"ON THE SPECTRAL PROPERTIES OF 2-ISOMETRIC AND RELATED OPERATORS ON A HILBERT SPACE."

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BY

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DECLARATION

This dissertation is my original work and has not been presented for a degree award in any University.

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ABSTRACT

We investigate the spectral properties of 2-isometric operators on a Hilbert Space.A bounded linear operator T is a 2-isometry if;

$$T^{*2}T^2 - 2T^*T + I = 0$$

2-isometric operators arose from the study of bounded linear transformations T of a complex Hilbert space that satisfy an identity of the form,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k = 0$$

for a positive integer m,such operators are said to be $m-isometries$.
The case $m = 1$,gives rise to the class of isometries on a Hilbert space which has been widely studied due to its fundamental importance in the theory of stochastic processes,the intrinsic problem of modelling the general contractive operator via its isometric dilation and many other areas in applied mathematics.
The case $m = 2$,is the class of 2-isometries on a Hilbert space,which contains the class of Brownian unitaries which play an essential role in the theory of non-stationary stochastic processes related to Brownian motion.Brownian motion or *Pedesis(Greek for leaping)* is the presumably random drifting of particles suspended in a fluid(a liquid or a gas) or the mathematical model used to describe such random movements,which is often called particle theory.
The mathematical model of brownian motion has several real world applications .An often quoted example is the stock market fluctuations.
It has been shown in [1],that the general 2-isometry has the form,$B = T|_M$,where B is the block form

$$B = \begin{pmatrix} V & \sigma E \\ 0 & U \end{pmatrix}$$

where $\sigma > 0$ is constant,V is an isometry,U is unitary,E is a Hilbert space isomorphism onto $kerV^*$ and M an invariant subspace for T.The operators B are refered to as Brownian unitaries of convariance σ.

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Dedication

I would like to dedicate this work to my parents for their support and prayers.My father Cyrus Kiratu for showing me the joy of pursuit of knowledge since i was a child,my mother Tabitha who sincerely raised me with caring and gently love.
George,Steve and Hope for being supportive and caring siblings.

This page is dedicated to my husband Harrison Ng'ang'a,whose dedication,love,persistent and confidence in me has taken the load off my shoulder .I owe him for not allowing his intelligence,passions and ambitions collide with mine.

LIST OF NOTATIONS

H, H_1, H_2; Hilbert Spaces.

$\mathbb{C}$; Space of complex numbers

l_2; Space of infinte square summable sequences

$C(\Omega)$; Space of complex valued functions that are continuous on Ω

$B(H)$; Banach algebra of bounded sequences on H

$R(T)$; Range of T

$N(T)$; Null space of T

$M \oplus N$;Direct sum of M and N

P_M; Orthogonal projection onto a closed subspace M

$M^{\perp}$; Orthogonal complement of M.

$\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$

$\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$

T^*; Adjoint of T

$|T| = (T^*T)^{\frac{1}{2}}$

$\|.\|$; Norm

$\langle , \rangle$; Inner product function

$\sigma(T)$; Spectrum of T

$\sigma_p(T)$; Point spectrum of T

$\sigma_c(T)$; Continuous Spectrum

$\sigma_r(T)$; Residual Spectrum

$\sigma_{ap}(T)$; Approximate Point Spectrum

$\sigma_{cp}(T)$; Compression spectrum

$\Pi_o(T)$; set of eigenvalues of finite multiplicity.

$\omega(T)$; Weyl spectrum.

S_{ω_n}; Weighted unilateral shift

S_{e_n}; Unilateral shift

$W(T)$; Numerical range of T

$w(T)$; Numerical radius

$r(T)$; Spectral radius

$c(T)$; Crawford number

INTRODUCTION

Origin of Spectral Theory

Spectral theory is one of the branches of functional analysis,which can be described as trying to "classify" linear operators.In order to understand its importance,we shall give a brief history of fuctional analysis.

Functional analysis is the branch of mathematics where vector spaces and operators(functions) on them are in focus.In linear algebra,the focus is on finite dimensional vector(linear) spaces over any field of scalars and the functions are viewed as matrices with scalar entries,but in functional analysis the vector spaces are infinite dimensional and not all operators can be represented by matrices.

Functional analysis has its origin in the theory of ordinary and partial differential equations which was used to solve several physical problems,which included the work of *Joseph Fourier(1768-1830)* on the theory of heat in which he wrote differential equations as integral equations.His work triggered not only the development of trigonometric series,which required mathematicians to consider what is a function and the meaning of convergence,this concieved Lebesque Integral which could accomodate broader functions compared to the classical Reimannian Integral.It also gave birth to the spectral theory which is a central concept of functional analysis.

In the beginning of the 20th century,functional analysis started to form a discipline of its own via integral equations.The first intensive study of integral equations was given by Swedish astronomer and mathematician *Ivar Fredholm* in a series of papers in the year 1900 to 1903,in which he developed a theory of "determinants" for integral equations of the form;

f(x)-$\lambda \int_a^b K(x,y)f(y)dy = g(x)$ where $K(x,y)$ is the *Kernel function.* (1)

Fredholm defined a "determinant" $D_K(\lambda)$ associated with the kernel λK,and showed that D_K is an entire function of λ.The roots of the equation $D_K(\lambda) = 0$ are called *eigenvalues* and the corresponding solution to the homogeneous equation $g(x) = 0$ are called *eigenfunctions*.
The work of Fredholm got immediate attention from mathematicians all over the world and *David Hilbert(1862-1943)* was one of the most enthusiastic,during the years 1904-1906,he published six papers on integral equations,in which he started transforming the integral equations to a finite system of equations under the restriction that the kernel function is symmetric. Both Fredholm and Hilbert studied eigenvalues in the sense that the operator $\lambda K - I$ is not invertible.As a result,spectral theory of operators was initiated and operators were classified in terms of their spectral properties on a Hilbert space.The restriction to this space was because linear operators on it are fairly concrete objects and the study of their spectrum shows how operators stretch the spaces in different factors and in mutually perpendicular directions.
Hilbert space refers to an infinite dimensional complete normed linear space which has an additional structure -called an inner product.The inner product is itself a generalization of the scalar product of elementary cartesian vector analysis .
The scalar product is usually defined in terms of the components of the vector,but in accordance with standard tactics in functional analysis ,the algebraic properties of the scalar product are taken as axioms in the abstract context.The presence of the scalar product enriches the geometrical properties of the space.

What is Spectral Theory?

Spectral theory can be described as trying to "classify" all linear operators which was motivated by the need to solve the linear equations $T(v) = w$ between Hilbert spaces.It was introduced by David Hilbert during his initial formulation of Hilbert space theory.The restriction to a Hilbert space occurs since Hilbert spaces are distinguished among Banach spaces by being closely linked to plane Euclidean geometry which is the correct description of our universe at many scales.

Finite dimensional linear algebra suggests that two linear maps $T_1, T_2 : H_1 \rightarrow H_2$ which are linked by the formula $T_2 o U_1 = T_2 o U_1$ for some invertible operators $U_i : H_i \rightarrow H_i$,which share many similar properties.This is because the U_i correspond to the changing of basis in H_i,which should be an operation that does not affect intrinsic properties of the operators,therefore it possible to diagonalize operators T_1 and T_2 using the change of basis matrix. As a result the "classification" problem is succesfully solved by the theory of eigenvalues,eigenspaces,minimal and characteristic polynomials,in which operators are represented by square matrices and eigenvalue decomposition is possible only when the operator is diagonalizable.

This interpretation fails in the case of an infinte dimensional Hilbert space since an operator may fail to have eigenvalues,so we need to replace the notion of eigenvalue with something more general;a complex number λ such that $T - \lambda I$ is not invertible,the set of all such λ is called the spectrum of T. In general spectral theory can be defined as the infinite dimensional version of diagonalization of a normal matrix(i.e a matrix that commutes with its adjoint).

Note:We define the spectrum of the operator T as the set of λ such that $T - \lambda I$ is not invertible.This means that a λ,in the sense of Fredholm is an eigenvalues iff $\frac{1}{\lambda}$ is an eigenvalue in our sense.

Structure of the Dissertation

In **Chapter 1**,we shall give some basic definitions and concepts in operator theory,especially properties of bounded linear operators since T is taken to be bounded.The notion of Invariant spaces will play a vital role in the decompositions of T.

In **Chapter 2**, we shall look at the spectrum of T and its partitions,the numerical range,maximal generalized inverse(Moore-Penrose inverse) of T and the reduced minimum modulus of T.

In **chapter 3**, we shall introduce the class of 2-isometric operators,their spectral properties.In particular,we show that the spectrum,numerical range and the Weyl spectrum of 2-isometry are equal to the closed unit disc.In addition we shall look at the von Neumann Wold Decomposition of 2-isometries,decomposition of an operator splits it into operators that are easy to understand,therefore plays a vital role in the theory of 2-isometries.

In **chapter four**,we are going to give other classes of Hilbert space operators related to 2-isometries.Finally we shall give a conclusion on suggested research topics that arose during our study.

Contents

Chapter 1

Preliminaries

1.1 Notations and terminologies

Definition 1.1.1. *Let H denote a vector space over the field of complex numbers.An inner product is a complex valued function $\langle , \rangle$ on $H \times H$ such that for all $f, g, h \in H$ and α a complex number,the following axioms hold:*

$\langle f, f \rangle \geq 0$ *and* $\langle f, f \rangle = 0$ *iff* $f = 0$

$\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$

$\langle f, g \rangle = \overline{\langle g, f \rangle}$ *where the bar denotes the complex conjugate.*

$\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$

A space H equiped with an inner product is known as a pre-hilbert space or an inner product space.

Example 1.1.2. *For an infinite dimensional complex vector space, the appropriate inner product is, with*

$f = (f_1, f_2, f_3, ...)$ *and* $g = (g_1, g_2, g_3, ...)$

$\langle f, g \rangle = \sum_{j=1}^{\infty} f_j \overline{g_j}$

Example 1.1.3. *An inner product may easily be constructed for the set of complex valued functions* $C(\Omega)$ *by setting* $\langle f, g \rangle = \int_{\Omega} f(x)\overline{g(x)}dx$

Theorem 1.1.4. *For any element* $x, y \in H$ *the following properties hold:*
$|\langle x, y \rangle| \leq \|f\| \, \|g\|$ *(Schwartz Inequality)*

$\|x + y\| \leq \|x\| + \|y\|$ *(Triangle inequality)*

$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad x \perp y$ *(Parallelogram law)*

Theorem 1.1.5. *A pre-hilbert space is a normed vector space with the norm,* $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$

A pre-hilbert space which is complete with respect to the norm is called a *Hilbertspace*

The symbol H will henceforth always denote a Hilbert space.
H is called a separable space if there exists vectors $v_1, v_2, ...$ which span a subspace dense in H.

Proposition 1.1.6. *Every Separable Hilbert space has an orthonormal basis.*

Recall:Let M denote any subset of H.Then the set of vectors orthogonal to M denoted $M^{\perp}$,meaning
$x \in M \quad , y \in M^{\perp} \quad \Rightarrow \quad \langle x, y \rangle = 0$

Theorem 1.1.7 (Projection Theorem). *Let M be a closed subspace of H. The $M^{\perp}$ is a closed subspace,*
and $H = M \oplus M^{\perp}$.

1.2 Invariant Subspaces

Sometimes properties of an operator $T \in B(H)$ can be determined rather easily by considering simpler operators which are restrictions of T to certain subspaces of H, known as invariant spaces.
We now present some elementary facts about invariant subspaces.

Definition 1.2.1. *Let $T \in B(H)$, a subspace M of H is invariant under T if $T(M) \subseteq M$.*

A subspace M of H reduces T if both M and $M^{\perp}$ are invariant under T.

If M is invariant under T, then relative to the decomposition $H = M \oplus M^{\perp}$, T can be written as

$$T = \begin{pmatrix} T|_M & X \\ 0 & Y \end{pmatrix}$$

for operator $X : M^{\perp} \to M$ and $Y : M^{\perp} \to M^{\perp}$ where $T|_M : M \to M$ is a restriction of T to M and $X = O$ iff M reduces T.

Definition 1.2.2. *A part of an operator is a restriction of it to an invariant subspace.*

A direct summand of an operator is a restriction of it to a reducing subspace.

Remark 1.2.3. *An operator is completely non-unitary if the restriction to any non-zero reducing subspace is not unitary. In particular, T has no non-zero unitary direct summand.*

Definition 1.2.4. *An operator $T \in B(H)$ is reducible if it has a non-trivial reducible subspace(equivalently,it has a proper non-zero direct summand),otherwise it is said to be irreducible.*

A unilateral shift of multiplicity one is irreducible.

1.3 Properties of bounded linear operators

Throughout this section H, H_1, H_2 denote Hilbert spaces over the complex plane and $B(H)$ denotes the Banach algebra of bounded operators on H.

Definition 1.3.1. *A function T which maps H_1 into H_2 is called a linear operator if for all $x, y \in H_1$ and α a complex number,*

$T(x + y) = T(x) + T(y)$ *and*

$T(\alpha x) = \alpha(T(x))$

Definition 1.3.2. *The linear operator $T : H_1, \to H_2$ is called bounded if;*

$\sup_{\|x\| \leq 1} \|Tx\| < \infty$

The norm of T ,written $\|T\|$ is given by;

$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$

Thus an operator is a bounded linear transformation of a non-zero complex Hilbert space into itself.

Example 1.3.3. *Let $S_r : l_2 \to l_2$ be defined by;*

$$S_r(a_1, a_2, ...) = (0, a_1, a_2, ...)$$

S_r is called the forward shift operator. S_r is linear and $\|S_r x\| = \|x\|$, $x \in l_2$. In particular ,$\|S_r\| = 1$

Proposition 1.3.4. *Let $T : H \to H$ be a non-zero linear operator. The following are equivalent;*

- *$R(T)$ is a closed subspace of H*
- *T is a bounded linear operator.*
- *$N(T)$ is a closed subspace of H*

Lemma 1.3.5. *Let T be an operator such that for all $x \in H, \|Tx\| \geq c\|x\|$, where c is a positive constant. Then $R(T)$ is closed.*

Proof

Let (y_n) be a convergent sequence of elements in $R(T)$ converging to y. Then $y_n = Tx_n$. For some (x_n). Since (y_n) is convergent, it is a Cauchy sequence. Now

$$\|x_n - x_m\| \leq \frac{1}{c}\|T(x_n - x_m)\| = \frac{1}{c}\|y_n - y_m\|$$

$\Rightarrow x_n$ is a Cauchy sequence and hence convergent to some element x. Then since T is continuous
$y = \lim y_n = \lim Tx_n = Tx$
$\Rightarrow (y_n) \to y \in R(T)$. Thus $R(T)$ contains its limit points, hence closed.

Definition 1.3.6. *If $T \in B(H)$ then its adjoint T^* is the unique operator in $B(H)$ that satisfies*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \, \forall x, y \in H$$

An operator $T \in B(H)$ is called self-adjoint if $T^* = T$.

Theorem 1.3.7. [9]
Let $T \in B(H)$, the following results hold;

1. *T^*T is a positive self adjoint operator.*

2. *$R(T)$ is closed iff $R(T^*T)$ is closed.*

3. *$\overline{R(T^*} = N(T)^\perp = N(T^*T)^\perp = \overline{R(T^*T)}$*

4. *$N(T$ and $N(T)^\perp$ are invariant under T^*T.*

Proof

We have $(T^*T)^* = T^*(T^*)^* = T^*T \Rightarrow T^*T$ is self adjoint.

To show that its positive consider,

$\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \geq 0$.Thus T^*T is positive,hence (1) holds.

To show that (2) holds,assume $R(T)$ is closed.

By defintion, $\|T^*T\| = \sup|\langle T^*Tx, \rangle| = \|T\|^2$.

Since $R(T)$ is closed, then $\|T\| \leq c\|x\|$ for some positive constant c,squaring both sides we have, $\|T\|^2 \leq c^2\|x\|$

Therefore $\|T^*T\| \leq c^2\|x\|^2$,taking $\|x\| = 1$ we have, $\|T^*T\| \leq c^2\|x\|$

By definition, $\|T^*T\| = \sup\|T^*Tx\|$,it follows that $\|T^*Tx\| \geq c^2\|x\|$

hence by *lemma*1.3.5 $R(T^*T)$ is closed.

Conversely,assume $R(T^*T)$ is closed,then by *prop.*1.3.2 we have T^*T is bounded,i.e there exists a positve constant b such that, $\|T^*T\| \leq b\|x\|$ since $\|T^*T\| = \|T\|^2$ we have, $\|T\| \leq \sqrt{b}\|x\|$,hence by *prop.*1.3.2 $R(T)$ is closed.

To prove (3),we first show that $\overline{R(T^*)} = N(T)^\perp$

Let $x \in N(T) \Rightarrow Tx = o$,take any $y \in H$
Then $0 = \langle Tx, y\rangle = \langle x, T^*y\rangle$
$\Rightarrow T^*y \in N(T)^\perp$
$\Rightarrow R(T^*) \subset N(T)^\perp$..............(*)

Conversely,let $y \in R(T^*) \Rightarrow \exists \quad z \in H | T^*z = y$ and $x \in N(T)$
Then $\langle x, y\rangle = \langle x, T^*z\rangle = \langle Tx, z\rangle = 0$
$\Rightarrow N(T) \subseteq R(T^*)^\perp \Rightarrow N(T)^\perp \subseteq R(T^*)$................(**)
From (*) and (**) and the fact that $R(T)$ is closed$\Leftrightarrow$ $R(T^*)$ is closed,we have $N(T)^\perp = \overline{R(T^*)}$.
Similarly $N(T^*T)^\perp = \overline{R(T*T)}$.

To show that (3) holds,it suffices to show that $N(T) = N(T^*T)$,therefore suppose that $x \in N(T) \Rightarrow Tx = 0$ applying T^* we have $T^*Tx = 0 \Rightarrow x \in N(T^*T). \Rightarrow N(T) \subseteq N(T^*T)$

Conversely,let $x \in N(T^*T) \Rightarrow T^*Tx = 0$,let $y \in H | Ty \neq 0$,then,

$0 = \langle T^*Tx, y\rangle = \langle Tx, Ty\rangle \Rightarrow Tx = 0 \Rightarrow x \in N(T)$,thus $N(T^*T) \subset N(T)$,it follows that $N(T^*T) = N(T)$,hence the proof.

To prove (4) we first define an invariant subspace,*a closed subspace M of H is said to be invariant under an operator T if* $T(M) \subset M$or $R(T|_M \subseteq M$,from (3),we have $N(T)^{\perp} = R(T^*T)$,it follows that $N(T)^{\perp}$ is invariant under T^*T.Therefore $x \in N(T)^{\perp} \Rightarrow T^*Tx \in N(T)^{\perp}$,taking the conjugate on both sides we have $\overline{x} \in N(T) \Rightarrow T^*T\overline{x} \in N(T)$,thus $N(T)$ is invariant under T^*T.Hence the proof.

1.4 Some classes of Hilbert space operator

Definition 1.4.1. *An operator $T \in B(H_1, H_2)$ is called invertible if there exists an operator $T^{-1} \in B(H_1, H_2$ such that;*

$T^{-1}Tx = x$ for every $x \in H_1$

$TT^{-1}y = y$ for every $y \in H_2$

The operator T^{-1} is called the inverse of T.

Theorem 1.4.2. *An operator $T \in B(H)$ is invertible if and only if the following properties hold;*

- *There exists a positive number c such that $\|Tx\| \geq c\|x\|$ for any $x \in H$.*
- *$R(T)$ is dense in H*

Theorem 1.4.3. *If $T \in B(H)$ is self-adjoint then $N(T)^{\perp} = \overline{R(T)}$. Thus by theorem 1.2.4 $H = N(T) \oplus \overline{R(T)}$*

Definition 1.4.4. *Any two complex Hilbert spaces H_1, H_2 of the same dimension are equivalent in the sense that there exists an operator $U \in B(H_1, H_2)$ such that U is surjective and $\|Ux\| = \|x\|$ for all $x \in H$*

An operator U which satisfies the property $\|Ux\| = \|x\|$ for all $x \in H$ is called an Isometry.

The forward shift operator $S_r e_i = e_{i+1}$ on l_2 is an example of an isometry.

Definition 1.4.5. *An operator $K \in B(H)$ is compact if for each sequence of unit vectors $\{x_n\}$ in H, the sequence $\{Kx_n\}$ has a convergent subsequence.*

Example 1.4.6. *The integral operator K where $f \in C(\Omega)$ and*

$Kf(x) = \int_{\Omega} K(x,y)f(y)dy$ *is compact.*

Definition 1.4.7. *A linear isometry which maps H onto itself is called a Unitary Operator.*

Lemma 1.4.8. *Let $T \in B(H)$ be unitary then;*

1. *T has an inverse T^{-1} which is unitary.*
2. *T is linear.*
3. *The adjoint operator coincides with its inverse.*

Example 1.4.9. *Let $a(t)$ be a Lebesgue measurable function on $[a,b]$ such that $|a(t)| = 1$ almost everywhere. The operator U defined on $L_2[a,b]$ by $(Uf)(t) = a(t)f(t)$ is unitary.*

Definition 1.4.10. *Let T_1 and T_2 be operators on H_1 and H_2 respectively, T_1 is said to be unitarily equivalent to T_2 if there exists a unitary operator $U \in B(H_1, H_2)$ such that $T_2 = UT_1U^*$.*

Definition 1.4.11. *An operator T acting on a Hilbert space H is said to be completely nonunitary if it has no non-trivial reducing subsapce N such that the restriction $T|_N$ of T to N is unitary.*

Definition 1.4.12. *An operator T is Fredholm if $R(T)$ is closed, $N(T)$ and $R(T)^{\perp}$ are finite dimensional.*
The index of T denoted by $i(T)$ is defined by

$i(T) = dim N(T) - dim R(T)^{\perp}$.
T is Weyl if it is Fredholm and of index zero.

Additional operators on a Hilbert space are defined as;

An operator $T \in B(H)$ is;

Normal if $T^*T = TT^*$

Normaloid if $r(T) = \|T\|$

Partial isometry if $TT^*T = T$

Quasinormal if T commutes with T^*T

Hyponormal if $T^*T \geq TT^*$

Quasi-hyponormal if $T^*\,[T^*, T]\,T$ is non-negative

Paranormal if $\|Tx\|^2 \leq \|T^2x\|\,\|x\|$ for all $x \in H$

m-isometry if $\sum_{k=0}^{m}(-1)^{m-k}\binom{n}{k}T^{*k}T^k = 0$

Dominant if $R(T - \alpha I) \subset R(T^* - \overline{\alpha} I)$ for a complex number α

Quasi-isometry if $T^{*2}T^2 = T^*T$

Positive if $\langle Tx, x\rangle \geq 0$ for all $x \in H$

Idempotent if $T^2 = T$

Nilpotent if $T^n = 0$ for some positive integer n.

Contraction if $\|T\| \leq 1$(Equivalently $\|Tx\| \leq \|x\|$ for all $x \in H$.)

Proper Contraction if $\|T\| \leq 1$

Strict Contraction if $\|T\| < 1$

These operators are related by the following inclusions.

Hyponormal$\subset$ Quasi-hyponormal$\subset$ Paranormal $\subset$ Normaloid

Strict Contraction $\subset$ Proper Contraction $\subset$ Contraction

Chapter 2

The Spectrum of an operator

2.1 Classification of the spectrum

Definition 2.1.1. *A complex number λ is said to be a regular value of an operator T if the operator $T - \lambda I$ is invertible.*

The resolvent set denoted by $\rho(T)$ is the set of regular values of T.

The set of all those λ ,which are not regular values of T is called the Spectrum of the operator T and is denoted by $\sigma(T)$.

Definition 2.1.2. *If there is a non-zero solution of the equation $Tx = \lambda x$, then λ is said to be an eigenvalue of T and x the eigenvector corresponding to the eigenvalue λ.*

The linear span of all eigenvectors corresponding to the eigenvalue λ is said to be an eigenspace of the operator T and is denoted by $N(T - \lambda I)$.

An element $x \in H$ such that $(T - \lambda I)^n x = 0$ for a positive integer n is said to be a principal vector corresponding to the eigenvalue λ.
The linear span of all principal vectors corresponding to an eigenvalue λ is said to be a principal space.

The dimension of a principal space is the multiplicity of the corresponding eigenvalue.

Proposition 2.1.3. *Eigenvalues of a square matrix* $T = [a_{jk}]_{j,k=1,2,\ldots n}$ *are the roots of the equation* $det(T - \lambda I) = 0$.

Example 2.1.4. *What are the eigenvalues and eigenvectors of*
$T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$
$det(T - \lambda I) = (1 - \lambda)(4 - \lambda) - 4 = 0$

$\Rightarrow \lambda^2 - 5\lambda = 0 \Rightarrow \lambda = 0 \text{ or } \lambda = 5$
Eigenvalues of T *are* $\lambda_1 = 0$ *and* $\lambda_2 = 5$.
For $\lambda_1 = 0$
$(T - 0I)x = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow y + 2z = 0$
and $2y + 4z = 0$

$\Rightarrow$ *if* $z = -1, y = 2$
Therefore the corresponding eigenvector is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
For $\lambda_2 = 5$
$(T - 5I)x = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -4y + 2z = 0$
$2y - z = 0 \Rightarrow$ *if* $y = 1, z = 2$
Therefore the corresponding eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The spectrum of an operator $T \in B(H)$ can be split into many disjoint parts.A classical partition comprises of the point spectrum,continuous spectrum and the residual spectrum.

Definition 2.1.5. *The point spectrum denoted by $\sigma_p(T)$, is the set of all eigenvalues of T*

$\sigma_p(T) = \{\lambda \in \mathbb{C} : N(T - \lambda I) \neq 0\}$

The continuous spectrum denoted by $\sigma_c(T)$ is defined as follows:$\lambda \in \sigma_c(T)$ iff $\lambda \in \frac{\sigma(T)}{\sigma_p(T)}$ and $R(T - \lambda I)$ is dense in H.

$\sigma_c(T)$=$\{\lambda \in \mathbb{C} : (T - \lambda I)^{-1}$ *is unbounded and* $\overline{R(T - \lambda I)} = H$

Example

On $L_2[0,1]$ define $T : L_2[0,1] \to L_2[0,1]$ with $Tx = t.x(t)$,then

$\sigma(T) = \sigma_c(T) = [0, 1]$

The residual spectrum is the set;$\sigma_r(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)^{-1}\}$ exists and $R(T - \lambda I) \neq H$

From the definition it follows that;

$\sigma_r = \frac{\sigma_p(T^*)^*}{\sigma_p(T)}$

Proposition 2.1.6. *$\sigma(T)$ is a closed set.*

Proposition 2.1.7. *$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$ holds,where$\sigma_p(T)$,$\sigma_c(T)$,$\sigma_r(T)$ are mutually disjoint parts of $\sigma(T)$.*

Definition 2.1.8. *The compression spectrum $\sigma_{cp}(T) = \left\{\lambda \in \mathbb{C} : \overline{R(T - \lambda I)} \subset H\right\}$*

The set $\sigma_{ap}(T)$ of all complex numbers λ such that there exists a sequence of unit vectors x_n such that $\|Tx_n - \lambda x_n\| \to 0$ as $n \to \infty$ is said to be approximate point spectrum.

Proposition 2.1.9. *$\sigma(T) = \sigma_{ap}(T) \cup \sigma_{cp}(T)$ holds,where $\sigma_{ap}(T)$ and $\sigma_{cp}(T)$ are not necessarily disjoint parts of the spectrum.*

Also $\sigma(T) = \sigma_r(T) \cup \sigma_{ap}(T)$ holds.

Proposition 2.1.10. *$\sigma_{ap}(T)$ is non-empty,includes the boundary $\partial(\sigma(T)$ of the spectrum*

$.\partial(\sigma(T) \subseteq \Pi(T)$.

Definition 2.1.11. *The Weyl spectrum denoted by $\omega(T) = \{\lambda \in \mathbb{C} | (T - \lambda I)^{-1}$ is not Weyl.*

Definition 2.1.12. *Let Ω be a non-empty set,the smallest convex set containing Ω denoted by $conv(\Omega)$ is known as the convex hull of Ω*

Theorem 2.1.13. *i) The entire spectrum of a self adjoint operator T is confined between its bounds*

$M = \sup_{\|x\|=1} |\langle Tx, x\rangle|$ *and* $m = \inf_{\|x\|=1} |\langle Tx, x\rangle|$.
ii) The bounds M and m of every self-adjoint operator belong to its spectrum.

Proof

Suppose $\lambda \notin [m, M]$ and $\lambda < m$ since $\langle Tx, x\rangle \geq m$,we have $\langle (T - \lambda I)x, x\rangle \geq m - \lambda$

But according to Shwartz inequality
$|\langle (T - \lambda I)x, x\rangle| \leq \|(T - \lambda)x\| \, \|x\|$

$\Rightarrow \|(T - \lambda I)x\| \geq m - \lambda$

$\Rightarrow \lambda \notin \sigma(T)$

Similarly for $\lambda > M$

By definition $\|T\| = M \Rightarrow \exists x_n \in H$ for which

$$\langle Tx_n, x_n\rangle \to M$$

$$\Rightarrow Tx_n \to Mx_n.$$

Therefore $0 \leq \|Tx_n - Mx_n\|^2 = \|(T - MI)x_n\|^2$

$$= \langle (T - MI)x_n, (T - MI)x_n\rangle$$

$$= \|Tx_n\|^2 - 2M\langle Tx_n, x_n\rangle + M^2$$

$$\leq 2M^2 - 2M\langle Tx_n, x_n\rangle \to 0$$

$$\Rightarrow M \in \sigma(T)$$

Similarly for $m \in \sigma(T)$.

Theorem 2.1.14. *Let $T \in B(H)$,then*

$$\sigma(T^*T) = (\sigma(T^*T)|_{N(T)} \cup (\sigma(T^*T)|_{N(T)^\perp}$$

Proof

From theorem 1.2.7,$H = N(T) \oplus N(T)^\perp$,relative to this decomposition and since both $N(T)$ and $N(T)^\perp$ are invariant under T^*T,we have

$$T^*T = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

Where $E_1 : N(T) \to N(T)$,and $E_2 : N(T)^\perp \to N(T)^\perp$

thus $\sigma(T^*T) = (\sigma(T^*T)|_{N(T)} \cup (\sigma(T^*T)|_{N(T)^\perp}$

Remark 2.1.15. *Since T^*T is a positve self adjoint operator and theorem 2.2.8 $\|T^*T\| = \|T\|^2$,$\sigma(T^*T) \subseteq [0, \|T\|^2]$*

2.2 The numerical range

Definition 2.2.1. *For an operator T,the numerical range $W(T)$ of T is a subset of the complex plane,given by,*
$W(T) = \{\langle Tx, x\rangle \,\|x \in H; \|x\| = 1\}$

The following properties of the numerical range are well known;

1. $W(aT + bI) = aW(T) + b$

2. $W(B \oplus C) = conv\{W(B) \cup W(C)\}$

3. $W(T)$ is a convex set(Hausdorff-Toeplitz).

4. $W(T)$ is bounded.

5. $W(T)$ is closed if $\dim(H) < \infty$

Definition 2.2.2. *The numerical radius $w(T)$of T is defined by;*
$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$.

The crawford number $c(T)$ of T defined by;
$c(T) = \inf\{|\lambda| : \lambda \in W(T)\}$.

The spectral radius $r(T)$ of an operator T is defined by
$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$,*which is the smallest circle on the complex plane $\mathbb{C}$ which contains the spectrum of T.*

Definition 2.2.3. *The essential numerical range $W_e(T)$ is defined as;*
$W_e(T) = \cap\overline{W(T+k)}$,*$K$ compact.*

Let $T = (T_1, T_2, ...T_n)$ be an n-tuple of operators acting on H.The joint numerical range of T is defined as;

$W_j(T) = (\langle T_1x, x\rangle, \langle T_2x, x\rangle, ... \langle T_nx, x\rangle)$.

Proposition 2.2.4. *Let $T \in B(H)$ then $\sigma_p(T) \subseteq W(T)$.*

proof

Suppose $\lambda \in \sigma_p(T) \Rightarrow \exists \quad x \neq 0 \quad \in H : \lambda x = Tx$

Therefore $\lambda = \lambda \langle x, x \rangle = \langle \lambda x, x \rangle = \langle Tx, x \rangle$

$\Rightarrow \lambda \in W(T)$

Therefore $\sigma_p \subseteq W(T)$

Corollary 2.2.5. $\sigma_p(T) \cup \sigma_r(T) \subseteq W(T)$

proof

Suppose $\lambda \in \sigma_p(T) \Rightarrow \lambda \in W(T)$

If $\lambda \in \sigma_r(T)$ then $\overline{\lambda} \in \sigma_p(T^*) \Rightarrow \lambda \in W(T^T)$ since $\frac{\sigma_p(T^*)^*}{\sigma_p(T)}$ Hence $\sigma_p(T) \cup \sigma_r(T) \subseteq W(T)$

Proposition 2.2.6. *Let $T \in B(H)$ then $\sigma(T) \subseteq \overline{W(T)}$.*

proof

Recall:$\sigma(T) = \sigma_r(T) \cup \sigma_{ap}(T)$

Suppose $\lambda \in \sigma_{ap}(T)$

$\Rightarrow 0 \leq |\lambda - \langle Tx_n, x_n \rangle|$

$= |\langle (T - \lambda I)x_n, x_n \rangle|$

$\leq \|(T - \lambda I)x_n\| \, \|x_n\| \rightarrow o$ as $n \rightarrow \infty$

$\Rightarrow \lambda \in \overline{W(T)}$

Therefore $\sigma_{ap}(T) \subseteq \overline{W(T)}$

$\Rightarrow \sigma(T) \subseteq \overline{W(T)}$

Theorem 2.2.7. *$conv(\sigma(T)) \subseteq \overline{W(T)}$. This follows from the proposition above.*

Definition 2.2.8. *An operator $T \in B(H)$ is said to be;*

- *Convexoid if $\overline{W(T)} = conv(\sigma(T)$*
- *Normaloid if $r(T) = \|T\|$*
- *Spectraloid if $w(T) = r(T)$.*

2.3 Spectral characterization of closed range operators

Definition 2.3.1. *A bounded linear operator $S \in B(H)$ is said to be a generalized inverse of $T \in B(H)$, if $TST = T$ and $STS = S$.*
The Generalized Spectrum is defined by replacing the notion of invertibility which appears in the classical definition of the spectrum by existence of analytic generalized inverse.

Definition 2.3.2. *For $T \in B(H)$, the minimum modulus is defined by the number*

$$\gamma(T) = \inf \left\{ \|Tx\| ; \|x\| = 1, x \in N(T)^{\perp} \right\}.$$

$\gamma(T) = \infty$ *if* $T = 0$
$\gamma(T) > 0$ *implies injectivity of* T, *the converse does not hold true in general.*

Definition 2.3.3. *The Maximal Generalized Inverse of T, denoted by T^+, is a unique linear operator with domain $D(T^+) = R(T) \oplus^{\perp} R(T)^{\perp}$ and $N(T^+) = R(T)^{\perp}$ satisfying the following properties,*

(i) $R(T) \subseteq D(T^+)$

(ii) $R(T^+) \subseteq D(T)$

(iii) $T^+Tx = P_{\overline{R(T^+)}}x$, *for all* $x \in D(T)$

(iv) $TT^+y = P_{\overline{R(T)}}y$, *for all* $y \in D(T^+)$

In general, an mxn matrix T has many generalized inverses unless $m = n$ and T is invertible. It is possible to add conditions to the definition of a generalized inverse so that there is always a unique generalized inverse.

T^+ is called a Moore-Penrose inverse of T if it satisfies;

- $TT^+T = T$ and $T^+TT^+ = T^+$

- $(TT^+)^* = TT^+$ and $(T^+T)^* = T^+T$

Proposition 2.3.4. [10]
Let $T \in B(H)$. Then we have the following,
(i) $R(T)$ *is closed, the* $\gamma(T) = \frac{1}{\|T^+\|}$

(ii) $\gamma(T^*T) = \gamma(T)^2$

proof(i)

Assume $R(T)$ is closed $\Rightarrow R(T) = D(T^+)$,by definition

$\|T^+\| = \sup\left\{\frac{\|T^+\|}{\|y\|} : 0 \neq y \in D(T^+)\right\}$

$= \sup\left\{\frac{\|T^+\|}{\|y\|} : 0 \neq y \in R(T)\right\}$(since $R(T) = D(T^+)$ and)

$=\sup\left\{\frac{\|x\|}{\|Tx\|} : 0 \neq x \in N(T)^{\perp}\right\}$(since $R(T^+) = N(T)^{\perp} \Rightarrow \exists 0 \neq x \in N(T)^{\perp}$: $T^+y = 0$)

$=\inf\left\{\frac{\|Tx\|}{\|x\|} : 0 \neq x \in N(T)^{\perp}\right\}^{-1}$

$=\gamma(T)^{-1}$

proof(ii)

$\gamma(T^*T) = \frac{1}{\|(T^*T)^+\|} = \frac{1}{\|T^+\|^2}$

$=\gamma(T)^2$ (since $\|T^*T\| = \|T\|^2$)

Proposition 2.3.5. [10]

For T∈ $B(H)$,the following statements are equivalent.

(i) $R(T)$ is closed.

(ii) $R(T^*)$ is closed

(iii) $\gamma(T) > 0$

(iv) T^+ is bounded.

(v) $\gamma(T) = \gamma(T^*)$

(vi) Let $\lambda \in (0, \infty)$. Then $\lambda \in \sigma(T) \Leftrightarrow \frac{1}{\lambda} \in \sigma(T^+)$

If T^{-1} exists,then $0 \neq \lambda \in \sigma(T) \Leftrightarrow \frac{1}{\lambda} \in \sigma(T^{-1})$

Definition 2.3.6. *An operator $T \in B(H)$ is said to be positive if $\langle Tx, x\rangle > 0$ for all $x \in H$.*

Proposition 2.3.7. [10]
Let $T \in B(H)$ be a positive operator. Then the following results hold,
(i) T^{+} is positive.

(ii) $\sigma(T)/\{0\} = \sigma(T_o)/\{o\}$ where $T_o = T|_{N(T)^{\perp}}$

(iii) $\sigma(T^{+})/\{0\} = \sigma(T_o^{-1})/\{o\}$

proof(i)
Let $T \in B(H)$ be a positive operator. Then T is a self adjoint operator.
Let $y = Tv + u$ where $u \in N(T)^{\perp}$ and $v \in R(T)^{\perp}$.

Since $D(T^{+}) = R(T) \oplus^{\perp} R(T)^{\perp}$ we have,

$\langle T^{+}y, y\rangle = \langle T^{+}y, Tu + v\rangle$

$= \langle T^{+}y, Tu\rangle + \langle T^{+}y, v\rangle$

$= \langle T^{+}y, Tu\rangle$ (since $R(T)^{\perp} = N(T^{+}), \langle T^{+}y, v\rangle = 0$)

$= \left\langle P|_{\overline{R(T)}}y, u\right\rangle = \langle Tu, u\rangle \geq 0$

$\Rightarrow T^{+}$ is positive.
proof(ii)
Since T is self adjoint,it is reducible by $N(T)$
i.e $T(N(T)) \subset N(T)$ and $T(N(T)^{\perp}) \subset N(T)^{\perp}$
by *theorem 2.2.9* we have $\sigma(T) = \sigma(T|_{N(T)) \cup \sigma(T|_{N(T)^{\perp}})}$

i.e $\sigma(T) = \{0\} \cup \sigma(T_o)$
hence $\sigma(T)/\{0\} = \sigma(T_o)/\{0\}$.

proof(iii)

Since T^+ is self adjoint,it is reducible by $N(T^+) = R(T)^\perp$

since $T^+|_{R(T)} = T_o^{-1}$,(ii) implies that

$\sigma(T^+)/\{0\} = \sigma(T_o^{-1})/\{0\}$

Theorem 2.3.8. [10]

Let T∈ $B(H)$ be a positive operator and

$d(T) = \inf\{|\lambda| : \lambda \in \sigma(T)/\{0\}\} = d(0, \sigma(T)/\{0\})$

Then $\gamma(T) = d(T)$

proof

Case 1:$\gamma(T) > 0$

If $\gamma(T) > 0$,then $R(T)$ is closed.In this case T_o^{-1} and T^+ are bounded,self adjoint operators with

$\|T_o^{-1}\| = \|T^+\| = \frac{1}{\gamma(T)}$

hence $\gamma(T) = \frac{1}{\|T^+\|}$

$= (\sup\{|\mu| : \mu \in \sigma(T_o^{-1})\})^{-1}$

$= (\sup\{(\lambda)^{-1} : 0 \neq \lambda \in \sigma(T_o)\})^{-1}$

$= \inf\{|\lambda| : 0 \neq \lambda \in \sigma(T_o)\} = d(T)$

Case 2:$\gamma(T) = 0$

Since T^+ is positive,$\gamma(T) = 0 \Rightarrow T^+$ is unbounded $\Rightarrow$ $\sigma(T^+)$ is unbounded.

$\Rightarrow$ for all $n = 1, 2, 3, .., \exists\lambda_n \in \sigma(T^+)$ such that

$\lambda_n > n \Rightarrow \frac{1}{\lambda} \in \sigma(T)$

and $\frac{1}{\lambda_n} \to 0$ as $n \to \infty$

Hence $d(T) = 0$

Theorem 2.3.9. [10

Suppose $T \in B(H)$ is a positive operator and 0 is an isolated spectral valued of T. Then 0 is an eigenvalue.

proof

Since 0 is an isolated spectral value,$d(T) > 0$ then by *proposition 2.4.5* $\gamma(T) > 0 \Rightarrow R(T)$ is closed.
If $0 \in \sigma_p(T)$,then $N(T) = \{0\} \Rightarrow R(T) = H$
making T one to one and onto,hence invertible,a contradiction.

Remark 2.3.10. *The converse of these theorem need not be true. To see this consider $T : l^2 \to l^2$ defined by*

$$T(x_1, x_2, x_3, x_4, x_5, ...) = (0, 2x_2, \frac{1}{3}x_3, 4x_4, \frac{1}{5}x_5, ...)$$

Here T is a positive operator. Since T is not one to one ,$0 \in \sigma_p(T)$ but it is not an isolated point of the spectrum point of the spectrum,

$$\sigma(T) = (o, 2, \frac{1}{3}, 4, \frac{1}{5}, ...)$$

Lemma 2.3.11. [10]
Let $T \in B(H)$ be self adjoint. Then $R(T)$ is closed iff 0 is not an accumulation point of $\sigma(T)$

proof

We know that $R(T)$ is closed iff $\gamma(T) > 0$ since $d(T) = \gamma(T)$,it follows that $d(T) > 0$,thus 0 is not an accumilation point of $\sigma(T)$

Chapter 3

2-isometric Operators

3.1 Properties of 2-isometric operators

Definition 3.1.1. *An operator $T \in B(H)$ is an m-isometry if it satisfies,*

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{2}{k}T^{*k}T^{k} = 0.................(*)$$

for some positive integer $m > 0$

The study of m-isometries originated from the study of bounded linear transformations T on a Hilbert space which satisfy

$$\sum_{k=0}^{m}(-1)^{k}\binom{2}{k}T^{*m-k}T^{k} = 0.................(**)$$

for some positive integer $m > 0$,T is said to be m-symmetric.
Agler et.al in [1] studied the properties of m-isometries and some of the basic properties included;m-isometry is an $m+1$-isometry,that m-isometries are bounded below and that their spectrum lies in the closed unit disc. We want to specialize for the case $m = 2$,this gives the class of 2-isometries.

Proposition 3.1.2. *If $T \in B(H)$ is a 2-isometry then,*
$\sum_{k=0}^{2} \binom{2}{k} \left\|T^k x\right\|^2 = 0$

proof

Assume T is a 2-isometry.Then (*) holds.

Therefore,$0 = \langle \sum_{k=0}^{2}(-1)^{2-k}\binom{2}{k}T^{*k}T^k x, x\rangle$

$= \sum_{k=0}^{2}(-1)^{2-k}\binom{2}{k} \langle T^{*k}T^k x, x\rangle$

$= \sum_{k=0}^{2}(-1)^{2-k}\binom{2}{k} \langle T^k x, T^k x\rangle$

$= \sum_{k=0}^{2}(-1)^{2-k}\binom{2}{k} \left\|T^k x\right\|^2$

Proposition 3.1.3. *Every isometry is a 2-isometry*

proof

Suppose $T \in B(H)$ is an isometry then,$T^*T = I$

Therefore $T^{*2}T^2 - 2T^*T + I$

$= T^*T^*TT - 2I + I$

$= T^*IT - I = 0$

It follows from (*) that T is a 2-isometry.

Remark 3.1.4. *As a result we have the inclusion*

unitary $\subset$ *isometry* $\subset$ 2 − *isometry*

Therefore if both T and T^* are 2-isometries then T is invertible and so must be unitary.In particular if T is an invertible 2-isometry,then T is an isometry.In general an $m+1$-isometry is m-isometry.

Lemma 3.1.5. [17]
Let T be a 2-isometry,then the following statements are equivalent;
(i) T is normal.
(ii) T is invertible
(iii) T is unitary.

Definition 3.1.6. *An operator $T \in B(H)$ is said to be unitarily equivalent to $S \in B(H)$ if there exist a unitary operator $U \in B(H)$ such that $U^*TU = S$.*

Theorem 3.1.7. [17]
Let T be a 2-isometry,
(i) If S is unitarily equivalent T,then S is a to 2-isometry.
(ii) If $M \subseteq H$ is an invariant subspace for T,then $T|_M$ is a 2-isometry.
(iii) If T commutes with an isometry S,then TS is a 2-isometry.

Proposition 3.1.8. [12]
A power of a 2-isometry is a 2-isometry.

Theorem 3.1.9. [12]
A power bounded 2-isometry is an isometry.

proof

Let T be a power bounded 2-isometry.Then there exists a positive real number M such that

$\|T^n\| \leq M$ for $n = 1, 2, 3, ...$

The definition of a 2-isometry yields,
$\|T^2\|^2 + 1 = \|T\|^2$

By induction we have

$\|T^{2^n}\|^2 = 2^n \|T\|^2 - 2^n$

It follows that, $\frac{M^2}{2^n} = \|T\|^2 - 1$, letting $n \to \infty$ we have $\|T\| = 1$Thus $\|T\| = \sup_{\|x\|=1} \|Tx\| \Rightarrow \|Tx\| \geq 1$

and since $T \in B(H)$, $\|Tx\| \leq \|T\| \|x\| \Rightarrow \|Tx\| \leq 1$

$\Rightarrow \|Tx\| = 1 = \|x\|$

Hence T is an isometry.

Definition 3.1.10. *A bounded linear operator is said to be regular if it can be written as a linear combination of positive operators.*

Proposition 3.1.11. [12]
Every self-adjoint 2-isometry is regular.

Lemma 3.1.12. [12] *Let $T \in B(H)$ be a non-unitary 2-isometry, then $\sigma_{ap}(T)$ lies on the unit circle $\partial\mathbb{D}$.*

proof
Assume T is a 2-isometry and let $\lambda \in \sigma_{ap}(T) \Rightarrow \exists \{x_n\} \in H| \|x_n\| = 1$

Such that $\|(T - \lambda I)\| \|x_n\| \to 0$ as $n \to \infty$

By induction, $\|(T^k - \lambda^k I)\| \|x_n\| \to 0$ as $n \to \infty$

Therefore $0 = \left\langle \sum_k^2 (-1)^{2-k} \binom{2}{k} T^{*k} T^k x_n, x_n \right\rangle$

$= \sum_k^2 (-1)^{2-k} \binom{2}{k} \langle T^{*k} T^k x_n, x_n \rangle$
$\to \sum_k^2 (-1)^{2-k} \binom{2}{k} \langle \lambda^k x_n, \lambda^k x_n \rangle = \sum_k^2 (-1)^{2-k} \binom{2}{k} |\lambda^k|^2 = (|\lambda^2| - 1)^2 \Rightarrow |\lambda| = 1$
Hence the result.

Theorem 3.1.13. [1]
Let $T \in B(H)$ be a non-unitary 2-isometry, then $\sigma(T) = \sigma_{ap}(T)$ is the closed unit disc $\overline{\mathbb{D}}$.

proof

Since $\partial(\sigma(T)) \subseteq \sigma_{ap}(T)$, then from the lemma above we have $\sigma_{ap}(T) \subseteq \sigma(T)$..........(i)

If $\lambda \notin \sigma_{ap}(T)$, then $\exists \epsilon > 0$ such that $\|Ty - \lambda y\| \geq \epsilon \|y\|$, for all $y \in H$ with $\|y\| \leq 1$

If $y \perp R(T - \lambda I)$ then
$0 = \langle (T - \lambda I)x, y \rangle = \langle x, (T^* - \overline{\lambda})y \rangle$
and therefore $T^*y - \overline{\lambda}y = 0 \Rightarrow y = 0$

It follows that $R(T - \lambda I) = \{o\}$. So that $H = \overline{R(T - \lambda I)}$. i.e., $T - \lambda I$ has bounded inverse so $\lambda \notin \sigma(T) \Rightarrow \sigma(T) \subseteq \sigma_{ap}(T)$.............(ii)

From (i) and (ii) equality holds. Since $\sigma_{ap}(T)$ lies on the unit circle $\partial\mathbb{D}$, it follows that $\sigma(T) = \sigma_{ap}(T) =$ the closed unit disc $\overline{\mathbb{D}}$.

Theorem 3.1.14. [12]
A non-unitary 2-isometry similar to a spectraloid operator is an isometry.

proof
Let T be a 2-isometry similar to a spectraloid operator A

Then $r(T^n) = r(A^n) = w(A^n)$ for $n = 1, 2, 3, \ldots$

Since $\sigma(T)$ is the closed unit disc $\overline{\mathbb{D}}$, it follows that $r(T) = 1 = w(A^n)$
By definition, $\|A^n\| \sup |\langle A^n x, x \rangle| = w(A^n) = 1$.

Therefore A is power bounded and similarity of T and A show that T is power bounded,it follows from *theorem 3.1.6* that T is an isometry.

Corollary 3.1.15. [12]
Let T be a non-unitary 2-isometry,then $1 \in \sigma(T^*T)$.

proof

Suppose $1 \notin \sigma(T^*T) \Rightarrow A = (T^*T - I)$ is invertible.

From the definition of a 2-isometry we have

$T^{*2}T^2 - T^*T = T^*T - I$

$\Rightarrow T^*(T^*T - I)T = T^*T - I$

$\Rightarrow T^*AT = A$

$\Rightarrow \sigma(T^*AT) = \sigma(A)$

which implies that T is similar to an isometry and so must be an isometry.This contradicts our assumption that $1 \notin \sigma(T^*T)$.

Theorem 3.1.16. [5]
Let T be a non-unitary 2-isometry. Then,
(i) $z \in \sigma_{ap}(T) \Rightarrow z^* \in \sigma_{ap}(T^*)$
(ii) $z \in \sigma_p(T) \Rightarrow z^* \in \sigma_p(T*)$
(iii) Eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

proof(i)
Let $z \in \sigma_{ap}(T)$,and $\{x_n\}$ a sequence of unit vectors in H.
Then $(T - zI)x_n \to 0 \Rightarrow (T^2 - z^2I)x_n \to 0$
$\Rightarrow (T^{*2}T^2 - z^2T^{*2})x_n \to 0$

Since T is a 2-isometry we have

$0 = T^{*2}T^2 - 2T^*T + I$
$= T^{*2}T^2x_n - 2T^*Tx_n + x_n$
$\rightarrow z^2T^{*2}x_n - zT^*x_n = (zT^* - I)^2x_n$

$\Rightarrow (T^* - z^*I)^2x_n \rightarrow 0$

$\Rightarrow z^* \in \sigma_{ap}(T^*)$

proof(ii)
let $z \in \sigma_p(T) \Rightarrow \exists x \neq o$ such that
$(T - zI)x = 0$
$\Rightarrow (T^{*2}T^2 - z^2T^{*2})x = 0$ and$(T^*T - zT^*)x = 0$

Since T is a 2-isometry we have,

$0 = T^{*2}T^2 - 2T^*T + I$

$= T^{*2}T^2x - 2T^*Tx + x$

$= z^2T^{*2}x - zT^*x = (zT^* - I)^2x = (T^* - z^*I)^2x$
$\Rightarrow z^* \in \sigma_p(T)$

proof(iii)

Let β and α be distinct eigenvalues corresponding to the eigenvectors x and y

i.e $\beta x = Tx$ and $\alpha y = Ty$

since is a 2-isometry we have ,

$0 = \langle (T^{*2}T^2 - 2T^*T + I)x, y\rangle$

$= \langle T^2x, T^2y\rangle - 2\langle Tx, Ty\rangle + \langle x, y\rangle$

$= \langle \beta^2 x, \alpha^2 y\rangle - 2\langle \beta x, \alpha y\rangle + \langle x, y\rangle$

$= (\beta^2\alpha^{*2} - 2\beta\alpha^* + 1)\langle x, y\rangle$

since $\beta \neq \alpha$,then $(\beta^2\alpha^{*2} - 2\beta\alpha^* + 1) \neq 0$

$\Rightarrow \langle x, y\rangle = 0$,thus x and y are orthogonal.

Theorem 3.1.17. [5]

Let T be a non-unitary 2-isometry.Then the numerical range of T is the closed unit disc $\overline{\mathbb{D}}$.

proof

from *theorem 2.3.7* we have $conv(\sigma(T)) \subseteq \overline{W(T)}$,we know the numerical range of a operator T is convex and since T is a 2-isometry $\sigma(T)$ isthe closed unit disc $\overline{\mathbb{D}}$,therefore $\overline{W(T)}$ is the closed unit disc $\overline{\mathbb{D}}$.

Since -1 and 1 are eigenvalues of T,this belongs to $W(T)$.Hence $W(T)$ is the closed unit disc $\overline{\mathbb{D}}$.

Lemma 3.1.18. [7, problem182]

If the difference between operators is compact,then their spectra ar the same except for eigenvalues.More explicitly,if $A - B$ is compact,and if $\lambda \in \sigma(A) - \Pi_o(A)$ then $\lambda \in \sigma(B)$

Definition 3.1.19. *Weyl's theorem refers to any theorem that characterizes the spectrum of an operator as a subset of the weyl spectrum,whose classical definition is*

$\omega(T) = \cap\sigma(T + K)$:$K$ *is compact.*

It has been shown in [7] *lemma 2.3 that for any bounded operator* T, $\omega(T)=\sigma(T)-\Pi_o(T)$

Theorem 3.1.20. [12]
The Weyl's theorem holds for 2-isometries.

proof
If T is unitary then it has no eigenvalues except 0,therefore the result holds.If T is non-unitary then $\sigma(T) = \overline{\mathbb{D}}$ implying that $\Pi_o(T) = \phi$,therefore the result holds.

Corollary 3.1.21. *The weyl spectrum of a 2-isometry is the closed unit disc.*

Remark 3.1.22. *From the properties of 2-isometries,it is identified that the spectrum,weyl spectrum,numerical range and approximate point spectrum are equal to the closed unit disc* $\overline{\mathbb{D}}$*,if* T *is non-unitary.*

Theorem 3.1.23. [5]
Let T *be a bounded self-adjoint operator on* H*.Then* T *has dense range if an only if* T *is a 2-isometric operator.*

Proof
Assume T has dense range $\Rightarrow \overline{R(T)} = H$
Since T is bounded,$R(T)$ is closed.
By *theorem 1.2.7*

$H = R(T) \oplus R(T)^{\perp}$

$= \overline{R(T)} \oplus \overline{R(T)^{\perp}}$

$\Rightarrow R(T)^{\perp} = \{0\}$

Since T is self-adjoint,
$N(T^*) = N(T) = R(T)^{\perp} = \{0\}$

By theorem 1.3.6 (3)
$N(T^*) = N(T^*T) = \overline{R(T^*T)^{\perp}} = R(T)^{\perp} = \{0\}$

Then $R(T^*T)^{\perp} = \{0\}$
It follows that $R(T^*T) = H$ and since $R(T^*T)$ is closed ,$\overline{R(T^*T)} = H \Rightarrow$
$T^*Tx = x$, for all $x \in H$

$\Rightarrow T^*T = I$
Thus T^*T is an isometry and hence a 2-isometry.

Conversely,assume T is a 2-isometric operator i.e
$T^{*2}T^2 - 2T^*T + T = 0$

Sice T is self-adjoint,we have $T^*T = I \Rightarrow T^*Tx = x$ for all $x \in H$

Clearly $N(T^*T) = \{0\}$ and since $H = N(T^*T) \oplus N(T^*T)^{\perp}$
We have,$N(T^*T)^{\perp} = H$
By *theorem 1.3.6 (3)*
$N(T^*T)^{\perp} = \overline{R(T^*T)} = \overline{R(T^*)} = H$
Since T is self-adjoint,$\overline{R(T^*)} = \overline{R(T)} = H$
Hence T has dense range.

Theorem 3.1.24. *Let T be the non-zero self-adjoint 2-isometric operator on H. Then 0 is not an accumulation(limit)point of $\sigma(T^*T)$*

Proof

Assume T is a non-zero self-adjoint 2-isometric operator,then T has dense range on H

Consider the operator

$A . T^*T|_{N(T)^\perp} : N(T)^\perp \to N(T)^\perp$

Since $N(T) = \{0\}$,A is injetive.

By theorem 1.3.6 (2)

$R(T)$is closed $\Leftrightarrow R(T^*T)$ is closed

$\Leftrightarrow A$ is injective.

$\Leftrightarrow 0 \notin \sigma(A)$

$\Leftrightarrow \exists r > 0 : \sigma(A) \subseteq [r, \infty]$

By *theorem 2.2.13* we have
$\sigma(T^*T) = (\sigma(T^*T)|_{N(T)} \cup (\sigma(T^*T)|_{N(T)^\perp})$

$\subset \{0\} \cup [r, \|T\|^2]$
Hence 0 is not an accumulation point of $\sigma(T^*T)$

3.2 The unilateral weighted shift operators

In this section we look at an example of 2-isometric operator and its spectral properties.Bermudez et. al. in [3] gave a characterization for weighted shift operators on a separable Hilbert Space which are m-isometries.We use the results obtained for the case when m=2.

Definition 3.2.1. *An operator S_r acting on a Hilbert space H is a unilateral shift if there exists a sequence of (pairwise) orthogonal subspaces $\{H_k : k > o\}$ such that $H = \oplus_{k=0}^{\infty} H_k$ and S_r maps each H_k isometrically onto H_{k+1}.In particular consider the Hilbert space l_2 of square summable sequences ,the unilateral shift is the operator U on l_2 defined by*
$U(\xi_0, \xi_1, \xi_2, ...) = (0, \xi_1, \xi_2, ...)$

Theorem 3.2.2. *If U is the unilateral shift then*
$\sigma(U) = \sigma(U^*) =$ the closed unit disc D,$\sigma_p(U) = \phi$,$\sigma_{ap}(U)$= Unit circle C,$\sigma_{ap}(U^*) = D$ and $\sigma_p(U^*)$is the open unit disc.

Proof

Consider the matrix representation of U given by

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0\ldots.. & \\ 0 & 1 & 0 & 0\ldots.. & \\ \ldots\ldots\ldots & & & & \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots.. & \end{pmatrix} \text{ and } U^* = \begin{pmatrix} 0 & 1 & 0 & \ldots.. & \\ 0 & 0 & 1 & 0\ldots.. & \\ 0 & 0 & 0 & 1\ldots.. & \\ \ldots\ldots\ldots & & & & \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots.. & \end{pmatrix}$$

Let $\lambda \in \sigma_p(U)$ the $Ux = \lambda x$,where $x = (\xi_0, \xi_1, \xi_2, ...)$ then

$(\lambda\xi_0, \lambda\xi_1, \lambda\xi_2, ...) = (0, \xi_1, \xi_2, ...)$ so that $0 = \lambda\xi_0 \Rightarrow \lambda = 0$,
and hence $\sigma_p(U) = \phi$.

If $U^*x = \lambda x$ where $x = (\xi_0, \xi_1, \xi_2, ...)$ then
$(\lambda\xi_0, \lambda\xi_1, \lambda\xi_2, ...) = (\xi_1, \xi_2, \xi_3, ...)$
So that $\xi_{n+1} = \xi_n$.If ξ_0,then $x = 0$ a contradiction,therefore a necessary and sufficient condition for $x \in l_2$ is that $|\lambda| < 1 \Rightarrow \sigma_p(U^*)$ is the open unit disc.

Since $\sigma_p(U) \subset \sigma(U)$ and $\sigma(U)$ is a closed set,it follows that $\sigma(U)$ is the closed unit disc,similarly $\sigma(U^*)$ is the closed unit disc.

Let $\lambda \in \sigma(U)$,Since U is an isometry then we have

$$||\lambda| - 1| \, \|x\| = |\|\lambda x - \| - \|Ux\|| \leq \|(\lambda I - U)x\| \, , \forall x \in H$$

If $\lambda \neq 1$,then $\lambda I - U$ is bounded below,a contradiction.Therefore $|\lambda|$=1 since $\partial\sigma(U) \subseteq \sigma_{ap}(U)$,it follows that $\sigma_{ap}(U)$ includes the unit circle.

For U^* the situation is different.Since $\sigma_p(U^*) \subseteq \sigma_{ap}(U^*)$and since $\sigma_p(U^*)$ is the open unit disc,it follows that $\sigma_{ap}(U^*)$ is the closed unit disc.

Since $\sigma(U) = \sigma_{ap}(U) \cup \sigma_{cp}(U)$.Then $\sigma_{cp}(U)$ is the open unit disk and

$\sigma_{cp}(U^*) = \phi$.

Definition 3.2.3. *A unilateral weighted shift operator is the product of the unilateral shift operator and a compatible diagonal operator.More explicitly suppose that $\{e_n\}$ is an orthornormal basis $(n = 0, 1, 2, ...)$ and suppose $\{\omega_n\}$ is a bounded sequence of complex numbers.A unilateral weighted shift operator S_ω is an operator of the form $S_r P$,where S_r is the unilateral shift and P is the diagonal operator with diagonal $\{\omega_n\}$ $(Pe_n = \omega_n e_n)$*

Proposition 3.2.4. *Let S_ω be the unilateral weighted shift operator on H with weight sequence $(\omega_n)_{n\geq1}$.If S_ω is a 2-isometry,the $\omega_n \neq 0$ for all $n > 1$*

Proof

Assume there exists a positive integer n such that $\omega_n = 0$,Since $S^k_{\omega_k} e_n = 0$ for all $k \geq 1$,we obtain

$$\sum_{k=0}^{2}(-1)^{2-k}\binom{2}{k}\left\|S^k_{\omega_k} e_n x\right\|^2 = 1 \qquad (3.1)$$

A contradiction since S_ω is a 2-isometry.Hence $\omega_n \neq 0$ for all $k \geq 1$.

Proposition 3.2.5. *Let S_ω be the unilateral weighted shift operator on H with weighted sequence $\omega_n \geq 1$. Then S_w is a 2-isometry iff*

$$\sum_{k=o}^{2}(-1)^{2-k}\binom{2}{k}\left|\omega_0...\omega_{n+k-1}\right|^2 = 0 \qquad (3.2)$$

For all $n \geq 1$, where $\omega_0 := 1$.

Proof

Let $x = \sum_{n=1}^{\infty} x_n e_n \in H$..Assume S_ω is a 2-isometry then

$$0 = \sum_{k=0}^{2}(-1)^{2-k}\binom{2}{k}\left\|S^k_\omega x_n\right\|^2$$
$$= \|x\|^2 + \sum_{k=1}^{2}(-1)^{2-k}\binom{2}{k}\sum_{n=1}^{\infty}\left|\omega_n..\omega_{n+k-1}\right|^2 |x_n|^2$$

Taking $x = e_n$ and multiplying by $|\omega_0...\omega_{n-1}|$,we have;

Corollary 3.2.7. *Let S_ω be a non-isometric unilateral weighted shift with weights $\{\omega_n\}$.If S_ω is a 2-isometry then the following assertions hold;*

1. *$\{|\omega_n|\}$ is a strictly decreasing sequence of real numbers converging to 1.*

2. *$\sqrt{2} > |\omega_n| > 1$ for each $n > 1$.*

proof

Suppose $|\omega_{n+1}| > |\omega_n|$

From the theorem above we

$|\omega_{n+1}|^2 = \frac{2|\omega_n|^2 - 1}{|\omega_n|^2}$
$= |\omega_n|^2 - \frac{(1-|\omega_n|^2)^2}{|\omega_n|^2}$ ....

Which implies that $0 \geq (1 - |\omega_n|^2)^2$ or $|\omega_n|^2 = 1 \Rightarrow |\omega_n| = 1$This contradicts (2) in *theorem 3.2.6*,therefore $\{|\omega_n|\}$ is strictly decreasing sequence of real numbers and so must be convergent and $|\omega_n| \to 1$.

To proof (2) we have

$|\omega_{n+1}|^2 + \frac{1}{|\omega_n|^2} = 2$
Since $|\omega_{n+1}| \leq |\omega_n|$

$\Rightarrow |\omega_n|^2 + \frac{1}{|\omega_n|^2} \geq 2$
$\Rightarrow |\omega_n| < \sqrt{2}$ since $|\omega_n| > 1$.
Then $\sqrt{2} > |\omega_n| > 1$ for each $n > 1$.

Remark 3.2.8. *The class of 2-isometries contains operators which are not bounded.*

Example

Consider the unilateral weighted shift (non-isometric),since $S_{\omega}^{n}e_{n} = \omega_{n}e_{n+1}$

$$\|S_{\omega}^{n}\| = \sup \|S_{\omega}^{n}e_{n}\|$$
$$= \sup |\omega_{n}|\|^{2} > 1$$

Since $|\omega_{n}| > 1$ for each $n > 1$

$\Rightarrow \|S_{\omega}^{n}\| > 1$

Since S_{ω} is a 2-isometry,we conclude that the class of 2-isometries contains operators which have arbitrarily large norm,hence not bounded.

3.3 A von Neumann-Wold Decomposition for 2-isometries

Decomposition means separation into "parts".As far as operators are concerned this usually is done by product(factorization) or by sum.
For instance,the polar decomposition says that every operator can be factorized as the product of a parial isometry and a non-negative operator.On the other hand ,the cartesian decomposition is one by(ordinary)sum:every operator T can be written as $T = Re(T) + Im(T)$,where $Re(T) = \frac{T+T^*}{2}$ and $Im(T) = \frac{1}{2}(T - T^*)$ are self-adjoint operators.
However,in this section we shall deal with decomposition by direct sums which do "isolate the parts",by restriction to contractions only,so that the appropriate decompositions will isolate unitary direct summands.

Recall:A contraction is an operator $T \in B(H)$ such that $||T|| \leq 1$.

Some of the well-known basic results on contractions include;

(i) T is a contraction iff T^* is a contraction
(ii) A contraction T converges strongly to an operator A if for $n \geq 1$ $||(T^{*n}T^n - A).x|| = o$ for every $x \in H$.Moreover,A is a non-negative contraction (i.e $0 \leq A \leq I$).

(iii) $||A|| = 1$ whenever $A \neq 0$.

(iv) $||T^n x|| \to \left\|A^{\frac{1}{2}}x\right\|$ for every $x \in H$.

(v) $N(A) = \{x \in H : T^n \to 0\}$
$N(I - A) = \{x \in H : ||T^n x|| = ||x|| \forall n \geq 1\}$
$= \{x \in H : ||Ax|| = ||x||\}$

(vi) $T^{*n}AT^n = A$ for every $n \geq 1$.(so that T is an isometry whenever $A = I$)

Definition 3.3.1. *An operator $T \in B(H)$ is uniformly stable if the power sequence $\{T^n\}_{n\geq 1}$ converges uniformly to the null operator(i.e $\|T^n\| \to 0$)*

It is strongly stable if $\{T^n\}_{n\geq 1}$ converges strongly to the null operator (i.e $\|T^n x\| \to 0$ for every $x \in H$

Definition 3.3.2. ***Nagy-Foias class of contractions***
Suppose $T^{*n}T^n \to A$ and $T^nT^{*n} \to A_*$

(i) $C_{\cdot 0}$,class of contractions whose adjoint is strongly stable (i.e $N(A_* = H$ and $A_* = 0$)

$C_{0\cdot}$,class of strongly stable contractions

(ii)$C_{1\cdot}$,if T is such that T^n does not converge to 0 for all $x \in H$(i.e if $N(A) = \{0\}$).

$C_{\cdot 1}$,if T^* is such that T^{*n} does not converge to 0 for all $x \in H$(i.e if $N(A_*) = \{0\}$).

All combinations are possible and these leads to classes $C_{00}, C_{01}, C_{10}, C_{11}$,defined by;
$T \in C_{00} \Leftrightarrow A = A_* = 0$
$T \in C_{01} \Leftrightarrow A = 0, N(A_* = \{0\})$

$T \in C_{10} \Leftrightarrow A_* = 0, N(A = \{0\})$

$T \in C_{11} \Leftrightarrow N(A) = N(A_* = \{0\})$

Remark 3.3.3. *If T is a strict contraction,then it is uniformly stable,and hence of class C_{00}. Thus a contraction not in C_{00} is necessarily non-strict.(i.e $T \in C_{00}$,then $\|T\| = 1$).In particular,contractions in $C_{1.}$ or in $C_{.1}$ are non-strict.*

Theorem 3.3.4. [4] ***Nagy-Foias Langer Decomposition***
Let TbeacontractiononaHilbertspaceHansset
M=N(I-A)$\cap N(I - A_*)$
M is a reducing subspace for T.Moreover,the decomposition,

$$T = C \oplus U \ on\ H = M \oplus M^{\perp}$$

such that $C := T|_{M^{\perp}}$ is complete non-unitary contraction and $U := T|_M$ is unitary.

Remark 3.3.5. *This type of decomposition exhibits a reducing subspace for a contraction that is the largest reducing subspace on which it is unitary.*
If T is an isometry(i.e $A = I$),then the completely non-unitary direct summand becomes a unilateral shift.
If A is non-zero projection,then the completely non-unitary direct summand is the direct sum of a strongly stable contraction and a unilateral shift.
Since an operator is a unilateral shift iff it is a completely non-unitary isometry,we get the following corollary for theorem 3.3.4.

Corollary 3.3.6. [4] ***von Neumann-Wold decomposition for isometries***
IfTisanisometryonahilbertspaceH, thenN(I-A_*) is a reducing subspace for T.Moreover the decomposition.

$$T = S_+ \oplus U \ on\ H = N(I - A_*) \oplus N(I - A_*)^{\perp}$$

is such that $S_+ := T|_{N(I-A_*)^{\perp}}$ is a unilateral shift and $U := T|_{N(I-A_*)}$ is unitary.

RECALL:An operator $T \in B(H)$ is pure if it has no normal direct summands,T is said to be completely non-normal.Since every unitary operator is normal,it follows that every completely non-normal operator is completely non-unitary.

Therefore to present a von Neumann-Wold decomposition for 2-isometries on a general Hilbert space,we show that for every 2-isometry we can find a pure 2-isometry which is unitarily equivalent to a unilateral shift so that the von Neumann-Wold decomposition for isometries holds for 2-isometries.

Definition 3.3.7. *An operator $T \in B(H)$ is said to be concave if*

$$\|T^2x\|^2 + \|x\|^2 \leq 2\|Tx\|^2 \, \forall x \in H$$

By definition T is a 2-isometry if,$\|T^2x\|^2+\|x\|^2 = 2\|Tx\|^2 \, \forall x \in H$,therefore every 2-isometry is concave. Also not that if $T \in B(H)$ is concave,the sequence $\|T^nx\|^2$is increasing,since it is both non-negative and concave,thus a concave operator is *expansive*,that is $\|Tx\| \geq \|x\|$ for $x \in H$

Proposition 3.3.8. [2]
Let $T \in B(H)$ be a concave operator,then the space $H_o = \cap_{k>0}T^k(H)$ is a reducing subspace for T and the restriction to this space is unitary.

Proof
Define an operator $L = (T^*T)^{-1}T^*$.Note $(T^*T)^{-1}$ exists since T is bounded.
Let $x \in H_o$,substituting L^2x for x in
$\|T^2x\|^2 + \|x\|^2 \leq 2\|Tx\|^2$,we see that
$\|L^2x\|^2 + \|x\|^2 \leq 2\|Lx\|^2$
Thus $L|_{H_o}$ is concave and therefore expansive.Since L is a contraction and the restriction $L|_{H_o}$ is an isometry,which means $T|H_o$ is an isometry which implies that H_o is invariant under T.

To show that H_o is invariant under T^*,define the defect operator by $D = (T^*T - I)^{\frac{1}{2}}$.
since T is an isometry(i.e $\|Tx\| = x$ for all $x \in H_o$),we have $Dx = 0$ for such x.Now $T^*Tx = x$ for $x \in H_o$,and we see that
$T^*x = T^*(T^*Tx)$
$= (T^*T)Lx$
$= Lx \in H_o$ if $x \in H_o$

Thus H_o is invariant under T^*.
It follows that H_o is invariant under T and T^* and hence a reducing subspace for T.Since T is isometric on H_o,the restriction $T|_{H_o} : H_o \to H_o$ is unitary.
Let $T \in B(H)$.A unitary operator $U \in B(K)$ defined on a larger Hilbert space K containing H as a closed subspace is called a unitary extension of T if $Tx = Ux$,for all $x \in H$

Remark 3.3.9. *To present a von Neumann-Wold decomposition for 2-isometries,let $T \in B(H)$ be a 2-isometry,consider the defect operator $D = (T^*T - I)^{\frac{1}{2}}$ since $T|_{H_o}$ is unitary, we have $D|_{H_o}$ is an isometry.Therefore from the identity of a 2-isometry,$\|DTx\|^2 = \|Dx\|^2$.Thus for a 2-isometry T the induced map $T_1 := \overline{R(D)} \to \overline{R(D)}$ defined by $T_1 : Dx \to DTx$ and continuity is an isometry.*
The von Neumann-Wold decomposition for 2-isometries,is based on the observation that if T_1 is a pure 2-isometry that is $H_o = \{0\}$.Therefore $H_o^{\perp} = H$ since T_1 is non-unitary, $T_1|_{H_o^{\perp}}$ is equivalent to the unilateral shift.As a result,the von Neumann-Wold decomposition for isometries holds for pure 2-isometries. It follows that T_1 can be decomposed into $T_1 = S_+ \oplus U$ $H = H_o \oplus H_o^{\perp}$,$U = T|_{H_o}$ and $S_+ = T|_{H_o^{\perp}}$

Chapter 4

Operators related to 2-isometries

In this chapter we shall look at Hilbert space operators which are related to two isometries and the conditions under which they are 2-isometric.

4.1 quasi-isometries

Definition 4.1.1. *A bounded linear operator T on a complex Hilbert space is said to a partial isometry provided that $\|Tx\| = \|x\|$for every $x \in N(T)^{\perp}$ and $TT^*T = T$ (i.e T^* is a generalized inverse of T)*
T is quasi-isometry if $T^{*2}T^2 = T^*T$

Lemma 4.1.2. [13]
Let T be an operator with right handed polar decomposition $T = UP$(U an isometry and P a projection).Then T is a quasi-isometry iff PU is a partial isometry with $N(PU) = N(U)$.

Theorem 4.1.3. [13]
If T is quasi-isometry and $\|T\| = 1$,T is hyponormal i.e $T^*T \geq TT^*$

Proof

Suppose T is a quasi-isometry,then

$\|Tx - T^*T^2x\| = \langle Tx - T^*T^2x, Tx - T^*T^2x\rangle$

$= \|Tx\|^2 + \|T^*T^2x\|^2 - 2Re\langle x, T^{*2}T^2x\rangle$

$= \|Tx\|^2 + \|T^*T^2x\|^2 - 2\|Tx\|^2 = 0(\text{since } \|T\| = 1)$

Thus $T^* = T^{*2}T$

Hence $T^* = T^{*2}T$,since $P^2 \leq I$,we find $U^*P^2U \leq U^*U \geq UU^* \geq UP^2U^*$,this leads to $PU^*T^*TUP \geq P(T^*T)P$

since $P^2(T^*T) = (T^*T)$ by *lemma 3.4.2*,P commutes with TT^*,therefore we have

$T^*T = T^{*2}T^2 \geq P(T^*T)P = P^2(TT^*) = TT^*$,thus T is hypornormal.

Corollary 4.1.4. [13]

If T is a quasi-isometry and quasi-nilpotent,then $T^n = 0$(i.e T is nilpotent).

Note:T is said to be quasi-nilpotent if $\sigma(T) = \{0\}$

Proof

Suppose T is quasi-isometry and quasi-nilpotent the $r(T) = 0$,since $\|T^n\| < 1$ for some positive integer n,since T^n is a quasi-isometry,$\|T^n\| = 1$.By *theorem 4.1.3* T^n is hyponormal $\Rightarrow \quad \|T^n\| = r(T^n) = o \quad \Rightarrow T^n = 0$

Remark 4.1.5. *From the results obtained we observe that,an operator T with polar decomposition $T = UP$*

(i)If a quasi-isometry T is quasi-nilpotent,then it becomes a 2-isometry.

(ii) If PU is a partial isometry with $N(PU) = N(U)$ and $\sigma(PU) = \{0\}$,then PU is a 2-isometry.

4.2 Composition Operator

Definition 4.2.1. *Let $(X, \sum, \lambda)$ be a sigma finite measure space and let $T : X \to X$ be a non-singular measurable transformation. The equation $C_T f = f \circ T \quad , f \in L^2(\lambda)$ defines a transformation on $L^2(\lambda)$ called the composition operator.*

Note: Every essentially bounded complex valued measurable function f_o induces a bounded operator Mf_o on L^2 which is defined by $Mf_o(f) = f_o f$ for every $f \in L^2(\lambda)$. Further $C_T^* C_T = f_o$ and $C_T^{*2} C_T^2 = f_o^2$.

Theorem 4.2.2. [14]
A composition operator C_T on $L^2(\lambda)$ is a quasi-isometry iff $f_o^2 = f_o$.

Proof
Suppose C_T is quasi-isometry,
$\Leftrightarrow C_T^{*2} C_T^2 = C_T^* C_T$

$\Leftrightarrow \langle (C_T^{*2} C_T^2 - C_T^* C_T) f, f \rangle = 0$ for every $f \in L^2(\lambda)$

$\Leftrightarrow \int_E (f_o^2 - f_o) |f|^2 \, d\lambda = 0$ for every $E \in \sum$

$\Leftrightarrow f_o^2 = f_o \quad a.e$

Example
Let $X = \mathbb{N}$, the set of all natural numbers and λ be the counting measure on it. Define $T : \mathbb{N} \to \mathbb{N}$ as
$T(n) = n$ if n is even.
$T(n) = n + 1$ if n is odd.
Since $f_o^2 = f_o \quad a.e$ for every n, C_T is a quasi-isometry.
Recall: A bounded linear operator is a m-isometry if,
$\sum_k^m (-1)^k mC_k T^{*(m-k)} T^{m-k} = 0$

Theorem 4.2.3. [14]
The composition operator C_T on $L^2(\lambda)$ is m-isometry iff
$\sum_k^m (-1)^k mC_k f_o^{m-k} = 0$

Proof

Suppose C_T is m-isometry

$\Leftrightarrow \sum_k^m (-1)^k mC_k C_T^{*(m-k)} C_T^{m-k} = 0$

$\Leftrightarrow \left\langle (\sum_k^m (-1)^k mC_k C_T^{*(m-k)} C_T^{m-k}) f, f \right\rangle = 0$for every $f \in L^2(\lambda)$

$\Leftrightarrow \int_E (\sum (-1)^k mC_k f_o^{m-k}) |f|^2 \, d\lambda = 0$ for every $E \in \sum$

$\Leftrightarrow \sum_k^m (-1)^k mC_k f_o^{m-k} = 0 \quad a.e$

Corollary 4.2.4. [14]
The composition operator C_T on $L^2(\lambda)$ is called 2-isometry iff $f_o^2 - 2f_o + 1 = 0$.

Example 4.2.5. *Let $X = \mathbb{N}$,and λ be the counting measure on it.*
Define $T : \mathbb{N} \to \mathbb{N}$
as $T(1) = 1, T(2) = 1$ and $T(n) = n - 1$ for all $n \geq 3$
since $f_o^2 - 2f_o + 1 = 0$ a.e for every n,C_T is 2-isometry but not an isometry.

4.3 A-2 isometric operators

Definition 4.3.1. [11] *Let$A \in B(H)$ be positive,$A \neq 0$,an operator $T \in B(H)$ is said to an A-contraction or an A-isometry if it satisfies the inequality;*

$$T^*AT \leq A$$

If T and T^* are A-isometries,we say that T is an A-unitary operator.

Example 4.3.2. *(i) Recall:T is a quasi-isometry if $T^{*2}T^2 = T^*T$,therefore a quasi-isometry is a T^*T-isometry.*
(ii) A 2-isometry can be written as
$T^*(T^*T - I)T = T^*T - I$
$\Rightarrow$A 2-isometry T is a $(T^*T - I)$-isometry.

Definition 4.3.3. *T is a pure A-contraction on H if T is an A-contraction and there exists no non-zero reducing subspace for A and T in which T is an A-isometry.*

Note that any positive operator $A \in B(H)$ defines a positive semi-definite sesquilinear form:

$$\langle , \rangle_A : HxH \to C$$
$$\langle x, y\rangle_A = \langle Ax, y\rangle \text{ for all } x, y \in H$$

Therefore, if T is an A-2-isometry then it satisfies;
$\|T^2x\|_A^2 + \|x\|_A^2 = 2\,\|\|Tx\|_A^2$.

Simple computation shows that if $A = I$ an A-2-isometry becomes a 2-isometry.

4.4 Operators of class Q

Definition 4.4.1. *An operator T is of class Q if*
$0 \le Q = T^{*2}T^2 - 2T^*T + I$
Equivalently $T \in Q$ if $\|Tx\|^2 \le \frac{1}{2}\left(\|T^2x\| + \|x\|^2\right)$ for every $x \in H$.

Lemma 4.4.2. [6]
For any real λ and any operator $T \in B(H)$,
$\lambda \|T^2x\| \|x\| \le \frac{1}{2} \|T^2\|^2 + \lambda^2 \|x\|^2$
and in particular if $\lambda = 1$
$\|T^2x\| \|x\| \le \frac{1}{2}\left(\|T^2\|^2 + \|x\|^2\right)$

Recall:An operator $T \in B(H)$ is paranormal if $\|Tx\|^2 \le \|T^2x\| \|x\|$ for every $x \in H$

Theorem 4.4.3. [6]
An operator T is paranormal iff $T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \ge 0$ holds for all $\lambda > 0$.

Remark 4.4.4. *Prop 4.4.3 implies that every operator of class Q is paranormal. Also note that,taking $\lambda = 1$ and if equality hold,we observe that T becomes a 2-isometry,thus we have the inclusion;*
2-isometries $\subset$ class Q $\subset$ Paranormal.

4.5 (m, p)-Isometries

An operator $T \in B(H)$ is called an (m, p)-Isometry if there exists an $m \in \mathbb{N}$ $m > 1$ and a $p \in [1, \infty]$ such that;

for all $x \in H$ $\sum_{k=0}^{\infty}(-1)^k \binom{m}{k} \|T^k x\|^p = 0$ [15]

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $(\mathbb{C}^2, \|.\|_2)$ is a $(3, 2)$-Isometry

Remark 4.5.1. *Note that all basic properties of m-isometries on a Hilbert space carry over (m, p)-Isometries on Banach spaces.*

Proposition 4.5.2. *If T is an invertible (m, p)-Isometry and m is even, T is an $(m - 1, p)$-Isometry.*

The result follows from Agler et.al[1], (proposition 1.23).

Definition 4.5.3. *An operator $T \in B(H)$ is called (m, ∞)-Isometry iff $\max_{k=0,\ldots m_{keven}} \|T^k x\| = \max_{k=0,\ldots m_{kodd}} \|T^k x\|$ for all $x \in H$*

Note that this definition does not imply evey (m, p)-Isometry is an (m, ∞)-Isometry.

Proposition 4.5.4. [15]
*If we have for $T \in B(H)$ that $\|T^m\|$ and $\|T^m x\| \geq \|T^k x\|$,
$k = 0, \ldots, m - 2$, for all $x \in H$ This condition is also necessary if $m = 2$*

Proof

The first part follows from the definition of a (m, ∞)-Isometry.
Let T be a $(2, p)$-Isometry. By definition $\|Tx\| = \max\{\|T^2 x\|, \|x\|\}$ for all $x \in H$.
Hence $\|Tx\| \geq \|T^2 x\|$ and $\|Tx\| \geq \|x\|$.
Furthermore, the defining equation holds for $x \in R(T)$, thus $\|T^2 x\| = \max\{\|T^3 x\|, \|Tx\|\}$ and therefore $\|T^2 x\| \geq \|Tx\|$. So we have equality which proves the statements.

Corollary 4.5.5. [15]
If $T \in B(H)$ is an (m, p)-Isometry and $(2, \infty)$-Isometry then it is an isometry.

Remark 4.5.6. *Since every isometry is a 2-isometry corollary 4.5.6 gives a necessary condition under which an (m,p)-Isometry becomes a 2-isometry.*

4.6 Conclusion

The study of spectral properties of 2-isometries and related operators on a Hilbert gave basic results on the "structure" of a 2-isometry.Since this class of operators has not be studied extensively,we would like to suggest possible areas that can be investigated in future.

(I)Bergman Shift operators that are m-isometries

Definition 4.6.1. *A Bergman space A^p is a function space consisting of functions that are analytic on $\mathbb{D}$ and satisfy;*
$\int_{\mathbb{D}} |f(z)|^p < \infty$ *for a non-zero positive integer p.*

We would like study conditions under which Bergman shift operators make contact with $m - isometries$ and consider the case when $m = 2$.

(II)Similarity and Quasi-similarity of 2-isometries.

It has been shown that every cyclic analytic 2-isometry can be represented as a multiplication by z on a Dirichlet-type space $D(\mu)$(μ denotes the finite Borel measure).This representation theorem can be used to investigate similarity and Quasi-similarity of 2-isometries.

(III)Hyponormality and Subnormality properties of 2-isometries.

We would like to investigate the relationship between a m-isometries and m-hyponormal operators.Also establish a relationship between a 2-isometry and subnormal operators(an operator that has a normal extension) this will establish a condition for a normal 2-isometry.

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