CURVATURE TENSORS IN SASAKIAN, P-SASAKIAN, LP-SASAKIAN AND $\eta-$EINSTEIN SASAKIAN MANIFOLDS

A THESIS SUBMITTED TO THE SCHOOL OF MATHEMATICS, UNIVERSITY OF NAIROBI, IN FULFILMENT OF THE DEGREE OF DOCTOR OF PHILOSOPHY IN PURE MATHEMATICS.

By
James Kiwanuka Katende
School of Mathematics
July 2011

Declaration and Approval

Declaration

I the undersigned declare that this thesis contains my own work. To the best of my knowledge, no portion of this work has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

Signature.......................................

Date : 05/AUG/2011

James Kiwanuka Katende

Approval

This thesis has been under our supervision and has our approval for submission.

Prof G. P. Pokhariyal
University of Nairobi
School of Mathematics
Signature________________
9/8/2011

Prof. Jairus Khalagai
University of Nairobi
School of Mathematics
Signature________________
9/8/2011

Dedication

To my Parents and my Wife Catherine

Acknowledgement

I wish to thank God for keeping me healthy and in a state of mind to do my research. Special thanks to DAAD for providing the financial support without which this work would not be possible. My Supervisors: Professor Pokhariyal, who introduced me to Differential Geometry, for his tireless encouragement and fatherly advice and professor Khalagai for his support and valuable comments and suggestions. My wife Catherine for keeping me ever aware that my thesis was incomplete, and her undying support and love. I am grateful to my Parents for instilling the academic spirit in me during my formative years and my brothers and sisters for all the encouragement and support. I wish to also thank the director, School of Mathematics and my colleagues for their support and encouragement. There are many other people who may not be mentioned here and I would like to thank them too.

Abstract

In chapter 1, the preliminaries and definitions are introduced. The notion on manifolds, differentiable manifolds, tensor and vector fields, connections and complex manifolds and curvature tensors are introduced. The spaces to be studied namely, Sasakian, Para-Sasakian, LP-Sasakian and $\eta-$Einstein Sasakian are defined. The literature review is also included in this chapter.

In Chapter 2, properties and representations of the W_3 and W_5 curvature tensor are studied in a Sasakian Manifold. The results obtained include the representation of the W_5 curvature tensor in a W_5-symmetric Sasakian manifold . It is also proved that W_3-flat Sasakian manifold is and $\eta-$Einstein Manifold and the representation of the Riemann curvature tensor in such a manifold is obtained. Expressions of the Ricci tensor and the Riemann curvature tensor in a W_5 flat sasakian manifold are derived. It is further shown that a W_5-flat Einstein Sasakian manifold is a flat manifold.

In Chapter 3, properties and representations of the W_3 and W_5 curvature tensor are studied in a Para-Sasakian Manifold. Characterisations of the W_5 and the W_3 curvature tensors under various conditions are derived. It is shown that W_3-flat P-Sasakian manifold is $\eta-$Einstein and an expression of the Riemann curvature tensor is obtained. It is shown that W_5-flat P-Sasakian manifold is an Einstein manifold and a corresponding expression of the Riemann curvature tensor is obtained. The Ricci tensor is also considered in a P-Sasakian manifold satisfying $W_5 \cdot S = 0$. An expression for the W_5 curvature tensor in a P-Sasakian manifold satisfying $W_5(\xi, X) \cdot W_5 = 0$ is derived and it is further proved that such a manifold is Ricci flat. It is also shown that a P- Sasakian manifold satisfying $W_5(\xi, X) \cdot S = 0$ or $W_5(\xi, X) \cdot R = 0$ is an $\eta-$Einstein Manifold.

In chapter 4, we study Lorentzian Para Sasakian manifolds that are $\phi - W_3$ flat, $\phi - W_5$ flat, W_3-flat and W_5-flat. It is shown that LP-Sasakian manifolds that are $\phi - W_3$ flat, $\phi - W_5$ flat or W_3-flat are $\eta-$Einstein. An expression for the Riemann curvature tensor and the Ricci tensor in a W_5-flat LP-Sasakian manifold is derived and further more, it is proved that such a manifold is a manifold of negative constant scalar curvature

In chapter 5, properties of the W_3 curvature tensors along with its symmetric and skew symmetric parts are studied in an $\eta-$Einstein Sasakian manifold. An expression for the W_3 curvature tensor and its symmetric and skew symmetric part in a W_3-symmetric $\eta-$Einstein Sasakian manifold is derived. It is shown that a W_3-flat $\eta-$Einstein Sasakian manifold is an Einstein Manifold . It is further shown that such a manifold is isometric to the unit sphere and is a manifold of negative contant scalar curvature.

Contents

[1]This chapter has been submitted to Kyungpook Mathematics Journal
[1]This chapter has been submitted to the balkan journal of geometry
[2]This chapter has been submitted to Quastiones Mathematicae
[3]This chapter has been submitted to Differential geometry-Dynamical Systems

Chapter 1

Preliminaries and Definitions

1.1 Introduction

In this study, we investigate curvature tensors in Sasakian manifolds, $P-$Sasakian manifolds, LP-Sasakian and η-Einstein Sasakian manifolds. Some of the preliminaries and basic concepts are discussed in this chapter.

1.1.1 Differentiable manifolds

A non empty paracompact Hausdorff space M is said to be an n-dimensional topological manifold, if every point $x \in M$ has an open neigbourhood U in M, that is homeomorphic to an open subspace of the n-dimensional euclidean space $\Re^n$.

Definition 1.1.1 *A chart on M is an embedding $\phi : U \longrightarrow \Re^n$ of an open subspace U of M into $\Re^n$ such that $\phi(U)$ is an open subspace of $\Re^n$.*

Let $p_i(t_1, t_2, ...t_n) = t_i \forall t \in \Re^n$, then for every chart $\phi : U \longrightarrow \Re^n$, the composition $\phi_i = p_i \circ \phi : U \longrightarrow \Re$ is called the ith coordinate of the point $x \in U$ with respect to ϕ. The chart $\phi : U \longrightarrow \Re^n$ is called the local coordinate system in U $\forall x \in U$ and the n real numbers $(t_1, t_2, ..., t_n) = (\phi_1(x), \phi_2(x), ..., \phi_n(x))$ are said to be the coordinates of the point x with respect to ϕ.

A function $f : W \longrightarrow \Re$, defined on a non empty space W of $\Re^n$, is said to be of class:

i) C^0 iff it is continous.

ii) $C^k, k = 1, 2, ..$ iff it has continous partial derivatives of all orders $r \leq k$.

iii) C^∞ or smooth if it is of class C^k for every positive integer k.

iv) C^ω if it is an analytic function.

Definition 1.1.2 *An atlas of class C^k is a collection α of charts on M, such that the domains of all the charts in α cover the n-manifold M; that is $\bigcup_{\phi\in\alpha} Dom(\phi) = M$ and for any two charts $\phi : U \rightarrow \Re^n$ and $\psi : W \rightarrow \Re^n$ with $U \cap W$ not empty, the function $f_{(\phi,\psi)}(t) = \psi(\phi^{-1}(t))$ is of class C^k.*

The function $f_{(\phi,\psi)}$ is called the connecting function of the two charts ϕ and ψ and $\forall x \in U \cap W$, we have $f_{(\phi,\psi)}(\phi(x)) = \psi(x)$. Hence $f(\phi, \psi)$ is called the transformation for the change of local coordinate system from ϕ to ψ.

Let $C^k(M)$ be the set of all atlases on M of class C^k. If $k \neq 0$, this set $C^k(M)$ may be empty. The relation $\sim$ on M, defined by $\alpha \sim \beta$ iff $\alpha \cup \beta$ is an atlas in $C^k(M)$ for any two atlases α and β in $C^k(M)$, is an equivalence relation in $C^k(M)$ partitioning it into disjoint equivalence classes. Each of these equivalence classes is called a differentiable structure. Two atlases are said to be compatible if their union is an atlas.

Definition 1.1.3 *An n-manifold M together with a given differentiable structure σ of class C^k on M, is called a differentiable n-manifold.*

Let X and Y be diferentiable m and n manifolds respectively of class C^k with differentiable structures ζ and η, where $k = 0, 1, ...\infty$. An arbitrary function $f : X \rightarrow Y$ is said to be differentiable of class $C^h, h \leq k$, if for every chart $\phi : U \rightarrow \Re^m$ in the maximal atlas of ζ and every chart $\psi : W \rightarrow \Re^n$ with $A = U \cap f^{-1}(W) \neq \emptyset$, the function $f_{(\phi,\psi)} : \phi(A) \rightarrow \Re^n$ defined by $f_{(\phi,\psi)}(t) = \psi(f(\phi^{-1}(t))\forall t \in \phi(A)$, where $\phi(A)$ is an open subspace of $\Re^m$, is of class C^h.

A differentiable curve of class C^k in M is a differentiable mapping of class C^k of a closed interval $[a, b]$ of $\Re$ into M, which is essentially the restriction of a differentiable function of class C^k of an open interval containing $[a, b]$ into M.

1.1.2 Tangent Vectors and Vector Fields

Let $x(t)$ be a curve of class C^1, $a \leq b \leq c$ such that $x(t_0) = p$. The vector tangent to the curve $x(t)$ at p is a projection $f : \Im(p) \rightarrow \Re$ defined by $X(f) = \frac{\mathrm{df(x(t))}}{\mathrm{dt}}|_{t_0}$ or Xf is the derivative of f in the direction of the curve $x(t)$ at $t = t_0$. The vector X satisfies the following properties:

(i) $X(af + bg) = aXf + bXg)$

(ii) $X(fg) = (Xf)g(p) + f(p)(Xg)$.

This set of functions X of $\Im(p)$ into $\Re$ form a real vector space of dimension n. The set of tangent vectors at $p \in M$ is denoted by $T_p(M)$ and is called the tangent space of M at p.

Definition 1.1.4 *A vector field X on a manifold M is an assignment of a vector X_p to each point p of M defined by $Xf(p) = X_pf$.*

A vector field is called differentiable, if Xf is differentiable for every differentiable function f.

1.1.3 Tensors

Let M be an n dimensional smooth manifold. A tensor of type (r, s) at p is an $(r+s)$ linear valued function on $(T_p)^r \otimes (T_p)^s$ and the vector space of this product is denoted by ${T_p}^r_s$.

Let V be a fixed vector space over a field F,then $T^r = V \otimes V \otimes ... \otimes V$ (r times tensor product) is called the contravariant tensor space of degree r . Similarly $T_s = V^* \otimes V^* \otimes ... \otimes V^*$ (s times tensor product) is called the covariant tensor space of degree s. By convention $T^1 = V, T_1 = V^*$ and $T^0 = T_0 = F$.

A mixed tensor space of type (r, s) or a tensor space of contravariant degree r and covariant degree s is the tensor product $T^r \otimes T^s = V \otimes V \otimes ... \otimes V \otimes V^* \otimes V^* \otimes ... \otimes V^*$. An element of T^r_s is called a tensor of type (r, s) or tensor of contravariant degree r and covariant degree s.

Let $T_p(M)$ be the tangent space to a manifold M at p and $T^r_s(p)$ a tensor space of type (r, s) at over T_p. A tensor field of type (r, s) on a subset N of M, is an assignment of a tensor $K_x \in {T_r}^s(x)$ to each point x of N.

Definition 1.1.5 *A tensor Q of type $(r, 0)$ is said to be*
i) symmetric in the h^{th} and k^{th} places if $S_{(h,k)}(Q) = Q$ and
ii) skew symmetric in the h^{th} and k^{th} places if $S_{(h,k)}(Q) = -Q$,
where $(1 \leq h < k \leq r)$ and $S_{(h,k)}$ is a linear mapping which interchanges the indices at the h^{th} and k^{th} places. A tensor of type $(r, 0)$ is said to be skew symmetric if (i) holds for all pairs of indices h, k and skew symetric if (ii) holds for all pairs of indices $h, k, (1 \leq h < k \leq r)$.

Definition 1.1.6 *The linear mapping $C : {T_s}^r \to T^{r-1}_{s-1}$ defined by $C(v_1 \otimes ... \otimes v_r \otimes v_1^* \otimes ... \otimes v_s^*) =< v_i, v_j^* > (v_1 \otimes ... \otimes v_{i-1} \otimes v_{i+1} \otimes .. \otimes v_r \otimes v_1^* \otimes ... \otimes v_{j-1}^* \otimes v_{j+1}^* \otimes .. \otimes v_s^*$, where $v_1, ...v_r \in V$ and $v_1^*, ..., v_s^* \in V^*$, is called a contraction.*

A contraction map lowers both contravariant and covariant degree by one . If the initial degrees are equal, successive contractions define a map down to $T_0^0 = \mathbb{R}$ but not uniquely. Indeed if both the covariant and contravariant degree equal k, there are $k!$ different contractions depending on how we pair the indices. By considering a basis for V, contraction can also be defined in terms of coordinates.

1.1.4 Connections

Let M be a C^∞ manifold. A connection, infinitesimal connection or covariant differentiation on M is an operator ∇ that assigns to each pair of C^∞ vector fields X, Y with domain A a C^∞ field $\nabla_X Y$ with domain A. If Z is a C^∞ field on A while f is a C^∞ real valued function on A, then ∇ satisfies the following properties:

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z \tag{1.1}$$
$$\nabla_{X+Y}(Z) = \nabla_X Z + \nabla_Y Z \tag{1.2}$$
$$\nabla_{fX}(Y) = f\nabla_X Y \tag{1.3}$$
$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y. \tag{1.4}$$

Let σ be a curve in M with tangent field T. A C^∞ vector field Y on σ is said to be parallel along σ if $\nabla_T Y = 0$, on σ. The curve σ is a geodesic if $\nabla_T T = 0$. Thus, a curve is a geodesic if it's tangent field is a parallel field along the curve.

The existence of many manifolds with connections has been illustrated by naturally induced hypersurfaces of $\mathbb{R}^n$.

1.1.5 Riemannian manifold

Let T_p be the tangent space at the point p of a differentiable manifold M. If we single out a real valued bilinear, symmetric and positive definite function g on the ordered pairs of tangent vectors at each point p in M, then M is called a Riemannian manifold and g is called the metric tensor of M. Thus, for two vectors X, Y in T_p, we have

i). $g(X,Y) \in \Re$

ii) $g(X,Y) = g(Y,X)$

iii) $g(aX+bY, Z) = ag(X,Z) + bg(Y,Z)$

iv) $g(X, X) > 0$

v) If X and Y are C^∞ fields with domain A, then $g(X, Y)$ is a C^∞ function on A.

1.1.6 Lie Brackets

Vector fields can be thought of as derivations on functions. For two vector fields X and Y it may not always be true that $X(Y(f)) = Y(X(f))$ for all f. This leads to the definition of the Lie brackets or commutators of two vector fields.

Definition 1.1.7 *The Lie bracket or commutator of two vector fields X and Y on a differentiable manifold M is the unique vector field, denoted by $[X, Y]$, defined by $[X, Y](f) = X(Y(f)) - YX(f)$, where $f : M \to \mathbb{R}$ is a smooth function.*

The Lie bracket is also a derivation and is a vector field. The lie bracket of two vector fields is in some way a measure of the failure of the two vector fields to commute. Two vectors X and Y are said to commute in a region, if their lie bracket vanishes in the region. A set of vector fields is said to commute if every pair in the set commutes.

1.1.7 Riemannian Connections

Definition 1.1.8 *A connection ∇ is said to be Riemannian if,*
i) ∇ is symmetric or torsion free that is $\nabla_X Y - \nabla_Y X = [X, Y]$
ii) g is covariant constant with respect to ∇ or $\nabla_X g = 0$.

Definition 1.1.9 *The torsion tensor of a connexion ∇ is a vector valued bilinear function T which assigns to each pair of C^∞ fields X, Y, with domain A, a C^∞ vector field $T(X, Y)$ with domain A defined by,*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{1.5}$$

If $T(X, Y) = 0$, then the connexion ∇ is said to be torsion free or symmetric.

1.1.8 Curvature tensor of a connexion

Let ∇ be a Riemannian connexion. The curvature tensor R of the connexion ∇ is a linear transformation valued function that assigns to each pair of vectors X and Y at a point p of M a linear transformation $R(X, Y)$ of the tangent space M_p at p into itself. It is called the Riemann Curvature tensor. We define $R(X,Y,Z)$ by embedding X,Y and Z into smooth fields about M and setting

$$R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_Z[X,Y]. \tag{1.6}$$

The curvature tensor is linear over the ring of smooth functions as coefficients on the right and it is skew symmetric in the first and second slot, that is $R(X,Y,Z) = -R(Y,X,Z)$ and if f is a smooth function $R(fX,Y,Z) = fR(X,Y,Z)$. The Riemann curvature tensor satisfies the identities:

$$R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) = 0 \text{ (cyclic property)} \tag{1.7}$$

$$(\nabla_X R)(Y,Z,W) + (\nabla_Y R)(Z,X,W) + (\nabla_Z R)(X,Y,W) = 0. \tag{1.8}$$

These are Known as Bianchi's first identity and second identity respectively.

The Riemann curvature tensor can be viewed as a measure of the failure of a manifold that admits a connection to have locally flat geometry in an affine space.

Definition 1.1.10 *Consider a smooth manifold M and let ∇ and $\overline{\nabla}$ be two connections on a manifold M. For two fields X and Y on M, we define the difference tensor B by*

$$B(X,Y) = \nabla_X Y - \overline{\nabla}_X Y. \tag{1.9}$$

Definition 1.1.11 *Two connexions ∇ and $\overline{\nabla}$ in a smooth manifold M are said to be projectively related if*

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(X)Y + \eta(Y)X, \tag{1.10}$$

where η is a 1- form and X and Y are vector fields in M.

Let us define

$$R(X,Y,Z,W) = g(R(X,Y,Z),W). \tag{1.11}$$

It is known that R satisfies the following properties:

i) R is skew symmetric in the first 2 slots as well as the last two slots.

ii) R satisfies Bianchi's first and second identities.

The Ricci transformation at a point p with respect to a pair of vectors X, $Y \in T_pM$, is the linear map $Q_{X,Y} : T_pM \to T_pM$ such that $W \to R(W,X,Y)$.

Definition 1.1.12 *The Ricci tensor $S(X,Y)$ is a symmetric contraction of the Riemann curvature tensor.*

The Ricci tensor of a manifold M at a point p can also be regarded as a bilinear map $S : T_pM \times T_pM \to Q$ defined by $S(X,Y) \to trQ_{X,Y}$, where $trQ_{X,Y}$ is the trace of the Ricci transformation $Q_{X,Y}$.

The Ricci curvature is given by $S(X,X)$ and it gives a measure of how curved a manifold is in each of the planes containing X and is therefore a some kind of "curvature along X".

Definition 1.1.13 *A manifold is said to be an Einstein manifold if $S(X,Y) = kg(X,Y)$, where k is the Einstein constant.*

Definition 1.1.14 *A Riemannian manifold is said to be flat, if the Riemman curvature tensor vanishes for arbitrary vector fields X, Y, Z and W; that is $R(X,Y,Z,W) = 0$.*

The Weyl projective curvature tensor W, the confomal curvature tensor V, the concircular curvature tensor C and the conharmonic curvature tensor L are defined respectively by:

$$
\begin{aligned}
W(X,Y,Z) &= R(X,Y,Z) - \frac{1}{n-1}\{S(X,Y)Z - S(Y,Z)X\} && (1.12)\\
V(X,Y,Z) &= \{R(X,Y,Z) - \frac{1}{n-2}S(Y,Z)X - S(X,Z)Y - g(X,Z)QY \\
&\quad +g(Y,Z)QX + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]\} && (1.13)\\
C(X,Y,Z) &= R(X,Y,Z) - \frac{r}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\} && (1.14)\\
L(X,Y,Z) &= R(X,Y,Z) \\
&\quad -\frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(X,Z)QY + g(Y,Z)QX\}, && (1.15)
\end{aligned}
$$

where Q is the symmetric endomorphism of a tangent vector to the Ricci tensor.

Based on the Weyl projective curvature tensor Pokhariyal and Mishra [23] and Pokhariyal [41] defined some new curvature tensors to study their geometric and physical properties. These tensors are listed below.

$$W_1(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\{g(X,T)S(Y,Z) - g(Y,T)S(X,Z)\} \quad (1.16)$$

$$W_2(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\{g(X,Z)S(Y,T) - g(Y,Z)S(X,T)\} \quad (1.17)$$

$$W_3(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\{g(Y,Z)S(X,T) - g(Y,T)S(X,Z) \quad (1.18)$$

$$W_4(X,Y,Z,T) = R(X,Y,Z,T)\frac{1}{n-1}\{g(X,Z)S(Y,T) - g(X,Y)S(Z,T)\} \quad (1.19)$$

$$W_5(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\{g(X,Y)S(Z,T) - g(X,T)S(Y,Z)\} \quad (1.20)$$

$$W_6(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\{g(X,T)S(Z,Y) - g(X,Z)S(Y,T)\} \quad (1.21)$$

$$W_7(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\{g(Y,Z)S(X,T) - g(X,T)S(Y,Z)\} \quad (1.22)$$

$$W_8(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\{g(Z,T)S(X,Y) - g(X,T)S(Y,Z)\} \quad (1.23)$$

$$W_9(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\{g(Z,T)S(X,Y) - g(Y,Z)S(X,T)\}. \quad (1.24)$$

Some properties of these tensors have been studied in different manifolds. In this study the W_3 and the W_5 curvature tensors are investigated in different manifolds.

1.2 Complex Manifolds

Definition 1.2.1 *A complex structure on a real vector space V is a linear endormorphism J of V such that $J^2 = -1$, where 1 stands for the identity transformation of V.*

A real vector space V with a complex structure J can be turned into a complex vector space by defining scalar multiplication by complex numbers as follows:

$$(a + ib)X = aX + bJX, \quad (1.25)$$

for $X \in V, a, b \in \Re$. The real dimension m of V must be even say $2n$ and n will then be the complex dimension of V. The complex space $\mathbb{C}^n$ can be identified with the real vector space $\Re^{2n}$. The cannonical complex structure of $\Re^{2n}$, in terms of the natural basis for $\Re^{2n}$ is given by the matrix

$$J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (1.26)$$

where I is the $n \times n$ identity matrix.

Definition 1.2.2 *An almost complex structure on a manifold M is a tensor field J which is at every point $x \in M$ an endomorphism of the tangent space T_xM, such that $J^2 = -1$, where 1 denotes the identity transformation of T_xM. A manifold with a fixed almost complex structure is called an almost complex manifold.*

1.2.1 Almost Sasakian and Sasakian Manifolds

Let us consider an n dimensional real differentiable manifold M of differentiability class C^{r+1} endowed with a vector valued linear function ϕ, a $1-$form η and a vector field ξ satisfying

$$\phi^2X + X = \eta(X)\xi, \tag{1.27}$$

for an arbitrary vector field X. Then the system (ϕ, ξ, η) is said to give an almost contact structure to M and M is called an almost contact manifold. From (1.27), we have

$$\begin{aligned} \phi\xi &= 0 && (1.28)\\ \eta(\phi X) &= 0 && (1.29)\\ \eta(\xi) &= 1. && (1.30) \end{aligned}$$

If there exists a metric tensor g in M satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{1.31}$$

Then M is an almost Grayan manifold. From (1.28)and (1.29) , we have

$$g(X, \xi) = \eta(X). \tag{1.32}$$

If we put

$$\varphi(X, Y) = g(\phi X, Y), \tag{1.33}$$

then from (1.27), (1.28) , (1.29) and (1.30), we have

$$\begin{aligned} \varphi(\phi X, \phi Y) = -g(\phi X, Y) = \varphi(X, Y) && (1.34)\\ \varphi(X, Y) + \varphi(Y, X) = 0 && (1.35)\\ \varphi(X, Y) = (\nabla_X\eta)(Y) - (\nabla_Y\eta)X = (d\eta)(X, Y), && (1.36) \end{aligned}$$

where ∇ is a Riemannian connexion, then M is an almost Sasakian manifold and d is the operator of the exterior derivative. In a Sasakian manifold, φ is closed:

$$(\nabla_X\varphi)(Y,Z) + (\nabla_Y\varphi)(Z,X) + (\nabla_Z\varphi)(X,Y) = 0. \tag{1.37}$$

An almost Sasakian manifold is said to be Sasakian, if ξ is a killing vector, that is

$$(\nabla_X\eta)(Y) + (\nabla_Y\eta)(X) = 0. \tag{1.38}$$

Therefore, in a Sasakian manifold, we have (Sasaki,1965)

$$\varphi(X,Y) = (\nabla_X\eta)Y. \tag{1.39}$$

An almost Sasakian manifold on which ξ is a Killing vector and $(\nabla_X\eta)(Y) = 0$, is called a k-contact Riemannian manifold. If on a k-contact Riemannian manifold

$$(\nabla_Z\varphi)(X,Y) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z) \Leftrightarrow (\nabla_Z\varphi)X = \eta(X)Z - g(X,Z)\xi. \tag{1.40}$$

holds, then the manifold is Sasakian.

1.2.2 Lorentzian Para-Sasakian manifold

In 1989, K. Matsumoto introduced the notion of LP-Sasakian manifold [33]. Then I. Mihai and R. Rosca introduced the same notion independently and they obtained several results on this manifold.

An n-dimensional differentiable manifold M is said to be a Lorentzian Para-Sasakian (LP-Sasakian) manifold, if it admits a $(1,1)$ tensor field ϕ, a C^∞ vector field ξ a C^∞ 1-form η, and a Lorentzian metric g, which satisfy:

$$\begin{aligned}
\eta(\xi) &= -1 && (1.41)\\
\phi(X) &= X + \eta(X)\xi && (1.42)\\
g(\phi X, \phi Y) &= g(X,Y) + \eta(X)\eta(Y) && (1.43)\\
g(X,\xi) &= \eta(X), \nabla_X\phi = \phi X && (1.44)\\
(\nabla_X\varphi)(Y) &= \{g(X,Y) + \eta(X)\eta(Y)\}\xi + \{X + \eta(X)\xi\}\eta(Y) && (1.45)
\end{aligned}$$

In an LP-sasakian manifold with with structure (ϕ, ξ, η, g) it can be seen that

$$\phi\xi = 0, \eta(\phi X) = 0 \tag{1.46}$$

$$\text{rank}(\phi) = n - 1 \tag{1.47}$$

1.2.3 η-Einstein Sasakian Manifolds

An $n = (2m + 1)$ Sasakian Manifold M is called an η-Einstein Sasakian manifold if the Ricci tensor satisfies

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y) \tag{1.48}$$

for some scalars α and β. This metric was introduced by Okumura (1962) and named by Sasaki(1965). Okumura assumed that α and β are functions but later on proved that they are constants when the manifold has dimension greater than 1. In particular $\alpha + \beta = n - 1 = (2m)$ and every $\eta-$Einstein Sasakian manifold is a manifold of constant scalar curvature $r = 2n(1 + \alpha)$. When $\beta = 0$ this reduces to an Einstein Sasakian manifold.

1.3 Literature Review

Differential geometry builds on the following disciplines as its prerequisites: the analytic geometry of Descartes and Calculus (Leibniz 1646-1716, Newton (1645-1727)

The first isolated results on curves and surfaces date from the eighteenth century. Gauss (1777-1785) transformed the theory of sufaces into its modern systematic mould. A foundation of intrinsic geometry independent of embedding was given by Riemann(1826-1866). Riemann also dropped the restriction to 3 dimensions.

Around the 20^{th} century the tensor calculus was developed as a powerful tool for differential geometry by Ricci and Levi Civita together with the general relativity of Einstein (1879-1955).This signalled the development of other geometric structures in differentiable manifolds.

Calculus of variatons is closely linked to differential geometry. In 1918, Finsler [3] wrote his dissertation in which this connection was used to construct a new metric differential geometry that has since developed considerably.

Investigations of problems that are at least in part of differential geometric interest by coordinate free methods not based on any analytic assumptions were pioneered by [3] A. D. Alesandrov(1955) and H. Bussemann (1958).

The chief aim of tensor calculus has been the investigation of relations which remain valid when we change from one coordinate system to another. This makes tensor calculus desirable as a mathematical tool for developing physical laws. Tensors also allow complex expressions to be represented in a compact way and thus simplifies the mechanics of development of theory.

A great deal of work has been done on tensors. Mishra and Pokhariyal(1970) studied various geometric and physical properties of the curvature tensors. They defined a new tensor W_2 based on the Weyl projetive curvature tensor and investigated its relativistic significance. Based on the same Weyl Projective curvature tensor, Pokhariyal (1971) has defined other tensors $W_3, W_4, W_5, W_6, W_7, W_8, W_9$. Some of the physical and geometric properties of these and other tensors in different manifolds have been studied. The results obtained in these manifolds are reviewed in the following sections.

1.3.1 On Sasakian Manifolds

Mishra (1969) Studied some properties of the Riemann curvature tensors as well as the the Weyl projective curvature tensor and the conharmonic curvature tensors in Sasakian manifolds. He showed that a concircular symmetric Sasakian manifold is a manifold of constant curvature and that the concircular and Riemann curvature tensors do not vanish in a Sasakian manifold.

Pokhariyal (1971) studied the properties of the Bochner curvature tensor in the Kahler Manifold, in particular the relationship between conharmonic recurrence, Bochner recurrence and Ricci reccurence in the Kahler manifold. Jakubowicz (1973) studied the classification of four dimensional Riemannian and Einstein spaces with different signatures.

B. Sinha and J. P. Sinha (1975) studied the properties of a Sasakian manifold with constant F-holomorphic sectional curvature in connection with Ricci tensor and a parallel field of null planes.

Sinha and Sharma (1979) have studied the structure induced on the hypersurface of a Sasakian manifold and subsequently its ifinitesimal variations in various modes and remarked that the discussions could be used to study infinitesimal deformations of the universe (with unified field structure) as a hypersuface of a five dimensional Sasakian manifold.

Olszak(1979) studied conditions under which nearly Sasakian non-Sasakian manifolds are 5-dimensional and showed that such manifolds are Einstenian.

Matsumoto (1980) investigated curvature preserving transformations of P-Sadakian manifolds. He showed that each curvature preserving infinitesimal transformation is necessarily an infinitesimal automorphism.

Khan(2006) studied Einstein Projective Sasakian Manifold. He showed that a projectively flat Sasakian manifold is an Einstein Manifold and is a manifold of constant curvature. He also showed that if an Einstein Sasakian Manifold is projectively flat, then it is locally Isometric with the unit sphere $S^n(1)$.

Tripathi and Dwivedi (2008) studied the structure of some classes of K-contact Manifolds. They showed that a $(2m+1)$ dimensional Sasakian Manifold is quasiprojectively flat if and only if it is locally isometric to the unit sphere $S^{2n+1}(1)$.

De, Jun and Gazi(2008) studied Sasakian Manifolds with quasi-conformal curvature tensor. It is proved that a Quasi-conformally flat Sasakian manifold is an $\eta-$Einstein Manifold and is necessarily locally isomorhpic to the unit sphere. They also showed that a compact orientable quasi-conformally flat Sasakian manifold can not admit a non isometric conformal transformation. They also proved that an $n-$dimensional Sasakian Manifold $(n > 3)$ is quasiconformally flat if and only if it is Quasi conformally semi-symmetric.

1.3.2 On $P-$Sasakian Manifolds

T.Adati and T. Miyazawa(1979) defined and studied P-Sasakian manifolds considered as special cases of an almost para-contact manifold. They investigated P-Sasakian manifolds which are Ricci recurrent, projectively recurrent and conformally recurrent. They showed that a Ricci symmetric P-Sasakian manifold is an Einstein manifold and give the Ricci tensor for this case. They also showed that a P-Sasakian manifold cannot be a Ricci-recurrent manifold.

Sato and Matsumoto(1979) studied P-Sasakian manifolds in which the Riemannian curvature tensor or the Ricci tensor with respect to the associated Riemannian metric satisfy certain conditions. They considered P-Sasakian manifolds whose Ricci tensor is recurrent and also showed that such a manifold does not exist and in general that a recurrent P-Sasakian manifold does not exist. They also proved that a Ricci parallel $P-$ Sasakian manifold is Einstenian and that a symmetric P-Sasakian manifold is a manifold of constant curvature.

Pokhariyal (1983) studied properties of the W_3 tensor in a Sasakian manifold. Bucki (1985) studied almost r-paracontact structures of P-Sasakian type and proved that an almost r-paracontact manifold of paracontact type cannot be compact.

Ozgur (2005) investigated a Weyl Pseudo- symmetric P-sasakian Manifolds. He showed that every $n-$dimensional P-Sasakian manifold $(n \geq 4)$ is a Weyl pseudo-symmetric manifold. He also studied some characterizations of the Ricci Tensor in P -Sasakian Manifolds satisfying various conditions.

Ozgur and Tripathi (2007) investigated the Concircular curvature tensor on $P-$Sasakian Manifolds. They showed that an n dimensional P-Sasakian manifold M satisfies $Z(\xi, X) \cdot R = 0$ if and only if either M is isometric to the hyperbolic space $H^n(-1)$ or M has a constant scalar curvature $r = n(n-1)$. They also showed that an $n-$dimensional P-Sasakian manifold M satisfies $Z(\xi, X) \cdot S = 0$ if and only if either M has scalar curvature $r = n(1-n)$ or M is an Einstein manifold with scalar curvature $r = n(1-n)$, where Z is the concircular curvature tensor and S is the Ricci operator, and R is the Riemann curvature tensor.

1.3.3 On $LP-$Sasakian Manifolds

Matsumoto and Mihai(1988) studied some properties of a transformation in a LP-Sasakian manifold and came up with some new results. Ki and Kim (1990) Studied Sasakian manifolds whose C-Bochner curvature tensor vanishes. They showed that such a manifold has constant scalar curvature and at most three constant Ricci curvatures provided that the square of the length of the Ricci tensor is constant.

Gebarowski (1991) has studied conformal collineations in a LP-Sasakian manifold and showed that any conformal collineation of an LP- Sasakian manifold is necessarily a conformal motion.

Pokhariyal (1996) Studied the symmetric and skew symmetric properties of the W_1 tensor in LP-Sasakian manifolds and showed tha a W_1 symmetric LP-Sasakian manifold is not W_1 flat. These tensors have been used to explain some Physical and geometric behaviours of the four dimensional space time, Kahler, Sasakian and other complex manifolds.

Tarafdar and Bhatacharya (2000) studied LP-Sasakian manifolds with conformally flat and quasiconformally flat curvature tensor. They showed in both cases that the manifold is isometric to the unit sphere $S^n(1)$.Ozgur (2003) considered $\varphi-$conformally flat, $\varphi-$conharmonically flat and $\varphi-$projectively flat $LP-$Sasakian manifolds. He showed that a $\varphi-$conformally flat $LP-$Sasakian manifold is an $\eta-$Einstein manifold and further, that a $\varphi-$conharmonically flat $LP-$Sasakian manifold is an $\eta-$Einstein manifold with zero scalar curvature and that a $\varphi-$projectively flat $LP-$Sasakian manifold is an Einstein manifold with constant scalar curvature.

Murathan et al (2006) studied certain classifications of the LP Sasakian Manifold which satisfy the conditions $P.C = 0$, $P.Z - Z.P = 0$, and $P.Z + Z.P = 0$, where P is the $v-$Weyl projective curvature tensor, Z is the concircular curvature tensor, and C is the Weyl conformal curvature tensor. They constructed some chararacterisations of the Ricci Tensor with $P.C = 0$. They also showed that for an $n-$dimensional $LP-$Sasakian manifold $(n > 3)$ M, $P.Z - Z.P = 0$, if and only if M is an $\eta-$Einstein Manifold, $P.Z + Z.P = 0$ if and only if M is an Einstein Manifold.

Venkatesh and Bagewadi (2008) studied concircular $\varphi-$recurrent $LP-$Sasakian manifold and showed that such a manifold is an Einstein manifold. They also showed that a $\varphi-$recurrent $LP-$Sasakian manifold having nonzero constant sectional curvature is locally concircular $\varphi-$symmetric.

1.3.4 On $\eta-$Einstein Sasakian Manifolds

Pokhariyal(2001) investigated the properties of the W_2 tensor along with its associated symmetric and skew symmetric tensors in η-Einstein Sasakian manifolds.

Zhang (2009) considered compact Sasakian Manifolds with constant scalar curvature. Under some positive curvature assumption, it was shown that such Sasakian metrics must be $\eta-$Einstein.

Chapter 2

A Study of Sasakian Manifolds[1]

In this chapter, we study some properties of the W_3 and W_5 curvature tensors in a Sasakian manifold. In particular, we consider characterisations of a W_5-symmetric, a W_3-flat and a W_5-flat Sasakian manifold. We consider the representation of the Riemman curvature tensor and the Ricci Tensor in W_3 flat Sasakian Manifolds. It is also Shown that a W_3 flat Einstein Sasakian manifold is locally isometric to the unit sphere and is a manifold of constant curvature.

2.1 Introduction

Let M_n be an n $(= 2m+1)$ dimensional real differential manifold. Let there exist a vector valued linear function ϕ, a smooth vector field ξ and a smooth 1 form η satisfying

$$\begin{aligned} \eta(\xi) &= 1 & (2.1)\\ \phi^2(X) + X &= \xi\eta(X), \quad \text{for an arbitrary vector field } X & (2.2)\\ \phi(\xi) &= 0 & (2.3)\\ \eta(\phi(X)) &= 0, \quad \text{for an arbitrary vector field } X & (2.4)\\ \text{rank}(\eta) &= 1. & (2.5) \end{aligned}$$

Then the manifold is said to have an almost contact structure (ϕ, ξ, η).

Let the contact manifold be endowed with a nonsingular metric tensor g. Let us define

$$\varphi(X, Y) = g(\phi(X), Y). \tag{2.6}$$

[1]This chapter has been submitted to Kyungpook Mathematics Journal

The manifold is then called an almost contact metric manifold or an almost grayan manifold if the following conditions hold[16],[24],[31]

$$\begin{aligned} \varphi(X,Y)+\varphi(Y,X) &= 0 && (2.7)\\ g(\xi,X) &= \eta(\xi) && (2.8)\\ g(\phi(X),\phi(Y)) &= g(X,Y)-\eta(X)\eta(Y). && (2.9) \end{aligned}$$

In an almost contact metric manifold, let

$$2\varphi(X,Y) = (\nabla_X\eta)(Y) - (\nabla_Y\eta)(X) \tag{2.10}$$

and

$$(\nabla_X\varphi)(Y,Z) + (\nabla_Y\varphi)(Z,X) + (\nabla_Z\varphi)(X,Y) = 0, \tag{2.11}$$

where ∇ is the Riemann connection. The manifold is then called an almost Sasakian manifold. If in an Almost Sasakian manifold η is a killing vector field:

$$(\nabla_X\eta)(Y) + (\nabla_Y\eta)(X) = 0, \tag{2.12}$$

the manifold is then called a Sasakian manifold or a K-contact manifold.

In a Sasakian manifold the following relations hold [15],[22]

$$\begin{aligned} g(\xi,X) &= \eta(X) && (2.13)\\ S(X,\xi) &= (n-1)\eta(X) && (2.14)\\ R(\xi,X,Y,\xi) &= g(X,Y)-\eta(X)\eta(Y) && (2.15)\\ R(\xi,X,\xi) &= -X+\eta(X)\xi && (2.16)\\ R(\xi,X,Y) &= g(X,Y)-\eta(Y)X && (2.17)\\ R(X,Y,Z,\xi) &= \eta(X)g(Y,Z)-\eta(Y)g(X,Z). && (2.18) \end{aligned}$$

Pokhariyal and Mishra[23] defined new curvature tensors W_3 and W_5 to study their properties in various manifolds and their relativistic significance. These tensors are given by

$$W_5(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}\{g(X,Z)S(Y,U) - g(Y,U)S(X,Z)\} \tag{2.19}$$

$$W_3(X,Y.Z.U) = R(X,Y,Z,U) + \frac{1}{n-1}\{g(Y,Z)S(X,U) - g(Y,U)S(X,Z)\}. \qquad (2.20)$$

In this chapter we study the characterisitics and properties of the W_3 and W_5 curvature tensors in Sasakian manifolds.

Theorem 2.1.1 *In a Sasakian manifold, the W_5 curvature tensor satisfies the following properties:*

$$\begin{aligned} W_5(X,Y,Z,\xi) &= \eta(X)g(Y,Z) - \frac{1}{n-1}\eta(Y)S(X,Z) & (2.21)\\ W_5(\xi,Y,Z,\xi) &= g(Y,Z) - \eta(Y)\eta(Z) & (2.22)\\ W_5(\xi,Y,Z,U) &= \eta(U)g(Y,Z) - 2g(Y,U)\eta(Z) + \frac{1}{n-1}\eta(Z)S(Y,U). & (2.23)\end{aligned}$$

Proof. Setting $U = \xi$ in (2.19) , we have

$$W_5(X,Y,Z,\xi) = R(X,Y,Z,\xi) + \frac{1}{n-1}\{g(X,Z)S(Y,\xi) - g(Y,\xi)S(X,Z)\}.$$

Using (2.8), (2.14) and (2.18) , we get

$$W_5(X,Y,Z,\xi) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z) + \frac{1}{n-1}\{g(X,Z)(n-1)\eta(Y) - \eta(Y)S(X,Z)\}$$

Simplifying and putting like terms together we get (2.21) . Setting $X = \xi$ in (2.21), we have

$$W_5(\xi,Y,Z,\xi) = R(\xi,Y,Z,\xi) + \frac{1}{n-1}\{g(\xi,Z)S(Y,\xi) - g(Y,\xi)S(\xi,Z)\}.$$

Using (2.8,) (2.14) and (2.15) , we get (2.22). Setting $X = \xi$ in (2.19) , we have

$$W_5(\xi,Y,Z,U) = R(\xi,Y,Z,U) + \frac{1}{n-1}\{g(\xi,Z)S(Y,U) - g(Y,U)S(\xi,Z)\}.$$

Using (2.8), (2.14) and (2.17) , we get (2.23). ■

Corollary 2.1.2 *In a Sasakian manifold the W_5 curvature tensor satisfies the following properties*

$$\begin{aligned} W_5(\phi X,Y,\phi Z,\xi) &= 0 & (2.24)\\ W_5(\xi,\phi Y,\phi Z,\xi) &= g(Y,Z) - \eta(Y)\eta(Z) & (2.25)\\ W_5(\xi,Y,\phi Z,\phi U) &= 0. & (2.26)\end{aligned}$$

Proof. Applying ϕ to X and Y in (2.21) and using (2.4), we get

$$W_5(\phi X, Y, \phi Z, \xi) = \eta(\phi X)g(Y, Z) - \frac{1}{n-1}\eta(\phi Y)S(X, Z) = 0.$$

Applying ϕ to Y and Z in (2.22) and using (2.4) and (2.9), we get

$$W_5(\xi, \phi Y, \phi Z, \xi) = g(\phi Y, \phi Z) - \eta(\phi Y)\eta(\phi Z) = g(Y, Z) - \eta(Y)\eta(Z).$$

Applying ϕ to Y and Z in (2.23) and using (2.4), we get

$$W_5(\xi, Y, \phi Z, \phi U) = \eta(\phi U)g(Y, \phi Z) - 2g(Y, \phi U)\eta(\phi Z) + \frac{1}{n-1}\eta(\phi Z)S(Y, \phi U) = 0,$$

as desired. ■

2.2 W_5−Symmetric Sasakian manifold

Definition 2.2.1 *A Sasakian manifold is said to be W_5 -symmetric if*

$$(\nabla_Y W_5)(Z, U, V) = 0. \tag{2.27}$$

Condition (2.27) is equivalent to

$$R(X, Y, W_5(Z, U, V), \xi) - W_5(R(X, Y, Z), U, V, \xi) - W_5(Z, R(X, Y, U), V, \xi) - W_5(Z, U, R(X, Y, V), \xi) = 0 \tag{2.28}$$

Setting $X = \xi$ in (2.28), we get

$$R(\xi, Y, W_5(Z, U, V), \xi) - W_5(R(\xi, Y, Z), U, V, \xi) - W_5(Z, R(\xi, Y, U), V, \xi) - W_5(Z, U, R(\xi, Y, V), \xi) = 0. \tag{2.29}$$

Expanding each term individually, we have

$$R(\xi, Y, W_5(Z, U, V), \xi) = \eta(\xi)g(W_5(Z, U, V), Y) - \eta(Y)W_5(Z, U, V, \xi).$$

Using (2.1) and (2.21), we have

$$R(\xi, Y, W_5(Z, U, V), \xi) = W_5(Z, U, V, Y) - \eta(Y)\eta(Z)g(U, V) + \frac{1}{n-1}\{\eta(Y)\eta(U)S(Z, V)\}.$$

Using (2.21) the second term becomes

$$W_5(R(\xi,Y,Z),U,V,\xi) = \eta(R(\xi,Y,Z))g(U,V) - \frac{1}{n-1}\eta(U)S(R(\xi,Y,Z),V).$$

Using (2.17), we get

$$W_5(R(\xi,Y,Z),U,V,\xi) = \{g(Y,Z) - \eta(Y)\eta(Z)\}\, g(U,V) - \frac{1}{n-1}\eta(U)S\,\{g(Y,Z)\xi - \eta(Z)Y, V\}.$$

Using Linearity of the Ricci tensor, we have

$$\begin{aligned} W_5(R(\xi,Y,Z),U,V,\xi) &= \{g(Y,Z) - \eta(Y)\eta(Z)\}\, g(U,V) \\ &\quad -\frac{1}{n-1}\{g(Y,Z)S(V,\xi) - \eta(Z)S(Y,V)\}. \end{aligned}$$

Using (2.14) this simplifies to

$$\begin{aligned} W_5(R(\xi,Y,Z),U,V,\xi) &= g(Y,Z)g(U,V) - \eta(Y)\eta(Z)g(U,V) \\ &\quad -\eta(U)\eta(V)g(Y,Z) + \frac{1}{n-1}\eta(U)\eta(Z)S(Y,V). \end{aligned}$$

Using (2.14) and (2.21), we have

$$\begin{aligned} W_5(Z,R(\xi,Y,U),V,\xi) &= \eta(Z)R(\xi,Y,U,V) - \frac{1}{n-1}\eta(R(\xi,Y,U))S(Z,V) \\ &= \eta(Z)\eta(V)g(Y,U) - \eta(Z)\eta(U)g(Y,V) \\ &\quad -\frac{1}{n-1}g(Y,U)S(Z,V) + \frac{1}{n-1}\eta(Y)\eta(U)S(Z,V). \end{aligned}$$

Using (2.8), (2.14) and (2.21) and linearity of the Ricci tensor yields

$$\begin{aligned}
W_5(Z,U,R(\xi,Y,V),\xi) &= \eta(Z)R(\xi,Y,V,U) - \frac{1}{n-1}\eta(U)S(Z,R(\xi,Y,V)) \\
&= \eta(Z)\left\{\eta(U)g(Y,V) - \eta(V)g(Y,U)\right\} \\
&\quad -\frac{1}{n-1}\eta(U)\left\{S(Z,g(Y,V)\xi - \eta(V)Y\right\} \\
&= \eta(Z)\eta(U)g(Y,V) - \eta(Z)\eta(V)g(Y,U) \\
&\quad -\frac{1}{n-1}\eta(U)\left\{g(Y,V)S(Z,\xi) - \eta(V)S(Y,Z)\right\} \\
&= \eta(Z)\eta(U)g(Y,V) - \eta(Z)\eta(V)g(Y,U) \\
&\quad -\frac{1}{n-1}\eta(U)\left\{g(Y,V)(n-1)\eta(Z) - \eta(V)S(Y,Z)\right\} \\
&= -\eta(Z)\eta(V)g(Y,U) + \frac{1}{n-1}\eta(U)\eta(V)S(Y,Z).
\end{aligned}$$

Substituting all these terms back into (2.29) and puting like terms together yields

$$\begin{aligned}
0 &= W_5(Z,U,V,Y) - g(Y,Z)g(U,V) + \eta(U)\eta(V)g(Y,Z) \\
&+ \eta(Z)\eta(U)g(Y,V) - \frac{1}{n-1}\{\eta(U)\eta(Z)S(Y,V) \\
&- g(Y,U)S(Z,V) + \eta(U)\eta(V)S(Y,Z)\}.
\end{aligned}$$

On rearrangement, we have

$$\begin{aligned}
W_5(Z,U,V,Y) &= g(Y,Z)g(U,V) - \eta(U)\eta(V)g(Y,Z) \\
&- \eta(Z)\eta(U)g(Y,V) + \frac{1}{n-1}\{\eta(U)\eta(Z)S(Y,V) \\
&- g(Y,U)S(Z,V) + \eta(U)\eta(V)S(Y,Z)\}.
\end{aligned}$$

Thus, we have proved the following theorem.

Theorem 2.2.2 *In a W_5 symmetric Sasakian manifold, the W_5 curvature tensor is given by*

$$\begin{aligned}
W_5(Z,U,V,Y) &= g(Y,Z)g(U,V) - \eta(U)\eta(V)g(Y,Z) \\
&- \eta(Z)\eta(U)g(Y,V) + \frac{1}{n-1}\{\eta(U)\eta(Z)S(Y,V) \\
&- g(Y,U)S(Z,V) + \eta(U)\eta(V)S(Y,Z)\}.
\end{aligned}$$

2.3 A Sasakian manifold Satisfying $W_3 = 0$

Definition 2.3.1 *A Sasakian manifold is said to be W_3–flat if $W_3(X, Y, Z, U)$=0, for arbitrary vector fields X, Y, Z, U.*

Definition 2.3.2 *A sasakian manifold is said to be an Einstein manifold if $R(X, Y) = kg(X, Y)$ where k is a scalar field.*

The W_3 curvature tensor is given by

$$W_3(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}\{g(Y,Z)S(X,U) - g(Y,U)S(X,Z)\}. \tag{2.30}$$

Theorem 2.3.3 *A W_3– flat Sasakian manifold is η–Einstein.*

Proof. Let M_n be a W_3–flat sasakian manifold then we have $W_3(X,Y,Z,U) = 0$ or

$$R(X,Y,Z,U) = \frac{1}{1-n}\{g(Y,Z)S(X,U) - g(Y,U)S(X,Z)\}. \tag{2.31}$$

Setting $U = \xi$ and using (2.14) and (2.18)yields

$$\eta(X)g(Y,Z) - \eta(Y)g(X,Z) = \frac{1}{1-n}\{g(Y,Z)(n-1)\eta(X) - \eta(Y)S(X,Z)\}.$$

Putting like terms together, we have

$$2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) = \frac{1}{n-1}\eta(Y)S(X,Z).$$

Setting $Y = \xi$ and using (2.1) and (2.13) , we get

$$2\eta(X)\eta(Z) - g(X,Z) = \frac{1}{n-1}S(X,Z). \tag{2.32}$$

Rearranging the terms, we get

$$S(X,Z) = -(n-1)g(X,Z) + 2(n-1)\eta(X)\eta(Z). \tag{2.33}$$

Hence a W_3–flat Sasakian manifold is η–Einstein as desired. ■

Corollary 2.3.4 *Let M_n be a W_3–flat Sasakian manifold. Then the curvature tensor is* *given by*

$$R(X,Y,Z,U) = g(Y,Z)\{g(X,U) - 2\eta(X)\eta(U)\} - g(Y,U)g(X,Z) + g(Y,U)\eta(X)\eta(Z). \tag{2.34}$$

Proof. Substituting (2.33) back into (2.31), we get (2.34). ■

Theorem 2.3.5 *In a W_3–flat Einstein Sasakian manifold the Ricci tensor is given by*

$$S(X,Y) = (1-n)g(X,Y). \tag{2.35}$$

Proof. Consider a W_3–flat Einstein Sasakian manifold. Then we have by definition

$$S(X,Y) = kg(X,Y), \tag{2.36}$$

where k is a scalar field. Since $W_3 = 0$, we have

$$R(X,Y,Z) = \frac{1}{n-1}\{YS(X,Z) - g(Y,Z)QX\}. \tag{2.37}$$

Using (2.36) we have

$$R(X,Y,Z) = \frac{1}{n-1}\{kg(X,Z)Y - g(Y,Z)kX\}.$$

Transvecting with U, we get

$$R(X,Y,Z,U) = \frac{k}{n-1}\{g(X,Z)g(Y,U) - g(Y,Z)g(X,U)\}.$$

Setting $X = U = \xi$ and using (2.4) and (2.18) yields

$$g(Y,Z) - \eta(Y)\eta(Z) = \frac{k}{n-1}\{\eta(Y)\eta(Z) - g(Y,Z)\}.$$

Putting like terms together, we get

$$\left(1 - \frac{k}{1-n}\right)\{g(Y,Z) - \eta(Y)\eta(Z)\} = 0.$$

which implies $k = 1-n$, since $g(Y,Z) \neq \eta(Y)\eta(Z)$.Hence the theorem is proved. ■

Corollary 2.3.6 *A W_3–flat Einstein Sasakian manifold is locally isometric to the unit* *sphere* $S^n(1)$.

Proof. Substituting(2.36)into (2.37) and transvecting with U yields

$$\begin{aligned} R(X,Y,Z,U) &= \frac{1}{1-n}\{g(Y,Z)(1-n)g(X,U) - g(Y,U)(1-n)g(X,Z)\} \\ &= g(Y,Z)g(X,U) - g(Y,U)g(X,Z). \end{aligned}$$

Hence a W_3-flat Einstein Sasakian manifold is locally isometric to the unit sphere $S^n(1)$. ∎

Corollary 2.3.7 *A W_3-flat Einstein Sasakian manifold is a manifold of constant curvature.*

Proof. Contracting (2.37) we get

$$r = n(1-n).$$

Hence the scalar curvature of a W_3-flat Einstein Sasakian manifold is constant. ∎

2.4 Sasakian Manifold with $W_5 = 0$.

Definition 2.4.1 *A Sasakian manifold is said to be W_5-flat if $W_5(X,Y,Z,U) = 0$, for arbitrary vector fields X, Y, Z and U.*

Theorem 2.4.2 *In a W_5-Flat Sasakian manifold the Ricci tensor is given by*

$$S(X,Y) = (n-1)\eta(X)\eta(Y). \tag{2.38}$$

Proof. The W_5 curvature tensor is given by

$$W_5(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}\{g(X,Z)S(Y,U) - g(Y,U)S(X,Z)\}. \tag{2.39}$$

Let the Sasakian manifold be W_5-flat. Setting $W_5(X,Y,Z,U) = 0$ in (2.39), we have

$$-R(X,Y,Z,U) = \frac{1}{n-1}\{g(X,Z)S(Y,U) - g(Y,U)S(X,Z)\}. \tag{2.40}$$

Setting $U = \xi$ and using (2.13), (2.14) and (2.18), we get

$$-\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} = \frac{1}{n-1}\{g(X,Z)(n-1)\eta(Y) - \eta(Y)S(X,Z)\}.$$

Setting $Y = \xi$ and using (2.1) and (2.13) yields

$$-\{\eta(X)\eta(Z) - g(X,,Z)\} = g(X,Z) - \frac{1}{n-1}S(X,Z).$$

On simplification, we get

$$S(X,Z) = (n-1)\eta(X)\eta(Z),$$

as desired. ■

Corollary 2.4.3 *In A W_5-Flat Sasakian manifold the curvature tensor is given by*

$$R(X,Y,Z,U) = g(Y,U)\eta(X)\eta(Z) - g(X,Z)\eta(Y)\eta(U). \tag{2.41}$$

Proof. Substituting(2.38) into(2.40), we have

$$-R(X,Y,Z,U) = \frac{1}{n-1}\{g(X,Z)(n-1)\eta(Y)\eta(U) - g(Y,U)(n-1)\eta(X)\eta(Z)\}.$$

Putting like terms together and simplifying yields (2.41). ■

Theorem 2.4.4 *A W_5-flat Einstein-Sasakian manifold is a flat manifold.*

Proof. Let M_n be a W_5-flat Einstein-Sasakian manifold. Then the ricci tensor is by definition

$$S(X,Y) = kg(X,Y), \tag{2.42}$$

where k is a scalar. But $W_5 = 0$, therefore we have

$$-R(X,Y,Z,U) = \frac{1}{n-1}\{g(X,Z)S(Y,U) - g(Y,U)S(X,Z)\}.$$

Using (2.42) we get

$$-R(X,Y,Z,U) = \frac{1}{n-1}\{g(X,Z)kg(Y,U) - g(Y,U)k(g(X,Z))\} = 0.$$

which implies

$$R(X,Y,Z,U) = 0.$$

Hence, a W_5-flat Einstein Sasakian manifold is a flat manifold and by extension is Ricci flat. ■

Chapter 3

Para Sasakian Manifolds Satisfying Certain Conditions[1]

In this Chapter we study some properties of the W_3 and W_5 curvature tensors in a Para Sasakian manifold. We consider characterisations of the curvature tensor and the Ricci tensor in a W_3-flat and a W_5-flat Para Sasakian Manifold. We also consider a Para Sasakian Manifold satisfying various conditions and derive their consequences. It is shown that a W_3 flat Para Sasakian manifold is $\eta-$Einstein and W_5-flat Para Sasakian manifold in an Einstein manifold.

3.1 Introduction

Let M be an n-dimensional contact manifold with contact form η, i.e, $\eta \wedge (d\eta)^n \neq 0$. A contact manifold admits a vector field ξ, called the characteristic vector field such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$, for every $X \in \chi(M)$. Moreover, M admits a Riemanian metric g and a tensor field ϕ of type $(1,1)$ such that

$$\phi^2 = I - \eta \otimes \xi, \tag{3.1}$$

$$g(X, \xi) = \eta(X), \tag{3.2}$$

$$g(X, \phi Y) = d\eta(X, Y). \tag{3.3}$$

We say that (ϕ, ξ, η, g) is a contact metric structure. A contact metric manifold is said

[1]This chapter has been submitted to the balkan journal of geometry

to be Sasakian if,

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{3.4}$$

in which case,

$$\begin{aligned} \nabla_X \xi &= -\phi X, & (3.5) \\ R(X,Y,\xi) &= \eta(Y)X - \eta(X)Y. & (3.6) \end{aligned}$$

We now define a structure similar to Sasakian Structure but with no contact. An $n-$dimensional differentiable manifold M is said to admit and almost paracontact Riemannian structure (ϕ, ξ, η, g) , if ϕ is a $(1,1)$ tensor field, ξ is a vector field , η is a $1-$form and g is a Riemanian metric on M such that,

$$\begin{aligned} \phi\xi &= 0 & (3.7) \\ \eta\phi &= 0 & (3.8) \\ \eta(\xi) &= 1 & (3.9) \\ g(\xi, X) &= \eta(X) & (3.10) \\ \phi^2 X &= X - \eta(X)\xi & (3.11) \\ g(\phi X, \phi Y) &= g(X,Y) - \eta(X)\eta(Y), & (3.12) \end{aligned}$$

for all vector fields $X, Y \in M$. The vector field ξ is the metric dual of η. If (ϕ, ξ, η, g) satisfy the equations,

$$\begin{aligned} d\eta &= 0 & (3.13) \\ \nabla_X \xi &= \phi X & (3.14) \\ (\nabla_X \phi) Y &= -g(X,Y) + \eta(X)\eta(Y), & (3.15) \end{aligned}$$

then M is called a Para$-$Sasakian manifold or briefly a P-sasakian manifold [2].

In a $P-$Sasakian manifold, the following relations hold[7], [8]

$$\begin{aligned}
S(X,\xi) &= (1-n)\,\eta\,(X) && (3.16)\\
\eta(R(X,Y,Z)) &= g(X,Z)\eta(Y) - g(Y,Z)\eta(X) && (3.17)\\
R(X,Y,\xi) &= \eta(X)Y - \eta(Y)X && (3.18)\\
R\,(\xi,X,Y) &= \eta(Y)X - g(X,Y)\xi && (3.19)\\
R(\xi,X,\xi) &= X - \eta(X)\xi && (3.20)\\
Q\xi &= (1-n)\xi && (3.21)\\
\eta(R(X,Y,\xi) &= 0 && (3.22)\\
\eta(R(\xi,X,Y)) &= \eta(X)\eta(Y) - g(X,Y), && (3.23)
\end{aligned}$$

for any vector fields $X, Y, Z \in \chi\,(M)$.

The $P-$Sasakian manifold is said to be $\eta-$Einstein if

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y), \qquad (3.24)$$

where S is the Ricci tensor and α and β are smooth functions on M.

Pokhariyal and Mishra [23] defined new curvature tensors

$$W_3(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1}\{g(Y,Z)QX - YS(X,Z)\} \qquad (3.25)$$

and

$$W_5(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1}\{g(X,Y)QZ - XRic(Y,Z)\}, \qquad (3.26)$$

to study some physical and geometric properties of these tensors . In this study we consider W_3-flat and W_5-flat $P-$ Sasakian manifolds. We further study W_3 and W_5 curvature tensors in $P-$Sasakian manifolds satisfying various conditions.

3.2 P-Sasakian manifolds

Theorem 3.2.1 *In a P-Sasakian Manifold, we have*

$$\begin{aligned}
W_5(X,Y,\xi) &= \eta(X)Y - g(X,Y)\xi && (3.27)\\
W_5(\xi,Y,Z) &= \eta(Z)Y - g(Y,Z)\xi + \frac{1}{n-1}\{\eta(Y)QZ - \xi S(Y,Z)\}. && (3.28)
\end{aligned}$$

Proof. Setting $Z = \xi$ in (3.26) and using (3.10), (3.16) and (3.18), we get

$$\begin{aligned} W_5(X,Y,\xi) &= R(X,Y,\xi) + \frac{1}{n-1}\{g(X,Y)Q\xi - XS(Y,\xi)\} \\ &= \eta(X)Y - \eta(Y)X + \frac{1}{n-1}\{g(X,Y)(1-n)\xi - X(1-n)\eta(Y)\} \\ &= \eta(X)Y - g(X,Y)\xi. \end{aligned} \tag{3.29}$$

Setting $Z = \xi$ in (3.28) and using (3.10), (3.16) and (3.19), we get

$$\begin{aligned} W_5(\xi,Y,Z) &= R(\xi,Y,Z) + \frac{1}{n-1}\{g(\xi,Y)QZ - \xi S(X,Z)\} \\ &= \eta(Z)Y - g(Y,Z)\xi + \frac{1}{n-1}\{\eta(Y)QZ - S(Y,Z)\xi\}. \end{aligned} \tag{3.30}$$

as desired. ∎

Theorem 3.2.2 *In a P-Sasakian Manifold, we have*

$$W_3(X,Y,\xi) = 2\eta(X)Y - \eta(Y)X + \frac{1}{n-1}\eta(Y)QX \tag{3.31}$$

$$W_3(\xi,Y,Z) = 2\eta(Z)Y - 2g(Y,Z)\xi \tag{3.32}$$

Proof. Setting $Z = \xi$ in (3.25), we get

$$W_3(X,Y,\xi) = R(X,Y,\xi) + \frac{1}{n-1}\{g(Y,\xi)QX - YS(X,\xi)\}.$$

Using (3.10), (3.16) and (3.18), we get

$$W_3(X,Y,\xi) = \eta(X)Y - \eta(Y)X + \frac{1}{n-1}\{\eta(Y)QX - Y(1-n)\eta(X)\}.$$

Putting like terms together and simplifying yields (3.31). Setting $X = \xi$ in (3.25), we get

$$W_3(\xi,Y,Z) = R(\xi,Y,Z) + \frac{1}{n-1}\{g(Y,Z)Q\xi - YS(\xi,Z)\}.$$

Using (3.10), (3.16) and (3.19) we get

$$W_3(\xi,Y,Z) = \eta(Z)Y - g(Y,Z)\xi + \frac{1}{n-1}\{g(Y,Z)(1-n)\xi - Y(1-n)\eta(Z)\}.$$

Putting like terms together and simplifying, we get (3.32). ∎

3.3 W_3-flat $P-$Sasakian manifolds

Definition 3.3.1 *The $P-$Sasakian manifold is said to be W_3-flat if it satisfies,*

$$W_3(X,Y,Z,U)=0,$$

for arbitrary vector fields X, Y, Z and U.

Theorem 3.3.2 *The W_3- flat P Sasakian manifold is $\eta-$Einstein.*

Proof. Putting $W_3=0$ implies that

$$R(X,Y,Z,U)=\frac{1}{n-1}\left\{g(Y,Z)S(X,U)-g(Y,U)S(X,Z)\right\}. \tag{3.33}$$

Setting $U=\xi$ in (3.33) and using (3.10) and (3.16)yields

$$g(X,Z)\eta(Y)-g(Y,Z)\eta(X)=\frac{1}{1-n}\left\{g(Y,Z)(1-n)\eta(X)-\eta(Y)S(X,Z)\right\}.$$

Setting $Y=\xi$ and using (3.9) and (3.10)yields

$$g(X,Z)-\eta(Z)\eta(X)=\eta(Z)\eta(X)-\frac{1}{1-n}S(X,Z).$$

Putting like terms together and simplifying, we get

$$S(X,Z)=(n-1)g(X,Z)-2(n-1)\eta(X)\eta(Z). \tag{3.34}$$

Hence, a W_3-flat $P-$Sasakian manifold is $\eta-$Einstein. ∎

Corollary 3.3.3 *In the W_3- flat P Sasakian manifold the Riemann curvature tensor is given by*

$$R(X,Y,Z,U)=g(Y,U)g(X,Z)-g(Y,Z)g(X,U)+2\left\{g(Y,Z)\eta(X)\eta(U)-g(Y,U)\eta(X)\eta(Z)\right\}. \tag{3.35}$$

Proof. Substituting (3.34) back into (3.33) yields

$$\begin{aligned} R(X,Y,Z,U) &= \frac{1}{n-1}\{g(Y,Z)(n-1)\left\{g(X,U)-2\eta(X)\eta(U)\right\} \\ &\quad -g(Y,U)(n-1)\left\{g(X,Z)-\eta(X)\eta(Z)\right\}\}. \end{aligned}$$

On simplifying and rearranging, we get (3.35). ■

3.4 W_5–flat P-Sasakian manifolds

Definition 3.4.1 *A P–Sasakian manifold is said to be W_5–flat, if it satisfies*

$$W_5(X, Y, Z, U) = 0.$$

Theorem 3.4.2 *A W_5 – flat P-Sasakian manifold is an Einstein manifold.*

Proof. $W_5 = 0$ implies that

$$R(X, Y, Z) = \frac{1}{1-n}\left\{g(X, Y)QZ - XS(Y, Z)\right\}. \tag{3.36}$$

Setting $X = \xi$ and using (3.10) and (3.17), we have

$$\eta(Z)Y - g(Y, Z)\xi = \frac{1}{1-n}\left\{\eta(Y)QZ - \xi S(Y, Z)\right\}.$$

Transvecting with ξ and using (3.9), (3.10) and (3.16) yields

$$\eta(Z)\eta(Y) - g(Y, Z) = \frac{1}{1-n}\left\{\eta(Y)(1-n)\eta(Z) - S(Y, Z)\right\}.$$

Putting like terms together and simplifying, we get

$$\begin{aligned} g(Y, Z) &= \frac{1}{1-n}S(Y, Z) \\ S(Y, Z) &= (1-n)g(Y, Z) \end{aligned} \tag{3.37}$$

as desired. ■

Corollary 3.4.3 *In a W_5 – flat P-Sasakian manifold the Riemann curvature tensor is given by*

$$R(X, Y, Z, U) = g(X, Y)g(Z, U) - g(X, U)g(Y, Z). \tag{3.38}$$

Proof. Substituting equation (3.37) into (3.36), we get

$$\begin{aligned} R(X, Y, Z) &= \frac{1}{1-n}\left\{g(X, Y)(1-n)Z - X(1-n)g(Y, Z)\right\} \\ &= g(X, Y)Z - Xg(Y, Z). \end{aligned}$$

Tranvecting with U, we get (3.38). ∎

3.5 P-Sasakian manifold satisfying $W_5 \cdot S = 0$

Theorem 3.5.1 *In a P−Sasakian manifold satisfying $W_5 \cdot S = 0$, we have $S^2(Y,Z) = (1-n)^2\eta(Y)\eta(Z)$.*

Proof. Suppose $W_5 \cdot S = 0$, then

$$S(W_5(U,X,Y),Z) + S(Y, W_5(U,X,Z)) = 0.$$

Setting $U = \xi$ and expanding each term individually, we get

$$\begin{aligned} S(W_5(\xi,X,Y),Z) &= \eta(Y)S(X,Z) - g(X,Y)S(\xi,Z) \\ &\quad + \frac{1}{n-1}\{\eta(X)S(QY,Z) - S(X,Y)S(\xi,Z)\}. \end{aligned}$$

Using (3.16), we have

$$\begin{aligned} S(W_5(\xi,X,Y),Z) &= \eta(Y)S(X,Z) - g(X,Y)(1-n)\eta(Z) \\ &\quad + \frac{1}{n-1}\{\eta(X)S^2(Y,Z) - S(X,Y)(1-n)\eta(Z)\} \\ &= \eta(Y)S(X,Z) - g(X,Y)(1-n)\eta(Z) + S(X,Y)\eta(Z) \\ &\quad + S(X,Y)\eta(Z) + \frac{1}{n-1}\eta(X)S^2(Y,Z) \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} S(Y,W_5(\xi,X,Z) &= \eta(Z)S(X,Y) - g(X,Z)S(\xi,Y) \\ &\quad + \frac{1}{n-1}\{\eta(X)S(QZ,Y) - S(X,Z)S(\xi,Y)\}. \end{aligned}$$

Again, using (3.16) yields

$$\begin{aligned} S(Y,W_5(\xi,X,Z) &= \eta(Z)S(X,Y) - g(X,Z)(1-n)\eta(Y) \\ &\quad + S(X,Z)\eta(Y) + \frac{1}{n-1}S^2(Y,Z). \end{aligned} \tag{3.40}$$

Setting $X = \xi$ and using (3.9), (3.10), (3.16) and (3.30), equations (3.39) and (3.40) respectively become

$$
\begin{aligned}
S(W_5(\xi,\xi,Y),Z) &= \eta(Y)S(\xi,Z) - g(\xi,Y)(1-n)\eta(Z) \\
&\quad + S(\xi,Y)\eta(Z) + \frac{1}{n-1}\eta(\xi)S^2(Y,Z) \\
&= \eta(Y)(1-n)\eta(Z) - \eta(Y)(1-n)\eta(Z) \\
&\quad + (1-n)\eta(Y)\eta(Z) + \frac{1}{n-1}S^2(Y,Z) \\
&= (1-n)\eta(Y)\eta(Z) + \frac{1}{n-1}S^2(Y,Z) \qquad (3.41)
\end{aligned}
$$

and

$$
\begin{aligned}
S(Y,W_5(\xi,\xi,Z) &= \eta(Z)S(\xi,Y) - g(\xi,Z)(1-n)\eta(Y) \\
&\quad + S(\xi,Z)\eta(Y) + \frac{1}{n-1}S^2(Y,Z) \\
&= \eta(Z)(1-n)\eta(Y) - (1-n)\eta(Y)\eta(Z) \\
&\quad + (1-n)\eta(Y)\eta(Z) + \frac{1}{n-1}S^2(Y,Z). \qquad (3.42)
\end{aligned}
$$

Adding (3.41) and (3.42) , we get

$$2\left\{(1-n)\eta(Y)\eta(Z) + \frac{1}{n-1}S^2(Y,Z)\right\} = 0$$

which simplifies to

$$S^2(Y,Z) = (1-n)^2\eta(Y)\eta(Z) \qquad (3.43)$$

as desired. ■

3.6 $P-$ Sasakian Manifolds Satisfying $W_5(\xi, X).W_5 = 0$

Theorem 3.6.1 *In a P-Sasakian manifold satisfying* $W_5(\xi, X).W_5 = 0$ *, the* W_5 *curvature tensor is given by*

$$\begin{aligned}
W_5(X,Y,U) &= -g(X,Y)U - \eta(X)g(Y,U)\xi + \frac{1}{n-1}\eta(U)\eta(X)QY \\
&\quad + g(U,X)Y - \frac{1}{n-1}Y Ric(U,X) + \frac{2}{n-1}\eta(U)S(X,Y)\xi \\
&\quad - \frac{1}{n-1}S(U,X)\eta(Y)\xi + \eta(U)\eta(X)Y - g(U,Y)\eta(X)\xi \\
&\quad - \frac{1}{n-1}\eta(U)\eta(X)QY + \frac{1}{n-1}\xi\eta(X)S(U,Y).
\end{aligned}$$

Proof. Suppose a P- Sasakian manifold satisfies $W_5(\xi, X).W_5 = 0$, then we have

$$0 = [W_5(\xi, U), W_5(X, Y)]\,\xi - W_5(W_5\,(\xi, U, X)\,, Y, \xi) - W_5\,(X, W_5(\xi, U, Y)\,, \xi).$$

Expanding the lie brackets yields

$$0 = W_5(\xi, U, W_5(X, Y, \xi)) - W_5(X, Y, W_5(\xi, U, \xi) - W_5(W_5\,(\xi, U, X)\,, Y, \xi) - W_5\,(X, W_5(\xi, U, Y)\,, \xi). \tag{3.44}$$

Expanding each term in (3.44) individually and using (3.29) and (3.30) , we get

$$\begin{aligned}
W_5(\xi, U, W_5(X,Y,\xi)) &= \eta(W_5(X,Y,\xi)U - g(W_5(X,Y,\xi),U)\xi \\
&\quad + \frac{1}{n-1}\{\eta(U)QW_5(X,Y,\xi) - \xi S(U, W_5(X,Y,\xi)\} \\
&= \{\eta(X)\eta(Y) - g(X,Y)\}U - \{\eta(X)g(Y,U) - g(X,Y)\eta(U)\}\,\xi \\
&\quad + \frac{1}{n-1}\{\eta(U)\,\{\eta(X)QY - g(X,Y)Q\xi\} \\
&\quad - \xi\,\{S(U, \eta(X)Y - g(X,Y)\xi\}\} \\
&= \eta(X)\eta(Y)U - g(X,Y)U - \eta(X)g(Y,U)\xi - g(X,Y)\eta(U)\xi \\
&\quad + \frac{1}{n-1}\{\eta(U)\,\{\eta(X)QY - g(X,Y)(1-n)\xi\} \\
&\quad - \eta(X)S(U,Y)\xi + g\,(X,Y)\,(1-n)\eta(U)\xi\} \\
&= \eta(X)\eta(Y)U - g(X,Y)U - \eta(X)g(Y,U)\xi - g(X,Y)\eta(U)\xi \\
&\quad + \frac{1}{n-1}\{\eta(U)\eta(X)QY - \eta(X)S(U,Y)\xi\}\,.
\end{aligned} \tag{3.45}$$

$$\begin{aligned}
W_5(X,Y,W_5(\xi,U,\xi) &= W_5(X,Y,U-\eta(U)\xi) \\
&= W_5(X,Y,U)-\eta(U)W_5(X,Y,\xi) \\
&= W_5(X,Y,U)-\eta(U)\left\{\eta(X)Y-g(X,Y)\xi\right\} \\
&= W_5(X,Y,U)-\eta(U)\eta(X)Y+g(X,Y)\eta(U)\xi. \qquad (3.46)
\end{aligned}$$

$$\begin{aligned}
W_5(W_5\left(\xi,U,X\right),Y,\xi) &= \eta(W_5(\xi,U,X)Y-g(W_5(\xi,U,X),Y)\xi \\
&= \left\{\eta(X)\eta(Y)-g(U,X)\eta(Y)+\frac{1}{n-1}\left\{\eta(U)(1-n)\eta(X)-Ric(U,X)\right\}\right\}Y \\
&\quad -\{\eta(X)g(U,Y)-g(U,X)\eta(Y) \\
&\quad +\frac{1}{n-1}\left\{\eta(U)Ric(X,Y)-\eta\eta(Y)Ric(U,X)\right\}\}\xi \\
&= -g(U,X)Y+\frac{1}{n-1}YRic(U,X)-\eta(X)g(U,Y)\xi+g(U,X)\eta(Y)\xi \\
&\quad -\frac{1}{n-1}\eta(U)Ric(X,Y)\xi+\frac{1}{n-1}\eta(Y)Ric(U,X). \qquad (3.47)
\end{aligned}$$

$$\begin{aligned}
W_5\left(X,W_5(\xi,U,Y)\,,\xi\right) &= \eta(X)W_5(\xi,U,Y)-W_5(\xi,U,Y,X)\xi \\
&= \eta(X)\left\{\eta(Y)U-g(U,Y)\xi+\frac{1}{n-1}\left\{\eta(U)QY-\xi S(U,Y)\right\}\right\} \\
&\quad -\{\eta(Y)g(U,X)\}-g\left(U,Y\right)\eta(X) \\
&\quad +\frac{1}{n-1}\left\{\eta(U)Ric(X,Y)-\eta(X)S(U,Y)\right\}\} \\
&= \eta(X)\eta(Y)U-g(U,Y)\xi+\frac{1}{n-1}\eta(U)QY \\
&\quad -\frac{1}{n-1}\xi S(U,Y)-\eta(Y)g(U,X)\xi+g(U,Y)\eta(X)\xi \\
&\quad -\frac{1}{n-1}\eta(U)S(X,Y)\xi+\frac{1}{n-1}\eta(X)S(U,Y)\xi. \qquad (3.48)
\end{aligned}$$

Substituting all these back into (3.44) and putting like terms together yields

$$\begin{aligned}
0 &= -W_5(X,Y,U)-g(X,Y)U-\eta(X)g(Y,U)\xi \\
&\quad +\frac{1}{n-1}\eta(U)\eta(X)QY+g(U,X)Y-\frac{1}{n-1}YS(U,X) \\
&\quad +\frac{2}{n-1}\eta(U)Ric(X,Y)\xi-\frac{1}{n-1}S(U,X)\eta(Y)\xi+\eta(U)\eta(X)Y \\
&\quad -g(U,Y)\xi-\frac{1}{n-1}\eta(U)QY+\frac{1}{n-1}\xi S(U,Y). \qquad (3.49)
\end{aligned}$$

Making W_5 the subject yields,

$$\begin{aligned}
W_5(X,Y,U) &= -g(X,Y)U - \eta(X)g(Y,U)\xi + \frac{1}{n-1}\eta(U)\eta(X)QY \\
&+g(U,X)Y - \frac{1}{n-1}YS(U,X) + \frac{2}{n-1}\eta(U)S(X,Y)\xi \\
&-\frac{1}{n-1}S(U,X)\eta(Y)\xi + \eta(U)\eta(X)Y - g(U,Y)\eta(X)\xi \\
&-\frac{1}{n-1}\eta(U)\eta(X)QY + \frac{1}{n-1}\xi\eta(X)S(U,Y),
\end{aligned} \tag{3.50}$$

as desired. ■

Corollary 3.6.2 *A P-Sasakian manifold satisfying $W_5(\xi,X).W_5 = 0$ is Ricci flat.*

Proof. Setting $U = \xi$ in (3.50) and using (3.10)and (3.16) yields,

$$\begin{aligned}
\eta(X)Y - g(X,Y)\xi &= -g(X,Y)\xi - \eta(X)\eta(Y)\xi + \eta(X)Y + \eta(X)Y \\
&-\eta(Y)\eta(X)\xi + \frac{1}{n-1}\{\eta(X)QY - Y(1-n)\eta(X) \\
&+2S(X,Y)\xi - (1-n)\eta(X)\eta(Y)\xi - \eta(X)QY \\
&+\xi(1-n)\eta(Y)\eta(X)\}.
\end{aligned}$$

Transvecting with ξ and using (3.9), (3.10) and rearranging, we get

$$\begin{aligned}
\eta(X)\eta(Y) - g(X,Y) &= -g(X,Y) - \eta(X)\eta(Y) + \eta(X)\eta(Y) \\
&+\eta(X)\eta(Y) - \eta(X)\eta(Y) - \eta(X)\eta(Y) \\
&+\eta(X)\eta(Y) + \eta(X)\eta(Y) + \eta(X)\eta(Y) \\
&-\eta(X)\eta(Y) + \frac{2}{n-1}S(X,Y).
\end{aligned}$$

This simplifies to,

$$\frac{2}{n-1}S(X,Y) = 0 \Rightarrow S(X,Y) = 0.$$

Thus, the theorem is proved. ■

3.7 $P-$ Sasakian Manifold Satisfying $W_5(\xi, X).S = 0$

In a $P-$Sasakian manifold, $W_5(\xi, X).S = 0$ implies that

$$S(W_5(\xi, X, Y), \xi) + S(Y, W_5(Y, W_5(\xi, X, \xi) = 0. \tag{3.51}$$

Expanding each term individually and using (3.16), we get

$$\begin{aligned} S(W_5(\xi, X, Y), \xi) &= (1-n)\eta\,(W_5(\xi, X, Y)) \\ &= (1-n)\eta(Y)\eta(X) - g(X, Y) + \frac{1}{n-1}\left\{\eta(X)(1-n)\eta(Y) - S(X, Y)\right\}. \end{aligned}$$

Putting like terms together and simplifying, we get

$$S(W_5(\xi, X, Y), \xi) = (n-1)g(X, Y) + Ric(X, Y) \tag{3.52}$$

$$\begin{aligned} S(Y, W_5(Y, W_5(\xi, X, \xi) &= S(Y, X - \eta(X)\xi) \\ &= S(Y, X) - (1-n)\eta(X)\eta(Y). \end{aligned} \tag{3.53}$$

Substituting (3.52) and (3.53) back into (3.51) and simplifying, we get

$$\begin{aligned} 2S(X, Y) &= -(n-1)g(X, Y) + (1-n)\eta(X)\eta(Y) \\ S(X, Y) &= \frac{(1-n)}{2}g(X, Y) + \frac{(1-n)}{2}\eta(X)\eta(Y). \end{aligned}$$

Thus, we have proved the following theorem

Theorem 3.7.1 *A P-Sasakian manifold satisfying the condition $W_5(\xi, X).S = 0$ is $\eta-$Einstein.*

3.8 $P-$Sasakian manifolds satisfying $W_5(\xi, U) \cdot R = 0$

Theorem 3.8.1 *A P-Sasakian manifold satisfying $W_5(\xi, U).R = 0$ is $\eta-$Einstein.*

Proof. $W_5(\xi, U) \cdot R = 0$ implies that

$$[W_5(\xi, U), R(X, Y)]\,V - R(W_5(\xi, U, X), Y, V) - R(X, W_5(\xi, U, Y), V) = 0. \tag{3.54}$$

Expanding the lie brackets, we get

$$W_5(\xi, U, R(X,Y,V)) - R(X,Y,W_5(\xi,U,V) - R(W_5(\xi,U,X),Y,V) - R(X, W_5(\xi,U,Y),V) = 0 \tag{3.55}$$

Expanding each term in (3.55) individually, we get

$$\begin{aligned} W_5(\xi,U,R(X,Y,V)) &= \eta(R(X,Y,V))U - R(X,Y,V,U)\xi \\ &\quad + \frac{1}{n-1}\{\eta(U)Q(R(X,Y,V)) - \xi S(R(X,Y,V),U)\} \\ &= g(X,V)\eta(Y)U - g(Y,V)\eta(X)U - R(X,Y,V,U)\xi \\ &\quad + \frac{1}{n-1}\eta(U)Q(R(X,Y,V)) - \frac{1}{n-1}\xi S(R(X,Y,V),U)\ . \end{aligned} \tag{3.56}$$

$$\begin{aligned} R(X,Y,W_5(\xi,U,V) &= R\left(X,Y,\eta(V)U - g(U,V)\xi + \frac{1}{n-1}\{\eta(U)QV - \xi S(U,V)\}\right) \\ &= \eta(V)R(X,Y,U) - g(U,V)R(X,Y,\xi) \\ &\quad + \frac{1}{n-1}\eta(U)R(X,Y,QV - \frac{1}{n-1}S(U,V)R(X,Y,\xi). \end{aligned}$$

Using (3.18) and simplifying, we get

$$\begin{aligned} R(X,Y,W_5(\xi,U,V)) &= \eta(V)R(X,Y,U) - g(U,V)\eta(X)Y - g(U,V)\eta(Y)X \\ &\quad + \frac{1}{n-1}\eta(U)R(X,Y,QV) - \frac{1}{n-1}S(U,V)\eta(X)Y \\ &\quad + \frac{1}{n-1}S(U,V)\eta(Y)X. \end{aligned} \tag{3.57}$$

$$\begin{aligned} R(W_5(\xi,U,X),Y,V) &= R\left(\eta(X)U - g(U,X)\xi + \frac{1}{n-1}\{\eta(U)QX - \xi S(U,X)\}, Y, V\right) \\ &= \eta(X)R(U,Y,V) - g(U,X)R(\xi,Y,V) + \frac{1}{n-1}\eta(U)R(QX,Y,V) \\ &\quad - \frac{1}{n-1}S(U,X)R(\xi,Y,V). \end{aligned}$$

Using (3.19) and simplifying, we get

$$\begin{aligned} R(W_5(\xi,U,X),Y,V) &= \eta(X)R(U,Y,V) - g(U,X)\eta(V)Y + g(U,X)g(Y,V)\xi \\ &\quad + \frac{1}{n-1}\eta(U)R(QX,Y,V) - \frac{1}{n-1}S(U,X)\eta(V)Y. \end{aligned} \tag{3.58}$$

$$
\begin{aligned}
R(X,W_5(\xi,U,Y),V) &= R\left(X,\eta(Y)U-g(U,Y)\xi+\frac{1}{n-1}\{\eta(U)QY-\xi S(U,Y)\},V\right)\\
&= \eta(Y)R(X,U,V)-g(U,Y)R(X,\xi,V)+\frac{1}{n-1}R(X,QY,V)\eta(U)\\
&\quad -\frac{1}{n-1}S(U,Y)R(X,\xi,V)\\
&= \eta(Y)R(X,U,V)+g(U,Y)\{\eta(V)X-g(X,V)\xi\}\\
&\quad +\frac{1}{n-1}R(X,QY,V)+\frac{1}{n-1}S(U,Y)\{\eta(V)X-g(X,V)\xi\}\\
&= \eta(Y)R(X,U,V)+g(U,Y)\eta(V)X-g(U,Y)g(X,V)\xi\\
&\quad +\frac{1}{n-1}R(X,QY,V)+\frac{1}{n-1}\eta(V)S(U,Y)X-\frac{1}{n-1}g(X,V)S(U,Y)\xi\ .
\end{aligned}
$$

Substituting all these back into equation (3.55) , we get

$$
\begin{aligned}
0 &= g(X,V)\eta(Y)U-g(Y,V)\eta(X)U-R(X,Y,V,U)\xi+\frac{1}{n-1}\eta(U)QR(X,Y,V)\\
&\quad -\frac{1}{n-1}\xi S(R(R(X,Y,V),U)-\eta(V)R(X,Y,U)+g(U,V)\eta(X)Y+g(U,V)\eta(Y)X\\
&\quad -\frac{1}{n-1}\eta(U)R(X,Y,QV)+\frac{1}{n-1}S(U,V)\eta(X)Y-\eta(X)R(U,Y,V)+g(U,X)\eta(V)Y\\
&\quad -g(U,X)g(Y,V)\xi-\frac{1}{n-1}\eta(U)R(QX,Y,V)+\frac{1}{n-1}S(U,X)\eta(V)Y\\
&\quad -\frac{1}{n-1}S(U,X)g(Y,V)\xi-\eta(Y)R(X,U,V)-g(U,Y)\eta(V)X+g(U,Y)g(X,V)\xi\\
&\quad -\frac{1}{n-1}R(X,QY,V)\eta(U)-\frac{1}{n-1}\eta(V)R(U,Y,X)+\frac{1}{n-1}g(X,V)S(U,Y)\xi. \qquad (3.60)
\end{aligned}
$$

Transvecting with ξ and using (3.9), (3.10)and putting like terms together and on re-arrangement, we get

$$
\begin{aligned}
0 &= -\left(1+\frac{1}{n-1}\right)R(X,Y,U,V)-\frac{1}{n-1}\eta(Y)\eta(V)g(X,U)+2g(U,V)\eta(X)\eta(Y)\\
&\quad -\frac{2}{n-1}S(X,V)\eta(U)\eta(Y)+\frac{2}{n-1}S(Y,V)\eta(U)\eta(X)+\frac{1}{n-1}S(U,V)\eta(X)\eta(Y)\\
&\quad -g(U,X)g(Y,V)+\frac{1}{n-1}S(U,X)\eta(V)\eta(Y)-\frac{1}{n-1}S(U,X)g(Y,V)-g(U,Y)g(X,V)\\
&\quad +\frac{1}{n-1}\eta(V)\eta(U)g(Y,X)-\frac{1}{n-1}\eta(V)\eta(X)S(U,Y)+\frac{1}{n-1g(X,V)}S(U,Y). \qquad (3.61)
\end{aligned}
$$

Setting $X = U = \xi$ and using (3.9) ,(3.10) and (3.16) and on simplification, we get

$$\left(1 - \frac{1}{n-1}\right)\eta(Y)\eta(V) + \left(1 + \frac{1}{n-1}\right)g(Y,V) = \frac{-2}{n-1}S(Y,V),$$

which simplifies to

$$S(Y,V) = \frac{1}{2}(2-n)\eta(Y)\eta(V) - \frac{1}{2}ng(Y,V). \tag{3.62}$$

Hence, the manifold is $\eta-$Einstein as stated. ■

Chapter 4

Lorentzian Para-Sasakian Manifolds[1]

In this chapter, we study characterisations of $\phi - W_3$ flat, $\phi - W_5$ flat, W_3 flat and W_5 flat Lorentzian Para-Sasakian Manifolds. It is shown that if a Lorentzian Para Sasakian manifold is $\phi - W_3$ flat, W_3 flat or $\phi - W_5$ flat , then it is η-Einstein. It is also shown that a W_3 flat Lorentzian Para Sasakian manifold is a manifold of negative constant curvature.

4.1 Introduction

Let M_n be an n-dimensional differentiable manifold equipped with a triple (ϕ, ξ, η) where ξ is a vector filed, ϕ is a (1,1) tensor field, η is a 1-form on M_n such that,

$$\eta(\xi) = -1, \tag{4.1}$$

$$\phi^2 X = X + \eta(X)\xi. \tag{4.2}$$

This implies that [40]

$$\eta \circ \phi = 0, \tag{4.3}$$

$$\phi\xi = 0 \tag{4.4}$$

$$rank(\phi) = n - 1. \tag{4.5}$$

Then M_n admits a Lorentzian metric g, such that

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{4.6}$$

[1]This chapter has been submitted to Quastiones Mathematicae

and M_n is said to admit a Lorentzian almost Para Contact structure (ϕ, ξ, η, g). Let $\varphi(X,Y) = g(X, \phi Y)$, then, we have

$$\begin{aligned} g(X,\xi) &= \eta(X) & (4.7) \\ \varphi(X,Y) \stackrel{\text{def}}{=} g(X,\phi Y) &= g(\phi X, Y) = \varphi(Y,X) & (4.8) \end{aligned}$$

and

$$(\nabla_X \varphi)(Y,Z) = g(Y, (\nabla_X \varphi)Z) = (\nabla_X \varphi)(Z,Y), \tag{4.9}$$

where ∇ is covariant differentiation with respect to g. The manifold M_n equiped with a Lorentzian almost Para Contact structure (ϕ, ξ, η, g) is said to be a *Lorentzian almost paracontact manifold.*

The Lorentzian almost paracontact manifold equiped with the structure (ϕ, ξ, η, g) is called a Lorentzian paracontact manifold if,

$$\varphi(X,Y) = \frac{1}{2}((\nabla_X \eta)Y + (\nabla_Y \eta)X). \tag{4.10}$$

The Lorentzian almost paracontact manifold M_n equiped with the structure (ϕ, ξ, η, g) is called a Lorentzian Para-Sasakian ($LP-$Sasakian) manifold if,

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X. \tag{4.11}$$

In an $LP-$Sasakian manifold the 1-form η is closed. Further on an $LP-$Sasakian manifold M_n with the structure (ϕ, ξ, η, g) the following relations hold[6],[9]

$$R(\xi, Y, Z) = g(Y,Z)\xi - \eta(Z)Y \tag{4.12}$$

$$R(X,Y,\xi) = \eta(Y)X - \eta(X)Y \tag{4.13}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{4.14}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{4.15}$$

for any X, Y, Z in the lie algebra $\chi(M_n)$ of vector fields in M_n.

Definition 4.1.1 *An $LP-$Sasakian manifold is said to be $\eta-$Einstein if its Ricci tensor satisfies*

$$S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y), \tag{4.16}$$

for any vector fields X, Y in M_n, where α and β are scalar fields on M_n.

Pokhariyal and Mishra[23] defined the W_3 and W_5 curvature tensors to study some of their properties in various manifolds. The W_3 and W_5 tensors are defined by

$$W_3(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}\{g(Y,Z)Ric(X,U) - g(Y,U)Ric(X,Z)\} \quad (4.17)$$

$$W_5(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}\{g(X,Z)S(Y,U) - g(Y,U)S(X,Z)\}. \quad (4.18)$$

4.2 Main results

In this section we study various properties of W_3 curvature and W_5 curvature tensors in $LP-$Sasakian manifolds..

4.2.1 The $\phi - W_3$ flat $LP-$Sasakian Manifold

We now consider $\phi - W_3$ flat $LP-$Sasakian manifolds.

The tangent space at each point $p \in M_n$ denoted by $T_p(M_n)$ can be decomposed into the direct sum
$T_p(M_n) = \phi(T_p(M_n)) \oplus L(\xi_p)$, where $L(\xi_p)$ is a 1-dimensional linear subspace of $T_p(M_n)$ generated by ξ_p. We have the following map:
$W_3 : \phi(T_p(M_n) \times T_p(M_n) \times T_p(M_n) \rightarrow \phi(T_p(M_n)) \oplus L(\xi_p)$.

It may be natural to consider the following special cases.

1. $W_3 : T_p(M_n) \times T_p(M_n) \times T_p(M_n) \rightarrow L(\xi_p)$, that is, the projection of the image of W_3 in $\phi(T_p(M))$ is zero.

2. $W_3 : T_p(M_n) \times T_p(M_n) \times T_p(M_n) \rightarrow \phi(T_p(M_n))$ that is, the projection of the image of W_3 in $L(\xi_p)$ is zero.

3. $W_3 : \phi(T_pM_n)) \times \phi(T_p(M_n)) \times \phi(T_p(M_n)) \rightarrow L(\xi_p)$, that is when W_3 is restricted to $\phi(T_p(M_n)) \times \phi(T_p(M_n)) \times \phi(T_p(M_n))$, the projection of the image of W_3 in $\phi(T_p(M_n))$ is zero. This condition is equivalent to [2]

$$\phi^2 W_3(\phi X, \phi Y)\phi Z = 0. \quad (4.19)$$

Definition 4.2.1 *A differentiable manifold* (M_n, g), $n > 3$, *is said to be* $\phi - W_3$ *flat if* $\phi^2 W_3(\phi X, \phi Y)\phi Z = 0$.

We now study some characteristics of a $LP-$Sasakian manifold satisfying condition (4.19).

Theorem 4.2.2 *A $\phi - W_3$ flat $LP-$Sasakian manifold is $\eta-$Einstein.*

Proof. Let (M_n, g) be a $\phi - W_3$ flat. The condition $\phi^2 W_3(\phi X, \phi Y)\phi Z = 0$ is equivalent to $g(W_3(\phi X, \phi Y, \phi Z), \phi U) = 0$ or $W_3(\phi X, \phi Y, \phi Z), \phi U) = 0$.From (4.17)this means that

$$-R(\phi X, \phi Y, \phi Z), \phi U) = \frac{1}{n-1}\{g(\phi(Y), \phi(Z))S(\phi X, \phi U) - g(\phi Y, \phi U)Ric\phi X, \phi Z). \quad (4.20)$$

Let $\{e_1, ...e_{n-1,}\xi\}$ be a local orthonormal basis of vector fields in M_n, then $\{\phi e_1, ...\phi e_{n-1,}\xi\}$ is also an orthonormal local basis. If we set $X = U = e_i$ and summing over i , we get

$$-\sum_1^{n-1} g(R(\phi e_i.\phi Y, \phi Z), \phi e_i) = \frac{1}{n-1}\sum_1^{n-1}\{g(\phi Y.\phi Z)S(\phi e_i, \phi e_i) - g(\phi Y, \phi e_i)S(\phi e_i, \phi Z). \quad (4.21)$$

It can be shown that [6],[9]

$$\sum_1^{n-1} g(R(\phi e_i.\phi Y, \phi Z), \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z) \quad (4.22a)$$

$$\sum_1^{n-1} S(\phi e_i, \phi e_i) = \tau + n - 1 \quad (4.22b)$$

$$\sum_1^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z) \quad (4.22c)$$

$$\sum_1^{n-1} g(\phi e_i, \phi e_i) = n + 1 \quad (4.22d)$$

$$\sum_1^{n-1} g(\phi e_i, \phi Z)g(\phi Y.\phi e_i) = g(\phi Y, \phi Z). \quad (4.22e)$$

Using these equations in (4.21) , we have

$$-\{S(\phi Y, \phi Z) + g(\phi Y, \phi Z)\} = \frac{1}{n-1}\{g(\phi Y, \phi Z)(\tau + n - 1) - S(\phi Y, \phi Z)\}.$$

Putting like terms together, we get

$$(\frac{1}{n-1} - 1)S(\phi Y, \phi Z) = (\frac{\tau + n - 1}{n-1})g(\phi Y, \phi Z),$$

which simplifies to

$$S(\phi Y, \phi Z) = \left(\frac{\tau + 2n - 2}{2 - n}\right) g(\phi Y, \phi Z).$$

Using (4.6) and (4.15), we have

$$\begin{aligned} S(X,Y) + (n-1)\eta(X)\eta(Y) &= \left(\frac{\tau + 2n - 2}{2 - n}\right) \{g(X,Y) + \eta(X)\eta(Y)\} \\ S(X,Y) &= \left\{\frac{\tau + 2n - 2}{2 - n}\right\} g(X,Y) - \left\{\frac{\tau + 2n - 2}{2 - n} + (n-1)\right\} \eta(X)\eta(Y). \end{aligned}$$

On simplification, we get

$$S(X,Y) = \left\{\frac{\tau + 2n - 2}{2 - n}\right\} g(X,Y) + \left\{\frac{\tau - n + n^2}{2 - n}\right\} \eta(X)\eta(Y).$$

Hence, a ϕW_3-flat $LP-$Sasakian manifold is $\eta-$Einstein. ∎

4.2.2 The $\phi - W_5$ flat $LP-$Sasakian Manifold

Definition 4.2.3 *A differentiable manifold (M_n, g), $n > 3$, is said to be $\phi - W_5$ flat if*

$$\phi^2 W_5(\phi X, \phi Y)\phi Z = 0. \tag{4.23}$$

We now study the characteristics of $LP-$Sasakian manifolds satisfying condition (4.23).

Theorem 4.2.4 *The $\phi - W_5$ flat $LP-$Sasakian manifold is $\eta-$Einstein*

Proof. Let (M_n, g) be a $\phi - W_5$ flat manifold . The condition

$$\phi^2 W_5(\phi X, \phi Y)\phi Z = 0,$$

is equivalent to

$$g(W_5(\phi X, \phi Y, \phi Z), \phi U) = 0$$

or

$$W_5(\phi X, \phi Y, \phi Z), \phi U) = 0.$$

For a $\phi - W_5$ flat LP-Sasakian Manifold, we have

$$W_5(\phi X, \phi Y, \phi Z, \phi U) = R(\phi X, \phi Y, \phi Z, \phi U) + \frac{1}{n-1}\{g(\phi X, \phi Z)S(\phi Y, \phi U) - g(\phi Y, \phi U)S(\phi X, \phi Z)\} =$$

This implies that

$$-R(\phi X, \phi Y, \phi Z, \phi U) = \frac{1}{n-1}\left\{g(\phi X, \phi Z)S(\phi Y, \phi U) - g(\phi Y, \phi U)S(\phi X, \phi Z)\right\}.$$

Let $\{e_1, ...e_{n-1,}\xi\}$ be a local orthonormal basis of vector fields in M_n, then $\{\phi e_1, ...\phi e_{n-1,}\xi\}$ is also an orthonormal local basis. If we set $X = U = e_i$ and summing over i, we get

$$\sum_1^{n-1} - (R(\phi e_i, \phi Y, \phi Z), \phi e_i) = \frac{1}{n-1}\left\{\sum_1^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) - \sum_1^{n-1} g(\phi Y, \phi e_i)S(\phi e_i, \phi Z\right\}.$$

Using equations $(4.22a)$ and $(4.22c)$ yields

$$-\left\{S(\phi Y, \phi Z) + g(\phi Y, \phi Z)\right\} = \frac{1}{n-1}\left\{S(\phi Y, \phi Z) - S(\phi Y, \phi Z)\right\}$$

Putting like terms together and on rearrangement, we have

$$S(\phi Y, \phi Z) = -g(\phi Y, \phi Z).$$

Using (4.6) and (4.15), we get

$$S(Y, Z) + (n-1)\eta(Y)\eta(Z) = -g(Y, Z) - \eta(Y)\eta(Z).$$

Putting like terms together and rearranging yields

$$S(Y, Z) = -g(Y, Z) - n\eta(Y)\eta(Z).$$

Hence, the $\phi - W_5$ flat $LP-$Sasakian manifold is $\eta-$Einstein. ■

4.2.3 The W_3-Flat $LP-$Sasakian Manifolds

Definition 4.2.5 *A manifold is said to be W_3-flat if $W_3(X, Y, Z, U) = 0$ for arbitrary vector fields X, Y, Z, U.*

Theorem 4.2.6 *The W_3-Flat $LP-$Sasakian Manifold is $\eta-$Einstein.*

Proof. The W_3 tensor is given by

$$W_3(X, Y, Z, U) = R(X, Y, Z, U) + \frac{1}{n-1}\{g(Y, Z)S(X, U) - g(Y, U)S(X, Z)\}.$$

On an $LP-$Sasakian manifold for which $W_3 = 0$, we have

$$R(X, Y, Z) = \frac{1}{n-1}\{YS(X, Z) - g(Y, Z)QX\}.$$

Setting $Z = \xi$ and using (4.7), (4.13) and (4.14), we have

$$\begin{aligned} \eta(Y)X - \eta(X)Y &= \frac{1}{n-1}\{YS(X, \xi) - \eta(Y)QX \\ \eta(Y)X - \eta(X)Y &= \frac{1}{n-1}\{Y(n-1)\eta(X) - \eta(Y)QX. \end{aligned}$$

Setting $Y = \xi$ and using (4.1), we have

$$-X - \eta(X)\xi = \frac{1}{n-1}\{(n-1)\eta(X)\xi + QX\}. \tag{4.24}$$

Making QX the subject, we have

$$QX = (-X - 2\eta(X)\xi)(n-1).$$

Simplifying further, we get

$$QX = (1-n)X - 2(n-1)\eta(X)\xi).$$

Hence, the manifold is $\eta-$Einstein. ■

4.2.4 The W_5-Flat $LP-$Sasakian Manifolds

Definition 4.2.7 *A manifold is said to be W_5-flat, if $W_5(X, Y, Z, U) = 0$, for arbitrary vector fields X, Y, Z, U.*

Theorem 4.2.8 *In A W_5-flat LP-Sasakian manifold the Ricci tensor is given by*

$$S(Y, U) = -(n-1)\eta(Y)\eta(U)$$

Proof. If an $LP-$ Sasakian manifold is W_5-flat (4.18)reduces to

$$-R(X, Y, Z) = \frac{1}{n-1}\{g(X, Z)QY - YS(Y, Z)\}. \tag{4.25}$$

Setting $X = \xi$ yields

$$-R(\xi, Y, Z) = \frac{1}{n-1} \{g(\xi, Z)QY - YS(\xi, Z)\}.$$

Using (4.7), (4.12) and (4.14) and transvecting by U yields

$$-\{g(Y,Z)\eta(U) - \eta(Z)g(Y,U)\} = \frac{1}{n-1} \{\eta(Z)S(Y,U) - (n-1)g(Y,U)\eta(Z)\}.$$

Putting like terms together yields

$$-g(Y,Z)\eta(U) = \frac{1}{n-1}\eta(Z)S(Y,U).$$

Setting $Z = \xi$ and using (4.1) and (4.7), we have

$$-\eta(Y)\eta(U) = \frac{1}{n-1}S(Y,U).$$

On rearranging, we get

$$S(Y,U) = -(n-1)\eta(Y)\eta(U), \tag{4.26}$$

as desired. ∎

Corollary 4.2.9 *The W_5–flat LP-Sasakian manifold is a manifold of negative constant scalar curvature.*

Proof. Contracting (4.26) yields

$$r = -(n-1), \tag{4.27}$$

as desired. ∎

Corollary 4.2.10 *In a W_5–flat LP-Sasakian manifold the Riemann curvature tensor is given by*

$$R(X,Y,Z,U) = -g(X,Z)\eta(Y)\eta(U) + \eta(X)\eta(Z)g(Y,U). \tag{4.28}$$

Proof. Putting (4.26) into (4.25) and transvecting with U we get

$$R(X,Y,Z,U) = \frac{1}{n-1} \{g(X,Z)(n-1)\eta(Y)\eta(U) - (n-1)\eta(X)\eta(Z)g(Y,U)\}.$$

On further simplification, we get(4.28). ∎

Chapter 5

Curvature Tensors in $\eta-$Einstein Sasakian Manifolds[1]

In this chapter, we study some properties the W_3 curvature tensor along with its symmetric and skew symmetric parts in an $\eta-$Einstein Sasakian manifold. We also consider a W_3-symmetric and a W_3-flat $\eta-$Einstein Sasakian Manifold. We show that a W_3-flat $\eta-$Einstein Sasakian Manifold is is an Einstein manifold and is isometric to the unit sphere.

5.1 Introduction

Let us consider an $n-$dimensional real differentiable manifold M_n. If there exists a vector valued linear function ϕ, a 1-form η and a vector field ξ satisfying

$$\phi^2 X + X = \eta(X)\xi, \tag{5.1}$$

for any arbitrary vector field X, then M_n is called an *almost contact manifold* and the structure (ϕ, ξ, η) is known as an *almost contact structure.* From (5.1) we have,

rank $\phi = n - 1$, n is odd and $\phi\xi = 0$. Also we have,

$$\begin{aligned} \eta(\xi) &= 1, & (5.2)\\ \eta(\phi X) &= 0. & (5.3) \end{aligned}$$

[1]This chapter has been submitted to Differential geometry-Dynamical Systems

If in addition M_n admits a metric tensor g satisfying,

$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y), \tag{5.4}$$

then M_n is called an almost grayan manifold. From (5.1) and (5.4) we have

$$g(X, \xi) = \eta(X). \tag{5.5}$$

Setting $\grave{\phi}(X, Y) = g(\phi X, Y)$, we have

$$\grave{\phi}(\phi X, \phi Y) = -g(X, \phi Y) = g(\phi X, Y) = \grave{\phi}(X, Y) \tag{5.6a}$$
$$\grave{\phi}(X, Y) + \grave{\phi}(Y, X) = 0. \tag{5.6b}$$

If in an almost Grayan manifold

$$\grave{\phi}(X, Y) = (D_X\eta)(Y) - (D_Y\eta)(X) = (d\eta)(X, Y), \tag{5.7}$$

where D is a Riemannian conection, then M_n is called an almost Sasakian manifold. In a Sasakian manifold ϕ is closed. An almost Sasakian manifold is said to be Sasakian if ξ is a Killing vector. That is,

$$(D_X\eta)(Y) + (D_Y\eta)(X) = 0. \tag{5.8}$$

Thus, in a Sasakian manifold

$$\grave{\phi}(X, Y) = (D_X\eta)(Y) \tag{5.9}$$

and

$$(D_X\grave{\phi})(Y, Z) = R(X, Y, Z, \xi), \tag{5.10}$$

where R is the curvature tensor of type (0,4) on M_n.In a Sasakian manifold the following properties hold[39] ,[52]

$$R(\xi, X, Y, \xi) = g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{5.11}$$
$$R(X, Y, Z, \xi) = \eta(R(X, Y, Z)) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z) \tag{5.12}$$
$$R(\xi, X, Y, Z) = \eta(Z)g(X, Y) - \eta(Y)g(X, Z) \tag{5.13}$$
$$R(\xi, Y, Z) = g(Y, Z)\xi - \eta(Z)Y \tag{5.14}$$
$$R(X, Y, \xi) = \eta(Y)X - \eta(X)Y, \tag{5.15}$$

where $R(X,Y,Z,U) = g(R(X,Y,Z),U)$. In a Sasakian manifold ,we also have

$$S(X,\xi) = g(r(X),\xi) = \eta(r(X)) = (n-1)\eta(X) \tag{5.16}$$

$$S(\phi X,Y) + S(X,\phi Y) = 0, \tag{5.17}$$

where S is the Ricci tensor.

Definition 5.1.1 *A sasakian manifold M_n is called an $\eta-$Einstein Sasakian Manifold if the Ricci tensor satisfies*

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y). \tag{5.18}$$

Mishra and Pokhariyal [23] defined the following tensor

$$W_3(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}\{g(Y,Z)S(X,U) - g(Y,U)S(X,Z)\}, \tag{5.19}$$

to study its relativistic and geometric properties.

5.2 Main Results

We now break W_3 into its symetric and skew symmetric parts in X and Y. We start with the symetric part G.

$$\begin{aligned}
G(X,Y,Z,U) &= \frac{1}{2}[W_3(X,Y,Z,U) + W_3(Y,X,Z,U)] \\
&= \frac{1}{2}\{R(X,Y,Z,U) + \frac{1}{n-1}\{g(Y,Z)S(X,U) - g(Y,U)S(X,Z)\} \\
&+ R(Y,X,Z,U) + \frac{1}{n-1}\{g(X,Z)S(Y,U) - g(X,U)S(Y,Z)\} \\
&= \frac{1}{2(n-1)}\{g(Y,Z)S(X,U) - g(Y,U)S(X,Z) \\
&\qquad + g(X,Z)S(Y,U) - g(X,U)S(Y,Z)\}.
\end{aligned} \tag{5.20}$$

Let H be the skew symetric part of W_3, then we have

$$H(X,Y,Z,U) = \frac{1}{2}[W_3(X,Y,Z,U) - W_3(Y,X,Z,U)]$$
$$= \frac{1}{2}\{R(X,Y,Z,U) + \frac{1}{n-1}\{g(Y,Z)S(X,U) - g(Y,U)S(X,Z) - R(Y,X,Z,U)$$
$$- \frac{1}{n-1}\{g(X,Z)S(Y,U) + g(X,U)S(Y,Z)$$
$$= R(X,Y,Z,U) + \frac{1}{2(n-1)}\{g(Y,Z)S(X,U) - g(Y,U)S(X,Z)$$
$$- g(X,Z)S(Y,U) + g(X,U)S(Y,Z)\} \tag{5.21}$$

Theorem 5.2.1 *In an $\eta-$Einstein Sasakian manifold, we have*

$$W_3(X,Y,Z,\xi) = 2\eta(X)g(Y,Z) - \eta(y)g(X,Z)$$
$$-\frac{1}{n-1}\{\alpha\eta(Y)g(X,Z) + \beta\eta(X)\eta(Y)\eta(Z)\} \tag{5.22}$$
$$W_3(\xi,Y,Z,U) = 2R(\xi,Y,Z,U) \tag{5.23}$$
$$W_3(\xi,Y,Z,\xi) = 2R(\xi,Y,Z,\xi) = 2\{g(Y,Z) - \eta(Y)\eta(Z)\}. \tag{5.24}$$

Proof. Using (5.19) and putting $U = \xi$, we have

$$W_3(X,Y,Z,\xi) = R(X,Y,Z,\xi) + \frac{1}{n-1}\{g(Y,Z)S(X,\xi) - g(Y,\xi)S(X,Z)\}.$$

Using (5.12), (5.16) ,(5.5) and (5.18) and putting the like terms together yields (5.22). Using (5.19), we have

$$W_3(\xi,Y,Z,U) = R(\xi,Y,Z,U) + \frac{1}{n-1}\{g(Y,Z)S(\xi,U) - g(Y,U)S(\xi,Z)\}.$$

Using $X = \xi$ in (5.13), (5.16) and putting the like terms together yields (5.23). Setting $U = \xi$ in (5.23) and using (5.11), we get (5.24). ■

Theorem 5.2.2 *In an $\eta-$Einstein Sasakian manifold, we have*

$$G(X,Y,Z,\xi) = \frac{1}{2}\left(1 - \frac{\alpha}{n-1}\right)\{\eta(X)g(Y,Z) + \eta(Y)g(X,Z)\} - \frac{\beta}{(n-1)}\eta(X)\eta(Y)\eta(Z). \tag{5.25}$$

$$G(\xi,Y,Z,U) = \frac{1}{2}\left(1 - \frac{\alpha}{(n-1)}\right)\{g(Y,Z)\eta(U) - \eta(Z)g(Y,U)\}. \tag{5.26}$$

$$G(\xi, Y, Z, \xi) = \frac{1}{2}\left(1 - \frac{\alpha}{n-1}\right)\{g(Y,Z) - \eta(Y)\eta(Z)\}. \tag{5.27}$$

Proof. Putting $U = \xi$ in (5.20) ,we have

$$G(X,Y,Z,\xi) = \frac{1}{2(n-1)}\{g(Y,Z)S(X,\xi) - g(Y,\xi)S(X,Z) + g(X,Z)S(Y,\xi) - g(X,\xi)S(Y,Z)\}.$$

Using (5.5), (5.12), (5.18) and simplifying, we get (5.25).
Putting $X = \xi$ in (5.20) and using (5.5) , (5.16) and (5.18) we get

$$\begin{aligned} G(\xi, Y, Z, U) &= \frac{1}{2}\{g(Y,Z)\eta(U) - g(Y,U)\eta(Z)\} + \frac{1}{2(n-1)}\{\eta(Z)[\alpha g(Y,U) + \beta\eta(Y)\eta(U)] \\ &\quad -\eta(U)[\alpha g(Y,Z) + \beta\eta(Y)\eta(Z)]\}. \end{aligned}$$

Putting the like terms together yields (5.26) .Setting $U = \xi$ in (5.26) and using (5.2) and (5.5) we get (5.27) ■

Theorem 5.2.3 *For the $\eta-$Einstein Sasakian manifold, we have*

$$H(X,Y,Z,\xi) = \frac{1}{2}\left(3 + \frac{\alpha}{2(n-1)}\right)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} \tag{5.28}$$

$$H(\xi,Y,Z,U) = \frac{1}{2}\left(3 + \frac{\alpha}{2(n-1)}\right)\{\eta(U)g(Y,Z) - \eta(Z)g(Y,U)\} \tag{5.29}$$

$$H(\xi,Y,Z,\xi) = \frac{1}{2}\left(3 + \frac{\alpha}{2(n-1)}\right)\{g(Y,Z) - \eta(Y)\eta(Z)\}. \tag{5.30}$$

Proof. Using (5.21) and setting $U = \xi$, we have

$$\begin{aligned} H(X,Y,Z,\xi) &= R(X,Y,Z,\xi) \\ &\quad + \frac{1}{2(n-1)}\{g(Y,Z)S(X,\xi) - g(Y,\xi)S(X,Z) \\ &\quad -g(X,Z)S(Y,\xi) + g(X,\xi)S(Y,Z)\}. \end{aligned}$$

Using (5.5) , (5.12) and (5.16), we get

$$\begin{aligned} H(X,Y,Z,\xi) &= \eta(X)g(Y,Z) - \eta(Y)g(X,Z) \\ &\quad + \frac{1}{2(n-1)}\{(n-1)g(Y,Z)\eta(X) - \eta(Y)S(X,Z) \\ &\quad -g(X,Z)(n-1)\eta(Y) + \eta(X)S(Y,Z)\}. \end{aligned}$$

Using (5.18) , we get

$$
\begin{aligned}
H(X,Y,Z,\xi) &= \eta(X)g(Y,Z)-\eta(Y)g(X,Z)+\frac{1}{2(n-1)}\{(n-1)g(Y,Z)\eta(X) \\
&\quad -\eta(Y)\left(\alpha g(X,Z)+\beta\eta(X)\eta(Z)\right)-g(X,Z)(n-1)\eta(Y) \\
&\quad +\eta(X)\left(\alpha g(Y,Z)+\beta\eta(Y)\eta(Z)\right)\}.
\end{aligned}
$$

Putting like terms together and simplifying, we get (5.28). Similarly using (5.21) and setting $X=\xi$, we have

$$
\begin{aligned}
H(\xi,Y,Z,U) &= R(\xi,Y,Z,U) \\
&\quad +\frac{1}{2(n-1)}\{g(Y,Z)S(\xi,U)-g(Y,U)S(\xi,Z)+g(\xi,Z)S(Y,U)-g(\xi,U)S(Y,Z)\}.
\end{aligned}
$$

Using (5.5), (5.13), (5.16) , (5.18) and simplifying yields (5.29). Setting $U=\xi$ in (5.29) and using (5.1) and (5.5) yields (5.30). ■

Theorem 5.2.4 *For the $\eta-$Einstein Sasakian manifold, we have*

$$
\begin{aligned}
W_3(\phi X,\phi Y,Z,\xi) &= 0 && (5.31)\\
W_3(X,\phi Y,\phi Z,\xi) &= 2\eta(X)g(Y,Z) && (5.32)\\
W_3(\xi,Y,\phi Z,\phi U) &= 0 && (5.33)\\
W_3(\xi,\phi Y,\phi Z,U) &= 2\{\eta(U)g(Y,Z)-\eta(U)\eta(Y)\eta(Z)\}. && (5.34)
\end{aligned}
$$

Proof. Aplying ϕ to X and Y in (5.22) and using (5.3), we get (5.31). Applying ϕ on Y and Z in (5.22) and using (5.3) yields (5.32). Applying ϕ on Z and U in (5.23) and using (5.3) yields (5.33). Applying ϕ to Y and Z in (5.23) and using (5.3) and (5.4), we get (5.34). ■

Theorem 5.2.5 *For the $\eta-$Einstein Sasakian manifold, we have*

$$
\begin{aligned}
&H(\phi X,\phi Y,Z,\xi)=0 && (5.35)\\
&H(\xi,Y,\phi Z,\phi U)=0 && (5.36)\\
&H(\xi,\phi Y,\phi Z,\xi)=\left(\frac{3}{2}-\frac{\alpha}{2(n-1)}\right)\{g(X,Y)-\eta(X)\eta(Y)\}. && (5.37)
\end{aligned}
$$

Proof. Aplying ϕ to X and Y in (5.28) and using (5.3), we get (5.35). Applying ϕ to U and Z in (5.29) and using (5.3), we get (5.36). Lastly, applying ϕ to Y and Z in

(5.30) and using (5.3) and (5.4), we get (5.37). ■

Theorem 5.2.6 *In an $\eta-$Einstein Sasakian manifold, we have*

$$G(\phi X, \phi Y, Z, \xi) = 0 \tag{5.38}$$

$$G(X, \phi Y, \phi Z, \xi) = \eta(X) g(Y, \phi Z) \tag{5.39}$$

$$G(\xi, \phi Y, \phi Z, \xi) = \frac{1}{2}\left\{1 - \frac{\alpha}{n-1}\right\}\{g(Y, Z) - \eta(Y)\eta(Z)\}. \tag{5.40}$$

Proof. *Aplying ϕ to X and Y in* (5.25) *and using* (5.3), *we get* (5.38). *Applying ϕ on Y and Z in* (5.25) *and using* (5.3) *yields* (5.39). *Aplying ϕ to Y and Z in* (5.27) *and using* (5.3) *and* (5.4), *we get* (5.40). ■

5.3 W_3 Symmetric η – Einstein Sasakian Manifold

An $\eta-$Einstein Sasakian manifold is said to be W_3 symmetric if

$$D_Y W_3(Z, U, V) = 0. \tag{5.41}$$

This implies that

$$R(X, Y, W_3(Z, U, V)) - W_3(R(X, Y, Z), U, V) - W_3(Z, R(X, Y, U), V) - W_3(Z, U, R(X, Y, V)) = 0. \tag{5.42}$$

Setting $X = \xi$ yields

$$\begin{aligned} &R(\xi, Y, W_3(Z, U, V), \xi) - W_3(R(\xi, Y, Z), U, V, \xi) \\ &-W_3(Z, R(\xi, Y, U), V, \xi) - W_3(Z, U, R(\xi, Y, V), \xi) = 0. \end{aligned} \tag{5.43}$$

Expanding each term individually yields

$$\begin{aligned} R(\xi, Y, W_3(Z, U, V), \xi) &= W_3(Z, U, V, Y) - \eta(Y)\eta(W_3(Z, U, V)) \\ &= W_3(Z, U, V, Y) - 2\eta(Y)\eta(Z)g(U, V) \\ &\quad + \left(1 + \frac{\alpha}{n-1}\right)\eta(U)\eta(Y)g(Z, V) + \frac{\beta}{n-1}\eta(Y)n(U)\eta(Z)\eta(V). \end{aligned}$$

Using (5.11), (5.13), (5.14) and (5.22)and simplifying we get the following equations

$$
\begin{aligned}
W_3(R(\xi,Y,Z),U,V,\xi) &= 2\eta(R(\xi,Y,Z))g(U,V)-\eta(U)g(R(\xi,Y,Z),V) \\
&\quad -\frac{1}{n-1}\{\alpha\eta(U)g(R(\xi,Y,Z),V)-\beta\eta(R(\xi,Y,Z))\eta(U)\eta(V)\} \\
&= 2g(Y,Z)g(U,V)-2\eta(Y)\eta(Z)g(U,V) \\
&\quad -\left(1+\frac{\alpha}{n-1}\right)\{\eta(U)\eta(V)g(Y,Z)-\eta(U)\eta(Z)g(Y,V)\} \\
&\quad -\frac{\beta}{n-1}\{g(Y,Z)\eta(Y)\eta(V)-\eta(Y)\eta(Z)\eta(U)\eta(V)\}.
\end{aligned}
$$

$$
\begin{aligned}
W_3(Z,R(\xi,Y,U),V,\xi) &= 2\eta(Z)g(R(\xi,Y,U),V)-\eta(R(\xi,Y,U))g(Z,V) \\
&\quad -\frac{1}{n-1}\{\alpha\eta(R(\xi,Y,U))g(Z,V)+\beta\eta(Z)\eta(R(\xi,Y,U))\eta(V)\} \\
&= 2\eta(Z)\eta(V)g(Y,U)-2\eta(U)\eta(Z)g(Y,V) \\
&\quad -\left(1+\frac{\alpha}{n-1}\right)\{g(Y,U)g(Z,V)-\eta(Y)\eta(U)g(Z,V)\} \\
&\quad -\frac{\beta}{n-1}\eta(Z)\eta(V)g(Y,U)+\frac{\beta}{n-1}\eta(Y)\eta(U)\eta(z)\eta(V).
\end{aligned}
$$

$$
\begin{aligned}
W_3(Z,U,R(\xi,Y,V),\xi) &= 2\eta(Z)g(U,R(\xi,Y,V))-\eta(U)g(R(\xi,Y,V),Z) \\
&\quad -\frac{1}{n-1}\{\alpha\eta(U)g(R(\xi,Y,V),Z)+\beta\eta(R(\xi,Y,V))\eta(U)\eta(Z)\} \\
&= 2\eta(Z)\eta(U)g(Y,V)-2\eta(Z)\eta(V)g(Y,U) \\
&\quad -\left(1+\frac{\alpha}{n-1}\right)\{\eta(U)\eta(Z)g(Y,V)-\eta(U)\eta(V)g(Y,Z)\} \\
&\quad -\frac{\beta}{n-1}g(Y,V)\eta(U)\eta(Z)+\frac{\beta}{n-1}\eta(Y)\eta(U)\eta(Z)\eta(V).
\end{aligned}
$$

Substituting these back into the equation (5.43) and putting like terms together and rearranging yields

$$\begin{aligned} W_3(Z,U,V,Y) &= 2g(Y,Z)g(U,V) - \left(1+\frac{\alpha}{n-1}\right) g(Y,U)g(Z,V) \\ &+\frac{\beta}{n-1}\{2\eta(U)\eta(Y)\eta(Z)\eta(V) - \eta(U)\eta(V)g(Y,Z) \\ &-\eta(U)\eta(Z)g(Y,V) - \eta(Z)\eta(V)g(Y,U)\}. \end{aligned}$$

Thus, we have proved the following result.

Theorem 5.3.1 *In a W_3-symmetric $\eta-$Einstein Sasakian manifold, the W_3 curvature tensor is given by*

$$\begin{aligned} W_3(Z,U,V,Y) &= 2g(Y,Z)g(U,V) - \left(1+\frac{\alpha}{n-1}\right) g(Y,U)g(Z,V) \\ &+\frac{\beta}{n-1}\{2\eta(U)\eta(Y)\eta(Z)\eta(V) - \eta(U)\eta(V)g(Y,Z) \\ &-\eta(U)\eta(Z)g(Y,V) - \eta(Z)\eta(V)g(Y,U)\}. \end{aligned}$$

Corollary 5.3.2 *In a G symmetric $\eta-$Einstein Sasakian Manifold G is given by*

$$\begin{aligned} G(X,Y,Z,U) &= \frac{1}{2}\{\left(1-\frac{\alpha}{n-1}\right) \{g(X,U)g(Y,Z) - g(Y,U)g(X,Z)\} \\ &+\frac{\beta}{n-1}4n(X)\eta(Y)\eta(Z)\eta(U) - 2\eta(Y)\eta(Z)g(X,U) \\ &-2\eta(X)\eta(Y)g(Z,U) - 2\eta(X)\eta(Z)g(Y,U)\}. \end{aligned} \tag{5.44}$$

Proof.

$$G(X,Y,Z,U) = \frac{1}{2}\{W_3(X,Y,Z,U) + W_3(Y,X,Z,U)\}$$

Using linearity of covariant differentiation and Theorem (5.3.2), we get (5.44) as desired. ∎

Corollary 5.3.3 *In a H symmetric $\eta-$Einstein Sasakian Manifold H is given by*

$$H(X,Y,Z,U) = \frac{1}{2}\left(3+\frac{\alpha}{n-1}\right)\{g(X,U)g(Y,Z) + g(Y,U)g(X,Z)\}. \tag{5.45}$$

Proof.

$$H(X,Y,Z,U) = \frac{1}{2}\{W_3(X,Y,Z,U) - W_3(Y,X,Z,U)\}$$

Using linearity of covariant differentiation and Theorem (5.3.1) and after some simplification we get (5.45) ■

5.4 W_3 Flat $\eta-$Einstein Sasakian manifolds

Definition 5.4.1 *A manifold is said to be W_3-flat if $W_3(X,Y,Z,U) = 0$ satisfied on the manifold.*

Theorem 5.4.2 *The W_3-flat $\eta-$Einstein Sasakian Manifold is an Einstien manifold.*

Proof. We have

$$W_3(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}\{g(Y,Z)S(X,U) - g(Y,U)S(X,Z)\}.$$

But $W_3 = 0$. Hence, we have

$$-R(X,Y,Z,U) = \frac{1}{n-1}\{g(Y,Z)S(X,U) - g(Y,U)S(X,Z)\}. \tag{5.46}$$

Using (5.18), we get

$$-R(X,Y,Z,U) = \frac{1}{n-1}\{g(Y,Z)\left[\alpha g(X,U) + \beta\eta(X)\eta(U)\right] - \eta(Y)\left[\alpha g(X,Z) - \beta\eta(X)\eta(Z)\right]\}.$$

Setting $U = \xi$ and using (5.2) and (5.5), we get

$$-\eta(X)g(Y,Z)+\eta(Y)g(X,Z) = \frac{1}{n-1}\{g(Y,Z)\left[\alpha\eta(X) + \beta\eta(X)\right] - \eta(Y)\left[\alpha g(X,Z) - \beta\eta(X)\eta(Z)\right]\}.$$

Setting $X = \xi$ yields

$$-g(Y,Z) + \eta(Y)\eta(Z) = \frac{1}{n-1}\{g(Y,Z)\left[\alpha+\beta\right] - \eta(Y)\eta(Z)\left[\alpha-\beta\right]\}. \tag{5.47}$$

Comparing coefficients of like terms, we get $\alpha + \beta = -(n-1)$, and $\alpha - \beta = -(n-1)$. Solving for α and β ,we get $\alpha = -(n-1)$ and $\beta = 0$. Substituting these into (5.18) yields

$$R(X,Y) = -(n-1)g(X,Y). \tag{5.48}$$

The theorem is thus proved. ■

Corollary 5.4.3 *A W_3-flat $\eta-$Einstein Sasakian manifold is isometric to the unit sphere.*

Proof. Substituting (5.48) into (5.46) yields

$$R(X,Y,Z,U) = \frac{1}{n-1}\left\{g(Y,Z)(n-1)g(X,U) - g(Y,U)(n-1)g(X,Z)\right\}.$$

On simplification, we get

$$R(X,Y,Z,U) = g(Y,Z)g(X,U) - g(Y,U)g(X,Z).$$

Hence, the manifold is isometric to the unit sphere. ■

Corollary 5.4.4 *A W_3-flat $\eta-$Einstein Sasakian manifold is a manifold of negative constant scalar curvature $r = -n(n-1)$*

Proof. Contracting (5.48) we have $r = -n(n-1)$. Hence, it is a manifold of negative constant scalar curvature. ■

References

1. Adati T. and Miyazawa T., On P-Sasakian manifolds admitting some parallel and recurrent tensors, tensor, N.S., 33, 287-292 (1979).

2. Adati T., and Miyazawa, T. : On P-Sasakian manifolds satisfying certain conditions, Tensor (N.S) 33, 173-178 (1979).

3. Aigwitz D, Differential geometry and Riemannian geometry, Academic press, (1965).

4. Asegawa I, Ricci parallel hypersufaces in a Sasakian space form, tensor, 52, 82-102, (1993)

5. Bucki A., Almost r-paracontact structures of P-Sasakian type, Tensor, N.S., 42, 189-197, (1988)

6. Cihan O: $\phi-$conformally flat Lorentzian Para Sasakian Manifolds, Radovi Matematicki, vol 12, 1-8 (2003).

7. Cihan O.: On A Class of Para Sasakian Manifolds, Turk J of Math, 29, 249-257, (2005).

8. Cihan O. and Mukut M. T.: On P-Sasakian manifolds satisfying certain conditions on the Concircular curvature Tensor, Turk J Math, 31, 171-179 (2007).

9. Cengizhan M. et al: On a Class of Lorentzian Para Sasakian Manifolds, Proc. of Estonian Acad. Sci. Phy. math., 55, 4, 210-219 (2006).

10. De U. C., Jun B.J. and Gazi K. A., Sasakian manifold with quasiconformal curvature tensor, Bull. Korean Math. Soc. 45 , No. 2, 313-319, (2008)

11. G.P. Pokhariyal, Study of new curvature tensor in a Sasakian manifold, N.S, 36 , 222-226 (1982)

12. Gebarowski A, On homogeneous conformally Ricci recurrent manifolds, Tensor, 50, 12-17, (1991)

13. Hu S.T. Differentiable manifolds, Holt rhinehart and Winston inc. 1969.

14. Jackubowicz A., Supplement to the classification of four dimensional Riemannian and Einstein spaces, Tensor, N.S., 27, 291-294, (1973)

15. Khan Q: On an Einstein Projective Sasakian manifold, Novi Sad J. Math., vol. 36, No 1., 97-102 (2006).

16. Ki U. and Kim H. S., Sasakian manifolds whose C-bohner curvature tensor vanishes, Tensor, N.S., 49, 32-39, (1990)

17. Kobayashi and Nomizu, Foundations of differential geometry, vol 1 and 2, John Wiley and sons inc, 1969.

18. Liviu Ornea and Misha V.:Sasakian structures on CR manifolds, Geom. Dedicata,125, 159-174 (2007).

19. Matsumoto K and Mihai I, On a certain transformation in a Lorentzian Para Sasakian manifold, Tensor, N.S., 47, 189-197, (1988)

20. Matsumoto K., Curvature preserving infinitesimal transformations of a P-Sasakian manifold, Tensor, N.S., 34, 35-38, (1980)

21. Matsumoto M amd Tano S, Kahlerian spaces with parallel or vanishing Bochner curvature tensor, Tensor, N. S. 27, 291-294, (1973)

22. Mishra R. S:On Sasakian manifolds, Indian Journal of mathematics, (1969).

23. Mishra R. S and Pokhariyal G. P.: Curvature Tensors and Their Relativistic significance, the Yokohama Mathematics, vol XVIII, 2, 106-108 (1970).

24. Mishra R. S., Almost complex and almost contact Submanifold, Tensor, N.S., 25, (1972).

25. Mishra R. S.,A differentiable manifold with F-structure of rank r, Tensor, N.S., 27, (369-378), (1973)

26. Mishra R. S., Structures of differentiable manifolds and their applications, Chandrama Prakshan, 1984.

27. Miyazawa T., On projectively recurrent spaces, Tensor, N.S., 32, 216-218, (1978)

28. Mukut M. T. and Mohit K. D: The structure of some classes of K-contact Manifolds, Proc. Indian Acad. Sci (Math Sci.) vol. 118, No. 3, 371-379 (2008).

29. Murathan C., On a class of Lorentzian para-Sasakian manifolds, Proc. Estonian Acad. Sci. Phys. Math., 2006, 55, 4, 210–219

30. Okumura M, Some remarks on a space with certain contact structure, Tohoku Math J., vol 14, No. 2, 135-145, (1962)

31. Olszak Z, Five dimensional nearly Sasakian manifolds, Tensor, N. S, 34, 273-276 (1979)

32. Olszak, Z, Nearly Sasakian manifolds, Tensor, N.S., 33, 277-286, (1979).

33. Ozgur C., $\varphi-$conformally flat Lorentzian para-Sasakian manifolds, Radovi Matematicki, Vol 12, 1-8 (2003)

34. Ozgur C. H, On a class of Para Sasakian Manifolds, Turk. Jour. Math, 29, 249-257, (2005)

35. Ozgur C. H. and Tripathi M. M., On P Sasakian Manifolds satisfying certain conditions on the concircular curvature tensors, Turk. Jour. Math, 31, 171-179 (2007)

36. Pokhariyal G. P and Mishra R. S, Curvature tensors and their relativistic significance(II): Yokohama math jour. vol XIX, No. 2, 97-103 (1971).

37. Pokhariyal G. P and Mishra R. S, Curvature tensors and their relativistic significance(II): Yokohama math jour. vol XXI, 115-119 (1973).

38. Pokhariyal G. P, Curvature tensors in a Riemannian manifold, The proceedings of the Indian academy of Sciences, vol XXIX, 1974.

39. Pokhariyal G. P : Curvature tensors on $\eta-$Einstein Sasakian manifolds, Balkan journal of Geometry and its Applications, vol.6, No.1, 45-50 (2001).

40. Pokhariyal G. P., Curvature tensors in a Lorentzian para Sasakian Manifold, Quastiones Mathematicae, 19(1-2), 129-136, (1996).

41. Pokhariyal G. P., Relativistic significance of curvature tensors, International journal of science and mathematics, vol 2, (1982).

42. Pokhariyal G.P, Study on a Sasakian manifold, Progress of mathematics, 17 1-8, (1983).

43. Pokhariyal G. P, Relativistic significance of curvature tensors, Internat. J. Math & Sci, vol5, No1, 133-139 (1982).

44. Sasaki S, Almost contact manifolds, Part 1, Lecture Notes, Tohoku University, (1965)

45. Sato I and Matsumoto K., On P-Sasakian manifolds satisfying certain conditions, Tensor, N. S., 33, 173-178(1979).

46. Shi Wei N., Curvature in 2-population ecological model, Indian jounal of mathematics, 1992.

47. Sinha B. B. and Sharma R., Infinitesimal variations of hypersufaces of a Sasakian manifold, Progress of mathematics, 13, (1979).

48. Sinha B. B. and Singh I. J. P., Some properties of the Sasakian manifold, Progress of Mathematics, vol. 9. No 2, (1975).

49. Tarafdar M. and Bhatacharya A.: On Lorenztian Para Sasakian Manifolds, Proceedings of the colloquium of differential geometry, 343-348 (25-30 July, 2000).

50. Tripathi M. M. and Dwivedi M. K., The structure of some classes of K-contact manifolds, Proc. Indian Acad. Sci. (Math. Sci.), 118 , No. 3, 371-379. (2008)

51. Venkatesh and Bagewadi, On concircular $\varphi-$recurrent $LP-$Sasakian Manifolds, Differential Geometry - Dynamical Systems, Vol.10, 312-319,(2008)

52. Yamata M: On an $\eta-$Einstein Sasakian manifold satisfying certain conditons, Tensor, N.S, 49, 305-309 (1990).

53. Yano K, Differential geometry of complex and almost complex spaces, Pergamon press, (1965).

54. Zhang X, Sasakian metrics with constant scalar curvature, Journal of mathematical Physics, (50) 103505 (2009)