\frac{1}{2}}(\chi\sqrt{\psi})(\psi)^{\frac{\lambda}{2}+\frac{1}{4}} \chi^{\lambda-\frac{1}{2}} \\
&= \frac{(\psi)^{\frac{\lambda}{2}+\frac{1}{4}} \chi^{\lambda-\frac{1}{2}}}{2^{\lambda-\frac{1}{2}}\Gamma(\lambda)\sqrt{\pi}} K_{\lambda-\frac{1}{2}}(\chi\sqrt{\psi})
\end{aligned}$$

4.2.2 Moments of Gamma distribution

First Moments

$$
\begin{aligned}
E(z) &= \int_0^\infty z \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)} z^{\lambda-1} e^{-\frac{1}{2}\psi z} dz \\
&= \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)} \int_0^\infty z^{\lambda+1-1} e^{-(\frac{\psi}{2})z} dz \\
&= \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)} \frac{\Gamma(\lambda+1)}{(\frac{\psi}{2})^{\lambda+1}} \\
&= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)(\frac{\psi}{2})} \\
&= \frac{\lambda\Gamma(\lambda)}{\Gamma(\lambda)(\frac{\psi}{2})} \\
&= \frac{2\lambda}{\psi}
\end{aligned}
$$

Second Moment

$$
\begin{aligned}
E(z^2) &= \int_0^\infty z^2 \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)} z^{\lambda-1} e^{-\frac{1}{2}\psi z} dz \\
&= \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)} \int_0^\infty z^{\lambda+2-1} e^{-(\frac{\psi}{2})z} dz \\
&= \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)} \frac{\Gamma(\lambda+2)}{(\frac{\psi}{2})^{\lambda+2}} \\
&= \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)(\frac{\psi}{2})^2} \\
&= \frac{(\lambda+1)\lambda\Gamma(\lambda)}{\Gamma(\lambda)(\frac{\psi}{2})^2} \\
&= \frac{4(\lambda+1)\lambda}{\psi^2}
\end{aligned}
$$

Third Moment

$$
\begin{aligned}
E(z^3) &= \int_0^{\infty} z^3 \frac{(\frac{\psi}{2})^{\lambda}}{\Gamma(\lambda)} z^{\lambda-1} e^{-\frac{1}{2}\psi z} \\
&= \frac{(\frac{\psi}{2})^{\lambda}}{\Gamma(\lambda)} \int_0^{\infty} z^{\lambda+3-1} e^{-(\frac{\psi}{2})z} dz \\
&= \frac{(\frac{\psi}{2})^{\lambda}}{\Gamma(\lambda)} \frac{\Gamma(\lambda+3)}{(\frac{\psi}{2})^{\lambda+3}} dz \\
&= \frac{\Gamma(\lambda+3)}{\Gamma(\lambda)(\frac{\psi}{2})^3} \\
&= \frac{(\lambda+2)(\lambda+1)\lambda\Gamma(\lambda)}{\Gamma(\lambda)(\frac{\psi}{2})^3} \\
&= \frac{8(\lambda+2)(\lambda+1)\lambda}{\psi^3}
\end{aligned}
$$

Fourth Moment

$$
\begin{aligned}
E(z^4) &= \int_0^{\infty} z^4 \frac{(\frac{\psi}{2})^{\lambda}}{\Gamma(\lambda)} z^{\lambda-1} e^{-\frac{1}{2}\psi z} dz \\
&= \frac{(\frac{\psi}{2})^{\lambda}}{\Gamma(\lambda)} \int_0^{\infty} z^{\lambda+4-1} e^{-(\frac{\psi}{2})z} dz \\
&= \frac{(\frac{\psi}{2})^{\lambda}}{\Gamma(\lambda)} \frac{\Gamma(\lambda+4)}{(\frac{\psi}{2})^{\lambda+4}} \\
&= \frac{\Gamma(\lambda+4)}{\Gamma(\lambda)(\frac{\psi}{2})^4} \\
&= \frac{(\lambda+3)(\lambda+2)(\lambda+1)\lambda\Gamma(\lambda)}{\Gamma(\lambda)(\frac{\psi}{2})^4} \\
&= \frac{16(\lambda+3)(\lambda+2)(\lambda+1)\lambda}{\psi^4}
\end{aligned}
$$

4.2.3 Convolution property

Convolution property is one of the properties of VG where it is said to be closed under convolution.
that is,

$$VG(x_1, \lambda_1, \psi) * VG(x_2, \lambda_2, \psi) = VG(x_1 + x_2, \lambda_1 + \lambda_2, \psi)$$

$$let, Y = (x_1 + x_2)$$

Based on MGF technique,

$$M_y(t) = M_{x_1}(t) * M_{x_2}(t)$$

Proof.

$$\begin{aligned} M_x(t) =& E[e^{tx}] \\ =& EE[e^{\frac{tx}{\lambda}}] \\ =& E[e^{\frac{tx}{\lambda}}] \\ =& e^{\mu t + \frac{1}{2}\lambda t^2} \\ =& \int_0^\infty e^{\mu t + \frac{1}{2}\lambda t^2} g(\lambda) d\lambda \end{aligned}$$

$$let, \mu = 0$$

$$M_x(t) = \int_0^\infty e^{\frac{1}{2}\lambda t^2} g(\lambda) d\lambda$$

$$
\begin{aligned}
M_x(t) &= \int_0^\infty e^{\frac{1}{2}\lambda t^2} g(\lambda) d\lambda \\
&= \int_0^\infty e^{\frac{1}{2}\lambda t^2} \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)} z^{\lambda-1} e^{-\frac{1}{2}\psi z} dz \\
&= \int_0^\infty e^{\frac{1}{2}\lambda t^2} \frac{(\frac{\psi}{2}) z^{\lambda-1}}{\Gamma(\lambda)} e^{-(\frac{\psi}{2})z} dz \\
&= \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)} \int_0^\infty z^{\lambda-1} e^{\frac{1}{2}\lambda t^2} e^{-(\frac{\psi}{2})z} dz \\
&= \frac{(\frac{\psi}{2})^\lambda}{\Gamma(\lambda)} \int_0^\infty z^{\lambda-1} e^{-(\frac{\psi}{2}-\frac{t^2}{2})z} dz \\
&= \frac{(\frac{\psi}{2})^\lambda \Gamma(\lambda)}{\Gamma(\lambda)(\frac{\psi}{2}+\frac{t^2}{2})^\lambda} \\
&= \frac{(\frac{\psi}{2})^\lambda}{(\frac{\psi}{2}+\frac{t^2}{2})^\lambda} \\
&= (\frac{\frac{\psi}{2}}{\frac{\psi}{2}+\frac{t^2}{2}})^\lambda \\
&= (\frac{\psi}{\psi+t^2})^\lambda
\end{aligned}
$$

for,VG to be closed under convolution,then

$$
\begin{aligned}
M_y(t) &= M_{x_1}(t) * M_{x_2}(t) \\
&= (\frac{\psi}{\psi+t^2})^{\lambda_1} * (\frac{\psi}{\psi+t^2})^{\lambda_2} \\
&= (\frac{\psi}{\psi+t^2})^{\lambda_1+\lambda_2}
\end{aligned}
$$

□

4.3 Construction based on the Second Approach

4.3.1 Based on Nadarajah's approach

$$
\begin{aligned}
f(x) =& \int_0^\infty \frac{1}{\sqrt{2\pi\lambda}} exp(\frac{-x}{\lambda}) \frac{\lambda^{\beta-1} exp(\frac{-\lambda}{\mu})}{\mu^\beta \Gamma(\beta)} d\lambda \\
=& \frac{1}{\mu^\beta \Gamma(\beta)\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\lambda}} exp(\frac{-x}{\lambda}) \lambda^{\beta-1} exp(\frac{-\lambda}{\mu}) d\lambda \\
=& \frac{1}{\mu^\beta \Gamma(\beta)\sqrt{2\pi}} \int_0^\infty \lambda^{\frac{-1}{2}} exp(\frac{-x}{\lambda}) \lambda^{\beta-1} exp(\frac{-\lambda}{\mu}) d\lambda \\
=& \frac{1}{\mu^\beta \Gamma(\beta)\sqrt{2\pi}} \int_0^\infty \lambda^{\beta-\frac{3}{2}} e^{-(\frac{\lambda}{\mu}+\frac{x}{\lambda})} d\lambda \\
=& \frac{1}{\mu^\beta \Gamma(\beta)\sqrt{2\pi}} \int_0^\infty \lambda^{\beta-\frac{3}{2}} e^{-\frac{1}{\mu}(\lambda+\frac{\mu x}{\lambda})} d\lambda
\end{aligned}
$$

$$
let, \lambda = \sqrt{\mu x}\, z, and, d\lambda = \sqrt{\mu x}\, dz
$$

$$
\begin{aligned}
=& \frac{1}{\mu^\beta \Gamma(\beta)\sqrt{2\pi}} \int_0^\infty (\sqrt{\mu x} z)^{\beta-\frac{3}{2}} e^{-\frac{1}{\mu}(\sqrt{\mu x} z+\frac{\mu x}{\sqrt{\mu x} z})} (\sqrt{\mu x}) dz \\
=& \frac{1}{\mu^\beta \Gamma(\beta)\sqrt{2\pi}} \int_0^\infty (\sqrt{\mu x})^{\beta-\frac{1}{2}} z^{\beta-\frac{3}{2}} e^{-\frac{\sqrt{\mu x}}{\mu}(z+\frac{1}{z})} dz \\
=& \frac{2(\sqrt{\mu x})^{\beta-\frac{1}{2}}}{\mu^\beta \Gamma(\beta)\sqrt{2\pi}} \frac{1}{2} \int_0^\infty z^{\beta-\frac{3}{2}} e^{-\frac{2\sqrt{\frac{x}{\mu}}}{2}(z+\frac{1}{z})} dz \\
=& \frac{2(\sqrt{\mu x})^{\beta-\frac{1}{2}}}{\mu^\beta \Gamma(\beta)\sqrt{2\pi}} K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x}{\mu}}) \\
=& \frac{\sqrt{2}\mu^{\frac{1}{2}(\beta-\frac{1}{2})} x^{\frac{1}{2}(\beta-\frac{1}{2})}}{\mu^\beta \Gamma(\beta)\sqrt{\pi}} K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x}{\mu}}) \\
=& \frac{\sqrt{2} x^{\frac{1}{2}(\beta-\frac{1}{2})}}{\Gamma(\beta)\sqrt{\pi}\mu^{(\frac{2\beta+1}{4})}} K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x}{\mu}})
\end{aligned}
$$

4.3.2 Moments of Gamma distribution

First Moment

$$
\begin{aligned}
E[\lambda] = &\int_0^\infty \lambda \frac{\lambda^{\beta-1} e^{\frac{-\lambda}{\mu}}}{\mu^\beta \Gamma(\beta)} d\lambda \\
=&\frac{\lambda^{\beta-1}}{\mu^\beta \Gamma(\beta)} \int_0^\infty \lambda^{\beta+1-1} e^{\frac{-\lambda}{\mu}} d\lambda \\
=&\frac{\Gamma(\beta+1)}{\mu^\beta \Gamma(\beta)(\frac{1}{\mu})^{\beta+1}} \\
=&\frac{\Gamma(\beta+1)}{\Gamma(\beta)\mu^\beta \mu^{-\beta-1}} \\
=&\frac{\Gamma(\beta+1)}{\Gamma(\beta)\mu^{-1}} \\
=&\frac{\mu(\beta)\Gamma(\beta)}{\Gamma(\beta)} \\
=&\mu\beta
\end{aligned}
$$

Second Momment

$$
\begin{aligned}
E[\lambda^2] = &\int_0^\infty \lambda^2 \frac{\lambda^{\beta-1} e^{\frac{-\lambda}{\mu}}}{\mu^\beta \Gamma(\beta)} d\lambda \\
=&\frac{\lambda^{\beta-1}}{\mu^\beta \Gamma(\beta)} \int_0^\infty \lambda^{\beta+2-1} e^{\frac{-\lambda}{\mu}} d\lambda \\
=&\frac{\Gamma(\beta+2)}{\mu^\beta \Gamma(\beta)(\frac{1}{\mu})^{\beta+2}} \\
=&\frac{\Gamma(\beta+2)}{\Gamma(\beta)\mu^\beta \mu^{-\beta-2}} \\
=&\frac{\Gamma(\beta+2)}{\Gamma(\beta)\mu^{-2}} \\
=&\frac{\mu^2(\beta+1)(\beta)\Gamma(\beta)}{\Gamma(\beta)} \\
=&\mu^2(\beta)(\beta+1)
\end{aligned}
$$

Third moment

$$
\begin{aligned}
E[\lambda^3] &= \int_0^\infty \lambda^3 \frac{\lambda^{\beta-1} e^{\frac{-\lambda}{\mu}}}{\mu^\beta \Gamma(\beta)} d\lambda \\
&= \frac{\lambda^{\beta-1}}{\mu^\beta \Gamma(\beta)} \int_0^\infty \lambda^{\beta+3-1} e^{\frac{-\lambda}{\mu}} d\lambda \\
&= \frac{\Gamma(\beta+3)}{\mu^\beta \Gamma(\beta)(\frac{1}{\mu})^{\beta+3}} \\
&= \frac{\Gamma(\beta+3)}{\Gamma(\beta)\mu^\beta \mu^{-\beta-3}} \\
&= \frac{\Gamma(\beta+3)}{\Gamma(\beta)\mu^{-3}} \\
&= \frac{\mu^3(\beta+2)(\beta+1)(\beta)\Gamma(\beta)}{\Gamma(\beta)} \\
&= \mu^3(\beta)(\beta+1)(\beta+2)
\end{aligned}
$$

Fourth moment

$$
\begin{aligned}
E[\lambda^4] &= \int_0^\infty \lambda^4 \frac{\lambda^{\beta-1} e^{\frac{-\lambda}{\mu}}}{\mu^\beta \Gamma(\beta)} d\lambda \\
&= \frac{\lambda^{\beta-1}}{\mu^\beta \Gamma(\beta)} \int_0^\infty \lambda^{\beta+4-1} e^{\frac{-\lambda}{\mu}} d\lambda \\
&= \frac{\Gamma(\beta+4)}{\mu^\beta \Gamma(\beta)(\frac{1}{\mu})^{\beta+4}} \\
&= \frac{\Gamma(\beta+4)}{\Gamma(\beta)\mu^\beta \mu^{-\beta-4}} \\
&= \frac{\Gamma(\beta+4)}{\Gamma(\beta)\mu^{-4}} \\
&= \frac{\mu^4(\beta+3)(\beta+2)(\beta+1)(\beta)\Gamma(\beta)}{\Gamma(\beta)} \\
&= \mu^4(\beta)(\beta+1)(\beta+2)(\beta+3)
\end{aligned}
$$

4.3.3 Convolution Property

Convolution property is one of the properties of VG where it is said to be closed under convolution.
that is,

$$VG(x_1,\beta_1,\mu)*VG(x_2,\beta_2,\mu) = VG(x_1+x_2,\beta_1+\beta_2,\mu)$$

$$let, Y = (x_1+x_2)$$

Based on MGF technique,

$$M_y(t) = M_{x_1}(t)*M_{x_2}(t)$$

Proof.

$$\begin{aligned} M_x(t) =& E[e^{tx}] \\ =& EE[e^{\frac{tx}{\lambda}}] \\ =& E[e^{\frac{tx}{\lambda}}] \\ =& e^{\mu t+\frac{1}{2}\lambda t^2} \\ =& \int_0^\infty e^{\mu t+\frac{1}{2}\lambda t^2} g(\lambda) d\lambda \end{aligned}$$

$$let, \mu = 0$$

$$
\begin{aligned}
M_x(t) &= \int_0^\infty e^{\frac{1}{2}\lambda t^2} g(\lambda) d\lambda \\
&= \int_0^\infty e^{\frac{1}{2}\lambda t^2} \frac{\lambda^{\beta-1} e^{\frac{-\lambda}{\mu}}}{\mu^\beta \Gamma(\beta)} d\lambda \\
&= \frac{1}{\mu^\beta \Gamma(\beta)} \int_0^\infty \lambda^{\beta-1} e^{\frac{1}{2}\lambda t^2} e^{\frac{-\lambda}{\mu}} d\lambda \\
&= \frac{1}{\mu^\beta \Gamma(\beta)} \int_0^\infty \lambda^{\beta-1} e^{\frac{1}{2}\lambda t^2 - \frac{\lambda}{\mu}} d\lambda \\
&= \frac{1}{\mu^\beta \Gamma(\beta)} \int_0^\infty \lambda^{\beta-1} e^{-}(\frac{1}{\mu} - \frac{t^2}{2})\lambda d\lambda \\
&= \frac{\Gamma(\beta)}{\mu^\beta \Gamma(\beta)(\frac{1}{\mu} - \frac{t^2}{2})^{\beta-1}} \\
&= \frac{(\frac{1}{\mu} - \frac{t^2}{2})}{\mu^\beta (\frac{1}{\mu} - \frac{t^2}{2})^\beta}
\end{aligned}
$$

For,VG to be closed under convolution,therefore,

$$
\begin{aligned}
M_y(t) =& M_{x_1}(t) * M_{x_2}(t) \\
&= \frac{(\frac{1}{\mu} - \frac{t^2}{2})}{\mu^{\beta_1} (\frac{1}{\mu} - \frac{t^2}{2})^{\beta_1}} * \frac{(\frac{1}{\mu} - \frac{t^2}{2})}{\mu^{\beta_2} (\frac{1}{\mu} - \frac{t^2}{2})^{\beta_2}} \\
&= \frac{(\frac{1}{\mu} - \frac{t^2}{2})}{\mu^{\beta_1+\beta_2} (\frac{1}{\mu} - \frac{t^2}{2})^{\beta_1} + \beta_2}
\end{aligned}
$$

□

5 Maximum likelihood Estimator

5.1 Introduction

In this chapter, the maximum likelihood estimator method is considered as a method of estimation. MLE's for the two distribution are derived based on both approaches used for construction of the distributions.
MLE method has the following steps,

1. Likelihood

$$L(\varphi) = \prod_{i=1}^{t} f(x_i; \varphi)$$

   where $\varphi = \{.\}$

2. Log Likelihood

$$\log L(\varphi) = \sum_{i=1}^{t} f(x_i; \varphi)$$

3. First partial derivative of the Log Likelihood with respect to the parameters being estimated

$$\frac{\partial}{\partial \varphi} \log L(\varphi)$$

4. Maximization of the derivative

$$\frac{\partial}{\partial \varphi} \log L(\varphi) = 0$$

5.2 Maximum likelihood estimator for NIG distribution

5.2.1 MLE for NIG distribution

Based on Nadarajah's approach

$$f(x;\mu,\phi)=\frac{\phi e^{\phi}}{\pi\sqrt{\mu\phi+2x}}K_1(\sqrt{\phi(\phi+\frac{2x}{\mu}}))$$

$$\begin{aligned}L(\varphi)&=\prod_{i=1}^{t}f(x_i;\varphi)\\&=\prod_{i=1}^{t}\frac{\phi e^{\phi}}{\pi\sqrt{\mu\phi+2x}}K_1(\sqrt{\phi(\phi+\frac{2x}{\mu}}))\\&=\frac{\phi^t e^{\phi t}}{\pi^t\prod_{i=1}^{t}\sqrt{(\phi\mu+2x_i)}}\prod_{i=1}^{t}K_1(\sqrt{\phi(\phi+\frac{2x}{\mu}}))\end{aligned}$$

$$\log L(\varphi)=t\log\phi+t\phi-t\log\pi-\sum_{i=1}^{t}\log(\sqrt{(\phi\mu+2x_i)})+\sum_{i=1}^{t}K_1(\sqrt{\phi(\phi+\frac{2x}{\mu}}))$$

$$\frac{\partial\log L(\varphi)}{\partial(\phi)}=\frac{t}{\phi}+t-\sum_{i=1}^{t}\frac{\partial}{\partial(\phi)}\log(\sqrt{\phi\mu+2X_i})+\sum_{i=1}^{t}\frac{\partial\log K_1(w)}{\partial(\phi)}$$

$$\frac{\partial}{\partial(\phi)}\log(\sqrt{\phi\mu+2X_i})=\frac{1}{(\sqrt{\phi\mu+2X_i})}\frac{\partial}{\partial\phi}(\sqrt{\phi\mu+2X_i})$$

$$let, v=(\phi\mu+2x_i)^{\frac{1}{2}}, and, u=\phi\mu+2x_i)\Rightarrow v=u^{\frac{1}{2}}$$

thus,

$$\frac{d(v)}{d(u)}=\frac{1}{2}u^{\frac{-1}{2}}, and, \frac{d(u)}{d(\phi)}=\mu$$

$$\frac{\partial}{\partial\phi}(\sqrt{\phi\mu+2X_i})=\frac{\mu}{2}(\phi\mu+2x_i)^{\frac{-1}{2}}$$

$$\frac{\partial \log K_1(w)}{\partial \phi} = \frac{1}{K_1(w)} \frac{\partial K_1(w)}{\partial \phi}$$

$$\frac{\partial \log K_1(w)}{\partial \phi} = \frac{1}{K_1(w)} \frac{\partial K_1(w)}{\partial w} \frac{\partial w}{\partial \phi}$$

where,

$$\begin{aligned} w =& \sqrt{\phi(\phi + \frac{2x_i}{\mu})} \\ =& (\phi^2 + \frac{\phi 2x_i}{\mu})^{\frac{1}{2}} \\ =& v^{\frac{1}{2}} \end{aligned}$$

$$v = (\phi^2 + \frac{\phi 2x_i}{\mu})$$

thus,

$$\frac{\partial w}{\partial v} = \frac{1}{2} v^{\frac{-1}{2}}$$

$$\frac{\partial v}{\partial \phi} = (2\phi + \frac{2x_i}{\mu})$$

$$\begin{aligned} \frac{\partial w}{\partial \phi} =& (2\phi + \frac{2x_i}{\mu}) \frac{1}{2} v^{\frac{-1}{2}} \\ =& (\phi + \frac{x_i}{\mu})(\phi^2 + \frac{\phi 2x_i}{\mu})^{\frac{-1}{2}} \\ =& (\phi + \frac{x_i}{\mu}) \frac{1}{w} \end{aligned}$$

Recall, equation 7 hankel function derivative,

$$\frac{\partial K_\lambda(w)}{\partial w} = -[\frac{1}{w} K_\lambda(w) + K_{\lambda-1}(w)]$$

$$\frac{\partial K_1(w)}{\partial w} = -[\frac{1}{w}K_1(w) + K_0(w)]$$

differentiating with respect to ϕ we get,

$$\begin{aligned}
\frac{\partial \log L(\varphi)}{\partial \phi} =& t(\frac{1}{\phi}+1) - \sum_{i=1}^{t}\frac{\mu}{2}(\phi\mu+2x_i)^{-1} + \sum_{i=1}^{t}\frac{\partial K_1(w)}{\partial w} \\
=& t(\frac{1}{\phi}+1) - \sum_{i=1}^{t}\frac{\mu}{2}(\phi\mu+2x_i)^{-1} + \sum_{i=1}^{t}\frac{1}{K_1(w)} - [\frac{1}{w}K_1(w)+K_0(w)](\phi+\frac{x_i}{\mu})\frac{1}{w} \\
=& t(\frac{1}{\phi}+1) - \sum_{i=1}^{t}\frac{\mu}{2}(\phi\mu+2x_i)^{-1} - \sum_{i=1}^{t}[\frac{1}{w}+\frac{K_0(w)}{K_1(w)}](\phi+\frac{x_i}{\mu})\frac{1}{w} \\
=& t(\frac{1}{\phi}+1) - \sum_{i=1}^{t}\frac{\mu}{2}(\phi\mu+2x_i)^{-1} - \sum_{i=1}^{t}[\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}} + \frac{K_0(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}{K_1(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}] \\
& (\phi+\frac{x_i}{\mu})\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}}) \\
=& t(\frac{1}{\phi}+1) - \frac{\mu}{2}\sum_{i=1}^{t}\frac{1}{(\phi\mu+2x_i)} - \sum_{i=1}^{t}[\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}} + \frac{K_0(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}{K_1(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}] \\
& (\phi+\frac{x_i}{\mu})\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}})
\end{aligned}$$

hence,

$$\sum_{i=1}^{t}[\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}} + \frac{K_0(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}{K_1(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}](\phi+\frac{x_i}{\mu})\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}} = t(\frac{1}{\phi}+1) - \frac{\mu}{2}\sum_{i=1}^{t}\frac{1}{(\phi\mu+2x_i)} - \sum_{i=1}^{t}[\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}}$$

differentiating with respect to μ,

$$\frac{\partial \log L(\varphi)}{\partial \mu} = -\sum_{i=1}^{t} \frac{\partial}{\partial \mu} \log(\sqrt{\phi\mu) + 2x_i} + \sum_{i=1}^{t} \frac{\partial}{\partial \mu} \log K_1(w)$$

$$\frac{\partial}{\partial \mu} \log(\sqrt{\phi\mu) + 2x_i}) = \frac{1}{(\sqrt{\phi\mu + 2x_i})} \frac{\partial}{\partial \mu}(\sqrt{\phi\mu) + 2x_i})$$

$$let, v = (\phi\mu + 2x_i)^{\frac{1}{2}} and, u = \phi\mu) + 2x_i \Rightarrow v = u^{\frac{1}{2}}$$

thus,

$$\frac{\partial v}{\partial u} = \frac{1}{2} u^{\frac{-1}{2}}, and, \frac{\partial u}{\partial \mu} = \phi$$

$$\begin{aligned} \frac{\partial v}{\partial \mu} &= \frac{\phi}{\mu} u^{\frac{-1}{2}} \\ &= \frac{\phi}{\mu} (\phi\mu) + 2x_i)^{\frac{-1}{2}} \end{aligned}$$

$$\frac{\partial}{\partial \mu} \log(v) = \frac{\phi}{2} (\phi\mu) + 2x_i)^{-1}$$

$$\begin{aligned} \frac{\partial}{\partial \mu} \log K_1(w) &= \frac{1}{K_1(w)} \frac{\partial K_1(w)}{\partial \mu} \\ &= \frac{1}{K_1(w)} \frac{\partial K_1(w)}{\partial \mu} \frac{\partial w}{\partial \mu} \end{aligned}$$

$$w = \sqrt{\phi(\phi + \frac{2x_i}{\mu})}$$
$$= (\phi^2 + \frac{\phi 2x_i}{\mu}^{\frac{1}{2}}$$

let,

$$w = v^{\frac{1}{2}} and, v = (\phi^2 + \frac{\phi 2x_i}{\mu}), \Rightarrow \frac{\partial w}{\partial v} = \frac{1}{2} v^{\frac{-1}{2}}$$

$$\frac{\partial v}{\partial \mu} = \frac{-\phi 2x_i}{\mu^2} \frac{1}{2} v^{\frac{-1}{2}}$$
$$= \frac{-\phi x_i}{\mu^2} w^{-1}$$

$$\frac{\partial K_1(w)}{\partial w} = -[\frac{1}{w} K_1(w) + K_0(w)]$$

$$\frac{\partial Log K_1(w)}{\mu} = \frac{-1}{K_1(w)} [\frac{1}{w} K_1(w) + K_0(w)] \frac{\partial w}{\partial \partial \mu}$$
$$= -[\frac{1}{w} + \frac{K_0(w)}{K_1(w)}] \frac{-\phi x_i}{\mu^2} w^{-1}$$
$$= \frac{\phi}{\mu^2} \frac{x_i}{w} [\frac{1}{w} + \frac{K_0(w)}{K_1(w)}]$$

therefore,

$$\frac{\partial Log L(\varphi)}{\partial \mu} = -\sum_{i=1}^{t} \frac{-\phi x_i}{\mu^2} w^{-1} + \sum_{i=1}^{t} \frac{\phi}{\mu^2} \frac{x_i}{w} [\frac{1}{w} + \frac{K_0(w)}{K_1(w)}]$$

$$\sum_{i=1}^{t} \frac{\phi}{\mu^2} \frac{x_i}{w} [\frac{1}{w} + \frac{K_0(w)}{K_1(w)}] = \sum_{i=1}^{t} \frac{-\phi}{2} w^{-1}$$

$$\sum_{i=1}^{t}\frac{\phi}{\mu^2}\frac{x_i}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}}[\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}}+\frac{K_0(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}{K_1(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}]=\sum_{i=1}^{t}\frac{-\phi}{2}(\phi\mu+2x_i)^{-1}$$

hence,

$$\sum_{i=1}^{t}\frac{x_i}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}}[\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}}+\frac{K_0(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}{K_1(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}]=\frac{\mu^2}{2}\sum_{i=1}^{t}\frac{1}{(\phi\mu+2x_i)}$$

The two equations for the maximum likelihood estimators are,

$$\sum_{i=1}^{t}[\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}}+\frac{K_0(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}{K_1(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}](\phi+\frac{x_i}{\mu})\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}}=t(\frac{1}{\phi}+1)-\frac{\mu}{2}\sum_{i=1}^{t}\frac{1}{(\phi\mu+2x_i)}-\sum_{i=1}^{t}[\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}}$$

and,

$$\sum_{i=1}^{t}\frac{x_i}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}}[\frac{1}{\sqrt{\phi(\phi+\frac{2x_i}{\mu})}}+\frac{K_0(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}{K_1(\sqrt{\phi(\phi+\frac{2x_i}{\mu})})}]=\frac{\mu^2}{2}\sum_{i=1}^{t}\frac{1}{(\phi\mu+2x_i)}$$

Based on Ole Barndoff

$$f(x)=\sqrt{\frac{\chi}{\pi}}e^{\sqrt{\chi\psi}}\sqrt{\frac{\psi}{(\chi+x^2)}}K_1(\sqrt{(\chi+x^2)\psi})$$

$$let,\psi=\alpha^2,and,\chi=\delta^2$$

$$f(x) = \frac{\delta}{\pi}\, e^{\delta\alpha}\, \sqrt{\frac{\alpha^2}{\delta^2+x^2}}\, K_1(\alpha\sqrt{\delta^2+x^2})$$

$$\begin{aligned}
L(\varphi) =& \prod_{i=1}^{t} f(x_i,\varphi) \\
=& \prod_{i=1}^{t} \frac{\alpha\delta}{\pi}\, e^{\delta\alpha}\, \frac{1}{\sqrt{\delta^2+x^2}}\, K_1(\alpha\sqrt{\delta^2+x^2}) \\
=& \frac{\alpha^t\ \delta^t}{\pi^t}\, e^{t\delta\alpha} \prod_{i=1}^{t} \frac{1}{\sqrt{\delta^2+x^2}} \prod_{i=1}^{t} K_1(\alpha\sqrt{\delta^2+x^2})
\end{aligned}$$

$$\log L(\varphi) = t\log\alpha + t\log\delta + t\log\pi + t\alpha\delta + \sum_{i=1}^{t} \log\frac{1}{\sqrt{\delta^2+x^2}} + \sum_{i=1}^{t} \log K_1(\alpha\sqrt{\delta^2+x^2})$$

differentiating with respect to α,

$$\frac{\partial \log L(\varphi)}{\partial\alpha} = \frac{t}{\alpha} + t\delta + \sum_{i=1}^{t} \frac{\partial}{\partial\alpha} \log K_1(\alpha\sqrt{\delta^2+x^2})$$

let,

$$w = \alpha\sqrt{\delta^2+x_i^2}, \Rightarrow \frac{\partial w}{\partial\alpha} = \sqrt{\delta^2+x_i^2}$$

Recall hankel function ,equation 7

$$\begin{aligned}
\frac{\partial \log K_1(w)}{\partial w} =& \frac{1}{K_1(w)}\, \frac{\partial K_1(w)}{\partial\alpha} \\
=& \frac{1}{K_1(w)}\, \frac{\partial K_1(w)}{\partial w}\, \frac{\partial w}{\partial\alpha}
\end{aligned}$$

$$\frac{\partial K_v(w)}{\partial w} = -[\frac{v}{w}K_v(w) + K_{v-1}(w)]\sqrt{\delta^2 + x_i^2}$$

$$\frac{\partial K_1(w)}{\partial w} = -[\frac{1}{w}K_1(w) + K_0(w)]\sqrt{\delta^2 + x_i^2}$$

$$\begin{aligned}\frac{\partial \log K_1(w)}{\partial w} =& \frac{1}{K_1(w)} \ -[\frac{1}{w}K_1 w) + K_0(w)]\sqrt{\delta^2 + x_i^2} \\ =& -\sqrt{\delta^2 + x_i^2}\ [\frac{1}{w} + \frac{K_0(w)}{K_1(w)}]\end{aligned}$$

maximizing,

$$\frac{\partial \log L(\varphi)}{\partial \alpha} = 0$$

$$\frac{t}{\alpha} + t\delta - \sum_{i=1}^{t}\sqrt{\delta^2 + x_i^2}\ [\frac{1}{\alpha\sqrt{\delta^2 + x_i^2}} + \frac{K_0(\alpha\sqrt{\delta^2 + x_i^2})}{K_1(\alpha\sqrt{\delta^2 + x_i^2})}] = 0$$

hence,

$$\sum_{i=1}^{t}\sqrt{\delta^2 + x_i^2}\ [\frac{1}{\alpha\sqrt{\delta^2 + x_i^2}} + \frac{K_0(\alpha\sqrt{\delta^2 + x_i^2})}{K_1(\alpha\sqrt{\delta^2 + x_i^2})}] = \frac{t}{\alpha} + t\delta$$

differentiating with respect to δ

$$\frac{\partial \log L(\varphi)}{\partial \delta} = \frac{t}{\delta} + t\alpha + \sum_{i=1}^{t}\frac{\partial}{\partial \delta}\log\frac{1}{\sqrt{\delta^2 + x_{i^2}}} + \sum_{i=1}^{t}\frac{\partial}{\partial \delta}\log K_1(\alpha\sqrt{\delta^2 + x^2})$$

$$
\begin{aligned}
\frac{\partial}{\partial\delta}\log\frac{1}{\sqrt{\delta^2+x_i^2}} &= \frac{-1}{2}\frac{\partial}{\partial\delta}\log(\delta^2+x_i^2) \\
&= \frac{-1}{2}\frac{1}{(}\delta^2+x_i^2)\ \frac{\partial}{\partial\delta}\ (\delta^2+x_i^2) \\
&= \frac{-2\delta}{2}(\frac{1}{\delta^2+x_i^2}) \\
&= (\frac{\delta}{\delta^2+x_i^2})
\end{aligned}
$$

$$
let, w=\alpha\sqrt{\delta^2+x_i^2}, \Rightarrow \frac{\partial w}{\partial\delta}=\alpha\delta(\delta^2+x_i^2)^{\frac{-1}{2}}
$$

Recall hankel function ,equation 7

$$
\begin{aligned}
\frac{\partial log K_1(w)}{\partial w} &= \frac{1}{K_1(w)}\ \frac{\partial K_1(w)}{\partial\alpha} \\
&= \frac{1}{K_1(w)}\ \frac{\partial K_1(w)}{\partial w}\ \frac{\partial w}{\partial\alpha} \\
&= \frac{-1}{K_1(w)}\ \frac{\partial K_1(w)}{\partial w}\ \alpha\delta(\delta^2+x_i^2)^{\frac{-1}{2}} \\
&= -[\frac{1}{w}+\frac{K_0(w)}{K_1(w)}]\ \frac{\alpha\delta}{\sqrt{\delta^2+x_i^2}}
\end{aligned}
$$

thus,

$$
\frac{\partial LogL(\varphi)}{\partial\delta}=\frac{t}{\delta}+t\alpha+\sum_{i=1}^{t}(\frac{\delta}{(\delta^2+x_i^2)})-\sum_{i=1}^{t}[\frac{1}{\alpha\sqrt{\delta^2+x_i^2}}+\frac{K_0(\alpha\sqrt{\delta^2+x_i^2})}{K_1(\alpha\sqrt{\delta^2+x_i^2})}]\ \frac{\alpha\delta}{\sqrt{\delta^2+x_i^2}}
$$

maximizes,

$$
\frac{\partial LogL(\varphi)}{\partial\delta}=0
$$

$$\frac{t}{\delta}+t\alpha+\sum_{i=1}^{t}(\frac{\delta}{(\delta^2+x_i^2})-\sum_{i=1}^{t}[\frac{1}{\alpha\sqrt{\delta^2+x_i^2}}+\frac{K_0(\alpha\sqrt{\delta^2+x_i^2})}{K_1(\alpha\sqrt{\delta^2+x_i^2})}]\ \frac{\alpha\delta}{\sqrt{\delta^2+x_i^2}}=0$$

$$\sum_{i=1}^{t}[\frac{1}{\alpha\sqrt{\delta^2+x_i^2}}+\frac{K_0(\alpha\sqrt{\delta^2+x_i^2})}{K_1(\alpha\sqrt{\delta^2+x_i^2})}]\ \frac{\alpha\delta}{\sqrt{\delta^2+x_i^2}}=\frac{t}{\delta}+t\alpha+\sum_{i=1}^{t}(\frac{\delta}{\delta^2+x_i^2})$$

The two equations for the maximum likelihood estimators are,

$$\sum_{i=1}^{t}\sqrt{\delta^2+x_i^2}\ [\frac{1}{\alpha\sqrt{\delta^2+x_i^2}}+\frac{K_0(\alpha\sqrt{\delta^2+x_i^2})}{K_1(\alpha\sqrt{\delta^2+x_i^2})}]=\frac{t}{\alpha}+t\delta$$

and,

$$\sum_{i=1}^{t}[\frac{1}{\alpha\sqrt{\delta^2+x_i^2}}+\frac{K_0(\alpha\sqrt{\delta^2+x_i^2})}{K_1(\alpha\sqrt{\delta^2+x_i^2})}]\ \frac{\alpha\delta}{\sqrt{\delta^2+x_i^2}}=\frac{t}{\delta}+t\alpha+\sum_{i=1}^{t}(\frac{\delta}{\delta^2+x_i^2})$$

5.2.2 Maximum Likelihood Estimator for Variance Gamma Distribution

Based on Nadarajah's approach

$$f(x,\mu,\beta)=\frac{\sqrt{2}\,x^{\frac{1}{2}(\beta-\frac{1}{2})}}{\Gamma(\beta)\ \sqrt{\pi}\ \mu^{(\frac{2\beta+1}{4})}}\ K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x}{\mu}})$$

$$\begin{aligned}L(\varphi)=&\prod_{i=1}^{t}\frac{\sqrt{2}\,x_i^{\frac{1}{2}(\beta-\frac{1}{2})}}{\Gamma(\beta)\ \sqrt{\pi}\ \mu^{(\frac{2\beta+1}{4})}}\ K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})\\=&\frac{2^{\frac{t}{2}}}{\pi^{\frac{t}{2}}\ (\Gamma(\beta))^t}\ \frac{1}{\mu^{(1+2\beta)\frac{t}{4}}}\ \prod_{i=1}^{t}x_i^{\frac{\beta}{2}-\frac{1}{4}}\prod_{i=1}^{t}K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})\end{aligned}$$

$$\log L(\varphi)=\frac{t}{2}\log 2-\frac{t}{2}\log\pi-t\log(\Gamma(\beta))-\frac{(1+2\beta)t}{4}\log\mu+(\frac{\beta}{2}-\frac{1}{4})\sum_{i=1}^{t}\log x_i+\sum_{i=1}^{t}\log K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})$$

differentiating with respect to μ

$$\frac{\partial \log L(\varphi)}{\partial \mu} = \frac{-t(1+2\beta)}{4\mu} + \sum_{i=1}^{t} \frac{\partial}{\partial \mu} \log K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})$$

$$\begin{aligned}\frac{\partial \log\, K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{\partial \mu} =& \frac{1}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}\ \frac{\partial}{\partial \mu} K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}}) \\ =& \frac{1}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}\ \frac{\partial}{\partial \omega} K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})\ \frac{\partial \omega}{\partial \mu}\end{aligned}$$

let,

$$\begin{aligned} w =& (2\sqrt{\frac{x_i}{\mu}}) \\ =& 2\ \sqrt{x_i}\ \mu^{\frac{-1}{2}} \end{aligned}$$

$$\frac{\partial w}{\partial \mu} = \frac{-\sqrt{x_i}}{\mu^{\frac{3}{2}}}$$

Recall,

$$\frac{\partial k_v(w)}{\partial w} = \frac{v}{w} K_v(w) - K_{v+1}(w)$$

But in our case ,

$$v = (\beta - \frac{1}{2}), and, w = (2\sqrt{\frac{x_i}{\mu}})$$

therefore,

$$\frac{\partial K_{\beta-\frac{1}{2}}(w)}{\partial w} = [\frac{\beta-\frac{1}{2}}{w} K_{\beta-\frac{1}{2}}(w) - K_{\beta+\frac{1}{2}}(w)]$$

$$\begin{aligned}\frac{\partial \log\, K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{\partial \mu} =& \frac{1}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}[\frac{\beta-\frac{1}{2}}{(2\sqrt{\frac{x_i}{\mu}})}\ K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})\ - K_{\beta+\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})]\frac{-\sqrt{x_i}}{\mu^{\frac{3}{2}}} \\ =& \frac{-\sqrt{x_i}}{\mu^{\frac{3}{2}}}[\frac{\beta-\frac{1}{2}}{(2\sqrt{\frac{x_i}{\mu}})}\ - \frac{K_{\beta+\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}]\end{aligned}$$

$$\frac{\partial \log L(\varphi)}{\partial \mu} = \frac{-t(1+2\beta)}{4\mu} + \sum_{i=1}^{t} \frac{-\sqrt{x_i}}{\mu^{\frac{3}{2}}}[\frac{\beta-\frac{1}{2}}{(2\sqrt{\frac{x_i}{\mu}})} \quad - \frac{K_{\beta+\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}]$$

$$\frac{\partial \log L(\varphi)}{\partial \mu} = 0$$

$$\frac{-t(1+2\beta)}{4\mu} + \frac{1}{\mu^{\frac{3}{2}}} \sum_{i=1}^{t} \sqrt{x_i}[-\frac{\beta-\frac{1}{2}}{(2\sqrt{\frac{x_i}{\mu}})} \quad + \frac{K_{\beta+\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}] = 0$$

$$\begin{aligned}\frac{t(1+2\beta)}{4\mu} &= \frac{1}{\mu^{\frac{3}{2}}} \sum_{i=1}^{t} \sqrt{x_i}[-\frac{\beta-\frac{1}{2}}{(2\sqrt{\frac{x_i}{\mu}})} \quad + \frac{K_{\beta+\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}] \\ &= \frac{1}{\mu^{\frac{3}{2}}} \sum_{i=1}^{t} \sqrt{x_i}[\frac{K_{\beta+\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})} - \frac{\beta-\frac{1}{2}}{(2\sqrt{\frac{x_i}{\mu}})}]\end{aligned}$$

$$\sum_{i=1}^{t} \sqrt{x_i}[\frac{K_{\beta+\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})} - \frac{\beta-\frac{1}{2}}{(2\sqrt{\frac{x_i}{\mu}})}] = \frac{t\ \mu^{\frac{3}{2}}\ (1+2\beta)}{4\mu}$$

differentiating with respect to (β)

$$\frac{\partial \log L(\varphi)}{\partial \beta} = \frac{-t \log \Gamma(\beta)}{\partial \beta} - \frac{\partial (1+2\beta)\frac{t}{4} \log \mu}{\partial \beta} + \frac{1}{2} \sum_{i=1}^{t} \log x_i + \sum_{i=1}^{t} \frac{\partial \log K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{\partial \beta}$$

$$
\begin{aligned}
-t\frac{\partial \log \Gamma\beta}{\partial\beta} =& t\frac{1}{\Gamma\beta}\frac{\partial\Gamma\beta}{\partial\beta} \\
=& -t\frac{1}{\Gamma\beta}\int_0^{\infty} e^{-y}y^{\beta-1}\ \log(y)\ d(y) \\
=& -t\frac{\Gamma\beta^{'}}{\Gamma\beta} \\
=& -t\psi(\beta)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial(1+2\beta)\frac{t}{4}\log\mu}{\partial\beta} =& \frac{\partial\log\mu(\frac{2\beta t}{4})}{\partial\beta} \\
=& \frac{t}{2}\log\mu
\end{aligned}
$$

$$
\frac{\partial \log K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{\partial\beta} = \frac{1}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}\ \frac{\partial K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{\partial\beta}
$$

$$
\frac{\partial \log L(\varphi)}{\partial\beta} = -t\psi(\beta) - \frac{t}{2}\log\mu + \frac{1}{2}\sum_{i=1}^{t}\log x_i + \sum_{i=1}^{t}\frac{1}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}\ \frac{\partial K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{\partial\beta}
$$

$$
\frac{\partial \log L(\varphi)}{\partial\beta} = 0
$$

$$
-t\psi(\beta) - \frac{t}{2}\log\mu + \frac{1}{2}\sum_{i=1}^{t}\log x_i + \frac{1}{K_{\beta-\sum_{i=1}^{t}\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}\ \frac{\partial K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{\partial\beta} = 0
$$

$$
\sum_{i=1}^{t}\frac{1}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}\frac{\partial K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{\partial\beta} = t\psi(\beta) + \frac{t}{2}\log\mu - \frac{1}{2}\sum_{i=1}^{t}\log x_i
$$

The two equations for the maximum likelihood estimators are,

$$\sum_{i=1}^{t}\sqrt{x_i}[\frac{K_{\beta+\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}-\frac{\beta-\frac{1}{2}}{(2\sqrt{\frac{x_i}{\mu}})}]=\frac{t\ \mu^{\frac{3}{2}}\ (1+2\beta)}{4\mu}$$

and,

$$\sum_{i=1}^{t}\frac{1}{K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}\frac{\partial K_{\beta-\frac{1}{2}}(2\sqrt{\frac{x_i}{\mu}})}{\partial\beta}=t\psi(\beta)+\frac{n}{2}\log\mu-\frac{1}{2}\sum_{i=1}^{t}\log x_i$$

Based on Ole Barndoff approach

$$\begin{aligned}
L(\varphi)=&\prod_{i=1}^{t}f(x_i,\varphi)\\
=&\prod_{i=1}^{t}\frac{1}{2^{\lambda-\frac{1}{2}}\sqrt{\pi}}\ \frac{1}{\Gamma\lambda}\ K_{\lambda-\frac{1}{2}}(x_i,\alpha)\ \alpha^{\lambda+\frac{1}{2}}\ x_i^{\lambda-\frac{1}{2}}\\
=&(\frac{1}{2^{\lambda-\frac{1}{2}}\sqrt{\pi}})^t\ (\frac{1}{\Gamma\lambda})^t\ \prod_{i=1}^{t}K_{\lambda-\frac{1}{2}}(x_i,\alpha)\ (\alpha^{\lambda+\frac{1}{2}})^t\ \prod_{i=1}^{t}x_i^{\lambda-\frac{1}{2}}\\
=&(\frac{1}{2^{\lambda-\frac{1}{2}}})^t\ (\frac{1}{\sqrt{\pi}})^t\ (\frac{1}{\Gamma\lambda})^t\ \prod_{i=1}^{t}K_{\lambda-\frac{1}{2}}(x_i,\alpha)\ (\alpha^{\lambda+\frac{1}{2}})^t\ \prod_{i=1}^{t}x_i^{\lambda-\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\log L(\varphi)=&(\lambda-\frac{1}{2})t\log 2-t\log\Gamma\lambda-t\log\sqrt{\pi}+(\lambda+\frac{1}{2})t\log\alpha+\sum_{i=1}^{t}\log K_{\lambda-\frac{1}{2}}(x_i,\alpha)+\sum_{i=1}^{t}\log x_i^{\lambda-\frac{1}{2}}\\
=&-t(\lambda-\frac{1}{2})\log 2-t\log\Gamma\lambda-t\log\sqrt{\pi}+t(\lambda+\frac{1}{2})\log\alpha+\sum_{i=1}^{t}\log K_{\lambda-\frac{1}{2}}(x_i,\alpha)\\
&+(\lambda-\frac{1}{2})\sum_{i=1}^{t}\log x_i
\end{aligned}$$

differentiating with respect to (α),

$$\frac{\partial\log L(\varphi)}{\partial\alpha}=\frac{t(\lambda+\frac{1}{2})}{\alpha}+\sum_{i=1}^{t}\frac{\partial}{\partial\alpha}\log K_{\lambda-\frac{1}{2}}(x_i,\alpha)$$

$$let, w = x_i, \alpha, \Rightarrow, \frac{\partial w}{\partial \alpha} = x_i$$

$$\begin{aligned}
\frac{\partial}{\partial \alpha} \log K_{\lambda-\frac{1}{2}}(x_i, \alpha) =& \frac{\partial}{\partial w} \log K_{\lambda-\frac{1}{2}}(x_i, \alpha)\ \frac{\partial w}{\partial \alpha} \\
=& \frac{1}{K_{\lambda-\frac{1}{2}}(x_i, \alpha)} \frac{\partial}{\partial w} K_{\lambda-\frac{1}{2}}(x_i, \alpha)\ \frac{\partial w}{\partial \alpha}
\end{aligned}$$

Recall,

$$\frac{\partial K_v(w)}{\partial w} = \frac{V}{w} K_v(w) - K_{v+1}(w)$$

therefore,

$$\begin{aligned}
\frac{\partial}{\partial \alpha} \log K_{\lambda-\frac{1}{2}}(x_i, \alpha) =& \frac{1}{K_{\lambda-\frac{1}{2}}(x_i, \alpha)}\ [\frac{\lambda - \frac{1}{2}}{\omega} K_{\lambda-\frac{1}{2}}(w) - K_{\lambda+\frac{1}{2}}(w)]\ \frac{\partial w}{\partial \alpha} \\
=& [\frac{\lambda - \frac{1}{2}}{w} - \frac{K_{\lambda+\frac{1}{2}}(w)}{K_{\lambda-\frac{1}{2}}(w)}]\ x_i
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \log L(\varphi)}{\partial \alpha} =& \frac{t(\lambda + \frac{1}{2})}{\alpha} + \sum_{i=1}^{t} x_i [\frac{\lambda - \frac{1}{2}}{w} - \frac{K_{\lambda+\frac{1}{2}}(w)}{K_{\lambda-\frac{1}{2}}(w)}] \\
=& \frac{t(\lambda + \frac{1}{2})}{\alpha} + \sum_{i=1}^{t} x_i [\frac{\lambda - \frac{1}{2}}{x_i, \alpha} - \frac{K_{\lambda+\frac{1}{2}}(x_i, \alpha)}{K_{\lambda-\frac{1}{2}}(x_i, \alpha)}]
\end{aligned}$$

$$\frac{\partial \log L(\varphi)}{\partial \alpha} = 0$$

$$\frac{t(\lambda + \frac{1}{2})}{\alpha} + \sum_{i=1}^{t} x_i [\frac{\lambda - \frac{1}{2}}{x_i, \alpha} - \frac{K_{\lambda+\frac{1}{2}}(x_i, \alpha)}{K_{\lambda-\frac{1}{2}}(x_i, \alpha)}] = 0$$

$$\sum_{i=1}^{t} x_i [\frac{\lambda - \frac{1}{2}}{x_i, \alpha} - \frac{K_{\lambda+\frac{1}{2}}(x_i, \alpha)}{K_{\lambda-\frac{1}{2}}(x_i, \alpha)}] = \frac{t(\lambda + \frac{1}{2})}{\alpha}$$

differentiating,with respect to,λ

$$\frac{\partial \log L(\varphi)}{\partial \lambda} = -t\log 2 - t\frac{\partial \log \Gamma\lambda}{\partial \lambda} + t\log\alpha + \sum_{i=1}^{t}\log x_i + \sum_{i=1}^{t}\frac{\partial}{\partial \lambda}\log K_{\lambda-\frac{1}{2}}(x_i,\alpha)$$

$$\begin{aligned} t\frac{\partial \log \Gamma\lambda}{\partial \lambda} =& t\frac{1}{\lambda}\frac{\partial}{\partial \lambda} \\ =& t\frac{\Gamma\lambda^{'}}{\Gamma\lambda} \\ =& t\psi(\lambda) \end{aligned}$$

hence,

$$\frac{\partial \log L(\varphi)}{\partial \lambda} = -t\log 2 - t\psi(\lambda) + t\log\alpha + \sum_{i=1}^{t}\log x_i + \sum_{i=1}^{t}\frac{\partial}{\partial \lambda}\log K_{\lambda-\frac{1}{2}}(x_i,\alpha)$$

$$\frac{\partial \log L(\varphi)}{\partial \lambda} = 0$$

$$-t\log 2 - t\psi(\lambda) + t\log\alpha + \sum_{i=1}^{t}\log x_i + \sum_{i=1}^{t}\frac{\partial}{\partial \lambda}\log K_{\lambda-\frac{1}{2}}(x_i,\alpha) = 0$$

$$\sum_{i=1}^{t}\frac{\partial}{\partial \lambda}\log K_{\lambda-\frac{1}{2}}(x_i,\alpha) = t\log 2 - t\psi(\lambda) + t\log\alpha - \sum_{i=1}^{t}\log x_i$$

The two equations for the maximum likelihood estimators are,

$$\sum_{i=1}^{t} x_i[\frac{\lambda-\frac{1}{2}}{x_i,\alpha} - \frac{K_{\lambda+\frac{1}{2}}(x_i,\alpha)}{K_{\lambda-\frac{1}{2}}(x_i,\alpha)}] = \frac{t(\lambda+\frac{1}{2})}{\alpha}$$

and,

$$\sum_{i=1}^{t}\frac{\partial}{\partial \lambda}\log K_{\lambda-\frac{1}{2}}(x_i,\alpha) = t\log 2 - t\psi(\lambda) + t\log\alpha - \sum_{i=1}^{t}\log x_i$$

6 Estimation Based on Method of Moments

6.1 Assymetric Variance Gamma

let q_t,denote the financial returns(log increments) where $t = 1, 2......$,
Two models have been discussed in this paper.These are:

i) Generalived Normal variance Mean Model(G. N .V.M)
where $q/v \sim N(a(b+v), c^2v+d^2)$ V⇒ denotes non negative random variable and a,b,c,d are constants

ii) Normal Variance Mean Model
let $\delta = ab, d = 0, c^2 = \sigma^2, a = \theta$
thus $q/v \sim N(\delta + \theta v, \sigma^2 v)$
where v is gamma distributed and q is variance gamma distributed

The general formula for central moments

$$M_k = E(X - E(X))^k, where, k = 2, 3, 4$$

Formula for skewness

$$\begin{aligned} \beta =& \frac{E(X-E(X))^3}{(E(X-E(X))^2)^{\frac{3}{2}}} \\ =& \frac{M_3}{(M_2)^{\frac{3}{2}}} \end{aligned}$$

$$\beta = \frac{M_3}{(M_2)^{\frac{3}{2}}} \quad (9)$$

Formula for Kurtosis

$$k = \frac{E(X-E(X))^4}{(E(X-E(X))^2)^2}$$
$$= \frac{M_3}{(M_2)^2}$$

$$k = \frac{M_3}{(M_2)^2} \tag{10}$$

The general formula for skewness

$$Skew(X) = \frac{E(X^3) - 3\mu^* E(X^2) + 2{\mu^*}^3}{{\sigma^*}^3}$$

For a normal distribution, $Skew(X) = 0$
hence,

$$E(X^3) = 3\mu^* E(X^2) - 2{\mu^*}^3 \tag{11}$$

The general formula for Kurtosis

$$Kurt(X) = \frac{E(X^4) - 4\mu^* E(X^3) + \sigma^* \mu^2 E(X^2) - 3{\mu^*}^4}{{\sigma^*}^4}$$

For a normal distribution, $Kurt(X) = 3$
hence,

$$E(X^4) = 3{\sigma^*}^4 + 4\mu^* E(X^3) - \sigma^* \mu^2 E(X^2) + 3{\mu^*}^4 \tag{12}$$

6.1.1 Normal Variance Mean Model

where,

$$q/v \sim N(\delta + \theta v, \sigma^2)$$

Central moments of Gamma distribution

$k = 2,3,4$ and $v \sim gamma\ distributed$,Assuming$E(v) = 1$

Second Gamma Central Moments

$$
\begin{aligned}
M_2 =& E(v - E(v))^2 \\
=& E(v-1)^2 \\
=& E(v^2 - 2v + 1) \\
=& E(v^2) - 1
\end{aligned}
$$

hence,

$$E(v^2) = M_2 + 1 \tag{13}$$

Third Gamma Central Moments

$$
\begin{aligned}
M_3 =& E(v - E(v))^3 \\
=& E(v-1)^3 \\
=& E(v^3 - 3v^2 + 3v - 1) \\
=& E(v^3) - 3E(v^2) + 2 \\
=& E(v^3) - 3(M_2 + 1) + 2 \\
=& E(v^3) - 3(M_2) - 3 + 2 \\
=& E(v^3) - 3(M_2) - 1
\end{aligned}
$$

hence,

$$E(v^3) = M_3 + 3M_2 + 1 \tag{14}$$

Fourth Gamma Central Moments

$$
\begin{aligned}
M_4 =& E(v-E(v))^4 \\
=& E(v-1)^4 \\
=& E(v^4-4v^3+6v^2-4v+1) \\
=& E(v^4)-4E(v^3)+6E(v^2)-4+1 \\
=& E(v^4)-4(M_3+3M_2+1)+6(M_2+1)-4+1 \\
=& E(v^4)-4M_3-12M_2-4+6M_2+6-4+1 \\
=& E(v^4)-4M_3-6M_2-1
\end{aligned}
$$

hence,

$$E(v^4) = M_4+4M_3+6M_2+1 \tag{15}$$

Raw moments of variance Gamma distribution

$$q/v \sim N(\delta+\theta v, \sigma^2 v)$$

and, $q \sim varianceGamma$, $E(v) = 1$

First variance Gamma Raw Moments

$$
\begin{aligned}
E(q) =& E(E(q/v)) \\
=& E(\delta+\theta v) \\
=& \delta+\theta E(v) \\
=& \delta+\theta
\end{aligned}
$$

$$E(q) = \delta+\theta \tag{16}$$

Second variance Gamma Raw Moments

$$\begin{aligned}
E(q^2) =& E(E(q^2/v)) \\
=& E(var(q/v) + (E(q/v))^2) \\
=& E(\sigma^2 v + (\delta + \theta v)^2) \\
=& \sigma^2 E(v) + E(\delta^2 + 2\theta\delta v + \theta^2 v^2) \\
=& \sigma^2 + \delta^2 + 2\theta\delta + \theta^2 E(v^2) \\
=& \sigma^2 + \delta^2 + 2\theta\delta + \theta^2 (M_2 + 1)
\end{aligned}$$

Second variance Gamma Central Moments

$$\begin{aligned}
Var(q) =& E(q^2) - (E(q))^2 \\
=& \sigma^2 + \delta^2 + 2\theta\delta + \theta^2 M_2 + \theta^2 - (\delta + \theta)^2 \\
=& \sigma^2 + \delta^2 + 2\theta\delta + \theta^2 M_2 + \theta^2 - \delta^2 - 2\delta\theta - \theta^2 \\
=& \sigma^2 + \theta^2 M_2
\end{aligned}$$

$$Var(q) = \sigma^2 + \theta^2 M_2 \tag{17}$$

Third variance Gamma Raw Moments

$E(q^3) = E(E(q^3/v))$
recall,equation(11)

$$\begin{aligned}
E(q^3) =& E(3\mu^* E(q^2/v) - 2\mu^{*^3}) \\
=& E(3(\delta + \theta v)(\sigma^2 v + (\delta + \theta v)^2) - 2(\delta + \theta v)^3) \\
=& E(3(\delta + \theta v)\ \sigma^2 v + (\delta + \theta v)^3)) \\
=& E(3\delta\sigma^2 v + 3\theta\sigma^2 v^2 + \delta^3 + 3\theta\delta^2 v + 3\theta^2\delta v^2 + \theta^3 v^3) \\
=& 3\delta\sigma^2 + 3\theta\sigma^2 E(v^2) + \delta^3 + 3\theta\delta^2 + 3\theta^2\delta E(v^2) + \theta^3 E(v^3)
\end{aligned}$$

Third variance Gamma Central Moments

$$\begin{aligned}
Q_3 =& E(q-E(q))^3\\
=& E(q^3-3q^2E(q)+3Y(E(q))^2-(E(q))^3)\\
=& E(q^3)-3E(q^2)(\delta+\theta)+3E(q)(E(q))^2-(E(q))^3\\
=& E(q^3)-3E(q^2)(\delta+\theta)+3(\delta+\theta)(\delta+\theta)^2-(\delta+\theta)^3\\
=& E(q^3)-3E(q^2)(\delta+\theta)+2(\delta+\theta)^3\\
=& E(q^3)-(3\sigma^2+\delta^2+2\theta\delta+\theta^2E(v^2))(\delta+\theta)+2(\delta+\theta)^3\\
=& 3\delta\sigma^2+3\theta\sigma^2E(v^2)+\delta^3+3\theta\delta^2+3\theta^2\delta E(v^2)+\theta^3E(v^3)-3\delta\sigma^2-3\delta^3-2\theta\delta^2-3\delta\theta^2E(v^2)\\
&-3\theta\sigma^2-3\theta\delta^2-6\theta^2\delta-3\theta^3E(v^2)+2\delta^3+6\theta\delta^2+6\theta^2\delta+2\theta^3\\
=& 3\theta\delta^2E(v^2)+\theta^3E(v^3-3\theta\delta^2-3\theta^3E(v^2)+2\theta^3)
\end{aligned}$$

recall,equation(13)

$$E(v^2)=M_2+1$$

and, recall,equation(13)

$$E(v^3)=M_3+3M_2+1$$

substitute to the equation,

$$\begin{aligned}
Q_3 =& 3\theta\delta^2(M_2+1)+\theta^3(M_3+3M_2+1)-3\theta\delta^2-3\theta^3(M_2+1)+2\theta^3)\\
=& 3\theta\delta^2M_2+\theta^3M_3
\end{aligned}$$

$$Q_3=3\theta\delta^2M_2+\theta^3M_3 \tag{18}$$

Fourth Variance Gamma Raw Moments

$E(q^4)=E(E(q^4/v))$
recall,equation(12)

$$E(X^4)=3\sigma^{*^4}+4\mu^*E(X^3)-\sigma^*\mu^2E(X^2)+3\mu^{*^4}$$

hence,

$$
\begin{aligned}
E(q^4/v) =& 3\sigma^{*^4} + 4\mu^* E(q^3/v) - \sigma^* \mu^2 E(q^2/v) + 3\mu^{*^4} \\
=& 3\sigma^{*^4} + 4\delta(E(q^3/v)) + 4\theta v E(q^3/v) + 3(\delta+\theta v)^4 - 6(\delta+\theta v)^2(\sigma^2 v + (\delta+\theta v)^2) \\
=& 3\sigma^{*^4} + 4\delta(3\delta\sigma^2 v + 3\theta\sigma^2 v^2 + \delta^3 + 3\theta\delta^2 v + 3\theta^2\delta v^2 + \theta^3 v^3) + 4\theta v(3\delta\sigma^2 v + 3\theta\sigma^2 v^2 + \delta^3 \\
& + 3\theta\delta^2 v + 3\theta^2\delta v^2 + \theta^3 v^3) - 6(\delta^2 + 2\theta\delta v + \theta^2 v^2)(\sigma^2 v + \delta^2 + 2\theta\delta v + \theta^2 v^2) + 3(\delta^4 \\
& + 4\theta\delta^3 v + 6\theta^3\delta^2 v^3 + \theta^4 v^4) \\
=& 3\sigma^{*^4} + 4(3\delta^2\sigma^2 v + 3\theta\delta\sigma^2 v^2 + \delta^4 + 3\theta\delta^2 v + 3\theta^2\delta^2 v^2 + \delta\theta^3 v^3) + 4(3\delta\theta\sigma^2 v^2 + 3\theta^2\sigma^2 v^3 \\
& + \delta^3\theta v + 3\delta^2\theta^2 v^2 + 3\delta\theta^3 v^3 + \theta^4 v^4) - 6(\delta^2\sigma^2 v + \delta^4 + 2\delta^3\theta v + \delta^2\theta^2 v^2) - 6(2\theta\delta\sigma^2 v^2 \\
& + 2\theta\delta^3 v + 4\theta^2\delta^2 v^2 + 2\theta^3\delta v^3) - 6(\theta^2\delta^2 v^3 + \delta^2\theta^2 v^2 + 2\delta\theta^3 v^3 + \theta^4 v^4) + 3(\delta^4 + 4\theta\delta^3 v \\
& + 6\theta^2\delta^2 v^2 + 4\theta^3\delta v^3 + \theta^4 v^4) \\
=& 3\sigma^4 v^2 + \delta^4 + 12\theta\delta\sigma^2 v^2 + 6\delta^2\sigma^2 v + 4\delta^3\theta v + 6\theta^2\delta^2 v^2 + 6\theta^2\sigma^2 v^3 + 4\theta^3\delta v^3 + \theta^4 v^4
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E(q^4) =& 3\sigma^4 E(v^2) + \delta^4 + 12\theta\delta\sigma^2 E(v^2) + 6\delta^2\sigma^2 + 4\delta^3\theta + 6\theta^2\delta^2 E(v^2) + 6\theta^2\sigma^2 E(v^3) + 4\theta^3\delta E(v^3) \\
& + \theta^4 E(v^4)
\end{aligned}
$$

Fourth Variance Gamma Central Moments

$$
\begin{aligned}
Q_4 =& E(q-E(q))^4 \\
=& E(q^4-4q^3E(Y)+6q^2(E(q))^2-4Y(E(q))^3+(E(q))^4) \\
=& E(q^4)-4E(q^3)(\delta+\theta)+6E(q^2)(\delta+\theta)^2-3(\delta+\theta)^4 \\
=& E(q^4)-4E(q^3)(\delta+\theta)+6E(q^2)(\delta^2+2\theta\delta+\theta^2)-3(\delta+\theta)^4 \\
=& E(q^4)-4(\delta+\theta)(3\delta\sigma^2+3\theta\sigma^2E(v^2)+\delta^3+3\theta\delta^2+3\theta^2\delta E(v^2)+\theta^3E(v^3))+6(\delta+\theta)^2 \\
& (\sigma^2+\delta^2+2\theta\delta+\theta^2E(v^2))-3(\delta^4+4\theta\delta^3+6\theta^2\delta^2+4\theta^3\delta+\theta^4) \\
=& 3\sigma^4E(v^2)+\delta^4+12\theta\delta\sigma^2E(v^2)+6\delta^2\sigma^2+4\delta^3\theta+6\theta^2\delta^2E(v^2)+6\theta^2\sigma^2E(v^3)+4\theta^3\delta E(v^3) \\
& +\theta^4E(v^4)-4(\delta+\theta)(3\delta\sigma^2+3\theta\sigma^2E(v^2)+\delta^3+3\theta\delta^2+3\theta^2\delta E(v^2)+\theta^3E(v^3))+6(\delta+\theta)^2 \\
& (\sigma^2+\delta^2+2\theta\delta+\theta^2E(v^2))-3(\delta^4+4\theta\delta^3+6\theta^2\delta^2+4\theta^3\delta+\theta^4) \\
=& 3\sigma^4E(v^2)+6\theta^2\sigma^2E(v^3)+\theta^4E(v^4)-12\theta^2\sigma^2E(v^2)-4\theta^4E(v^3)-3\theta^4+6\theta^4E(v^2)+6\theta^2\sigma^2 \\
=& 3\sigma^4E(v^2)+6\theta^2\sigma^2(E(v^3)-2E(v^2)+1)+\theta^4(E(v^4)-4E(v^3)+6E(v^2)-3) \\
=& 3\sigma^4(M_2+1)+6\theta^2\sigma^2(M_3+3M_2+1-2(M_2+1))+\theta^4(M_4+4M_3+6M_2+1-4(M_3+3M_2+1) \\
& +6(M_2+1)-3) \\
=& 3\sigma^4(M_2+1)+6\theta^2\sigma^2(M_3+M_2+\theta^4M_4)
\end{aligned}
$$

$$
Q_4 = 3\sigma^4(M_2+1)+6\theta^2\sigma^2(M_3+M_2+\theta^4M_4) \tag{19}
$$

Skewness Variance Gamma

recall,equation(9) therefore,

$$
\beta = \frac{3\theta\delta^2M_2+\theta^3M_3}{(\sigma^2+\theta^2M_2)^{\frac{3}{2}}} \tag{20}
$$

Kurtosis Variance Gamma

recall,equation(10) therefore,

$$
k = \frac{3\sigma^4(M_2+1)+6\theta^2\sigma^2(M_3+M_2+\theta^4M_4)}{(\sigma^2+\theta^2M_2)^2} \tag{21}
$$

Parameters

- $\hat{\sigma^2} = s^2$
- $\hat{v} = \frac{\hat{k}}{3} - 1$
- $\hat{\theta} = \frac{\hat{\beta} s}{k-3}$
- $\hat{\delta} = \hat{\mu} - \theta$

6.2 Normal Variance Model

let X_t,denote the financial returns(log increments) where $t = 1, 2......,$
and,
$X/\lambda \sim N((\mu + \beta\lambda), \sigma^2\lambda)$
where λ is gamma distributed and X is variance gamma distributed let,$\mu = \beta = 0, and, \sigma^2 = 1$

Gamma Distribution Moments

Recall,

$$E(\lambda) = \mu\beta$$
$$E(\lambda^2) = \mu^2(\beta)(\beta + 1)$$
$$E(\lambda^3) = \mu^3(\beta)(\beta + 1)(\beta + 2)$$
$$E(\lambda^2) = \mu^4(\beta)(\beta + 1)(\beta + 2)(\beta + 3)$$

Central moments of Variance Gamma distribution

First Central moment

$$\begin{aligned} E(X) =& E(E(X/\lambda)) \\ =& E(\mu + \beta\lambda) \\ =& \mu + \beta E(\lambda) \\ =& 0 \end{aligned}$$

Second Raw moment

$$\begin{aligned}E(X^2) =&E(E(X^2/\lambda))\\ =&E(var(X/\lambda)+(E(X/\lambda)^2)\\ =&E(\sigma^2\lambda+0)\\ =&\sigma^2E(\lambda)\\ =&E(\lambda)\end{aligned}$$

Second Central moment

$$\begin{aligned}var(X) =&E(X^2)-(E(X))^2\\ =&E(\lambda)-0\\ =&E(\lambda)\\ =&\mu\beta\end{aligned}$$

Third Raw moment

recall,equation(11) thus,

$$\begin{aligned}E(X^3) =&E(E(X^3/\lambda))\\ =&E(3\mu^*E(X^2/\lambda)-2\mu^{*^3})\\ =&E(3(\mu+\beta\lambda)\sigma^2\lambda-2(\mu+\beta\lambda)^3)\\ =&0\end{aligned}$$

Fourth Raw moment

Recall,equation(12) thus,

$$\begin{aligned} E(X^4) =& E(E(X^4/\lambda)) \\ =& E(3\sigma^{*^4} + 4(\mu+\beta\lambda)E(X^3/\lambda) - \sigma^*\mu^*E(X^2) + 3\mu^{*^4}) \\ =& E(3\sigma^4\lambda^2) \\ =& 3\sigma^4 E(\lambda^2) \\ =& 3E(\lambda^2) \\ =& 3(\mu^2\beta(\beta+1)) \end{aligned}$$

$$Skewness = Skew(X) = 0$$

$$\begin{aligned} kurtosis = k =& \frac{3(\mu^2\beta(\beta+1))}{\mu\beta} \\ =& \frac{3(\beta+1)}{\beta} \end{aligned}$$

6.3 Normal Inverse Gaussian Distribution

6.3.1 Assymetric Normal Inverse Gaussian

Let q_t,denote the financial returns(log increments) where $t = 1, 2......$,

Normal Variance Mean

Assymetric Normal inverse Gaussian

let the conditional distribution is normally distributed takin the form $q/v \sim N(\mu+\beta v, \sigma^2 v)$ where v is Inverse Gaussian distributed taking the form $v \sim IG(\alpha, \lambda)$ and q is Normal Inverse Gaussian distributed

Symetric Normal Inverse Gaussian

Let$\beta = 0$ the conditional distribution is normally distributed takin the form $q/v \sim N(\mu, \sigma^2 v)$
where v is Inverse Gaussian distributed taking the form $v \sim IG(\alpha, \lambda)$ and q is Normal Inverse Gaussian distributed

Normal Variance

Let$\beta = 0, \mu = 0, \sigma^2 = 1$ the conditional distribution is normally distributed taking the form $q/v \sim N(0, v)$
where v is Inverse Gaussian distributed taking the form $v \sim IG(\alpha, \lambda)$ and q is Normal Inverse Gaussian distributed

6.3.2 Central Moments of Normal Inverse Gaussian

- $E(q) = \mu + \frac{\lambda\beta}{\alpha}$
- $var(q) = \frac{\lambda\delta^2}{\alpha^3}$
- $Skewness = a = \frac{3\beta}{\delta(\alpha^{\lambda})^{\frac{1}{2}}}$
- $Kurtosis = b = \frac{3(1+4\frac{\beta^2}{\delta^2})}{\alpha^{\lambda}}$

6.3.3 Parameters

- $\hat{\alpha} = \frac{3}{s\sqrt{3b_2 - 5\overline{a_1}^2}}$
- $\hat{\beta} = \frac{\overline{a_1} s \hat{\alpha}}{\hat{\beta^2} + \hat{\alpha^2}}$
- $\hat{\lambda} = \frac{s^2 \hat{\alpha}}{\hat{\beta^2} + \hat{\alpha^2}}$
- $\hat{\mu} = \overline{x} + \frac{\hat{\beta\lambda}}{\hat{\alpha}}$

7 Data Analysis

The analysis of data was done by fitting Standard and Poor's from January 1977 to December 1981 on the respective distribution.

Table 1. Descriptive statistics

Stocks Return data	sample size	samplemean	σ	sample skewness	sample kurtosis
S and P 500	1262	0.000106	0.008032	-0.022525	4.215883

The table above gives the descriptive statistics for the parameter estimates.The parameter estimates were estimated using raw data readings on S and p 500 Index as from 3^{rd} January 1977 to 31^{st} December 1981.

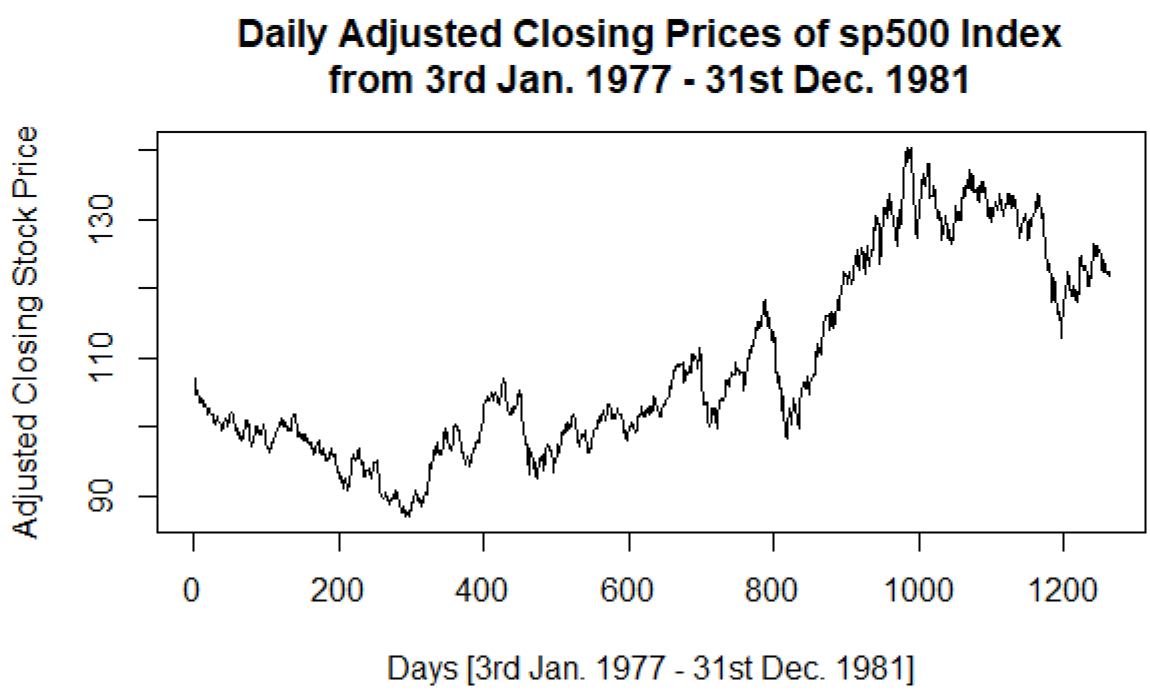

Figure 3. Graph for the adjusted closing prices

The figure above gives a vivid description on the stock movement based on the adjusted closing prices for S and p 500 Index as from 3^{rd} January 1977 to 31^{st} December 1981.

7.1 Variance Gamma Distribution

Tables extracted from the results

Table 2. Parameters Description

S and P 500	θ	δ	σ
skewed	-0.00014305	0.00024905	0.008032
symetric	0	0.000106	0.00803
N.V	0	0	1

Table 2: above gives a description of the parameter estimates for Variance Gamma distribution.The estimates are given based on the specified three models Skewed,Symmetric and N.V.

Goodness of fit

Analysis for Goodness of fit was based on the log likelihood and AIC.

Table 3. Goodness of fit

Distribution	Skewed VG	Symmetric VG	Nad VG
Log likehood	4315.2145216	4314.5013559	4314.103
AIC	-8622.4290432	-8623.0027117	-8624.206

Table 3: gives values of Goodness of fit based on the three models.The analysis was done based on the log likelihood and AIC test. Skewed Variance Gamma has a better goodness of fit as compared to the others since it gives a greater log likelihood value.Based on AIC, Variance Gamma Distribution constructed(Nadarajah's approach) gives a lower value as compared to Skewed and Symmetric hence can be taken as the best model for fitting stock returns.

Graphs

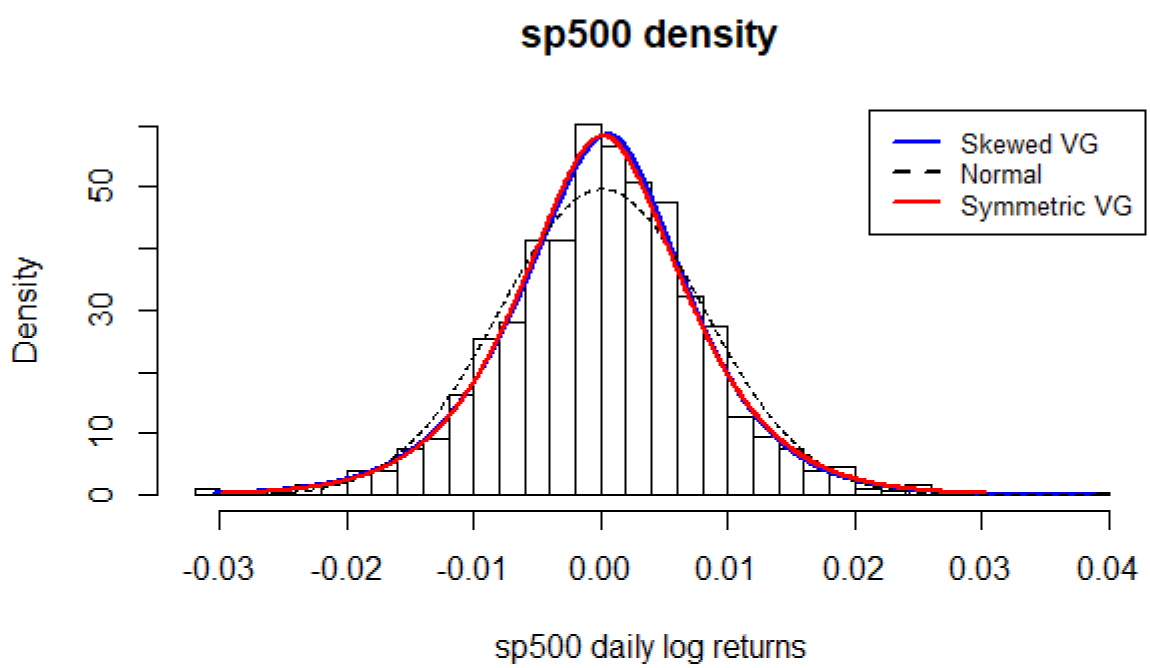

Figure 4. Skewed and symmetric

The graph above gives a comparison of the Skewed VG, Symmetric VG and Normal distribution. Both Skewed VG and Symmetric VG are highly peaked at the center as compared to the normal distribution .Though Skewed VG tends to be slightly higher than Symmetric.

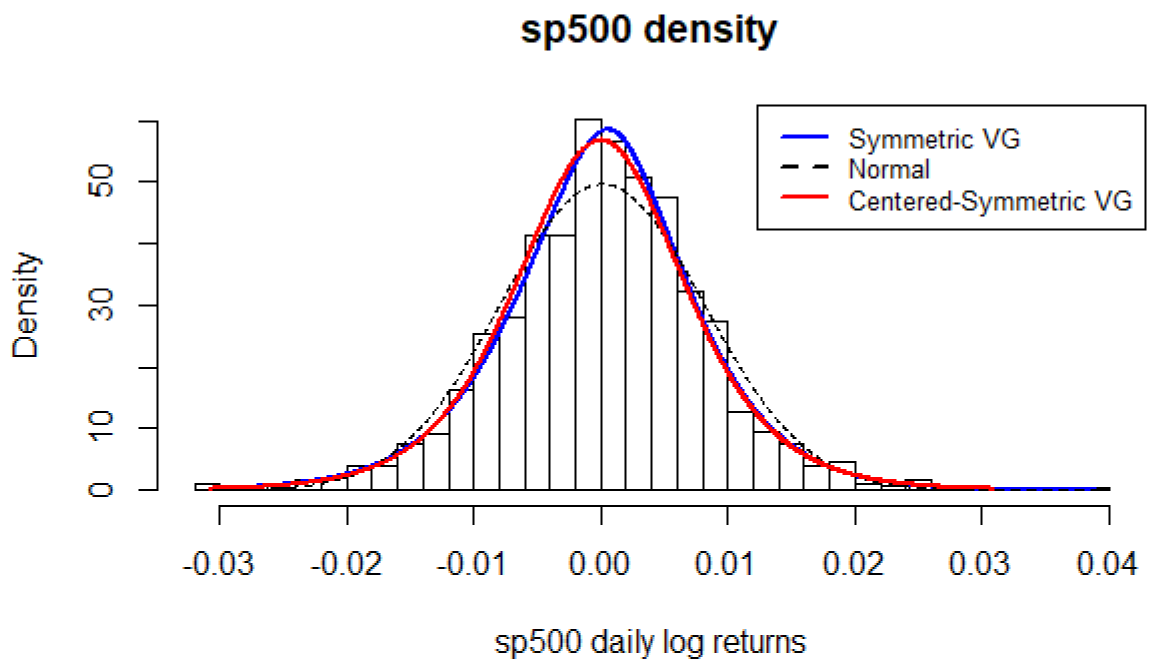

Figure 5. Symmetric and Centered - symmetric

The graph above gives a comparison of the Symmetric VG, Centered -Symmetric VG(Nadarajah's Approach) and Normal distribution. Both Symmetric VG and Centered -Symmetric VG(Nadarajah's Approach) are highly peaked at the center as compared to the normal distribution .Though Symmetric VG tends to be slightly higher than Centered -Symmetric.

7.2 Normal Inverse Gaussian Distribution

7.2.1 Tables extracted from the results

Table 4. Parameters Description

S and P 500	β	λ	α	μ
skewed	-11.2594	0.01211	188.7253	0.00083
symetric	0	0.01223	189.5669	0.00018
N.V	0	0.01233	191.2142	0

Table 4: above gives a description of the parameter estimates for Normal Inverse Gaussian Distribution.The estimates are given based on the specified three models Skewed,Symmetric and N.V.

Goodness of fit

Analysis for Goodness of fit was based on the log likelihood and AIC.

Table 5. Goodness of fit

NIG	Skewed	Symmetric	N.V
Log likehood	4315.0755799	4314.5836310	4314.2525
AIC	-8622.1511597	-8623.1672619	-8624.5045

Table 5: gives values of Goodness of fit based on the three models.The analysis was done based on the log likelihood and AIC test. Skewed NIG has a better goodness of fit as compared to the others since it gives a greater log likelihood value.Based on AIC, NIG Distribution constructed(Nadarajah's approach) gives a lower value as compared to Skewed and Symmetric hence can be taken as the best model for fitting stock returns.

Graphs

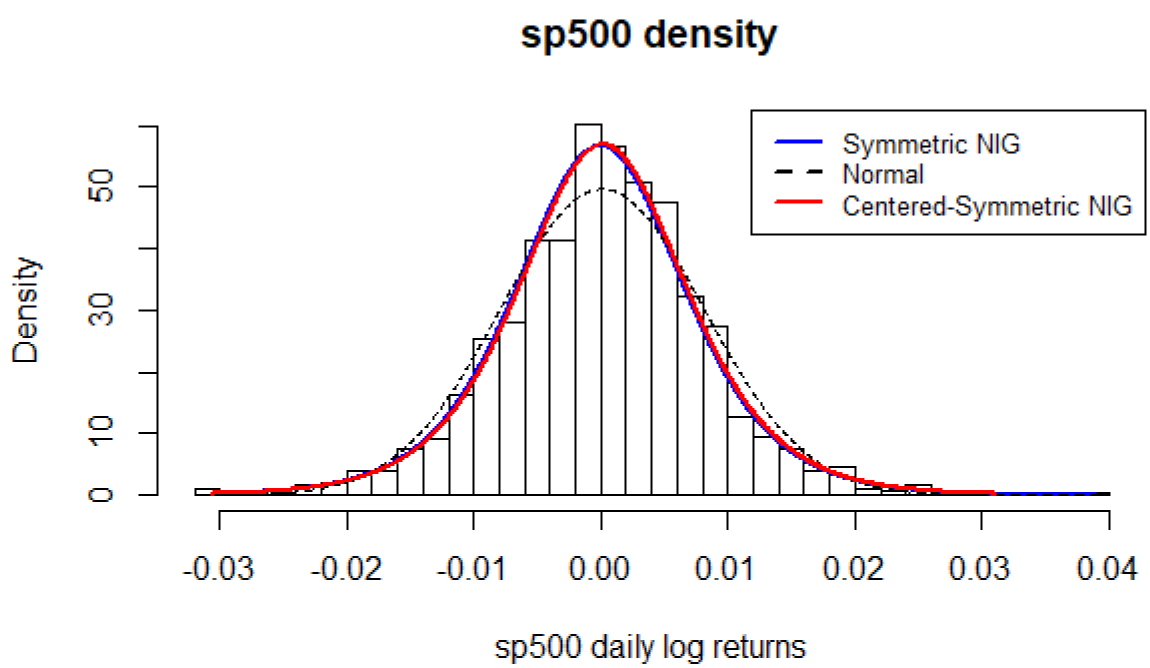

Figure 6. Symmetric and Centered - symmetric

The graph above gives a comparison of the Symmetric NIG, Centered -Symmetric NIG (Nadarajah's Approach) and Normal distribution. Both Symmetric NIG and Centered -Symmetric NIG(Nadarajah's Approach) are highly peaked at the center as compared to the normal distribution .

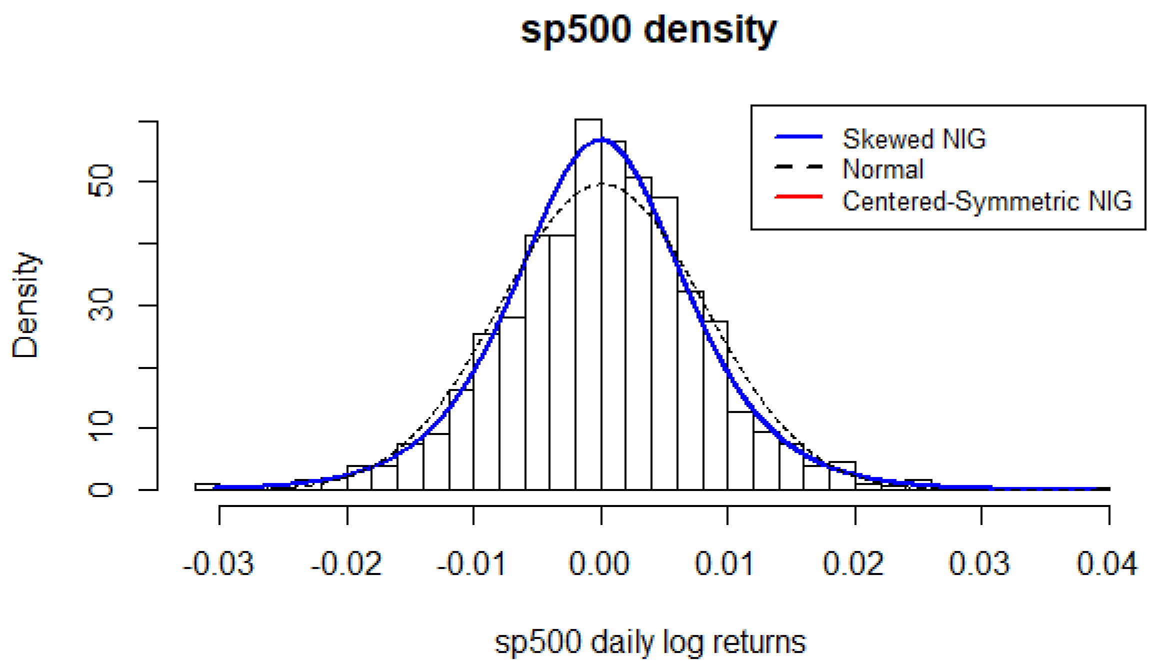

Figure 7. Skewed and Centered - symmetric

The graph above gives a comparison of the Skewed NIG, Centered -Symmetric NIG (Nadarajah's Approach) and Normal distribution. Both Skewed NIG and Centered - Symmetric NIG(Nadarajah's Approach) are highly peaked at the center as compared to the normal distribution .

8 Conclusion and Recommendations

8.1 Conclusion

According to the analysis done on the three Models Asymetric(Skewed), Symmetric and the distribution constructed using Nadarajah's approach on the respective Distibutions that is NIG and VG.

The log likelihood test shows that the Skewed Variance Gamma distribution gives the best goodness of fit since it has a greater log likelihood value as compared to the other distributions.According to AIC test Nadarajah's Approach Normal Inverse Gaussian distribution tends to be the best model for fitting stock returns this is because it gives a lower value as compared to the others.

8.2 Recommendation

1. Estimation of the respective distributions can be done based on EM algorithm
2. Study can be done to investigate whether or not there is a relation between Variance Gamma distribution and Normal Inverse Gaussian distribution with an absorbing Brownian Motion.

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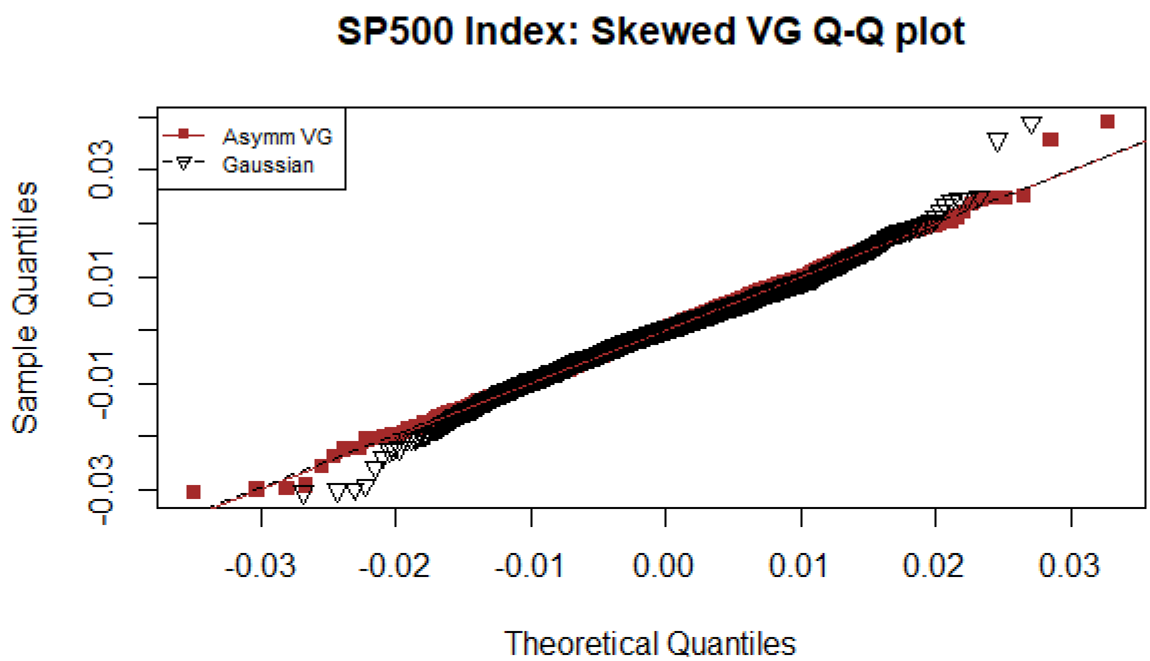
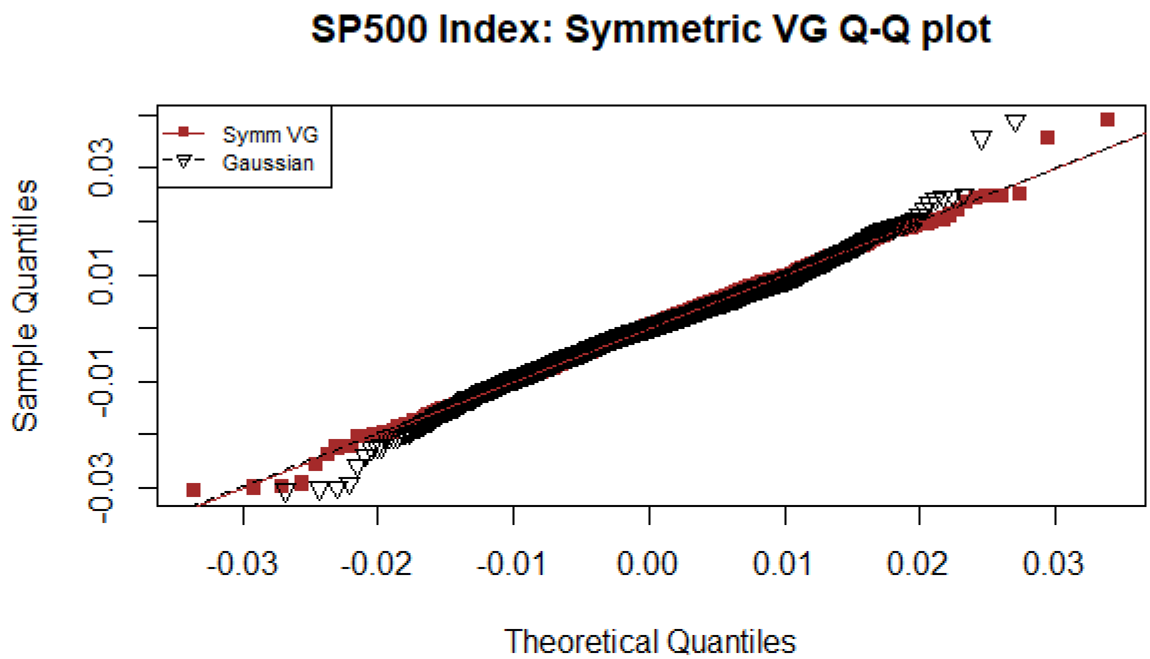
9 Appendix

9.1 QQ Plots





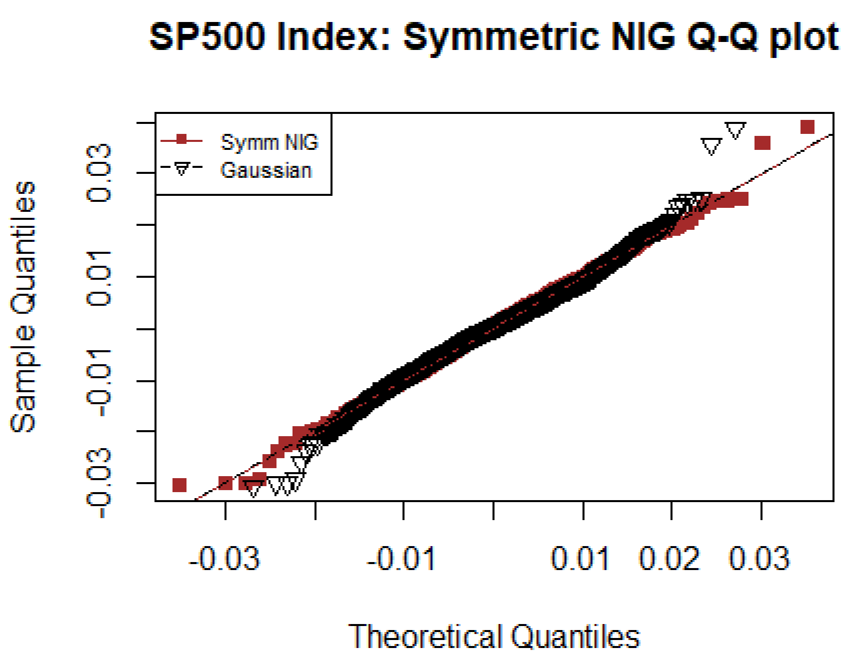


Figure 8. Comparison of the distributions

Graphs

Functions and Properties

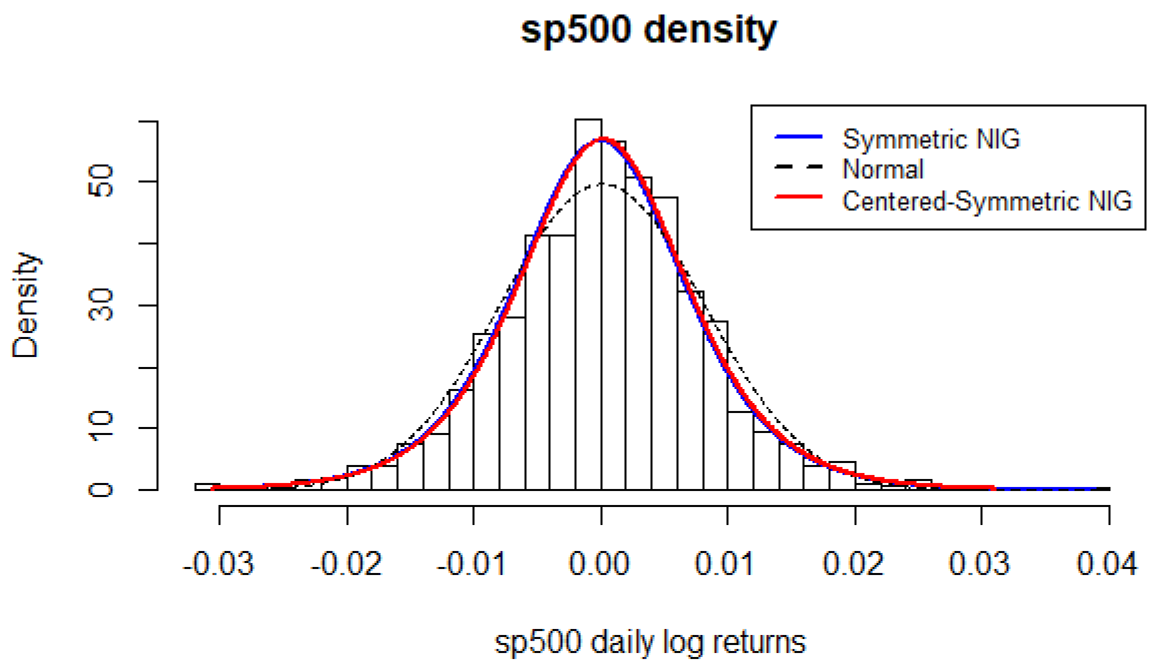


Figure 9. Comparison of the distributions

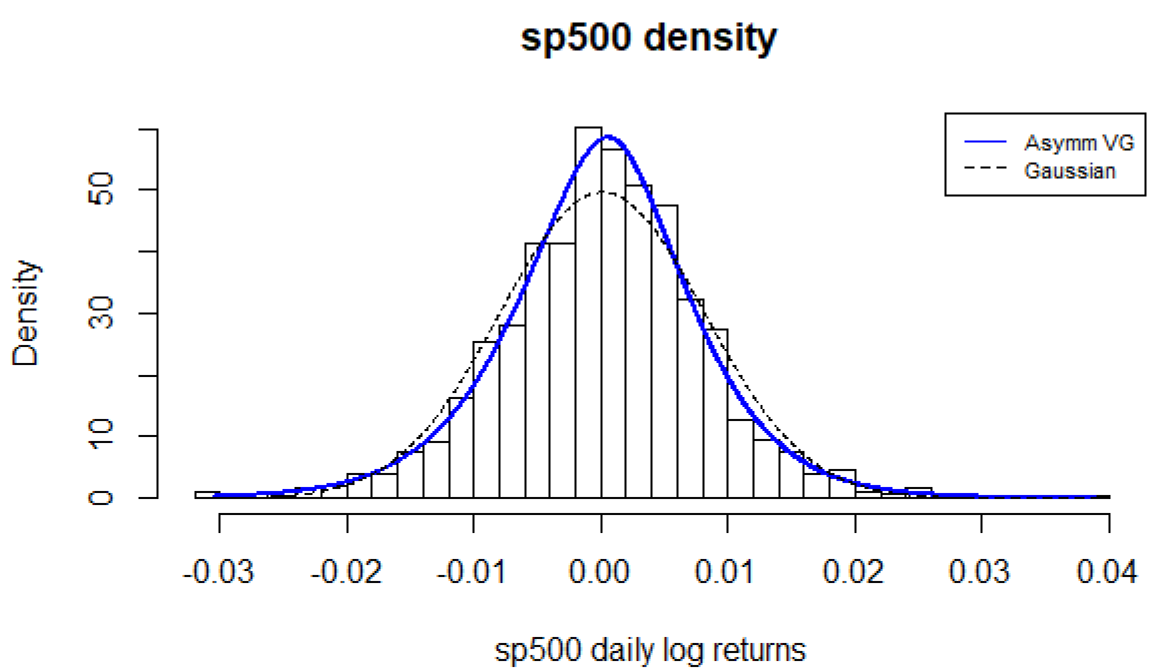


Figure 10. Assymetric and Normal distribution graph

$$\int_0^\infty \frac{\beta^\alpha}{\Gamma\alpha} e^{-\beta\alpha} x^{\alpha-1} dx = 1$$

$$\frac{\Gamma\alpha}{\beta^\alpha} = \int_0^\infty e^{-\beta\alpha} x^{\alpha-1} dx$$

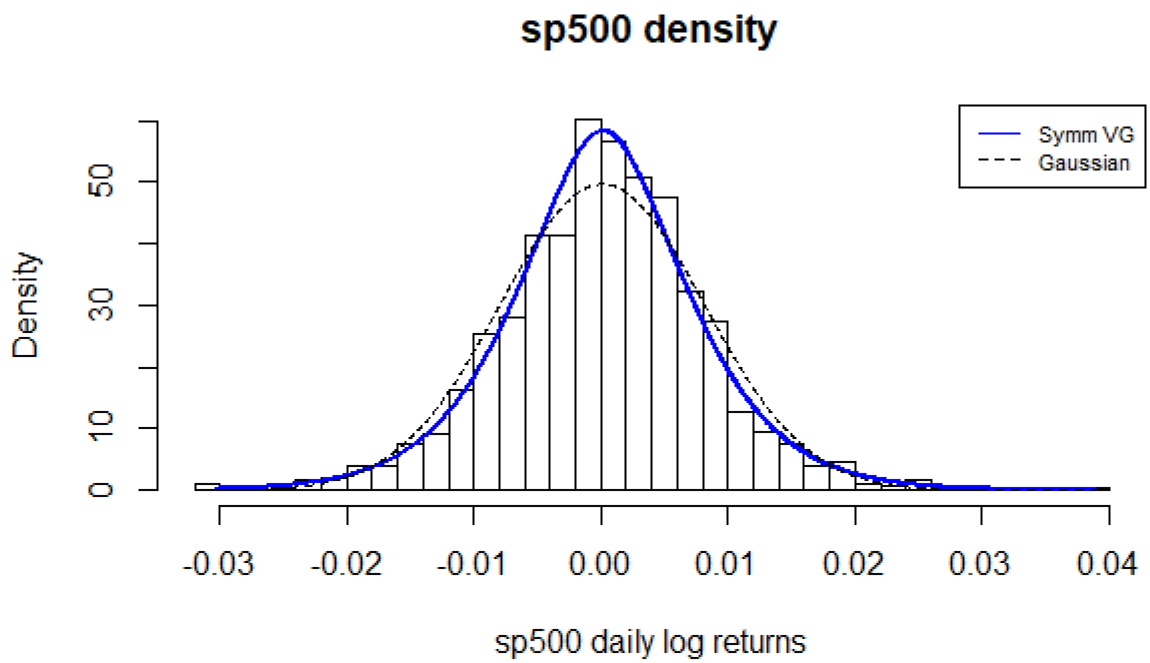


Figure 11. Symetric and Normal distribution graph

skewNIgnormempskeVG

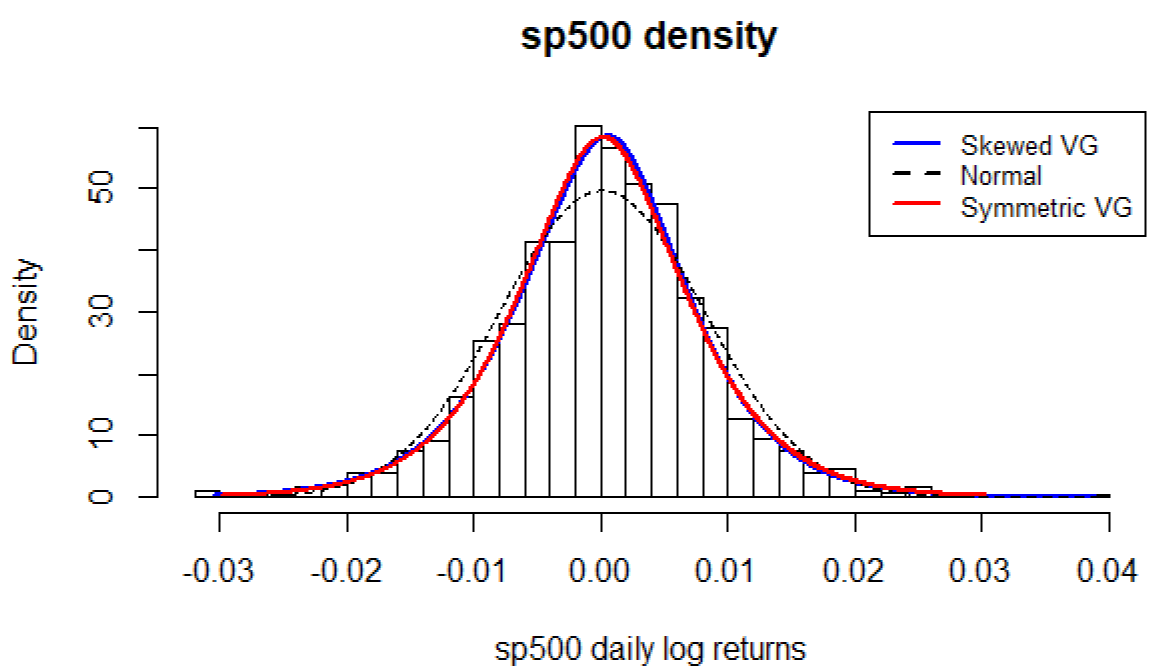


Figure 12. Comparison of the distributions

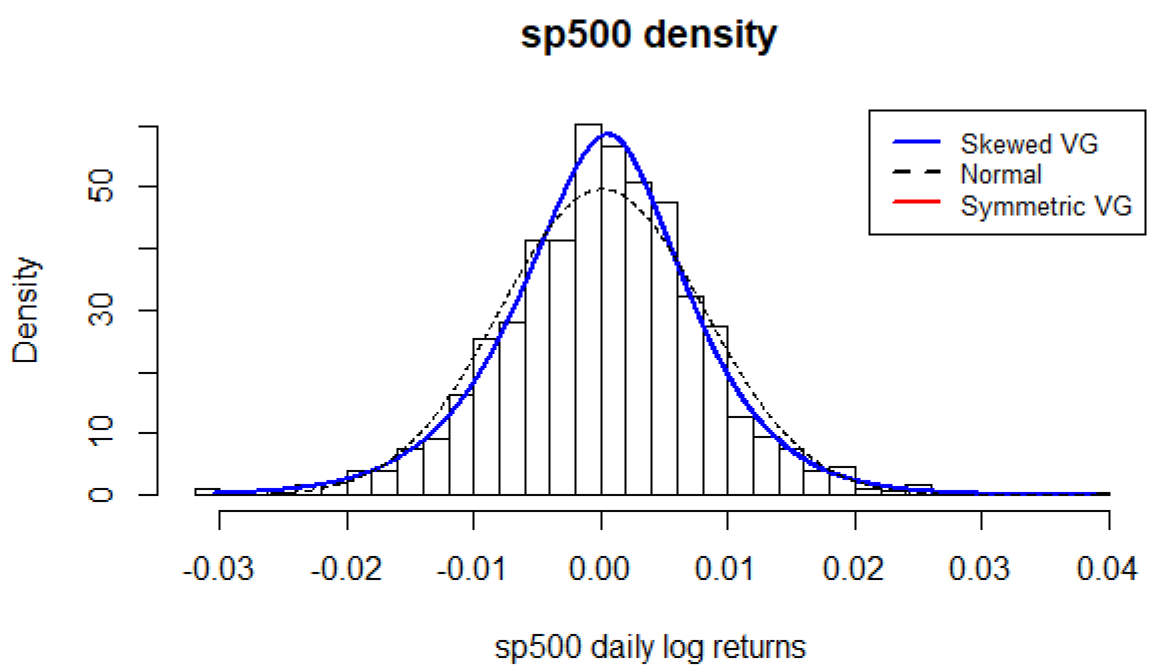


Figure 13. Comparison of the distributions

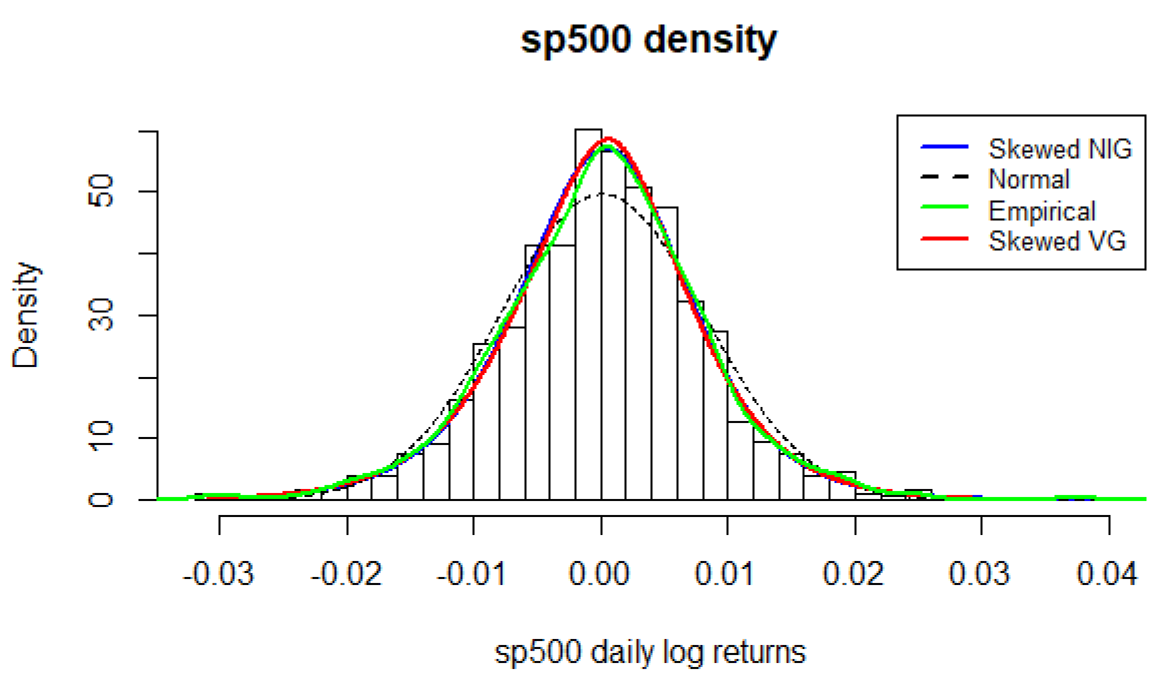


Figure 14. Comparison of the distributions

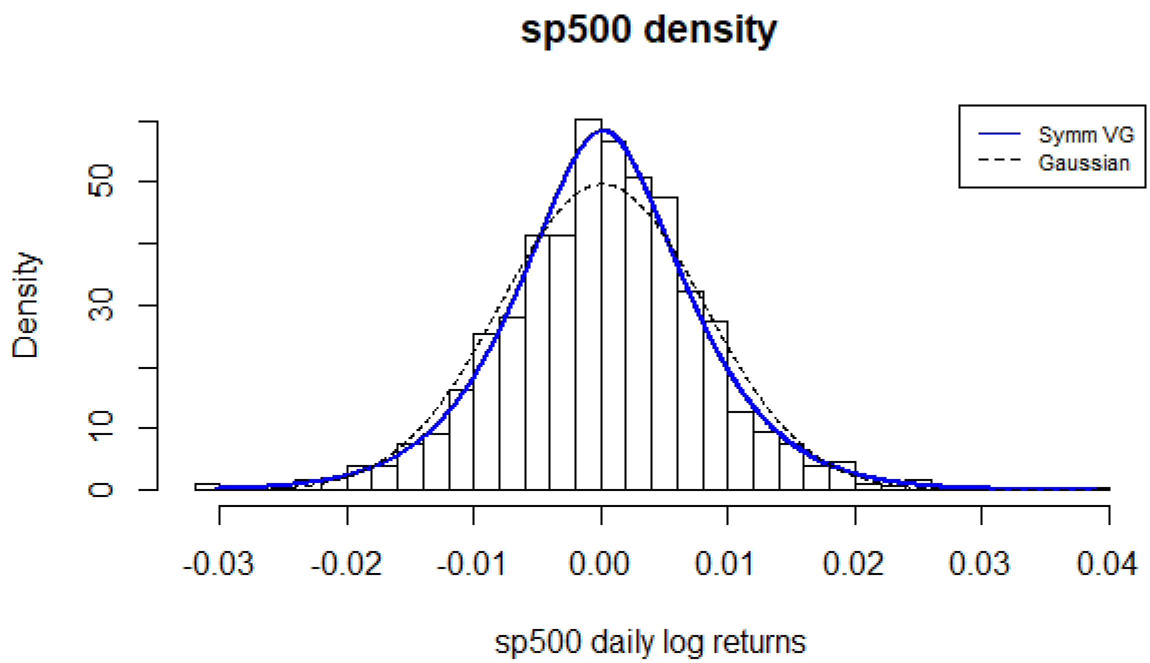


Figure 15. Comparison of the distributions

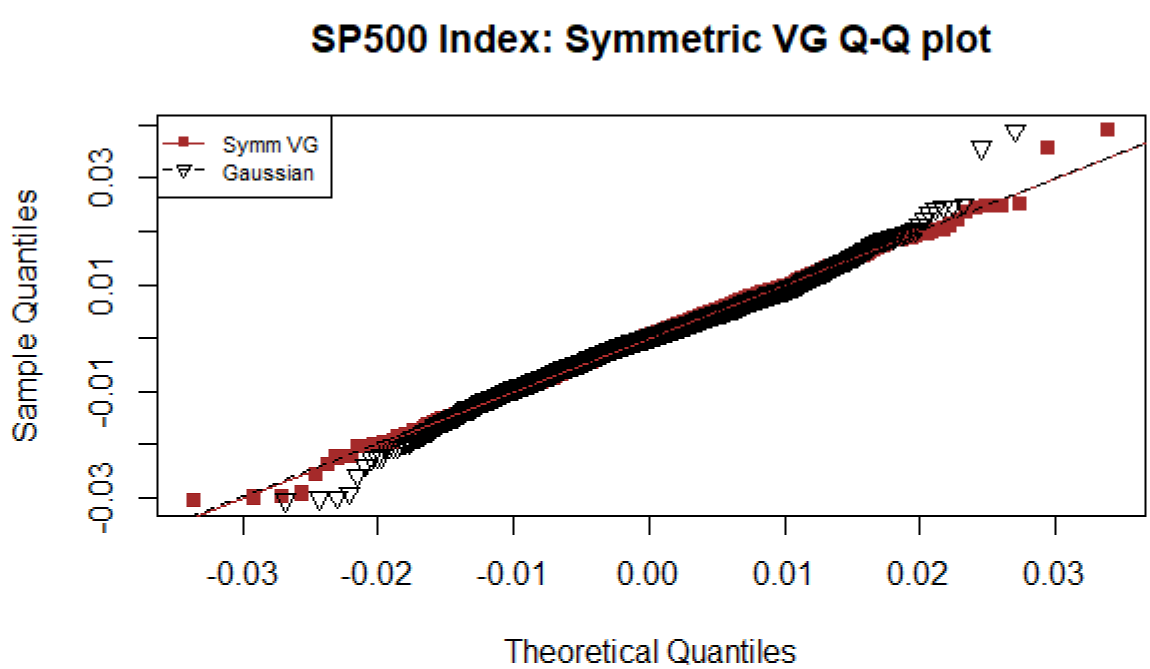


Figure 16. Comparison of the distributions