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Currency Option Valuation using Esscher and Fourier Transforms

Research Report in Mathematics, Number 4, 2019

Simon Muoria Kamau

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Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Actuarial Science

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Simon Muoria Kamau

School of Mathematics
College of Biological and Physical Sciences
Chiromo, off Riverside Drive
30197-00100 Nairobi, Kenya

Master Thesis
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Abstract

The size and volatility of the exchange rate market warrants the use of currency options in order to hedge the exchange rate risk. This study sought to build a robust pricing model that takes into account some of the stylized facts of exchange rates reported in literature. The normal inverse Gaussian distribution is chosen to model the exchange rate returns as: one, its higher order moments exist, thus can capture skewness and excess kurtosis unlike the two-parameter normal distribution; and two, since it is a normal mixture, the aggregational Gaussianity property holds. To price options written on these exchange rates under the risk-neutral measure, a change of measure is required as this proposed model renders the market incomplete, which implies the existence of a range of equivalent martingale measures (by the fundamental theorem of asset pricing). To choose a unique martingale measure, we apply the Esscher measure. Pricing equations are, thus, derived as the expected discounted value of the payoffs with respect to this risk-neutral measure. The fast Fourier transform is then applied to compute option prices due to its computational efficiency. From the results presented herein, the NIG distribution results in a better fit of the empirical distribution of the exchange rate returns and the corresponding option pricing model, the Esscher-NIG model, outperforms the Black-Scholes model in pricing performance.

Keywords: FFT; Esscher transform; leptokurtic returns; normal mixtures; stochastic processes; equivalent martingale measures; Fourier inversion; option pricing; non-normality.

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Declaration and Approval

I, the undersigned, declare that this dissertation is my original work and to the best of my knowledge, has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

SIMON MUORIA KAMAU
Reg No. I56/7891/2017

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

Dr. Ivivi J. Mwaniki,
School of Mathematics,
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: jimwaniki@uonbi.ac.ke

Dedication

To my parents, Dan & Ann, and my family in its entirety.

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Simon Muoria Kamau

Nairobi, 2019.

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Abbreviations and Acronyms

AIC	Akaike Information Criterion
ATM	At-the-money
BS-73	Black-Scholes model of 1973
CAD	Canadian Dollar
CBK	Central Bank of Kenya
CDF	Cumulative Distribution Function
CHF	Swiss Franc
DM	Deutsche Mark
EMM	Equivalent Martingale Measure
EUR	Euro
FFT	Fast Fourier Transform
GBM	Geometric Brownian Motion
GBP	British Pound
GHD	Generalised Hyperbolic Distribution
GK83	Garman & Kohlhagen pricing model of 1983
ITM	In-the-money
JPY	Japanese Yen
MGF	Moment Generating Function
NIG	Normal inverse Gaussian
OTC	Over-the-counter
OTM	Out-of-the-money
PDF	Probability Density Function
SNB	Swiss National Bank

1 Introduction

1.1 Background

1.1.1 Foreign Exchange Rates

The foreign exchange market is a highly volatile and liquid market with a daily turnover averaging $5.1 trillion in 2016 [3]. It is an OTC market with trading taking place 24 hours a day. Although trading is conducted throughout the day, liquidity tends to be higher during standard working hours.

An exchange rate is the price of one currency in terms of another currency, Encyclopaedia of Finance (2013). These two currencies form a pair which is denoted by a label containing each currency's trading symbol. For example, the label USD/KES denotes how many units of the Kenyan shilling (KES) are required to buy one U.S. dollar (USD). A rate of 100.03 for USD/KES implies that one US dollar is worth KES 100.03. In the label USD/KES, the first tag (USD) is the foreign currency and the second (KES) is the domestic currency.

Some currencies may be pegged (fixed) to another currency, while others are allowed to float freely. For example, the SNB fixed the CHF to the EUR in 2011 by setting an exact exchange rate of 1.2 CHF for one EUR. This pegging was, however, abandoned in 2015. Currencies of major developed countries are free floating; hence, can be traded freely. Therefore, their values are determined by the market forces of demand and supply.

Foreign exchange rates movements have direct and indirect effects on the economy. These movements affect prices of imports and exports, trade competitiveness, domestic inflation, tourism, international reserves held by central banks, as well as domestic debt repayments written in foreign currency. Countries with high inflation rates have high interest rates and depreciating currencies. A strong KES makes exports more expensive whereas a weak KES makes exports cheaper and imports more expensive.

Before defining and constructing models for the exchange rates, we, first, define and outline some common properties of exchange rates.

1.1.2 Stylized Facts and Statistical Properties of Exchange Rates

Empirical studies of various financial instruments and markets have found certain statistical properties that seem to be common across the instruments/markets. These properties, popularly termed as *stylized facts*, have been reported in both developed and emerging markets.
With regard to financial asset returns, stylized facts are often exhibited in different asset classes such as stock prices, exchange rates and market indices. A general review of these properties for various asset returns is provided in [22].
We, however, restrict ourselves to the case of foreign exchange (FX, henceforth) rates returns.

i. Absence of linear autocorrelations in returns: Rogalski & Vinso (1978) reported that weekly FX rates changes were serially uncorrelated for both fixed and floating regimes.

Friedman & Vandersteel (1982) reported the absence of autocorrelations on daily FX rates fluctuations for nine currencies written against the USD for the period 1973 - 1979.

Hsieh (1988) reported presence of serial correlations, but after adjusting for heteroskedasticity, no serial correlations were reported on four of the five daily FX data considered in the study.

Cheung et. al (2012) tested this hypothesis of serial uncorrelatedness by conducting tests that allowed for conditional heteroskedasticity of unknown form on daily FX returns of 82 foreign currencies against the EUR for the period 1999 - 2010. Majority of the currencies (58 out of the 82) reported no significant autocorrelations.

ii. Heavy tails: past empirical studies [16, 22, 28, 35, 38] have shown that the (unconditional) distribution of daily logarithmic exchange rate returns is asymmetric and exhibits fatter tails than the Gaussian distribution.

iii. Volatility clustering: large (daily, weekly and even monthly) price variations tend to be followed by price variations of similar magnitude.

iv. Aggregational Gaussianity: the shape of the distribution changes with time. Excess kurtosis generally decreases as the time scale Δt is increased and the distribution seems to converge to the normal distribution [16, 22].

v. Intermittency: returns exhibit variability specified by a presence of erratic jumps (discontinuities).

vi. Conditional heavy tails: less fatter tails (as compared to the unconditional distribution) are witnessed even after accounting for volatility clustering in returns. For instance, after accounting for the time-varying variance using an ARCH model, Hsieh (1988) reported that standardized residuals still displayed degrees of leptokurtosis. Similarly, Jorion (1988) reported jumps/discontinuities in the distribution of weekly exchange rates even after explicitly allowing for heteroskedasticity.

1.1.3 Currency Options

A currency option is a contract that gives the buyer the right, but not the obligation, to purchase or sell a specific amount of currency on or before a specified future date and at a specified exchange rate. A European-style option can only be exercised on the maturity date itself, whereas American options can be exercised before or on the maturity date.

A call option gives the right to purchase a unit of foreign currency with the domestic currency whereas a put, the right to sell. The option writer is the party promising to deliver the underlying asset. The buyer ("holder") purchases the option from the writer (seller). The writer issues the option and he/she is compelled to deliver the primitive security if the buyer decides to fulfil the obligation.

An option's payoff is its value at expiry. The payoff from a long position in a European call is $\max(G_T - X, 0)$; where, G_T is the price of the primitive asset at maturity, T is the expiration date, X is the strike price, and the *max* function represents the optionality. The payoff function can also be written as $(G_T - X)^+$ and the option is exercised if $G_T > X$. For each $t \in [0, T]$, a call option is ITM, if $G_t > X$; ATM, if $G_t = X$; and OTM, if $G_t < X$. For a similar position in a put, the pay-off is $\max(X - G_T, 0)$ and the option is exercised if $X > G_T$.

Currency options are appealing to investors as they can be used to hedge currency risk (adverse movements in FX rates). An investor being paid in a foreign currency at some future time can hedge his/her risk by purchasing puts on the foreign currency which mature at the same time. Thus, the investor benefits from any favourable FX movements and is protected from any unfavourable movements by the position held in the put. Options are also advantageous as they have reduced transaction costs as compared to trading the underlying asset itself [11].

FX options are transacted in both OTC markets and on organized exchanges. While organized exchanges are regularized with the traded instruments being possibly standardized, OTC markets are not. OTC transactions are carried out between several commercial and investment banks. The trading volume of OTC FX options, as reported in [3], was $254 billion in 2016.

The pricing model by [12, 44] has been the benchmark model for option pricing since its inception and has been applied extensively in this context. However, the model cannot be applied to the valuation of FX options due to the presence of multiple interest rates. In light of this, [9, 29, 33], independently, developed a pricing formula for FX options by modifying the BS-73 to allow for the two interest rates. However, the same major assumptions from Black-Scholes formula still hold; notably, the normality assumption (log returns of exchange rates are Gaussian) and the assumption of constant volatility. These assumptions, through numerous studies on the statistical properties of exchange rate returns as outlined in Section 1.1.2, have long since been invalidated.

Currency pricing formulae have been developed which tackle these assumptions. For example, Bollen et al. (2000) proposed a regime switching model, Daal & Madan (2005) proposed the variance-gamma model, and Miyahara & Moriwaki (2009) proposed the GSP & MEMM model.

1.2 Statement of Problem

Empirical studies have found evidence against the use of Brownian motion to model foreign exchange rates as this model implies normality of log returns (the two-parameter normal distribution which cannot capture extreme values adequately).

In developing models for FX rates, studies have sought to construct models that take into account some of the stylized facts mentioned before. With regard to the pricing of currency options, however, a great deal of scholarly work has concentrated on the properties that are in direct contradiction to the assumptions underlying the Garman & Kohlhagen pricing model of 1983 (GK83, henceforth); notably, that the underlying asset price process is governed by a GBM implying that the distribution of FX log returns is normal (contradicts property ii.) and two, that volatility is constant (contradicts property iii.).

Given that the distribution of the exchange rate changes exhibits fatter tails than the normal distribution, the empirical distribution of these changes could be explained by either considering a stationary process with heavier tails than the Gaussian such as the generalised hyperbolic distribution (or any of its subclasses); or a normal distribution with time-varying parameters such as an ARCH process. For example, in order to factor in the leptokurtic nature of log returns (fat tails), distributions with finite higher-order moments are often considered as the normal distribution (a two-parameter model) cannot capture extreme values adequately. Cont (2001) suggested that an appropriate distribution should have atleast four parameters: a parameter describing the decay of the tails, a location parameter, a scale parameter, and an asymmetry parameter. Second, volatility clustering illustrates the fact that volatility (of the primitive asset) is not constant but varies with time and may at times switch between phases of either high or low volatility, hence, the need to model it as a stochastic variable. The family of ARCH models and regime-switching models are popular candidates in this respect.

Thus, this study proposes to develop analytically tractable pricing formulae for currency options that take into account some of these shortcomings. However, it is worth noting that constructing models that consider the stylized facts earlier mentioned, presents a new set of challenges when it comes to risk-neutral pricing as the BS-73 framework essentially breaks down. In the BS-73 model, the market is complete and hence there is exactly one EMM - the "natural" EMM. The proposed models presented here, however, imply an incomplete market setting where there exists a wide range of EMMs. Thus, a prudent method to choose a unique EMM is required to render the market complete (by Theorem B.1.3 and B.1.4).

1.3 Objectives

1.3.1 General Objective

The main objective of this project is to develop pricing formulae for currency options using an Esscher-transformed martingale measure and the fast Fourier transform given that the distribution of the log-returns of the underlying security is normal inverse Gaussian.

1.3.2 Specific Objectives

The general objective will be achieved by considering the following specific objectives:

- To model (unconditional moments of) exchange rate returns via the normal inverse Gaussian distribution.
- To construct Esscher transforms and the risk-neutral Esscher measure.
- To derive currency option pricing formulae via Esscher martingale measures and the fast Fourier transform.
- To compare the fit of the NIG pricing model constructed to the benchmark (GK83).

1.4 Significance of Study

The behaviour of exchange rates can be perceived by analysing the distribution governing the rates.
According to the National Treasury Annual Public Debt Management Report (2018), Kenya's foreign debt as at the end of June 2018 was 2.57 trillion which constituted fifty point nine percent of the total public debt of KShs. 5.05 trillion. Seventy-one point seven percent of this external debt was denominated in U.S. dollars, fourteen point nine percent in the Euro, six point one seven percent in the Chinese Yuan, four point three percent in the Japanese Yen, two point six percent in Sterling Pound and zero point three percent was denominated in other currencies. Understanding the behaviour of the exchange rates would be important to the government when repayment instalments fall due as it can create buffer reserves to meet such future repayments.

The size and volatility of the FX market warrants the utilization of currency options in order to hedge the FX risk. This study seeks to build a robust pricing model that takes into account some of the stylized facts outlined in Section 1.1.2. Normal variance-mean mixtures are chosen to model the exchange rates as their higher-order moments exist, thus can capture skewness and kurtosis unlike the two-parameter normal distribution. To price options written on these exchange rates under the no-arbitrage approach, a change of measure is required. This is achieved via the Esscher measure. Pricing equations are, thus, derived as the expected discounted value of the payoffs w.r.t. this measure.

Therefore, this study seeks to add to the literature on currency option pricing by constructing analytically tractable pricing formulae and equations that consider the stylized facts of exchange rates mentioned in Section 1.1.2 that the GK83 does not consider, consequently, reducing mispricing.

2 Literature Review

2.1 Introduction

This chapter outlines and reviews past relevant literature on models for foreign exchange rates and the valuation of currency options.

2.2 Models for the Exchange Rates

It is essential to develop an appropriate model for the returns of the underlying variable (in our case FX returns). This is because FX returns are key components in: one, assets' proportion allocations in international portfolios and two, in pricing currency options. For example, in considering mean-variance analysis of international portfolios of assets, a distribution such as the symmetric stable Paretian may not be suitable as the second moment (which is usually a measure of risk) does not exist [27]. In comparison to stock returns, exchange rates are more leptokurtic. Employing the Student's t distribution, Rogalski & Vinso (1978) showed that exchange rates had lower degrees of freedom than stock returns. In the Student's t distribution, the degrees of freedom measure peakedness and the smaller the values, the higher the tail probability of the distribution. Similarly, Jorion (1988) reported the presence of jumps in both exchange rates and stock market indices with the jumps being more pronounced in the former as compared to the latter.

Daily and weekly financial returns exhibit significant skewness and kurtosis, thus, discrediting the normal distribution as a suitable model for the returns. Positive excess kurtosis implies that the distribution governing the returns is more peaked and fat-tailed than the normal distribution. Alternative distributions, other than the normal distribution, to model the FX returns have been proposed with larger emphasis on (scale) mixture distributions, even for time-varying parameters [16].

Rogalski & Vinso (1978) tested the suitability of the Student's t distribution to describe the behaviour of weekly FX rate changes for both fixed (1962 - 1971) and floating (1973 - 1975) periods. Using likelihood ratio tests, they concluded that the Student's t distribution (with 3.92 degrees of freedom) best described the rates during the floating period whereas the stable Paretian was a better fit during fixed rate periods. However, considering a similar but longer floating period (1974-77), Calderon-Rossell and Ben-Horim (1982) found no unique distribution that described the behaviour of daily FX rate changes on 14 currencies.

Boothe & Glassman (1987) compared the empirical fits of three alternative distributions: the Student, mixture of two normals and the Stable Paretian distribution to daily changes [2nd Jan. 1973 - 8th Aug. 1984] in FX rates using chi-square goodness of fit tests. The currencies considered were: GBP, CAD, DM and JPY all written on the USD. Daily and weekly returns exhibited positive excess kurtosis. The kurtosis decreased as the time interval increased, where in some cases, monthly returns exhibited kurtosis. Quarterly returns, however, showed no evidence of kurtosis implying no deviations from normality. They concluded that the Student and the mixture of two normals provided the best fits for daily data, with evidence that the distribution parameters vary over time.

Akgiray & Booth (1988) proposed a jump-diffusion process to model daily exchange rate changes and compared the empirical fit of the model to the stable, normal and the mixture of five normals. The data used in their study were the daily exchange rates of the GBP, the French franc and the DM relative to the USD, spanning a period of nine years but divided into three different sub-periods corresponding to the U.S. monetary policy regimes over the period 1976 - 1985. Their results showed that the mixed jump-diffusion process performed better than the other models, with its resulting parameters being dependent on the monetary policy regime in force.

Tucker & Pond (1988) conducted statistical pairwise comparisons of four distributions on daily exchange rates of the USD against the GBP, CAD, DM, French franc, CHF and JPY for the period 1980 - 1984. Using likelihood ratio tests, the scaled-*t* performed better than the stable distribution whereas a mixture of three normals performed better than the scaled-*t* for five of the the six currencies considered. Using the Schwarz approximation, the mixed-jump model performed better than the compound normal distributions. From this, they concluded that the mixed-jump model described the behaviour of all six exchange rates better than the aforementioned models.

Bollen et. al (2000) fitted regime-switching models to weekly spot exchange rates for the GBP, the JPY and the DM, all written against the USD. Their results indicated that a regime-switching model performed better than a single regime model.

Barndorff-Nielsen & Prause (2001) fitted the normal inverse Gaussian distribution to intra-day (three-hour) exchange rate data for the USD/German Mark. This model, they reported, described the behaviour of exchange rates better than the normal distribution.

Corlu & Corlu (2014) proposed the generalized lambda distribution (GLD) as a model for daily exchange rate returns and compared its performance with the Skewed *t*, the unbounded Johnson family of distributions and the NIG. Their data consisted of nine currencies written on the USD for the period 2006 - 2011. Overall results obtained by use of visual plots, Kolmogorov–Smirnov and A-D tests indicated that the aforementioned four distributions performed similarly. However, in terms of value at risk (VaR) and expected shortfall performance, the NIG, Skewed t and GLD captured the tail behaviour better than the unbounded Johnson family and the normal distribution.

Nadarajah et. al (2015) compared the empirical performance of eight distributions fitted to daily exchange rate returns: asymmetric Student's t, the generalised hyperbolic distribution, the hyperbolic distribution, Student's t and the four distributions in [24]. Using the same exchange rate data as [24], but for an extended period (3 January 2000 to 2 January 2015), results indicated that the Student's t and the skewed t performed better.

2.3 Currency Option Valuation Models

The Black-Scholes formula cannot be applied to the valuation of currency options due to the presence of multiple interest rates. Further, assuming that the investor holding a foreign currency invests it in a short-term currency bond instead of placing it in an account that bears no interest, a return equivalent to the riskless rate is guaranteed. Hence, valuing a currency option is analogous to the valuation of an option written on a dividend-paying stock [9].

Taking this into consideration, [9, 29, 33], independently, developed a pricing formula for FX options by modifying the BS-73 to account for the two interest rates. However, the pricing formula developed is, by definition, a generalization of the B-S formula. Consequently, the same assumptions from the BS-73 still hold; notably, the assumption that the underlying currency spot price process evolution is governed by a GBM and the assumption that the volatility and the interest rates are constant.

These assumptions have been discredited through empirical studies conducted on both exchange rate and option data. Past studies on exchange rate data [16, 28, 35] have reported the time-varying property of moments of the FX returns. And, whereas, the GK83 model implies that the volatility is constant, studies on market option data have revealed the fact that volatility varies with K and the time to expiration and reproduces what has been termed in literature as the volatility smile. Further studies contrasting actual market options data and GK83 option prices have reported inconsistencies with the aforementioned assumptions. For example, it has been documented that the GK83 underprices short-dated OTM currency options [40] and overprices ITM options [25]. Models correcting these inconsistencies have been developed and we highlight a few.

Bollen & Rasiel (2003) compared the performance of regime-switching, GARCH and jump-diffusion option valuation models for OTC currency options to the Garman-Kohlhagen pricing formula. They found that the former models outperformed the latter model with the jump-diffusion model performing better than the other models.

Daal & Madan (2005) applied the variance-gamma model to value currency options and compared its performance relative to the modified BS-73 (GK83) and the currency option equivalent of the jump-diffusion model. Generally, all models overpriced the deutsche mark foreign currency options, but the overpricing was noted to be less for the VG model. Comparatively, the VG model outperformed the GK83 and the modified JD model, with the highest improvement noted for short-dated options. Further, adding a diffusion element in the VG model, they reported, did not improve its performance as its contribution was insignificant. In considering the time-to-maturity bias, short-dated calls were reported to be consistently overpriced compared to medium- and long-dated options.

Miyahara & Moriwaki (2009) introduced the geometric stable process and minimal entropy martingale measure (GSP & MEMM, hereafter) model and applied it in pricing currency options. From the in-sample and out-of-sample analysis, the GSP & MEMM model outperformed other models such as the Black-Scholes model, the finite moment log stable model, as well as Esscher-transformed distribution models such as the scaled-*t*, the normal inverse Gaussian and the variance-gamma distributions.

Aduda & Weke (2011) extended Duan's GARCH option pricing model of 1995 to the case of currency options and compared the model performance relative to the GK83. Their results illustrated the importance of accounting for the conditional heteroscedasticity inherent in the underlying asset price process when constructing currency option pricing models.

3 A model for the Exchange Rates

3.1 Introduction

This chapter considers the construction of a model for the unconditional moments of exchange rate returns under a heavy-tailed distribution - the normal inverse Gaussian distribution.

By proving - through normality tests - that the distribution of the daily logarithmic changes of exchange rates has fatter tails than the Gaussian, the use of the aforementioned distribution is, thus, justified. This distribution has positive excess kurtosis implying heavier tails than the symmetric-mesokurtic Gaussian distribution.

Skewness, *Skew(w)*, measures the extent of a distribution's asymmetry while kurtosis, *Kurt(w)*, measures tail heaviness. By definition,

$$Skew(w) = E\left[\left(\frac{W-\mu_w}{\sigma_w}\right)^3\right] \qquad \text{and} \qquad Kurt(w) = E\left[\left(\frac{W-\mu_w}{\sigma_w}\right)^4\right]$$

Thus, defining $Kurt(w)-3$ as the excess kurtosis, clearly, the excess kurtosis of a Gaussian variable is zero.
Calculating the $Skew(w)$ and $Kurt(w)$ values and comparing them directly with those of the normal distribution is not a prudent way to test for normality, hence, the need for a statistical test. We employ the Jarque-Bera (J-B) goodness-of-fit test under the joint hypothesis of both skewness and excess kurtosis being zero.

3.2 Normal Variance-Mean Mixtures

A random variable Y has a normal variance-mean distribution if

$$Y = \mu + bW + \sigma\sqrt{W}Z \tag{3.2.1}$$

where, $Z \sim N(0,1)$, W is a positive random variable independent of Z and μ, b and $\sigma > 0$. Clearly, the conditional distribution of Y given W is

$$Y|W \sim N(\mu + bw, \sigma^2 w) \tag{3.2.2}$$

3.3 Modified Bessel Function of the Third Kind and its Properties

The modified Bessel function of the third kind with index ϑ (in integral form) is given by,

Definition 1

$$K_\vartheta(w) = \frac{1}{2}\int_0^\infty y^{\vartheta-1}e^{-\frac{w}{2}(y+y^{-1})}dy \tag{3.3.1}$$

Definition 2

$$K_\vartheta(w) = \frac{1}{2}\left(\frac{w}{2}\right)^\vartheta \int_0^\infty t^{-\vartheta-1}e^{-t-\frac{w^2}{4t}}dt \tag{3.3.2}$$

Definition 3

$$K_\vartheta(w) = \left(\frac{w}{2}\right)^\vartheta \frac{\Gamma(\frac{1}{2})}{\Gamma(\vartheta+\frac{1}{2})}\int_1^\infty (t^2-1)^{\vartheta-1/2}e^{-wt}dt \tag{3.3.3}$$

And has the following properties:

Property 3.3.1 (Symmetry). $K_\vartheta(w) = K_{-\vartheta}(w)$

Proof.
In equation 3.3.1, let

$$y = \frac{1}{z} \qquad \Rightarrow \qquad dy = -\frac{1}{z^2}dz$$

Thus,

$$\begin{aligned} K_\vartheta(w) =& \frac{1}{2}\int_\infty^0 \left(\frac{1}{z}\right)^{\vartheta-1} e^{-\frac{w}{2}(z^{-1}+z)}\left(-\frac{1}{z^2}\right)dz \\ =& \frac{1}{2}\int_0^\infty z^{-\vartheta-1}e^{-\frac{w}{2}(z^{-1}+z)}dz \\ =& K_{-\vartheta}(w) \end{aligned}$$

□

Property 3.3.2. $\dfrac{\partial}{\partial w}K_\vartheta(w) = -\dfrac{1}{2}\left[K_{\vartheta+1}(w) + K_{\vartheta-1}(w)\right]$

Proof.

$$\frac{\partial}{\partial w}K_{\vartheta}(w) = \frac{\partial}{\partial w}\left[\frac{1}{2}\int_0^{\infty} y^{\vartheta-1}e^{-\frac{w}{2}(y+y^{-1})}dy\right]$$

$$= \frac{1}{2}\int_0^{\infty} y^{\vartheta-1}\frac{\partial}{\partial w}\left[e^{-\frac{w}{2}(y+y^{-1})}\right]dy$$

$$= \frac{1}{2}\int_0^{\infty} y^{\vartheta-1}\left[-\frac{1}{2}(y+y^{-1})\right]e^{-\frac{w}{2}(y+y^{-1})}dy$$

$$= -\frac{1}{2}\left[\frac{1}{2}\int_0^{\infty} y^{\vartheta+1-1}e^{-\frac{w}{2}(y+y^{-1})}dy + \frac{1}{2}\int_0^{\infty} y^{\vartheta-1-1}e^{-\frac{w}{2}(y+y^{-1})}dy\right]$$

$$= -\frac{1}{2}\left[K_{\vartheta+1}(w) + K_{\vartheta-1}(w)\right]$$

□

Property 3.3.3. $\frac{\partial}{\partial w}K_{\vartheta}(w) = \frac{\vartheta}{w}K_{\vartheta}(w) - K_{\vartheta+1}(w)$

Proof.
From definition 2 of the modified Bessel function (3.3.2), we have

$$\frac{\partial}{\partial w}K_{\vartheta}(w) = \frac{\partial}{\partial w}\left[\frac{1}{2}\left(\frac{w}{2}\right)^{\vartheta}\int_0^{\infty} t^{-\vartheta-1}e^{-t-\frac{w^2}{4t}}dt\right]$$

$$= \frac{1}{2}\left[\frac{\vartheta}{2}\left(\frac{w}{2}\right)^{\vartheta-1}\int_0^{\infty} t^{-\vartheta-1}e^{-t-\frac{w^2}{4t}}dt + \left(\frac{w}{2}\right)^{\vartheta}\int_0^{\infty} t^{-\vartheta-1}e^{-t-\frac{w^2}{4t}}\left(\frac{-2w}{4t}\right)dt\right]$$

$$= \frac{1}{2}\left[\frac{\vartheta}{w}\left(\frac{w}{2}\right)^{\vartheta}\int_0^{\infty} t^{-\vartheta-1}e^{-t-\frac{w^2}{4t}}dt - \left(\frac{w}{2}\right)^{\vartheta+1}\int_0^{\infty} t^{-\vartheta-1-1}e^{-t-\frac{w^2}{4t}}dt\right]$$

$$= \frac{\vartheta}{w}\left[\frac{1}{2}\left(\frac{w}{2}\right)^{\vartheta}\int_0^{\infty} t^{-\vartheta-1}e^{-t-\frac{w^2}{4t}}dt\right] - \left[\frac{1}{2}\left(\frac{w}{2}\right)^{\vartheta+1}\int_0^{\infty} t^{-(\vartheta+1)-1}e^{-t-\frac{w^2}{4t}}dt\right]$$

$$= \frac{\vartheta}{w}K_{\vartheta}(w) - K_{\vartheta+1}(w)$$

□

Property 3.3.4. $K_{\vartheta+1}(w) = \frac{2\vartheta}{w} K_{\vartheta}(w) + K_{\vartheta-1}(w)$

Proof.
Equating the derivatives obtained in property 3.3.2 and property 3.3.3, we have

$$-\frac{1}{2}\Big[K_{\vartheta+1}(w) + K_{\vartheta-1}(w)\Big] = \frac{\vartheta}{w} K_{\vartheta}(w) - K_{\vartheta+1}(w)$$

Rearranging yields,

$$K_{\vartheta+1}(w) = \frac{2\vartheta}{w} K_{\vartheta}(w) + K_{\vartheta-1}(w)$$

□

Property 3.3.5. $K_{1/2}(w) = \sqrt{\frac{\pi}{2w}} e^{-w}$

Proof.
In definition 3 of the modified Bessel function (3.3.3), let $\vartheta = \frac{1}{2}$.

$$\begin{aligned}
\Rightarrow \qquad K_{1/2}(w) &= \left(\frac{w}{2}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2})}{\Gamma(1)} \int_1^\infty (t^2-1)^0 e^{-wt} dt \\
&= \sqrt{\frac{w\pi}{2}} \int_1^\infty e^{-wt} dt \\
&= \sqrt{\frac{w\pi}{2}} \cdot \left[\frac{1}{w} e^{-w}\right] \\
&= \sqrt{\frac{\pi}{2w}} e^{-w}
\end{aligned}$$

□

3.4 Generalised Inverse Gaussian Distribution

The generalised inverse Gaussian (GIG, henceforth) distribution has PDF

$$f_{GIG}(y; \vartheta, \xi, \phi) = \begin{cases} \dfrac{(\phi/\xi)^{\vartheta/2}}{2K_{\vartheta}(\sqrt{\xi\phi})} y^{\vartheta-1} e^{-\frac{1}{2}\left(\phi y + \xi y^{-1}\right)}, & y > 0 \\ 0, \text{ elsewhere} \end{cases}$$

where, $K_{\vartheta}(\cdot)$ is the modified Bessel function given by equation 3.3.1.

3.4.1 Construction

Following parametrization by Jørgensen (1982) and from equation 3.3.1,

$$K_\vartheta(w) = \frac{1}{2}\int_0^\infty y^{\vartheta-1} e^{-\frac{w}{2}(y+y^{-1})} dy$$

Let $w = \sqrt{\xi\phi}$ and $y = \sqrt{\frac{\phi}{\xi}}z \quad \Rightarrow \quad dy = \sqrt{\frac{\phi}{\xi}}dz$
Thus,

$$\begin{aligned}
K_\vartheta(\sqrt{\xi\phi}) =& \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{\phi}{\xi}}z\right)^{\vartheta-1} e^{-\frac{\sqrt{\xi\phi}}{2}\left(\sqrt{\frac{\phi}{\xi}}z+\sqrt{\frac{\xi}{\phi}}z^{-1}\right)} \sqrt{\frac{\phi}{\xi}}dz \\
=& \frac{1}{2}\left(\sqrt{\frac{\phi}{\xi}}\right)^{\vartheta} \int_0^\infty z^{\vartheta-1} e^{-\frac{1}{2}\left(\phi z+\xi z^{-1}\right)} dz
\end{aligned}$$

Dividing both sides by the term on the LHS yields,

$$1 = \frac{1}{2K_\vartheta(\sqrt{\xi\phi})}\left(\sqrt{\frac{\phi}{\xi}}\right)^{\vartheta} \int_0^\infty z^{\vartheta-1} e^{-\frac{1}{2}\left(\phi z+\xi z^{-1}\right)} dz$$

$$1 = \int_0^\infty \frac{(\phi/\xi)^{\vartheta/2}}{2K_\vartheta(\sqrt{\xi\phi})} z^{\vartheta-1} e^{-\frac{1}{2}\left(\phi z+\xi z^{-1}\right)} dz$$

Thus,

$$f_{GIG}(z;\vartheta,\xi,\phi) = \begin{cases} \dfrac{(\phi/\xi)^{\vartheta/2}}{2K_\vartheta(\sqrt{\xi\phi})} z^{\vartheta-1} e^{-\frac{1}{2}\left(\phi z+\xi z^{-1}\right)}, & z > 0 \\ 0, \text{ elsewhere} & \end{cases}$$

is a PDF; where, $\vartheta \in \mathbb{R}$ and $(\xi,\phi) \in \Theta_\vartheta$ where

$$\Theta_\vartheta = \begin{cases} \{(\xi,\phi): \quad \xi \geq 0, \quad \phi > 0\} & \text{if } \vartheta > 0 \\ \{(\xi,\phi): \quad \xi > 0, \quad \phi > 0\} & \text{if } \vartheta = 0 \\ \{(\xi,\phi): \quad \xi > 0, \quad \phi \geq 0\} & \text{if } \vartheta < 0 \end{cases}$$

Let GIG_ϑ denote the GIG distributions, we thus write

$$GIG_\vartheta = \left\{GIG(\vartheta,\xi,\phi) : (\xi,\phi) \in \Theta_\vartheta\right\}$$

Some special cases of the GIG include: the gamma distribution when $(\xi = 0,\ \phi > 0)$, the inverse Gaussian when $\vartheta = -\frac{1}{2}$, the inverse gamma when $(\phi = 0,\ \vartheta < 0)$ and the reciprocal inverse Gaussian when $\vartheta = \frac{1}{2}$.

3.4.2 Properties

Property 3.4.1. ***Moments***

$$E(Y^m) = \left(\sqrt{\frac{\phi}{\xi}}\right)^m \frac{K_{\vartheta+m}\left(\sqrt{\xi\phi}\right)}{K_{\vartheta}\left(\sqrt{\xi\phi}\right)}$$

Proof.

$$\begin{aligned}
E(Y^m) &= \int_0^\infty y^m \frac{(\phi/\xi)^{\vartheta/2}}{2K_\vartheta(\sqrt{\xi\phi})} y^{\vartheta-1} e^{-\frac{1}{2}\left(\phi y+\xi y^{-1}\right)} dy \\
&= \frac{(\phi/\xi)^{\vartheta/2}}{2K_\vartheta(\sqrt{\xi\phi})} \int_0^\infty y^{\vartheta+m-1} e^{-\frac{1}{2}\left(\phi y+\xi y^{-1}\right)} dy \\
&= \frac{(\phi/\xi)^{\vartheta/2}}{2K_\vartheta(\sqrt{\xi\phi})} \frac{2K_{\vartheta+m}(\sqrt{\xi\phi})}{(\phi/\xi)^{(\vartheta+m)/2}} \int_0^\infty \frac{(\phi/\xi)^{(\vartheta+m)/2}}{2K_{\vartheta+m}(\sqrt{\xi\phi})} y^{\vartheta+m-1} e^{-\frac{1}{2}\left(\phi y+\xi y^{-1}\right)} dy \\
&= \frac{(\phi/\xi)^{\vartheta/2}}{(\phi/\xi)^{(\vartheta+m)/2}} \frac{K_{\vartheta+m}(\sqrt{\xi\phi})}{K_\vartheta(\sqrt{\xi\phi})} \\
&= \frac{1}{(\phi/\xi)^{m/2}} \frac{K_{\vartheta+m}(\sqrt{\xi\phi})}{K_\vartheta(\sqrt{\xi\phi})} \\
&= \left(\frac{\xi}{\phi}\right)^{m/2} \frac{K_{\vartheta+m}(\sqrt{\xi\phi})}{K_\vartheta(\sqrt{\xi\phi})} \\
&= \left(\sqrt{\frac{\xi}{\phi}}\right)^m \frac{K_{\vartheta+m}(\sqrt{\xi\phi})}{K_\vartheta(\sqrt{\xi\phi})}, \qquad m \in \mathbb{R}
\end{aligned}$$

□

Property 3.4.2. ***Moment Generating Function***
The MGF of $Y \sim GIG(\vartheta, \xi, \phi)$ is

$$M_Y(t) = \left(\frac{\phi}{\phi-2t}\right)^{\vartheta/2} \frac{K_\vartheta\left(\sqrt{\xi(\phi-2t)}\right)}{K_\vartheta(\sqrt{\xi\phi})}$$

Proof.

$$
\begin{aligned}
M_Y(t) =& E[e^{tY}] \\
=& \int_0^\infty e^{ty} \frac{(\phi/\xi)^{\vartheta/2}}{2K_\vartheta(\sqrt{\xi\phi})} y^{\vartheta-1} e^{-\frac{1}{2}\left(\phi y+\xi y^{-1}\right)} dy \\
=& \frac{(\phi/\xi)^{\vartheta/2}}{2K_\vartheta(\sqrt{\xi\phi})} \int_0^\infty y^{\vartheta-1} e^{-\frac{1}{2}\left(\phi y+\xi y^{-1}-2ty\right)} dy \\
=& \frac{(\phi/\xi)^{\vartheta/2}}{2K_\vartheta(\sqrt{\xi\phi})} \frac{2K_\vartheta\left(\sqrt{\xi(\phi-2t)}\right)}{\left(\frac{\phi-2t}{\xi}\right)^{\vartheta/2}} \int_0^\infty \frac{\left(\frac{\phi-2t}{\xi}\right)^{\vartheta/2}}{2K_\vartheta\left(\sqrt{\xi(\phi-2t)}\right)} y^{\vartheta-1} e^{-\frac{1}{2}\left((\phi-2t)y+\xi y^{-1}\right)} dy \\
=& \frac{(\phi/\xi)^{\vartheta/2}}{K_\vartheta(\sqrt{\xi\phi})} \frac{K_\vartheta\left(\sqrt{\xi(\phi-2t)}\right)}{\left(\frac{\phi-2t}{\xi}\right)^{\vartheta/2}} \\
=& \left(\frac{\phi}{\phi-2t}\right)^{\vartheta/2} \frac{K_\vartheta\left(\sqrt{\xi(\phi-2t)}\right)}{K_\vartheta(\sqrt{\xi\phi})}
\end{aligned}
$$

□

3.5 Generalised Hyperbolic Distribution

The generalised hyperbolic distributions (GHDs, henceforth) are variance-mean mixtures of normal distributions where the mixing distributions are GIGs.
Letting $\sigma^2 = 1$ in equation 3.2.2, we have

$$Y|W \sim N(\mu + bw, w)$$

If $W \sim GIG(\vartheta, \xi, \phi)$, then $Y \sim GH(\vartheta, \xi, \phi, b, \mu)$.

3.5.1 Construction: The Barndorff-Nielsen Approach

$$f(y) = \int_w f(y,w)dw = \int_w f(y|w)g(w)dw$$

$$f(y|w) = \begin{cases} \frac{1}{\sqrt{2\pi w}} e^{-\frac{1}{2w}[y-(\mu+bw)]^2}, & y \in \mathbb{R} \\ 0, \text{ elsewhere} \end{cases}$$

$$g(w) \sim GIG(\vartheta, \xi, \phi) = \begin{cases} \frac{(\phi/\xi)^{\vartheta/2}}{2K_\vartheta(\sqrt{\xi\phi})} w^{\vartheta-1} e^{-\frac{1}{2}\left(\phi w+\xi w^{-1}\right)}, & w > 0 \\ 0, \text{ elsewhere} \end{cases}$$

$$f(y) = \int_w f(y|w)g(w)dw$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi w}} e^{-\frac{1}{2w}[y-(\mu+bw)]^2} \frac{(\phi/\xi)^{\vartheta/2}}{2K_\vartheta(\sqrt{\xi\phi})} w^{\vartheta-1} e^{-\frac{1}{2}\left(\phi w+\xi w^{-1}\right)} dw$$

$$= \frac{(\phi/\xi)^{\vartheta/2}}{2\sqrt{2\pi}K_\vartheta(\sqrt{\xi\phi})} \int_0^\infty \frac{1}{\sqrt{w}} e^{-\frac{1}{2w}[y-(\mu+bw)]^2} w^{\vartheta-1} e^{-\frac{1}{2}\left(\phi w+\xi w^{-1}\right)} dw$$

$$= \frac{(\phi/\xi)^{\vartheta/2}}{2\sqrt{2\pi}K_\vartheta(\sqrt{\xi\phi})} \int_0^\infty w^{\vartheta-\frac{1}{2}-1} e^{-\frac{1}{2w}[y-(\mu+bw)]^2-\frac{1}{2}\left(\phi w+\xi w^{-1}\right)} dw$$

$$= \frac{(\phi/\xi)^{\vartheta/2}}{2\sqrt{2\pi}K_\vartheta(\sqrt{\xi\phi})} \int_0^\infty w^{\vartheta-\frac{1}{2}-1} e^{-\frac{1}{2}\left[\left((y-\mu)\cdot\frac{1}{w}-b\right)^2+\phi w+\xi w^{-1}\right]} dw$$

$$= \frac{(\phi/\xi)^{\vartheta/2} e^{b(y-\mu)}}{2\sqrt{2\pi}K_\vartheta(\sqrt{\xi\phi})} \int_0^\infty w^{\vartheta-\frac{1}{2}-1} e^{-\frac{1}{2}\left\{[(y-\mu)^2+\xi]\cdot\frac{1}{w}+[b^2+\phi]w\right\}} dw$$

$$= \frac{(\phi/\xi)^{\vartheta/2} e^{b(y-\mu)}}{2\sqrt{2\pi}K_\vartheta(\sqrt{\xi\phi})} 2K_{\vartheta-\frac{1}{2}}\left(\sqrt{[b^2+\phi][(y-\mu)^2+\xi]}\right)\left(\frac{(y-\mu)^2+\xi}{b^2+\phi}\right)^{\frac{1}{2}(\vartheta-\frac{1}{2})} \times$$

$$\int_0^\infty \frac{w^{\vartheta-\frac{1}{2}-1} e^{-\frac{1}{2}\left\{[(y-\mu)^2+\xi]\cdot\frac{1}{w}+[b^2+\phi]w\right\}}}{2K_{\vartheta-\frac{1}{2}}\left(\sqrt{[b^2+\phi][(y-\mu)^2+\xi]}\right)\left(\frac{(y-\mu)^2+\xi}{b^2+\phi}\right)^{\frac{1}{2}(\vartheta-\frac{1}{2})}} dw$$

$$= \frac{(\phi/\xi)^{\vartheta/2} e^{b(y-\mu)}}{2\sqrt{2\pi}K_\vartheta(\sqrt{\xi\phi})} 2K_{\vartheta-\frac{1}{2}}\left(\sqrt{[b^2+\phi][(y-\mu)^2+\xi]}\right)\left(\frac{(y-\mu)^2+\xi}{b^2+\phi}\right)^{\frac{1}{2}(\vartheta-\frac{1}{2})}$$

$$= \frac{(\phi/\xi)^{\vartheta/2} e^{b(y-\mu)}}{\sqrt{2\pi}K_\vartheta(\sqrt{\xi\phi})} K_{\vartheta-\frac{1}{2}}\left(\sqrt{[b^2+\phi][(y-\mu)^2+\xi]}\right)\left(\frac{(y-\mu)^2+\xi}{b^2+\phi}\right)^{\frac{1}{2}(\vartheta-\frac{1}{2})}$$

$$f_{GH}(y;\vartheta,\xi,\phi,b,\mu) = \begin{cases} \dfrac{(\phi/\xi)^{\frac{\vartheta}{2}}}{\sqrt{2\pi}} \cdot \dfrac{K_{\vartheta-\frac{1}{2}}\left(\sqrt{[b^2+\phi][\xi+(y-\mu)^2]}\right)}{K_\vartheta(\sqrt{\xi\phi})} \left(\dfrac{\xi+(y-\mu)^2}{b^2+\phi}\right)^{\frac{1}{2}(\vartheta-\frac{1}{2})} e^{b(y-\mu)}, & y \in \mathbb{R} \\ 0, \text{ elsewhere} & \end{cases} \tag{3.5.1}$$

This was the result obtained in Barndorff-Nielsen (1977).

3.5.2 Parametrizations

1st Parametrization: $(a,b) - parametrization$

This parametrization is obtained by setting $\xi = \delta^2$ and $\phi = a^2 - b^2$ in equation 3.5.1. Thus, the PDF of the one-dimensional GHD, under the (a,b) - parametrization, is given by

$$f_{GH}(y;\vartheta,a,b,\delta,\mu) = \begin{cases} \dfrac{(a^2-b^2)^{\vartheta/2}}{\sqrt{2\pi}a^{\vartheta-1/2}\delta^{\vartheta}K_{\vartheta}\left(\delta\sqrt{(a^2-b^2)}\right)} \cdot \dfrac{K_{\vartheta-1/2}\left(a\sqrt{\delta^2+(y-\mu)^2}\right)}{\left(\sqrt{\delta^2+(y-\mu)^2}\right)^{1/2-\vartheta}} e^{b(y-\mu)}, \; y \in \mathbb{R} \\ 0, \text{ elsewhere} \end{cases} \tag{3.5.2}$$

where, $\mu \in \mathbb{R}$, the functions $K_{\vartheta}(w)$ and $K_{\vartheta-1/2}(w)$ are the modified Bessel functions of the third kind (given by equation 3.3.1) with orders ϑ and $\vartheta - \frac{1}{2}$, respectively, and

$$\delta \geq 0, |b| < a \qquad \text{if } \vartheta > 0$$

$$\delta > 0, |b| < a \qquad \text{if } \vartheta = 0$$

$$\delta > 0, |b| \leq a \qquad \text{if } \vartheta < 0$$

$\vartheta \in \mathbb{R}$ characterizes certain sub-classes. When $\vartheta = 1$, we have the hyperbolic distribution, and the normal inverse Gaussian, which is the main focus of this study, arises when $\vartheta = -\frac{1}{2}$.

Similar results were obtained in Prause (1999).

Under this parametrization, the MGF of the GHD is

$$M_{GH}(t) = e^{\mu t}\left(\frac{a^2-b^2}{a^2-(b+t)^2}\right)^{\vartheta/2}\frac{K_{\vartheta}(\delta\sqrt{a^2-(b+t)^2})}{K_{\vartheta}(\delta\sqrt{a^2-b^2})}, \quad |b+t| < a.$$

Proof.

For ease of notation, let

$$h(\vartheta,a,b,\delta) = \frac{(a^2-b^2)^{\vartheta/2}}{\sqrt{2\pi}a^{\vartheta-1/2}\delta^{\vartheta}K_{\vartheta}\left(\delta\sqrt{(a^2-b^2)}\right)}$$

$$
\begin{aligned}
M_Y(t) =& E[e^{tY}] \\
=& h(\vartheta,a,b,\delta)\int_{-\infty}^{\infty} e^{(b+t)y-b\mu}\frac{K_{\vartheta-1/2}\left(a\sqrt{\delta^2+(y-\mu)^2}\right)}{(\sqrt{\delta^2+(y-\mu)^2})^{1/2-\vartheta}}dy \\
=& e^{\mu t}\frac{h(\vartheta,a,b,\delta)}{h(\vartheta,a,b+t,\delta)}\int_{-\infty}^{\infty} h(\vartheta,a,b+t,\delta)e^{(b+t)y-(b+t)\mu}\frac{K_{\vartheta-1/2}\left(a\sqrt{\delta^2+(y-\mu)^2}\right)}{(\sqrt{\delta^2+(y-\mu)^2})^{1/2-\vartheta}}dy \\
=& e^{\mu t}\frac{h(\vartheta,a,b,\delta)}{h(\vartheta,a,b+t,\delta)} \\
=& e^{\mu t}\left(\frac{a^2-b^2}{a^2-(b+t)^2}\right)^{\vartheta/2}\frac{K_\vartheta(\delta\sqrt{a^2-(b+t)^2})}{K_\vartheta(\delta\sqrt{a^2-b^2})}, \quad |b+t|<a.
\end{aligned}
$$

□

And, the characteristic function is given by

$$
\psi_{GH}(u) = e^{i\mu u}\left(\frac{a^2-b^2}{a^2-(b+iu)^2}\right)^{\vartheta/2}\frac{K_\vartheta(\delta\sqrt{a^2-(b+iu)^2})}{K_\vartheta(\delta\sqrt{a^2-b^2})}
$$

Proof.
The proof follows from the MGF proof by noting that,

$$
\psi_{GH}(u) = M_{GH}(iu)
$$

where, $i=\sqrt{-1}$. □

The mean and the variance of the GHD is given by

$$
E[Y] = \mu + \frac{\delta b}{\sqrt{a^2-b^2}}\frac{K_{\vartheta+1}(\delta\sqrt{a^2-b^2})}{K_\vartheta(\delta\sqrt{a^2-b^2})}
$$

$$
Var[Y] = \delta^2\left(\frac{K_{\vartheta+1}(\delta\sqrt{a^2-b^2})}{\delta\sqrt{a^2-b^2}K_\vartheta(\delta\sqrt{a^2-b^2})} + \frac{b^2}{a^2-b^2}\left[\frac{K_{\vartheta+2}(\delta\sqrt{a^2-b^2})}{K_\vartheta(\delta\sqrt{a^2-b^2})} - \left(\frac{K_{\vartheta+1}(\delta\sqrt{a^2-b^2})}{K_\vartheta(\delta\sqrt{a^2-b^2})}\right)^2\right]\right)
$$

The term in $\left(\cdot\right)$ is scale- and location-invariant.

Although the parametrization $(\vartheta, a, b, \delta, \mu)$ is commonly used in literature, other scale- and location-invariant parametrizations of the GHDs have also been considered.

2$^{\text{nd}}$ Parametrization:	$(\zeta, \rho) - parametrization$	$\zeta = \delta\sqrt{a^2 - b^2}, \quad \rho = b/a$
3$^{\text{rd}}$ Parametrization:	$(\upsilon, \xi) - parametrization$	$\upsilon = (1+\zeta)^{-1/2}, \quad \xi = \upsilon\rho$
4$^{\text{th}}$ Parametrization:	$(\bar{a}, \bar{b}) - parametrization$	$\bar{a} = a\delta, \quad \bar{b} = b\delta$

The parameters μ & δ describe the location and the scale, respectively, whereas b describes the skewness. Increasing υ or decreasing ζ or $\bar{a}$ reflect an increase in the kurtosis [48].

For symmetric distributions, $b = \bar{b} = \rho = \xi = 0$.

Other Parametrizations

Breymann & Lüthi (2013), following McNeil et. al (2005), used the $(\vartheta, \xi, \phi, \mu, \sigma, \gamma)$-parametrization where,

$$b = \frac{\gamma}{\sigma^2}, \quad \delta = \sigma\sqrt{\xi} \quad \text{and} \quad a = \sqrt{\frac{\phi}{\sigma^2} + b^2}$$

The $(\vartheta, \bar{a}, \mu, \sigma, \gamma)$-parametrization is derived by setting,

$$\bar{a} = \sqrt{\phi\xi} \quad \text{and} \quad \sqrt{\frac{\xi}{\phi}\frac{K_{\vartheta+1}(\sqrt{\phi\xi})}{K_{\vartheta}(\sqrt{\phi\xi})}} = 1,$$

which implies that

$$\phi = \bar{a}\frac{K_{\vartheta+1}(\bar{a})}{K_{\vartheta}(\bar{a})}, \qquad \xi = \bar{a}\frac{K_{\vartheta}(\bar{a})}{K_{\vartheta+1}(\bar{a})}$$

Fitting the GHDs in the *ghyp R* package is done under this parametrization.

The GHDs have been applied extensively in financial modelling due to their link to Lévy processes. For every GHD, a Lévy process can be constructed such that, over a fixed time interval, the value of the increment has the specified GHD. This is only possible because the generalized hyperbolic law is a infinitely divisible distribution as demonstrated in [5].

We employ the NIG in modelling our exchange rate return data as it has more tractable probabilistic properties than the GHD itself and its other sub-classes. See, Barndorff-Nielsen & Prause (2001) on application to exchange rate data.

3.6 Normal Inverse Gaussian Distribution

The NIG distribution is a variance-mean mixture of a Gaussian distribution with the inverse Gaussian as the mixing distribution. As earlier noted, the inverse Gaussian distribution is a special case of the GIG with $\vartheta = -\frac{1}{2}$.
Since GHDs have the GIGs as the mixing distributions, it follows that NIG is a special case of the GHD with $\vartheta = -\frac{1}{2}$, i.e., $GH(-\frac{1}{2}, \xi, \phi, b, \mu) \sim NIG(\xi, \phi, b, \mu)$.

3.6.1 Construction

Let $\vartheta = -\frac{1}{2}$ in equation 3.5.1, thus

$$f_{NIG}(y) = \frac{(\phi/\xi)^{1/2(-1/2)}}{\sqrt{2\pi}} \frac{K_{-1}\left(\sqrt{[b^2+\phi][\xi+(y-\mu)^2]}\right)}{K_{-1/2}(\sqrt{\xi\phi})} \left(\frac{\xi+(y-\mu)^2}{b^2+\phi}\right)^{-\frac{1}{2}} e^{b(y-\mu)}$$

By the symmetry property of the Bessel function (Property 3.3.1),

$$f_{NIG}(y) = \frac{(\phi/\xi)^{-1/4}}{\sqrt{2\pi}} \frac{K_{1}\left(\sqrt{[b^2+\phi][\xi+(y-\mu)^2]}\right)}{K_{1/2}(\sqrt{\xi\phi})} \left(\frac{\xi+(y-\mu)^2}{b^2+\phi}\right)^{-\frac{1}{2}} e^{b(y-\mu)}$$

By Property 3.3.5,

$$K_{1/2}(\sqrt{\xi\phi}) = \sqrt{\frac{\pi}{2}} \left(\frac{1}{\xi\phi}\right)^{1/4} e^{-\sqrt{\xi\phi}}$$

Thus,

$$\begin{aligned} f_{NIG}(y) =& \frac{(\phi/\xi)^{-1/4}}{\sqrt{2\pi}} \frac{K_{1}\left(\sqrt{[b^2+\phi][\xi+(y-\mu)^2]}\right)}{\sqrt{\frac{\pi}{2}}\left(\frac{1}{\xi\phi}\right)^{1/4} e^{-\sqrt{\xi\phi}}} \left(\frac{\xi+(y-\mu)^2}{b^2+\phi}\right)^{-\frac{1}{2}} e^{b(y-\mu)} \\ =& \frac{\xi^{1/2}}{\pi} K_1\left(\sqrt{[b^2+\phi][\xi+(y-\mu)^2]}\right) \left(\frac{\xi+(y-\mu)^2}{b^2+\phi}\right)^{-\frac{1}{2}} e^{b(y-\mu)+\sqrt{\xi\phi}} \end{aligned}$$

Thus,

$$f_{NIG}(y;\xi,\phi,b,\mu) = \begin{cases} \dfrac{\sqrt{\xi}}{\pi}\sqrt{\dfrac{b^2+\phi}{\xi+(y-\mu)^2}} K_1\left(\sqrt{[b^2+\phi][\xi+(y-\mu)^2]}\right) e^{\sqrt{\xi\phi}+b(y-\mu)}, & y \in \mathbb{R} \\ 0, \text{ elsewhere} \end{cases} \tag{3.6.1}$$

As before, we set $\xi = \delta^2$ and $\phi = a^2 - b^2$ (in equation 3.6.1) to obtain the NIG distribution under the first parametrization.

Under this parametrization, a random variable Y is said to follow a NIG distribution with parameters (a,b,δ,μ), i.e., $Y \sim NIG(a,b,\delta,\mu)$ if its PDF is of the form

$$f_{NIG}(y;a,b,\delta,\mu) = \begin{cases} \dfrac{a\delta K_1\left(a\sqrt{\delta^2+(y-\mu)^2}\right)}{\pi\sqrt{\delta^2+(y-\mu)^2}} e^{\delta\sqrt{a^2-b^2}+b(y-\mu)}, & y \in \mathbb{R} \\ 0, \text{ elsewhere} \end{cases} \tag{3.6.2}$$

where, $\mu \in \mathbb{R}, 0 \leq |b| \leq a$ and $\delta > 0$. $K_1(\cdot)$ is the modified Bessel function of the third kind (given by equation 3.3.1) with index one. a is the steepness parameter, b the asymmetry parameter, δ the scale parameter and μ the location parameter. a and b determine the shape.
Simplifying (3.6.2) further,

$$f_{NIG}(y;a,b,\delta,\mu) = \frac{a\delta}{\pi} \frac{K_1\left(a\delta\sqrt{1+\frac{(y-\mu)^2}{\delta^2}}\right)}{\delta\sqrt{1+\frac{(y-\mu)^2}{\delta^2}}} e^{\delta\sqrt{a^2-b^2}-b\mu} e^{by}$$

$$= \frac{a}{\pi} e^{\delta\sqrt{a^2-b^2}-b\mu} \left(\sqrt{1+\left(\frac{y-\mu}{\delta}\right)^2}\right)^{-1} K_1\left(a\delta\sqrt{1+\left(\frac{y-\mu}{\delta}\right)^2}\right) e^{by}$$

Let $q(c) = \sqrt{1+c^2}$ and $h(a,b,\mu,\delta) = \dfrac{a}{\pi} e^{\delta\sqrt{a^2-b^2}-b\mu}$ where, $c = \frac{y-\mu}{\delta}$ and $h(a,b,\mu,\delta)$ is the norming constant.

$$\Rightarrow \qquad f_{NIG}(y;a,b,\delta,\mu) = \begin{cases} h(a,b,\mu,\delta)\left[q(c)\right]^{-1} K_1\left(a\delta q(c)\right) e^{by}, & y \in \mathbb{R} \\ 0, \text{ elsewhere} \end{cases}$$

where, $\mu \in \mathbb{R}, \delta \in \mathbb{R}_+, 0 \leq |b| \leq a$.
The NIG PDF in Barndorff-Nielsen (1997, 1998) is of this form.

3.6.2 Properties

Property 3.6.1 (MGF).

$$M_{NIG}(t) = e^{\mu t - \delta\left[\sqrt{a^2-(b+t)^2}-\sqrt{a^2-b^2}\right]}, \quad |b+t| < a. \tag{3.6.3}$$

Proof.

$$\begin{aligned}
M_Y(t) =& E[e^{tY}] \\
=& \int_{-\infty}^{\infty} e^{ty} h(a,b,\mu,\delta)\left[q\left(\frac{y-\mu}{\delta}\right)\right]^{-1} K_1\left(a\delta q\left(\frac{y-\mu}{\delta}\right)\right)e^{by}dy \\
=& h(a,b,\mu,\delta)\int_{-\infty}^{\infty} e^{(b+t)y}\left[q\left(\frac{y-\mu}{\delta}\right)\right]^{-1} K_1\left(a\delta q\left(\frac{y-\mu}{\delta}\right)\right)dy \\
=& \frac{h(a,b,\mu,\delta)}{h(a,b+t,\mu,\delta)}\int_{-\infty}^{\infty} h(a,b+t,\mu,\delta)\left[q\left(\frac{y-\mu}{\delta}\right)\right]^{-1} K_1\left(a\delta q\left(\frac{y-\mu}{\delta}\right)\right)e^{(b+t)y}dy \\
=& \frac{h(a,b,\mu,\delta)}{h(a,b+t,\mu,\delta)} \\
=& \frac{e^{\delta\sqrt{a^2-b^2}-b\mu}}{e^{\delta\sqrt{a^2-(b+t)^2}-(b+t)\mu}} = e^{\mu t}\frac{e^{\delta\sqrt{a^2-b^2}}}{e^{\delta\sqrt{a^2-(b+t)^2}}}, \quad |b+t| < a.
\end{aligned}$$

□

Property 3.6.2.

$$\psi_{NIG}(u) = e^{i\mu u}e^{-\delta\left[\sqrt{a^2-(b+iu)^2}-\sqrt{a^2-b^2}\right]} \tag{3.6.4}$$

Proof.
The proof is similar to the proof to Property 3.6.1, by noting that

$$\psi_{NIG}(u) = M_{NIG}(iu)$$

where, $i = \sqrt{-1}$. □

Property 3.6.3 (Moments). *From equation 3.6.3, any integer order moments of NIG can be derived.*
The first four central moments are given by:

$$\begin{aligned}
E[Y] &= \mu + \frac{\delta b}{\sqrt{a^2 - b^2}} \\
Var[Y] &= \delta \frac{a^2}{(\sqrt{a^2 - b^2})^3} \\
Skew[Y] &= 3 \frac{b}{a\sqrt{\delta\sqrt{a^2 - b^2}}} \\
Kurt[Y] &= 3 + 3\left(1 + 4\left(\frac{b}{a}\right)^2\right)\frac{1}{\delta\sqrt{a^2 - b^2}}
\end{aligned}$$

Property 3.6.4 (Scaling property).

Blæsild (1981) showed that if $Y \sim GH(\vartheta, a, b, \delta, \mu)$, then, for $c > 0$ and $d \in \mathbb{R}$, $X = cY + d$ is also a GHD, i.e., $X \sim GH(\vartheta^{\oplus}, a^{\oplus}, b^{\oplus}, \delta^{\oplus}, \mu^{\oplus})$ where, $\vartheta^{\oplus} = \vartheta$, $a^{\oplus} = a/|c|$, $b^{\oplus} = b/|c|$, $\delta^{\oplus} = \delta|c|$ and $\mu^{\oplus} = c\mu + d$.
Since the GHD nests the NIG as a special case, this property holds for NIG as well.

Property 3.6.5 (Convolution property).

$$NIG(a, b, \delta_X, \mu_X) * NIG(a, b, \delta_Y, \mu_Y) = NIG(a, b, \delta_X + \delta_Y, \mu_X + \mu_Y)$$

Proof.

If X and Y are two independent random variables, then

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

Thus, if $X \sim NIG(a, b, \delta_X, \mu_X)$ and $Y \sim NIG(a, b, \delta_Y, \mu_Y)$, then

$$\begin{aligned}
M_{X+Y}(t) =& e^{\mu_X t} \frac{e^{\delta_X \sqrt{a^2 - b^2}}}{e^{\delta_X \sqrt{a^2 - (b+t)^2}}} \cdot e^{\mu_Y t} \frac{e^{\delta_Y \sqrt{a^2 - b^2}}}{e^{\delta_Y \sqrt{a^2 - (b+t)^2}}} \\
=& e^{(\mu_X + \mu_Y)t} \frac{e^{(\delta_X + \delta_Y)\sqrt{a^2 - b^2}}}{e^{(\delta_X + \delta_Y)\sqrt{a^2 - (b+t)^2}}}
\end{aligned}$$

Hence, $X + Y \sim NIG(a, b, \delta_X + \delta_Y, \mu_X + \mu_Y)$ □

In general, if $Y_1, \ldots, Y_m$ are independent NIG random variables with common parameters a and b but with different μ_i's and δ_i's (for $i = 1, \ldots, m$), then $Y_1 + \cdots + Y_m \sim$ NIG$(a, b, \sum_{i=1}^m \delta_i, \sum_{i=1}^m \mu_i)$.

Property 3.6.6 (Infinite divisibility).

The GIG distribution is infinitely divisible [5]. Consequently, a (normal) mixture with an infinitely divisible mixing distribution is also infinitely divisible. Thus, GHDs are infinitely divisible [5]. Since the NIG is a subclass of the GHD, it follows that NIG distributions are also infinitely divisible. For a complete definition of infinite divisibility, see, Lemma B.1.1 and Lemma B.1.2.

Property 3.6.7 (Semi-heavy tails).

NIG has semi-heavy tails [6, 8], specifically,

$$f_{NIG}(y;a,b,\delta,\mu) \sim const.|y|^{-\frac{3}{2}} e^{-a|y|+by} \quad as \quad y \to \pm\infty$$

Proof. See, [6]. □

3.6.3 Parameter Estimation

Parameters are estimated via maximum likelihood.
Assuming $y_1, \ldots, y_n$ are independent (and identically distributed), model parameters are obtained by maximizing

$$\log L(\Theta) = \sum_{i=1}^{n} \log f(y_i;\Theta)$$

where, Θ represents the set of model parameters.
The log-likelihood function of a NIG variable, say, $Y \sim NIG(a,b,\delta,\mu)$ is

$$\begin{aligned} \log L(\Theta) =& n\log(a) + n\log(\delta) - n\log(\pi) + n\left[\delta\sqrt{a^2-b^2} - b\mu)\right] + b\sum_{i=1}^{n} y_i \\ & - \frac{1}{2}\sum_{i=1}^{n} \log\left[\delta^2 + (y_i-\mu)^2\right] + \sum_{i=1}^{n} \log K_1\left(a\sqrt{\delta^2+(y_i-\mu)^2}\right) \end{aligned}$$

Due to the complexity in maximizing this likelihood induced by the Bessel function, a maximizing algorithm as suggested in [41] is regarded. In particular, parameter estimation is done using the *ghyp R* package by [18], see, Appendix C.1.

4 On Esscher and Fourier Transforms in Currency Option Pricing

4.1 Introduction

We first define terms used continually in this chapter. For this, the reader is referred to Appendix B.1 and for a detailed exposition, see, [10, 17, 23, 47, 50, 51].

4.2 Esscher Transforms

The Esscher transform was first postulated by Fredrik Esscher in 1932 with applications in risk theory. Gerber & Shiu (1994a, 1994b, 1996) applied Esscher transforms in option pricing.

This section and the immediate section that follows provide extensions to the Gerber-Shiu approach by deriving valuation formulae for currency options. We provide an alternate derivation of the Garman-Kohlhagen formula (GK83) under this framework and, following Prause (1999), an extension to pricing formulae with the NIG distribution discussed in Section 3.6. The NIG is chosen because, other than the variance-gamma, it is the only member of the GHD family that has the convolution property, thus its Esscher-transformed density is obtained in closed form.

4.2.1 The Esscher Transform of a Random Variable

Let Y be a random variable with PDF $f(y)$ and $\varphi \in \mathbb{R}$, such that the MGF of Y given by

$$M_Y(\varphi) = E(e^{\varphi Y}) = \int_{-\infty}^{\infty} e^{\varphi y} f(y) dy,$$

exists.

$$\Rightarrow \qquad 1 = \int_{-\infty}^{\infty} \frac{e^{\varphi y} f(y)}{M_y(\varphi)} dy$$

Thus,

$$f(y;\varphi) = \frac{e^{\varphi y} f(y)}{M_Y(\varphi)}$$

is a PDF. This density is the Esscher transform (parametrized by φ) of the initial distribution.

The corresponding MGF is given by

$$\begin{aligned}
M_Y(\mathfrak{z};\varphi) &= \int_{-\infty}^{\infty} e^{\mathfrak{z}y} f(y;\varphi)dy \\
&= \int_{-\infty}^{\infty} e^{\mathfrak{z}y} \frac{e^{\varphi y} f(y)}{M_Y(\varphi)} dy = \frac{1}{M_Y(\varphi)} \int_{-\infty}^{\infty} e^{(\mathfrak{z}+\varphi)y} f(y)dy \\
&= \frac{M_Y(\mathfrak{z}+\varphi)}{M_Y(\varphi)}
\end{aligned}$$

Proposition 4.2.1. *The Esscher transform of an Esscher transform is again an Esscher transform.*

Proof. See, Appendix B.2. $\square$

4.2.2 Esscher Transforms of Stochastic Processes

Let $\{S_t\}_{t\geq 0}$ denote the spot exchange rate price process. Further, let $\{Y(t)\}_{t\geq 0}$ be a stochastic process with independent and stationary increments such that $S_t = S_0 e^{Y_t}$ with $Y_0 = 0$.

By lemma B.1.1, $Y(t)$ has an infinitely divisible distribution. Let its CDF and MGF be denoted, respectively, by

$$F(y,t) = Prob[Y(t) \leq y], \qquad M(\mathfrak{z},t) = E[e^{\mathfrak{z}Y(t)}].$$

If $M(\mathfrak{z},t)$ is continuous at $t = 0$, then by stationarity and by lemma B.1.2,

$$M(\mathfrak{z},t) = [M(\mathfrak{z},1)]^t \tag{4.2.1}$$

Further, if $Y(t)$ has density $f(y,t)$ then

$$M(\mathfrak{z},t) = E(e^{\mathfrak{z}Y(t)}) = \int_{-\infty}^{\infty} e^{\mathfrak{z}y} f(y,t)dy \tag{4.2.2}$$

From equation 4.2.2,

$$1 = \int_{-\infty}^{\infty} \frac{e^{\mathfrak{z}y}}{M(\mathfrak{z},t)} f(y,t)dy$$

Thus,

$$f(y,t;\varphi) = \frac{e^{\varphi y} f(y,t)}{\int_{-\infty}^{\infty} e^{\varphi x} f(x,t)dx} = \frac{e^{\varphi y} f(y,t)}{M(\varphi,t)} \tag{4.2.3}$$

is a PDF.
The corresponding MGF is

$$\begin{aligned} M(\mathfrak{z},t;\varphi) &= \int_{-\infty}^{\infty} e^{\mathfrak{z}y} f(y,t;\varphi)dy \\ &= \int_{-\infty}^{\infty} e^{\mathfrak{z}y} \frac{e^{\varphi y} f(y,t)}{M(\varphi,t)} dy = \frac{1}{M(\varphi,t)} \int_{-\infty}^{\infty} e^{(\mathfrak{z}+\varphi)y} f(y,t)dy \\ &= \frac{M(\mathfrak{z}+\varphi,t)}{M(\varphi,t)} \end{aligned} \tag{4.2.4}$$

And,

$$M(\mathfrak{z},t;\varphi) = [M(\mathfrak{z},1;\varphi)]^t$$

Extending definition B.1.18 to a stochastic process $Y(t)$ setting, we have the Esscher measure, $\mathbb{P}_{Y_t}^{\varphi}$, defined by

$$d\mathbb{P}_{Y_t}^{\varphi} = \frac{e^{\varphi Y_t}}{M(\varphi,t)} d\mathbb{P}_{Y_t} = \frac{e^{\varphi Y_t}}{[M(\varphi,1)]^t} d\mathbb{P}_{Y_t}$$

provided $M(\varphi,t)$ exists.
Clearly,

$$\frac{d\mathbb{P}_{Y_t}^{\varphi}}{d\mathbb{P}_{Y_t}} = \frac{e^{\varphi Y_t}}{M(\varphi,t)}$$

Thus,

Lemma 4.2.1. $\dfrac{e^{\varphi Y_t}}{M(\varphi,t)}$ *is a Radon-Nikodým derivative of* $\mathbb{P}_{Y_t}^{\varphi}$ *w.r.t.* $\mathbb{P}_{Y_t}$.

Proof.
For ease of notation, let $Z = \dfrac{e^{\varphi Y_t}}{M(\varphi,t)}$.
To show that Z is indeed a Radon-Nikodým derivative, we require $Z > 0$ and $E^{\mathbb{P}}(Z) = 1$.

Provided $M(\varphi,t)$ exists, $Z > 0$, because of the exponential term $e^{\varphi Y_t}$.

Also,

$$E^{\mathbb{P}}(Z) = E^{\mathbb{P}}\left[\frac{e^{\varphi Y_t}}{M(\varphi,t)}\right] = \frac{1}{M(\varphi,t)} E^{\mathbb{P}}[e^{\varphi Y_t}] = 1$$

□

Thus, Z is a Radon-Nikodým derivative. From Theorem B.1.2, $\mathbb{P}_{Y_t}^{\varphi}$ is equivalent to $\mathbb{P}_{Y_t}$, in that they have the same null sets.

4.3 Risk-Neutral Pricing with Esscher Transforms

4.3.1 Equivalent Martingale Measure and Market Completeness

We find $\varphi = \varphi^*$ such that the discounted exchange rate price process $\left\{e^{-(r_d - r_f)t} S_t\right\}_{t \geq 0}$ is a $\mathbb{P}^{\varphi^*}$-martingale.
In particular,

$$S_0 = E^{\mathbb{P}^{\varphi^*}}[e^{-(r_d - r_f)t} S_t]$$

But, $S_t = S_0 e^{Y_t}$, ergo

$$1 = e^{-(r_d - r_f)t} E^{\mathbb{P}^{\varphi^*}}[e^{Y(t)}] \qquad \Rightarrow \qquad e^{(r_d - r_f)t} = E^{\mathbb{P}^{\varphi^*}}[e^{Y(t)}]$$

Thus,

$$e^{(r_d - r_f)t} = M(1, t; \varphi^*) \tag{4.3.1}$$

From $M(\mathfrak{z}, t; \varphi) = [M(\mathfrak{z}, 1; \varphi)]^t$, we set $t = 1$ as the solution does not depend on t.

$$\therefore \qquad e^{r_d - r_f} = M(1, 1; \varphi^*) \qquad \Rightarrow \qquad r_d - r_f = \log[M(1, 1; \varphi^*)] \tag{4.3.2}$$

The MGF of the Esscher-transformed distribution, $M(1, 1; \varphi^*)$, is derived, having made distributional assumptions about the stochastic process governing the exchange rates. We solve for φ^* given that r_d and r_f can be obtained from market data or by making sufficient assumptions about them.

Lemma 4.3.1. *The value of φ^* obtained is unique.*

Proof.
Following Gerber & Shiu (1994b), consider the function,

$$g(\varphi) = \log[M(1, 1; \varphi)] = \log\left[\frac{M(1+\varphi, 1)}{M(\varphi, 1)}\right]$$

Thus,

$$M(1+\varphi, 1) = E(e^{(1+\varphi)Y(1)}) \qquad \text{and} \qquad M(\varphi, 1) = E(e^{\varphi Y(1)})$$

$$\Rightarrow \qquad \frac{d}{d\varphi} M(1+\varphi, 1) = E\left(\frac{d}{d\varphi} e^{(1+\varphi)Y(1)}\right) = E(Y(1); 1+\varphi) \cdot M(1+\varphi, 1)$$

$$\frac{d}{d\varphi} M(\varphi, 1) = E\left(\frac{d}{d\varphi} e^{\varphi Y(1)}\right) = E(Y(1); \varphi) \cdot M(\varphi, 1)$$

Also,

$$\frac{d^2}{d\varphi^2} M(\varphi, 1) = \frac{d}{d\varphi}[E(Y(1); \varphi) \cdot M(\varphi, 1)] = M(\varphi, 1)\left[\left(E(Y(1); \varphi)\right)^2 + \frac{d}{d\varphi} E(Y(1); \varphi)\right]$$

Hence,

$$Var(Y(1);\varphi) = \frac{d}{d\varphi}E(Y(1);\varphi)$$

Thus, $E(Y(1);\varphi)$ is an increasing function in φ.
Using chain and quotient differentiation rules, it can be shown that,

$$\begin{aligned} g\prime(\varphi) =& \frac{M(\varphi,1)}{M(1+\varphi,1)}\left[\frac{E[Y(1);1+\varphi]\cdot M(1+\varphi,1)\cdot M(\varphi,1) - E[Y(1);\varphi]\cdot M(1+\varphi,1)\cdot M(\varphi,1)}{[M(\varphi,1)]^2}\right] \\ =& E[Y(1);1+\varphi] - E(Y(1);\varphi) \end{aligned}$$

Hence, $g\prime(\varphi)$ is positive which implies that $g(\varphi)$ is an increasing function. Thus, the solution to equation 4.3.2, which is $g(\varphi) = r_d - r_f$, is unique. □

The parameter φ^* is the risk-neutral Esscher transform and the corresponding EMM is the risk-neutral Esscher measure.

4.3.2 Risk-Neutral Pricing with Esscher Transforms

The price of a derivative is calculated as the expectation of its discounted payoffs under the risk-neutral measure.
In particular, the price of a derivative with payoff function $g(S_T)$ is computed as

$$E^{\mathbb{P}^{\varphi^*}}[e^{-r_dT}g(S_T)] = e^{-r_dT}\int g(S_0e^{Y_T})d\mathbb{P}^{\varphi^*}_{Y_T} \tag{4.3.3}$$

where, $\mathbb{P}^{\varphi^*}$ and $\mathbb{P}^{\varphi^*}_{Y_T}$ denote the risk-neutral Esscher measure and the distribution of Y_T, respectively.
The price for a European currency call, C_0, with $g(S_T) = \max(S_T - K, 0)$ is

$$\begin{aligned} C_0 =& E^{\mathbb{P}^{\varphi^*}}[e^{-r_dT}g(S_T)] \\ =& e^{-r_dT}\int_{-\infty}^{\infty} max(S_T - K, 0)d\mathbb{P}^{\varphi^*}_{Y_T} \end{aligned}$$

We now wish to get rid of the $max(\cdot)$ function in the integral above. We accomplish this by noting that a call is almost surely exercised when it closes ITM which implies a positive payoff, otherwise the payoff is zero. Using $S_T = S_0e^{Y_T}$, we have

$$\begin{aligned} S_T &> K \\ S_0e^{Y_T} &> K \\ Y_T &> \log({}^K/_{S_0}) \\ Y_T &> \varsigma, \qquad \text{where,} \quad \varsigma = \log({}^K/_{S_0}) \end{aligned}$$

For $Y_T \leq \varsigma$, the payoff is certainly zero. Using this fact, we can therefore, partition the integral such that:

$$C_0 = e^{-r_d T}\left[\int_{-\infty}^{\varsigma} 0 \cdot d\mathbb{P}_{Y_T}^{\varphi^*} + \int_{\varsigma}^{\infty} (S_T - K) d\mathbb{P}_{Y_T}^{\varphi^*}\right]$$

For ease of notation and WLOG, let $Y_T = y$. Thus, $\mathbb{P}_y^{\varphi^*}$ denotes the Esscher-transformed distribution of Y_T. Since $\mathbb{P}_y^{\varphi^*}$ and $\mathbb{P}_y$ are equivalent, by definition B.1.15, we may define

$$\frac{e^{\varphi^* y}}{M(\varphi^*, T)}$$

as a Radon-Nikodým derivative of $\mathbb{P}_y^{\varphi^*}$ w.r.t. $\mathbb{P}_y$.

$$\therefore \qquad \frac{d\mathbb{P}_y^{\varphi^*}}{d\mathbb{P}_y} = \frac{e^{\varphi^* y}}{M(\varphi^*, T)} \qquad \Rightarrow \qquad d\mathbb{P}_y^{\varphi^*} = \frac{e^{\varphi^* y}}{M(\varphi^*, T)} d\mathbb{P}_y$$

Since $\mathbb{P}_y$ is the distribution of Y_T, it follows that

$$d\mathbb{P}_y = f(y, T) dy$$

Thus,

$$d\mathbb{P}_y^{\varphi^*} = \frac{e^{\varphi^* y}}{M(\varphi^*, T)} d\mathbb{P}_y = \frac{e^{\varphi^* y}}{M(\varphi^*, T)} f(y, T) dy = f(y, T; \varphi^*) dy$$

And,

$$\begin{aligned} C_0 =& e^{-r_d T} \int_{\varsigma}^{\infty} (S_T - K) d\mathbb{P}_y^{\varphi^*} \\ =& e^{-r_d T} \int_{\varsigma}^{\infty} (S_T - K) f(y, T; \varphi^*) dy \\ =& e^{-r_d T} \int_{\varsigma}^{\infty} [S_0 e^y - K] f(y, T; \varphi^*) dy \\ =& S_0 e^{-r_d T} \int_{\varsigma}^{\infty} e^y f(y, T; \varphi^*) dy - K e^{-r_d T} \int_{\varsigma}^{\infty} f(y, T; \varphi^*) dy \end{aligned}$$

It can be shown that

$$e^y f(y, T; \varphi^*) = e^{(r_d - r_f)T} f(y, T; \varphi^* + 1)$$

Proof.

$$e^y f(y,T;\varphi^*) = \frac{e^{(\varphi^*+1)y} f(y,T)}{M(\varphi^*,T)}$$

Using equation 4.2.3,

$$e^{(\varphi^*+1)y} f(y,T) = M(\varphi^*+1,T) f(y,T;\varphi^*+1)$$

Thus,

$$e^y f(y,T;\varphi^*) = \frac{M(\varphi^*+1,T)}{M(\varphi^*,T)} f(y,T;\varphi^*+1)$$

From equation 4.2.4 and 4.3.1, it follows that

$$\frac{M(\varphi^*+1,T)}{M(\varphi^*,T)} = M(1,T;\varphi^*) = e^{(r_d-r_f)T}$$

$$\Rightarrow \qquad e^y f(y,T;\varphi^*) = e^{(r_d-r_f)T} f(y,T;\varphi^*+1). \qquad \square$$

Thus, the call pricing formula w.r.t. the risk-neutral Esscher measure is

$$C_0 = S_0 e^{-r_f T} \int_{\varsigma}^{\infty} f(y,T;\varphi^*+1)dy - Ke^{-r_d T} \int_{\varsigma}^{\infty} f(y,T;\varphi^*)dy \tag{4.3.4}$$

where, $f(y,T;\varphi^*)$ is the risk-neutral Esscher-transformed PDF of Y_T.

4.3.3 Esscher-Normal Model Prices: An alternative derivation of the GK83

Let $\Phi(y;\mu,\sigma^2)$ denote the Gaussian distribution function (of a random variable Y) with mean μ and variance σ^2. Further, let $\{Y(t)\}$ be a Wiener process with drift.
It follows that the CDF of $\{Y(t)\}$ is

$$F(y,t) = \Phi(y;\mu t,\sigma^2 t)$$

and its MGF

$$M(\mathfrak{z},t) = e^{[(\mu\mathfrak{z}+\frac{1}{2}\sigma^2\mathfrak{z}^2)t]}$$

Consequently,

$$M(\mathfrak{z},t;\varphi) = \frac{M(\mathfrak{z}+\varphi,t)}{M(\varphi,t)} = e^{[(\mu+\varphi\sigma^2)\mathfrak{z}+\frac{1}{2}\sigma^2\mathfrak{z}^2]t}$$

Hence, the Esscher transform (parameter φ) is also a Wiener process with modified parameter $\mu^M = \mu + \varphi\sigma^2$ and CDF

$$F(y,t;\varphi) = \Phi[y;(\mu+\varphi\sigma^2)t,\sigma^2 t] = \Phi[y;\mu^M t,\sigma^2 t] \tag{4.3.5}$$

Setting $\varphi = \varphi^*$ and $t = \mathfrak{z} = 1$, we have

$$M(1,1;\varphi^*) = e^{[(\mu+\varphi^*\sigma^2)+\frac{1}{2}\sigma^2]}$$

But, $e^{r_d - r_f} = M(1,1;\varphi^*)$,

$$\Rightarrow \qquad r_d - r_f = (\mu + \varphi^*\sigma^2) + \frac{1}{2}\sigma^2$$

Thus,

$$\varphi^* = \frac{r_d - r_f - \mu - \frac{1}{2}\sigma^2}{\sigma^2} \tag{4.3.6}$$

If $f(w;\Theta)$ is a PDF of a continuous random variable W with parameter set Θ, then, by definition,

$$1 = \int_{-\infty}^{\infty} f(w;\Theta)dw = \int_{-\infty}^{w} f(w;\Theta)dw + \int_{w}^{\infty} f(w;\Theta)dw$$

Thus,

$$1 - F(w;\Theta) = \int_{w}^{\infty} f(w;\Theta)dw$$

where, $F(w;\Theta) = \int_{-\infty}^{w} f(w;\Theta)dw$ is the CDF.

From equation 4.3.4, we have

$$C_0 = S_0 e^{-r_f T}\int_{\varsigma}^{\infty} f(y,T;\varphi^*+1)dy - Ke^{-r_d T}\int_{\varsigma}^{\infty} f(y,T;\varphi^*)dy$$

$$= S_0 e^{-r_f T}\Big[1 - F\big(\varsigma,T;\varphi^*+1\big)\Big] - Ke^{-r_d T}\Big[1 - F\big(\varsigma,T;\varphi^*\big)\Big]$$

From equation 4.3.5, and letting $\mu^* = \mu + \varphi^*\sigma^2$,

$$\begin{aligned} F\big(\varsigma,T;\varphi^*\big) &= \Phi\Big[\varsigma;(\mu+\varphi^*\sigma^2)T, \sigma^2 T\Big] \\ &= \Phi\Big[\varsigma;\mu^* T, \sigma^2 T\Big] \end{aligned}$$

$$\begin{aligned} F\big(\varsigma,T;\varphi^*+1\big) &= \Phi\Big[\varsigma;(\mu+(\varphi^*+1)\sigma^2)T, \sigma^2 T\Big] \\ &= \Phi\Big[\varsigma;(\mu^*+\sigma^2)T, \sigma^2 T\Big] \end{aligned}$$

Making φ^* the subject of the formula in $\mu^* = \mu + \varphi^*\sigma^2$, we have

$$\varphi^* = \frac{\mu^* - \mu}{\sigma^2}$$

But from equation 4.3.6,

$$\varphi^* = \frac{r_d - r_f - \mu - \frac{1}{2}\sigma^2}{\sigma^2}$$

Thus, equating the two,

$$\begin{aligned} \frac{\mu^* - \mu}{\sigma^2} =& \frac{r_d - r_f - \mu - \frac{1}{2}\sigma^2}{\sigma^2} \\ \Rightarrow \qquad \mu^* =& r_d - r_f - \frac{1}{2}\sigma^2 \end{aligned}$$

Therefore,

$$F\left(\varsigma, T; \varphi^*\right) = \Phi\left[\varsigma; (r_d - r_f - \frac{1}{2}\sigma^2)T, \sigma^2 T\right]$$

and

$$F\left(\varsigma, T; \varphi^* + 1\right) = \Phi\left[\varsigma; (r_d - r_f + \frac{1}{2}\sigma^2)T, \sigma^2 T\right]$$

$\therefore$ The call pricing formula is given by

$$C_0 = S_0 e^{-r_f T}\left\{1 - \Phi\left[\varsigma; (r_d - r_f + \frac{1}{2}\sigma^2)T, \sigma^2 T\right]\right\} - K e^{-r_d T}\left\{1 - \Phi\left[\varsigma; (r_d - r_f - \frac{1}{2}\sigma^2)T, \sigma^2 T\right]\right\}$$

We can express the above distribution functions in terms of the standard normal, by noting that, for $Y \sim \Phi(y; \mu, \sigma^2)$, standardization implies $\dfrac{Y - \mu}{\sigma} \sim \Phi(z; 0, 1)$ where, $Z = \dfrac{Y - \mu}{\sigma}$.

For ease of notation, we write $\Phi(z; 0, 1)$ as $\Phi(z)$.

Thus, $\quad \dfrac{\varsigma - (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \sim \Phi(z_1) \quad$ and $\quad \dfrac{\varsigma - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \sim \Phi(z_2)$

where,

$$Z_1 = \frac{\varsigma - (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \qquad \text{and} \qquad Z_2 = \frac{\varsigma - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

But,

$$1 - \Phi(z) = \Phi(-z) \qquad \text{where, as before,} \qquad \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du$$

Therefore,

$$C_0 = S_0 e^{-r_f T}\Phi(-z_1) - K e^{-r_d T}\Phi(-z_2)$$

By letting,

$$d_1 = -z_1 = \frac{-\varsigma + (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = -z_2 = \frac{-\varsigma + (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

we obtain the Garman-Kohlhagen currency pricing formula, A.3.3.
Specifically,

$$C_0 = S_0 e^{-r_f T}\Phi(d_1) - K e^{-r_d T}\Phi(d_2) \tag{4.3.7}$$

4.3.4 Esscher-NIG Model Prices

From equation 3.6.3 and equation 4.2.1,

$$M_{NIG}(\mathfrak{z},t) = [M_{NIG}(\mathfrak{z},1)]^t = e^{\mu\mathfrak{z}t}\frac{e^{\delta t\sqrt{a^2-b^2}}}{e^{\delta t\sqrt{a^2-(b+\mathfrak{z})^2}}} \tag{4.3.8}$$

Since $M(\mathfrak{z},t;\varphi) = \dfrac{M(\mathfrak{z}+\varphi,t)}{M(\varphi,t)}$, it can be shown that

$$M_{NIG}(\mathfrak{z},t;\varphi) = e^{\mu\mathfrak{z}t}\frac{e^{\delta t\sqrt{a^2-(b+\varphi)^2}}}{e^{\delta t\sqrt{a^2-(b+\mathfrak{z}+\varphi)^2}}} \tag{4.3.9}$$

Proposition 4.3.1. *The Esscher transform (parameter φ) of a NIG $(a,b,t\delta,t\mu)$ distribution is again a NIG $(a,b^*,t\delta,t\mu)$ distribution, but with modified parameter $b^* = b+\varphi$.*

Proof. Using moment generating functions, we have

$$\begin{aligned} M_{NIG}(\mathfrak{z},t;\varphi) =& \frac{M_{NIG}(\mathfrak{z}+\varphi,t)}{M_{NIG}(\varphi,t)} \\ =& \frac{e^{\mu(\mathfrak{z}+\varphi)t+\delta t\sqrt{a^2-b^2}-\delta t\sqrt{a^2-(b+\mathfrak{z}+\varphi)^2}}}{e^{\mu\varphi t+\delta t\sqrt{a^2-b^2}-\delta t\sqrt{a^2-(b+\varphi)^2}}} \\ =& e^{\mu\mathfrak{z}t}\frac{e^{\delta t\sqrt{a^2-(b+\varphi)^2}}}{e^{\delta t\sqrt{a^2-(b+\mathfrak{z}+\varphi)^2}}} \qquad \text{where,} \quad |b+\varphi| < a \quad \text{and} \quad |b+\mathfrak{z}+\varphi| < a. \end{aligned}$$

□

As before, we find $\varphi = \varphi^*$ such that discounted exchange rate price process is a $\mathbb{P}^{\varphi^*}$-martingale.
Thus, setting $\varphi = \varphi^*$ and $t = \mathfrak{z} = 1$ in equation 4.3.9, we have

$$M_{NIG}(1,1;\varphi^*) = e^{\mu}\frac{e^{\delta\sqrt{a^2-(b+\varphi^*)^2}}}{e^{\delta\sqrt{a^2-(b+\varphi^*+1)^2}}} = e^{\mu+\delta\sqrt{a^2-(b+\varphi^*)^2}-\delta\sqrt{a^2-(b+\varphi^*+1)^2}} \tag{4.3.10}$$

But, $e^{r_d - r_f} = M(1,1;\varphi^*)$,

$$\Rightarrow \qquad r_d - r_f = \log[M_{NIG}(1,1;\varphi^*)]$$
$$= \mu + \delta\sqrt{a^2 - (b+\varphi^*)^2} - \delta\sqrt{a^2 - (b+\varphi^*+1)^2}$$
$$= \mu + \delta\left[\sqrt{a^2 - (b+\varphi^*)^2} - \sqrt{a^2 - (b+\varphi^*+1)^2}\right]$$

$$\therefore \qquad \frac{r_d - r_f - \mu}{\delta} = \sqrt{a^2 - (b+\varphi^*)^2} - \sqrt{a^2 - (b+\varphi^*+1)^2} \tag{4.3.11}$$

We solve for φ^* in equation 4.3.11 using root finding methods (see Appendix C.2) given that all other variables in the equation can either be observed from the market or estimated from data.

Finally, prices are calculated directly using equation 4.3.4. Since the risk-neutral Esscher-transformed NIG density is obtained in closed form, equation 4.3.4 reduces to a numerical integration problem which is given in a more complete form by equation 4.3.12.

$$C_0 = S_0 e^{-r_f T}\int_{\varsigma}^{\infty} \mathbf{nig}(y,T;a,b+\varphi^*+1,T\delta,T\mu)dy - Ke^{-r_d T}\int_{\varsigma}^{\infty} \mathbf{nig}(y,T;a,b+\varphi^*,T\delta,T\mu)dy \tag{4.3.12}$$

where, $\mathbf{nig}(y,T;a,\bullet,T\delta,T\mu)$ is the risk-neutral Esscher-transformed NIG PDF of Y_T.

4.4 Fourier Transforms in Currency Option Pricing

As earlier noted, the pricing formula given by equation 4.3.4 can be evaluated through direct numerical integration provided that the risk-neutral Esscher-transformed density can be obtained in closed form. However, most risk-neutral densities are complicated or not known in closed form whereas their characteristic functions are simple. Thus, Fourier methods are attractive as one can invert the characteristic function to obtain the risk-neutral PDF, consequently derive option pricing formulae.

Here, we consider option pricing via the Fourier inversion method proposed by [20].

4.4.1 Option Prices by Fourier Inversion

Carr & Madan (1999) derived a numerical approach for valuing options - provided that the characteristic function is analytically known - by employing the fast Fourier transform (FFT, henceforth).

Definition 4.4.1 (Fourier transform). *For an absolutely integrable function $h(z)$, its Fourier transform is defined as*

$$\psi(w) = \int_{-\infty}^{\infty} e^{iwz} h(z)dz$$

Definition 4.4.2 (Inverse Fourier transform). *If $h(z)$ is square integrable and given $\psi(w)$, $h(z)$ can be recovered via*

$$h(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwz}\psi(w)dw$$

Let the characteristic function of $s_T = \log(S_T)$ be

$$\psi_{s_T}(u) = E[e^{iu\log(S_T)}] = \int_{-\infty}^{\infty} e^{ius} f(s,T;\varphi^*)ds$$

where, $f(s,T;\varphi^*)$ is the risk-neutral Esscher-transformed PDF of s_T.
Further, let $C(\kappa)$ denote the price of a call option that matures at T, with exercise price, K.
Thus,

$$C(\kappa) = e^{-r_d T}\int_{\kappa}^{\infty}(e^s - e^{\kappa})f(s,T;\varphi^*)ds \tag{4.4.1}$$

where, $\kappa = \log(K)$.
Equation 4.4.1 is, however, not square-integrable since as $\kappa \to -\infty$, $C(\kappa) \to S_0$.

Proof.

$$\begin{aligned}
\lim_{\kappa\to-\infty} C(\kappa) =& e^{-r_d T}\int_{\kappa\to-\infty}^{\infty}(e^s - e^{\kappa\to-\infty})f(s,T;\varphi^*)ds \\
=& \int_{-\infty}^{\infty} e^{-r_d T} S_T f(s,T;\varphi^*)ds \\
=& E^{\mathbb{P}^{\varphi^*}}\left[e^{-r_d T} S_T\right] \\
=& S_0
\end{aligned}$$

□

Hence, to transform the pricing formula into square-integrable form, a dampening factor ϖ is considered, and the transformed call price $c(\kappa)$ is

$$c(\kappa) = e^{\varpi\kappa}C(\kappa) \qquad \text{where} \quad \varpi > 0$$

Define the Fourier transform of $c(\kappa)$ as

$$\begin{aligned}
\rho_{s_T}(w) &= \int_{-\infty}^{\infty} e^{iw\kappa}c(\kappa)d\kappa \\
&= \int_{-\infty}^{\infty} e^{iw\kappa}\int_{\kappa}^{\infty} e^{\varpi\kappa}e^{-r_d T}(e^s - e^{\kappa})f(s,T;\varphi^*)dsd\kappa \\
&= \int_{-\infty}^{\infty} e^{-r_d T} f(s,T;\varphi^*)\int_{-\infty}^{s}(e^{s+\varpi\kappa} - e^{(1+\varpi)\kappa})e^{iw\kappa}d\kappa ds \\
&= \int_{-\infty}^{\infty} e^{-r_d T} f(s,T;\varphi^*)\int_{-\infty}^{s} e^{s+\varpi\kappa+iw\kappa} - e^{(1+\varpi+iw)\kappa}d\kappa ds \\
&= \int_{-\infty}^{\infty} e^{-r_d T} f(s,T;\varphi^*)\left[\frac{e^{(\varpi+1+iw)s}}{\varpi+iw} - \frac{e^{(\varpi+1+iw)s}}{\varpi+1+iw}\right]ds
\end{aligned}$$

But,

$$e^{(\varpi+1+iw)s} = e^{\frac{i^2}{i^2}(\varpi+1+iw)s} = e^{\left[\frac{(\varpi+1)i-w}{i^2}\right]is} = e^{[w-(\varpi+1)i]is}$$

Thus,

$$\begin{aligned}\rho_{s_T}(w) &= \int_{-\infty}^{\infty} e^{-r_d T} e^{[w-(\varpi+1)i]is} \left(\frac{1}{(\varpi)^2+\varpi-w^2+i(2\varpi+1)w} \right) f(s,T;\varphi^*)ds \\ &= \frac{e^{-r_d T}\psi_{s_T}\big(w-(\varpi+1)i\big)}{(\varpi)^2+\varpi-w^2+i(2\varpi+1)w}\end{aligned} \qquad \textbf{(4.4.2)}$$

Call prices are then obtained by,

$$C(\kappa) = \frac{e^{-\varpi\kappa}}{2\pi}\int_{-\infty}^{\infty} e^{-iw\kappa}\rho_{s_T}(w)dw = \frac{e^{-\varpi\kappa}}{\pi}\int_{0}^{\infty} e^{-iw\kappa}\rho_{s_T}(w)dw \qquad \textbf{(4.4.3)}$$

Choice of optimal ϖ

For $c(\kappa)$ to be square-integrable, $\rho_{s_T}(0)$ must be finite.
From equation 4.4.2, this requires that $\psi_{s_T}\big(-(\varpi+1)i\big)$ is finite which implies

$$E(S_T^{\varpi+1}) < \infty$$

Following Lord & Kahl (2007), we choose ϖ such that

$$\varpi^* = \underset{\varpi\in\{\varpi_{min},\varpi_{max}\}}{\arg\min} |e^{-\kappa\varpi}\psi_{s_T}\big(-(\varpi+1)i\big)|$$

or, equivalently,

$$\varpi^* = \underset{\varpi\in\{\varpi_{min},\varpi_{max}\}}{\arg\min} \Psi(\varpi,\kappa) \qquad \text{where,} \qquad \Psi(\varpi,\kappa) = \left[-\varpi\kappa + \frac{1}{2}\log\left(\psi_{s_T}\big(-((\varpi+1)i)^2\big)\right)\right] \qquad \textbf{(4.4.4)}$$

Obtaining the optimal ϖ entails finding the minimum of Ψ, thus, we solve

$$\frac{\partial\Psi(\varpi,\kappa)}{\partial\varpi} = 0$$

The solution to equation 4.4.4 shows that Ψ has a local minimum in $\varpi^* \in (\varpi_{min}, -1)$, $\varpi^* \in (-1,0)$, and $\varpi^* \in (0,\varpi_{max})$. An upper bound on ϖ is thus obtained. Carr & Madan (1999) recommended a quarter of this upper bound as an appropriate choice of ϖ.

Implementation of the Fourier Inverse Transform: Normal & NIG

In the Carr-Madan approach, call prices are determined w.r.t. the risk-neutral PDF of s_T. In contrast, the valuation formula given by equation 4.3.4 considers the risk-neutral PDF of the stochastic process, Y_T.
The relationship between these two approaches can be established using the linear transformation property of both the normal distribution and the NIG distribution.
As before, let $S_T = S_0 e^{Y_T}$. Rearranging,

$$\log(S_T) = \log(S_0) + Y_T \tag{4.4.5}$$

Normal

The choice of ϖ is obtained via

$$\varpi = \frac{d_1}{\sigma\sqrt{T}} - 1 \tag{4.4.6}$$

Proof. See, Appendix B.2. □

We now have a value of ϖ for every κ, thus, we choose the maximum value among these as the upper bound on ϖ. A quarter of this upper bound, thus, serves as our choice of ϖ.

Option prices in the normal model are obtained via

$$C(\kappa) = \frac{e^{-\varpi\kappa}}{\pi} \int_0^\infty e^{-iw\kappa} \rho_{s_T}(w) dw \tag{4.4.7}$$

where,

$$\rho_{s_T}(w) = \frac{e^{-r_d T} e^{[iw+\varpi+1][\log(S_0)+(r_d-r_f-\sigma^2/2)T]-\frac{1}{2}\sigma^2 T[w-(\varpi+1)i]^2}}{(\varpi)^2+\varpi-w^2+i(2\varpi+1)w}$$

NIG

Similarly, if $Y_T \sim NIG(a, b, T\delta, T\mu)$, then by property 3.6.4,

$$s_T \sim NIG(a, b, T\delta, T\mu + \log(S_0))$$

By equation 3.6.4, the characteristic function of Y_T is

$$\psi_{Y_T}(u, T; \varphi) = e^{i\mu u T} \frac{e^{\delta T\sqrt{a^2-(b+\varphi)^2}}}{e^{\delta T\sqrt{a^2-(b+iu+\varphi)^2}}}$$

Thus, for s_T,

$$\psi_{s_T}(u,T;\varphi) = e^{iu(\mu T+\log(S_0))}\frac{e^{\delta T\sqrt{a^2-(b+\varphi)^2}}}{e^{\delta T\sqrt{a^2-(b+iu+\varphi)^2}}}$$

$$\psi_{s_T}\big(w-(\varpi+1)i,T;\varphi\big) = e^{[\mu T+\log(S_0)][iw+\varpi+1]}\frac{e^{\delta T\sqrt{a^2-(b+\varphi)^2}}}{e^{\delta T\sqrt{a^2-(b+iw+\varpi+1+\varphi)^2}}} \tag{4.4.8}$$

And,

$$\psi_{s_T}\big(-(\varpi+1)i,T;\varphi\big) = e^{(\varpi+1)[\mu T+\log(S_0)]}\frac{e^{\delta T\sqrt{a^2-(b+\varphi)^2}}}{e^{\delta T\sqrt{a^2-(b+\varpi+1+\varphi)^2}}}$$

For the choice of optimal ϖ,

$$\begin{aligned}
\frac{\partial\Psi(\varpi,\kappa)}{\partial\varpi} =& \frac{\partial}{\partial\varpi}\Big[-\varpi\kappa+\frac{1}{2}\log\big(\psi_{s_T}\big(-((\varpi+1)i)^2\big)\Big] \\
=& -\kappa+\frac{\partial}{\partial\varpi}\Big[\log\big(\psi_{s_T}\big(-((\varpi+1)i)\big)\Big] \\
=& -\kappa+\frac{\partial}{\partial\varpi}\Bigg[(\varpi+1)[\mu T+\log(S_0)]+\Big(\delta T\sqrt{a^2-(b+\varphi)^2}\Big) \\
& -\Big(\delta T\sqrt{a^2-(b+\varpi+1+\varphi)^2}\Big)\Bigg] \\
=& -\kappa+\mu T+\log(S_0)-\frac{\partial}{\partial\varpi}\Big[\delta T\sqrt{a^2-(b+\varpi+1+\varphi)^2}\Big] \\
=& -\kappa+\mu T+\log(S_0)+\frac{\delta T(b+\varpi+1+\varphi)}{\sqrt{a^2-(b+\varpi+1+\varphi)^2}}
\end{aligned}$$

We, thus, solve for ϖ in equation 4.4.9 through a simple root finding method (see Appendix C.2).

$$-\kappa+\mu T+\log(S_0)+\frac{\delta T(b+\varpi+1+\varphi)}{\sqrt{a^2-(b+\varpi+1+\varphi)^2}} = 0 \tag{4.4.9}$$

As before, we find the upper bound on ϖ; a quarter of which serves as our choice of ϖ.

Option prices in the NIG model are, thus, obtained via

$$C(\kappa) = \frac{e^{-\varpi\kappa}}{\pi}\int_0^\infty e^{-iw\kappa}\rho_{s_T}(w)dw$$

where,

$$\rho_{s_T}(w) = \frac{e^{-r_dT}\psi_{s_T}\big(w-(\varpi+1)i,T;\varphi\big)}{(\varpi)^2+\varpi-w^2+i(2\varpi+1)w},$$ and $\psi_{s_T}\big(w-(\varpi+1)i,T;\varphi\big)$ is as defined in eqn. 4.4.8.

Approximating C(κ) using the trapezoid and Simpson's rule

The integral (4.4.3) can be computed directly. Alternatively, integral approximation techniques may be applied. Here, we consider the trapezoid and Simpson's rule approximations, see, [Hirsa & Neftci (2014), Chapter 22].

Suppose we intend to evaluate the integral

$$\int_{w_0}^{w_1} h(w)dw$$

where, $h(w)$ is a function (such as in Figure 4.4.1).
The trapezoid rule approximates this integral by regarding the area under the curve as a trapezium. Thus,

$$\int_{w_0}^{w_1} h(w)dw \approx \frac{\Delta w}{2}\Big[h(w_0)+h(w_1)\Big] \quad \text{where,} \quad \Delta w = w_1 - w_0$$

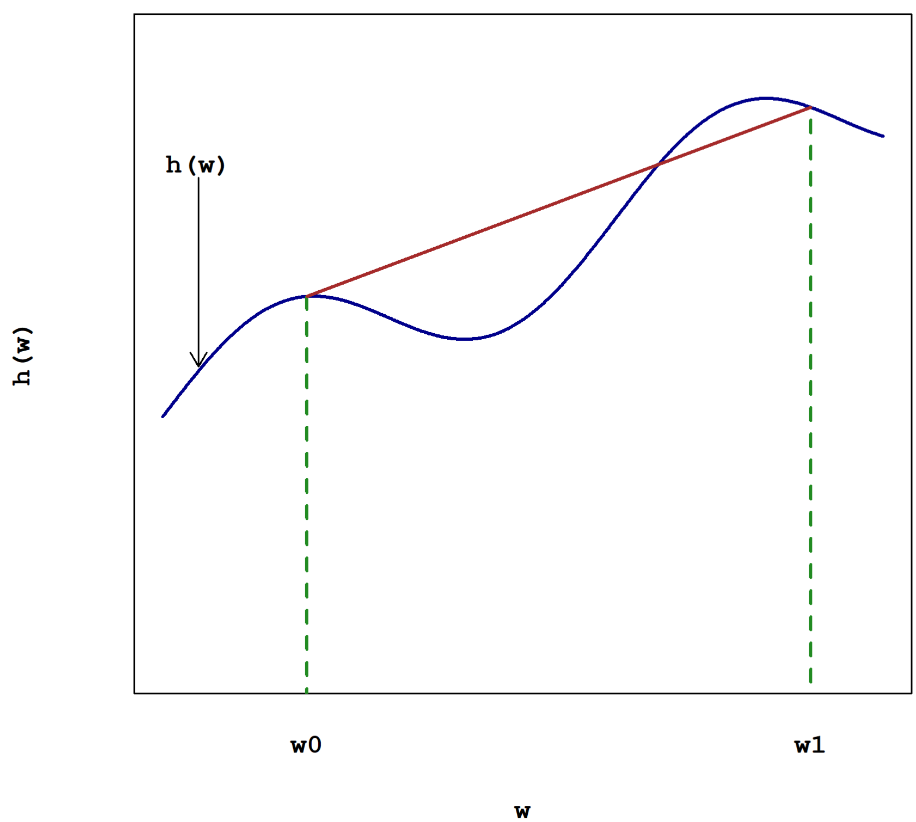

Figure 4.4.1. h(w)=w+2 sin(w): single trapezium

The accuracy of the approximation can be improved by considering smaller partitions and applying the trapezoid rule to these partitions.

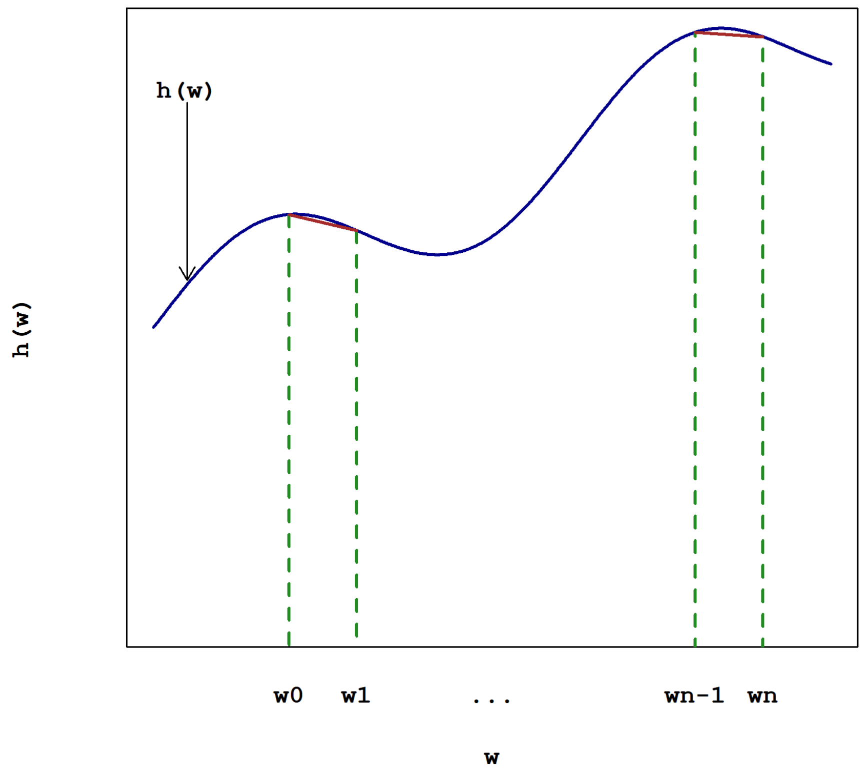

Figure 4.4.2. h(w)=w+2 sin(w): several trapezia

Suppose we form a uniform grid with n equal intervals with $\eta = \Delta w = \frac{w_n - w_0}{n}$.
The required approximation is, thus, obtained as a summation of the areas of the trapezia, i.e.,

$$
\begin{aligned}
\int_{w_0}^{w_n} h(w)dw \approx & \frac{\eta}{2}\sum_{m=1}^{n}\left[h(w_{m-1})+h(w_m)\right] \\
= & \frac{\eta}{2}\left[h(w_0)+2h(w_1)+2h(w_2)+2h(w_3)+\cdots+2h(w_{n-1})+h(w_n)\right] \\
= & \frac{\eta}{2}\left[h(w_0)+h(w_n)\right]+\eta\left[\sum_{m=1}^{n-1}h(w_m)\right]
\end{aligned}
$$

Hence, as n increases the accuracy improves. This improvement in the accuracy of the approximation can be seen by a casual comparison of Figure 4.4.1 and Figure 4.4.2.

Again, from Figure 4.4.1 and Figure 4.4.2, it is clear that the trapezoid rule is not very accurate, especially for curves.

The Simpson's rule estimates the area under a curve between two points (say, w_0 and w_2) in terms of three ordinates, i.e.,

$$\int_{w_0}^{w_2} h(w)dw \approx \frac{\Delta w}{3}\Big[h(w_0)+4h(w_1)+h(w_2)\Big]$$

where, $\Delta w = \frac{w_2-w_0}{2}$.

Generally, in a uniform grid with n equal intervals with $\eta = \Delta w = \frac{w_n-w_0}{n}$,

$$\begin{aligned}
\int_{w_0}^{w_n} h(w)dw \approx & \frac{\eta}{3}\sum_{m=1}^{n/2}\Big[h(w_{2m-2})+4h(w_{2m-1})+h(w_{2m})\Big] \\
= & \frac{\eta}{3}\Big[h(w_0)+4h(w_1)+2h(w_2)+4h(w_3)+2h(w_4)\cdots+4h(w_{n-1})+h(w_n)\Big] \\
= & \frac{\eta}{3}\Big[h(w_0)+h(w_n)\Big]+\frac{4\eta}{3}\sum_{m=1}^{n/2}h(w_{2m-1})+\frac{2\eta}{3}\sum_{m=1}^{n/2-1}h(w_{2m})
\end{aligned}$$

where, n is even, and $w_m = w_0 + m\Delta w$, for $m = 0,1,2,\ldots,n$.
Recall equation 4.4.3,

$$C(\kappa) = \frac{e^{-\varpi\kappa}}{\pi}\int_0^{\infty} e^{-iw\kappa}\rho_{s_T}(w)dw$$

where, $\quad \rho_{s_T}(w) = \dfrac{e^{-r_dT}\psi_{s_T}\big(w-(\varpi+1)i\big)}{(\varpi)^2+\varpi-w^2+i(2\varpi+1)w}$

As before, we let n denote the number of intervals, $\eta = \Delta w$ the distance between integration points and further let $w_m = \eta(m-1)$, for $m = 1,2,\ldots,n+1$.
Thus, setting B as an upper bound for the integral and applying the trapezoid rule to equation 4.4.3,

$$\begin{aligned}
C(\kappa) = & \frac{e^{-\varpi\kappa}}{\pi}\int_0^{\infty} e^{-iw\kappa}\rho_{s_T}(w)dw \\
\approx & \frac{e^{-\varpi\kappa}}{\pi}\int_0^{B} e^{-iw\kappa}\rho_{s_T}(w)dw \\
\approx & \frac{e^{-\varpi\kappa}}{\pi}\Big[e^{-iw_1\kappa}\rho_{s_T}(w_1)+2e^{-iw_2\kappa}\rho_{s_T}(w_2)+\cdots+2e^{-iw_n\kappa}\rho_{s_T}(w_n)+e^{-iw_{n+1}\kappa}\rho_{s_T}(w_{n+1})\Big]\frac{\eta}{2}
\end{aligned}$$

We disregard the last term as the terms decay exponentially, thus, we obtain

$$C(\kappa) \approx \frac{e^{-\varpi\kappa}}{\pi}\sum_{m=1}^{n} e^{-iw_m\kappa}\rho_{s_T}(w_m)d_m$$

where, $d_m = \frac{\eta}{2}(2-\delta_{m-1})$ and

$$\delta_m = \begin{cases} 1 & \text{for} \quad m=0 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for $m = 1,2,\ldots,n+1$ with $\eta = \Delta w$ and $w_m = \eta(m-1)$, applying Simpson's rule to equation 4.4.3 yields,

$$\begin{aligned} C(\kappa) =& \frac{e^{-\varpi\kappa}}{\pi}\int_0^\infty e^{-iw\kappa}\rho_{s_T}(w)dw \\ \approx& \frac{e^{-\varpi\kappa}}{\pi}\int_0^B e^{-iw\kappa}\rho_{s_T}(w)dw \\ \approx& \frac{e^{-\varpi\kappa}}{\pi}\Big[e^{-iw_1\kappa}\rho_{s_T}(w_1) + 4e^{-iw_2\kappa}\rho_{s_T}(w_2) + 2e^{-iw_3\kappa}\rho_{s_T}(w_3) + \cdots \\ & + 4e^{-iw_n\kappa}\rho_{s_T}(w_n) + e^{-iw_{n+1}\kappa}\rho_{s_T}(w_{n+1})\Big]\frac{\eta}{2} \end{aligned}$$

As before, we disregard the last term to obtain,

$$C(\kappa) \approx \frac{e^{-\varpi\kappa}}{\pi}\sum_{m=1}^{n} e^{-iw_m\kappa}\rho_{s_T}(w_m)d_m \tag{4.4.10}$$

where, $d_m = \frac{\eta}{3}(3+(-1)^m - \delta_{m-1})$ and

$$\delta_m = \begin{cases} 1 & \text{for} \quad m=0 \\ 0 & \text{otherwise} \end{cases}$$

4.4.2 Fast Fourier Transform

While performing the numerical integration in (4.4.3) is adequate, it is not highly efficient. The FFT is an efficient algorithm for computing the sum

$$w(\kappa) = \sum_{m=1}^{n} e^{-i\frac{2\pi}{n}(m-1)(\kappa-1)}v(m) \qquad \text{for } \kappa = 1,2,\ldots,n, \tag{4.4.11}$$

where, n is a power of 2.

The pricing formula given by equation 4.4.3 can be converted into FFT form by considering a set of strikes near $\kappa = 0$ which corresponds to ATM call options. Employing a λ-spacing, the values for κ are

$$\kappa_\varepsilon = -b + \lambda(\varepsilon - 1) \quad \text{for} \quad \varepsilon = 1,\ldots,u \tag{4.4.12}$$

where, $b = \frac{1}{2}u\lambda$.
Substituting 4.4.12 into 4.4.10,

$$\begin{aligned} C(\kappa_\varepsilon) &\approx \frac{e^{-\varpi\kappa_\varepsilon}}{\pi}\sum_{m=1}^{n} e^{-iw_m\kappa_\varepsilon}\rho_{s_T}(w_m)d_m \quad \text{for} \quad \varepsilon = 1,\ldots,u \\ &\approx \frac{e^{-\varpi\kappa_\varepsilon}}{\pi}\sum_{m=1}^{n} e^{-iw_m[-b+\lambda(\varepsilon-1)]}\rho_{s_T}(w_m)d_m \end{aligned}$$

Noting that $w_m = \eta(m-1)$

$$C(\kappa_\varepsilon) \approx \frac{e^{-\varpi\kappa_\varepsilon}}{\pi}\sum_{m=1}^{n} e^{-i\lambda\eta(m-1)(\varepsilon-1)}e^{ibw_m}\rho_{s_T}(w_m)d_m$$

To complete the FFT conversion we note, from equation 4.4.11, that $\lambda\eta = \frac{2\pi}{n}$.
Thus, the general pricing formula is

$$C(\kappa_\varepsilon) \approx \frac{e^{-\varpi\kappa_\varepsilon}}{\pi}\sum_{m=1}^{n} e^{-i\frac{2\pi}{n}(m-1)(\varepsilon-1)}e^{ibw_m}\rho_{s_T}(w_m)d_m \tag{4.4.13}$$

where, $d_m = \frac{\eta}{3}(3+(-1)^m - \delta_{m-1})$ and

$$\delta_m = \begin{cases} 1 & \text{for} \quad m = 0 \\ 0 & \text{otherwise} \end{cases}$$

Remark. *For a single strike price, i.e., for $\varepsilon = 1$, equation 4.4.13 reduces to equation 4.4.10.*

FFT Implementation

In the application to pricing, following [20], we set η=0.25 and n=4096.
And, in the $\rho_{s_T}(w)$ function in equation 4.4.13, we set

$$\rho_{s_T}(w) = \frac{e^{-r_dT}e^{[iw+\varpi+1][\log(S_0)+(r_d-r_f-\sigma^2/2)T]-\frac{1}{2}\sigma^2T[w-(\varpi+1)i]^2}}{(\varpi)^2+\varpi-w^2+i(2\varpi+1)w}$$

in the normal model, and

$$\rho_{s_T}(w) = \frac{e^{-r_dT}\psi_{s_T}\left(w-(\varpi+1)i,T;\varphi\right)}{(\varpi)^2+\varpi-w^2+i(2\varpi+1)w}, \quad \text{with} \quad \psi_{s_T}\left(w-(\varpi+1)i,T;\varphi\right) \text{ as defined in eqn. 4.4.8.}$$

in the NIG model.

The corresponding R codes are provided in Appendix C.2 and Appendix C.3.

5 Data Analysis

5.1 Data Description and Summary

Secondary data consisting of closing prices of four spot exchange rates were obtained from the CBK and the Bank of England. These rates consist of four currency pairs: EUR/KES, EUR/GBP, USD/JPY, and USD/KES for the period 02/01/2008 - 31/12/2018. Data cleaning was carried out, thus, outliers were excluded from the study.

Daily log-returns are computed using these rates. By letting R_t denote the log-returns (first differences of daily log-prices of the exchange rates), then

$$R_t = \log\left(\frac{S_t}{S_{t-\Delta t}}\right)$$

where, Δt is the time scale which can range from minutes to months. Here, we set Δt=1 day.

Log-returns are used as they eliminate the unit root behaviour intrinsic to S_t and thus achieving stationarity. Also, due to the additivity property of log-returns, multi-period returns are calculated as the sum of one-period returns. Logarithmic changes also eliminate the problem of asymmetry when the choice of the domestic currency is changed [1, 16].

The basic summary statistics of daily logarithmic changes of exchange rates were computed and the results presented in Table 5.1.1. A plot of the spot FX prices is presented in Figure 5.1.1.

The results in Table 5.1.1 indicate that FX returns are, clearly, not Gaussian. If their distribution was normal, the skewness and excess kurtosis would both be zero. This seems not to hold, however, as the daily skewness and excess kurtosis significantly differ from zero, indicating heavier tails than that of a normal distribution. Standard normality tests (Shapiro-Wilk) - with H_0: FX log-returns follow a Gaussian distribution - further confirm this, at the 5% level.

Table 5.1.1. Basic summary statistics of daily FX log-returns

Basic Statistics	EUR/KES	EUR/GBP	USD/JPY	USD/KES
From	02/01/2008	02/01/2008	02/01/2008	02/01/2008
To	31/12/2018	31/12/2018	31/12/2018	31/12/2018
n	2770	2779	2779	2770
Minimum	-0.040941	-0.027182	-0.046042	-0.050000
Maximum	0.050891	0.062180	0.036994	0.044466
Lower Quartile	-0.003681	-0.003192	-0.003430	-0.000748
Upper Quartile	0.003581	0.003146	0.003640	0.001065
Mean	0.000082	0.000067	0.000001	0.000171
Median	0.000066	0.000000	0.000000	0.000094
Variance	0.000055	0.000033	0.000045	0.000020
Std. Deviation	0.007396	0.005759	0.006743	0.004435
Skewness	0.237936	0.612518	-0.212117	0.030582
Excess Kurtosis	4.600547	6.645884	4.504226	26.825796
Shapiro-Wilk statistic	0.9437	0.96021	0.95432	0.69173
J-B Statistic	2474.7966	5298.9134	2375.6892	83190.4328

Figure 5.1.1. Spot exchange rates

5.2 Fitting the Models to Exchange Rate Data

5.2.1 Parameter Estimation

The model specified in Section 3.6 was fitted to the daily log-return exchange rate data summarised in Table 5.1.1.

Having made a simplified assumption that the returns were i.i.d., parameter estimates for the NIG distribution were obtained through maximum likelihood via the *ghyp R* package. Similarly, for comparison purposes, the parameters of the statistical process of the exchange rates under the normality assumption (GK83) were also estimated. The resulting NIG parameter estimates are presented in Table 5.2.1 and the normal parameter estimates for $\hat{\mu}$ and $\hat{\sigma}$ are simply the corresponding mean and std. deviation values in Table 5.1.1.

Table 5.2.1. NIG parameter estimates

	$\bar{a}$	μ	σ	γ
EUR/KES	0.6052146000	0.0000203363	0.0073806810	0.00006364924
USD/KES	0.0376666700	0.0001029319	0.0048867840	0.00006877645
USD/JPY	0.8214528732	0.0001825712	0.0067033686	-0.00018227990
EUR/GBP	1.2735595099	-0.0003979956	0.0056773692	0.00046545770

5.2.2 Model Comparison

The goodness of fit of the models was compared via visual plots and statistical measures (the log likelihood function (logL) and the AIC).
The AIC is defined by

$$AIC = 2l - 2\ ln\ L(\hat{\Theta})$$

where, l is the no. of estimated parameters and $L(\hat{\Theta})$ denotes the maximum value of the model's $L(\Theta)$.
The visual plots considered were Q-Q plots and plots comparing the empirical kernel distribution and the corresponding normal and NIG densities. These plots are presented in Figure 5.2.1 and Figure 5.2.2, respectively. From these plots, the deviation from normality is clear. Additionally, the NIG model appears to describe the exchange rates returns adequately.

Further, the results in Table 5.2.2 clearly illustrate the superior performance of the NIG model over the normal model.

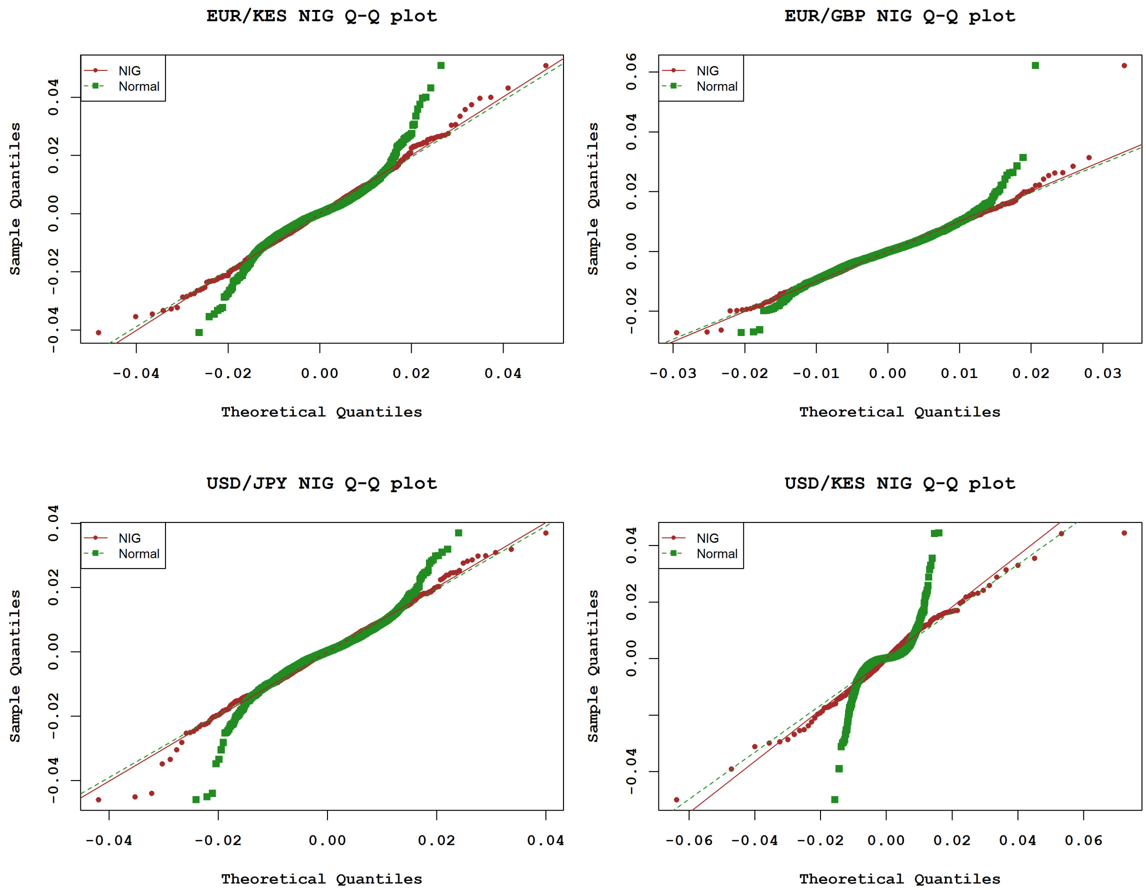

Figure 5.2.1. Q-Q plots

Table 5.2.2. Log Likelihood and AIC Values

	logL		AIC	
	Normal	NIG	Normal	NIG
EUR/KES	9662.093	9910.723	-19320.19	-19813.45
USD/KES	11078.51	12667.870	-22153.03	-25327.74
USD/JPY	9950.255	10138.890	-19896.51	-20269.78
EUR/GBP	10388.39	10526.520	-20772.78	-21045.03

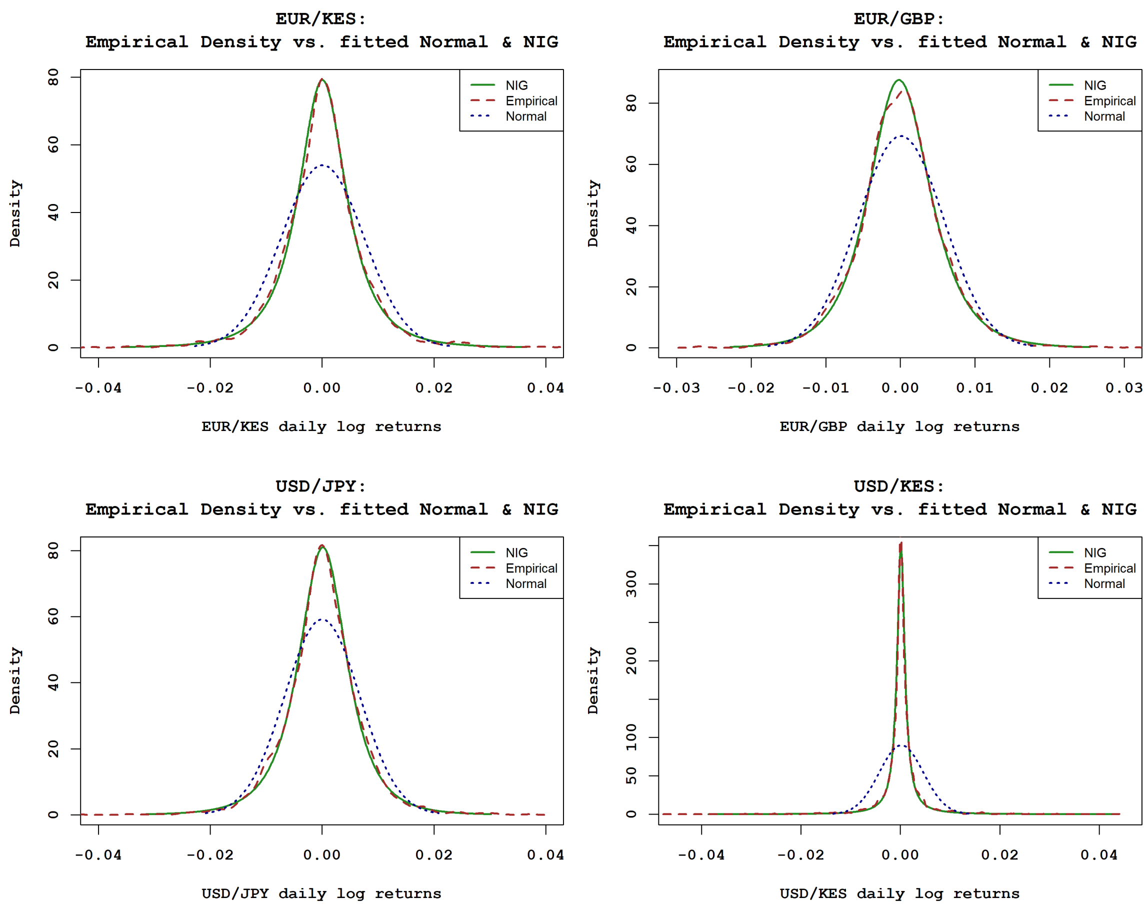

Figure 5.2.2. Empirical kernel density vs. normal, NIG densities

5.3 Option Pricing

5.3.1 Currency Option Pricing

Due to the unavailability of currency options market data (options trading is yet to be introduced in Kenya), we considered a simulation of options written on the EUR/KES exchange rate. We, therefore, set our model parameters as: $S = 100$, $r_d =$ 0.05 p.a., $r_f =$ 0.02 p.a., $K = 70, 71, 72, \ldots, 130$ and the time to expiration as 30 days, 90 days & 254 days.

For the NIG model, using the parameter estimates in Table 5.2.1 and solving equation 4.3.11 (see Appendix C.2), the risk-neutral Esscher parameter φ^* was found to be -2.020077.

The dampening coefficient, ϖ, for the normal model was obtained by solving equation 4.4.6. Similarly, ϖ_{NIG} values were obtained by solving equation 4.4.9 (see Appendix C.2). The results for the different maturities are presented in Table 5.3.1.

Table 5.3.1. Dampening coefficient values for normal and NIG models

	Time to expiration (days)		
	30	90	254
ϖ_{Normal}	54.2244	17.99507	6.298982
ϖ_{NIG}	21.97653	11.82649	4.539671

Finally, option prices were computed via equations 4.3.4, 4.4.3 and 4.4.10 for both the normal and the NIG model. In the FFT model, we set η=0.25 and n=4096. An extract of the prices, for $T - t = 30$ days, is presented in Table 5.3.2.

For the class of GK83 models (see, Table 5.3.2), the similarity in resulting prices is clear. The same is also evident for the class of NIG models. We may, therefore, focus on a representative from each class. Thus, subsequent analysis considers the original GK83 (closed-form formula) and the NIG-FFT model.

The pricing performance of the NIG model relative to the GK83 model was compared through price difference plots (the GK83 prices minus the NIG prices), see, Figure 5.3.1.

It was earlier noted (see, Proposition 4.3.1) that the Esscher transform of the NIG variable was still NIG, hence, leptokurtic (i.e. can capture jumps/large price movements). Therefore, when pricing under this model, the probability of an option OTM finishing ITM is high, in which case, the NIG model option price will be greater than the GK83 price. This also holds for an ITM option. GK83 prices, however, are relatively higher for ATM options than the NIG model prices. Deep ITM and deep OTM option prices are generally independent of the pricing model. This is because, in these cases, the integrals in equation 4.3.4 are close to zero or one. Similar results are reported in Eberlein (2001).

Another approach to testing the pricing performance of our models is to compare the model-generated prices and market prices directly. However, due to the unavailability of currency options market data, this was not possible. We, therefore, utilized stock option data to calibrate our models.

5.3.2 Model Performance: Observed Market Prices versus Model Prices

To further determine the performance of our pricing model (4.3.4) and due to the unavailability of currency option market data, we employed the readily available stock option data having made valid assumptions and adjustments. By setting $r_f = 0$ in equation 4.3.4, we obtain the relevant pricing relations for stock options. The use of these data is justified as both stock and exchange rate returns possess certain common properties (stylized

Table 5.3.2. EUR/KES option prices: S=100, r_d=0.05 p.a., r_f=0.02 p.a., T-t=30 days

	K	NIG-D.I.	NIG-FFT	GK83	FFT-GK83	CM-GK83
1	70	30.0017716	30.0017714	30.0017716	30.0017718	30.0017716
2	71	29.0018306	29.0018305	29.0018306	29.0018307	29.0018307
3	72	28.0018897	28.0018895	28.0018897	28.0018897	28.0018897
4	73	27.0019487	27.0019486	27.0019487	27.0019487	27.0019487
5	74	26.0020078	26.0020076	26.0020078	26.0020078	26.0020078
⋮	⋮	⋮	⋮	⋮	⋮	⋮
26	95	5.1975947	5.1975946	5.1958861	5.1958861	5.1958861
27	96	4.3270717	4.3270716	4.3285078	4.3285078	4.3285078
28	97	3.5185276	3.5185275	3.5240228	3.5240228	3.5240228
29	98	2.7873296	2.7873295	2.7969417	2.7969417	2.7969417
30	99	2.1463102	2.1463101	2.1590557	2.1590557	2.1590557
31	100	1.6035409	1.6035409	1.6175919	1.6175919	1.6175919
32	101	1.1608895	1.1608895	1.1740970	1.1740970	1.1740970
33	102	0.8138057	0.8138056	0.8243263	0.8243263	0.8243263
34	103	0.5523607	0.5523607	0.5591280	0.5591280	0.5591280
35	104	0.3631548	0.3631548	0.3660312	0.3660312	0.3660312
36	105	0.2314909	0.2314909	0.2311053	0.2311053	0.2311052
⋮	⋮	⋮	⋮	⋮	⋮	⋮
57	126	0.0000003	0.0000003	0.0000000	0.0000000	0.0000012
58	127	0.0000001	0.0000001	0.0000000	0.0000000	0.0000011
59	128	0.0000001	0.0000001	0.0000000	0.0000000	0.0000009
60	129	0.0000000	0.0000000	0.0000000	0.0000000	0.0000007
61	130	0.0000000	0.0000000	0.0000000	0.0000000	0.0000005

Notes:

1). NIG-D.I. refers to prices obtained via equation (4.3.12).

2). CM-GK83 refers to prices obtained via equation (4.4.7).

facts), see, Cont (2001).
In this respect, secondary data were obtained as follows: data for the respective historical stock prices (Google) were obtained from Yahoo Finance and the stock option data (Google) were obtained from SPX Options section on MarketWatch.

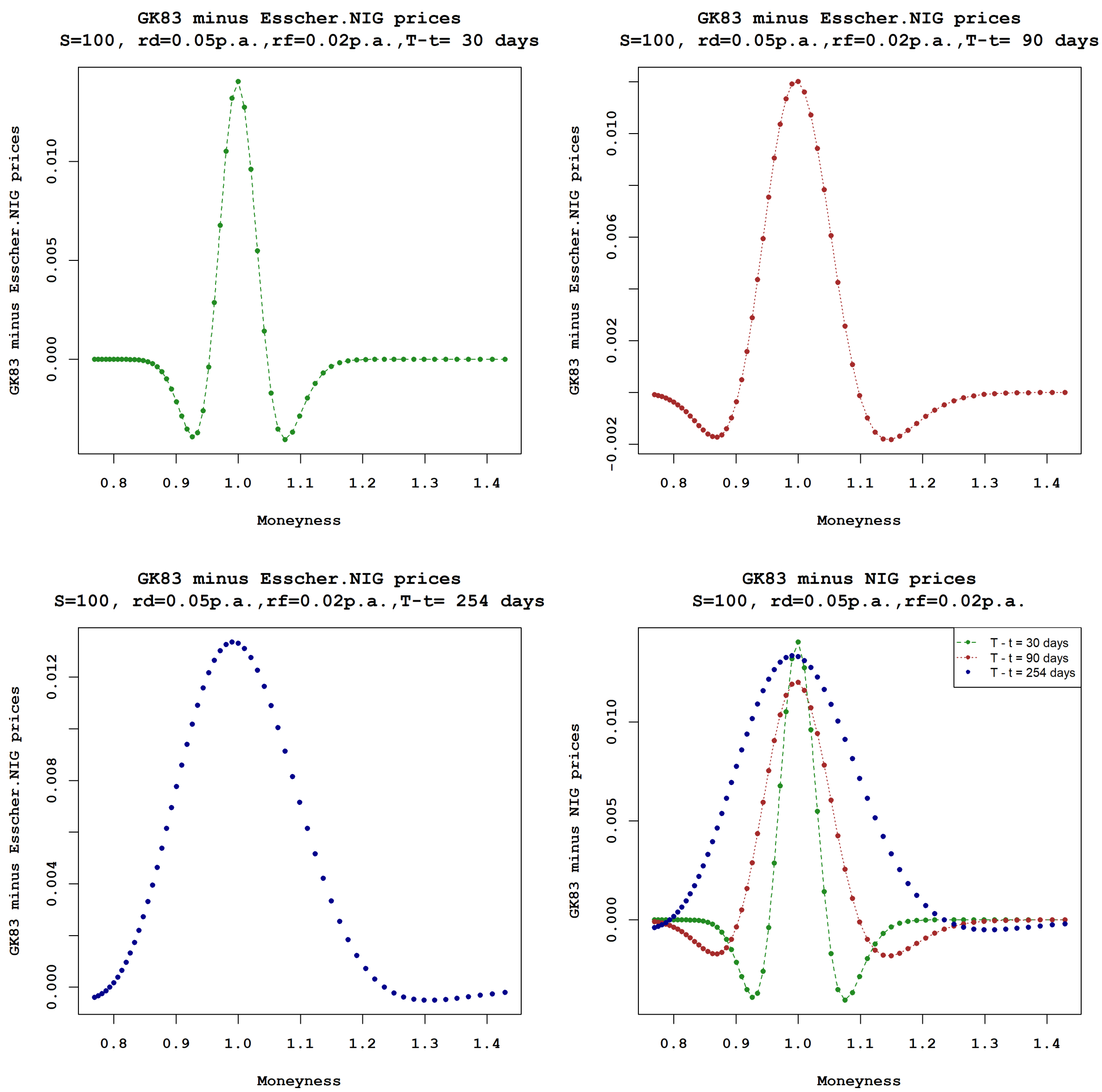

Figure 5.3.1. EUR/KES option, GK83 minus NIG prices: S=100, r_d=0.05,r_f=0.02 p.a.

Data cleaning was carried out; outliers and option pairs with zero bid and ask prices were excluded from the study.

For the historical stock data, daily adjusted closing prices of Google Inc. as from 9th April 2014 to 9th April 2019 were used to determine daily log returns. The basic summary statistics of these returns are compiled in Table 5.3.3.

The option data obtained from MarketWatch were selected based on maturities of 16 days (short-dated), 72 days (medium-dated) and 163 days (long-dated), yielding 118, 119 and 78, as the number of observations, respectively.

Table 5.3.3. Basic summary statistics of daily log returns of Google stock

Basic Statistics	
From	09/04/2014
To	09/04/2019
n	1258
Minimum	-0.054645
Maximum	0.148872
Lower Quartile	-0.006163
Upper Quartile	0.008226
Mean	0.000603
Median	0.000489
Variance	0.000214
Standard deviation	0.014641
Skewness	0.732633
Excess Kurtosis	10.408932
Shapiro-Wilk statistic	0.92524
J-B Statistic	5815.2506

Table 5.3.4. Parameter estimates (Google Stock) under $\mathbb{P}$

	Distributions	
	Normal	NIG
$\bar{a}$	-	0.6663438626
μ	0.0006025148	0.0012371478
σ	0.0146413627	0.0144051773
γ	-	-0.0006340821
LogL	3529.147	3648.727
AIC	-7054.295	-7289.455

However, the results in Figure 5.3.1 clearly show the independence of deep ITM and deep OTM option prices on a particular model. Thus, to conduct our analysis efficiently, we focused on options near the money as opposed to the entire set. Consequently, the number of observations of this new data set becomes 75, 65 & 55, respectively.

It was further assumed that the "true" market prices of calls and puts were obtained by averaging their respective bid and ask prices.

Black-Scholes call prices were calculated using equation 4.3.7, having set $r_f = 0$. The model parameters were obtained as follows: the current stock price (as at 9th April, 2019) was \$1,197; r_d was assumed to be 5% p.a.; $\hat{\sigma}$ was obtained from the historical stock data (as the estimated std. deviation value reported in Table 5.3.3); the strike prices, K, and the time to maturity were obtained from the market option data.

For the NIG model, using the parameter estimates in Table 5.3.4 and solving equation 4.3.11 (see Appendix C.2), the risk-neutral Esscher parameter φ^* was found to be -3.37251.

Using formulae 4.4.6 and 4.4.9, the dampening coefficient (ϖ) values for the different expiration dates for both models were obtained and are presented in Table 5.3.5.

Table 5.3.5. Dampening coefficient values for normal and NIG models

	Time to expiration (days)		
	16	72	163
ϖ_{Normal}	6.036947	4.140444	1.760404
ϖ_{NIG}	9.70277	2.485894	1.024037

Option prices were computed via equations 4.3.4, 4.4.3 and 4.4.10 for both the normal and the NIG model. In the FFT model, we set η=0.25 and n=4096. An extract of the prices, for $T-t = 16$ days, is presented in Table 5.3.6.

As before, and from Table 5.3.6, models in the same class (i.e. with the same distribution for the underlying asset price process) provide similar prices. We may, therefore, focus on a representative from each class. Thus, subsequent analysis focuses on the BS-73 model and the NIG-FFT model.

For each of the three data sets with time to maturity 16, 72 and 163 days, a plot of the market prices and the model prices as functions of strike price was made and is presented in Figure 5.3.2. The main results can be summarised as follows. First, from Figure 5.3.2, both models overprice at-the-money call options at all maturities. Second, the two models overprice short-dated options near the money but the overpricing is slightly less for the NIG model. Third, for both medium- and long-dated options, the two models underprice ITM options and overprice OTM options. Additionally, there is a time-to-maturity bias in both models, in that short-dated options are overpriced relative to medium- and long-dated options. Similar findings are reported in Daal & Madan (2005).

Table 5.3.6. Model Comparison: Google option prices with S=1197, r_d=0.05 p.a., T-t=16 days.

	NIG-D.I.	NIG-FFT	B-S	FFT-B.S	CM-B.S
1	99.4986579	99.4987126	99.2713203	99.2713203	99.2713242
2	94.9120388	94.9120928	94.6941888	94.6941888	94.6941888
3	90.3835929	90.3836463	90.1801400	90.1801400	90.1801400
4	85.9193900	85.9194426	85.7354932	85.7354932	85.7354932
5	81.5257657	81.5258176	81.3667195	81.3667195	81.3667195
⋮	⋮	⋮	⋮	⋮	⋮
71	0.3834521	0.3834529	0.3870598	0.3870598	0.3870598
72	0.2761503	0.2761509	0.2709584	0.2709584	0.2709584
73	0.1979559	0.1979564	0.1877130	0.1877130	0.1877130
74	0.1413122	0.1413125	0.1287086	0.1287086	0.1287086
75	0.1005004	0.1005007	0.0873571	0.0873571	0.0873571

Notes:

1). NIG-D.I. refers to prices obtained via equation (4.3.12) with r_f=0.

2). B-S refers to prices obtained via the Black-Scholes model.

Further, the performance of the NIG and the BS-73 model based on observed market data (Google options) was measured using the following indicators:

$$RMSE(\$) = \sqrt{\sum_{k=1}^{n} \frac{(C_k^{market} - C_k^{model})^2}{n}}$$
$$ARPE(\%) = \frac{1}{n}\sum_{k=1}^{n} \frac{|C_k^{market} - C_k^{model}|}{C_k^{market}} \times 100$$
$$APE(\%) = \frac{1}{n}\sum_{k=1}^{n} \frac{|C_k^{market} - C_k^{model}|}{\overline{C^{market}}} \times 100$$

where, n denotes the total number of options and $\overline{C^{market}}$ is the mean option price.

The pricing errors were computed and are compiled in Table 5.3.7. The results presented therein indicate the superior performance of the NIG-FFT model over the BS-73 under all the categories. Thus, the NIG-FFT model appears to reduce mispricing.

Following Daal & Madan (2005), we further subset the option data into moneyness categories. We defined a call option as ITM if $S/K > 1.01$, ATM if $S/K \in [0.99, 1.01]$ and OTM if $S/K < 0.99$. Figure 5.3.3 shows the prices for short- and medium-dated options. Pricing error indicators for these options are given in Table 5.3.8 and Table 5.3.9.

Table 5.3.7. Pricing errors for Google calls: S=$1197, r_d=0.05 p.a.

	$T-t$	$\overline{C^{market}}$	RMSE($)	ARPE(%)	APE(%)
	16 days	26.5410			
Black-Scholes			6.0144	187.4655	18.7347
Esscher-NIG Model			5.6269	173.8068	17.5664
	72 days	112.9492			
Black-Scholes			6.7759	25.6065	5.2244
Esscher-NIG Model			6.3667	23.7937	4.9336
	163 days	140.9718			
Black-Scholes			8.9608	14.9168	5.2767
Esscher-NIG Model			8.5290	14.0564	5.0514

Notes: All figures correct to 4 d.p.

Table 5.3.8. Pricing errors for Google calls: S=$1197, r_d=0.05 p.a., $T-t$ = 16 days

		Black-Scholes	Esscher-NIG Model
RMSE($)	ITM	3.0262	2.8841
	ATM	8.8056	8.2477
	OTM	6.3926	5.9411
ARPE(%)	ITM	4.1635	3.9604
	ATM	48.6126	45.5219
	OTM	370.9966	343.6643
APE(%)	ITM	1.63822	1.5671
	ATM	46.4597	43.5185
	OTM	165.1034	153.2620

Notes:

1) All figures correct to 4 d.p.

2) $\overline{C^{market}}$=18.9278, 146.4945, 3.2063 for ATM, ITM and OTM, respectively.

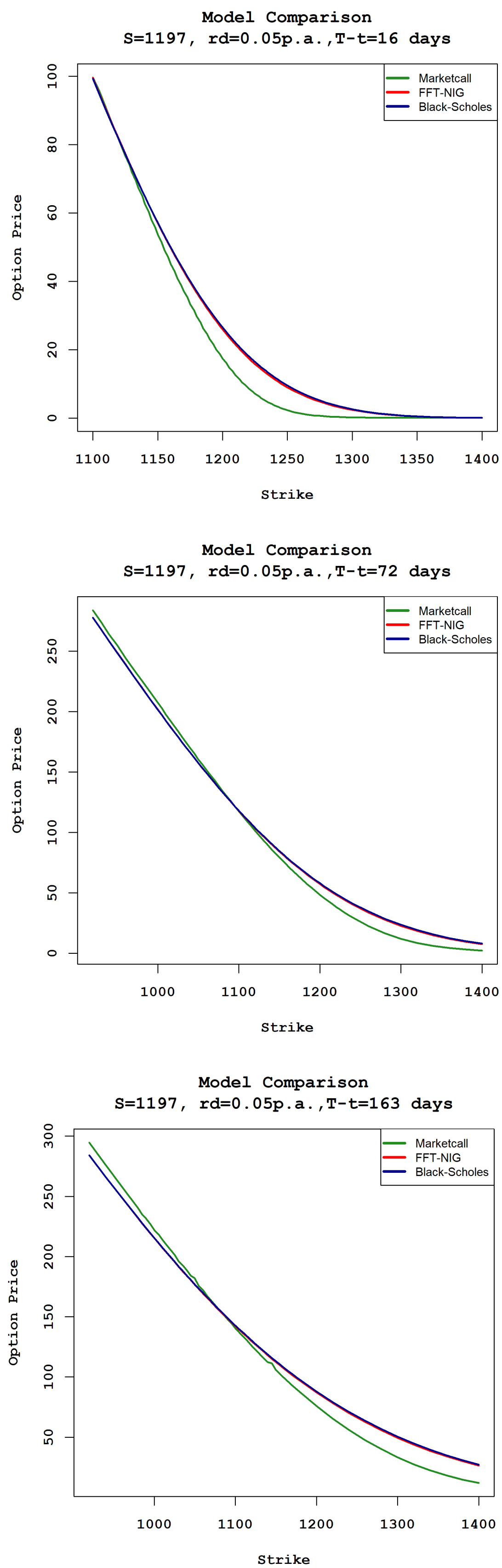

Figure 5.3.2. Model Comparison: S=1197, r_d=0.05 p.a.

Table 5.3.9. Pricing errors for Google calls: S**=\$1197,** r_d**=0.05 p.a.,** $T - t$ **= 72 days**

		Black-Scholes	Esscher-NIG Model
RMSE(\$)	ITM	4.6487	4.5139
	ATM	9.3344	8.6648
	OTM	6.8200	6.3526
ARPE(%)	ITM	2.5793	2.4478
	ATM	18.8319	17.4812
	OTM	145.6978	132.1024
APE(%)	ITM	1.4714	1.4329
	ATM	18.7037	17.3605
	OTM	51.7266	47.9962

Notes:

1) All figures correct to 4 d.p.

2) $\overline{C^{market}}$ = 49.8500, 290.7519, 9.3514 for ATM, ITM and OTM, respectively.

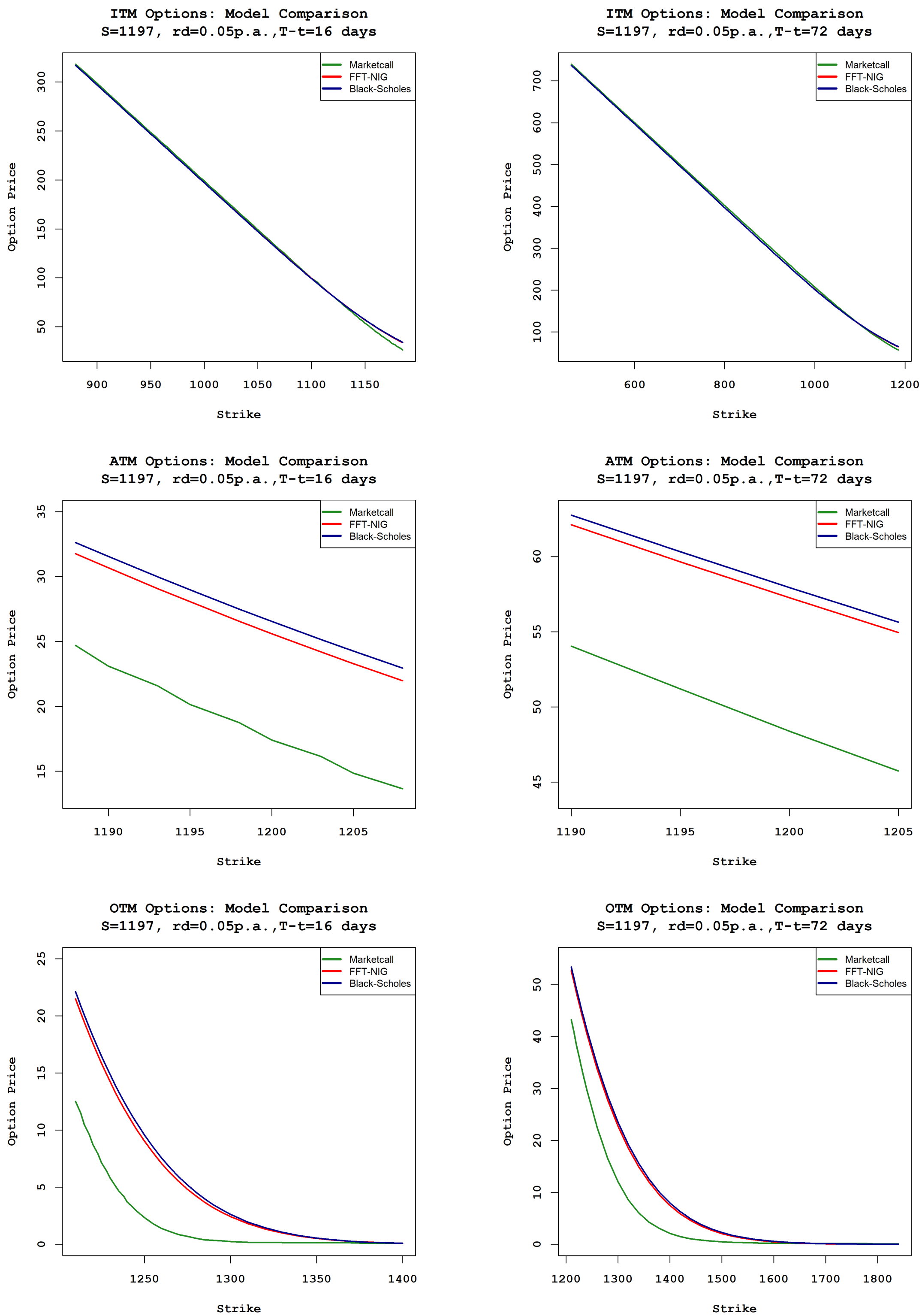

Figure 5.3.3. Model Comparison: S=1197, r_d=0.05 p.a., $T-t$ = 16 days and 72 days

6 Conclusions and Recommendations

6.1 Conclusions

We developed a model (the Esscher-NIG-FFT model) to price currency options and examined whether it improved the pricing performance when compared to the GK83. The model was constructed by first, fitting the NIG distribution to exchange rate data. Consequently, the Esscher measure was constructed and option prices were obtained under this measure, using both direct integration techniques and the Carr-Madan FFT, which allows the pricing of options for a whole range of strikes in one step.

The main results can be summarised as follows. First, the NIG distribution appears to describe the exchange rate behaviour better than the Gaussian distribution. Second, both option pricing models overprice ATM call options at all maturities. Third, the two models overprice short-dated options near the money but the overpricing is slightly less for the NIG model. Fourth, for both medium- and long-dated options, the two models underprice ITM options and overprice OTM options. Additionally, there is a time-to-maturity bias in both models, in that short-dated options are overpriced relative to medium- and long-dated options, implying that mispricing decreases with time to maturity. Similar findings are reported in Daal & Madan (2005).

The results presented herein indicate the superior pricing performance of the NIG model over the BS-73. Thus, the NIG-FFT model appears to reduce mispricing. It can therefore be concluded that failing to account for stylized facts can lead to mispricing of currency options.

As a limitation of our study, stock option data were used, due to the unavailability of currency option market data, to calibrate the models.

6.2 Future Research

This thesis focused solely on European-type options. Future studies may look at other option types such as American options, Asian options, or barrier options. Also, only one pricing mechanism, the FFT, was considered, further studies may make use of other mechanisms such as the lattice framework due to its efficiency especially for path dependent options.
The models outlined in this thesis do not account for conditional hetereskedasticity, thus, extensions to models that allow for time-varying volatility is recommended.

Bibliography

[1] Aduda, J. A. & Weke, P. G. O. (2011). A comparison of the classical Black-Scholes model and the GARCH option pricing model for currency options. *ICASTOR Journal of Mathematical Sciences*, 5(2): pp. 267-284.

[2] Akgiray, V. & Booth, G. G. (1988). Mixed diffusion-jump process modeling of exchange rate movements. *The Review of Economics and Statistics*, Vol. 70, No. 4, pp. 631-637.

[3] Bank for International Settlements (2016). *BIS Triennial Central Bank Survey 2016*. Available at: www.bis.org/publ/rpfx16.htm [Accessed March 2019]

[4] Barndorff-Nielsen, O. E. (1977). Exponentially decreasing distributions for the logarithm of particle size. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 353, No. 1674, pp. 401-419.

[5] Barndorff-Nielsen, O. E. & Halgreen, C. (1977). Infinite divisibility of the hyperbolic and generalized inverse Gaussian distributions. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 38, pp. 309-311.

[6] Barndorff-Nielsen, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics*, Vol. 24, No. 1, pp. 1-13.

[7] Barndorff-Nielsen, O. E. (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics*, 2, pp. 41–68.

[8] Barndorff-Nielsen, O. E. & Prause, K. (2001). Apparent scaling. *Finance and Stochastics*, 5, pp. 103–113.

[9] Biger, N. & Hull, J. (1983). The valuation of currency options. *Financial Management*, Vol. 12, No. 1, pp. 24-28.

[10] Björk, T. (2009). *Arbitrage Theory in Continuous Time*. Third Edition. Oxford University Press.

[11] Black, F. (1975). Fact and fantasy in the use of options. *Financial Analysts Journal*, Vol. 31, No. 4, pp. 36-41+61-72.

[12] Black, F. & Scholes, M. (1973). The pricing of options and corporate liabilities. *The Journal of Political Economy*, Vol. 81, No. 3, pp. 637-654.

[13] Blæsild, P. (1981). The two-dimensional hyperbolic distribution and related distributions, with an application to Johannsen's bean data. *Biometrika*, 68, pp. 251-263.

[14] Bollen, N. P. B., Gray, S. F. & Whaley, R. E. (2000). Regime switching in foreign exchange rates: Evidence from currency option prices. *Journal of Econometrics*, 94, pp. 239-276.

[15] Bollen, N. P. B. & Rasiel, E. (2003). The performance of alternative valuation models in the OTC currency options market. *Journal of International Money and Finance*, 22, pp. 33-64.

[16] Boothe, P. & Glassman, D. (1987). The statistical distribution of exchange rates: empirical evidence and economic implications. *Journal of International Economics*, 22, pp. 297-319.

[17] Breiman, L. (1992). *Probability*. Society for Industrial and Applied Mathematics (SIAM).

[18] Breymann, W. & Lüthi, D. (2013). ghyp: A package on generalized hyperbolic distributions. *Manual for R Package ghyp*.

[19] Calderon-Rossell, J. R. & Ben-Horim, M. (1982). The behavior of foreign exchange rates. *Journal of International Business Studies*, Vol. 13, No. 2, pp. 99-111.

[20] Carr, P. & Madan, D. B. (1999). Option valuation using the fast Fourier transform. *Journal of Computation Finance*, Vol. 2, No. 4, pp. 61-73.

[21] Cheung, A. W.-K. , Su, J.-J. and Choo, A. K. (2012). Are exchange rates serially correlated? New evidence from the Euro FX markets. *Review of Financial Economics*, 21, pp. 14-20.

[22] Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, Vol. 1, pp. 223-236.

[23] Cont, R. & Tankov, P. (2004). *Financial Modelling with Jump Processes*. First Edition. Chapman & Hall/CRC.

[24] Corlu, C. G. & Corlu, A. (2014). Modelling exchange rate returns: which flexible distribution to use? *Quantitative Finance*, Vol. 15, No. 11, pp. 1851-1864, doi: 10.1080/14697688.2014.942231.

[25] Daal, E. A. & Madan, D. B. (2005). An empirical examination of the variance-gamma model for foreign currency options. *The Journal of Business*, Vol. 78, No. 6, pp. 2121-2152.

[26] Eberlein, E. (2001). Application of Generalized Hyperbolic Lévy Motions to Finance. In: Barndorff-Nielsen O. E., Resnick S. I., Mikosch T. (eds). *Lévy Processes*, pp. 319-336. Birkhäuser, Boston, MA. https://doi.org/10.1007/978-1-4612-0197-7_14.

[27] Fama, E. F. & Roll, R. (1968). Some properties of symmetric stable distributions. *Journal of the American Statistical Association*, Vol. 63, No. 323, pp. 817-836.

[28] Friedman, D. & Vandersteel, S. (1982). Short-run fluctuations in foreign exchange rates: Evidence from the data 1973-79. *Journal of International Economics*, 13, pp. 171-186.

[29] Garman, M. B. & Kohlhagen, S. W. (1983). Foreign currency option values. *Journal of International Money and Finance*, 2, pp. 231-237.

[30] Gerber, H. U. & Shiu, E. S. W. (1994a). Option pricing by Esscher transforms. *Transactions of the Society of Actuaries*, XLVI, pp. 99-140; Discussion pp. 141-191.

[31] Gerber, H. U. & Shiu, E. S. W. (1994b). Martingale approach to pricing perpetual American options. *ASTIN Bulletin*, Vol. 24, No. 2, pp. 195-220.

[32] Gerber, H. U. & Shiu, E. S. W. (1996). Actuarial bridges to dynamic hedging and option pricing. *Insurance: Mathematics and Economics*, 18, pp. 183-218.

[33] Grabbe, J. O. (1983). The pricing of call and put options on foreign exchange. *Journal of International Money and Finance*, 2, pp. 239-253.

[34] Hirsa, A. & Neftci, S. N. (2014). *An Introduction to the Mathematics of Financial Derivatives*. Third Edition. Academic Press/Elsevier.

[35] Hsieh, D. A. (1988). The statistical properties of daily foreign exchange rates: 1974 - 1983. *Journal of International Economics*, 24, pp. 129-145.

[36] Hubalek, F. & Sgarra, C. (2006). Esscher transforms and the minimal entropy martingale measure for exponential Lévy models. *Quantitative Finance*, 6(2), pp. 125-145. https://doi.org/10.1080/14697680600573099

[37] Hull, J. C. (2002). *Options, Futures, and Other Derivatives*. Fifth Edition. Prentice Hall.

[38] Humala, A. & Rodriguez, G. (2013). Some stylized facts of return in the foreign exchange and stock markets in Peru. *Studies in Economics and Finance*, Vol. 30, Issue: 2, pp. 139-158. https://doi.org/10.1108/10867371311325444

[39] Jørgensen, B. (1982). *Statistical Properties of the Generalized Inverse Gaussian Distribution*. Springer-Verlag.

[40] Jorion, P. (1988). On jump processes in the foreign exchange and stock markets. *The Review of Financial Studies*, Vol. 1, No. 4, pp. 427-445.

[41] Karlis, D. (2002). An EM type algorithm for maximum likelihood estimation of the normal–inverse Gaussian distribution. *Statistics & Probability Letters*, 57, pp. 43-52.

[42] Lord, R. & Kahl, C. (2007). Optimal Fourier inversion in semi-analytical option pricing. *Tinbergen Institute Discussion Paper No. 2006-066/2*. http://dx.doi.org/10.2139/ssrn.921336

[43] McNeil, A. J., Frey, R. & Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press.

[44] Merton, R. C. (1973). Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, Vol. 4, No. 1, pp. 141-183.

[45] Miyahara, Y. & Moriwaki, N. (2009). Option pricing based on geometric stable processes and minimal entropy martingale measures. In Kijima, M. et al. (Eds), *Recent Advances in Financial Engineering: Proceedings of the 2008 Daiwa International Workshop on Financial Engineering*, pp. 119-133. World Scientific, Singapore.

[46] Nadarajah, S., Afuecheta, E. & Chan, S. (2015). A note on "Modelling exchange rate returns: which flexible distribution to use?", *Quantitative Finance*, doi: 10.1080/14697688.2015.1032997.

[47] Øksendal, B. (2000). *Stochastic Differential Equations: An Introduction with Applications*. Fifth Edition. Springer-Verlag.

[48] Prause, K. (1999). The Generalized Hyperbolic Model: Estimation, Financial Derivatives, and Risk Measures. *PhD Thesis*, University of Freiburg.

[49] Rogalski, R. J. & Vinso, J. D. (1978). Empirical properties of foreign exchange rates. *Journal of International Business Studies*, Vol. 9, No. 2, pp. 69-79.

[50] Schoutens, W. (2003). *Lévy Processes in Finance: Pricing Financial Derivatives*. First Edition. John Wiley & Sons, Ltd.

[51] Shreve, S. E. (2004). *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer.

[52] The National Treasury of Kenya (2018). *Annual Public Debt Management Report*. Available at:
http://www.treasury.go.ke/economy/category/157-annual-debt-management.html
[Accessed March 2019]

[53] Tsay, R. S. (2013). *An Introduction to Analysis of Financial Data with R*. First Edition. John Wiley & Sons, Inc.

[54] Tucker, A. L. & Pond, L. (1988). The probability distribution of foreign exchange price changes: tests of candidate processes. *The Review of Economics and Statistics*, Vol. 70, No. 4, pp. 638-647.

A Option Pricing in the Black-Scholes World

A.1 Notations

$$
\begin{aligned}
S &= \text{Spot exchange rate (value of one unit of foreign currency in domestic currency)}\\
T-t &= \text{Time to expiration}\\
r_d &= \text{risk-free rate in the domestic country}\\
r_f &= \text{risk-free rate in the foreign country}\\
S^* &= \text{Spot stock price}\\
K &= \text{Strike price}\\
\mu &= \text{drift of the spot currency price}\\
\sigma &= \text{volatility of the spot price}\\
\alpha &= \text{expected rate of return on a security}\\
\delta &= \text{standard deviation of the security}\\
\Phi(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du
\end{aligned}
$$

A.2 Black-Scholes Model

A.2.1 Model Construction

Define $C^*(t,S_t^*)$ and $P^*(t,S_t^*)$ as European call and put options on a non-dividend paying stock, respectively. At maturity, T, the value of the contracts are:

$$C^*(T,S_T^*) = \max(S_T^* - K, 0) \qquad \text{and} \qquad P^*(T,S_T^*) = \max(K - S_T^*, 0)$$

where, as usual K is the strike price and S_T^* is the (spot) price of the underlying asset at maturity. Assuming the Black-Scholes market set up where:

- The underlying asset price process follows a Geometric Brownian Motion
- The volatility of the underlying asset and the risk-free rate are known and constant
- No arbitrage possibilities and no transaction costs in trading the stock or the option
- The stock does not pay dividends
- Short selling is allowed, security trading is continuous, and assets are perfectly divisible.

Under these assumptions, and considering a market with two assets: a risk-free asset, $B(t)$, with price process, $dB(t) = rB(t)dt$, and a stock with price process, $dS^*(t) = \alpha S^*(t)dt + \sigma S^*(t)dW(t)$, where, $W(t)$ is a standard Wiener process; and a derivative, where $C^*(t, S_t^*)$ is the value of the derivative; then a portfolio consisting of the underlying stock and the derivative can be constructed, say, $V(t) = C^*(t, S_t^*) + \Delta S^*(t)$, where Δ denotes the number of units of the underlying asset.
Using Itô's lemma, the dynamics of V can be derived whose risk can be eliminated by a delta hedge. Since the portfolio is essentially risk-free, its return should be equal to the return on a risk-free asset, that is, $dV(t) = rV(t)dt$. Therefore, equating the dynamics of the hedged (risk-free) portfolio with those of the risk-free asset, a PDE is obtained,

$$C_t^* + rS^*C_s^* + \frac{1}{2}\sigma^2(S^*)^2C_{ss}^* - rC^* = 0 \tag{A.2.1}$$

and the $\mathbb{Q}$-dynamics of S^* are given by $dS^*(t) = rS^*(t)dt + \sigma S^*(t)dW(t)$.
The PDE given by equation (A.2.1) is the Black-Scholes PDE. The solution of the PDE, given the boundary condition $C^*(T, S_t^*) = \max(S^* - K, 0)$, gives the Black-Scholes pricing formula for a European call option on a non-dividend paying stock.

A.2.2 Black-Scholes Call Option Pricing Formula

The Black-Scholes option pricing formula for a European call on a non-dividend paying stock is given by:

$$C(t, S_t^*) = S_t^*\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{A.2.2}$$

where:

$$d_1 = \frac{\log(S_t^*/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}; \quad d_2 = \frac{\log(S_t^*/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

A.2.3 Black-Scholes Put Option Pricing Formula

The pricing formula for a European put option can be derived using equation A.2.2 and the put-call parity relation which is given by:

$$C(t, S_t^*) + Ke^{-r(T-t)} = P(t, S_t^*) + S_t^* \tag{A.2.3}$$

Making $P(t, S_t^*)$ the subject of the formula, we have

$$P(t, S_t^*) = C(t, S_t^*) + Ke^{-r(T-t)} - S_t^* \tag{A.2.4}$$

But $C(t,S_t^*)$ is as defined in equation A.2.2.
Therefore, making the substitution:

$$\begin{aligned}P(t,S_t^*) =& S_t^*\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) + Ke^{-r(T-t)} - S_t^* \\ =& S_t^*[\Phi(d_1) - 1] - Ke^{-r(T-t)}[\Phi(d_2) - 1] \\ =& - S_t^*[1 - \Phi(d_1)] + Ke^{-r(T-t)}[1 - \Phi(d_2)]\end{aligned}$$

But, $1 - \Phi(x) = \Phi(-x)$

$$\Rightarrow P(t,S_t^*) = -S_t^*\Phi(-d_1) + Ke^{-r(T-t)}\Phi(-d_2) \tag{A.2.5}$$

where, d_1 and d_2 are defined as before.

A.3 Garman-Kohlhagen Currency Pricing Formula

A.3.1 Model Construction

The Black-Scholes formula cannot be applied directly to price currency options as these options involve multiple interest rates. Using this inadequacy as the basis, Garman & Kohlhagen (1983) extended the Black-Scholes formula to accommodate the multiple interest rates and derived valuation formulae for European call and put FX options having made the following assumptions:

- The currency spot price process follows a Geometric Brownian Motion; i.e., the dynamics of S, under objective probability measure $\mathbb{P}$, are given by $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$, where W is the standard Weiner process.
- Markets are frictionless.
- No arbitrage possibilities.
- Volatility and interest rates (in both domestic and foreign markets) are constant.

Consider an option with domestic price $C(t,S_t)$ on the exchange rate S. Using the Black-Scholes hedge, a portfolio consisting of the option and Δ units of the foreign currency can be constructed for a domestic investor. The value of this portfolio (denominated in the domestic currency) is $V(t) = C(t,S_t) + \Delta S(t)$ and, by linearity, the change in V is given by $dV = dC + \Delta dS$; where, by Itô's lemma, $dC = C_t dt + C_s dS + \frac{1}{2}C_{ss}(dS)^2$ and $dS = \mu S dt + \sigma S dW$.
Assuming further that all holdings of the foreign currency are invested in a foreign risk-free asset,

$$\begin{aligned}dV =& dC + \Delta dS \\ =& C_t dt + C_s dS + \frac{1}{2}\sigma^2 S^2 C_{ss} dt + \Delta dS + r_f \Delta S dt\end{aligned}$$

where the last term on the RHS is the interest earned on the foreign currency.
The risk in this portfolio can be completely hedged away by letting $\Delta = -C_s$. Thus, the return on this portfolio should be the same as that of the domestic risk-free asset of the form $dV = r_d V dt$.
Therefore, letting $\Delta = -C_s$ and setting

$$r_d V dt = C_t dt + C_s dS + \frac{1}{2}\sigma^2 S^2 C_{ss} dt + \Delta dS + r_f \Delta S dt$$

We have

$$C_t + (r_d - r_f) S C_s + \frac{1}{2}\sigma^2 S^2 C_{ss} - r_d C = 0 \tag{A.3.1}$$

And the $\mathbb{Q}$-dynamics of S are given by:

$$dS(t) = (r_d - r_f) S(t) dt + \sigma S(t) dW(t) \tag{A.3.2}$$

The solution of the PDE, equation A.3.1, given the boundary condition, $C(T, S_t) = max(S - K, 0)$, gives the Garman-Kohlhagen currency pricing formula for a European call option.

A.3.2 Garman-Kohlhagen Currency Pricing Formula: European Call

The Garman-Kohlhagen currency call pricing formula is

$$C(t, S_t) = e^{-r_f(T-t)} S_t \Phi(d_1) - K e^{-r_d(T-t)} \Phi(d_2) \tag{A.3.3}$$

where:

$$d_1 = \frac{\log(S_t/K) + (r_d - r_f + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}; \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

A.3.3 Garman-Kohlhagen Currency Pricing Formula: European Put

Since valuing FX options is analogous to pricing a dividend-paying stock option (with the dividend-yield equal to the foreign risk-free interest rate), the corresponding pricing formula for a European put option can be derived using equation A.3.3 and the put-call parity relation for a dividend-paying stock given by equation A.3.4.

$$C(t, S_t) + K e^{-r(T-t)} = P(t, S_t) + S_t e^{-q(T-t)} \tag{A.3.4}$$

Substituting $C(t, S_t)$ in equation A.3.4 with the RHS of equation A.3.3 and the interest rates, we have:

$$\begin{aligned} P(t, S_t) =& e^{-r_f(T-t)} S_t \Phi(d_1) - K e^{-r_d(T-t)} \Phi(d_2) + K e^{-r_d(T-t)} - S_t e^{-r_f(T-t)} \\ =& - S_t e^{-r_f(T-t)} [1 - \Phi(d_1)] + K e^{-r_d(T-t)} [1 - \Phi(d_2)] \\ =& - S_t e^{-r_f(T-t)} \Phi(-d_1) + K e^{-r_d(T-t)} \Phi(-d_2) \end{aligned}$$

where, d_1 and d_2 are defined as before.

B Appendix for Chapter 4

B.1 Notations, Definitions and Modelling Assumptions

Notations

$S =$ Spot exchange rate (value of one unit of foreign currency in domestic currency)

$T =$ Maturity date

$r_d =$ risk-free rate in the domestic country

$r_f =$ risk-free rate in the foreign country

$K =$ Strike price

Definitions

Definition B.1.1. *If Ω is a given non-empty set, then a σ-algebra $\mathfrak{F}$ on Ω is a family of subsets of Ω with the following properties:*

(i) $\varnothing \in \mathfrak{F}$

(ii) $\Omega \in \mathfrak{F}$

(iii) if $A \in \mathfrak{F}$, then $A^{\complement} \in \mathfrak{F}$, where $A^{\complement} = \Omega \backslash A$ is the complement of A in Ω.
From (ii), $\Rightarrow \Omega^{\complement} = \varnothing \in \mathfrak{F}$

(iv) $A_1, A_2, \ldots, \in \mathfrak{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathfrak{F}$

The pair $(\Omega, \mathfrak{F})$ is called a measurable space.

Definition B.1.2. *Given a measurable space $(\Omega, \mathfrak{F})$, a probability measure $\mathbb{P}$ on $(\Omega, \mathfrak{F})$ is a function that, to every set $A \in \mathfrak{F}$, assigns a number in [0,1], called the probability of A, precisely, $\mathbb{P} : \mathfrak{F} \to [0,1]$ such that*

(i) $\mathbb{P}(\varnothing) = 0, \quad \mathbb{P}(\Omega) = 1 \quad$ *and* $\quad 0 \leq \mathbb{P}(A) \leq 1$

(ii) (countable additivity) for any sequence $A_1, A_2, \ldots, \in \mathfrak{F}$ of pairwise disjoint sets (that is, $A_i \cap A_j = \varnothing$ if $i \neq j$)

$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

The triplet $(\Omega, \mathfrak{F}, \mathbb{P})$ is called a probability space.

Definition B.1.3. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. If a set $A \in \mathfrak{F}$ satisfies $\mathbb{P}(A) = 1$, we say that the event A occurs almost surely and write $\mathbb{P} - a.s.$*

Definition B.1.4. *A stochastic process, $\{X_t\}$ for $t \in [0, \infty)$, is a family of random variables defined on a complete probability space, $(\Omega, \mathfrak{F}, \mathbb{P})$ and assuming values in $\mathbb{R}$.*

Definition B.1.5. *A process $\{X_t\}_{t \geq 0}$ has independent increments if for any t and $u > 0$, the distribution of $[X(t+u) - X(t)]$ is independent of the distribution of $[X(s), s \leq t]$.*

Definition B.1.6. *A process $\{X_t\}_{t \geq 0}$ has stationary increments if the distribution of $X(t+u) - X(t)$, $u \geq 0$, does not depend on t.*

Lemma B.1.1. *Let $\{X_t\}_{t \geq 0}$ be a process with stationary, independent increments; then $X(t)$ has an infinitely divisible distribution for every $t \geq 0$. Thus, for every n, the law of $X(t)$ can be expressed as the sum of n i.i.d. random variables with $X_{t/n}$ as a common law [50].*

Lemma B.1.2. *Let $\{X_t\}$ be a process with stationary, independent increments such that the characteristic function of $X(t)$, $\psi_t(u)$, is continuous at $t = 0$ for every u; then $X(t)$ is continuous in probability and $\psi_t(u) = [\psi_1(u)]^t$.*

Proof. See Breiman (1992, Proposition 14.18) □

Definition B.1.7. *A stochastic process W is a Wiener process if:*

(i) $W(0) = 0$

(ii) The process W has independent increments, i.e. if $r < s \leq t < u$, then $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables.

(iii) For $s < t$, the stochastic variable $W(t) - W(s) \sim N\big(0, t-s\big)$

(iv) For $t > 0$, $W(t) \sim N(0, t)$

(v) W has continuous trajectories.

Definition B.1.8. *Let Ω be a non-empty set. Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a σ-algebra $\mathfrak{F}_t$. Assume further that if $s \leq t$, then every set in $\mathfrak{F}_s$ is also in $\mathfrak{F}_t$.*
A filtration is a non-decreasing family $\mathbb{F} = \{\mathfrak{F}_t : 0 \leq t \leq T\}$ of sub-σ-algebras of $\mathfrak{F}$:

$$\mathfrak{F}_s \subseteq \mathfrak{F}_t \subseteq \mathfrak{F}_T \subseteq \cdots \subseteq \mathfrak{F} \qquad \text{for} \quad 0 \leq s \leq t \leq T$$

where $\mathfrak{F}_t$ represents the information available at time t, and the filtration $\mathbb{F} = \{\mathfrak{F}_t : 0 \leq t \leq T\}$ represents the information flow evolving with time.
If filtration is added to the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, we have a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{F}, \mathbb{P})$.

Definition B.1.9. *Let X be a random variable defined on a non-empty sample space Ω. Let $\mathfrak{G}$ be a σ-algebra of subsets of Ω. If every set in $\sigma(X)$ is also in $\mathfrak{G}$, we say that X is $\mathfrak{G}$-measurable. That is, a random variable X is $\mathfrak{G}$-measurable iff the information in $\mathfrak{G}$ is sufficient to determine the value of X.*

Definition B.1.10. *Let Ω be a non-empty sample space equipped with filtration $\mathbb{F}$. A stochastic process $\{X_t\}_{t\in[0,T]}$ is adapted to the filtration $\mathbb{F}$ if for each t, X_t is $\mathfrak{F}_t$-measurable.*

Definition B.1.11. *A stochastic process $X = (X_t : t \geq 0)$ is a martingale relative to $(\mathbb{F}, \mathbb{P})$ if*

(i) X *is* $\mathbb{F}$*-adapted,*

(ii) $E(|X_t|) < \infty \qquad \forall t \geq 0,$

(iii) $E(X_t|\mathfrak{F}_s) = X_s, \quad \forall s,t \quad \text{with} \quad s \leq t \quad (\mathbb{P}-a.s.)$

Theorem B.1.1. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and let Y be an almost surely positive random variable with $E^{\mathbb{P}}(Y) = 1$. Define*

$$\mathbb{Q}(A) = \int_A Y(\omega) d\mathbb{P}(\omega) \qquad \forall A \in \mathfrak{F}. \tag{B.1.1}$$

Then $\mathbb{Q}$ is a probability measure. Further, if X is a positive random variable, then

$$E^{\mathbb{Q}}[X] = \int_\Omega X(\omega) d\mathbb{Q}(\omega) = E^{\mathbb{P}}[XY]$$

If Y is almost surely strictly positive, we also have

$$E^{\mathbb{P}}[Z] = E^{\mathbb{Q}}\left[\frac{Z}{Y}\right]$$

for every positive random variable Z.

Proof. See Shreve (2004, Theorem 1.6.1) □

Definition B.1.12 (Absolute continuity)**.** *Given two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $\mathfrak{F}$, $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ on $\mathfrak{F}$ if $\forall\, A \in \mathfrak{F}$,*

$$\mathbb{P}(A) = 0 \qquad \Rightarrow \qquad \mathbb{Q}(A) = 0$$

This is written as $\mathbb{Q} \ll \mathbb{P}$.
If $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$, then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent and we write $\mathbb{Q} \sim \mathbb{P}$.

Definition B.1.13 (Equivalent measures). *Given two probability measures $\mathbb{P}$ and $\mathbb{Q}$ defined on the same Ω with $A \in \mathfrak{F}$, $\mathbb{Q}$ is equivalent to $\mathbb{P}$ iff*

$$\mathbb{P}(A) = 0 \qquad \Longleftrightarrow \qquad \mathbb{Q}(A) = 0$$

or

$$\mathbb{P}(A) = 1 \qquad \Longleftrightarrow \qquad \mathbb{Q}(A) = 1$$

That is, $\mathbb{P}$ and $\mathbb{Q}$ have the same null sets (events which cannot occur under $\mathbb{P}$ cannot also occur under $\mathbb{Q}$ and vice versa).

Definition B.1.14 (Equivalent Martingale Measure). *A probability measure $\mathbb{Q}$ is an equivalent martingale measure and hence a risk-neutral probability measure if*

(i) $\mathbb{Q} \sim \mathbb{P}$

(ii) the discounted underlying asset price process is a martingale under $\mathbb{Q}$

Definition B.1.15 (Radon-Nikodým Derivative). *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, let $\mathbb{Q}$ be another probability measure on $(\Omega, \mathfrak{F})$ that is equivalent to $\mathbb{P}$, and let Y be an almost surely positive random variable that relates $\mathbb{P}$ and $\mathbb{Q}$ via equation B.1.1. Then Y is called the Radon-Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, i.e.,*

$$Y = \frac{d\mathbb{Q}}{d\mathbb{P}}.$$

Theorem B.1.2 (Radon-Nikodým Theorem). *Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures defined on $(\Omega, \mathfrak{F})$. Then there exists an almost surely positive random variable Y such that $E^{\mathbb{P}}(Y) = 1$ and*

$$\mathbb{Q}(A) = \int_A Y(\omega) d\mathbb{P}(\omega) \qquad \forall A \in \mathfrak{F}.$$

Definition B.1.16. *An arbitrage is defined to be a trading strategy which has no initial cost, zero risk of a loss in the future and a non-zero probability of a future profit.*

Theorem B.1.3 (First fundamental theorem of asset pricing). *The market is arbitrage free iff there exists an equivalent martingale measure.*

Proof. See Shreve (2004, Theorem 5.4.7) □

Definition B.1.17. *A market model is complete if every derivative security can be hedged.*

Theorem B.1.4 (Second fundamental theorem of asset pricing). *The market is complete iff the martingale measure $\mathbb{Q}$ is unique.*

Proof. See Shreve (2004, Theorem 5.4.9 and Lemma C.1) □

Definition B.1.18 (Hubalek & Sgarra (2006)). *Given a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, a random variable X and a parameter φ, the Esscher transform $\mathbb{P}^{\varphi}$, sometimes also called the Esscher measure, is defined by*

$$d\mathbb{P}^{\varphi} = \frac{e^{\varphi X}}{E[e^{\varphi X}]} d\mathbb{P} = \frac{e^{\varphi X}}{M_X(\varphi)} d\mathbb{P}$$

provided $M_X(\varphi)$ exists.

This change of measure is also referred to as *exponential tilting*.

Modelling Assumptions

We make the following assumptions:

- Markets are frictionless.
- No arbitrage possibilities.
- r_d and r_f are constant.
- Trading is continuous.
- No taxes, transaction costs and no restriction on borrowing.
- All securities are perfectly divisible.

B.2 Proofs

Proof to Proposition 4.2.1.

$$M_Y(\mathfrak{z};\varphi) = \int_{-\infty}^{\infty} e^{\mathfrak{z}y} f(y;\varphi)dy$$

$$\Rightarrow \qquad 1 = \int_{-\infty}^{\infty} \frac{e^{\mathfrak{z}y} f(y;\varphi)}{M_Y(\mathfrak{z};\varphi)} dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{\mathfrak{z}y} f(y;\varphi)}{M_Y(\mathfrak{z}+\varphi)} M_Y(\varphi) dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{\mathfrak{z}y} M_Y(\varphi)}{M_Y(\mathfrak{z}+\varphi)} \frac{e^{\varphi y} f(y)}{M_Y(\varphi)} dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{(\mathfrak{z}+\varphi)y} f(y)}{M_Y(\mathfrak{z}+\varphi)} dy$$

Thus,

$$f(y;\mathfrak{z}+\varphi) = \frac{e^{(\mathfrak{z}+\varphi)y} f(y)}{M_Y(\mathfrak{z}+\varphi)}$$

is a PDF which is an Esscher transform (parameter $\mathfrak{z}+\varphi$). □

Proof to Equation 4.4.6.
Applying Itô's lemma, the solution to the stochastic differential equation (A.3.2) is

$$S_T = S_0 e^{\left(r_d - r_f - \frac{\sigma^2}{2}\right)T + \sigma W_T}$$

where, W_T is a standard Wiener process with, $W_T \sim \Phi(w_T; 0, T)$.
Clearly, Y_T is a Wiener process with drift, i.e., $Y_T = \left(r_d - r_f - \sigma^2/2\right)T + \sigma W_T$ and

$$Y_T \sim \Phi\left[y_T; (r_d - r_f - \sigma^2/2)T, \sigma^2 T\right]$$

From equation 4.4.5, it follows that

$$\log(S_T) = \log(S_0) + (r_d - r_f - \sigma^2/2)T + \sigma W_T$$

Thus, letting $s_T = \log(S_T)$, we have

$$s_T \sim \Phi\left[s_T; \log(S_0) + (r_d - r_f - \sigma^2/2)T, \sigma^2 T\right]$$

The characteristic function of a normal random variable, say, Y is given by

$$\psi_Y(u) = e^{iu\mu - \frac{1}{2}\sigma^2 u^2}$$

Thus, for Y_T

$$\psi_{Y_T}(u) = e^{iu[(r_d - r_f - \sigma^2/2)T] - \frac{1}{2}\sigma^2 T u^2}$$

For s_T,

$$\psi_{s_T}(u) = e^{iu[\log(S_0) + (r_d - r_f - \sigma^2/2)T] - \frac{1}{2}\sigma^2 T u^2}$$

$$\begin{aligned}\psi_{s_T}\left(w - (\varpi + 1)i\right) =& e^{i[w-(\varpi+1)i][\log(S_0)+(r_d-r_f-\sigma^2/2)T] - \frac{1}{2}\sigma^2 T[w-(\varpi+1)i]^2} \\ =& e^{[iw+\varpi+1][\log(S_0)+(r_d-r_f-\sigma^2/2)T] - \frac{1}{2}\sigma^2 T[w-(\varpi+1)i]^2}\end{aligned}$$

And,

$$\psi_{s_T}\left(-(\varpi+1)i\right) = e^{[\varpi+1][\log(S_0)+(r_d-r_f-\sigma^2/2)T] - \frac{1}{2}\sigma^2 T[\varpi+1]^2}$$

Thus, for the choice of optimal ϖ, we solve

$$\begin{aligned}\frac{\partial \Psi(\varpi, k)}{\partial \varpi} =& \frac{\partial}{\partial \varpi}\left[-\varpi k + \frac{1}{2}\log\left(\psi_{s_T}\left(-((\varpi+1)i)^2\right)\right]\right. \\ =& -k + \frac{\partial}{\partial \varpi}\left[\log\left(\psi_{s_T}\left(-((\varpi+1)i)\right)\right]\right. \\ =& -k + \log(S_0) + (r_d - r_f - \sigma^2/2)T - \frac{1}{2}\sigma^2 T[2\varpi + 2] \\ =& -k + \log(S_0) + (r_d - r_f - \sigma^2/2)T - \sigma^2 T[\varpi + 1] \\ =& 0\end{aligned}$$

Thus,

$$\begin{aligned}\varpi + 1 =& \frac{\log(S_0) + (r_d - r_f - \sigma^2/2)T - k}{\sigma^2 T} \\ =& \frac{\log(S_0) + (r_d - r_f - \sigma^2/2)T - \log(K)}{\sigma^2 T} \\ =& \frac{\log\left(\frac{S_0}{K}\right) + (r_d - r_f - \sigma^2/2)T}{\sigma^2 T} \\ =& \frac{d_1}{\sigma\sqrt{T}}\end{aligned}$$

And,

$$\varpi = \frac{d_1}{\sigma\sqrt{T}} - 1$$

□

C Sample R Codes

C.1 Fitting the NIG Model to Exchange Rate Data

```
rm(list=ls())
setwd("C:/Users/Simon/Desktop/FX_data/KES")
library(car)
library(fBasics)
### EUR
EUR<- read.csv("KESeuro.csv", header=T,sep=",")
#View(EUR)
dim(EUR)
str(EUR)
head(EUR)
tail(EUR)
names(EUR)
summary(EUR)
#png("eurkesspot.png",width = 6,height=6,units ='in', res=300)
plot(ts(EUR$Mean),type="l",lty=1,col="forestgreen",family="mono",
     main="EUR/KES Exchange Rate\nfrom 2nd. Jan 2008 to 31st Dec. 2018",
     xlab="Days [2nd. Jan 2008 - 31st Dec. 2018]",
     ylab="KES-Euro Exchange Rate")
#dev.off()
EURkesrtn<-diff(log(EUR$Mean)) # Compute changes
basicStats(EURkesrtn)
mean(EURkesrtn)
hisvol<-sqrt(sum((EURkesrtn-mean(EURkesrtn))^2)*1/(length(EURkesrtn)-1))
Box.test(EURkesrtn,type="Ljung-Box")
### Fitting GHDS
library(ghyp)
##Normal
eurkesnorm<-fit.gaussuv(data=EURkesrtn)
summary(eurkesnorm)
##NIG
eurkesnig<-fit.NIGuv(data=EURkesrtn,silent=TRUE)
summary(eurkesnig)
par(mfrow=c(1,1))
#########
hist(eurkesnig,gaussian=T,ghyp.col="forestgreen",legend.cex=0.7,lty=0,
    main="EUR/KES: Empirical Density vs. fitted Normal & NIG",ghyp.lty="
    solid",ghyp.lwd=2,col="blue4",xlab="EUR/KES daily log returns")
lines(density(EURkesrtn),lwd=2,type="l",col="brown",lty=2,pch=0)
legend("topright", c("NIG","Empirical","Normal"),
       cex=0.8,lty=c(1,2,3),lwd=c(2,2,2),col=c("forestgreen","brown","
          blue4"))
#########
#png("eurkesnigqq.png",width = 6,height=6,units ='in', res=300)
qqghyp(eurkesnig,gaussian=T,plot.legend=T,legend.cex=0.7,ghyp.col="brown
    ",ghyp.pch = 20,gauss.pch=15,gauss.col = "forestgreen",
       main="EUR/KES NIG Q-Q plot",font=2,font.lab=2,family="mono")
#dev.off()
```

```
####################
mean(eurkesnig)
logLik(eurkesnig)
logLik(eurkesnorm)
AIC(eurkesnorm)
AIC(eurkesnig)
coef(eurkesnig)
### Parameters
eurkeslambda<-coef(eurkesnig)$lambda;eurkeslambda
eurkesalpha.bar<-coef(eurkesnig)$alpha.bar;eurkesalpha.bar
eurkesmu<-coef(eurkesnig)$mu;eurkesmu
eurkessigma<-coef(eurkesnig)$sigma;eurkessigma
eurkesgamma<-coef(eurkesnig)$gamma;eurkesgamma
#### Parametrization 1
eurkesbeta<-eurkesgamma/(eurkessigma^2);eurkesbeta
eurkesdelta<- eurkessigma*(sqrt(eurkesalpha.bar));eurkesdelta
eurkesalpha<-sqrt((eurkesalpha.bar/(eurkessigma^2))+(eurkesbeta^2));
   eurkesalpha
```

C.2 Esscher-NIG Prices

NIG Esscher Parameter [Equation 4.3.11]

```
##Finding the risk-neutral Esscher parameter theta
source("C:/Users/Simon/Desktop/currency_options/underlyingasset.R")
source("C:/Users/Simon/Desktop/currency_options/optiondataimport.R")
thetafn<-function(theta){((sqrt(googalpha^2-(googbeta+theta)^2)) - (sqrt
   (googalpha^2-(googbeta+theta+1)^2))) -((rd-rf-googmu)/googdelta)}
theta<-uniroot(thetafn,lower=-4,upper=4)$root;theta
##Parameters for the risk-neutral PDF
alpha<-googalpha;alpha
delta<-googdelta*Tt;delta
mu<- googmu*Tt;mu
beta<-googbeta+theta;beta
betanw<-beta+1;betanw
```

NIG dampening coefficient [Equation 4.4.9]

```
source("C:/Users/Simon/Desktop/currency_options/NIGesscherparameter.R")
smu<-log(S)+mu;smu
##Risk-neutral NIG characteristic function for log(ST)
characEsscherNIG<-function(v){
  ((exp(1i*v*smu))*(exp(delta*(sqrt(alpha^2-beta^2)))))/(exp(delta*(sqrt
     (alpha^2-(beta+(1i*v))^2))))
}
characEsscherNIG(0)
##Dampening coefficient
allalphasNIG<-rep(0,length(k))
for (j in 1:length(k)) {
  optimalalpha<-function(fualpha){-k[j]+smu+((delta*(fualpha+betanw))/(
     sqrt((alpha^2)-(betanw+fualpha)^2)))}
  allalphasNIG[j]<-uniroot(optimalalpha,lower=-100,upper=100)$root
}
falphaNIG<-0.25*(allalphasNIG[which.max(allalphasNIG)]);falphaNIG
```

Esscher-NIG D.I. Prices [Equation 4.3.12]

```
EsscherNIGprices<-function(rd,rf,S,K,Tt,alpha,beta,mu,delta,betanw,theta
    ){
source("C:/Users/Simon/Desktop/currency_options/NIGesscherparameter.R")
  w<- function(x){
  alpha*(sqrt(delta^2+(x-mu)^2))
}
#w(0.02)
nigpdf<-function(x){(((alpha*delta*(besselK(w(x),1,expon.scaled = F)))/
    pi)*(exp(beta*(x-mu)))*exp(delta*(sqrt(alpha^2-beta^2))))/(sqrt(
    delta^2+ (x-mu)^2))
}
nigpdf2<-function(x){(((alpha*delta*(besselK(w(x),1,expon.scaled = F)))/
    pi)*(exp(betanw*(x-mu)))*exp(delta*(sqrt(alpha^2-betanw^2))))/(sqrt(
    delta^2+ (x-mu)^2))
}
I1fn<-function(K){K*exp(-rd*Tt)*integrate(nigpdf,log(K/S),Inf)$value}
newI1<-sapply(K,I1fn)
I2fn<-function(K){S*exp(-rf*Tt)*integrate(nigpdf2,log(K/S),Inf)$value}
newI2<-sapply(K,I2fn)
EsscherNIGprices<-newI2-newI1
return(EsscherNIGprices)
}
EsscherNIGprices.<-EsscherNIGprices(rd,rf,S,K,Tt,alpha,beta,mu,delta,
    betanw,theta)
```

Esscher-NIG-FFT Prices [Equation 4.4.13]

```
fftNIGprices<-function(rd,r_f,S,k,Tt,falphaNIG,n,eta){
  source("C:/Users/Simon/Desktop/currency_options/characteristicNIG+
     alpha.R")
  eta<-0.25
  n<-4096
  js<-seq(1,n,length=n);tail(js)
  jminus1s<-seq(0,n-1,length=n)
  vj<-seq(0,n-1,length=n)*eta;vj  ##vjs (j-1)*eta
  dirac<-seq(0,0,length=n)
  dirac[1]<-1
  wj<-(eta/3)*(3+(-1)^js-dirac)
  gtvNIG<-function(v){
    ((exp(-(rd-r_f)*Tt))*characEsscherNIG(v-(falphaNIG+1)*1i))/((
       falphaNIG)^2+falphaNIG-(v^2)+(1i*v*((2*falphaNIG)+1)))
  }
  fftNIGprices<-sapply(k, function(k){(exp(-falphaNIG*k)/pi)*(sum(Re((
     exp(-1i*k*vj))*gtvNIG(vj)*wj)))*exp(-(r_f*Tt))})
  return(fftNIGprices)
}
##########
allNIGprices<-as.data.frame(cbind(K,EsscherNIGprices.,fftNIGprices))
View(allNIGprices)
#require(xtable)
#xtable(allNIGprices,auto=T)
```

C.3 Normal Models Prices

Normal dampening coefficient [Equation 4.4.6]

```
source("C:/Users/Simon/Desktop/currency_options/underlyingasset.R")
source("C:/Users/Simon/Desktop/currency_options/optiondataimport.R")
sigma<-hisvol
k<-log(K)
fmu<-log(S)+((rd-rf-(0.5*sigma^2))*Tt)
characnormal<-function(v){
      exp(1i*v*fmu)*exp(-(0.5*Tt*(sigma^2)*(v)^2))
    }
characnormal(0)
### Optimal alpha
d1<-((log(S/K))+((rd-rf+((sigma^2)*0.5))*(Tt)))/(sigma*sqrt(Tt))
allalphasGK83<-(d1/(sigma*sqrt(Tt)))-1;allalphasGK83
falphaGK83<-0.25*(allalphasGK83[which.max(allalphasGK83)]);falphaGK83
```

All Normal Models Prices [Equations 4.3.7, 4.4.7, 4.4.13]

```
source("C:/Users/Simon/Desktop/currency_options/characteristicnormal+
   alpha.R")

##GK83

GK83call<-function(S,Tt,K,rd,rf,sigma){
  d1<-((log(S/K))+((rd-rf+((sigma^2)*0.5))*(Tt)))/(sigma*sqrt(Tt))
  d2<-((d1)-(sigma*Tt^0.5))
  S*exp(-rf*Tt)*pnorm(d1)-K*exp(-rd*Tt)*pnorm(d2)
}
GK83prices<-GK83call(S,Tt,K,rd,rf,sigma)

## Fourier inversion

library(elliptic)
fourierNormalprices<-function(rd,rf,S,k,Tt,falphaGK83){
  gtv<-function(v){
  ((exp(-(rd-rf)*Tt))*characnormal(v-(falphaGK83+1)*1i))/((falphaGK83)
      ^2+falphaGK83-(v^2)+(1i*v*((2*falphaGK83)+1)))
}
fourierprice<-function(v){
  gtv(v)*exp((-1)*1i*v*k)*exp(-falphaGK83*k)*(1/pi)
}
xvec<-function(k){myintegrate(function(v){
  gtv(v)*exp((-1)*1i*v*k)*exp(-falphaGK83*k)*(1/pi)
},0,Inf)}
fourierNormalprices<-exp(-(rf*Tt))*Re(sapply(k,xvec))
return(fourierNormalprices)
}
fourierNormalprices<-fourierNormalprices(rd,rf,S,k,Tt,falphaGK83)
```

```
## FFT
fftNormalprices<-function(rd,rf,S,k,Tt,falphaGK83,n,eta){
  eta<-0.25
  n<-4096
  js<-seq(1,n,length=n);tail(js)
  jminus1s<-seq(0,n-1,length=n)
  vj<-seq(0,n-1,length=n)*eta;vj  ##vjs (j-1)*eta
  dirac<-seq(0,0,length=n)
  dirac[1]<-1
  wj<-(eta/3)*(3+(-1)^js-dirac)
  gtv<-function(v){
      ((exp(-(rd-rf)*Tt))*characnormal(v-(falphaGK83+1)*1i))/((
         falphaGK83)^2+falphaGK83-(v^2)+(1i*v*((2*falphaGK83)+1)))
  }
  fftNormalprices<-sapply(k, function(k){(exp(-falphaGK83*k)/pi)*(sum(Re
     ((exp(-1i*k*vj))*gtv(vj)*wj)))*exp(-(rf*Tt))})
  return(fftNormalprices)
}
fftNormalprices<-fftNormalprices(rd,rf,S,k,Tt,falphaGK83,n,eta)
###############
allnormalprices<-as.data.frame(cbind(GK83prices,fftNormalprices,
   fourierNormalprices))
View(allnormalprices)
#require(xtable)
#xtable(allnormalprices,auto=T)
```