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Sums of Exponential Random Variables

Research Report in Mathematics, Number 49 of 2019

Wilfred Makori Steki

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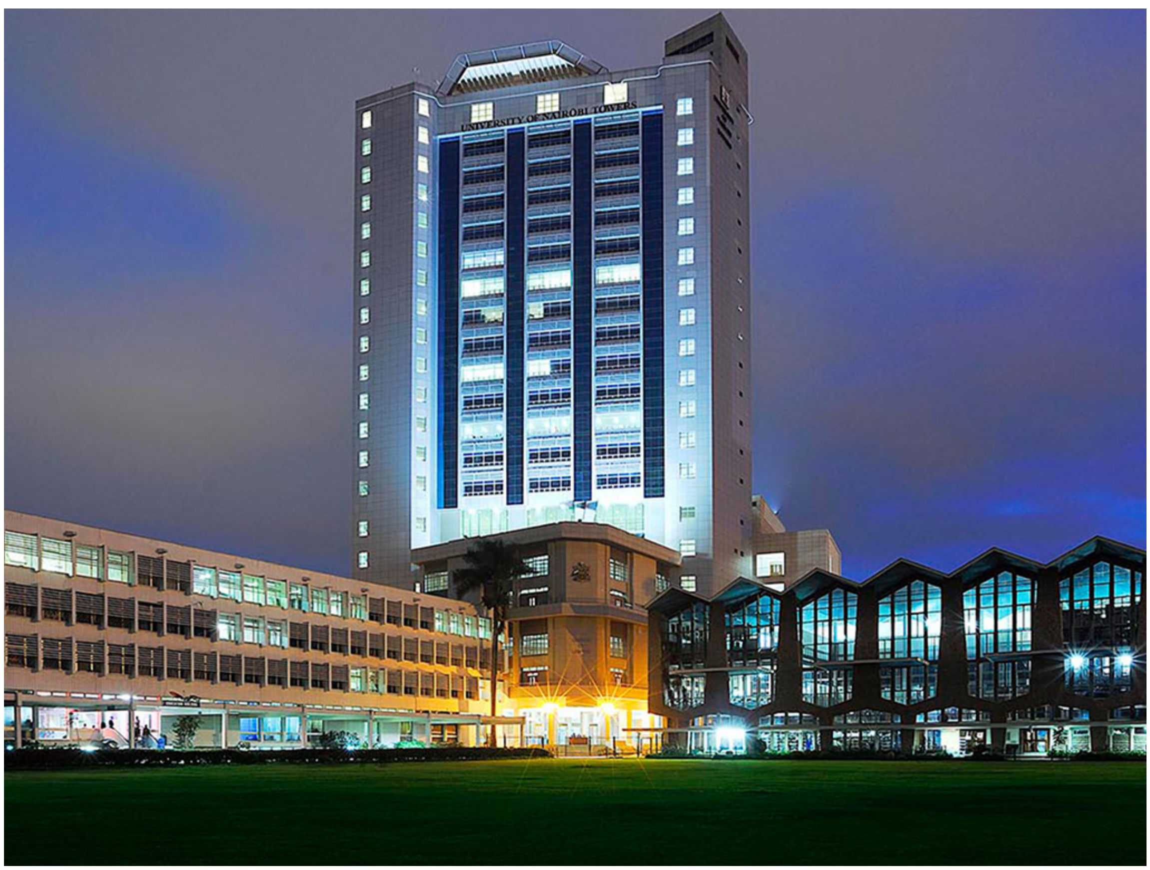

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Mathematical Statistics

Master Project in Mathematical Statistics **University of Nairobi**
September, 2019

Sums of Exponential Random Variables

Research Report in Mathematics, Number 49 of 2019

Wilfred Makori Steki

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Master Thesis

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Mathematical Statistics

Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

Exponential distribution is one of the continuous probability distributions that in most cases has been used in the analysis of Poisson processes and is the most widely used in statistical studies of reliability applications. In this project we are aiming at computing the sums of exponential random variables that have found a wide range of applications in real life mathematical modelling. In many processes involving waiting time of services, exponential distributions plays a significant role in making responsible statistical inferences for significant system output. In this study we have constructed different distributions for the sums of exponential random variables considering various cases where the parameter rates may be independent and identical or distinct.

The generalization of the sums of exponential random variables with independent and identical parameter describes the intervals until n counts occur in the Poisson process. This forms an Erlang random variable as well proved in this project as well as hypo-exponential random variable for the case of independent and distinct parameters. The results obtained indicates variation effects depending on the sample size of the distribution and nature of parameter rates on the efficiency of the estimation techniques chosen in comparing respective outputs in applications.

Estimation of properties is determined using the method of moments and maximum likelihood estimation for some cases attempted. Owing to the relationship of exponential distribution to Poisson process, a study on the compound mixed Poisson distribution have also been provided. We have also considered to derive the probability density functions for hypo-exponential distribution for the general cases where the model parameters form arithmetic and geometric sequences.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature	Date

WILFRED MAKORI STEKI
Reg No. I56/8296/2017

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature	Date

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Dedication

I dedicate this project to the Almighty God, my Creator, source and foundation of inspiration for granting me the strength, knowledge and wisdom to undertake this quest. I also dedicate this work to my dear wife Stellah Kemunto who has been my pillar of strength in life. To my Mum and Dad who first taught me the value of education and endurance. To my best friends Maria and Rima for their support of every kind throughout this programme. To my lovely sons Eugene and Linus my source of joy and happiness on earth who in one way or the other have been affected during the time I was carrying out this task. Thank you all and God bless.

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With deepest gratitude I thank the Almighty God for His gracious gift of life, good health, knowledge and strength granted to me throughout this academic journey.

I am indebted to the Director School of Mathematics, Prof. Patrick G. O. Weke of this great institution, University of Nairobi that has impacted greatly to shape my destiny for his exemplary leadership and to ensure that I completed this task on time.

My utmost heartfelt gratitude and thanks goes to my exceptional veteran project supervisor Prof. J.A.M Ottieno, an astute man I always admire for his tremendous enormous priceless assistance, scholarly advice and academic guidance throughout my academic life at the University of Nairobi. Truly, I have no words to express it enough for his work for me to get to this far. I will never forget you sir. God bless you.

I appreciate the entire academic and non-academic staff of the School of Mathematics especially for those I had the opportunity to interact with during my course. Their experienced research skills has really broadened my scope of knowledge and critical thought.

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Finally, I thank all my classmates that I was fortunate to meet and be part of their class at the right time. Your motivation, support and comfort added great impetus in my life. I will never forget you people.

God bless you all.

Wilfred Makori Steki

Nairobi, 2019.

Contents

1 INTRODUCTION

1.1 Background Information

This project embarks on the some important aspects of sums of $n \geq 2$ exponential random variables, a very important continuous distribution in Applied Mathematics and Statistics.

The sum of exponential random variables in particular have found wide range of applications in mathematical modelling in so many real life domains including insurance [Minkova, 2010], communications and computer science [Trivedi, 2002], Markov processes [Kordecki, 1997], reliability and performance evaluation [Bolch et al. 2006] and spatial diversity [Khuong and Kong, 2006].

Nadarajah (2008) presented a review of some the results on sums of random variables which provided a useful point of reference for this research. In many studies of market segmentation, these distributions helps to understand and make relevant statistical inferences in relation to human capital placement, predict the process trends and quality services provided for maximum clientèle satisfaction to determine the survival and success of service providers.

In nature, independent distributions may have identical or distinct parameter rates which determines the selection of the distribution and parameter estimation. For cases of independent and identically distributed exponential random variables, it has has been noted that the sum of $n \geq 2$ exponential random variables forms an Erlang distribution. On the other hand, the sum of $n \geq 2$ independent non-identical exponential random variables with distinct parameter rates is forms hypo-exponential distribution. There are also cases where the distribution may have both identical and distinct parameter rates for the sum of n exponential random variables.

Historical background dates back to the introduction of the gamma family by Karl Pearson (1895) and works of A. K. Erlang (1905) who was working on telephone traffic congestion. Following several studies over time, observations between two successive Poisson occurrences indicated to follow exponential distribution hence establishing the relationship of exponential distribution with Poisson Process. A number of extensions on this area have been made including by Weibull (1951) and Berrettoni (1964) on the use of exponential distributions in industrial quality control. Balakrishnan and Basu (1995) also pointed out exponential distributions as a special case of the gamma distribution. Lai et al (2006) used the distribution in the study of reliability and survival analysis.

This study is divided into seven chapters. Rest of this chapter gives the definitions, notations and terminologies used in this work. It also describes the problem statement, objectives of this current study, related work done so far and a few descriptions on the applications of the sums of exponential distributions studied in this project.

Chapter 2 deals with the construction of the probability distribution functions and cumulative distribution functions for a fixed sum of independent and identically distributed exponential random variables. The chapter concludes by computing some of the properties related to the distributions obtained.

Chapter 3 describes compound distribution in relation to exponential distribution using Laplace Transform. In this wok we also included the study of their relationship of the sums of independent and identically distributed exponential random variables compounded with some of the discrete distributions.

Chapter 4 focused on the compound mixed Poisson distribution as well as a discrete Erlang mixture resulted from the consideration of the random sum of independent and identically distributed exponential random variables. It also extends to study various compound mixed Poisson distributions and their desirable properties useful in various modelling processes.

Chapter 5 embarked on hypo-exponential random variables in cases where the random variables have distinct parameter rates with non-identical parameters. In this study we make use of convolution approach, Laplace Transform and moment generating functions techniques and also compute some of their related properties. Eventually, we also presented the probability distribution functions for the models constructed when the parameter rates forms an arithmetic sequence and geometric sequence.

Chapter 6 gives a brief description of some of the real life applications of both Erlang distribution and hypo-exponential distribution.

Chapter 7 concluded the study and giving suggested areas for future studies.

1.2 Definitions, Notations and Terminologies

Let $X_1, X_2, \cdots, X_n$ be independent exponential random variables with identical or distinct parameter rates λ_i, for $i = 1, 2, \cdots, n$. This distribution can also be expressed as $X_i \sim \text{Exp}(\lambda_i)$.

We define the sums of exponential random variables as

$$S_n = X_1 + X_2 + \cdots + X_n$$
$$= \sum_{i=1}^{n} X_i$$

The sum, S_n is termed as an Erlang distribution when all parameter rates of X_i are identical otherwise called hypo-exponential distribution.

The sum S_n may be fixed for $n \in N$ and $n \geq 2$ or random for N is also a random variable and independent of $X_i's$.

The following are some of the notations used in this study:

1. X_i - Exponentially distributed random variable
2. λ_i - Parameter rate for the exponentially distributed random variable X_i
3. S_n - Sum of exponential random variables
4. n - Number of X_i exponential random variables in the distribution
5. cdf - cumulative distribution function
6. pdf - probability distribution function
7. $f(x)$ - pdf of the random variables X
8. $F(x)$ - cdf of the random variables X
9. $L\{\cdot\}$ - Laplace transform
10. $i.i.d$ - independent and identically distributed
11. $M_X(t)$ - The moment generating function of X
12. $g(\lambda)$ - Probability density function of a mixing distribution
13. $\prod_{i=1}^{n}$ - product of all parameter rates
14. mgf - moment generating function

1.3 Problem Statement

Nadarajah (2008), presented some of the review of results on sums of random variables. Many of the research works in these area have concentrated on the discrete random variables than continuous distributions. Exponential distribution is one of the most applied continuous distribution with very wide applications in applied mathematical modelling. This called for the selection for the research and study.

It also called for the study of their related properties and estimation of some of the distributions so far obtained including the resulting effects of the mixing distribution on the compound distributions resulted from exponential distributions.

1.4 Objectives

Main Objective

The main aim of this dissertation is to present methods of finding the distribution of the sums of $n \geq 2$ independent exponentially distributed random variables with identical and with distinct parameter rates.

Specific Objectives

1. To derive the general case for the fixed sums of independent and identically distributed exponential random variables and its properties.
2. To establish the results of Compound mixed Poisson distributions.
3. To derive the general case of sums of exponential random variables with distinct parameters and its properties.
4. To present the probability distribution functions for the sums of exponential random variables with distinct parameters forming arithmetic or geometric sequences.

1.5 Methodology

The following methods were adopted for use in this project on the sums of exponential random variables:

(i) Convolution Technique

(ii) Laplace Transform

(iii) Moment generating function

(iv) Arithmetic sequence and Geometric sequence

1.6 Applications

Sum of exponential random variables provides a very wide range of applications in real life testing and reliability studies:

1.6.1 Communication and Computer Science

Sums of exponential random variables according to Hans and Lars (2007) forms necessary basic tools for Erlang intra-node computer programming. These distributions helps through the sending and receiving constructs between links and monitors to build robust applications to survive in the process.

1.6.2 Markov Processes

Continuous time Markov chains provide very useful models in predictability of performance in various systems. It also provides very flexible, powerful and efficient means to describe and analyse dynamic system properties.

1.6.3 Insurance Application

The sums of exponential random variables provides insurance risk theory knowledge necessary for the evaluation of the ruin probabilities and claims distribution . Useful computational formulas are made available for risk assessment and insurance claims modelling.

1.6.4 Reliability and Performance Evaluation

Efficiency of a system depends on the quality and kind of functions that can be performed by the system. Sums of exponential random variables explains the reliability of a system determined by independent components forming up the system.

2 LITERATURE REVIEW

2.1 Introduction

Sums of exponential random variables have been a focal point of interest by researchers owing to its wide application in real life modelling. The pioneer work was provided by A.K. Erlang (1909) a tele-traffic engineer.

Saralees Nadarajah (2008) provides a brief review of results on sums of random variables in sums of exponential random variables. In his paper he provided a review of the known results on sums of exponential, gamma, lognormal, Rayleigh and Weibull random variables. Further, he sited a few examples of applications in the wireless communications where these sums of random variables are useful. This review served us with the need to advance in this area of sums of exponential random variables.

Khaledi and Kochar (2011) also provides a review paper on the convolution of independent random variables due to its typical applications in real life areas. Most popular applications of the reviewed area include in reliability theory (Bon and Paltanea, 1999), in actuarial sciences (Kaas et al, 2001) and non-parametric goodness-of-fit (Serfling, 1980). Gamma distribution has been widely used to model total insurance claim distributions through its right skewness, non-negative and uni-modal properties (Furman, 2008). In their paper they pointed out that several works only concentrated on the convolutions of exponential random variables due to the complicated nature of gamma distributions that provides probability models for the waiting times.

Withers and Nadarajah (2011) studied compound Poisson-gamma random variable taking into account of its properties including estimationby the methods of moments and maximum likelihood estimation.

In this paper we consider three conditions that parameter rates of an exponential distribution may take to determine the sum of n random variables. These include:

(i) Where the sum of exponential random variables have independent and identical parameter rates

(ii) Where the sum of exponential random variables have independent and distinct parameter rates

(iii) Where the sum of exponential random variables have identical parameters and others have distinct parameters in the distribution.

2.2 Exponential Sum for Independent and Identical Parameter Rates

In this area the earliest recorded study was done by A.K. Erlang (1909) working on telephone traffic congestion giving rise to Erlang distributions. Since then a number of studies have been done in this area including by Molina, Lindley, Bailey, Kendall, Bhat, Conway, Neuts and Satly among many others.

Bartlett (1965) used the first and second moments of gamma random variables to evaluate an excess loss ratio premiums and came into conclusion to have agreed with the the risk-theoretic loss distribution. Bowers (1966)in his paper followed through several observations made by various authors at the time to study the distribution of the total claims in the risk-theory following the assumption made of using incomplete gamma distribution.

Jasiulewicz and Kordecki (2003) computed a formula for the case of independent and identical random variables resulting to Erlang distribution. They used Laplace transform to derive the formula.

Kundu and Gupta (2003) discussed the convenient expression of gamma distribution using the generalized exponential random variable for the shape parameter lying between 0 and 1. Through acceptance-rejection principle the method was compared with the most popular Ahrens and Dieter method and the method proposed by Best.

Akkouchi (2005) gives a simple formula for the sum of n gamma random variables to solve the complicated one given by Mathai (1982) for the case of sum of exponential random variables. Akkouchi's method involved the use of elementary computations expressing the results in multiple integrations that included generalized beta function.

Hu and Beaulieu (2005) provides a simple and an accurate closed-form for approximation of the cumulative density functions and probability density functions of the sum of independent and identically distributed Rayleigh random variables. Rayleigh and exponential random variables are regarded as special cases of gamma distribution.

Doyle (2006) computed the convolution of two exponential densities for identical random variables only. The paper also went ahead to present the properties of the distribution with the generalized distribution of the exponential random variable though did not provide the proof to the case.

Rajic (2013) estimated the parameters for two independent random variables from gamma distribution using the maximum likelihood estimation method and asymptotic distribution to construct confidence intervals. In addition they performed a simulation study to show the consistency property on the maximum likelihood estimators.

Rather and Rather (2017) in their research article demonstrated a generalized exponential distribution as a special case of the gamma random variables. This paper discussed the moment generating functions of the newly introduced distribution in relation to the one proposed by Weibull(1951). In conclusion, the paper introduced distribution as a special case of k-generalized exponential random variables.

2.3 Exponential Sum for Independent and Distinct Parameter Rates

Hypo-exponential distribution is characterized by sum of several independent exponential random variables having different parameter rates, Amari and Misira (1997) and Akkouchi (2008). Its application have found wide significance including tele-traffic engineering and queueing system. Mathai (1982) studies the sum of exponential distribution to provide with the formula.

Khalid et al(2001) modelled hypo-exponential distribution with three different parameters in his work on the study of the input-output of a multiple processor system. He also used Erlang distribution on the same to measure the performance of the system.

The convolution of exponential random variables with distinct parameter rates were given by the works of Sheldon M. Ross (2003). However, he did not consider into the properties of the distributions obtained.

Moreover, in the domain of reliability and performance evaluation of systems and software, some authors used arithmetic and geometric parameters, such as Gaudion and Ledoux (2007), Jelinski and Moranda (1972) and Moranda (1979).

Mohamed Akkouch (2008) used convolution of exponential random variables to get the general case for the distribution with different parameter rates. In his work he considered the method applied by Kordecki (2003) in deriving to the result through Laplace Transform technique. He used the generalization results provided by Sen and Balakrishnan (1999) that were also done by Jesiulewicz and Kordecki (2003)in establishing the formula without any conditions on the parameter rates.

Steifano and Stephen (2008) considered an alternative technique to obtain the probability density function of the sum of exponential random variable. In their work they considered the logarithmic relationship between beta and exponential distribution functions and the Wilks' integral representation for the products of independent beta random variables by providing a closed-form expression for the distribution of the sum of independent random variables.

Maode (2009) in his paper attempted to fit Coxian distribution to study the exact time for the service and hypo-exponential distribution to study the service time for an interrupted process in an optical burst switching network.

Samimi and Azmi (2009) presented an approximate method in evaluating the cumulative density functions for the sum of independent non-identical random variables. This method proposed is based on the convergent infinite series technique. However, in this paper involved complicated integrals to compute the probability density functions. This approximation method is used to evaluate the use of sum of independent random variables in analysing the system performance.

Sheldon Ross (2010) used mathematical induction method to obtain the general case for exponential distribution and some of its properties. In their work, they based on the convolutions of exponential random variables for non-identical parameters termed as hypo-exponential random variables.

Smaili et al (2013) used the moment generating function and the Laplace transform technique to derive the hypo-exponential distribution with n distinct parameters. They also used these two techniques to obtain properties.

Daskalakis (2013) on non identical independent random variables also investigated the sum of the distributions and their properties.

Oguntunde et al (2014) on their work obtained the distribution through convolution approach the sum of two exponentially distributed random variables.The paper also provided statistical properties on the resulting model of the two random variables. They went further to obtained the first four moments and cumulants as well as the mean, the variance, skewness and kurtosis of the distribution.

2.4 Exponential Sum for both Identical and Distinct Parameter Rates

Khuong and Kong (2006) used characteristic function approach to propose their simple way of determining the distribution for sum of n exponential random variables. In their work they considered a case where some of the parameter rates may be identical or have different parameters. There is no any suggested general formula so far for this particular case since k random variables have the same mean and $n-k$ remaining random variables have different mean.

3 A FIXED SUM OF INDEPENDENT AND IDENTICALLY DISTRIBUTED EXPONENTIAL RANDOM VARIABLES

3.1 Introduction

In this chapter we will consider the fixed sum of n independent and identically distributed exponential random variables, establish their probability density function and provide some of their related properties.

In this case let us consider the distribution $X_i's$ where $i = 1, 2, \cdots, n$ exponential random variables with equal or identical parameters that is $\lambda_i = \lambda_j$ for all $\lambda_1, \lambda_2, \cdots, \lambda_n$. Therefore, this distribution be represented as:

$$S_n = X_1 + X_2 + \cdots + X_n \tag{3.1.1}$$

3.2 Construction Using Convolution Approach

3.2.1 Sum of two Exponential Random Variables

In the section we need to consider the sums of two exponential random variables with independent and identical parameters.

Let

$$S_2 = X_1 + X_2 \tag{3.2.1}$$

Let $F_i(x_i)$ and $G(s_2)$ be the cumulative distribution function for X_i and S_2 respectively.

Cumulative Distribution Function

From the definition the cumulative distribution function is given by:

$$
\begin{aligned}
G(s_2) &= Prob[S_2 \leq s_2] \\
&= Prob\{X_1 + X_2 \leq s_2\} \\
&= Prob\{X_2 \leq s_2 - x_1\} \\
&= Prob\{0 \leq X_1 \leq \infty, 0 \leq X_2 \leq s_2 - x_1\} \\
&= Prob\{0 \leq X_1 \leq s_2, 0 \leq X_2 \leq s_2 - x_1\} \\
&= \int_0^{s_2} \int_0^{s_2 - x_1} f(x_1) f(x_2) dx_2 dx_1 \\
&= \int_0^{s_2} \lambda e^{-\lambda x_1} \left[\int_0^{s_2 - x_1} \lambda e^{-\lambda x_2} dx_2 \right] dx_1 \\
&= \int_0^{s_2} \lambda e^{-\lambda x_1} \left[-e^{-\lambda x_2} \right]_0^{s_2 - x_1} dx_1 \\
&= \int_0^{s_2} \lambda e^{-\lambda x_1} \left[1 - e^{-\lambda s_2 + \lambda x_1} \right] dx_1 \\
&= \int_0^{s_2} \left(\lambda e^{-\lambda x_1} - \lambda e^{-\lambda s_2} \right) dx_1 \\
&= \int_0^{s_2} \lambda e^{-\lambda x_1} dx_1 - \int_0^{s_2} \lambda e^{-\lambda s_2} dx_1 \\
&= \left[-e^{-\lambda x_1} \right]_0^{s_2} - \left[x_1 \lambda e^{-\lambda s_2} \right]_0^{s_2} \\
\therefore G(s_2) &= 1 - e^{-\lambda s_2} - s_2 \lambda e^{-\lambda s_2} \qquad (3.2.2)
\end{aligned}
$$

Therefore from the above equation (3.2.2) it can be concluded that

$$
\lim_{s_2 \to \infty} G(s_2) = 1
$$

and

$$
\lim_{s_2 \to 0} G(s_2) = 0 \qquad (3.2.3)
$$

Probability Distribution Function

From equation (3.2.2) therefore

$$
\begin{aligned}
g(s_2) &= \frac{d}{ds_2} G(s_2) \\
&= \frac{d}{ds_2} \left[1 - e^{-\lambda s_2} - s_2 \lambda e^{-\lambda s_2} \right] \\
&= \lambda e^{-\lambda s_2} + \lambda s_2 \lambda e^{-\lambda s_2} - \lambda e^{-\lambda s_2} \\
&= \lambda^2 s_2 e^{-\lambda s_2} \qquad (3.2.4)
\end{aligned}
$$

as required.

3.2.2 Sum of three Exponential Random Variables

In the section we need to consider the sums of three exponential random variables with independent and identical parameters.

Let

$$\begin{aligned} S_3 &= X_1 + X_2 + X_3 \\ &= S_2 + X_3 \end{aligned} \tag{3.2.5}$$

Let $F_i(x_i)$ and $G(s_3)$ be the cumulative distribution function for X_i and S_3 respectively.

Cumulative Distribution Function

From the definition the cumulative distribution function is given by:

$$\begin{aligned} G(S_3) &= Prob[S_3 \le s_3] \\ &= Prob\{S_2 + X_3 \le s_3\} \\ &= Prob\{X_3 \le s_3 - s_2\} \\ &= Prob\{0 \le s_2 \le \infty, 0 \le X_3 \le s_3 - s_2\} \\ &= Prob\{0 \le s_2 \le s_3, 0 \le X_3 \le s_3 - s_2\} \\ &= \int_0^{s_3}\int_0^{s_3-s_2} g_2(s_2) f(x_3) dx_3 ds_2 \\ &= \int_0^{s_3} \lambda^2 s_2 e^{-\lambda s_2}\left[\int_0^{s_3-s_2} \lambda e^{-\lambda x_3} dx_3\right] ds_2 \\ &= \int_0^{s_3} \lambda^2 s_2 e^{-\lambda s_2}\left[-e^{-\lambda x_3}\right]_0^{s_3-s_2} ds_2 \\ &= \int_0^{s_3} \lambda^2 s_2 e^{-\lambda s_2}\left[1 - e^{-\lambda s_3 + \lambda s_2}\right] ds_2 \\ &= \int_0^{s_3} \lambda^2 s_2 e^{-\lambda s_2} ds_2 - \int_0^{s_3} \lambda^2 s_2 e^{-\lambda s_3} ds_2 \\ &= -s_3 \lambda e^{-\lambda s_3} - e^{-\lambda s_3} + 1 - \frac{1}{2} s_3^2 \lambda^2 e^{-\lambda s_3} \\ \therefore G(S_3) &= 1 - s_3 \lambda e^{-\lambda s_3} - \frac{1}{2} s_3^2 \lambda^2 e^{-\lambda s_3} - e^{-\lambda s_3} \end{aligned} \tag{3.2.6}$$

Therefore from the above equation (3.2.6) it can be concluded that

$$\lim_{s_3 \to \infty} G(s_3) = 1$$

and

$$\lim_{s_3 \to 0} G(s_3) = 0 \tag{3.2.7}$$

Probability Distribution Function

From equation (3.2.6) therefore

$$\begin{aligned} g(s_3) &= \frac{d}{ds_3} G(s_3) \\ &= \frac{d}{ds_3}\left[1 - s_3\lambda e^{-\lambda s_3} - \frac{1}{2}s_3^2\lambda^2 e^{-\lambda s_3} - e^{-\lambda s_3}\right] \\ &= -\lambda e^{-\lambda s_3} + s_3\lambda^2 e^{-\lambda s_3} - s_3\lambda^2 e^{-\lambda s_3} + \frac{1}{2}s_3^2\lambda^3 e^{-\lambda s_3} + \lambda e^{-\lambda s_3} \\ &= \frac{\lambda^3}{2} e^{-\lambda s_3} s_3^2 \\ &= \frac{\lambda^3}{\Gamma(3)} e^{-\lambda s_3} s_3^{3-1} \end{aligned} \tag{3.2.8}$$

as required.

3.2.3 Sum of four Exponential Random Variables

In the section we need to consider the sums of four exponential random variables with independent and identical parameters.

Let

$$\begin{aligned} S_4 &= X_1 + X_2 + X_3 + X_4 \\ &= S_3 + X_4 \end{aligned} \tag{3.2.9}$$

Let $F_i(x_i)$ and $G(s_4)$ be the cumulative distribution function for X_i and S_4 respectively.

Cumulative Distribution Function

From the definition the cumulative distribution function is given by:

$$\begin{aligned} G(S_4) &= Prob[S_4 \leq s_4] \\ &= Prob\{S_3 + X_4 \leq s_4\} \\ &= Prob\{X_4 \leq s_4 - S_3\} \\ &= Prob\{0 \leq s_3 \leq \infty, 0 \leq X_4 \leq s_4 - S_3\} \\ &= Prob\{0 \leq s_3 \leq s_4, 0 \leq X_4 \leq s_4 - S_3\} \\ &= \int_0^{s_4}\int_0^{s_4 - s_3} g(s_3) f(x_4)\, dx_4\, ds_3 \end{aligned}$$

$$
\begin{aligned}
G(S_4) &= \int_0^{s_4} \frac{\lambda^3}{2} e^{-\lambda s_3} s_3^2 \left[\int_0^{s_4 - s_3} \lambda e^{-\lambda x_4} dx_4 \right] ds_3 \\
&= \int_0^{s_4} \frac{\lambda^3}{2} e^{-\lambda s_3} s_3^2 \left[-e^{-\lambda x_4} \right]_0^{s_4 - s_3} ds_3 \\
&= \int_0^{s_4} \frac{\lambda^3}{2} e^{-\lambda s_3} s_3^2 \left[1 - e^{-\lambda s_4 + \lambda s_3} \right] ds_3 \\
&= \int_0^{s_4} \frac{\lambda^3}{2} e^{-\lambda s_3} s_3^2 ds_3 - \int_0^{s_4} \frac{\lambda^3}{2} e^{-\lambda s_3} s_3^2 e^{-\lambda s_4} ds_3 \\
\therefore G(S_4) &= -s_4^2 \frac{\lambda^2}{2} e^{-\lambda s_4} - s_4 \lambda e^{-\lambda s_4} - e^{-\lambda s_4} + 1 - \lambda^3 e^{-\lambda s_4} \frac{s_4^3}{6} \qquad \textbf{(3.2.10)}
\end{aligned}
$$

Therefore from the above equation (3.2.10) it can be concluded that

$$
\lim_{s_4 \to \infty} G(s_4) = 1
$$

and

$$
\lim_{s_4 \to 0} G(s_4) = 0 \qquad \textbf{(3.2.11)}
$$

Probability Distribution Function

From equation (3.2.10) therefore

$$
\begin{aligned}
g(s_4) &= \frac{d}{ds_4} G(s_4) \\
&= \frac{d}{ds_4} \left[-s_4^2 \frac{\lambda^2}{2} e^{-\lambda s_4} - s_4 \lambda e^{-\lambda s_4} - e^{-\lambda s_4} + 1 - \lambda^3 e^{-\lambda s_4} \frac{s_4^3}{6} \right] \\
&= -s_4 \lambda^2 e^{-\lambda s_4} + \frac{s_4^2 \lambda^3 e^{-\lambda s_4}}{2} - \lambda e^{-\lambda s_4} + s_4 \lambda^2 e^{-\lambda s_4} + \lambda e^{-\lambda s_4} - \frac{s_4^2 \lambda^3 e^{-\lambda s_4}}{2} + \frac{s_4^3 \lambda^4 e^{-\lambda s_4}}{6} \\
&= \frac{s_4^3 \lambda^4 e^{-\lambda s_4}}{6} \\
&= \frac{\lambda^4}{\Gamma(4)} e^{-\lambda s_4} s_4^{4-1} \qquad \textbf{(3.2.12)}
\end{aligned}
$$

as required.

3.2.4 Sum of five Exponential Random Variables

In the section we need to consider the sums of five exponential random variables with independent and identical parameters.

Let

$$\begin{aligned} S_5 &= X_1 + X_2 + X_3 + X_4 + X_5 \\ &= S_4 + X_5 \end{aligned} \quad \textbf{(3.2.13)}$$

Let $F_i(x_i)$ and $G(s_5)$ be the cumulative distribution function for X_i and S_5 respectively.

Cumulative Distribution Function

From the definition the cumulative distribution function is given by:

$$\begin{aligned} G(S_5) &= Prob[S_5 \leq s_5] \\ &= Prob\{S_4 + X_5 \leq s_5\} \\ &= Prob\{X_5 \leq s_5 - s_4\} \\ &= Prob\{0 \leq s_4 \leq \infty, 0 \leq X_5 \leq s_5 - s_4\} \\ &= Prob\{0 \leq s_4 \leq s_5, 0 \leq X_5 \leq s_5 - s_4\} \\ &= \int_0^{s_5} \int_0^{s_5 - s_4} g(s_4) f(x_5) dx_5 ds_4 \\ &= \int_0^{s_5} \frac{\lambda^4}{\Gamma(4)} e^{-\lambda s_4} s_4^3 \left[\int_0^{s_5 - s_4} \lambda e^{-\lambda x_5} dx_5 \right] ds_4 \\ &= \int_0^{s_5} \frac{\lambda^4}{\Gamma(4)} e^{-\lambda s_4} s_4^3 \left[-e^{-\lambda x_5} \right]_0^{s_5 - s_4} ds_4 \\ &= \int_0^{s_5} \frac{\lambda^4}{\Gamma(4)} e^{-\lambda s_4} s_4^3 \left[1 - e^{-\lambda s_5 + \lambda s_4} \right] ds_4 \\ &= \int_0^{s_5} \frac{\lambda^4}{\Gamma(4)} e^{-\lambda s_4} s_4^3 ds_4 - \int_0^{s_5} \frac{\lambda^4}{\Gamma(4)} e^{-\lambda s_5} s_4^3 ds_4 \end{aligned}$$

$$\therefore G(S_5) = -\frac{s_5^3 \lambda^3 e^{-\lambda s_5}}{\Gamma(4)} - \frac{3 s_5^2 \lambda^2 e^{-\lambda s_5}}{\Gamma(4)} - \frac{6 s_5 \lambda e^{-\lambda s_5}}{\Gamma(4)} - \frac{6 e^{-\lambda s_5}}{\Gamma(4)} + \frac{6}{\Gamma(4)} - \frac{s_5^4 \lambda^4 e^{-\lambda s_5}}{4\Gamma(4)} \quad \textbf{(3.2.14)}$$

Therefore from the above equation (3.2.14) it can be concluded that

$$\lim_{s_5 \to \infty} G(s_5) = 1$$

and

$$\lim_{s_5 \to 0} G(s_5) = 0 \quad \textbf{(3.2.15)}$$

Probability Distribution Function

From equation (3.2.14) therefore

$$\begin{aligned}
g(s_5) &= \frac{d}{ds_5}G(s_5) \\
&= \frac{d}{ds_5}\left[-\frac{s_5^3\lambda^3e^{-\lambda s_5}}{\Gamma(4)} - \frac{3s_5^2\lambda^2e^{-\lambda s_5}}{\Gamma(4)} - \frac{6s_5\lambda e^{-\lambda s_5}}{\Gamma(4)} - \frac{6e^{-\lambda s_5}}{\Gamma(4)} + \frac{6}{\Gamma(4)} - \frac{s_5^4\lambda^4e^{-\lambda s_5}}{4\Gamma(4)}\right] \\
&= -\frac{3s_5^2\lambda^3e^{-\lambda s_5}}{\Gamma(4)} + \frac{s_5^3\lambda^4e^{-\lambda s_5}}{\Gamma(4)} - \frac{6s_5\lambda^2e^{-\lambda s_5}}{\Gamma(4)} + \frac{3s_5^2\lambda^3e^{-\lambda s_5}}{\Gamma(4)} \\
&- \frac{6\lambda e^{-\lambda s_5}}{\Gamma(4)} + \frac{6s_5\lambda^2e^{-\lambda s_5}}{\Gamma(4)} + \frac{6\lambda e^{-\lambda s_5}}{\Gamma(4)} - \frac{4s_5^3\lambda^4e^{-\lambda s_5}}{4\Gamma(4)} + \frac{s_5^4\lambda^5e^{-\lambda s_5}}{4\Gamma(4)} \\
&= \frac{s_5^4\lambda^5e^{-\lambda s_5}}{(5-1)\Gamma(5-1)} \\
&= \frac{\lambda^5}{\Gamma(5)}e^{-\lambda s_5}s_5^{5-1} \qquad \textbf{(3.2.16)}
\end{aligned}$$

as required.

3.2.5 Generalization case of Sum of Exponential Random Variables

In the section we need to consider the generalization of sums of n exponential random variables with independent and identical parameters.

Proposition 1

Let

$$S_n = X_1 + X_2 + \cdots + X_n \qquad \textbf{(3.2.17)}$$

for $n \geq 2$ be the sum of n exponential random variables, then the pdf for the distribution will be given by:

$$g(s_n) = \frac{\lambda^n}{\Gamma(n)}e^{-\lambda s_n}s_n^{n-1} \text{ where } \lambda, s_n > 0 \text{ and } n = 1, 2, 3, \cdots \qquad \textbf{(3.2.18)}$$

Proof

Let $F_i(x_i)$ and $G(s_n)$ be the cumulative distribution function for X_i and S_n respectively.

Cumulative Distribution Function

From the definition the cumulative distribution function is given by:

$$
\begin{aligned}
G(S_n) &= Prob[S_n \le s_n] \\
&= Prob\{S_{n-1} + X_n \le s_n\} \\
&= Prob\{X_n \le s_n - s_{n-1}\} \\
&= Prob\{0 \le s_{n-1} \le \infty, 0 \le X_n \le s_n - s_{n-1}\} \\
&= Prob\{0 \le s_{n-1} \le s_n, 0 \le X_n \le s_n - s_{n-1}\} \\
&= \int_0^{s_n} \int_0^{s_n - s_{n-1}} g(s_{n-1}) f(x_n) dx_n ds_{n-1} \\
&= \int_0^{s_n} \frac{\lambda^{n-1}}{\Gamma(n-1)} e^{-\lambda s_{n-1}} s_{n-1}^{n-2} \left[\int_0^{s_n - s_{n-1}} \lambda e^{-\lambda x_n} dx_n \right] ds_{n-1} \\
&= \int_0^{s_n} \frac{\lambda^{n-1}}{\Gamma(n-1)} e^{-\lambda s_{n-1}} s_{n-1}^{n-2} \left[-e^{-\lambda x_n} \right]_0^{s_n - s_{n-1}} ds_{n-1} \\
&= \int_0^{s_n} \frac{\lambda^{n-1}}{\Gamma(n-1)} e^{-\lambda s_{n-1} 4} s_{n-1}^{n-1} \left[1 - e^{-\lambda s_n + \lambda s_{n-1}} \right] ds_{n-1} \\
\therefore G(S_n) &= \int_0^{s_n} \frac{\lambda^{n-1}}{\Gamma(n-1)} e^{-\lambda s_{n-1}} s_{n-1}^{n-2} ds_{n-1} - \int_0^{s_n} \frac{\lambda^{n-1}}{\Gamma(n-1)} e^{-\lambda s_n} s_{n-1}^{n-2} ds_{n-1} \qquad \textbf{(3.2.19)}
\end{aligned}
$$

Solving the above equation (3.2.19)and considering equation (3.2.14)then it can be concluded to follow the same condition and therefore

$$
\lim_{s_n \to \infty} G(s_n) = 1
$$

and

$$
\lim_{s_n \to 0} G(s_n) = 0 \qquad \textbf{(3.2.20)}
$$

Probability Distribution Function

From equation (3.2.19) and considering equation (3.2.16) then

$$
\begin{aligned}
g(s_n) &= \frac{d}{ds_n} G(s_n) \\
&= \frac{\lambda^n}{\Gamma(n)} e^{-\lambda s_n} s_n^{n-1} \text{ where } \lambda, s_n > 0 \text{ and } n = 1, 2, 3, \cdots \qquad \textbf{(3.2.21)}
\end{aligned}
$$

Hence the prove which is an Erlang Distribution with parameters n and λ.

3.3 Properties of the Distribution

3.3.1 Moments

For

$$f_{s_n}(s) = \frac{\lambda^n e^{-\lambda s_n} s^{n-1}}{\Gamma(n)} \tag{3.3.1}$$

The r^{th} moment for the fixed sum of exponential random variables are given by:

$$\begin{aligned}
E(s_n^r) &= \int_0^\infty s_n^r f(s_n) ds_n \\
&= \int_0^\infty s_n^r \frac{\lambda^n e^{-\lambda s_n} s^{n-1}}{\Gamma(n)} ds_n \\
&= \frac{\lambda^n}{\Gamma(n)} \int_0^\infty s_n^r e^{-\lambda s_n} s_n^{n-1} ds_n
\end{aligned}$$

Let

$$z = \lambda s_n \Rightarrow s_n = \frac{z}{\lambda}$$

$$dz = \lambda ds_n$$

Therefore

$$\begin{aligned}
E(s_n^r) &= \frac{1}{\Gamma(n)} \int_0^\infty \left(\frac{z}{\lambda}\right)^r \lambda^n e^{-z} \left(\frac{z}{\lambda}\right)^{n-1} \frac{1}{\lambda} dz \\
&= \frac{1}{\Gamma(n)} \int_0^\infty \frac{z^r}{\lambda^r} \lambda^n e^{-z} \frac{z^{n-1}}{\lambda^n} dz \\
&= \frac{1}{\lambda^r \Gamma(n)} \int_0^\infty z^{r+n-1} e^{-z} dz \\
&= \frac{\Gamma(r+n)}{\lambda^r \Gamma(n)} \\
&= \frac{(r+n-1)!}{\lambda^r (n-1)!}
\end{aligned} \tag{3.3.2}$$

for positive value of integers of r and n.

Thus

$$E(s_n^r) = \frac{(r+n-1)!}{\lambda^r (n-1)!}$$

Hence,

$$E(s_n) = \frac{(n)!}{\lambda(n-1)!} = \frac{n}{\lambda} = \mu \tag{3.3.3}$$

$$E(s_n^2) = \frac{(n+1)!}{\lambda^2 (n-1)!} = \frac{n(n+1)}{\lambda^2} = \frac{n^2+n}{\lambda^2} = \mu\left(\frac{n+1}{\lambda}\right) \tag{3.3.4}$$

Thus,

$$\begin{aligned} Var(s_n) = \mu_2 &= E[s_n - \mu]^2 = \sigma^2 \\ &= E(s_n^2) - 2\mu E(s_n) + \mu^2 \\ &= E(s_n^2) - \mu^2 \\ &= \frac{n^2+n}{\lambda^2} - \frac{n^2}{\lambda^2} \\ &= \frac{n}{\lambda^2} \end{aligned} \tag{3.3.5}$$

Therefore,

$$\sigma^2 = \frac{n}{\lambda^2}$$

$$\sigma = \frac{\sqrt{n}}{\lambda}$$

Also,

$$\begin{aligned} E(s_n^3) &= \frac{(n+2)!}{\lambda^3 (n-1)!} \\ &= \frac{n(n+1)(n+2)}{\lambda^3} \\ &= \frac{n^3+3n^2+2n}{\lambda^3} \end{aligned}$$

$$\begin{aligned}
\therefore \mu_3 &= E[s_n-\mu]^3 \\
&= E(s_n^3) - 3\mu E(s_n^2) + 3\mu^2 E(s_n) - \mu^3 \\
&= \frac{n(n+1)(n+2)}{\lambda^3} - 3.\frac{n}{\lambda}.\frac{n^2+n}{\lambda^2} + 3.\frac{n^2}{\lambda^2}.\frac{n}{\lambda} - \frac{n^3}{\lambda^3} \\
&= \frac{n^3+3n^2+2n}{\lambda^3} - \frac{3n^3+3n^2}{\lambda^3} + \frac{3n^3}{\lambda^3} - \frac{n^3}{\lambda^3} \\
&= \frac{2n}{\lambda^3} \qquad \textbf{(3.3.6)}
\end{aligned}$$

Therefore

$$\begin{aligned}
E(s_n^4) &= \frac{n(n+1)(n+2)(n+3)}{\lambda^4} = \frac{(n+3)!}{\lambda^4(n-1)!} \\
\therefore \mu &= E[s_n-\mu]^4 \\
&= E(s_n^4) - 4\mu E(s_n^3) + 6\mu^2 E(s_n^2) - 4\mu^3 E(s_n) + \mu^4 \\
&= \frac{(n+3)!}{\lambda^4(n-1)!} - 4.\frac{n}{\lambda}\left(\frac{n^3+3n^2+2n}{\lambda^3}\right) + 6\left(\frac{n^2}{\lambda^2}\right)\left(\frac{n+n}{\lambda^2}\right) - 4\left(\frac{n^3}{\lambda^3}\right)\left(\frac{n}{\lambda}\right) + \frac{n^4}{\lambda^4} \\
&= \frac{n^4+6n^2+11n^2+6n}{\lambda^4} - \frac{4n^4+12n^3+8n^2}{\lambda^4} + \frac{6n^4+6n^3}{\lambda^4} - \frac{4n^4}{\lambda^4} + \frac{n^4}{\lambda^4} \\
&= \frac{3n^3+6n}{\lambda^4} \\
&= \frac{3n(n+2)}{\lambda^4} \qquad \textbf{(3.3.7)}
\end{aligned}$$

3.3.2 Moment Generating Function

Let,

$$S_N = X_1 + X_2 + \cdots + X_N$$

Let $S_n = y$ where $n \in N$
when

$$g(y;n,\lambda) = \frac{\lambda^n}{\Gamma(n)} e^{-\lambda y} y^{n-1} \qquad \textbf{(3.3.8)}$$

The mgf of the fixed sum of exponential random variables is given by;

$$
\begin{aligned}
M_Y(t) &= E\left[e^{ty}\right] \\
&= \int_0^\infty e^{ty} g(y) dy \\
&= \int_0^\infty e^{ty} \frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y} dy \\
&= \frac{\lambda^n}{(n-1)!} \int_0^\infty y^{n-1} e^{-y(\lambda - t)} dy
\end{aligned}
$$

Let,

$$
\begin{aligned}
u &= y(\lambda - t) \\
\Rightarrow y &= \frac{u}{\lambda - t} \\
du &= (\lambda - t) dy
\end{aligned}
$$

$$
\begin{aligned}
M_Y(t) &= \frac{\lambda^n}{(n-1)!} \int_0^\infty \left(\frac{u}{\lambda - t}\right)^{n-1} e^{-u} \frac{du}{(\lambda - t)} \\
&= \frac{\lambda^n}{(n-1)!} \int_0^\infty \frac{u^{n-1}}{\lambda - t^{n-1}} . \frac{1}{\lambda - t} e^{-u} du \\
&= \left(\frac{\lambda}{\lambda - t}\right)^n \int_0^\infty \frac{u^{n-1}}{\Gamma(n)} e^{-u} du \\
&= \left(\frac{\lambda}{\lambda - t}\right)^n
\end{aligned}
\tag{3.3.9}
$$

3.3.3 Mode

Let,

$$S_N = X_1 + X_2 + \cdots + X_N$$

Let $S_n = y$ where $n \in N$
when

$$g(y; n, \lambda) = \frac{\lambda^n}{\Gamma(n)} e^{-\lambda y} y^{n-1}$$

The mode of the model can be computed if the following condition is maintained:

$$\frac{d}{dy} g(y) = 0$$

Therefore

$$\frac{d}{dy}\left[\frac{\lambda^n}{\Gamma(n)}e^{-\lambda y}y^{n-1}\right] = 0$$
$$-\frac{\lambda^{n-1}}{\Gamma(n)}e^{-\lambda y}y^{n-1} + \frac{\lambda^n}{\Gamma(n)}e^{-\lambda y}(n-1)y^{n-2} = 0$$
$$\frac{\lambda^{n-1}}{\Gamma(n)}e^{-\lambda y}y^{n-1} = \frac{\lambda^n}{\Gamma(n)}e^{-\lambda y}(n-1)y^{n-2}$$
$$\lambda^{-1} = (n-1)y^{-1}$$
$$\therefore y = \lambda(n-1) \tag{3.3.10}$$

3.4 Parameter Estimation

In the classical method for obtaining point estimators for unknown parameters, we will consider the following methods of estimation as follows:

3.4.1 Methods of Moments

Let $S_n = X_1 + X_2 + \cdots + X_n$ be the fixed sum of independent and identically distributed exponential random variables from Erlang (n, λ).

Let

$$S_n = Y$$

The first two moments about origin are given by

$$\mu_1 = E[Y] = \frac{n}{\lambda}$$
$$\mu_2 = E[Y^2] = \frac{n^2+n}{\lambda^2}$$

From the distribution the sample moments can be obtained as follows:

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

and

$$M_2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$$

Now, using the methods of moments, we obtain,

$$
\begin{aligned}
M_1 &= \mu_1 \\
\Rightarrow \frac{1}{n}\sum_{i=1}^{n} Y_i &= \frac{n}{\lambda} \\
\Rightarrow \hat{\lambda} &= \frac{n}{\overline{Y}} \qquad (3.4.1)
\end{aligned}
$$

and

$$M_2 = \mu_2$$

This implies

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} Y_i^2 &= \frac{n^2+n}{\lambda^2} \\
&= \frac{n^2+n}{n}.\overline{Y} \\
&= (n+1)\overline{Y} \qquad (3.4.2)
\end{aligned}
$$

For solving n, we obtain

$$
\begin{aligned}
n\overline{Y} &= \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \overline{Y} \\
&= \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - n\overline{Y} \\
2n\overline{Y} &= \sum_{i=1}^{n} Y_i^2 \\
\therefore \hat{n} &= \frac{1}{2\overline{Y}}\sum_{i=1}^{n} Y_i^2 = \frac{S^2}{2\overline{Y}} \qquad \text{where } S^2 = \sum_{i=1}^{n} Y_i^2 \qquad (3.4.3)
\end{aligned}
$$

Therefore, the method of moments estimators of n and λ are

$$
\begin{aligned}
\hat{\lambda} &= \frac{n}{\overline{Y}} \\
\hat{n} &= \frac{S^2}{2\overline{Y}}
\end{aligned}
$$

3.4.2 Maximum Likelihood Estimation

Let $S_1, S_2, \cdots, S_k$ be the fixed sums of independent and identically distributed exponential random variables resulting to Erlang distribution with the pdf given by:

$$f(s;k,\lambda) = \frac{\lambda^k}{\Gamma(k)} e^{-\lambda s} s^{k-1}; 0 < s < \infty; \lambda > 0 \tag{3.4.4}$$

The likelihood function is given by:

$$\begin{aligned} L(\underline{s};k,\lambda) &= \prod_{i=1}^{n} \left[\frac{\lambda^k}{\Gamma(k)} e^{-\lambda s_i} s_i^{k-1} \right] \\ &= \frac{\lambda^{nk}}{(\Gamma(k))^n} e^{-\lambda \sum_{i=1}^{n} s_i} s_i^{k-1} \end{aligned}$$

The log likelihood function is given by;

$$log(L(\underline{s};k,\lambda)) = nklog\lambda - nlog\Gamma(k) - \lambda \sum_{i=1}^{n} s_i + (k-1) \sum_{i=1}^{n} s_i$$

Taking the partial derivative of the log likelihood function with respect to λ and setting it to 0

$$\frac{\delta}{\delta\lambda} log(L(\underline{s};k,\lambda)) = \frac{nk}{\lambda} - \sum_{i=1}^{n} s_i = 0$$

$$\begin{aligned} \therefore \frac{nk}{\lambda} &= \sum_{i=1}^{n} s_i \\ &= n\bar{s} \\ \Rightarrow \hat{\lambda} &= \frac{k}{\bar{s}} \end{aligned} \tag{3.4.5}$$

Taking the partial derivative of the log likelihood function with respect to k and setting it to 0

$$\frac{\delta}{\delta k} log(L(\underline{s};k,\lambda)) = nlog\lambda - n\frac{\Gamma'(k)}{\Gamma(k)} + \sum_{i=1}^{n} s_i = 0$$

$$\Rightarrow \frac{n\Gamma'(k)}{\Gamma(k)} = nlog\lambda + \sum_{i=1}^{n} s_i$$
$$\therefore \frac{\Gamma'(\hat{k})}{\Gamma(\hat{k})} = log\hat{\lambda} + \bar{s} \tag{3.4.6}$$

as required.

4 COMPOUND DISTRIBUTIONS IN TERMS OF LAPLACE TRANSFORM

4.1 Introduction

Let

$$S_N = X_1 + X_2 + \cdots + X_N \tag{4.1.1}$$

where $X_i' s$ are independent and identically distributed exponential random variables and N is also a random variable independent of the $X_i' s$.

4.2 Expectation Approach

In terms of Laplace Transform and Probability generating function

$$\begin{aligned} L_{S_N}(s) &= E\left[e^{-sS_N}\right] \\ &= EE\left[e^{-sS_N|N}\right] \\ &= E\{E(e^{-s(X_1+X_2+\cdots+X_N)})\} \\ &= E\{E(e^{-sX_1})E(e^{-sX_2})\cdots E(e^{-sX_N})\} \\ &= E\{L_{X_i}(S)\}^N \\ &= F_N[L_{X_i}(S)] \end{aligned} \tag{4.2.1}$$

which is called a compound distribution, where

$$F_N(S) = E(S^N) \tag{4.2.2}$$

which is the probability generating function of N.
Therefore

$$L_{S_N}(S) = F_N\left(\frac{\lambda}{\lambda+\beta}\right) \tag{4.2.3}$$

4.3 Considering Compound Distributions

(i) If N is Poisson, then $L_{S_N}(S)$ becomes a compound Poisson distribution. Suppose N is Poisson with parameter α, then

$$\begin{aligned} L_{S_N}(S) &= e^{-\alpha\left(1-L_{X_i}(S)\right)} \\ &= e^{-\alpha\left(1-\frac{\lambda}{\lambda+\beta}\right)} \\ &= e^{-\alpha\left(\frac{\lambda+\beta-\lambda}{\lambda+\beta}\right)} \\ &= e^{-\frac{\alpha\beta}{\lambda+\beta}} \end{aligned} \tag{4.3.1}$$

(ii) If N is Bernoulli, then $L_{S_N}(S)$ becomes a compound Bernoulli distribution. Suppose N is Bernoulli with parameter p, then

$$L_{S_N}(S) = q + p[L_{X_i}(S)]$$

where $q = 1 - p$
Therefore

$$\begin{aligned} L_{S_N}(S) &= (1-p) + p\left(\frac{\lambda}{\lambda+\beta}\right) \\ &= (1-p) + \frac{p\lambda}{\lambda+\beta} \\ &= \frac{(1-p)(\lambda+\beta)+p\lambda}{\lambda+\beta} \\ &= \frac{\lambda+\beta-p\lambda-p\beta+p\lambda}{\lambda+\beta} \\ &= \frac{\lambda+(1-p)\beta}{\lambda+\beta} \end{aligned} \tag{4.3.2}$$

(iii) If N is Binomial, then $L_{S_N}(S)$ becomes a compound Binomial distribution. Suppose N is Binomial with parameters n and p, then

$$L_{S_N}(S) = [q + pL_{X_i}(S)]^n$$

where $q = 1 - p$
Therefore

$$\begin{aligned} L_{S_N}(S) &= \left[1 - p + \frac{p\lambda}{\lambda + \beta}\right]^n \\ &= \left[\frac{\lambda + (1-p)\beta}{\lambda + \beta}\right]^n \end{aligned} \tag{4.3.3}$$

This implies that when $n = 1$, we obtain the results of Bernoulli distribution.

(iv) If N is Geometric, then $L_{S_N}(S)$ becomes a compound Geometric distribution. Suppose N is Geometric with parameter p, then

$$L_{S_N}(S) = \frac{pL_{S_N}}{1 - qL_{S_N}}$$

if $|L_{S_N}(S)| < q^{-1}$, and where where $q = 1 - p$
Therefore

$$\begin{aligned} L_{S_N}(S) &= \frac{p\left(\frac{\lambda}{\lambda+\beta}\right)}{1 - (1-p)\left(\frac{\lambda}{\lambda+\beta}\right)} \\ &= \frac{\frac{p\lambda}{\lambda+\beta}}{1 - \frac{\lambda - \lambda p}{\lambda+\beta}} \\ &= \left(\frac{p\lambda}{\lambda + \beta}\right)\left(\frac{\lambda + \beta}{\beta + p\lambda}\right) \\ &= \frac{p\lambda}{\beta + p\lambda} \end{aligned} \tag{4.3.4}$$

(v) If N is Negative Binomial, then $L_{S_N}(S)$ becomes a compound Negative Binomial distribution. Suppose N is Negative Binomial with parameter p, then

$$L_{S_N}(S) = \frac{pL_{S_N}}{1 - qL_{S_N}}$$

if $|L_{S_N}(S)| < q^{-1}$, and $p+q=1$
Therefore

$$
\begin{aligned}
L_{S_N}(S) &= \left[\frac{p\left(\frac{\lambda}{\lambda+\beta}\right)}{1-(1-p)\left(\frac{\lambda}{\lambda+\beta}\right)}\right]^n \\
&= \left[\frac{\left(\frac{p\lambda}{\lambda+\beta}\right)}{1-\frac{\lambda-p\lambda}{\lambda+\beta}}\right]^n \\
&= \left[\left(\frac{p\lambda}{\lambda+\beta}\right)\left(\frac{\lambda+\beta}{\beta+p\lambda}\right)\right]^n \\
&= \left[\frac{p\lambda}{\beta+p\lambda}\right]^n \qquad \textbf{(4.3.5)}
\end{aligned}
$$

This implies that when $n=1$ we obtain the results of Geometric distribution.

5 COMPOUND MIXED POISSON DISTRIBUTIONS AS DISCRETE MIXTURES

5.1 Introduction

Let

$$S_N = X_1 + X_2 + \cdots + X_N$$

where $X_i's$ are independent and identically distributed random variables and N is also a random variable independent of $X_i's$.
According to Klugman, Panjer and Willmot (2008), the distribution of N is called primary distribution and that of $X_i's$ is called secondary distribution.
The distribution of S_N is called a compound distribution.
If N is mixed Poisson distributed, then S_N is compound mixed Poisson distribution.
The compound distribution of S_N can be expressed as a discrete mixture as shown below:
Let

$$h(y) = Prob\{S_N = y \tag{5.1.1}$$

Then

$$\begin{aligned} h(y) &= Prob\{X_1 + X_2 + \cdots + X_N = y\} \\ &= \sum_n Prob\{X_1 + X_2 + \cdots + X_N = y, N = n\} \\ &= \sum_n Prob\{X_1 + X_2 + \cdots + X_N = y | N = n\} Prob\{N = n\} \\ &= \sum_n Prob\{X_1 + X_2 + \cdots + X_N = y\} p_n\} \\ &= \sum_n f_y^* p_n \end{aligned} \tag{5.1.2}$$

where $f_y^* p_n$ is the n^{th} - fold convolution.

5.2 Continuous Mixed Distribution

Let,

$$\begin{aligned} P_n &= Prob\{N = n\} \\ &= \int_0^\infty Prob\{N = n, \wedge = \lambda\} d\lambda \\ &= \int_0^\infty Prob\{N = n | \wedge = \lambda\} Prob\{\wedge = \lambda\} d\lambda \\ &= \int_0^\infty Prob\{N = n | \wedge = \lambda\} g(\lambda) d\lambda \end{aligned} \quad \textbf{(5.2.1)}$$

Where
$Prob\{N = n|\wedge = \lambda\}$ = the conditional pmf or pdf = the continuous mixing distribution and
Pn = a continuous mixture or continuous mixed distribution.
If
$Prob\{N = n|\wedge = \lambda\} = \frac{e^{-\lambda}\lambda^n}{n!}, n = 0, 1, 2, \cdots$
is a Poisson distribution (pmf),
then

$$P_n = \int_0^\infty \frac{e^{-\lambda}\lambda^n}{n!} g(\lambda) d\lambda \quad \textbf{(5.2.2)}$$

which is a continuous mixed Poisson pmf with

$$E(N) = EE(N|\wedge) = E(\wedge) \quad \textbf{(5.2.3)}$$

and

$$\begin{aligned} VarN &= VarE(N|\wedge) + EVar(N|\wedge) \\ &= Var\wedge + E(\wedge) \end{aligned} \quad \textbf{(5.2.4)}$$

Sarguta (2017) obtained continuous Poisson mixtures for various cases of mixing distributions.

5.3 Discrete Erlang Mixture Based on continuous Mixed Poisson Distributions

In this study, X_i is exponentially distributed with parameter λ. Therefore

$$\begin{aligned} f_y^* &= Prob\{X_1 + X_2 + \cdots + X_n = y\} \\ &= \frac{\lambda^n}{\Gamma(n)} e^{-\lambda y} y^{n-1}, \;\; y > 0, n = 1, 2, \cdots \end{aligned} \tag{5.3.1}$$

This is an Erlang distribution which is a gamma distribution with parameters n being a positive integer and $\lambda > 0$. The discrete mixing distribution P_n is a continuous Poisson mixture.

Therefore the distribution of S_N is

$$\begin{aligned} h(y) &= Prob\{S_N = y\} \\ &= \sum_{n=1}^{\infty} \left\{ \frac{\lambda^n}{\Gamma(n)} e^{-\lambda y} y^{n-1} \int_0^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} g(\lambda) d\lambda \right\} \\ &= \sum_{n=1}^{\infty} \left\{ \frac{\lambda^n}{\Gamma(n)} \frac{e^{-\lambda y}}{\Gamma(n+1)} y^{n-1} E(\wedge^n e^{-\wedge}) \right\} \end{aligned} \tag{5.3.2}$$

The problem is to obtain $Prob\{S_N = y\}$ for various cases of continuous mixing distributions, $g(\lambda)$.

$$E[S_N] = E(N)E(X) = E(\wedge)(EX) \tag{5.3.3}$$

and

$$\begin{aligned} Var[S_N] &= VarN[E(X_i)]^2 + E(N)VarX \\ &= [Var\wedge + E(\wedge)][E(X_i)]^2 + E(\wedge)VarX \\ &= Var\wedge [E(X_i)]^2 + E(\wedge)[E(X_i)]^2 + E(\wedge)VarX \end{aligned} \tag{5.3.4}$$

Therefore

$$Var[S_N] = Var\wedge [E(X_i)]^2 + E(\wedge)\left\{[E(X_i)]^2 + VarX\right\}$$

Since $X_i \sim exp(\lambda)$, then

$$E[S_N] = \frac{E(\wedge)}{\lambda} \tag{5.3.5}$$

$$Var[S_N] = \frac{1}{\lambda}\{Var\wedge + 2E(\wedge)\} \tag{5.3.6}$$

Remark:
In actuarial science, N is the number of claims and X_i is the amount of the i^{th} claim (severity). S_N is the total or aggregate claim or loss.
Therefore,
$Prob\{S_N = y\}$ =the probability of aggregate loss $y > 0$.
The probability of number of claims is

$$\begin{aligned} Prob\{S_N = 0\} &= 1 - Prob\{S_N > 0\} \\ &= 1 - \int_0^\infty Prob\{S_N = y\}dy \end{aligned} \tag{5.3.7}$$

5.4 Compound Mixed Poisson Distribution

5.4.1 Poisson-Exponential Mixing Distribution

$$g(\lambda) = \beta e^{-\beta\lambda}, \lambda > 0; \beta > 0 \tag{5.4.1}$$

Therefore, the mean

$$E(\wedge) = \frac{1}{\beta} \tag{5.4.2}$$

and variance,

$$Var\wedge = \frac{1}{\beta^2} \tag{5.4.3}$$

$$\begin{aligned}
E(\wedge^n e^{-\wedge}) &= \int_0^\infty \lambda^n e^{-\lambda} \beta e^{-\beta\lambda} d\lambda \\
&= \beta \int_0^\infty \lambda^n e^{-(1+\beta)\lambda)} d\lambda \\
&= \frac{\beta\Gamma(n+1)}{(1+\beta)^{n+1}} \\
\therefore Prob\{S_N = y\} &= \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma(n)} \frac{e^{-\lambda y}}{\Gamma(n+1)} y^{n-1} \beta \frac{\Gamma(n+1)}{(1+\beta)^{n+1}} \\
&= \beta e^{-\lambda y} \sum_{n=1}^{\infty} \frac{\lambda^n y^{n-1}}{\Gamma(n)(1+\beta)^{n+1}} \\
&= \frac{\lambda\beta e^{-\lambda y}}{(1+\beta)^2} \sum_{n=1}^{\infty} \left(\frac{\lambda y}{1+\beta}\right)^{n-1} \frac{1}{(n+1)!} \\
&= \frac{\lambda\beta e^{-\lambda y}}{(1+\beta)^2} .exp\left\{\left(\frac{\lambda y}{1+\beta}\right)\right\} \\
&= \frac{\lambda\beta}{(1+\beta)^2} .exp\left\{-\lambda y\left(1-\frac{1}{1+\beta}\right)\right\} \\
&= \frac{\lambda\beta}{(1+\beta)^2} exp\left(-\frac{\beta\lambda y}{1+\beta}\right), y > 0
\end{aligned} \tag{5.4.4}$$

Therefore

$$\begin{aligned}
\therefore Prob\{S_N = 0\} &= 1 - Prob\{S_N > 0\} \\
&= 1 - \int_0^\infty Prob\{S_N = y\} dy \\
&= 1 - \int_0^\infty \frac{\lambda\beta}{(1+\beta)^2} exp\left(-\frac{\beta\lambda y}{1+\beta}\right) dy \\
&= 1 - \frac{\lambda\beta}{(1+\beta)^2} . \frac{1+\beta}{\lambda\beta} \\
&= 1 - \frac{1}{1+\beta} \\
&= \frac{\beta}{1+\beta}
\end{aligned} \tag{5.4.5}$$

5.4.2 Poisson-Gamma Mixing Distribution

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1}, \lambda > 0, \alpha > 0, \beta > 0 \tag{5.4.6}$$

Mean,

$$E(\wedge) = \frac{\alpha}{\beta} \tag{5.4.7}$$

and variance,

$$Var(\wedge) = \frac{\alpha}{\beta^2} \tag{5.4.8}$$

Therefore

$$\begin{aligned}
E[\wedge^n e^{-\wedge}] &= \int_0^\infty \lambda^n e^{-\lambda} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{n+\alpha-1} e^{-(1+\beta)\lambda} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(n+\alpha)}{(1+\beta)^{n+\alpha}} \\
\therefore h(y) &= \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma(n)} \frac{e^{-\lambda y} y^{n-1}}{\Gamma(n+1)} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(n+\alpha)}{(1+\beta)^{n+\alpha}} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n)\Gamma(n+1)} \frac{\lambda^n e^{-\lambda y} y^{n-1}}{(1+\beta)^{n+\alpha}} \\
&= \beta^\alpha \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} \frac{\lambda^n}{\Gamma(n)} \frac{e^{-\lambda y} y^{n-1}}{(1+\beta)^{n+\alpha}} \\
&= \beta^\alpha \sum_{n=1}^{\infty} \binom{\alpha+n-1}{n} \frac{\lambda^n}{\Gamma(n)} \frac{e^{-\lambda y} y^{n-1}}{(1+\beta)^{n+\alpha}}, y > 0 \qquad (5.4.9) \\
\therefore \int_0^\infty h(y)dy &= \beta^\alpha \sum_{n=1}^{\infty} \binom{\alpha+n-1}{n} \frac{\lambda^n}{\Gamma(n)} \frac{1}{(1+\beta)^{n+\alpha}} \int_0^\infty y^{n-1} e^{-\lambda y} dy \\
&= \beta^\alpha \sum_{n=1}^{\infty} \binom{\alpha+n-1}{n} \frac{\lambda^n}{\Gamma(n)} \frac{1}{(1+\beta)^{n+\alpha}} \frac{\Gamma(n)}{\lambda^n} \\
&= \beta^\alpha \sum_{n=1}^{\infty} \binom{\alpha+n-1}{n} \frac{1}{(1+\beta)^{n+\alpha}} \\
&= \left(\frac{\beta}{(1+\beta)}\right)^\alpha \sum_{n=1}^{\infty} \binom{\alpha+n-1}{n} \left(\frac{1}{1+\beta}\right)^n \\
&= \left(\frac{\beta}{(1+\beta)}\right)^\alpha \sum_{n=1}^{\infty} (-1)^n \binom{-\alpha}{n} \left(\frac{1}{1+\beta}\right)^n
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\beta}{(1+\beta)}\right)^{\alpha} \sum_{n=1}^{\infty} \binom{-\alpha}{n} \left(-\frac{1}{1+\beta}\right)^{n} \\
&= \left(\frac{\beta}{(1+\beta)}\right)^{\alpha} \left[\left(1-\frac{1}{1+\beta}\right)^{-\alpha} - 1\right] \\
\therefore \int_{0}^{\infty} h(y)dy &= \left(\frac{\beta}{(1+\beta)}\right)^{\alpha} \left\{\left(\frac{\beta}{1+\beta}\right)^{-\alpha} - 1\right\} \\
&= 1 - \left(\frac{\beta}{1+\beta}\right)^{\alpha} \\
\therefore h(0) &= \left(\frac{\beta}{1+\beta}\right)^{\alpha} \qquad (5.4.10)
\end{aligned}
$$

5.4.3 Poisson-Transmuted Exponential Mixing Distribution

Transmuted probability distribution is given by

$$F(x) = (1+\alpha)G(x) - \alpha[G(x)]^2, -1 < \alpha < 1 \qquad (5.4.11)$$

where $G(x)$ and $F(x)$ are the old and new cumulative distributions functions respectively.

$$f(x) = (1+\alpha)g(x) - 2\alpha G(x)g(x), -1 < \alpha < 1 \qquad (5.4.12)$$

where $g(x)$ and $f(x)$ are the old and new probability distribution functions respectively.
For an exponential distribution,
$g(x) = \lambda e^{-\lambda x}$ and $G(x) = 1 - \lambda e^{-\lambda x}, x > 0; \lambda > 0$
Therefore, the transmuted exponential distribution becomes

$$
\begin{aligned}
f(x) &= (1+\alpha)\lambda e^{-\lambda x} - 2\alpha(1-\lambda e^{-\lambda x})\lambda e^{-\lambda x} \\
&= \lambda e^{-\lambda x}\{(1+\alpha) - 2\alpha(1-e^{-\lambda x})\} \\
&= \lambda e^{-\lambda x}\{1+\alpha-2\alpha+2\alpha e^{-\lambda x})\} \\
&= \lambda e^{-\lambda x}\{1-\alpha+2\alpha e^{-\lambda x}\} \\
&= (1-\alpha)\lambda e^{-\lambda x} + 2\alpha\lambda e^{-2\lambda x}, x > 0; \lambda > 0; -1 < \alpha < 1 \qquad (5.4.13)
\end{aligned}
$$

which is a finite mixture of an exponential distribution with λ with another exponential distribution with parameter 2λ.
We shall thus denote the transmuted exponential mixing distribution as:

$$g(\lambda) = (1-\alpha)\theta e^{-\theta\lambda} + 2\alpha\theta e^{-2\theta\lambda}, \lambda > 0; \alpha > 0 \tag{5.4.14}$$

Therefore mean is given by:

$$\begin{aligned}
E(\wedge) &= \int_0^\infty \lambda g(\lambda) d\lambda \\
&= \int_0^\infty \left\{(1-\alpha)\theta\lambda e^{-\theta\lambda}\right\} d\lambda + \int_0^\infty 2\alpha\theta\lambda e^{-2\theta\lambda} d\lambda \\
&= (1-\alpha)\theta \frac{\Gamma(2)}{\theta^2} + 2\alpha\theta \frac{\Gamma(2)}{(2\theta)^2} \\
&= \frac{(1-\alpha)}{\theta} + \frac{\alpha}{2\theta} \\
&= \frac{2-2\alpha+\alpha}{2\theta} \\
&= \frac{2-\alpha}{2\theta}
\end{aligned} \tag{4.4.15}$$

This implies

$$\begin{aligned}
E(\wedge^2) &= (1-\alpha)\int_0^\infty \lambda^2 e^{-\theta\lambda} d\lambda + 2\alpha\theta \int_0^\infty \lambda^2 e^{-2\theta\lambda} d\lambda \\
&= (1-\alpha)\theta \frac{\Gamma(3)}{\theta^3} + 2\alpha\theta \frac{\Gamma(3)}{(2\theta)^3} \\
&= \frac{2(1-\alpha)}{\theta^2} + \frac{2\alpha}{(2\theta)^2} \\
&= \frac{8-8\alpha+2\alpha}{4\theta^2} \\
&= \frac{4-3\alpha}{2\theta^2}
\end{aligned} \tag{5.4.16}$$

Variance,

$$\begin{aligned}
Var\wedge &= E(\wedge^2) - [E(\wedge)]^2 \\
&= \frac{4-3\alpha}{2\theta^2} - \left(\frac{2-\alpha}{2\theta}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4-3\alpha}{2\theta^2} - \left(\frac{4-4\alpha+\alpha^2}{4\theta^2}\right) \\
\therefore Var\wedge &= \frac{8-6\alpha-4+4\alpha-\alpha^2}{4\theta^2} \\
&= \frac{4-2\alpha-\alpha^2}{4\theta^2}, 0<\alpha<1 \qquad \textbf{(5.4.17)}
\end{aligned}
$$

Next,

$$
\begin{aligned}
E(\wedge^n e^{-\wedge}) &= \int_0^\infty \lambda^n e^{-\lambda}(1-\alpha)\theta e^{-\theta\lambda} d\lambda + \int_0^\infty \lambda^n e^{-\lambda} 2\alpha\theta e^{-2\theta\lambda} d\lambda \\
&= (1-\alpha)\theta \int_0^\infty \lambda^n e^{-(1+\theta)\lambda} d\lambda + 2\alpha\theta \int_0^\infty \lambda^n e^{-(1+2\theta)\lambda} d\lambda \\
&= (1-\alpha)\theta \frac{\Gamma(n+1)}{(1+\theta)^{n+1}} + 2\alpha\theta \frac{\Gamma(n+1)}{(1+2\theta)^{n+1}} \qquad \textbf{(5.4.17)} \\
\therefore h(y) &= Prob\{S_N = y\} \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma(n)} \frac{e^{-\lambda y}}{\Gamma(n+1)} y^{n-1} \left\{ \frac{(1-\alpha)\theta\Gamma(n+1)}{(1+\theta)^{n+1}} + \frac{2\alpha\theta\Gamma(n+1)}{(1+2\theta)^{n+1}} \right\} \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma(n)} e^{-\lambda y} y^{n-1} \frac{(1-\alpha)\theta}{(1+\theta)^{n+1}} + \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma(n)} e^{-\lambda y} y^{n-1} \frac{2\alpha\theta}{(1+2\theta)^{n+1}} \\
&= \frac{\lambda(1-\alpha)\theta e^{-\lambda y}}{(1+\theta)^2} \sum_{n=1}^{\infty} \frac{(\lambda y)^{n-1}}{(n-1)!(1+\theta)^{n-1}} + \frac{2\alpha\theta\lambda e^{-\lambda y}}{(1+2\theta)^2} \sum_{n=1}^{\infty} \frac{(\lambda y)^{n-1}}{(n-1)!(1+2\theta)^{n-1}} \\
&= \frac{\lambda(1-\alpha)\theta e^{-\lambda y}}{(1+\theta)^2} e^{\frac{\lambda y}{1+\theta}} + \frac{2\alpha\theta\lambda e^{-\lambda y}}{(1+2\theta)^2} e^{\frac{\lambda y}{1+2\theta}} \\
&= \frac{\lambda(1-\alpha)\theta}{(1+\theta)^2} exp\left\{-\lambda y\left(1-\frac{1}{1+\theta}\right)\right\} + \frac{2\alpha\theta\lambda}{(1+2\theta)^2} exp\left\{-\lambda y\left(1-\frac{1}{1+2\theta}\right)\right\} \\
&= \frac{\lambda(1-\alpha)\theta}{(1+\theta)^2} exp\left\{-\frac{\theta\lambda y}{1+\theta}\right\} + \frac{2\alpha\theta\lambda}{(1+2\theta)^2} exp\left\{-\frac{2\theta\lambda y}{1+2\theta}\right\} \\
&= \theta\left[\frac{\lambda(1-\alpha)}{(1+\theta)^2} exp\left\{-\frac{\theta\lambda y}{1+\theta}\right\} + \frac{2\alpha\lambda}{(1+2\theta)^2} exp\left\{-\frac{2\theta\lambda y}{1+2\theta}\right\}\right] \qquad \textbf{(5.4.19)}
\end{aligned}
$$

as obtained by Bhati et al (2016).

Next,

$$\begin{aligned}\int_0^\infty h(y)dy &= \frac{\theta(1-\alpha)\lambda}{(1+\theta)^2}\int_0^\infty e^{-\frac{\theta\lambda y}{1+\theta}}dy + \frac{2\alpha\lambda\theta}{(1+2\theta)^2}\int_0^\infty e^{-\frac{2\theta\lambda y}{1+2\theta}}dy \\ &= \left[\frac{\theta(1-\alpha)\lambda}{(1+\theta)^2}\right]\left[\frac{1+\theta}{\theta\lambda}\right] + \left[\frac{2\alpha\lambda\theta}{(1+2\theta)^2}\right]\left[\frac{1+2\theta}{2\theta\lambda}\right] \\ \therefore \int_0^\infty h(y)dy &= \frac{1-\alpha}{1+\theta} + \frac{\alpha}{1+2\theta} \end{aligned} \tag{5.4.20}$$

Therefore probability of no claim is

$$h(0) = \frac{1-\alpha}{1+\theta} + \frac{\alpha}{1+2\theta} \tag{5.4.21}$$

5.4.4 Poisson-Lindley Mixing Distribution

$$g(\lambda) = \frac{\theta^2}{1+\theta}(1+\lambda)e^{-\theta\lambda}, \lambda > 0, \theta > 0 \tag{5.4.22}$$

Mean,

$$\begin{aligned} E(\wedge) &= \frac{\theta^2}{1+\theta}\int_0^\infty \lambda(1+\lambda)e^{-\theta\lambda}d\lambda \\ &= \frac{\theta^2}{1+\theta}\int_0^\infty \lambda e^{-\theta\lambda}d\lambda + \int_0^\infty \lambda^2 e^{-\theta\lambda}d\lambda \\ &= \frac{\theta^2}{1+\theta}\left(\frac{\Gamma(2)}{\theta^2} + \frac{\Gamma(3)}{\theta^3}\right) \\ &= \frac{\theta^2}{1+\theta}\left(\frac{1}{\theta^2} + \frac{2}{\theta^3}\right) = \frac{1}{1+\theta}\left(1 + \frac{2}{\theta}\right) = \frac{\theta+2}{\theta(1+\theta)} \end{aligned} \tag{5.4.23}$$

$$\begin{aligned} E(\wedge^2) &= \frac{\theta^2}{1+\theta}\int_0^\infty \lambda^2(1+\lambda)e^{-\theta\lambda}d\lambda \\ &= \frac{\theta^2}{1+\theta}\int_0^\infty \lambda^2 e^{-\theta\lambda}d\lambda + \int_0^\infty \lambda^3 e^{-\theta\lambda}d\lambda \\ &= \frac{\theta^2}{1+\theta}\left\{\frac{\Gamma(3)}{\theta^3} + \frac{\Gamma(4)}{\theta^4}\right\} \\ &= \frac{\theta^2}{1+\theta}\left\{\frac{2}{\theta^3} + \frac{6}{\theta^4}\right\} = \frac{1}{1+\theta}\left\{\frac{2}{\theta} + \frac{6}{\theta^2}\right\} \end{aligned} \tag{5.4.24}$$

Therefore variance is given by:
$Var\wedge = E(\wedge^2) - [E(\wedge)]^2$

$$\begin{aligned}
Var\wedge &= \frac{1}{1+\theta}\left(\frac{2}{\theta}+\frac{6}{\theta^2}\right)-\frac{1}{(1+\theta)^2}\left(1+\frac{2}{\theta}\right)^2\\
&= \frac{1}{1+\theta}\left(\frac{2}{\theta}+\frac{6}{\theta^2}\right)-\frac{1}{(1+\theta)^2}\left(1+\frac{4}{\theta}+\frac{4}{\theta^2}\right)\\
&= \frac{1}{(1+\theta)^2}\left\{\frac{2(1+\theta)}{\theta}+\frac{6(1+\theta)}{\theta^2}-1-\frac{4}{\theta}-\frac{4}{\theta^2}\right\}\\
&= \frac{1}{(1+\theta)^2}\left\{\frac{2}{\theta}+2+\frac{6}{\theta^2}+\frac{6}{\theta}-1-\frac{4}{\theta}-\frac{4}{\theta^2}\right\}\\
&= \frac{1}{(1+\theta)^2}\left\{(2-1)+\left(\frac{2}{\theta}+\frac{6}{\theta}-\frac{4}{\theta}\right)+\left(\frac{6}{\theta^2}-\frac{4}{\theta^2}\right)\right\}\\
&= \frac{1}{(1+\theta)^2}\left(1+\frac{4}{\theta}+\frac{2}{\theta^2}\right) \qquad \textbf{(5.4.25)}\\
E(\wedge^n e^{\wedge}) &= \int_0^\infty \lambda^n e^{-\lambda}\frac{\theta^2}{1+\theta}(1+\lambda)e^{-\theta\lambda}d\lambda\\
&= \frac{\theta^2}{1+\theta}\int_0^\infty \lambda^n(1+\lambda)e^{-\theta\lambda-\lambda}d\lambda\\
&= \frac{\theta^2}{1+\theta}\int_0^\infty \lambda^n(1+\lambda)e^{-(1+\theta)\lambda}d\lambda\\
&= \frac{\theta^2}{1+\theta}\int_0^\infty \lambda^n e^{-(1+\theta)\lambda}d\lambda+\int_0^\infty \lambda^{n+1}e^{-(1+\theta)\lambda}d\lambda\\
&= \frac{\theta^2}{1+\theta}\left\{\frac{\Gamma(n+1)}{(1+\theta)^{n+1}}+\frac{\Gamma(n+2)}{(1+\theta)^{n+2}}\right\}\\
&= \frac{\theta^2\Gamma(n+1)}{(1+\theta)^{n+2}}\left\{1+\frac{n+1}{1+\theta}\right\}\\
&= \frac{\theta^2\Gamma(n+1)}{(1+\theta)^{n+3}}(n+\theta+2) \qquad \textbf{(5.4.26)}
\end{aligned}$$

Therefore

$$\begin{aligned}
h(y) = Prob\{S_N = y\} &= \sum_{n=1}^{\infty}\frac{\lambda^n}{\Gamma(n)}.\frac{e^{-\lambda y}y^{n-1}}{\Gamma(n+1)}.\frac{\theta^2\Gamma(n+1)(n+\theta+2)}{(1+\theta)^{n+3}}\\
\therefore h(y) &= \sum_{n=1}^{\infty}\frac{\lambda^n e^{-\lambda y}y^{n-1}\theta^2(n+\theta+2)}{\Gamma(n)(1+\theta)^{n+3}}\\
&= \frac{\theta^2 e^{-\lambda y}}{(1+\theta)^3}\sum_{n=1}^{\infty}\frac{\lambda^n y^{n-1}}{(1+\theta)^n}\frac{(n+\theta+2)}{(n-1)!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\theta^2 e^{-\lambda y}}{(1+\theta)^3}\left\{\sum_{n=1}^{\infty}\left(\frac{\lambda}{1+\theta}\right)^n y^{n-1}\left(\frac{n-1+\theta+3}{(n-1)!}\right)\right\} \\
&= \frac{\theta^2 e^{-\lambda y}}{(1+\theta)^3}\left\{\sum_{n=1}^{\infty}\left(\frac{\lambda}{1+\theta}\right)^n y^{n-1}\left(\frac{1}{(n-2)!}+\frac{\theta+3}{(n-1)!}\right)\right\}, y>0 \\
\therefore h(y) &= \frac{\theta^2 e^{-\lambda y}}{(1+\theta)^3}\left\{\sum_{n=1}^{\infty} y\left(\frac{\lambda}{1+\theta}\right)^n \frac{y^{n-2}}{(n-2)!}+\sum_{n=1}^{\infty}\left(\frac{\lambda}{1+\theta}\right)^n y^{n-1}\frac{(\theta+3)}{(n-1)!}\right\} \\
&= \frac{\theta^2 e^{-\lambda y}}{(1+\theta)^3}\left\{\left(\frac{\lambda}{1+\theta}\right)^2 y e^{\frac{\lambda y}{1+\theta}}+\frac{\lambda}{1+\theta}(\theta+3)e^{\frac{\lambda y}{(1+\theta)}}\right\} \\
&= \frac{\theta^2 e^{-\lambda y}}{(1+\theta)^3}\left\{\left(\frac{\lambda}{1+\theta}\right)^2 y+\frac{\lambda}{1+\theta}(\theta+3)\right\}e^{\frac{\lambda y}{(1+\theta)}} \\
&= \frac{\theta^2}{(1+\theta)^3}\left\{\left(\frac{\lambda}{1+\theta}\right)^2 y+\frac{\lambda}{1+\theta}(\theta+3)\right\}e^{-\lambda y\left(1-\frac{1}{1+\theta}\right)} \\
&= \frac{\theta^2\lambda}{(1+\theta)^5}\left\{\lambda y+(\theta+1)(\theta+3)\right\}e^{-\frac{\lambda y}{1+\theta}}, y>0 \qquad \textbf{(5.4.27)} \\
\therefore \int_0^{\infty} h(y) &= \frac{\lambda\theta^2}{(1+\theta)^5}\left\{\lambda\int_0^{\infty} y e^{-\frac{\theta\lambda}{1+\theta}y}dy+(\theta+1)(\theta+3)\int_0^{\infty} e^{-\frac{\theta\lambda}{1+\theta}y}dy\right\} \\
&= \frac{\lambda\theta^2}{(1+\theta)^5}\left\{\lambda\frac{(1+\theta)^2}{(\theta\lambda)^2}+(1+\theta)(3+\theta)\frac{(1+\theta)}{\theta\lambda}\right\} \\
&= \frac{\lambda\theta^2}{(1+\theta)^3}\left\{\frac{\lambda}{(\theta\lambda)^2}+\frac{3+\theta}{\theta\lambda}\right\} \\
&= \frac{1+\theta(3+\theta)}{(1+\theta)^3} \\
&= \frac{1+3\theta+\theta^2}{(1+\theta)^3} \qquad \textbf{(5.4.28)} \\
\therefore h(0) &= Prob\{S_N=0\} \\
&= 1-\frac{1+3\theta+\theta^2}{(1+\theta)^3} \\
&= \frac{(1+\theta)^3-(1+2\theta+\theta^2+\theta)}{(1+\theta)^3} \\
\therefore h(0) &= \frac{(1+\theta)^3-[(1+\theta)^2+\theta]}{(1+\theta)^3} \\
&= \frac{(1+\theta)^3-(1+\theta)^2-\theta}{(1+\theta)^3} \\
&= \frac{(1+\theta)^2(1+\theta-1)-\theta}{(1+\theta)^3}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\theta(1+\theta)^2-\theta}{(1+\theta)^3} \\
&= \frac{\theta}{(1+\theta)^3}[(1+\theta+1)(1+\theta-1)] \\
&= \frac{\theta^2(\theta+2)}{(1+\theta)^3}
\end{aligned} \tag{5.4.29}
$$

5.4.5 Poisson-Generalized Three Parameter Lindley Mixing distribution

Consider the following finite mixture:

$$
g(\lambda) = \omega_1 g_1(\lambda) + \omega_2 g_2(\lambda) \tag{5.4.31}
$$

where $\omega_1+\omega_2=1$, $\omega>0, \omega_2>0$

Suppose

$$
\omega_1 = \frac{\theta}{\theta+\gamma} \Rightarrow \omega_2 1 = \frac{\gamma}{\theta+\gamma}
$$

$$
\therefore g(\lambda) = \frac{\theta}{\theta+\gamma} g_1(\lambda) + \frac{\gamma}{\theta+\gamma} g_2(\lambda) \tag{5.4.31}
$$

If
$g_1(\lambda) \sim \Gamma(\alpha,\theta)$ and $g_2(\lambda) \sim \Gamma(\alpha+1,\theta)$,

Then

$$
\begin{aligned}
g(\lambda) &= \frac{\theta}{\theta+\gamma} . \frac{\theta^\alpha}{\Gamma(\alpha)} e^{-\theta\lambda}\lambda^{\alpha-1} + \frac{\gamma}{\theta+\gamma}\frac{\theta^{\alpha+1}}{\Gamma(\alpha+1)} e^{-\theta\lambda}\lambda^{(\alpha+1)-1} \\
&= \frac{\theta^{\alpha+1}}{\theta+\gamma}\left\{\frac{e^{-\theta\lambda}}{\Gamma(\alpha)}\lambda^{\alpha-1} + \frac{\gamma}{\Gamma(\alpha+1)} e^{-\theta\lambda}\lambda^{\alpha}\right\} \\
&= \frac{\theta^{\alpha+1}}{\theta+\gamma}\left\{\frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} + \frac{\gamma\lambda^{\alpha}}{\Gamma(\alpha+1)}\right\} e^{-\theta\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\theta^{\alpha+1}}{\theta+\gamma}\left\{\frac{1}{\Gamma(\alpha)} + \frac{\gamma\lambda}{\Gamma(\alpha+1)}\right\}\lambda^{\alpha-1}e^{-\theta\lambda} \\
&= \frac{\theta^{\alpha+1}}{(\theta+\gamma)\Gamma(\alpha+1)}\{\alpha+\gamma\lambda\}\lambda^{\alpha-1}e^{-\theta\lambda} \quad \text{for } \lambda>0, \alpha>0, \gamma>0 \text{and} \theta>0 \qquad (5.4.32)
\end{aligned}
$$

This is a 3-parameter generalized Lindley distribution with the following special cases:
(i) $\alpha = \gamma = 1$,

$$
g(\lambda) = \frac{\theta^2}{1+\theta}(1+\lambda)e^{-\theta\lambda}, \quad \lambda>0, \theta>0 \qquad (5.4.33)
$$

which is the one-parameter Lindley distribution.
(ii)$\gamma = 1$

$$
g(\lambda) = \frac{\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}(\alpha+\lambda)\lambda^{\alpha-1}e^{-\theta\lambda} \qquad (5.4.34)
$$

This is a 2-parameter generalized Lindley distribution as obtained by Zakerzadeh and Dollati(2010).
(iii) $\alpha = 1$,
we have a 2-parameter generalized Lindley distribution given by

$$
g(\lambda) = \frac{\theta^2}{\theta+\gamma}(1+\gamma\lambda)e^{-\theta\lambda}, \lambda>0, \gamma, \theta>0 \qquad (5.4.35)
$$

as obtained by Bhati et al (2015).

Mean

$$
\begin{aligned}
E(\wedge) &= \int_0^\infty \frac{\theta^{\alpha+1}}{(\theta+\gamma)\Gamma(\alpha+1)}\{\alpha\lambda+\gamma\lambda^2\}\lambda^{\alpha-1}e^{-\theta\lambda}d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta+\gamma)\Gamma(\alpha+1)}\left\{\alpha\int_0^\infty \lambda^{\alpha}e^{-\theta\lambda} + \gamma\int_0^\infty \lambda^{\alpha+1}e^{-\theta\lambda}d\lambda\right\}
\end{aligned}
$$

$$= \frac{\theta^{\alpha+1}}{(\theta+\gamma)\Gamma(\alpha+1)} \left\{ \alpha \frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}} + \gamma \frac{\Gamma(\alpha+2)}{\theta^{\alpha+2}} \right\}$$

$$= \left\{ \frac{\alpha\theta^{\alpha+1}}{(\theta+\gamma)\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}} + \frac{\gamma\theta^{\alpha+1}}{(\theta+\gamma)\Gamma(\alpha+1)} \frac{\Gamma(\alpha+2)}{\theta^{\alpha+2}} \right\}$$

$$= \frac{\theta^{\alpha+1}\Gamma(\alpha+1)}{\theta+\gamma\Gamma(\alpha+1)} \frac{1}{\theta^{\alpha+1}} \left\{ \alpha + \frac{\gamma(\alpha+1)}{\theta} \right\}$$

$$= \frac{1}{\theta+1} \left\{ \alpha + \frac{\gamma(\alpha+1)}{\theta} \right\} \quad \textbf{(5.4.36)}$$

$$E[\wedge^n e^{-\wedge}] = \int_0^\infty \lambda^n e^{-\lambda} \frac{\theta^{\alpha+1}}{(\theta+\gamma)\Gamma(\alpha+1)} (\alpha+\gamma\lambda)\lambda^{\alpha-1} e^{-\theta\lambda} d\lambda$$

$$= \frac{\theta^{\alpha+1}}{(\theta+\gamma)\Gamma(\alpha+1)} \int_0^\infty \lambda^n e^{-\lambda} (\alpha+\gamma\lambda)\lambda^{\alpha-1} e^{-\theta\lambda} d\lambda$$

$$\therefore E[\wedge^n e^{-\wedge}] = \frac{\theta^{\alpha+1}}{(\theta+\gamma)\Gamma(\alpha+1)} \int_0^\infty \lambda^{n+\alpha-1} (\alpha+\gamma\lambda) e^{-(1+\theta)\lambda} d\lambda$$

$$= \frac{\theta^{\alpha+1}}{(\theta+\gamma)\Gamma(\alpha+1)} \left\{ \frac{\alpha\Gamma(n+\alpha)}{(1+\theta)^{n+\alpha}} + \frac{\gamma\Gamma(n+\alpha+1)}{(1+\theta)^{n+\alpha+1}} \right\}$$

$$= \frac{\theta^{\alpha+1}\Gamma(n+\alpha)}{(\theta+\gamma)\Gamma(\alpha)(1+\theta)^{n+\alpha}} \left\{ \alpha + \frac{\gamma(n+\alpha)}{(1+\theta)} \right\}$$

$$= \frac{\theta^{\alpha+1}\Gamma(n+\alpha)\{\alpha(1+\theta)+\gamma(n+\alpha)\}}{(\theta+\gamma)\Gamma(\alpha+1)(1+\theta)^{n+\alpha+1}}$$

$$= \frac{\theta^{\alpha+1}\Gamma(n+\alpha)\{\alpha+\alpha\theta+\gamma n+\gamma\alpha\}}{(\theta+\gamma)\Gamma(\alpha+1)(1+\theta)^{n+\alpha+1}}$$

$$= \frac{\theta^{\alpha+1}\Gamma(n+\alpha)}{(\theta+\gamma)\Gamma(\alpha+1)} \left\{ \frac{(\alpha+\gamma n)+(\theta+\gamma)\alpha}{(1+\theta)^{n+\alpha+1}} \right\}$$

$$= \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \left\{ \alpha + \frac{\alpha+\gamma n}{\theta+\gamma} \right\} \left(\frac{1}{1+\theta} \right)^n \left(\frac{\theta}{1+\theta} \right)^{\alpha+1} \quad \textbf{(5.4.37)}$$

$$\therefore Prob\{S_N = y\} = \sum_{n=1}^\infty \frac{\lambda^n}{\Gamma(n)} \frac{e^{-\lambda y} y^{n-1}}{\Gamma(n+1)} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \left\{ \alpha + \frac{\alpha+\gamma n}{\theta+\gamma} \right\} \left(\frac{1}{1+\theta} \right)^n \left(\frac{\theta}{1+\theta} \right)^{\alpha+1}$$

$$\therefore \int_0^\infty h(y)dy = \sum_{n=1}^\infty \frac{\lambda^n}{\Gamma(n)} \frac{\Gamma(n)}{\lambda^n} \frac{1}{\Gamma(n+1)} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)} \left\{ \alpha + \frac{\alpha+\gamma n}{\theta+\gamma} \right\} \left(\frac{1}{1+\theta} \right)^n \left(\frac{\theta}{1+\theta} \right)^{\alpha+1}$$

$$= \sum_{n=1}^\infty \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha+1)} \left\{ \alpha + \frac{\alpha+\gamma n}{\theta+\gamma} \right\} \left(\frac{1}{1+\theta} \right)^n \left(\frac{\theta}{1+\theta} \right)^{\alpha+1}$$

$$= \sum_{n=1}^\infty \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} \left\{ 1 + \frac{1}{\theta+\gamma} + \frac{\gamma n}{\alpha(\theta+\gamma)} \right\} \left(\frac{1}{1+\theta} \right)^n \left(\frac{\theta}{1+\theta} \right)^{\alpha+1}$$

$$= \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} \left\{ \frac{\alpha(\theta+\gamma)+\alpha+\gamma n}{\alpha(\theta+\gamma)} \right\} \left(\frac{1}{1+\theta}\right)^{n} \left(\frac{\theta}{1+\theta}\right)^{\alpha+1}$$

$$= \frac{\alpha(\theta+\gamma)+\alpha}{\alpha(\theta+\gamma)} \sum_{n=1}^{\infty} \binom{\alpha+n-1}{n} \left(\frac{1}{1+\theta}\right)^{n} \left(\frac{\theta}{1+\theta}\right)^{\alpha+1}$$

$$+ \sum_{n=1}^{\infty} \binom{\alpha+n-1}{n} \gamma n \left(\frac{\theta}{1+\theta}\right)^{\alpha+1}$$

$$= \left(1+\frac{1}{\theta+\gamma}\right) \left(\frac{\theta}{1+\theta}\right)^{\alpha+1} \sum_{n=1}^{\infty} \binom{\alpha+n-1}{n} \left(\frac{1}{1+\theta}\right)^{n}$$

$$+ \left(\frac{\theta}{1+\theta}\right)^{\alpha+1} \sum_{n=1}^{\infty} \binom{\alpha+n-1}{n} n \left(\frac{1}{1+\theta}\right)^{n}$$

$$= \left(1+\frac{1}{\theta+\gamma}\right) \left(\frac{\theta}{1+\theta}\right)^{\alpha+1} \left[\left(1-\frac{1}{1+\theta}\right)^{-\alpha} - 1\right]$$

$$+ \gamma \left(\frac{\theta}{1+\theta}\right)^{\alpha+1} \sum_{n=1}^{\infty} \frac{n\Gamma(\alpha+n)}{n!\Gamma(\alpha)} \left(\frac{1}{1+\theta}\right)^{n}$$

$$= \left(1+\frac{1}{\theta+\gamma}\right) \left(\frac{\theta}{1+\theta}\right)^{\alpha+1} \left[\left(\frac{\theta}{1+\theta}\right)^{-\alpha} - 1\right]$$

$$+ \gamma \left(\frac{\theta}{1+\theta}\right)^{\alpha+1} \sum_{n=1}^{\infty} \frac{n\Gamma(\alpha+n)}{(n-1)!\Gamma(\alpha)} \left(\frac{1}{1+\theta}\right)^{n}$$

$$\therefore \int_0^{\infty} h(y)dy = \left(1+\frac{1}{\theta+\gamma}\right) \left(\frac{\theta}{1+\theta}\right) \left(1-\left(\frac{\theta}{1+\theta}\right)^{\alpha}\right)$$

$$+ \gamma \left(\frac{\theta}{1+\theta}\right)^{\alpha+1} \frac{\alpha}{1+\theta} \sum_{n=1}^{\infty} \binom{\alpha+n-1}{n-1} \left(\frac{1}{1+\theta}\right)^{n-1}$$

$$= \left(1+\frac{1}{\theta+\gamma}\right) \frac{\theta}{1+\theta} \left[1-\left(\frac{\theta}{1+\theta}\right)^{\alpha}\right]$$

$$+ \gamma \left(\frac{\theta}{1+\theta}\right)^{\alpha+1} \frac{\alpha}{1+\theta} \sum_{n=1}^{\infty} \binom{(\alpha+1)+(n-1)-1}{n-1} \left(\frac{1}{1+\theta}\right)^{n-1} \quad \textbf{(5.4.38)}$$

$$\therefore \int_0^{\infty} h(y)dy = \left(1+\frac{1}{\theta+\gamma}\right) \frac{\theta}{1+\theta} \left[1-\left(\frac{\theta}{1+\theta}\right)^{\alpha}\right]$$

$$+ \gamma \left(\frac{\theta}{1+\theta}\right)^{\alpha+1} \frac{\alpha}{1+\theta} \sum_{n=1}^{\infty} (-1)^{n-1} \binom{\alpha+1+n-1-1}{n-1} \left(\frac{1}{1+\theta}\right)^{n-1}$$

$$= \left(1+\frac{1}{\theta+\gamma}\right) \left[\frac{\theta}{1+\theta} - \left(\frac{\theta}{1+\theta}\right)^{\alpha+1}\right]$$

$$+ \gamma \left(\frac{\theta}{1+\theta}\right)^{\alpha+1} \frac{\alpha}{1+\theta} \left(1-\frac{1}{1+\theta}\right)^{-(\alpha+1)} \quad \textbf{(5.4.39)}$$

as required.

6 HYPO-EXPONENTIAL RANDOM VARIABLES WITH DISTINCT PARAMETERS

6.1 Introduction

In this chapter we will consider the sum of $n \geq 2$ exponential random variables with distinct parameters, obtain their distributions and study the properties.

For this case let us consider the distribution $X_i^{\prime}s$, where $i = 1, 2, \cdots, n$ independent exponential random variables with parameters λ_i, for $i = 1, 2, \cdots, n$ with $\lambda_i \neq \lambda_j$, and $i \neq j$ written as $\lambda_1, \lambda_2, \cdots, \lambda_n$. These kind of sum of exponential random variables, $\sum_{i=1}^{n} X_i$, are called hypo-exponential random variables.

Let

$$S_n = X_1 + X_2 + \cdots + X_n \tag{6.1.1}$$

where $X_{i's}$ are independent exponential random variables with parameters λ_i for $i = 1, 2, \cdots, n$.

Consequently, we will also consider circumstances where these hypo-exponential random variables with distinct parameters form an arithmetic sequences and geometric sequences and hence derive some of their general results.

6.2 Hypo-exponential Distribution of two Independent Random Variables with Distinct Parameters

6.2.1 Construction Using Convolution Approach

In this section we consider the distributions of the hypo-exponential random variables with two different parameters. Therefore let

$$S_2 = X_1 + X_2 \tag{6.2.1}$$

Let $F_i(x_i)$ and $G(s_2)$ be the cumulative distribution function for X_i and S_2 respectively.

Cumulative Density Function

From the definition the cumulative density function is given by:

$$
\begin{aligned}
G(s_2) &= P(S_2 \leq s_2) \\
&= Prob(X_1 + X_2 \leq s_2) \\
&= Prob(X_2 \leq s_2 - x_1) \\
&= Prob(0 \leq X_1 < \infty, 0 \leq X_2 < s_2 - x_1) \\
&= Prob(0 \leq X_1 < s_2, 0 \leq X_2 < s_2 - x_1) \\
&= \int_0^{s_2} \int_0^{s_2 - x_1} f_1(x_1) f_2(x_2) dx_2 dx_1 \\
&= \int_0^{s_2} f_1(x_1) \left[\int_0^{s_2 - x_1} f_2(x_2) dx_2 \right] dx_1 \\
&= \int_0^{s_2} \lambda_1 e^{-\lambda_1 x_1} \left[\int_0^{s_2 - x_1} \lambda_2 e^{-\lambda_2 x_2} dx_2 \right] dx_1 \\
&= \int_0^{s_2} \lambda_1 e^{-\lambda_1 x_1} \left[-e^{-\lambda_2 x_2} \right]_0^{s_2 - x_1} dx_1 \\
&= \int_0^{s_2} \lambda_1 e^{-\lambda_1 x_1} \left[1 - e^{-\lambda_2 s_2 + \lambda_2 x_1} \right] dx_1 \\
&= \int_0^{s_2} \left[\lambda_1 e^{-\lambda_1 x_1} - \lambda_1 e^{-\lambda_2 s_2} e^{x_1(\lambda_2 - \lambda_1)} \right] dx_1 \\
&= \int_0^{s_2} \lambda_1 e^{-\lambda_1 x_1} dx_1 - \lambda_1 e^{-\lambda_2 s_2} \int_0^{s_2} e^{x_1(\lambda_2 - \lambda_1)} dx_1 \\
&= 1 - e^{-\lambda_1 s_2} - \lambda_1 e^{-\lambda_2 s_2} \left[\frac{e^{(\lambda_2 - \lambda_1) s_2} - 1}{\lambda_2 - \lambda_1} \right] \\
&= 1 - e^{-\lambda_1 s_2} + \frac{\lambda_1 e^{-\lambda_2 s_2}}{\lambda_2 - \lambda_1} - \frac{\lambda_1 e^{-\lambda_1 s_2}}{\lambda_2 - \lambda_1} \\
\therefore G(s_2) &= 1 + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 s_2} - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 s_2} \qquad \textbf{(6.2.2)}
\end{aligned}
$$

Therefore from the above equation (6.2.2) it can be concluded that

$$\lim_{s_2 \to \infty} G(s_2) = 1$$

and

$$\lim_{s_2 \to 0} G(s_2) = 0 \qquad \textbf{(6.2.3)}$$

Probability Density Function

From equation (6.2.2) we can derive the probability density function of the hypo-exponential random variables with two parameters. This follows that:

$$
\begin{aligned}
g(s_2) &= \frac{d}{ds_2} G(s_2) \\
&= \frac{d}{ds_2}\left[1 + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 s_2} - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 s_2}\right] \\
&= \left(\frac{\lambda_1}{\lambda_2 - \lambda_1}\right) - \lambda_2 e^{-\lambda_2 s_2} + \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right) \lambda_1 e^{-\lambda_1 s_2} \\
&= \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right) \lambda_2 e^{-\lambda_2 s_2} + \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right) \lambda_1 e^{-\lambda_1 s_2}
\end{aligned}
$$

$$
\text{Hence } g(s_2) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 s_2} - e^{-\lambda_2 s_2}\right) \quad \text{where } s_2 > 0,\ \lambda_2 > \lambda_1 > 0 \qquad \textbf{(6.2.4)}
$$

as required.

Hazard Function

From the definition, the hazard function is given by:

$$
h(s_2) = \frac{g(s_2)}{1 - G(s_2)}
$$

Using the equation (5.2.4)

$$
\begin{aligned}
h(s_2) &= \frac{\lambda_1 \lambda_2 (e^{-\lambda_1 s_2} - e^{-\lambda_2 s_2})}{\lambda_2 - \lambda_1} \cdot \frac{\lambda_2 - \lambda_1}{\lambda_2 e^{-\lambda_1 s_2} - \lambda_1 e^{-\lambda_2 s_2}} \\
&= \lambda_1 \lambda_2 \left[\frac{e^{-\lambda_1 s_2} - e^{-\lambda_2 s_2}}{\lambda_2 e^{-\lambda_1 s_2} - \lambda_1 e^{-\lambda_2 s_2}}\right] \qquad \textbf{(6.2.5)}
\end{aligned}
$$

Validity of the Model

The model $g(s_2)$ is said to be valid if it satisfies the condition,

$$
\int_0^\infty g(s_2) ds_2 = 1
$$

Proof

From the equation (6.2.4)

$$
g(s_2) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 s_2} - e^{-\lambda_2 s_2}\right)
$$

$$\begin{aligned}\Rightarrow \int_0^\infty g(s_2)ds_2 &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\int_0^\infty \left(e^{-\lambda_1 s_2}-e^{-\lambda_2 s_2}\right)ds_2\\ &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[\frac{1}{\lambda_2}e^{-\lambda_2 s_2}-\frac{1}{\lambda_1}e^{-\lambda_1 s_2}\right]_0^\infty\\ &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2}\right]\\ &= 1 \end{aligned} \quad \textbf{(6.2.6)}$$

As required, hence the model is said to be valid.

6.2.2 Construction Using Moment Generating Function Approach

The moment generating function for hypo-exponential random variables S_2 is given by:

$$M_{s_2}(t) = E(e^{ts_2}); \qquad \text{where} \quad S_2 = X_1 + X_2 \quad \textbf{(6.2.7)}$$

From the moment generating function properties then

$$\begin{aligned} M_{s_2}(t) &= M_{X_1+X_2}(t)\\ &= E(e^{tx_1})\,\mathbf{.}\,E(e^{tx_2})\\ &= \left(\frac{\lambda_1}{\lambda_1-t}\right)\left(\frac{\lambda_2}{\lambda_2-t}\right)\\ \Rightarrow M_{s_2}(t) &= \frac{\lambda_1\lambda_2}{(\lambda_1-t)(\lambda_2-t)}\\ &= \lambda_1\lambda_2[(\lambda_1-t)(\lambda_2-t)]^{-1}\\ &= \lambda_1(\lambda_1-t)^{-1}\lambda_2(\lambda_2-t)^{-1}\\ &= \prod_{i=1}^{2}\lambda_i[\lambda_i-t]^{-1} \end{aligned} \quad \textbf{(6.2.8)}$$

6.2.3 Properties of the distribution

Moments

The r^{th} raw moment of the hypo-exponential random variables with two parameters can be given by:

$$E[S_2^r] = \frac{d^r M_{s_2}(t)}{dt^r}|_{t=0}$$

But from equation (6.2.7) we have

$$\begin{aligned} M_{s_2}(t) &= M_{X_1+X_2}(t) \\ &= E(e^{tx_1}) \cdot E(e^{tx_2}) \end{aligned}$$

From this the first four moments can be obtained as follows:

First moment

$$\begin{aligned} E[S_2] &= M'_{s_2}(0) \\ &= \frac{d}{dt}\lambda_1\lambda_2[(\lambda_1-t)^{-1}(\lambda_2-t)^{-1}]|_{t=0} \\ &= \lambda_1\lambda_2\left[(-1)(\lambda_1-t)^{-2}(-1)(\lambda_2-t)^{-1}+(-1)(\lambda_2-t)^{-2}(-1)(\lambda_1-t)^{-1}\right]|_{t=0} \\ &= \lambda_1\lambda_2\left[(\lambda_1-t)^{-2}(\lambda_2-t)^{-1}+(\lambda_2-t)^{-2}(\lambda_1-t)^{-1}\right]|_{t=0} \\ &= \lambda_1\lambda_2\left[\frac{1}{\lambda_1^2\lambda_2}+\frac{1}{\lambda_2^2\lambda_1}\right] \\ &= \frac{1}{\lambda_1}+\frac{1}{\lambda_2} \\ &= \sum_{i=1}^{2}\lambda_i^{-i} \end{aligned} \quad \textbf{(6.2.9)}$$

Second moment

$$\begin{aligned} E[S_2^2] &= M''_{s_2}(0) \\ &= \frac{d}{dt}\lambda_1\lambda_2\left[(\lambda_1-t)^{-2}(\lambda_2-t)^{-1}+(\lambda_2-t)^{-2}(\lambda_1-t)^{-1}\right]|_{t=0} \\ &= \lambda_1\lambda_2\left[2(\lambda_1-t)^{-3}(\lambda_2-t)^{-1}+(\lambda_2-t)^{-2}(\lambda_1-t)^{-2}+2(\lambda_2-t)^{-3}(\lambda_1-t)^{-1}\right]|_{t=0} \\ &+\lambda_1\lambda_2\left[(\lambda_1-t)^{-2}(\lambda_2-t)^{-2}\right]|_{t=0} \\ &= \lambda_1\lambda_2\left[2(\lambda_1)^{-3}(\lambda_2)^{-1}+(\lambda_2)^{-2}(\lambda_1)^{-2}+2(\lambda_2)^{-3}(\lambda_1)^{-1}+(\lambda_1)^{-2}(\lambda_2)^{-2}\right] \\ &= \lambda_1\lambda_2\left[\frac{2}{\lambda_1^3\lambda_2}+\frac{2}{\lambda_1^2\lambda_2^2}+\frac{2}{\lambda_1\lambda_2^3}\right] \\ &= 2\left[\frac{1}{\lambda_1^2}+\frac{1}{\lambda_1\lambda_2}+\frac{1}{\lambda_2^2}\right] \\ &= 2\sum_{i=0}^{2}\lambda_1^{-(2-i)}\lambda_2^{-i} \end{aligned} \quad \textbf{(6.2.10)}$$

Third moment

$$
\begin{aligned}
E[S_2^3] &= M_{s_2}^3(0) \\
&= \frac{d}{dt} 2\lambda_1\lambda_2\left[(\lambda_1 - t)^{-3}(\lambda_2 - t)^{-1} + (\lambda_2 - t)^{-2}(\lambda_1 - t)^{-2} + (\lambda_2 - t)^{-3}(\lambda_1 - t)^{-1}\right]|_{t=0} \\
&= 2\lambda_1\lambda_2\left[3(\lambda_1 - t)^{-4}(\lambda_2 - t)^{-1} + (\lambda_2 - t)^{-2}(\lambda_1 - t)^{-3} + 2(\lambda_1 - t)^{-3}(\lambda_2 - t)^{-2}\right]|_{t=0} \\
&+ 2\lambda_1\lambda_2\left[2(\lambda_2)^{-3}(\lambda_1 - t)^{-2} + 3(\lambda_2 - t)^{-4}(\lambda_1 - t)^{-1} + (\lambda_1 - t)^{-2}(\lambda_2 - t)^{-3}\right]|_{t=0} \\
&= 2\lambda_1\lambda_2\left[3(\lambda_1)^{-4}(\lambda_2)^{-1} + 3(\lambda_2)^{-2}(\lambda_1)^{-3} + 3(\lambda_2)^{-3}(\lambda_1)^{-2} + 3(\lambda_2)^{-4}(\lambda_1)^{-1}\right] \\
&= 6\left[\frac{1}{\lambda_1^3} + \frac{1}{\lambda_1^2\lambda_2} + \frac{1}{\lambda_1\lambda_2^2} + \frac{1}{\lambda_2^3}\right] \\
&= 3!\left[\frac{1}{\lambda_1^3} + \frac{1}{\lambda_1^2\lambda_2} + \frac{1}{\lambda_1\lambda_2^2} + \frac{1}{\lambda_2^3}\right] \\
&= 3!\left[\lambda_1^{-3} + \lambda_1^{-2}\lambda_2 + \lambda_1\lambda_2^{-2} + \lambda_2^{-3}\right]
\end{aligned}
$$

$$
\therefore E[S_2^3] = 3!\sum_{i=0}^{3} \lambda_1^{-(3-i)}\lambda_2^{-i} \qquad \textbf{(6.2.11)}
$$

Fourth moment

$$
\begin{aligned}
E[S_2^4] &= M_{s_2}^4(0) \\
&= \frac{d}{dt} 6\lambda_1\lambda_2\left[(\lambda_1 - t)^{-4}(\lambda_2 - t)^{-1} + (\lambda_2 - t)^{-2}(\lambda_1 - t)^{-3} + (\lambda_2 - t)^{-3}(\lambda_1 - t)^{-2}\right]|_{t=0} \\
&+ \frac{d}{dt} 6\lambda_1\lambda_2\left[(\lambda_2)^{-4}(\lambda_1 - t)^{-1}\right]|_{t=0} \\
&= 6\lambda_1\lambda_2\left[4(\lambda_1 - t)^{-5}(\lambda_2 - t)^{-1} + (\lambda_2 - t)^{-2}(\lambda_1 - t)^{-4} + 2(\lambda_2 - t)^{-3}(\lambda_1 - t)^{-3}\right]|_{t=0} \\
&+ 6\lambda_1\lambda_2\left[3(\lambda_1)^{-4}(\lambda_2 - t)^{-4}(\lambda_2 - t)^{-2} + 3(\lambda_2 - t)^{-4}(\lambda_1 - t)^{-2}\right]|_{t=0} \\
&+ 6\lambda_1\lambda_2\left[2(\lambda_1 - t)^{-3}(\lambda_2 - t)^{-3} + 4(\lambda_2 - t)^{-5}(\lambda_1 - t)^{-1} + (\lambda_1 - t)^{-2}(\lambda_2 - t)^{-4}\right]|_{t=0} \\
&= 6\lambda_1\lambda_2\left[4\lambda_1^{-5}\lambda_2^{-1} + 4\lambda_1^{-4}\lambda_2^{-2} + 4\lambda_1^{-3}\lambda_2^{-3} + 4\lambda_1^{-2}\lambda_2^{-4} + 4\lambda_1^{-1}\lambda_2^{-5}\right] \\
&= 24\lambda_1\lambda_2\left[\frac{1}{\lambda_1^5\lambda_2} + \frac{1}{\lambda_1^4\lambda_2^2} + \frac{1}{\lambda_1^3\lambda_2^3} + \frac{1}{\lambda_1^2\lambda_2^4} + \frac{1}{\lambda_1\lambda_2^5}\right] \\
&= 24\left[\frac{1}{\lambda_1^5\lambda_2} + \frac{1}{\lambda_1^4\lambda_2^2} + \frac{1}{\lambda_1^3\lambda_2^3} + \frac{1}{\lambda_1^2\lambda_2^4} + \frac{1}{\lambda_1\lambda_2^5}\right] \\
&= 4!\left[\lambda_1^{-4} + \lambda_1^{-3}\lambda_2^{-1} + \lambda_1^{-2}\lambda_2^{-2} + \lambda_1^{-1}\lambda_2^{-3} + \lambda_2^{-4}\right]
\end{aligned}
$$

$$
\therefore E[S_2^4] = 4!\sum_{i=0}^{4} \lambda_1^{-(4-i)}\lambda_2^{-i} \qquad \textbf{(6.2.12)}
$$

This implies that the r^{th} raw moment for S_2 can be generally expressed as

$$
E[S_2^r] = r!\sum_{i=0}^{r} \lambda_1^{-(r-i)}\lambda_2^{-i} \qquad \textbf{(6.2.13)}
$$

Therefore from the above properties of the hypo-exponential random variables with two different parameters the following can be obtained:

Mean

$$E[S_2] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \tag{6.2.14}$$

Variance

$$\begin{aligned}
Var[S_2] &= E[S_2^2] - (E[S_2])^2 \\
&= 2\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2^2}\right) - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)^2 \\
&= \frac{2}{\lambda_1^2} + \frac{2}{\lambda_1\lambda_2} + \frac{2}{\lambda_2^2} - \frac{1}{\lambda_1^2} - \frac{2}{\lambda_1\lambda_2} - \frac{1}{\lambda_2^2} \\
&= \frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}
\end{aligned} \tag{6.2.15}$$

Mode

The mode for the model is given when

$$\frac{d}{ds_2} g(s_2) = 0$$

This implies that

$$\begin{aligned}
\frac{d}{dS_2}\left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)\lambda_2 e^{-\lambda_2 s_2} &= -\left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)\lambda_1 e^{-\lambda_1 s_2} \\
-\left(\frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\right)\lambda_2 e^{-\lambda_2 s_2} &= \left(\frac{\lambda_2\lambda_1}{\lambda_2 - \lambda_1}\right)\lambda_1 e^{-\lambda_1 s_2} \\
\Rightarrow -\lambda_1 e^{-\lambda_2 s_2} &= \lambda_2 e^{-\lambda_1 s_2} \\
-ln\lambda_1 + \lambda_2 s_2 &= ln\lambda_2 - \lambda_1 s_2 \\
\lambda_1 s_2 + \lambda_2 s_2 &= ln\lambda_2 + ln\lambda_1 \\
\therefore s_2 &= \frac{ln\lambda_2 + ln\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned} \tag{6.16}$$

Asymptotic Behaviour of the Model

In seeking the asymptotic behaviour of the model formed in equation (6.2.4) the we consider the behaviour of the model when $s_2 \to 0$ and as $s_2 \to \infty$. That is

$$\lim_{s_2 \to 0} g(s_2) \text{ and } \lim_{s_2 \to \infty} g(s_2)$$

Therefore,

$$\begin{aligned}\lim_{s_2\to 0} g(s_2) &= \lim_{s_2\to 0}\left[\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 s_2}-e^{-\lambda_2 s_2}\right)\right]\\ &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\lim_{s_2\to 0} e^{-\lambda_1 s_2}-\lim_{s_2\to 0} e^{-\lambda_2 s_2}\right)\\ &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}(1-1)\\ &= 0\end{aligned}$$

and also

$$\begin{aligned}\lim_{S_2\to \infty} g(s_2) &= \lim_{S_2\to \infty}\left[\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 s_2}-e^{-\lambda_2 s_2}\right)\right]\\ &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\lim_{s_2\to \infty} e^{-\lambda_1 s_2}-\lim_{s_2\to \infty} e^{-\lambda_2 s_2}\right)\\ &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}(0-0)\\ &= 0\end{aligned} \tag{6.2.17}$$

From the two results it confirms that from the equation (6.2.4) the model has only one mode that is Uni-modal model.

Cumulant Generating Function

The cumulant generating function of a random variable S_2 can be obtained by:

$$\begin{aligned}C_{s_2}(t) &= log[M_{s_2(t)}]\\ &= log\left\{\frac{\lambda_1\lambda_2}{(\lambda_1-t)(\lambda_2-t)}\right\}\end{aligned}$$

Using Maclaurin series through expansion the following equation can be obtained:

$$C_{s_2}(t) = \sum_{i=1}^{\infty}(i-1)!\left[\left(\frac{1}{\lambda_1}\right)^i+\left(\frac{1}{\lambda_2}\right)^i\right]\frac{t^i}{i!} \tag{6.2.18}$$

From the definition of cumulants, the cumulant K_i of a random variable S_2 can be obtained using the coefficients of $\frac{t^i}{i!}$ in equation (6.2.18).

Hence,

$$K_i = (i-1)!\left[\left(\frac{1}{\lambda_1}\right)^i + \left(\frac{1}{\lambda_2}\right)^i\right] \quad \textbf{(6.2.19)}$$

Therefore the first four cumulants can be given as follows:

$$\begin{aligned} K_1 &= (1-1)!\left[\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right] \\ &= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \end{aligned}$$

$$\begin{aligned} K_2 &= (2-1)!\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2\right] \\ &= \left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 \end{aligned}$$

$$\begin{aligned} K_3 &= (3-1)!\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3\right] \\ &= 2\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3\right] \end{aligned}$$

$$\begin{aligned} K_4 &= (4-1)!\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4\right] \\ &= 6\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4\right] \end{aligned}$$

From the above results the following conclusions can be made:

(i) K_1 which forms the first cumulant gives the mean of the random variable S_2.

(ii) K_2 which forms the second cumulant gives the variance of the random variable S_2.

(iii) Skewness

$$= \frac{K_3}{K_2^{\frac{3}{2}}}$$

$$= \frac{2\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2\right]^{3/2}} \quad \textbf{(6.2.20)}$$

(iv) Kurtosis

$$\begin{aligned} &= \frac{K_4}{K_2^2} \\ &= \frac{6\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2\right]^2} \end{aligned} \quad \textbf{(6.2.21)}$$

6.2.4 Estimation Using Method of Moments

Let us consider a hypo-exponential distribution with two parameter rates λ_1 and λ_2 respectively. Assuming that $\lambda_1 < \lambda_2$ and considering a random sample of size n denoted by $S_{12}, S_{22}, S_{23}, \cdots, S_{2n}$ to indicate the total observed service time.

From subsection (6.2.3)the first two raw moments fr the sample m_1 and m_2 are given as:

$$m_1 = E[S_2] \quad \text{and} \quad m_2 = E[S_2^2]$$

Therefore using equation (6.2.14) and (6.2.15) we get:

$$\text{Mean} = E[S_2] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = m_1$$

and

$$\begin{aligned} \text{Variance} &= Var[S_2] \\ &= E[S_2^2] - (E[S_2])^2 \\ &= \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \\ &= m_2 - m_1^2 \end{aligned}$$

Let us now denote $\frac{1}{\lambda_i} = t_i$ where t_i represents the time taken in i^{th} stage.

Therefore from the assumption $\lambda_1 < \lambda_2$ it implies that $t_1 > t_2$.

Substituting the values of λ_1 and λ_2, the following equations are obtained:

$$t_1 + t_2 = m_1 \Rightarrow t_2 = m_1 - t_1 \quad \ldots\ldots\ldots\ldots \quad \textbf{(i)}$$

and

$$\begin{aligned} t_1^2 + t_2^2 &= m_2 - m_1^2 \quad \ldots\ldots\ldots\ldots \quad \textbf{(ii)} \\ \Rightarrow t_1^2 + (m_1 - t_1)^2 &= m_2 - m_1^2 \\ \Rightarrow t_1^2 + m_1^2 - 2m_1t_1 + t_1^2 &= m_2 - m_1^2 \\ \Rightarrow 2t_1^2 - 2m_1t_1 + 2m_1^2 - m_2 &= 0 \end{aligned}$$

Using the quadratic equation formula we get

$$t_1 = \frac{m_1 \pm \sqrt{2m_2 - 3m_1^2}}{2}$$

From equation (i) above then we get

$$\begin{aligned} t_2 &= m_1 - \frac{m_1 \pm \sqrt{2m_2 - 3m_1^2}}{2} \\ &= \frac{m_1 \mp \sqrt{2m_2 - 3m_1^2}}{2} \end{aligned}$$

Therefore from the assumption that $t_1 > t_2$ then the estimated values will be given as:

$$(\hat{t_1}, \hat{t_2}) = \left(\frac{m_1 + \sqrt{2m_2 - 3m_1^2}}{2}, \frac{m_1 - \sqrt{2m_2 - 3m_1^2}}{2} \right)$$

Now substituting the values of t_1 and t_2, therefore the parameter rates for the hypo-exponential distribution with two parameter rates can be estimated as:

$$(\hat{\lambda_1}, \hat{\lambda_2}) = \left(\frac{2}{m_1 + \sqrt{2m_2 - 3m_1^2}}, \frac{2}{m_1 - \sqrt{2m_2 - 3m_1^2}} \right)$$

6.2.5 The Case of Arithmetic Sequence for two distinct Parameters

Using the equation (6.2.4) and given that the model parameters takes an arithmetic sequence that is $\lambda_2 = \lambda_1 + d$ where d is the common difference, Therefore

$$\begin{aligned}
\text{From } g(s_2) &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1 s_2} + \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{-\lambda_2 s_2} \\
\Rightarrow g(s_2) &= \frac{\lambda_1\lambda_2}{(\lambda_1+d)-\lambda_1}e^{-\lambda_1 s_2} + \frac{\lambda_1\lambda_2}{\lambda_1-(\lambda_1+d)}e^{-\lambda_2 s_2} \\
&= \frac{\lambda_1\lambda_2}{d}e^{-\lambda_1 s_2} + \frac{\lambda_1\lambda_2}{-d}e^{-\lambda_2 s_2} \\
&= \lambda_1\lambda_2\left(\frac{e^{-\lambda_1 s_2}}{d} - \frac{e^{-\lambda_2 s_2}}{d}\right)
\end{aligned} \tag{6.2.24}$$

6.2.6 The Case of Geometric Sequence for two distinct Parameters

From the equation (6.2.4) if the model parameters takes the form of geometric sequence with the a common ratio r given by $\frac{\lambda_2}{\lambda_1}$ implying that $\lambda_2 = r\lambda_1$. Therefore

$$\begin{aligned}
\text{From } g(s_2) &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1 s_2} + \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{-\lambda_2 s_2} \\
&= \left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\lambda_1 e^{-\lambda_1 s_2} + \left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\lambda_2 e^{-\lambda_2 s_2} \\
\Rightarrow g(s_2) &= \left(\frac{r\lambda_1}{r\lambda_1-\lambda_1}\right)\lambda_1 e^{-\lambda_1 s_2} + \left(\frac{\lambda_1}{\lambda_1-r\lambda_1}\right)\lambda_2 e^{-\lambda_2 s_2} \\
&= \frac{\lambda_1 e^{-\lambda_1 s_2}}{1-r^{-1}} + \frac{\lambda_2 e^{-\lambda_2 s_2}}{1-r}
\end{aligned} \tag{6.2.25}$$

6.3 Hypo-exponential Distribution of three independent random variables with distinct parameters

6.3.1 Construction Using Convolution Approach

In this sub-section we want to consider the distributions for the sum of three independent exponential random variables with distinct parameters.

Let

$$\begin{aligned}
S_3 &= X_1 + X_2 + X_3 \\
S_3 &= S_2 + X_3, \qquad \text{where} \quad S_2 = X_1 + X_2
\end{aligned} \tag{6.3.1}$$

Let $F_i(x_i)$ and $G(s_3)$ be the cumulative distribution function for X_i and S_3 respectively.

Cumulative Density Function

From the definition the cumulative density function is given by:

$$
\begin{aligned}
G(S_3) &= Prob(S_3 \leq s_3) \\
&= Prob(S_2 + X_3 \leq s_3) \\
&= Prob(X_3 \leq s_3 - s_2) \\
&= Prob(0 \leq s_2 \leq s_3, 0 \leq X_3 \leq s_3 - s_2) \\
&= \int_0^{s_3} \int_0^{s_3 - s_2} g(s_2) f(x_3) dx_3 ds_2 \\
&= \int_0^{s_3} g(s_2) \left[\int_0^{s_3 - s_2} f(x_3) dx_3 \right] ds_2 \\
&= \int_0^{s_3} g(s_2) \left[\int_0^{s_3 - s_2} \lambda_3 e^{-\lambda_3 x_3} dx_3 \right] ds_2 \\
&= \int_0^{s_3} g(s_2) \left[1 - e^{-\lambda_3 (s_3 - s_2)} \right] ds_2 \\
&= \int_0^{s_3} \left[g(s_2) - g(s_2) e^{-\lambda_3 s_3 + \lambda_3 s_2)} \right] ds_2 \\
&= \left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \right) \int_0^{s_3} \left(e^{-\lambda_1 s_2} - e^{-\lambda_2 s_2} \right) ds_2 - \left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \right) \int_0^{s_3} \left(e^{-\lambda_1 s_2} - e^{-\lambda_2 s_2} \right) e^{-\lambda_3 s_3 + \lambda_3 s_2} ds_2 \\
&= \left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \right) \int_0^{s_3} e^{-\lambda_1 s_2} - \left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \right) \int_0^{s_3} e^{-\lambda_2 s_2} ds_2 \\
&- \left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \right) e^{-\lambda_3 s_3} \int_0^{s_3} e^{(\lambda_3 - \lambda_1) s_2} ds_2 + \left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \right) e^{-\lambda_3 s_3} \int_0^{s_3} e^{(\lambda_3 - \lambda_2) s_2} ds_2 \\
&= \left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \right) \left(\frac{1 - e^{-\lambda_1 s_3}}{\lambda_1} \right) - \left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \right) \left(\frac{1 - e^{-\lambda_2 s_3}}{\lambda_2} \right) \\
&- \left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \right) e^{-\lambda_3 s_3} \left(\frac{e^{(\lambda_3 - \lambda_1) s_3} - 1}{\lambda_3 - \lambda_1} \right) + \left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \right) e^{-\lambda_3 s_3} \left(\frac{e^{(\lambda_3 - \lambda_2) s_3} - 1}{\lambda_3 - \lambda_2} \right) \\
&= \frac{\lambda_2}{\lambda_2 - \lambda_1} - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 s_3} - \frac{\lambda_1}{\lambda_2 - \lambda_1} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 s_3} - \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} e^{-\lambda_1 s_3} \\
&+ \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} e^{-\lambda_3 s_3} + \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} e^{\lambda_2 s_3} - \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2} e^{-\lambda_3 s_3} \\
&= 1 + \frac{-\lambda_2(\lambda_3 - \lambda_1) - \lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} e^{-\lambda_1 s_3} + \frac{\lambda_1(\lambda_2 - \lambda_3) - \lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} e^{-\lambda_2 s_3} \\
&+ \frac{\lambda_1 \lambda_2 (\lambda_3 - \lambda_2) - \lambda_1 \lambda_2 (\lambda_3 - \lambda_1)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} e^{-\lambda_3 s_3}
\end{aligned}
$$

$$\therefore G(S_3) = 1 - \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_1}\right)e^{-\lambda_1 s_3} - \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_2}\right)e^{-\lambda_2 s_3}$$
$$- \left(\frac{\lambda_1}{\lambda_1 - \lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2 - \lambda_3}\right)e^{-\lambda_3 s_3} \qquad \textbf{(6.3.2)}$$

Therefore, from the above equation (6.3.2) it can be concluded that

$$\lim_{s_3 \to \infty} G(s_3) = 1$$
$$\text{and}$$
$$\lim_{s_3 \to 0} G(s_3) = 0 \qquad \textbf{(6.3.3)}$$

Probability Density Function

The probability density function of the sum of three independent exponential random variables can be obtained from the derivative of $G(s_3)$ in equation (6.3.2) with respect to s_3. That is:

$$\begin{aligned}
g(s_3) &= \frac{d}{ds_3}\left[G(s_3)\right] \\
&= \frac{d}{ds_3}\left[1 - \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_1}\right)e^{-\lambda_1 s_3} - \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_2}\right)e^{-\lambda_2 s_3}\right] \\
&\quad - \frac{d}{ds_3}\left[\left(\frac{\lambda_1}{\lambda_1 - \lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2 - \lambda_3}\right)e^{-\lambda_3 s_3}\right] \\
&= \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_1}\right)\lambda_1 e^{-\lambda_1 s_3} + \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_2}\right)\lambda_2 e^{-\lambda_2 s_3} \\
&\quad + \left(\frac{\lambda_1}{\lambda_1 - \lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2 - \lambda_3}\right)\lambda_3 e^{-\lambda_3 s_3} \\
\therefore g(s_3) &= \sum_{i=1}^{3}\left\{\lambda_i e^{-\lambda_i s_3}\prod_{j=1, j\neq i}^{3}\left(\frac{\lambda_j}{\lambda_j - \lambda_i}\right)\right\} \quad \text{where } s_3 > 0,\ \lambda_3 > \lambda_2 > \lambda_1 > 0 \qquad \textbf{(6.3.4)}
\end{aligned}$$

Hazard Function

From the definition, the hazard function is given by:

$$h(s_3) = \frac{g(s_3)}{1-G(s_3)}$$

But

$$1-G(s_3) = 1-1+\left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_1}\right)e^{-\lambda_1 s_3}+\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_2}\right)e^{-\lambda_2 s_3} + \left(\frac{\lambda_1}{\lambda_1-\lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_3}\right)e^{-\lambda_3 s_3}$$

$$\begin{aligned}
&= \left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_1}\right)e^{-\lambda_1 s_3}+\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_2}\right)e^{-\lambda_2 s_3} \\
&\quad + \left(\frac{\lambda_1}{\lambda_1-\lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_3}\right)e^{-\lambda_3 s_3} \\
&= \sum_{i=1}^{3} e^{-\lambda_i s_3} \prod_{j=1, j\neq i}^{3}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right) \\
\Rightarrow h(s_3) &= \frac{\sum_{i=1}^{3}\left\{\lambda_i e^{-\lambda_i s_3}\prod_{j=1,j\neq i}^{3}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\right\}}{\sum_{i=1}^{3} e^{-\lambda_i s_3}\prod_{j=1,j\neq i}^{3}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)} \\
&= \sum_{i=1}^{3}\lambda_i \qquad \textbf{(6.3.5)}
\end{aligned}$$

Validity of the Model

The model $g(s_3)$ is said to be valid if it satisfies the condition

$$\int_0^\infty g(s_3)ds_3 = 1$$

Proof

$$\begin{aligned}
\int_0^\infty g(s_3)ds_3 &= \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\int_0^\infty\left\{(\lambda_3-\lambda_2)e^{-\lambda_1 s_3}-(\lambda_3-\lambda_1)e^{-\lambda_2 s_3}\right\}ds_3 \\
&\quad + \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\int_0^\infty(\lambda_2-\lambda_1)e^{-\lambda_3 s_3}ds_3 \\
&= \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\left[\frac{\lambda_3-\lambda_2}{\lambda_1}-\frac{\lambda_3-\lambda_1}{\lambda_2}+\frac{\lambda_2-\lambda_1}{\lambda_3}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} - \frac{\lambda_1\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} + \frac{\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \\
&= \frac{\lambda_2\lambda_3(\lambda_3-\lambda_2)-\lambda_1\lambda_3(\lambda_3-\lambda_1)+\lambda_1\lambda_2(\lambda_2-\lambda_1)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \\
&= \frac{\lambda_2\lambda_3^2-\lambda_2^2\lambda_3-\lambda_1\lambda_3^2-\lambda_1^2\lambda_3+\lambda_1\lambda_2^2-\lambda_1^2\lambda_2}{\lambda_2\lambda_3^2-\lambda_2^2\lambda_3-\lambda_1\lambda_3^2-\lambda_1^2\lambda_3+\lambda_1\lambda_2^2-\lambda_1^2\lambda_2} \\
&= 1
\end{aligned}
$$

Alternatively

$$
\begin{aligned}
\int_0^\infty g(s_3)ds_3 &= \int_0^\infty \sum_{i=1}^{3}\left\{\lambda_i e^{-\lambda_i s_3}\prod_{j=1,j\neq i}^{3}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\right\}ds_3 \\
&= \sum_{i=1}^{3}\lambda_i\prod_{j=1,j\neq i}^{3}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\int_0^\infty e^{-\lambda_i s_3}ds_3 \\
&= \sum_{i=1}^{3}\lambda_i\prod_{j=1,j\neq i}^{3}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\cdot\frac{1}{\lambda_i} \\
&= \sum_{i=1}^{3}\prod_{j=1,j\neq i}^{3}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right) \\
&= \left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_1}\right)+\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_2}\right) \\
&+ \left(\frac{\lambda_1}{\lambda_1-\lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_3}\right) \\
&= \frac{\lambda_2\lambda_3(\lambda_3-\lambda_2)-\lambda_1\lambda_3(\lambda_3-\lambda_1)+\lambda_1\lambda_2(\lambda_2-\lambda_1)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_2)} \\
&= \frac{\lambda_2\lambda_3^2-\lambda_2^2\lambda_3-\lambda_1\lambda_3^2-\lambda_1^2\lambda_3+\lambda_1\lambda_2^2-\lambda_1^2\lambda_2}{\lambda_2\lambda_3^2-\lambda_2^2\lambda_3-\lambda_1\lambda_3^2-\lambda_1^2\lambda_3+\lambda_1\lambda_2^2-\lambda_1^2\lambda_2} \\
&= 1
\end{aligned}
\qquad \textbf{(6.3.6)}
$$

As required, hence the model is said to be valid.

6.3.2 Construction Using Moment Generating Function Approach

The moment generating function for the sum of three independent exponential random variables S_3 is given by

$$M_{s_3}(t) = E(e^{ts_3}); \qquad \text{where} \quad s_3 = X_1 + X_2 + X_3 \qquad \textbf{(6.3.7)}$$

From the moment generating function properties then

$$\begin{aligned}
M_{s_3}(t) &= M_{X_1+X_2+X_3}(t) \\
&= E(e^{tx_1})\,\boldsymbol{.}\,E(e^{tx_2})\,\boldsymbol{.}\,E(e^{tx_3}) \\
&= \left(\frac{\lambda_1}{\lambda_1 - t}\right)\left(\frac{\lambda_2}{\lambda_2 - t}\right)\left(\frac{\lambda_3}{\lambda_3 - t}\right) \\
\Rightarrow M_{s_3}(t) &= \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1 - t)(\lambda_2 - t)(\lambda_3 - t)} \\
&= \lambda_1\lambda_2\lambda_3[(\lambda_1 - t)(\lambda_2 - t)(\lambda_3 - t)]^{-1} \\
&= \lambda_1(\lambda_1 - t)^{-1}\lambda_2(\lambda_2 - t)^{-1}\lambda_3(\lambda_3 - t)^{-1} \\
&= \prod_{i=1}^{3}\lambda_i[\lambda_i - t]^{-1}
\end{aligned} \tag{6.3.8}$$

6.3.3 Properties of the distribution

Moments

The r^{th} raw moment of the sum of three independent exponential random variables can be obtained by:

$$E[S_3^r] = \frac{d^r M_{s_3}(t)}{dt^r}|_{t=0}$$

From equation (6.3.7) we have

$$\begin{aligned}
M_{s_3}(t) &= M_{X_1+X_2+X_3}(t) \\
&= E(e^{tx_1})\,\boldsymbol{.}\,E(e^{tx_2})\,\boldsymbol{.}\,E(e^{tx_3})
\end{aligned}$$

Therefore, the first four moments can be obtained as follows:

First moment

$$\begin{aligned}
E[S_3] &= M'_{s_3}(0) \\
&= \frac{d}{dt}\lambda_1\lambda_2\lambda_3[(\lambda_1 - t)^{-1}(\lambda_2 - t)^{-1}(\lambda_3 - t)^{-1}]|_{t=0} \\
&= \lambda_1\lambda_2\lambda_3\left[(\lambda_1 - t)^{-2}(\lambda_2 - t)^{-1}(\lambda_3 - t)^{-1} + (\lambda_2 - t)^{-2}(\lambda_1 - t)^{-1}(\lambda_3 - t)^{-1}\right]|_{t=0} \\
&+ \lambda_1\lambda_2\lambda_3\left[(\lambda_3 - t)^{-2}(\lambda_1 - t)^{-1}(\lambda_2 - t)^{-1}\right]|_{t=0} \\
&= \lambda_1\lambda_2\lambda_3\left[\frac{1}{\lambda_1^2\lambda_2\lambda_3} + \frac{1}{\lambda_2^2\lambda_1\lambda_3} + \frac{1}{\lambda_1\lambda_2\lambda_3^2}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \\
&= \sum_{i=1}^{3} \lambda_i^{-1} \qquad \textbf{(6.3.9)}
\end{aligned}
$$

Second moment

$$
\begin{aligned}
E[S_3^2] &= M''_{s_3}(0) \\
&= \frac{d}{dt}\lambda_1\lambda_2\lambda_3 \left[(\lambda_1-t)^{-2}(\lambda_2-t)^{-1}(\lambda_3-t)^{-1} + (\lambda_2-t)^{-2}(\lambda_1-t)^{-1}(\lambda_3-t)^{-1}\right]|_{t=0} \\
&+ \frac{d}{dt}\lambda_1\lambda_2\lambda_3 \left[(\lambda_3-t)^{-2}(\lambda_1-t)^{-1}(\lambda_2-t)^{-1}\right]|_{t=0} \\
&= 2\lambda_1\lambda_2\lambda_3 \left[(\lambda_1-t)^{-3}(\lambda_2-t)^{-1}(\lambda_3-t)^{-1} + (\lambda_2-t)^{-2}(\lambda_1-t)^{-2}(\lambda_3-t)^{-1}\right]|_{t=0} \\
&+ 2\lambda_1\lambda_2\lambda_3 \left[(\lambda_3-t)^{-2}(\lambda_1-t)^{-2}(\lambda_2-t)^{-1} + (\lambda_2-t)^{-3}(\lambda_1-t)^{-1}(\lambda_3-t)^{-1}\right]|_{t=0} \\
&+ 2\lambda_1\lambda_2\lambda_3 \left[(\lambda_3-t)^{-2}(\lambda_2-t)^{-2}(\lambda_1-t)^{-1} + (\lambda_3-t)^{-3}(\lambda_1-t)^{-1}(\lambda_2-t)^{-1}\right]|_{t=0} \\
&= 2\left[\lambda_1^{-2} + \lambda_1^{-1}\lambda_2^{-1} + \lambda_2^{-2} + \lambda_1^{-1}\lambda_3^{-1} + \lambda_2^{-1}\lambda_3^{-1} + \lambda_3^{-2}\right] \\
&= 2\left[\sum_{i=0}^{2}\lambda_1^{-(2-i)}\lambda_2^{-i} + \sum_{i=1}^{2}\lambda_1^{-(2-i)}\lambda_3^{-i} + \lambda_2^{-1}\lambda_3^{-1}\right] \qquad \textbf{(6.3.10)}
\end{aligned}
$$

Third moment

$$
\begin{aligned}
E[S_3^3] &= M_{s_3}^3(0) \\
&= 2\frac{d}{dt}\lambda_1\lambda_2\lambda_3 \left[(\lambda_1-t)^{-3}(\lambda_2-t)^{-1}(\lambda_3-t)^{-1} + (\lambda_2-t)^{-2}(\lambda_1-t)^{-2}(\lambda_3-t)^{-1}\right]|_{t=0} \\
&+ 2\frac{d}{dt}\lambda_1\lambda_2\lambda_3 \left[(\lambda_3-t)^{-2}(\lambda_1-t)^{-2}(\lambda_2-t)^{-1} + (\lambda_2-t)^{-3}(\lambda_1-t)^{-1}(\lambda_3-t)^{-1}\right]|_{t=0} \\
&+ 2\frac{d}{dt}\lambda_1\lambda_2\lambda_3 \left[(\lambda_3-t)^{-2}(\lambda_2-t)^{-2}(\lambda_1-t)^{-1} + (\lambda_3-t)^{-3}(\lambda_1-t)^{-1}(\lambda_2-t)^{-1}\right]|_{t=0} \\
&= \lambda_1\lambda_2\lambda_3 \left[3(\lambda_1-t)^{-4}(\lambda_2-t)^{-1}(\lambda_3-t)^{-1} + (\lambda_2-t)^{-2}(\lambda_1-t)^{-3}(\lambda_3-t)^{-1}\right]|_{t=0} \\
&+ \lambda_1\lambda_2\lambda_3 \left[3(\lambda_3-t)^{-2}(\lambda_1-t)^{-3}(\lambda_2-t)^{-1} + 3(\lambda_2-t)^{-3}(\lambda_1-t)^{-2}(\lambda_3-t)^{-1}\right]|_{t=0} \\
&+ \lambda_1\lambda_2\lambda_3 \left[3(\lambda_3-t)^{-2}(\lambda_2-t)^{-2}(\lambda_1-t)^{-2} + 3(\lambda_3-t)^{-3}(\lambda_1-t)^{-1}(\lambda_2-t)^{-1}\right]|_{t=0} \\
&+ \lambda_1\lambda_2\lambda_3 \left[3(\lambda_2-t)^{-4}(\lambda_1-t)^{-1}(\lambda_3-t)^{-1} + 3(\lambda_3-t)^{-2}(\lambda_2-t)^{-3}(\lambda_1-t)^{-1}\right]|_{t=0} \\
&+ \lambda_1\lambda_2\lambda_3 \left[3(\lambda_3-t)^{-3}(\lambda_2-t)^{-2}(\lambda_1-t)^{-1} + 3(\lambda_3-t)^{-4}(\lambda_1-t)^{-1}(\lambda_2-t)^{-1}\right]|_{t=0} \\
&= 6\left[\lambda_1^{-3} + \lambda_1^{-2}\lambda_2^{-1} + \lambda_1^{-2}\lambda_3^{-1} + \lambda_1^{-1}\lambda_2^{-2} + \lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1}\right] \\
&+ 6\left[\lambda_1^{-1}\lambda_3^{-2} + \lambda_2^{-3} + \lambda_2^{-2}\lambda_3^{-1} + \lambda_2^{-1}\lambda_3^{-2} + \lambda_3^{-3}\right] \\
&= 3!\left[\sum_{i=0}^{3}\lambda_1^{-(3-i)}\lambda_2^{-i} + \sum_{i=1}^{3}\lambda_1^{-(3-i)}\lambda_3^{-i} + \sum_{i=1}^{2}\lambda_2^{-(3-i)}\lambda_3^{-i} + \prod_{i=1}^{3}\lambda_i^{-1}\right] \qquad \textbf{(6.3.11)}
\end{aligned}
$$

Fourth moment

$$
\begin{aligned}
E[S_3^4] &= M_{s_3}^4(0) \\
&= 2\lambda_1\lambda_2\lambda_3 \frac{d}{dt}\left[3(\lambda_1-t)^{-4}(\lambda_2-t)^{-1}(\lambda_3-t)^{-1} + (\lambda_2-t)^{-2}(\lambda_1-t)^{-3}(\lambda_3-t)^{-1}\right]|_{t=0} \\
&+ 2\lambda_1\lambda_2\lambda_3 \frac{d}{dt}\left[3(\lambda_3-t)^{-2}(\lambda_1-t)^{-3}(\lambda_2-t)^{-1} + 3(\lambda_2-t)^{-3}(\lambda_1-t)^{-2}(\lambda_3-t)^{-1}\right]|_{t=0} \\
&+ 2\lambda_1\lambda_2\lambda_3 \frac{d}{dt}\left[3(\lambda_3-t)^{-2}(\lambda_2-t)^{-2}(\lambda_1-t)^{-2} + 3(\lambda_3-t)^{-3}(\lambda_1-t)^{-1}(\lambda_2-t)^{-1}\right]|_{t=0} \\
&+ 2\lambda_1\lambda_2\lambda_3 \frac{d}{dt}\left[3(\lambda_2-t)^{-4}(\lambda_1-t)^{-1}(\lambda_3-t)^{-1} + 3(\lambda_3-t)^{-2}(\lambda_2-t)^{-3}(\lambda_1-t)^{-1}\right]|_{t=0} \\
&+ 2\lambda_1\lambda_2\lambda_3 \frac{d}{dt}\left[3(\lambda_3-t)^{-3}(\lambda_2-t)^{-2}(\lambda_1-t)^{-1} + 3(\lambda_3-t)^{-4}(\lambda_1-t)^{-1}(\lambda_2-t)^{-1}\right]|_{t=0} \\
&= 4!\left[\lambda_1^{-4} + \lambda_1^{-3}\lambda_2^{-1} + \lambda_1^{-2}\lambda_2^{-2} + \lambda_1^{-1}\lambda_2^{-3} + \lambda_2^{-4}\right] \\
&+ 4!\left[\lambda_1^{-3}\lambda_3^{-1} + \lambda_1^{-2}\lambda_3^{-2} + \lambda_1^{-1}\lambda_3^{-3} + \lambda_3^{-4}\right] \\
&+ 4!\left[\lambda_2^{-3}\lambda_3^{-1} + \lambda_2^{-2}\lambda_3^{-2} + \lambda_2^{-1}\lambda_3^{-3}\right] \\
&+ 4!\left[\lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1}\right] \\
&= 4!\left[\sum_{i=0}^{4}\lambda_1^{-(4-i)}\lambda_2^{-i} + \sum_{i=1}^{4}\lambda_1^{-(4-i)}\lambda_3^{-i} + \sum_{i=1}^{3}\lambda_2^{-(4-i)}\lambda_3^{-i} + \prod_{i=1}^{4}\lambda_i^{-1}\right] \qquad (6.3.12)
\end{aligned}
$$

This implies the r^{th} raw moment for S_3 can be expressed by

$$
E[S_3^r] = r!\left[\sum_{i=0}^{r}\lambda_1^{-(r-i)}\lambda_2^{-i} + \sum_{i=1}^{r}\lambda_1^{-(r-i)}\lambda_3^{-i} + \sum_{i=1}^{r-1}\lambda_2^{-(r-i)}\lambda_3^{-i} + \prod_{i=1}^{r}\lambda_i^{-1}\right] \qquad (6.3.13)
$$

Therefore other properties for the sum of three independent exponential random variables with distinct parameters can be obtained as follows:

Mean

$$
E[S_3] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \qquad (6.3.14)
$$

Variance

$$
\begin{aligned}
Var[S_3] &= E[S_3^2] - (E[S_3])^2 \\
&= 2\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3^2}\right) - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)^2 \\
&= \frac{2}{\lambda_1^2} + \frac{2}{\lambda_1\lambda_2} + \frac{2}{\lambda_2^2} + \frac{2}{\lambda_1\lambda_3} + \frac{2}{\lambda_2\lambda_3} + \frac{2}{\lambda_3^2} - \frac{1}{\lambda_1^2} - \frac{2}{\lambda_1\lambda_2} - \frac{2}{\lambda_1\lambda_3} - \frac{2}{\lambda_2\lambda_3} - \frac{1}{\lambda_2^2} - \frac{1}{\lambda_3^2} \\
&= \frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} - \frac{1}{\lambda_3^2} \qquad (6.3.15)
\end{aligned}
$$

Mode

The mode for the model is given when the following condition is maintained

$$\frac{d}{ds_3}g(s_3) = 0$$

This implies that

$$\begin{aligned}
\frac{d}{ds_3}g(s_3) &= -\lambda_1\left(\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\right)e^{-\lambda_1 s_3} - \lambda_2\left(\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}\right)e^{-\lambda_2 s_3} \\
&\quad -\lambda_3\left(\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right)e^{-\lambda_1 s_3} \\
\Rightarrow \frac{d}{ds_3}(\lambda_3-\lambda_2)e^{-\lambda_1 s_3} &= \frac{d}{ds_3}(\lambda_3-\lambda_1)e^{-\lambda_2 s_3} - \frac{d}{ds_3}(\lambda_2-\lambda_1)e^{-\lambda_3 s_3} \\
-\lambda_1(\lambda_3-\lambda_2)e^{-\lambda_1 s_3} &= -\lambda_2(\lambda_3-\lambda_1)e^{-\lambda_2 s_3} + \lambda_3(\lambda_2-\lambda_1)e^{-\lambda_3 s_3} \\
-ln\lambda_1 + ln(\lambda_3-\lambda_2) - \lambda_1 s_3 &= -ln\lambda_2 + ln(\lambda_3-\lambda_1) - \lambda_2 s_3 + ln\lambda_3 + ln(\lambda_2-\lambda_1) - \lambda_3 s_3 \\
-\lambda_1 s_3 + \lambda_2 s_3 + \lambda_3 s_3 &= ln\lambda_1 - ln(\lambda_3-\lambda_2) - ln\lambda_2 + ln(\lambda_3-\lambda_1) + ln\lambda_3 + ln(\lambda_2-\lambda_1) \\
s_3(\lambda_2+\lambda_3-\lambda_1) &= ln\lambda_1 - ln\lambda_3 + ln\lambda_2 - ln\lambda_2 + ln\lambda_3 - ln\lambda_1 + ln\lambda_3 + ln\lambda_2 - ln\lambda_1 \\
\Rightarrow s_3 &= \frac{ln\lambda_2 + ln\lambda_3 - ln\lambda_1}{\lambda_2+\lambda_3-\lambda_1} \qquad \textbf{(6.3.16)}
\end{aligned}$$

Asymptotic Behaviour of the Model

In seeking the asymptotic behaviour of the model formed in equation (6.3.4) then we consider the behaviour of the model when $s_3 \to 0$ and as $s_3 \to \infty$. That is

$$\lim_{s_2\to 0} g(s_2) \text{ and } \lim_{s_2\to\infty} g(s_2)$$

This implies that

$$\begin{aligned}
\lim_{S_3\to 0} g(s_3) = &\left[\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right] \\
&\lim_{S_3\to 0}\left[(\lambda_3-\lambda_2)e^{-\lambda_1 s_3} - (\lambda_3-\lambda_1)e^{-\lambda_2 s_3} + (\lambda_2-\lambda_1)e^{-\lambda_3 s_3}\right] \\
&= \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\left[(\lambda_3-\lambda_2) - (\lambda_3-\lambda_1) + (\lambda_2-\lambda_1)\right] \\
&= 0
\end{aligned}$$

Alternatively

$$\begin{aligned}
\lim_{S_3\to 0} g(s_3) &= \sum_{i=1}^{3}\left\{\lambda_i \lim_{S_3\to 0} e^{-\lambda_i s_3}\prod_{j=1,j\neq i}^{3}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\right\}\\
&= \sum_{i=1}^{3}\left\{\lambda_i \prod_{j=1,j\neq i}^{3}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\right\}\\
&= \left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_1}\right)\lambda_1 + \left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_2}\right)\lambda_2\\
&+ \left(\frac{\lambda_1}{\lambda_1-\lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_3}\right)\lambda_3\\
&= \frac{\lambda_1\lambda_2\lambda_3[(\lambda_3-\lambda_2)-(\lambda_3-\lambda_1)+(\lambda_2-\lambda_1)]}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_2)}\\
&= \frac{\lambda_1\lambda_2\lambda_3^2-\lambda_1\lambda_2^2\lambda_3-\lambda_1\lambda_2\lambda_3^2+\lambda_1^2\lambda_2\lambda_3+\lambda_1\lambda_2^2\lambda_3-\lambda_1^2\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_2)}\\
&= 0
\end{aligned}$$

and also

$$\begin{aligned}
\lim_{S_3\to\infty} g(s_3) &= \left[\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right]\\
&\lim_{S_3\to\infty}\left[(\lambda_3-\lambda_2)e^{-\lambda_1 s_3}-(\lambda_3-\lambda_1)e^{-\lambda_2 s_3}+(\lambda_2-\lambda_1)e^{-\lambda_3 s_3}\right]\\
&= \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}(0)\\
&= 0
\end{aligned} \tag{6.3.17}$$

From the two results it confirms that from the equation (6.3.4) the model has only one mode that is Uni-modal model.

Cumulant Generating Function

The cumulant generating function of a random variable S_3 can be obtained by:

$$\begin{aligned}
C_{s_3}(t) &= log[M_{s_3(t)}]\\
&= log\left\{\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-t)(\lambda_2-t)(\lambda_3-t)}\right\}
\end{aligned}$$

Using Maclaurin series through expansion the following equation can be obtained:

$$C_{s_3}(t) = \sum_{i=1}^{\infty}(i-1)!\left[\left(\frac{1}{\lambda_1}\right)^i+\left(\frac{1}{\lambda_2}\right)^i+\left(\frac{1}{\lambda_3}\right)^i\right]\frac{t^i}{i!} \tag{6.3.18}$$

From the definition of cumulants, the cumulant K_i of a random variable S_3 can be obtained using the coefficients of $\frac{t^i}{i!}$ in equation (6.3.18).

Therefore,

$$K_i = (i-1)!\left[\left(\frac{1}{\lambda_1}\right)^i + \left(\frac{1}{\lambda_2}\right)^i + \left(\frac{1}{\lambda_3}\right)^i\right] \tag{6.3.19}$$

Therefore the first four cumulants can be obtained as follows:

$$\begin{aligned} K_1 &= (1-1)!\left[\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right] \\ &= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \end{aligned}$$

$$\begin{aligned} K_2 &= (2-1)!\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \left(\frac{1}{\lambda_3}\right)^2\right] \\ &= \left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \left(\frac{1}{\lambda_3}\right)^2 \end{aligned}$$

$$\begin{aligned} K_3 &= (3-1)!\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3\right] \\ &= 2\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3 + \left(\frac{1}{\lambda_3}\right)^3\right] \end{aligned}$$

$$\begin{aligned} K_4 &= (4-1)!\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4 + \left(\frac{1}{\lambda_3}\right)^4\right] \\ K_4 &= 6\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4 + \left(\frac{1}{\lambda_3}\right)^4\right] \end{aligned}$$

From the above results the following conclusions can be made:

(i) K_1 which forms the first cumulant gives the mean of the random variable S_3.

(ii) K_2 which forms the second cumulant gives the variance of the random variable S_3.

(iii) Skewness

$$
\begin{aligned}
&= \frac{K_3}{K_2^{\frac{3}{2}}} \\
&= \frac{2\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \left(\frac{1}{\lambda_3}\right)^2\right]^{3/2}} \qquad \textbf{(6.3.20)}
\end{aligned}
$$

(iv) Kurtosis

$$
\begin{aligned}
&= \frac{K_4}{K_2^2} \\
&= \frac{6\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4 + \left(\frac{1}{\lambda_3}\right)^4\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \left(\frac{1}{\lambda_3}\right)^2\right]^{2}} \qquad \textbf{(6.3.21)}
\end{aligned}
$$

6.3.4 The Case of Arithmetic Sequence of three Parameters

Using the equation (6.3.4) and given that the model parameters takes an arithmetic sequence that is $\lambda_2 - \lambda_1 = \lambda_3 - \lambda_2 = d$ where d is the common difference. It then implies that $\lambda_j - \lambda_i = (j-i)d$. Then

$$
\begin{aligned}
g(s_3) &= \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_1}\right)\lambda_1 e^{-\lambda_1 s_3} + \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_2}\right)\lambda_2 e^{-\lambda_2 s_3} \\
&\quad + \left(\frac{\lambda_1}{\lambda_1 - \lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2 - \lambda_3}\right)\lambda_3 e^{-\lambda_3 s_3} \\
\Rightarrow g(s_3) &= \left(\frac{\lambda_2}{(\lambda_1 + d) - \lambda_1}\right)\left(\frac{\lambda_3}{(\lambda_1 + 2d) - \lambda_1}\right)\lambda_1 e^{-\lambda_1 s_3} \\
&\quad + \left(\frac{\lambda_1}{\lambda_1 - (\lambda_1 + d)}\right)\left(\frac{\lambda_3}{(\lambda_1 + 2d) - (\lambda_1 + d)}\right)\lambda_2 e^{-\lambda_2 s_3} \\
&\quad + \left(\frac{\lambda_1}{\lambda_1 - (\lambda_1 + 2d)}\right)\left(\frac{\lambda_2}{(\lambda_1 + d) - (\lambda_1 + 2d)}\right)\lambda_3 e^{-\lambda_3 s_3} \\
&= \lambda_1\lambda_2\lambda_3\left(\frac{e^{-\lambda_1 s_3}}{2d^2}\right) + \lambda_1\lambda_2\lambda_3\left(\frac{e^{-\lambda_2 s_3}}{-d^2}\right) \\
&\quad + \lambda_1\lambda_2\lambda_3\left(\frac{e^{-\lambda_3 s_3}}{2d^2}\right)
\end{aligned}
$$

$$= \lambda_1\lambda_2\lambda_3\left[\left(\frac{e^{-\lambda_1 s_3}}{2d^2}\right)+\left(\frac{e^{-\lambda_2 s_3}}{-d^2}\right)+\left(\frac{e^{-\lambda_3 s_3}}{2d^2}\right)\right]$$

$$= \prod_{i=1}^{3}\lambda_i\left\{\sum_{i=1}^{3}\frac{e^{-\lambda_i s_3}}{(-1)^{i-1}(i-1)!(3-i)!d^{3-1}}\right\}$$

$$= \prod_{i=1}^{3}\lambda_i\left\{\sum_{i=1}^{3}\frac{e^{-\lambda_i s_3}}{\gamma_{i,3}}\right\} \qquad \text{where} \quad \gamma_{i,3} = (-1)^{i-1}(i-1)!(3-i)!d^{3-1} \tag{6.3.21}$$

6.3.5 The Case of Geometric Sequence of three Parameters

If the model parameters takes the form of geometric sequence with the a common ratio r. That is $\frac{\lambda_2}{\lambda_1} = \frac{\lambda_3}{\lambda_2} = r$. This implies that $\lambda_2 = r\lambda_1$ and $\lambda_3 = r^2\lambda_1$.

Therefore, from equation (6.3.4) we get

$$g(s_3) = \left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_1}\right)\lambda_1 e^{-\lambda_1 s_3} + \left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_2}\right)\lambda_2 e^{-\lambda_2 s_3} + \left(\frac{\lambda_1}{\lambda_1-\lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_3}\right)\lambda_3 e^{-\lambda_3 s_3}$$

$$\Rightarrow g(s_3) = \left(\frac{r\lambda_1}{r\lambda_1-\lambda_1}\right)\left(\frac{r^2\lambda_1}{r^2\lambda_1-\lambda_1}\right)\lambda_1 e^{-\lambda_1 s_3} + \left(\frac{\lambda_1}{\lambda_1-r\lambda_1}\right)\left(\frac{r^2\lambda_1}{r^2\lambda_1-r\lambda_1}\right)\lambda_2 e^{-\lambda_2 s_3} + \left(\frac{\lambda_1}{\lambda_1-r^2\lambda_1}\right)\left(\frac{r\lambda_1}{r\lambda_1-r^2\lambda_1}\right)\lambda_3 e^{-\lambda_3 s_3}$$

$$= \left(\frac{1}{1-r^{-1}}\right)\left(\frac{1}{1-r^{-2}}\right)\lambda_1 e^{-\lambda_1 s_3} + \left(\frac{1}{1-r}\right)\left(\frac{1}{1-r^{-1}}\right)\lambda_2 e^{-\lambda_2 s_3} + \left(\frac{1}{1-r^2}\right)\left(\frac{1}{1-r}\right)\lambda_3 e^{-\lambda_3 s_3}$$

$$= \frac{\lambda_1 e^{-\lambda_1 s_3}}{(1-r^{-1})(1-r^{-2})} + \frac{\lambda_2 e^{-\lambda_2 s_3}}{(1-r)(1-r^{-1})} + \frac{\lambda_3 e^{-\lambda_3 s_3}}{(1-r^2)(1-r)}$$

$$= \sum_{i=1}^{3}\frac{\lambda_i e^{-\lambda_i s_3}}{\prod_{j=1,j\neq i}^{3}(1-r^{i-j})} \tag{6.3.22}$$

6.4 Hypo-exponential Distribution of four independent random variables with distinct parameters

6.4.1 Construction Using Convolution Approach

In this sub-section we need to consider the distributions for the sum of four independent random variables with distinct parameters. Therefore let

$$\begin{aligned} S_4 &= X_1 + X_2 + X_3 + X_4 \\ S_4 &= S_3 + X_4, \qquad \text{where} \quad S_3 = X_1 + X_2 + X_3 \end{aligned} \tag{6.4.1}$$

Let $F_i(x_i)$ and $G(s_4)$ be the cumulative distribution function for X_i and S_4 respectively.

Cumulative Density Function

From the definition the cumulative density function is given as follows:

$$\begin{aligned} G(S_4) &= Prob(S_4 \leq s_4) \\ &= Prob(S_3 + X_4 \leq s_4) \\ &= Prob(X_4 \leq s_4 - s_3) \\ &= Prob(0 \leq s_3 \leq s_4, 0 \leq X_4 \leq s_4 - s_3) \\ &= \int_0^{s_4} \int_0^{s_4 - s_3} g(s_3) f(x_4) dx_4 ds_3 \\ &= \int_0^{s_4} g(s_3) \left[\int_0^{s_4 - s_3} f(x_4) dx_4 \right] ds_3 \\ &= \int_0^{s_4} g(s_3) \left[\int_0^{s_4 - s_3} \lambda_4 e^{-\lambda_4 s_4} dx_4 \right] ds_3 \\ &= \int_0^{s_4} g(s_3) \left[1 - e^{-\lambda_4 (s_4 - s_3)} \right] ds_3 \\ &= \int_0^{s_4} \left[g(s_3) - g(s_3) e^{-\lambda_4 (s_4 - s_3)} \right] ds_3 \\ &= \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right) \left(\frac{\lambda_3}{\lambda_3 - \lambda_1} \right) \lambda_1 \int_0^{s_4} e^{-\lambda_1 s_3} ds_3 + \left(\frac{\lambda_1}{\lambda_1 - \lambda_2} \right) \left(\frac{\lambda_3}{\lambda_3 - \lambda_2} \right) \lambda_2 \int_0^{s_4} e^{-\lambda_2 s_3} ds_3 \\ &+ \left(\frac{\lambda_1}{\lambda_1 - \lambda_3} \right) \left(\frac{\lambda_2}{\lambda_2 - \lambda_3} \right) \lambda_3 \int_0^{s_4} e^{-\lambda_3 s_3} ds_3 \\ &- \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right) \left(\frac{\lambda_3}{\lambda_3 - \lambda_1} \right) \lambda_1 e^{-\lambda_4 s_4} \int_0^{s_4} e^{(\lambda_4 - \lambda_1) s_3} ds_3 \end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_2}\right)\lambda_2 e^{-\lambda_4 s_4}\int_0^{s_4} e^{(\lambda_4-\lambda_2)s_3}ds_3\\
&-\left(\frac{\lambda_1}{\lambda_1-\lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_3}\right)\lambda_3 e^{-\lambda_4 s_4}\int_0^{s_4} e^{(\lambda_4-\lambda_3)s_3}ds_3\\
\therefore G(S_4) =& \left\{\frac{\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\right\}\left[1-e^{-\lambda_1 s_4}\right]+\left\{\frac{\lambda_1\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}\right\}\left[1-e^{-\lambda_2 s_4}\right]\\
&+\left\{\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right\}\left[1-e^{-\lambda_3 s_4}\right]\\
&-\left\{\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}\right\}\left[e^{-\lambda_1 s_4}-e^{-\lambda_4 s_4}\right]\\
&-\left\{\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}\right\}\left[e^{-\lambda_2 s_4}-e^{-\lambda_4 s_4}\right]\\
&-\left\{\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}\right\}\left[e^{-\lambda_3 s_4}-e^{-\lambda_4 s_4}\right]\\
&=\frac{\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}-\frac{\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}e^{-\lambda_1 s_4}\\
&+\frac{\lambda_1\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}-\frac{\lambda_1\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}e^{-\lambda_2 s_4}\\
&+\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}-\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}e^{-\lambda_3 s_4}\\
&-\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}e^{-\lambda_1 s_4}+\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}e^{-\lambda_4 s_4}\\
&-\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}e^{-\lambda_2 s_4}+\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}e^{-\lambda_4 s_4}\\
&-\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}e^{-\lambda_3 s_4}+\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}e^{-\lambda_3 s_4}e^{-\lambda_4 s_4}\\
\Rightarrow G(S_4) =& 1-\left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_1}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_1}\right)e^{-\lambda_1 s_4}\\
&-\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_2}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_2}\right)e^{-\lambda_2 s_4}\\
&-\left(\frac{\lambda_1}{\lambda_1-\lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_3}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_3}\right)e^{-\lambda_3 s_4}\\
&-\left(\frac{\lambda_1}{\lambda_1-\lambda_4}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_4}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_4}\right)e^{-\lambda_4 s_4} \qquad \textbf{(6.4.2)}
\end{aligned}
$$

as required.

Therefore, from the above equation (6.4.2) it can be concluded that

$$\lim_{s_4 \to \infty} G(s_4) = 1$$

and

$$\lim_{s_4 \to 0} G(s_4) = 0 \tag{6.4.3}$$

Probability Density Function

The probability density function of the sum of four independent exponential random variables can be obtained through finding the derivative of $G(s_4)$ from equation (6.4.2) with respect to s_4. This follows that:

$$\begin{aligned}
g(s_4) &= \frac{d}{ds_4}\left[G(s_4)\right] \\
&= \frac{d}{ds_4}1 - \frac{d}{ds_4}\left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_1}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_1}\right)e^{-\lambda_1 s_4} \\
&- \frac{d}{ds_4}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_2}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_2}\right)e^{-\lambda_2 s_4} \\
&- \frac{d}{ds_4}\left(\frac{\lambda_1}{\lambda_1-\lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_3}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_3}\right)e^{-\lambda_3 s_4} \\
&- \frac{d}{ds_4}\left(\frac{\lambda_1}{\lambda_1-\lambda_4}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_4}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_4}\right)e^{-\lambda_4 s_4} \\
&= \left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_1}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_1}\right)\lambda_1 e^{-\lambda_1 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_2}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_2}\right)\lambda_2 e^{-\lambda_2 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1-\lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_3}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_3}\right)\lambda_3 e^{-\lambda_3 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1-\lambda_4}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_4}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_4}\right)\lambda_4 e^{-\lambda_4 s_4} \\
\Rightarrow g(s_4) &= \sum_{i=1}^{4}\left\{\lambda_i e^{-\lambda_i s_4}\prod_{j=1, j\neq i}^{4}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\right\} \quad \text{where } s_4 > 0,\ \lambda_4 > \lambda_3 > \lambda_2 > \lambda_1 > 0
\end{aligned} \tag{6.4.4}$$

as required.

Hazard Function

From the definition, the hazard function is obtained as follows:

$$h(s_4) = \frac{g(s_4)}{1 - G(s_4)}$$

But

$$\begin{aligned}
1 - G(s_4) &= 1 - 1 + \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_1}\right)\left(\frac{\lambda_4}{\lambda_4 - \lambda_1}\right)e^{-\lambda_1 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_2}\right)\left(\frac{\lambda_4}{\lambda_4 - \lambda_2}\right)e^{-\lambda_2 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - \lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2 - \lambda_3}\right)\left(\frac{\lambda_4}{\lambda_4 - \lambda_3}\right)e^{-\lambda_3 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - \lambda_4}\right)\left(\frac{\lambda_2}{\lambda_2 - \lambda_4}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_4}\right)e^{-\lambda_4 s_4} \\
&= \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_1}\right)\left(\frac{\lambda_4}{\lambda_4 - \lambda_1}\right)e^{-\lambda_1 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_2}\right)\left(\frac{\lambda_4}{\lambda_4 - \lambda_2}\right)e^{-\lambda_2 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - \lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2 - \lambda_3}\right)\left(\frac{\lambda_4}{\lambda_4 - \lambda_3}\right)e^{-\lambda_3 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - \lambda_4}\right)\left(\frac{\lambda_2}{\lambda_2 - \lambda_4}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_4}\right)e^{-\lambda_4 s_4} \\
&= \sum_{i=1}^{4}\left\{e^{-\lambda_i s_4}\prod_{j=1, j\neq i}^{4}\left(\frac{\lambda_j}{\lambda_j - \lambda_i}\right)\right\} \\
\Rightarrow h(s_4) &= \frac{\sum_{i=1}^{4}\left\{\lambda_i e^{-\lambda_i s_4}\prod_{j=1, j\neq i}^{4}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\right\}}{\sum_{i=1}^{4}\left\{e^{-\lambda_i s_4}\prod_{j=1, j\neq i}^{4}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\right\}} \\
&= \sum_{i=1}^{4}\lambda_i \qquad \textbf{(6.4.5)}
\end{aligned}$$

Validity of the Model

The model $g(s_3)$ is said to be valid if it satisfies the condition

$$\int_0^\infty g(s_4)ds_4 = 1$$

Proof

$$\begin{aligned}\int_0^\infty g(s_4)ds_4 &= \int_0^\infty \sum_{i=1}^{4} \lambda_i e^{-\lambda_i s_4} \prod_{j=1, j\neq i}^{4} \left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right) ds_4 \\ &= \sum_{i=1}^{4} \prod_{j=1, j\neq i}^{4} \left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right) \lambda_i \int_0^\infty e^{-\lambda_i s_4} ds_4 \\ &= \sum_{i=1}^{4} \prod_{j=1, j\neq i}^{4} \left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right) \\ &= 1 \end{aligned} \tag{6.4.6}$$

As required, hence the model is said to be valid.

6.4.2 Construction Using Moment Generating Function Approach

The moment generating function for the sum of four independent exponential random variables S_4 with distinct parameters is given by

$$M_{s_4}(t) = E(e^{ts_4}); \qquad \text{where} \quad s_4 = X_1 + X_2 + X_3 + X_4 \tag{6.4.7}$$

Using the moment generating function properties then

$$\begin{aligned} M_{s_4}(t) &= M_{X_1+X_2+X_3+X_4}(t) \\ &= E(e^{tx_1}) \,.\, E(e^{tx_2}) \,.\, E(e^{tx_3}) \,.\, E(e^{tx_4}) \\ &= \left(\frac{\lambda_1}{\lambda_1-t}\right)\left(\frac{\lambda_2}{\lambda_2-t}\right)\left(\frac{\lambda_3}{\lambda_3-t}\right)\left(\frac{\lambda_4}{\lambda_4-t}\right) \\ \Rightarrow M_{s_4}(t) &= \frac{\lambda_1\lambda_2\lambda_3\lambda_4}{(\lambda_1-t)(\lambda_2-t)(\lambda_3-t)(\lambda_4-t)} \\ &= \lambda_1\lambda_2\lambda_3\lambda_4[(\lambda_1-t)(\lambda_2-t)(\lambda_3-t)(\lambda_4-t)]^{-1} \\ &= \lambda_1(\lambda_1-t)^{-1}\lambda_2(\lambda_2-t)^{-1}\lambda_3(\lambda_3-t)^{-1}\lambda_4(\lambda_4-t)^{-1} \\ &= \prod_{i=1}^{4} \lambda_i[\lambda_i-t]^{-1} \end{aligned} \tag{6.4.8}$$

6.4.3 Properties of the Distribution

Moments

The r^{th} raw moment of the sum of four independent exponential random variables can be obtained by:

$$E[S_4^r] = \frac{d^r M_{s_4}(t)}{dt^r}|_{t=0}$$

From equation (6.4.7) we have

$$\begin{aligned} M_{s_3}(t) &= M_{X_1+X_2+X_3+X_4}(t) \\ &= E(e^{tx_1}) \,.\, E(e^{tx_2}) \,.\, E(e^{tx_3}) \,.\, E(e^{tx_4}) \end{aligned}$$

Therefore,the first four moments can be obtained as follows:

First moment

$$\begin{aligned} E[S_4] &= M'_{s_4}(0) \\ &= \frac{d}{dt}\lambda_1\lambda_2\lambda_3\lambda_4 \left[(\lambda_1 - t)^{-1}(\lambda_2 - t)^{-1}(\lambda_3 - t)^{-1}(\lambda_4 - t)^{-1}\right]|_{t=0} \\ &= \lambda_1\lambda_2\lambda_3\lambda_4 \left[(\lambda_1 - t)^{-2}(\lambda_2 - t)^{-1}(\lambda_3 - t)^{-1}(\lambda_4 - t)^{-1}\right]|_{t=0} \\ &+ \lambda_1\lambda_2\lambda_3\lambda_4 \left[(\lambda_2 - t)^{-2}(\lambda_1 - t)^{-1}(\lambda_3 - t)^{-1}(\lambda_4 - t)^{-1}\right]|_{t=0} \\ &+ \lambda_1\lambda_2\lambda_3\lambda_4 \left[(\lambda_3 - t)^{-2}(\lambda_1 - t)^{-1}(\lambda_2 - t)^{-1}(\lambda_4 - t)^{-1}\right]|_{t=0} \\ &+ \lambda_1\lambda_2\lambda_3\lambda_4 \left[(\lambda_4 - t)^{-2}(\lambda_1 - t)^{-1}(\lambda_2 - t)^{-1}(\lambda_3 - t)^{-1}\right]|_{t=0} \\ &= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} \\ &= \sum_{i=1}^{4} \lambda_i^{-1} \end{aligned} \tag{6.4.9}$$

Second moment

$$\begin{aligned} E[S_4^2] &= M\text{''}_{s_4}(0) \\ &= \frac{d}{dt}\lambda_1\lambda_2\lambda_3\lambda_4 \left[(\lambda_1 - t)^{-2}(\lambda_2 - t)^{-1}(\lambda_3 - t)^{-1}(\lambda_4 - t)^{-1}\right]|_{t=0} \\ &+ \frac{d}{dt}\lambda_1\lambda_2\lambda_3\lambda_4 \left[(\lambda_2 - t)^{-2}(\lambda_1 - t)^{-1}(\lambda_3 - t)^{-1}(\lambda_4 - t)^{-1}\right]|_{t=0} \\ &+ \frac{d}{dt}\lambda_1\lambda_2\lambda_3\lambda_4 \left[(\lambda_3 - t)^{-2}(\lambda_1 - t)^{-1}(\lambda_2 - t)^{-1}(\lambda_4 - t)^{-1}\right]|_{t=0} \\ &+ \frac{d}{dt}\lambda_1\lambda_2\lambda_3\lambda_4 \left[(\lambda_4 - t)^{-2}(\lambda_1 - t)^{-1}(\lambda_2 - t)^{-1}(\lambda_3 - t)^{-1}\right]|_{t=0} \end{aligned}$$

$$
\begin{aligned}
&= 2\lambda_1\lambda_2\lambda_3\lambda_4\left[(\lambda_1-t)^{-3}(\lambda_2-t)^{-1}(\lambda_3-t)^{-1}(\lambda_4-t)^{-1}\right]|_{t=0}\\
&+ 2\lambda_1\lambda_2\lambda_3\lambda_4\left[(\lambda_2-t)^{-2}(\lambda_1-t)^{-2}(\lambda_3-t)^{-1}(\lambda_4-t)^{-1}\right]|_{t=0}\\
&+ 2\lambda_1\lambda_2\lambda_3\lambda_4\left[(\lambda_3-t)^{-2}(\lambda_1-t)^{-2}(\lambda_2-t)^{-1}(\lambda_4-t)^{-1}\right]|_{t=0}\\
&+ 2\lambda_1\lambda_2\lambda_3\lambda_4\left[(\lambda_4-t)^{-2}(\lambda_1-t)^{-2}(\lambda_2-t)^{-1}(\lambda_3-t)^{-1}\right]|_{t=0}\\
&+ 2\lambda_1\lambda_2\lambda_3\lambda_4\left[(\lambda_2-t)^{-3}(\lambda_1-t)^{-1}(\lambda_3-t)^{-1}(\lambda_4-t)^{-1}\right]|_{t=0}\\
&+ 2\lambda_1\lambda_2\lambda_3\lambda_4\left[(\lambda_3-t)^{-2}(\lambda_2-t)^{-2}(\lambda_1-t)^{-1}(\lambda_4-t)^{-1}\right]|_{t=0}\\
&+ 2\lambda_1\lambda_2\lambda_3\lambda_4\left[(\lambda_4-t)^{-2}(\lambda_2-t)^{-2}(\lambda_1-t)^{-1}(\lambda_3-t)^{-1}\right]|_{t=0}\\
&+ 2\lambda_1\lambda_2\lambda_3\lambda_4\left[(\lambda_3-t)^{-3}(\lambda_1-t)^{-1}(\lambda_2-t)^{-1}(\lambda_4-t)^{-1}\right]|_{t=0}\\
&+ 2\lambda_1\lambda_2\lambda_3\lambda_4\left[(\lambda_4-t)^{-2}(\lambda_3-t)^{-2}(\lambda_1-t)^{-1}(\lambda_2-t)^{-1}\right]|_{t=0}\\
&+ 2\lambda_1\lambda_2\lambda_3\lambda_4\left[(\lambda_4-t)^{-3}(\lambda_1-t)^{-1}(\lambda_2-t)^{-1}(\lambda_3-t)^{-1}\right]|_{t=0}\\
&= 2\left[\frac{1}{\lambda_1^2}+\frac{1}{\lambda_1\lambda_2}+\frac{1}{\lambda_1\lambda_3}+\frac{1}{\lambda_1\lambda_4}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_2\lambda_3}+\frac{1}{\lambda_2\lambda_4}+\frac{1}{\lambda_3^2}+\frac{1}{\lambda_3\lambda_4}+\frac{1}{\lambda_4^2}\right]\\
&= 2\left[\sum_{i=0}^{2}\lambda_1^{-(2-i)}\lambda_2^{-i}+\sum_{i=1}^{2}\lambda_1^{-(2-i)}\lambda_3^{-i}+\sum_{i=1}^{2}\lambda_1^{-(2-i)}\lambda_4^{-i}+\lambda_2^{-1}\lambda_3^{-1}+\lambda_2^{-1}\lambda_4^{-1}+\lambda_3^{-1}\lambda_4^{-1}\right]
\end{aligned}
\tag{6.4.10}
$$

Third moment

Considering the trend from the previous sections the third moment can be given as

$$
\begin{aligned}
E[S_4^3] &= M_{s_4}^3(0)\\
&= 3!\left[\sum_{i=0}^{3}\lambda_1^{-(3-i)}\lambda_2^{-i}+\sum_{i=1}^{3}\lambda_1^{-(3-i)}\lambda_3^{-i}+\sum_{i=1}^{3}\lambda_1^{-(3-i)}\lambda_4^{-i}+\sum_{i=1}^{2}\lambda_2^{-(3-i)}\lambda_3^{-i}\right]\\
&+ 3!\left[\sum_{i=1}^{2}\lambda_2^{-(3-i)}\lambda_4^{-i}+\sum_{i=1}^{2}\lambda_3^{-(3-i)}\lambda_4^{-i}+\prod_{i=1}^{3}\lambda_i^{-1}\right]
\end{aligned}
\tag{6.4.11}
$$

Fourth moment

Considering the trend from the previous sections the fourth moment can be given as

$$
\begin{aligned}
E[S_4^4] &= M_{s_4}^4(0)\\
&= 4!\left[\sum_{i=0}^{4}\lambda_1^{-(4-i)}\lambda_2^{-i}+\sum_{i=1}^{4}\lambda_1^{-(4-i)}\lambda_3^{-i}+\sum_{i=1}^{4}\lambda_1^{-(4-i)}\lambda_4^{-i}+\sum_{i=1}^{3}\lambda_2^{-(4-i)}\lambda_3^{-i}\right]\\
&+ 4!\left[\sum_{i=1}^{3}\lambda_2^{-(4-i)}\lambda_4^{-i}+\sum_{i=1}^{3}\lambda_3^{-(4-i)}\lambda_4^{-i}+\prod_{i=1}^{4}\lambda_i^{-1}\right]
\end{aligned}
\tag{6.4.12}
$$

This implies the r^{th} raw moment for S_4 can be expressed by

$$\begin{aligned}
E[S_4^r] &= M_{s_4}^r(0) \\
&= r!\left[\sum_{i=0}^{r}\lambda_1^{-(r-i)}\lambda_2^{-i} + \sum_{i=1}^{r}\lambda_1^{-(r-i)}\lambda_3^{-i} + \sum_{i=1}^{r}\lambda_1^{-(r-i)}\lambda_4^{-i} + \sum_{i=1}^{r-1}\lambda_2^{-(r-i)}\lambda_3^{-i}\right] \\
&+ r!\left[\sum_{i=1}^{r-1}\lambda_2^{-(r-i)}\lambda_4^{-i} + \sum_{i=1}^{r-1}\lambda_3^{-(r-i)}\lambda_4^{-i} + \prod_{i=1}^{r}\lambda_i^{-1}\right] \qquad \textbf{(6.4.13)}
\end{aligned}$$

Therefore other properties for the sum of four independent exponential random variables can be obtained as follows:

Mean

$$E[S_4] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} \qquad \textbf{(6.4.14)}$$

Variance

$$\begin{aligned}
Var[S_4] &= E[S_4^2] - (E[S_4])^2 \\
&= 2\left[\frac{1}{\lambda_1^2} + \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_1\lambda_4} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2\lambda_4} + \frac{1}{\lambda_3^2} + \frac{1}{\lambda_3\lambda_4} + \frac{1}{\lambda_4^2}\right] \\
&- \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\right)^2 \\
&= \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} + \frac{1}{\lambda_4^2} \qquad \textbf{(6.4.15)}
\end{aligned}$$

Mode

The mode for the model $g(s_4)$ can be obtained when

$$\frac{d}{ds_4}g(s_4) = 0$$

This implies that

$$
\begin{aligned}
\frac{d}{ds_4}g(s_4) &= -\lambda_1\left(\frac{\lambda_1\lambda_2\lambda_3\lambda_4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}\right)e^{-\lambda_1 s_4}\\
&-\lambda_2\left(\frac{\lambda_1\lambda_2\lambda_3\lambda_4}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}\right)e^{-\lambda_2 s_4}\\
&-\lambda_3\left(\frac{\lambda_1\lambda_2\lambda_3\lambda_4}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}\right)e^{-\lambda_3 s_4}\\
&-\lambda_4\left(\frac{\lambda_1\lambda_2\lambda_3\lambda_4}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}\right)e^{-\lambda_4 s_4}\\
&= -\lambda_1(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)e^{-\lambda_1 s_4}+\lambda_2(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)(\lambda_4-\lambda_3)e^{-\lambda_2 s_4}\\
&-\lambda_3(\lambda_2-\lambda_1)(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)e^{-\lambda_3 s_4}+\lambda_4(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)e^{-\lambda_4 s_4}\\
&= -ln\lambda_1 - ln(\lambda_3-\lambda_2) - ln(\lambda_4-\lambda_2) - ln(\lambda_4-\lambda_3) + \lambda_1 s_4\\
&+ ln\lambda_2 + ln(\lambda_3-\lambda_1) + ln(\lambda_4-\lambda_1) + ln(\lambda_4-\lambda_3) - \lambda_2 s_4\\
&- ln\lambda_3 - (\lambda_2-\lambda_1) - ln(\lambda_4-\lambda_1) - ln(\lambda_4-\lambda_2) + \lambda_3 s_4\\
&+ ln\lambda_4 + ln(\lambda_2-\lambda_1) + ln(\lambda_3-\lambda_1) + ln(\lambda_3-\lambda_2) - \lambda_4 s_4\\
&= -ln\lambda_1 - 2ln(\lambda_4-\lambda_2) - 2ln(\lambda_4-\lambda_3) + \lambda_1 s_4 + ln\lambda_2 - \lambda_2 s_4\\
&+ 2ln(\lambda_3-\lambda_1) - ln\lambda_3 + \lambda_3 s_4 + ln\lambda_4 - \lambda_4 s_4
\end{aligned}
$$

$$
\Rightarrow s_4 = \frac{ln\lambda_2 + ln\lambda_4 - ln\lambda_1 - ln\lambda_3 - 2ln(\lambda_4-\lambda_2) - 2ln(\lambda_4-\lambda_3) + 2ln(\lambda_3-\lambda_1)}{\lambda_2+\lambda_4-\lambda_1-\lambda_3} \quad \textbf{(6.4.16)}
$$

Asymptotic Behaviour of the Model

In this section from equation (6.4.4) we consider the behaviour of the model $g(s_4)$ when $s_4 \to 0$ and as $s_4 \to \infty$. That is

$$
\lim_{s_2\to 0} g(s_2) \text{ and } \lim_{s_2\to\infty} g(s_2)
$$

This implies that

$$
\begin{aligned}
\lim_{S_4\to 0} g(s_4) &= \sum_{i=1}^{4}\lambda_i \lim_{S_4\to 0} e^{-\lambda_i s_4}\prod_{j=1,j\neq i}^{4}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\\
&= \sum_{i=1}^{4}\lambda_i \prod_{j=1,j\neq i}^{4}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\\
&= 0
\end{aligned}
$$

and also

$$\lim_{S_4\to\infty} g(s_4) = \sum_{i=1}^{4} \lambda_i \lim_{S_4\to\infty} e^{-\lambda_i s_4} \prod_{j=1,j\neq i}^{4} \left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)$$
$$= 0 \tag{6.4.17}$$

From the two results it confirms that from the equation (6.4.4) the model has only one mode that is Uni-modal model.

Cumulant Generating Function

The cumulant generating function of a random variable S_4 can be obtained by:

$$C_{s_4}(t) = log[M_{s_4(t)}]$$
$$= log\left\{\frac{\lambda_1\lambda_2\lambda_3\lambda_4}{(\lambda_1-t)(\lambda_2-t)(\lambda_3-t)(\lambda_4-t)}\right\}$$

Using Maclaurin series through expansion the following equation can be obtained:

$$C_{s_4}(t) = \sum_{i=1}^{\infty}(i-1)!\left[\left(\frac{1}{\lambda_1}\right)^i + \left(\frac{1}{\lambda_2}\right)^i + \left(\frac{1}{\lambda_3}\right)^i + \left(\frac{1}{\lambda_4}\right)^i\right]\frac{t^i}{i!} \tag{6.4.18}$$

From the definition of cumulants, the cumulant K_i of a random variable S_4 can be obtained using the coefficients of $\frac{t^i}{i!}$ in equation (6.4.17).

Therefore,

$$K_i = (i-1)!\left[\left(\frac{1}{\lambda_1}\right)^i + \left(\frac{1}{\lambda_2}\right)^i + \left(\frac{1}{\lambda_3}\right)^i + \left(\frac{1}{\lambda_4}\right)^i\right] \tag{6.4.19}$$

Therefore the first four cumulants can be obtained as follows:

$$K_1 = (1-1)!\left[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\right]$$
$$= \frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}$$

$$K_2 = (2-1)!\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \left(\frac{1}{\lambda_3}\right)^2 + \left(\frac{1}{\lambda_4}\right)^2\right]$$
$$= \left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \left(\frac{1}{\lambda_3}\right)^2 + \left(\frac{1}{\lambda_4}\right)^2$$

$$K_3 = (3-1)!\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3 + \left(\frac{1}{\lambda_3}\right)^3 + \left(\frac{1}{\lambda_4}\right)^3\right]$$
$$= 2\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3 + \left(\frac{1}{\lambda_3}\right)^3 + \left(\frac{1}{\lambda_4}\right)^3\right]$$

$$K_4 = (4-1)!\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4 + \left(\frac{1}{\lambda_3}\right)^4 + \left(\frac{1}{\lambda_4}\right)^4\right]$$
$$= 6\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4 + \left(\frac{1}{\lambda_3}\right)^4 + \left(\frac{1}{\lambda_4}\right)^4\right]$$

From the above results the following conclusions can be made:

(i) K_1 which forms the first cumulant gives the mean of the random variable S_4.

(ii) K_2 which forms the second cumulant gives the variance of the random variable S_4.

(iii) Skewness

$$= \frac{K_3}{K_2^{\frac{3}{2}}}$$
$$= \frac{2\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3 + \left(\frac{1}{\lambda_3}\right)^3 + \left(\frac{1}{\lambda_4}\right)^3\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \left(\frac{1}{\lambda_3}\right)^2 + \left(\frac{1}{\lambda_4}\right)^2\right]^{3/2}} \qquad \textbf{(6.4.20)}$$

(iv) Kurtosis

$$= \frac{K_4}{K_2^2}$$
$$= \frac{6\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4 + \left(\frac{1}{\lambda_3}\right)^4 + \left(\frac{1}{\lambda_4}\right)^4\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \left(\frac{1}{\lambda_3}\right)^2 + \left(\frac{1}{\lambda_4}\right)^2\right]^2} \quad \textbf{(6.4.21)}$$

6.4.4 The Case of Arithmetic Sequence of four Parameters

Using the equation (6.4.4) and given that the model parameters takes an arithmetic sequence that is

$$\lambda_2 - \lambda_1 = \lambda_3 - \lambda_2 = \lambda_4 - \lambda_3 = d$$

where d is the common difference.

Therefore

$$\lambda_3 - \lambda_1 = \lambda_4 - \lambda_2 = 2d$$

This then implies that

$$\lambda_j - \lambda_i = (j-i)d \quad \textbf{(6.4.24)}$$

Therefore

$$\begin{aligned}
g(s_4) = &\left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_1}\right)\left(\frac{\lambda_4}{\lambda_4 - \lambda_1}\right)\lambda_1 e^{-\lambda_1 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_2}\right)\left(\frac{\lambda_4}{\lambda_4 - \lambda_2}\right)\lambda_2 e^{-\lambda_2 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - \lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2 - \lambda_3}\right)\left(\frac{\lambda_4}{\lambda_4 - \lambda_3}\right)\lambda_3 e^{-\lambda_3 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - \lambda_4}\right)\left(\frac{\lambda_2}{\lambda_2 - \lambda_4}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_4}\right)\lambda_4 e^{-\lambda_4 s_4} \\
\Rightarrow g(s_4) = &\left(\frac{\lambda_2}{(\lambda_1 + d) - \lambda_1}\right)\left(\frac{\lambda_3}{(\lambda_1 + 2d) - \lambda_1}\right)\left(\frac{\lambda_4}{(\lambda_1 + 3d) - \lambda_1}\right)\lambda_1 e^{-\lambda_1 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - (\lambda_1 + d)}\right)\left(\frac{\lambda_3}{(\lambda_1 + 2d) - (\lambda_1 + d)}\right)\left(\frac{\lambda_4}{(\lambda_1 + 3d) - (\lambda_1 + d)}\right)\lambda_2 e^{-\lambda_2 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - (\lambda_1 + 2d)}\right)\left(\frac{\lambda_2}{(\lambda_1 + d) - (\lambda_1 + 2d)}\right)\left(\frac{\lambda_4}{(\lambda_1 + 3d) - (\lambda_1 + 2d)}\right)\lambda_3 e^{-\lambda_3 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1 - (\lambda_1 + 3d)}\right)\left(\frac{\lambda_2}{(\lambda_1 + d) - (\lambda_1 + 3)}\right)\left(\frac{\lambda_3}{(\lambda_1 + 2d) - (\lambda_1 + 3d)}\right)\lambda_4 e^{-\lambda_4 s_4}
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_1\lambda_2\lambda_3\lambda_4\left(\frac{e^{-\lambda_1 s_4}}{6d^3}\right) + \lambda_1\lambda_2\lambda_3\lambda_4\left(\frac{e^{-\lambda_2 s_4}}{-2d^3}\right) \\
&+ \lambda_1\lambda_2\lambda_3\lambda_4\left(\frac{e^{-\lambda_3 s_4}}{-2d^3}\right) + \lambda_1\lambda_2\lambda_3\lambda_4\left(\frac{e^{-\lambda_4 s_4}}{-6d^3}\right) \\
&= \lambda_1\lambda_2\lambda_3\lambda_4\left[\left(\frac{e^{-\lambda_1 s_4}}{2d^2}\right) + \left(\frac{e^{-\lambda_2 s_4}}{-d^2}\right) + \left(\frac{e^{-\lambda_3 s_4}}{2d^2}\right) + \left(\frac{e^{-\lambda_4 s_4}}{2d^2}\right)\right] \\
&= \prod_{i=1}^{4}\lambda_i\left\{\sum_{i=1}^{4}\frac{e^{-\lambda_i s_4}}{(-1)^{i-1}(i-1)!(4-i)!d^{4-1}}\right\} \\
&= \prod_{i=1}^{4}\lambda_i\left\{\sum_{i=1}^{4}\frac{e^{-\lambda_i s_4}}{\gamma_{i,4}}\right\} \qquad \text{where} \quad \gamma_{i,4} = (-1)^{i-1}(i-1)!(4-i)!d^{4-1}
\end{aligned}
\tag{6.4.25}
$$

6.4.5 The Case of Geometric Sequence of four Parameters

If the model parameters for $n=4$ takes the form of geometric sequence with the a common ratio r that is $\frac{\lambda_2}{\lambda_1} = \frac{\lambda_3}{\lambda_2} = \frac{\lambda_4}{\lambda_3} = r$. This implies then $\lambda_2 = r\lambda_1, \lambda_3 = r^2\lambda_1,$ and $\lambda_4 = r^3\lambda_1$.

Therefore

$$
\begin{aligned}
g(s_4) &= \left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_1}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_1}\right)\lambda_1 e^{-\lambda_1 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_2}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_2}\right)\lambda_2 e^{-\lambda_2 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1-\lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_3}\right)\left(\frac{\lambda_4}{\lambda_4-\lambda_3}\right)\lambda_3 e^{-\lambda_3 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1-\lambda_4}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_4}\right)\left(\frac{\lambda_3}{\lambda_3-\lambda_4}\right)\lambda_4 e^{-\lambda_4 s_4} \\
\Rightarrow g(s_4) &= \left(\frac{r\lambda_1}{r\lambda_1-\lambda_1}\right)\left(\frac{r^2\lambda_1}{r^2\lambda_1-\lambda_1}\right)\left(\frac{r^3\lambda_1}{r^3\lambda_1-\lambda_1}\right)\lambda_1 e^{-\lambda_1 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1-r\lambda_1}\right)\left(\frac{r^2\lambda_1}{r^2\lambda_1-r\lambda_1}\right)\left(\frac{r^3\lambda_1}{r^3\lambda_1-r\lambda_1}\right)\lambda_2 e^{-\lambda_2 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1-r^2\lambda_1}\right)\left(\frac{r\lambda_1}{r\lambda_1-r^2\lambda_1}\right)\left(\frac{r^2\lambda_1}{r^3\lambda_1-r^2\lambda_1}\right)\lambda_3 e^{-\lambda_3 s_4} \\
&+ \left(\frac{\lambda_1}{\lambda_1-r^3\lambda_1}\right)\left(\frac{r\lambda_1}{r\lambda_1-r^3\lambda_1}\right)\left(\frac{r^2\lambda_1}{r^2\lambda_1-r^3\lambda_1}\right)\lambda_4 e^{-\lambda_4 s_4}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\lambda_1 e^{-\lambda_1 s_4}}{(1-r^{-1})(1-r^{-2})(1-r^{-3})} + \frac{\lambda_2 e^{-\lambda_2 s_4}}{(1-r)(1-r^{-1})(1-r^{-2})} \\
&+ \frac{\lambda_3 e^{-\lambda_3 s_4}}{(1-r^{-2})(1-r)(1-r^{-1})} + \frac{\lambda_4 e^{-\lambda_4 s_4}}{(1-r^{3})(1-r^{2})(1-r)} \\
&= \sum_{i=1}^{4} \frac{\lambda_i e^{-\lambda_i s_4}}{\prod_{j=1,j\neq i}^{4}(1-r^{i-j})} \\
&= \sum_{i=1}^{4} \frac{\lambda_i e^{-\lambda_i s_4}}{P_{i,4}} \qquad \text{where} \quad P_{i,4} = \prod_{j=1,j\neq i}^{4}(1-r^{i-j}) \qquad \textbf{(6.4.26)}
\end{aligned}
$$

6.5 Hypo-exponential Distribution for a fixed sum of n independent random variables with distinct parameters

6.5.1 Construction Using Convolution Approach

Extending from the previous sections for $N = n$ fixed sum of independent exponential random variables to suggest the general formula of getting the distributions for $n \geq 2$ random variables.

Proposition 2

Let

$$S_n = X_1 + X_2 + \cdots + X_n \qquad \textbf{(6.5.1)}$$

for $n \geq 2$ then the pdf will be given by:

$$
\begin{aligned}
g(s_n) &= \sum_{i=1}^{n} \left\{ \lambda_i e^{-\lambda_i s_n} \prod_{j=1,j\neq i}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right) \right\} \qquad \textbf{(6.5.2a)} \\
&= \sum_{i=1}^{n} \left\{ \left(\prod_{j=1,j\neq i}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i} \right) \lambda_i e^{-\lambda_i s_n} \right\} \\
&= \sum_{i=1}^{n} C_{i,n} \lambda_i e^{-\lambda_i s_n} \qquad \textbf{(6.5.2b)}
\end{aligned}
$$

where

$$C_{i,n} = \prod_{j=1,j\neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \qquad \textbf{(6.5.3)}$$

Proof

Using the induction formula for the case where $N = n$ and let consider $n+1$ arbitrary independent exponential X_i random variables with different parameters λ_i, for $i = 1, 2, \cdots, n+1$.

Now let

$$\begin{aligned} S_{n+1} &= X_1 + X_2 + \cdots + X_n + X_{n+1} \\ &= S_n + X_{n+1} \qquad \text{where} \quad S_n = X_1 + X_2 + \cdots + X_n \end{aligned} \tag{6.5.4}$$

Let $F_i(x_i)$ and $G(s_{n+1})$ be the cumulative distribution functions for X_i and S_n respectively.

Cumulative Density Function

Here we use the same method as in the previous section to get the cdf for the distribution S_{n+1}. This follows that

$$\begin{aligned} G(S_{n+1}) &= Prob(S_{n+1} \leq s_{n+1}) \\ &= Prob(S_n + X_{n+1} \leq s_{n+1}) \\ &= Prob(X_{n+1} \leq s_{n+1} - s_n) \\ &= Prob(0 \leq s_n \leq s_{n+1}, 0 \leq X_{n+1} \leq s_{n+1} - s_n) \\ \Rightarrow G(S_{n+1}) &= \int_0^{s_{n+1}} \int_0^{s_{n+1}-s_n} g(s_n) f(x_{n+1}) dx_{n+1} ds_n \\ &= \int_0^{s_{n+1}} g(s_n) \left[\int_0^{s_{n+1}-s_n} f(x_{n+1}) dx_{n+1} \right] ds_n \\ &= \int_0^{s_{n+1}} g(s_n) \left[\int_0^{s_{n+1}-s_n} \lambda_{n+1} e^{-\lambda_{n+1} s_{n+1}} dx_{n+1} \right] ds_n \\ &= \int_0^{s_{n+1}} g(s_n) \left[1 - e^{-\lambda_{n+1}(s_{n+1}-s_n)} \right] ds_n \\ &= \int_0^{s_{n+1}} \left[g(s_n) - g(s_n) e^{-\lambda_{n+1}(s_{n+1}-s_n)} \right] ds_n \end{aligned}$$

But from equation (6.5.2b)

$$g(s_n) = \sum_{i=1}^{n} C_{i,n} \lambda_i e^{-\lambda_i s_n}$$

Therefore

$$
\begin{aligned}
G(S_{n+1}) &= \sum_{i=1}^{n} C_{i,n} \int_0^{s_{n+1}} \lambda_i e^{-\lambda_i s_n} ds_n - \sum_{i=1}^{n} C_{i,n} e^{-\lambda_{n+1} s_{n+1}} \int_0^{s_{n+1}} \lambda_i e^{(-\lambda_{n+1}-\lambda_i)s_n} ds_n \\
&= \sum_{i=1}^{n} C_{i,n} \left[1 - e^{-\lambda_i s_{n+1}}\right] - \sum_{i=1}^{n} C_{i,n} e^{-\lambda_{n+1} s_{n+1}} \left[\frac{e^{(\lambda_{n+1}-\lambda_i)s_{n+1}}}{\lambda_{n+1}-\lambda_i}\right] \\
&= \sum_{i=1}^{n} C_{i,n} - \sum_{i=1}^{n} C_{i,n} e^{-\lambda_i s_{n+1}} - \sum_{i=1}^{n} C_{i,n} \frac{\lambda_i e^{-\lambda_i s_{n+1}}}{\lambda_{n+1}-\lambda_i} + \sum_{i=1}^{n} C_{i,n} \frac{\lambda_i e^{-\lambda_{n+1} s_{n+1}}}{\lambda_{n+1}-\lambda_i} \\
&= \sum_{i=1}^{n} C_{i,n} - \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_i}\right) e^{-\lambda_i s_{n+1}} + \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_i}{\lambda_{n+1}-\lambda_i}\right) e^{-\lambda_{n+1} s_{n+1}}
\end{aligned}
$$

From equations (6.3.6) and (6.4.6) it is indicates that

$$\sum_{i=1}^{n} C_{i,n} = 1$$

Hence the result

$$G(S_{n+1}) = 1 - \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_i}\right) e^{-\lambda_{n+1} s_{n+1}} - \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_i}{\lambda_i-\lambda_{n+1}}\right) e^{-\lambda_{n+1} s_{n+1}} \qquad \textbf{(6.5.5)}$$

Therefore, from the general cdf expression in equation (6.5.5) it can be concluded that

$$
\begin{aligned}
\lim_{s_{n+1}\to\infty} G(s_n) &= 1 \\
\text{and} \\
\lim_{s_{n+1}\to 0} G(s_n) &= 0 \qquad \textbf{(6.5.6)}
\end{aligned}
$$

Probability Density Function

From equation (6.5.5) we can now derive the pdf for the general sum of $n+1$ independent exponential random variables $S_n + 1$. This follows that

$$
\begin{aligned}
g(s_{n+1}) &= \frac{d}{ds_{n+1}} \left[G(s_{n+1})\right] \\
&= \frac{d}{ds_{n+1}} \left[1 - \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_i}\right) e^{-\lambda_i s_{n+1}} - \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_i}{\lambda_i-\lambda_{n+1}}\right) e^{-\lambda_{n+1} s_{n+1}}\right] \\
&= \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_i \lambda_{n+1}}{\lambda_{n+1}-\lambda_i}\right) e^{-\lambda_i s_{n+1}} + \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_{n+1} \lambda_i}{\lambda_i-\lambda_{n+1}}\right) e^{-\lambda_{n+1} s_{n+1}}
\end{aligned}
$$

$$= \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} \right) \lambda_i e^{-\lambda_i s_{n+1}} + \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_i}{\lambda_i - \lambda_{n+1}} \right) \lambda_{n+1} e^{-\lambda_{n+1} s_{n+1}}$$

$$= \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} \right) \lambda_i e^{-\lambda_i s_{n+1}} + K_{n+1} \lambda_{n+1} e^{-\lambda_{n+1} s_{n+1}} \quad \textbf{(6.5.7)}$$

where

$$K_{n+1} = \sum_{i=1}^{n} C_{i,n} \left(\frac{\lambda_i}{\lambda_i - \lambda_{n+1}} \right) \quad \textbf{(6.5.8)}$$

But

$$C_{i,n} = \prod_{j=1, j\neq i}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right) = \left(\frac{\lambda_1}{\lambda_1 - \lambda_i} \right) \left(\frac{\lambda_2}{\lambda_2 - \lambda_i} \right) \cdots \left(\frac{\lambda_n}{\lambda_n - \lambda_i} \right)$$

Therefore

$$\begin{aligned} C_{i,n} \left(\frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} \right) &= \left(\frac{\lambda_1}{\lambda_1 - \lambda_i} \right) \left(\frac{\lambda_2}{\lambda_2 - \lambda_i} \right) \cdots \left(\frac{\lambda_n}{\lambda_n - \lambda_i} \right) \left(\frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} \right) \\ &= \prod_{j=1, j\neq i}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i}, \text{for} \quad j = 1, 2, \cdots, n+1 \\ &= C_{i,n+1} \end{aligned}$$

This implies that the formula (6.5.7) becomes

$$g(s_{n+1}) = \sum_{i=1}^{n} C_{i,n+1} \lambda_i e^{-\lambda_i s_{n+1}} + K_{n+1} \lambda_{n+1} e^{-\lambda n+1 s_{n+1}} \quad \textbf{(6.5.9)}$$

where K_{n+1} is a constant which does not depend on s_{n+1}.

Equation (6.5.4) can be rewritten as

$$S_{n+1} = X_1 + S_n \qquad \text{where} \quad S_n = X_2 + X_3 + \cdots + X_{n+1} \quad \textbf{(6.5.10)}$$

Using the proposition 2 the pdf for the distribution S_n in the above equation (6.5.10) can be given by

$$\begin{aligned} g(s_n) &= \sum_{i=2}^{n+1} \left\{ \lambda_i e^{-\lambda_i s_n} \prod_{j=2, j\neq i}^{n+1} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right) \right\} \\ &= \sum_{i=2}^{n+1} C^*_{i,n+1} \lambda_i e^{-\lambda_i s_n} \end{aligned} \tag{6.5.11}$$

where

$$C^*_{i,n+1} = \prod_{j=2, j\neq i}^{n+1} \frac{\lambda_j}{\lambda_j - \lambda_i}$$

Therefore the cdf of the distribution can also be given by:

$$\begin{aligned} G(s_{n+1}) &= Prob(S_{n+1} \leq s_{n+1}) \\ &= Prob(S_n + X_1 \leq s_{n+1}) \\ &= Prob(X_1 \leq s_{n+1} - s_n) \\ &= Prob(0 \leq s_n \leq s_{n+1}, 0 \leq X_1 \leq s_{n+1} - s_n) \\ &= \int_0^{s_{n+1}} \int_0^{s_{n+1}-s_n} g(s_n) f(x_1) dx_1 ds_n \\ &= \int_0^{s_{n+1}} g(s_n) \left[\int_0^{s_{n+1}-s_n} f(x_1) dx_1 \right] ds_n \\ &= \int_0^{s_{n+1}} g(s_n) \left[\int_0^{s_{n+1}-s_n} \lambda_1 e^{-\lambda_1 x_1} dx_1 \right] ds_n \\ &= \int_0^{s_{n+1}} g(s_n) \left[1 - e^{-\lambda_1 (s_{n+1} - s_n)} \right] ds_n \\ &= \left[\int_0^{s_{n+1}} g(s_n) - g(s_n) e^{-\lambda_1 (s_{n+1} - s_n)} \right] ds_n \\ &= \sum_{i=2}^{n+1} C^*_{i,n+1} \int_0^{s_{n+1}} \lambda_i e^{-\lambda_i s_n} ds_n - \sum_{i=2}^{n+1} C^*_{i,n+1} \lambda_i e^{-\lambda_i s_{n+1}} \int_0^{s_{n+1}} e^{(\lambda_1 - \lambda_i) s_n} ds_n \\ &= \sum_{i=2}^{n+1} C^*_{i,n+1} \left[1 - e^{-\lambda_i s_{n+1}} \right] - \sum_{i=2}^{n+1} C^*_{i,n+1} \lambda_i \left[\frac{e^{(\lambda_1 - \lambda_i) s_{n+1}} - 1}{\lambda_1 - \lambda_i} \right] \\ &= \sum_{i=2}^{n+1} C^*_{i,n+1} - \sum_{i=2}^{n+1} C^*_{i,n+1} e^{-\lambda_i s_{n+1}} - \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_i}{\lambda_1 - \lambda_i} \right) e^{-\lambda_i s_{n+1}} \\ &+ \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_i}{\lambda_1 - \lambda_i} \right) e^{-\lambda_1 s_{n+1}} \\ \therefore G(s_{n+1}) &= 1 - \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_1}{\lambda_1 - \lambda_i} \right) e^{-\lambda_i s_{n+1}} - \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_i}{\lambda_i - \lambda_1} \right) e^{-\lambda_1 s_{n+1}} \end{aligned} \tag{6.5.12}$$

Similarly as $s_{n+1} \to \infty$ it can be deduced that

$$\lim_{s_{n+1}\to\infty} G(s_{n+1}) = 1$$

From the above equation (6.5.12) the pdf of $G_{s_{n+1}}$ can be given as:

$$\begin{aligned}
g(s_{n+1}) &= \frac{d}{ds_{n+1}} G(s_{n+1}) \\
&= \frac{d}{ds_{n+1}} \left[1 - \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_1}{\lambda_1 - \lambda_i} \right) e^{-\lambda_i s_{n+1}} - \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_i}{\lambda_i - \lambda_1} \right) e^{-\lambda_1 s_{n+1}} \right] \\
&= \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_i \lambda_1}{\lambda_1 - \lambda_i} \right) e^{-\lambda_i s_{n+1}} + \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_1 \lambda_i}{\lambda_i - \lambda_1} \right) e^{-\lambda_1 s_{n+1}} \\
&= \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_1}{\lambda_1 - \lambda_i} \right) \lambda_i e^{-\lambda_i s_{n+1}} + \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_i}{\lambda_i - \lambda_1} \right) \lambda_1 e^{-\lambda_1 s_{n+1}} \\
&= \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_1}{\lambda_1 - \lambda_i} \right) \lambda_i e^{-\lambda_i s_{n+1}} + K_1 \lambda_1 e^{-\lambda_1 s_{n+1}} \qquad \textbf{(6.5.13)}
\end{aligned}$$

where

$$K_1 = \sum_{i=0}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_i}{\lambda_i - \lambda_1} \right)$$

Comparing the equations (6.5.9) to (6.5.13), then

$$\sum_{i=1}^{n} C_{i,n+1} \lambda_i e^{-\lambda_i s_{n+1}} + K_{n+1} \lambda_{n+1} e^{-\lambda n+1 s_{n+1}} = \sum_{i=2}^{n+1} C^*_{i,n+1} \left(\frac{\lambda_1}{\lambda_1 - \lambda_i} \right) \lambda_i e^{-\lambda_i s_{n+1}} + K_1 \lambda_1 e^{-\lambda_1 s_{n+1}}$$

But,

$$\begin{aligned}
C^*_{i,n+1} \left(\frac{\lambda_1}{\lambda_1 - \lambda_i} \right) &= \left[\prod_{j=2, j\neq i}^{n+1} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right) \right] \left(\frac{\lambda_1}{\lambda_1 - \lambda_i} \right) \\
&= \left[\left(\frac{\lambda_2}{\lambda_2 - \lambda_i} \right) \left(\frac{\lambda_3}{\lambda_3 - \lambda_i} \right) \cdots \left(\frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} \right) \right] \left(\frac{\lambda_1}{\lambda_1 - \lambda_i} \right) \\
&= \left(\frac{\lambda_1}{\lambda_1 - \lambda_i} \right) \left(\frac{\lambda_2}{\lambda_2 - \lambda_i} \right) \cdots \left(\frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} \right) \\
&= C_{i,n+1}
\end{aligned}$$

This implies that

$$\sum_{i=1}^{n} C_{i,n+1}\lambda_i e^{-\lambda_i s_{n+1}} + K_{n+1}\lambda_{n+1}e^{-\lambda_{n+1}s_{n+1}} = \sum_{i=2}^{n+1} C_{i,n+1}\lambda_i e^{-\lambda_i s_{n+1}} + K_1\lambda_1 e^{-\lambda_1 s_{n+1}}$$

that is

$$\begin{aligned} C_{1,n+1}\lambda_1 e^{-\lambda_1 s_{n+1}} \\ +\sum_{i=2}^{n} C_{i,n+1}\lambda_i e^{-\lambda_i s_{n+1}} + K_{n+1}\lambda_{n+1}e^{-\lambda_{n+1}s_{n+1}} &= \sum_{i=2}^{n} C_{i,n+1}\lambda_i e^{-\lambda_i s_{n+1}} \\ &\quad + C_{n+1,n+1}\lambda_{n+1}e^{-\lambda_{n+1}s_{n+1}} + K_1\lambda_1 e^{-\lambda_1 s_{n+1}} \\ C_{1,n+1}\lambda_1 e^{-\lambda_1 s_{n+1}} + K_{n+1}\lambda_{n+1}e^{-\lambda_{n+1}s_{n+1}} &= C_{n+1,n+1}\lambda_{n+1}e^{-\lambda_{n+1}s_{n+1}} + K_1\lambda_1 e^{-\lambda_1 s_{n+1}} \end{aligned}$$

Comparing the coefficients of $\lambda_{n+1}e^{-\lambda_{n+1}s_{n+1}}$, we get

$$K_{n+1} = C_{n+1,n+1}$$

Also comparing the coefficients of $\lambda_1 e^{-\lambda_1 s_{n+1}}$, we get

$$K_1 = C_{1,n+1}$$

Thus

$$\begin{aligned} g(s_{n+1}) &= \sum_{i=1}^{n} C_{i,n+1}\lambda_i e^{-\lambda_i s_{n+1}} + K_{n+1}\lambda_{n+1}e^{-\lambda_{n+1}s_{n+1}} \\ &= \sum_{i=1}^{n} C_{i,n+1}\lambda_i e^{-\lambda_i s_{n+1}} + C_{n+1,n+1}\lambda_{n+1}e^{-\lambda_{n+1}s_{n+1}} \\ &= \sum_{i=1}^{n+1} C_{i,n+1}\lambda_i e^{-\lambda_i s_{n+1}} \\ &= \sum_{i=1}^{n+1} \prod_{j=1,j\neq i} \left(\frac{\lambda_j}{\lambda_j - \lambda_i}\right)\lambda_i e^{-\lambda_i s_{n+1}} \qquad \textbf{(6.5.14a)} \end{aligned}$$

Alternatively,

$$\begin{aligned}
g(s_{n+1}) &= \sum_{i=2}^{n+1} C^*_{i,n+1}\left(\frac{\lambda_1}{\lambda_1-\lambda_i}\right)\lambda_i e^{-\lambda_i s_{n+1}} + K_1\lambda_1 e^{-\lambda_1 s_{n+1}} \\
&= \sum_{i=2}^{n+1} C_{i,n+1}\lambda_i e^{-\lambda_i s_{n+1}} + C_{1,n+1}\lambda_1 e^{-\lambda_1 s_{n+1}} \\
&= C_{1,n+1}\lambda_1 e^{-\lambda_1 s_{n+1}} + \sum_{i=2}^{n+1} C_{i,n+1}\lambda_i e^{-\lambda_i s_{n+1}} \\
&= \sum_{i=1}^{n+1} C_{i,n+1}\lambda_i e^{-\lambda_i s_{n+1}} \qquad \textbf{(6.5.14b)}
\end{aligned}$$

hence the result.

Hazard Function

From the definition, the hazard function of the sum of n independent exponential random variables s_n is defined by:

$$\begin{aligned}
h(s_n) &= \frac{g(s_n)}{1-G(s_n)} \\
&= \frac{\sum_{i=1}^{n} C_{i,n}\lambda_i e^{-\lambda_i s_n}}{\sum_{i=1}^{n} C_{i,n} e^{-\lambda_i s_n}} \\
&= \sum_{i=1}^{n} \lambda_i \qquad \textbf{(6.5.15)}
\end{aligned}$$

Validity of the Model

For the model $g(s_n)$ to be said valid if it satisfies the condition

$$\int_0^\infty g(s_n)ds_n = 1$$

Proof

$$\begin{aligned}
\int_0^\infty g(s_n)ds_n &= \int_0^\infty \sum_{i=1}^{n}\lambda_i e^{-\lambda_i s_n}\prod_{j=1,j\neq i}^{n}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)ds_n \\
&= \sum_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\lambda_i\int_0^\infty e^{-\lambda_i s_n}ds_n \\
&= \sum_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(\frac{\lambda_j}{\lambda_j-\lambda_i}\right) \\
&= 1 \qquad \textbf{(6.5.16)}
\end{aligned}$$

As required, hence the model is said to be valid.

6.5.2 Construction Using Moment Generating Function Approach

The moment generating function for the fixed sum of n independent exponential random variables, s_n can be obtained by:

$$M_{s_n}(t) = E(e^{ts_n}); \qquad \text{where} \quad s_n = X_1 + X_2 + \cdots + X_n \tag{6.5.17}$$

From the moment generating function properties then

$$\begin{aligned}
M_{s_n}(t) &= M_{X_1+X_2+\cdots+X_n}(t) \\
&= E(e^{tx_1}) . E(e^{tx_2}) . \cdots . E(e^{tx_n}) \\
&= \left(\frac{\lambda_1}{\lambda_1 - t}\right)\left(\frac{\lambda_2}{\lambda_2 - t}\right)\cdots\left(\frac{\lambda_n}{\lambda_n - t}\right) \\
\Rightarrow M_{s_n}(t) &= \frac{\lambda_1\lambda_2\cdots\lambda_n}{(\lambda_1 - t)(\lambda_2 - t)\cdots(\lambda_n - t)} \\
&= (\lambda_1\lambda_2\cdots\lambda_n)[(\lambda_1 - t)(\lambda_2 - t)\cdots(\lambda_n - t)]^{-1} \\
&= \lambda_1(\lambda_1 - t)^{-1}\lambda_2(\lambda_2 - t)^{-1}\cdots\lambda_n(\lambda_n - t)^{-1} \\
\therefore M_{s_n}(t) &= \prod_{i=1}^{n}\lambda_i[\lambda_i - t]^{-1}
\end{aligned} \tag{6.5.18}$$

6.5.3 Properties of the Distribution

Moments

The r^{th} raw moment of an independent exponential random variable can be given by:

$$E[S_n^r] = \frac{d^r M_{s_n}(t)}{dt^r}|_{t=0}$$

But from equation (6.5.17) and following the trend from the previous sections on moments we can now obtain the first four moments as follows:

First moment

$$\begin{aligned}
E[S_n] &= M'_{s_n}(0) \\
&= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots + \frac{1}{\lambda_n} \\
&= \sum_{i=1}^{n}\lambda_i^{-1}
\end{aligned} \tag{6.5.19}$$

Second moment

$$
\begin{aligned}
E[S_n^2] &= M\text{''}_{s_n}(0) \\
&= 2\left[\sum_{i=0}^{2}\lambda_1^{-(2-i)}\lambda_2^{-i} + \sum_{i=1}^{2}\lambda_1^{-(2-i)}\lambda_3^{-i} + \sum_{i=1}^{2}\lambda_1^{-(2-i)}\lambda_4^{-i} + \cdots + \sum_{i=1}^{2}\lambda_1^{-(2-i)}\lambda_n^{-i}\right] \\
&+ 2\left[\lambda_2^{-1}\lambda_3^{-1} + \lambda_2^{-1}\lambda_4^{-1} + \cdots + \lambda_2^{-1}\lambda_n^{-1} + \cdots + \lambda_{n-1}^{-1}\lambda_n^{-1}\right] \qquad \textbf{(6.5.20)}
\end{aligned}
$$

Third moment

$$
\begin{aligned}
E[S_n^3] &= M_{s_n}^3(0) \\
&= 3!\left[\sum_{i=0}^{3}\lambda_1^{-(3-i)}\lambda_2^{-i} + \sum_{i=1}^{3}\lambda_1^{-(3-i)}\lambda_3^{-i} + \cdots + \sum_{i=1}^{3}\lambda_1^{-(3-i)}\lambda_n^{-i}\right] \\
&+ 3!\left[\sum_{i=1}^{2}\lambda_2^{-(3-i)}\lambda_3^{-i} + \cdots + \sum_{i=1}^{2}\lambda_2^{-(3-i)}\lambda_n^{-i}\right] \\
&+ \cdots + 3!\left[\sum_{i=1}^{2}\lambda_{n-1}^{-(3-i)}\lambda_n^{-i} + \prod_{i=1}^{n}\lambda_i^{-1}\right] \qquad \textbf{(6.5.21)}
\end{aligned}
$$

Fourth moment

$$
\begin{aligned}
E[S_n^4] &= M_{s_n}^4(0) \\
&= 4!\left[\sum_{i=0}^{4}\lambda_1^{-(4-i)}\lambda_2^{-i} + \sum_{i=1}^{4}\lambda_1^{-(4-i)}\lambda_3^{-i} + \cdots + \sum_{i=1}^{4}\lambda_1^{-(4-i)}\lambda_n^{-i}\right] \\
&+ 4!\left[\sum_{i=1}^{3}\lambda_2^{-(4-i)}\lambda_3^{-i} + \cdots + \sum_{i=1}^{3}\lambda_2^{-(4-i)}\lambda_n^{-i}\right] \\
&+ \cdots + 4!\left[\sum_{i=1}^{2}\lambda_{n-1}^{-(4-i)}\lambda_n^{-i} + \prod_{i=1}^{n}\lambda_i^{-1}\right] \qquad \textbf{(6.5.22)}
\end{aligned}
$$

This implies the r^{th} moment for s_n can be expressed by

$$
\begin{aligned}
E[S_n^r] &= M_{s_n}^r(0) \\
&= r!\left[\sum_{i=0}^{r}\lambda_1^{-(r-i)}\lambda_2^{-i} + \sum_{i=1}^{r}\lambda_1^{-(r-i)}\lambda_3^{-i} + \cdots + \sum_{i=1}^{r}\lambda_1^{-(r-i)}\lambda_n^{-i}\right] \\
&+ r!\left[\sum_{i=1}^{(r-1)}\lambda_2^{-(r-i)}\lambda_3^{-i} + \cdots + \sum_{i=1}^{(}r-1)\lambda_2^{-(r-i)}\lambda_n^{-i}\right] \\
&+ \cdots + r!\left[\sum_{i=1}^{2}\lambda_{n-1}^{-(r-i)}\lambda_n^{-i} + \prod_{i=1}^{n}\lambda_i^{-1}\right] \qquad \textbf{(6.5.23)}
\end{aligned}
$$

Therefore other properties for the sum of n independent exponential random variables can be obtained as follows:

Mean

$$E[S_4] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n} \tag{6.5.24}$$

Variance

$$\begin{aligned} Var[S_4] &= E[S_4^2] - (E[S_4])^2 \\ &= \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \cdots + \frac{1}{\lambda_n^2} \end{aligned} \tag{6.5.25}$$

Asymptotic Behaviour of the Model

In determining the asymptotic behaviour from equation (6.5.2b)we consider the behaviour of the model $g(s_n)$ when $s_n \to 0$ and as $s_n \to \infty$. That is

$$\lim_{s_n \to 0} g(s_n) \text{ and } \lim_{s_n \to \infty} g(s_n)$$

Therefore,

$$\begin{aligned} \lim_{S_n \to 0} g(s_n) &= \sum_{i=1}^{n} \lambda_i \lim_{S_n \to 0} e^{-\lambda_i s_n} \prod_{j=1, j \neq i}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right) \\ &= \sum_{i=1}^{n} \lambda_i \prod_{j=1, j \neq i}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right) \\ &= 0 \end{aligned}$$

and also

$$\begin{aligned} \lim_{S_n \to \infty} g(s_n) &= \sum_{i=1}^{n} \lambda_i \lim_{S_n \to \infty} e^{-\lambda_i s_n} \prod_{j=1, j \neq i}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right) \\ &= 0 \end{aligned} \tag{6.5.26}$$

From the two results it confirms that from the equation (6.5.2b) the model has only one mode that is Uni-modal model.

Cumulant Generating Function

The cumulant generating function of a random variable S_n can be obtained by:

$$
\begin{aligned}
C_{s_n}(t) &= log[M_{s_n(t)}] \\
&= log\left\{\frac{\lambda_1\lambda_2\cdots\lambda_n}{(\lambda_1-t)(\lambda_2-t)\cdots(\lambda_n-t)}\right\}
\end{aligned}
$$

Using Maclaurin series through expansion the following equation can be obtained:

$$
C_{s_n}(t) = \sum_{i=1}^{\infty}(i-1)!\left[\left(\frac{1}{\lambda_1}\right)^i + \left(\frac{1}{\lambda_2}\right)^i + \cdots + \left(\frac{1}{\lambda_n}\right)^i\right]\frac{t^i}{i!} \qquad \textbf{(6.5.27)}
$$

From the definition of cumulants, the cumulant K_i of a random variable S_n can be obtained using the coefficients of $\frac{t^i}{i!}$ in equation (5.5.27).

Hence,

$$
K_i = (i-1)!\left[\left(\frac{1}{\lambda_1}\right)^i + \left(\frac{1}{\lambda_2}\right)^i + \cdots + \left(\frac{1}{\lambda_n}\right)^i\right] \qquad \textbf{(6.5.28)}
$$

Therefore the first four cumulants can be obtained as follows:

$$
\begin{aligned}
K_1 &= (1-1)!\left[\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n}\right] \\
&= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n}
\end{aligned}
$$

$$
\begin{aligned}
K_2 &= (2-1)!\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \cdots + \left(\frac{1}{\lambda_n}\right)^2\right] \\
&= \left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \cdots + \left(\frac{1}{\lambda_n}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
K_3 &= (3-1)!\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3 + \cdots + \left(\frac{1}{\lambda_n}\right)^3\right] \\
&= 2\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3 + \cdots + \left(\frac{1}{\lambda_n}\right)^3\right]
\end{aligned}
$$

$$
\begin{aligned}
K_4 &= (4-1)!\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4 + \cdots + \left(\frac{1}{\lambda_n}\right)^4\right] \\
&= 6\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4 + \cdots + \left(\frac{1}{\lambda_n}\right)^4\right]
\end{aligned}
$$

From the above results the following conclusions can be made:

(i) K_1 the first cumulant gives the mean of the random variable S_n.

(ii) K_2 the second cumulant gives the variance of the random variable S_n.

(iii) Skewness

$$= \frac{K_3}{K_2^{\frac{3}{2}}} = \frac{2\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3 + \cdots + \left(\frac{1}{\lambda_n}\right)^3\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \cdots + \left(\frac{1}{\lambda_n}\right)^2\right]^{3/2}} \quad \textbf{(6.5.29)}$$

(iv) Kurtosis

$$= \frac{K_4}{K_2^2} = \frac{6\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4 + \cdots + \left(\frac{1}{\lambda_n}\right)^4\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2 + \cdots + \left(\frac{1}{\lambda_n}\right)^2\right]^2} \quad \textbf{(6.5.30)}$$

6.5.4 The Case of Arithmetic Sequence of A Fixed sum with distinct Parameters

Following the trend of the first four distinct parameters sum of $n \geq 2$ exponential random variables with the parameters are forming an arithmetic sequence.

From equation (6.4.24) that is

$$\lambda_j - \lambda_i = (j-i)d$$

where d is the common difference in the arithmetic sequence of parameters, it follows that the general formula can be suggested as:

$$g(s_n) = \prod_{i=1}^{n} \lambda_i \left\{ \sum_{i=1}^{n} \frac{e^{-\lambda_i s_n}}{(-1)^{i-1}(i-1)!(n-i)!d^{n-1}} \right\}$$
$$= \prod_{i=1}^{n} \lambda_i \left\{ \sum_{i=1}^{n} \frac{e^{-\lambda_i s_n}}{\gamma_{i,n}} \right\}$$
$$\text{where} \quad \gamma_{i,n} = (-1)^{i-1}(i-1)!(n-i)!d^{n-1} \quad \textbf{(6.5.31)}$$

6.5.5 The Case of Geometric Sequence of A Fixed sum with distinct Parameters

Using the general equation for the fixed sum of $n \geq 2$ independent exponential random variables with distinct parameters and then from equation (5.4.26), it follows to suggest the general formula as:

$$\begin{aligned} g(s_n) &= \sum_{i=1}^{n} \left\{ \lambda_i e^{-\lambda_i s_n} \prod_{j=1, j\neq i}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right) \right\} \\ &= \sum_{i=1}^{n} \left\{ \frac{\lambda_i e^{\lambda_i s_n}}{\prod_{j=1, j\neq i}^{n} (1 - r^{i-j})} \right\} \\ &= \sum_{i=1}^{n} P_{i,n} \lambda_i e^{\lambda_i s_n} \qquad \text{where} \quad P_{i,n} = \frac{1}{\prod_{j=1, j\neq i}^{n} (1 - r^{i-j})} \end{aligned} \tag{6.5.32}$$

Hence the result.

7 APPLICATIONS OF SUMS OF EXPONENTIAL RANDOM VARIABLES

7.1 Applications of Erlang Distributions

In an Erlang distribution it is assumed that the rate parameters of the process are independent and identically distributed. Therefore, if the number of phases in a process is known and the rates assumed to be equal, then the coefficient of variation C for the total time of the process satisfies:

$$\frac{1}{\sqrt{n}} \leq c\frac{1}{\sqrt{n-1}} \qquad \text{f or n} \in \mathbb{Z}^{+}$$

Therefore an Erlang distribution of n distribution gives a better fit for the total time of the process (Adan, 2001). The following are some of the application examples that can be modelled through an Erlang distributions:

7.1.1 Times of waiting

When independent events with an average rate occur in an exponential process, the times of waiting between the n number of exponential occurrences of such events results to an Erlang distribution.

If the time of waiting is modelled as an exponential distribution having the same parameter rates $\lambda > 0$, then the total waiting time for an n^{th} turns will form the sum of n exponential random variables with identical parameter rates λ. Therefore, if we let $X_1, X_2, \cdots, X_n$ of the successful waiting times and $X_i \sim exp(\lambda)$ where $i = 1, 2, \cdots, n$ then total waiting time for n^{th} turn will be given by $S_n = X_1 + X_2 + \cdots + X_n$ which is the sum of n exponential random variables with similar parameter rates λ hence resulting in an Erlang distribution $S_n \sim Erlang(n, \lambda)$.

For instance, in a call reception centre, the Erlang distribution can be used to measure an average time within the call comparing with the expected time for each incoming call. The information that can be deduced from the traffic load measured within the Erlang may be very important to inform determination of the human resources at the call reception centre for optimal result for instance in the call customer care service or emergence call reception centres in medical service providers.

In the business economic market Erlang distributions may be helpful to determine the inter-purchasing time to inform about the business trend and time for re-stocking in anticipation of avoiding losses or wastage.

In the banking sector the distribution may help to determine the number of employees that may be likely required to serve its customers more better and be able to set service target.

7.1.2 Stochastic Processes

In the long-run of the sum of n independent and identically distributed exponential random variables, the rate at which events are occurring results to the reciprocal of the expectation of the sum S_n, which is $\frac{\lambda}{n}$. The age specific for the distribution gets monotonic in x for $n > 1$ as it is increasing from 0 for $x = 0$ to λ as $x \to \infty$.

7.1.3 Insurance

Erlang distribution is useful in modelling insurance and financial risks management in insurance companies.It can be used to understand whether the insurance company is making profit or loss by knowing sum total of compensation in a particular period of time and hence enable the company to make right decisions to enable its success or survival in the market.

7.1.4 Machine Repairing System

In the process of repairing a machine, it requires some sequential steps to followed. If the time of repair the sequential steps takes exponential distribution at each step with similar rates then the total time for repairing the machine can be modelled in an Erlang distribution.

7.1.5 Computer Science

In computer programming several blocks are sequentially processed one after the other with the time spending at each block following an exponential distribution with the same independent rate. The total time spent in the compilation of all the computer programmes requires models it to form an Erlang distribution.

7.1.6 Tele-traffic Engineering in tele-communication network

In case arrival of call and service time at a customer care service call centre follows an exponential distribution with the same parameter rate λ and if we want to find the probability of the least time t taken by n people to call. Let S_i to represent the interval call arrival time between $(i-1)^{th}$ and i^{th} call. S_i also follows and exponential distribution with the same parameter rate λ. Therefore, the total time S of receiving n successful calls forms the sum of n exponential random variables that is $S = S_1 + S_2 + \cdots + S_n$ which forms an Erlang distribution.

7.2 Application of Hypo-Exponential Distributions

In service provisions where the process forms exponential distribution with different parameter rates, then Hypo-exponential distribution forms the best distribution model for the total service time of the process. A hypo-exponential distribution forming the Erlang distribution generalization fits in many life service processes with more versatile uses.

7.2.1 Reliability and Performance Evaluation

A compound system performs different functions whereby its efficiency is determined by the quality and the functions its components. This describes the reliability of the system.

In a compound system each component has attached the following conditions:
(i) the probability of failure
(ii)the rate of failure
(iii)the distribution of time of failure and
(iv) the steady state and instantaneous unavailability of the component.

All these conditions makes the assumption that the failure and repair events of each component is independent. Another assumption is that there is simple logical relationship between the system and its components that forms it up. Kordecki and Szajowki(1990) gives more properties of the combinatorial nature of such generalized reliability structures.

In this case, a good example is the performance of the computer hard disk which contains three components in it that is the disk seek, disk latency and disk transfer. Data processing time between the components takes exponential distribution with distinct parameter rates hence making hypo-exponential distribution the best model in this case.

The survival time in the performance of each component in the system form the reliability of the system. If one component in the system fails to perform therefore the reliability of the whole system fails.

7.2.2 Computer Science and Communication

In many daily life experiences it has been observed overtime that the service time for an input-output operations in computer system (Trivedi, 2002) more often displays hypo-exponential distribution. The transfer of information in the two phases in the computer which includes the control and data transfer operations takes the hypo-exponential distribution with different parameters.

The response time of the time of the router or web service depends on the performance of the metrics. The percentiles of the time of the metrics provides a better understanding of the mean value of the system.

In computer communication, the percentile takes hypo-exponential distribution that can be seen as a generalized Erlang distribution where from each phase i with distinct rate λ_i (Bushara and Perros, 2011).

The CDF of the hypo-exponential is given by:

$$c = \sum_{0}^{n} \left\{ (1 - c^{-\lambda_i s_n}) \prod_{j=1, j \neq i}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i} \right\}$$

then the c^{th} percentile gives the arithmetic sum of the sum of the individual percentile that determines the service time.

A computer input-output operation involves two processes that is the control operation and data transfer operation that works in a sequence. Therefore the service time of these sequential processes can be modelled in a two phases hypo-exponential distribution since each phase has distinct rate in operation.

7.2.3 Heavy Traffic Modelling in Queuing Systems

In heavy traffic customer flows these distributions helps to model the queuing systems for instance in modern communication systems where a lot of people are involved, transportation and even computer systems with heavy traffic. It helps to determine the performance of the systems, queuing length and waiting time distribution in the system. The asymptotic behaviours of the various models provides and understanding on the significance of the service systems.

7.2.4 Risk Measures

In actuarial applications, these distributions help to model insurance performance in order to avoid losses. Risk managers use these distributions too in determining prices, make reservations and responsible risk decisions for an organization.

7.2.5 Aggregate Loss Models

Insurance companies survives due to their ability to sum individual risks to an aggregate risk that is manageable for the company. This enables the insurance to determine the premium that attractive to the customers to avoid losses or get trapped in debts.

8 CONCLUSION

This project has considered sum of n fixed exponential random variables with identical parameters as well as for the distinct parameters. The study provided general cases and some of the properties able to be established.

A simple general method of obtaining the probability density function for a fixed sum of independent and identically distributed exponential random variables forming and Erlang distribution has been described. Statistical properties for the distribution have been done and proof made through the moment generating function for an Erlang distribution. Also by methods of moments and maximum likelihood estimation method estimators for the parameters of an Erlang distribution were obtained.

By use of Laplace transform we developed compound distribution of exponential random variables in relation with various discrete distribution. This provided several compound distributions that may be useful in various applications. Compound distributions obtained include: Poisson, Bernoulli, Binomial, Geometric and Negative Binomial distributions.

In this project we have proposed compound mixed Poissons distributions with desirable properties for modelling claim frequencies. Compound mixing Poisson described include the Poisson-Exponential, Poisson-Gamma,Poisson-Transmuted Exponential, Poisson-Lindley, and Poisson-Generalized three parameter Lindley mixing distributions. The results obtained may be used to derive formulae to determine total claims density when the distribution comes from the exponential random variables.

Hypo-exponential distribution for the random variables with distinct parameters have been studied thereby establishing the cumulative density function and probability density function through use of convolution approach and moment generating function technique. Some of the properties related to hypo-exponential distribution were provided progressively using two parameters in the sum of exponential distribution to the general case proposed as a simple formula that can be used.

The probability density function for the hypo-exponential distribution models when the parameters form arithmetic sequences and geometric sequences were also derived.

Finally, provided are some of the applications of the distribution studied here in mathematical modelling. The scope of application is enormous that only a few were dealt with.

8.1 Scope of Future Works

Following this interesting study and need for extensive application of sum of exponential random variables, the following areas have been observed and proposed for further studies:

(i) The computation of the k^{th} derivative of the sum of exponential random variables.

(ii) The pdf for models for both Erlang and hypo-exponential random variables when some of the parameters are identical and others are not identical in the distribution and their related properties.

(iii) Parameter estimation for the general cases of hypo-exponential distribution.

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