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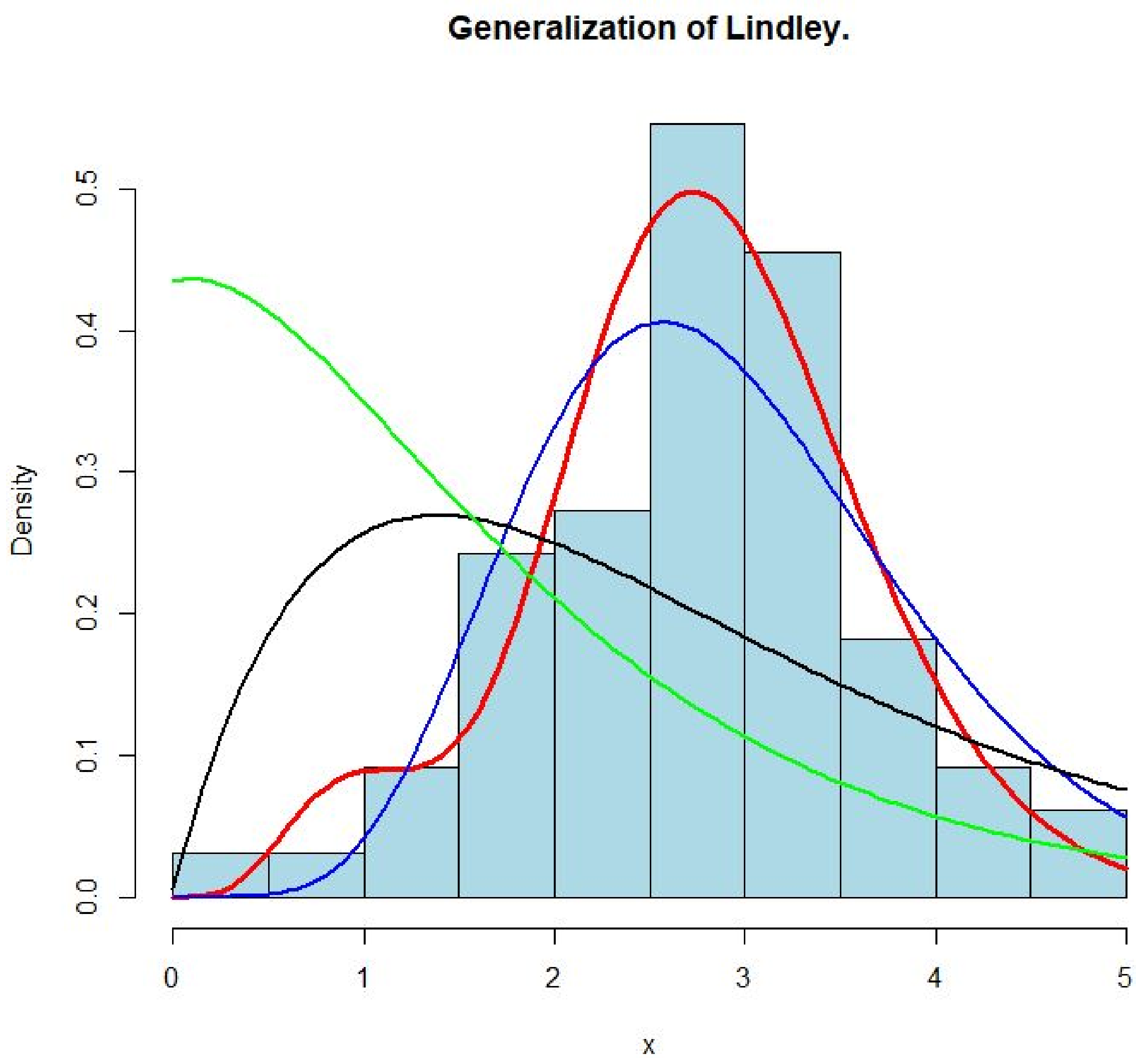
A RE-EXAMINATION OF A FIVE PARAMETER LINDLEY DISTRIBUTION.

Research Report in Mathematics,Number 04, 2020

MKAWE MUSEVENI YOHANA.

December, 2020



Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Mathematical Statistics.

Master Project in Mathematics **University of Nairobi**
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A RE-EXAMINATION OF A FIVE PARAMETER LINDLEY DISTRIBUTION.

Research Report in Mathematics,Number 04, 2020

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Master Thesis

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Mathematical Statistics.

Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

In this project,Generalized Five Parameter Lindley distribution(G5L) has been proposed as a new generalization of Lindley capable of modeling different shapes with increasing,decreasing,constant and bathtub failure rates.It has been constructed using finite mixture as the tool and Gamma as the special function.Construction of G5L,has been expressed in terms of Pdf,Cdf,Survival function,Hazard function,Reverse hazard function,Residual lifetime function,Mean residual life function,Equilibrium distribution,Survival function of equilibrium distribution and Hazard function of equilibrium distribution.G5L was applied to real data from Nichols and Padgett on breaking stress of Carbon fibers(2006) to draw shapes and test goodness of fit.Parameters were then estimated using Moment matching method(Mom) and Maximum likelihood estimation(MLE).The results obtained proved that G5L provided better fit than other generalizations of Lindley hence more flexible.

Therefore,G5L is recommended for modeling lifetime data that fails to follow other generalizations of Lindley because of its extensiveness.

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Declaration and Approval

I the undersigned declare that this project is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature   Date

MKAWE MUSEVENI YOHANA.
Reg No. I56/11412/2018

In my capacity as a supervisor of the candidate's project, I certify that this project report has my approval for submission.

Signature   Date

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Dedication

I hereby dedicate this entire project work to Almighty God for His mercy and grace He bestowed in me throughout research period.

Contents

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MKAWE MUSEVENI YOHANA.

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1 GENERAL INTRODUCTION.

1.1 Background Information.

Historical background of G5L dates back to the introduction of a distribution by use of finite mixture (Lindley,1958) as a counter example to exp (θ) which was by then widely used in Statistics.
Exponential distribution(θ) had a **limitation** of not being able to model data with varying failure rates since it has constant hazard function equivalent to its parameter value.

Ghitany et al,2008 applied Lindley distribution to lifetime data sets(100 banking customers awaiting for service provision) and concluded that Lindley is a superior model to exponential distribution.In applied sciences e.g. Finance,Medicine,Engineering e.t.c modeling of lifetime data is of great essence .Despite other forms of distributions being used to model such data,the quality of procedure involved relies on assumed probability model and Statistical methods used.Some of real lifetime data does not follow standard probability model and this remains a challenge.A one parameter Lindley and its generalizations has widely been used to model lifetime data obtained from Medicine(Shanker and Mishra,2015),Engineering(Al-Babtain,2015),Service provision rate(Ghitany et al,2008) e.t.c.

The generalization of Lindley to a five parameter by finite mixture is flexible enough to model different types of lifetime data with different forms of failure rate.It accommodates decreasing,increasing,constant and bathtub failure rates.

1.2 Definitions,Notations and Abbreviations.

Definition of Terms.

1. **Survival function**:-Is the probability of a system not failing prior to x.
   $Pr[X > x] = R(x) = 1 - Pr[X = x] = 1 - F(x)$

2. **Hazard function**:-is the failure rate of a device/system given that random variable X has an assigned pdf.
   $h(x)=\lim_{\Delta x}\frac{Pr[x<X<X+\Delta X/X>x]}{\Delta x}=\lim_{\Delta x}\frac{F[x+\Delta x]-F(x)}{\Delta x R(x)}=\frac{f(x)}{R(x)}$
   It provides information about a small interval after time $x(x+\Delta x)$.It may increase,decrease, constant or bathtub shaped depending on the values of parameters involved.

3. **Reverse Hazard Rate function**:-is the probability of one observing an outcome in the neighborhood of variable x,given that the outcome is not more than x.
   $r(x)=\lim_{\Delta x}\frac{Pr[x<X<x+\Delta x/X\leq x]}{\Delta x}=\lim_{\Delta x}\frac{F[x+\Delta x]-F(x)}{\Delta x F(x)}=\frac{f(x)}{F(x)}$

4. **Residual lifetime/Excess loss**:-is an additional measure of lifetime process given that a system has survived till a given time of x.It is obtained as a ratio of integration of a p.d.f to its Survival function.

5. **Mean Residual Lifetime/Mean Excess Loss**:-is an expectation of a random variable x representing the remaining lifetime of a process.It is a ratio of integration of expectation of a p.d.f to its Survival function. $m(x) = E)(X) = E[X - x/X > x] = \frac{\int_x^\infty R(t)dt}{R(x)}$ or $= \frac{\int_x^\infty tf(t)dt}{R(x)} - x$;
   we subtract x since we are only interested in additional lifetime above x,but the integral $[x,\infty]$ of tf(t) includes the whole expectation.

6. **Equilibrium distribution**:- is a mathematical function used to evaluate if a lifetime process preceding a certain random variable,x at a given time,t equals the mean of the distribution.It is constructed by obtaining a ratio of Survival function of a distribution to the mean of the same distribution.

Notations.

1. x for a sample random variable.

2. X for a population random variable.

3. E(x^r) for the r^{th} moment of distribution.

4. f(x) for Probability mass function(pmf).

5. f(x/ε) for a conditional property.

6. α for parameter alpha.
7. θ for parameter theta.
8. β for parameter beta.
9. n as a parameter.
10. k as another parameter.
11. γ_a(b)=$\frac{\gamma(a,b)}{\Gamma(a)} = \frac{1}{\Gamma(a)}\int_0^b x^{a-1}e^{-x}dx$ represents lower incomplete gamma function ratio.
12. Γ_a(b)=$\frac{\Gamma(a,b)}{\Gamma(a)} = \frac{1}{\Gamma(a)}\int_0^b x^{a-1}e^{-x}dx$ represents upper incomplete gamma function ratio.
13. $\psi(.)$ is a diagamma function defined as $\psi(a) = \frac{d}{da}ln\Gamma(a) = \frac{\Gamma'(a)}{\Gamma(a)}$.
14. **Generalization** is the addition of parameters to an existing model so as to allow flexibility of the model to accommodate data.

Terminologies.

1. Pdf represents Probability Density Function.
2. Pmf represents Probability Mass Function.
3. Cdf represents Cumulative Distribution Function.
4. MLE represents Maximum Likelihood Estimation.
5. MOM for Method of Moments.
6. MRL represents Mean Residual Lifetime.
7. G5L ; for Generalized Five Parameter Lindley Distribution.
8. G4L(1) ; Generalized Four Parameter Lindley Distribution, version 1.
9. G4L(2) ; Generalized Four Parameter Lindley Distribution, version 2.
10. G3L(1) ; Generalized Three Parameter Lindley Distribution, version 1.
11. G3L(2) ; Generalized Three Parameter Lindley Distribution,version 2.
12. G2L(1) ; Generalized Two Parameter Lindley Distribution,version 1.
13. G2L(2) ; Generalized Two Parameter Lindley Distribution,version 2.
14. LD ; Lindley Distribution.
15. QLD; Quasi Lindley Distribution.

1.3 Research Problems.

1. Studies have not shown clearly whether adding more parameters implies better modeling.Therefore,there is a motivation to draw pdf shapes to illustrate and compare the generalized Lindely distribution models.

2. Despite the fact that Five parameter Lindley has been constructed, its equilibrium distributions have not been obtained hence we need to construct its equilibrium distribution.

3. There is a drive to investigate the relationship between Mean Residual Lifetime(MRL) and Hazard function of equilibrium distribution of Five parameter Lindley distribution.

1.4 Objectives.

1.4.1 General Objective.

The main objective is to Re-Examine Generalized Five Parameter Lindley distribution.

1.4.2 Specific Objectives.

1. To explore use of Gamma distribution.

2. To investigate properties of G5L distribution.

3. To construct MRL of G5L.

4. To construct Hazard function of equilibrium distribution of G5L.

5. To estimate parameters of G5L using MoM and MLE.

6. To illustrate best density shape of G5L,G2L(1),G3L(2) and LD.

1.5 Methodology.

Mathematical tools we have used include;

i. Elementary Calculus.

   - Differentiation.
   - Integration.

ii. Special functions.

- Gamma distribution. From gamma function;$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt \Rightarrow 1 = \frac{\int_0^\infty t^{\alpha-1}e^{-t}dt}{\Gamma(\alpha)}$
  $f(x;\alpha) = \begin{cases} \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}, x > 0, \alpha > 0 \\ 0, otherwise \end{cases}$;where f($\lambda;\alpha$) is a pdf of One parameter Gamma distribution.
- Upper Incomplete gamma function.
  $IG_{upper} = \int_x^\infty t^{\alpha-1}e^{-t}dt$
- Lower Incomplete gamma function.
  $IG_{lower} = \int_0^x t^{\alpha-1}e^{-t}dt$
- Upper Incomplete Gamma Ratio function.
  $IGR_{upper} = \frac{\int_x^\infty t^{\alpha-1}e^{-t}dt}{\Gamma(\alpha)}$
- Lower Incomplete Gamma Ratio function.
  $IGR_{lower} = \frac{\int_0^x t^{\alpha-1}e^{-t}dt}{\Gamma(\alpha)}$
- $IG_{low} + IG_{upper} = \Gamma(\alpha)$
- $IGR_{low} + IGR_{upper} = 1$
- $IGR[0;1]$

iii. Moment Generating Function.

1.6 Significance of the Study

a.) Insurance industry to study **frequency of claims** within specified period of time.

b.) In the field of medicine to study increase of microorganisms within a cell size.

c.) Evaluation of **material reliability** (Engineering):to measure system performance.We can obtain the probability of a system surviving;System fails whenever an applied stress is greater than its strength.

2 LITERATURE REVIEW.

2.1 Introduction.

This chapter provides related literature useful in undertaking our research.Literature consists of researcher's critique and comparisons from other studies related to our area of interest.They include methodologies used,theoretical or conceptual framework,relationships and differences between studies.Critical aspects of reviewing the literature in clear and systematic way from existing studies has been done and missing gaps identified.

2.2 One Parameter Lindley Distribution.

Lindley(1958) originally proposed a One -Parameter distribution called Lindley distribution.This distribution is a finite mixture of Exponential(θ) and Gamma(2,θ) distributions where θ represents scale parameter.Ghitany et al(2008) investigated Lindley further by explaining its various properties.Some of the properties discussed include;pdf,cdf,shape,mode, moments and related measures,cumulants,failure rate and mean residual life,mean deviations,measure of inequality and uncertainty(Lorenz curve and Entropies),stochastic ordering,extreme order statistics,sums,products and ratios.One parameter was estimated using method of moments and maximum likelihood method.The model was applied on survival data.Simulations and applications on real data shows that Lindley is superior over exponential distribution.Therefore, mathematical properties for Lindley(LD) are more flexible than those of exponential distribution.
A discrete version of Lindley distribution (LD) was later suggested by Deniz and Ojeda(2011). A pmf was introduced by discretizing the continuous failure model of the Lindley distribution.The resulting model was over-dispersed and competitive with Poisson distribution in fitting automobile claims.

2.3 Two Parameter Lindley Distribution.

Shanker et al (2012),introduced one version of two parameter Lindley.This is a generalization of Lindley Distribution for Modeling Waiting and Survival Times Data,G2L(1).Its pdf is obtained by finite mixture of exponential (θ) and gamma($2,\theta$) with additional parameter (α) originating from weight,$\left(w_1 = \frac{\theta}{\theta+\alpha}\right)$.Statistical properties such as Moments,Failure rate function,Mean Residual lifetime function and Stochastic ordering have been discussed.The Failure rate function,Mean Residual life function and Stochastic ordering shows flexibility over Lindley and Exponential distribution.The new distribution was then fitted to waiting times data set and goodness of fit test carried out.This distribution

provided closer fit compared to Lindley.
Another version of two parameter Lindley distribution (Shanker and Mishra,2013) of which Lindley distribution is its particular case was introduced,G2L(1).It is constructed by finite mixture of exponential (θ) and gamma$(2,\theta)$.The second parameter (α) is obtained from weight,$\left(w_1 = \frac{\alpha\theta}{\alpha\theta+1}\right)$.Its properties that were discussed included;mode,Moments and related measures,Failure Rate function,Mean Residual lifetime function and Stochastic ordering.The Method of moments and maximum likelihood method as technique for estimating its parameters α and θ were discussed.This distribution was fitted on a data set with an aim of testing its goodness of fit.Two parameter Lindley distribution model proved to be a better fit than Lindley.
Two parameter Quasi Lindley distribution(QLD) which is a generalization of Lindley distribution was proposed by,Shanker and Mishra,2013.Its pdf was constructed by finite mixture of gamma$(2,\theta)$and exponential(θ) with a weight of $\left(w_1 = \frac{\alpha}{\alpha+1}\right)$.Statistical properties such as Failure rate,Moments,Mean Residual and Stochastic ordering were discussed.It was discovered that the Failure Rate Function,Mean Residual lifetime function and Stochastic ordering for this distribution is more flexible than Lindley and Exponential.MLE method was used to estimate its parameters and the goodness of fit provided closer values than those from Lindley.There exists another type of two parameter Lindley distribution,G2L(2) which is a generalization of Lindley distribution introduced by Al-Babtain(2015) as a submodel of Five Parameter Lindley distribution.It can also be constructed by finite mixture of gamma$(1,\theta)$ and gamma$(2,\theta)$ with a weight of $\left(w_1 = \frac{\theta k}{\theta k+1}\right)$ .Its statistical properties and estimation of its parameters by maximum Likelihood method can be obtained.

2.4 Three Parameter Lindley Distribution.

Zakerzadeh and Dolati (2009),introduced three parameter generalization of Lindley distribution,G3L(1).Its pdf was constructed by finite mixture of gamma $(,1,\theta)$ and gamma$(2,\theta)$ with additional parameters α and β originating from the weights.Further,the pdf of this distribution can be expressed as a two component mixture of gamma(α,θ) and gamma$(\alpha+1,\theta)$ with a weight of $w_1 = \frac{\theta}{\beta+\theta}$ where β is our third parameter.It accommodates Exponential,Gamma and Lindley distribution as its special cases.Various statistical properties discussed include;behaviour of the density and hazard rate function,distribution of sums,Stochastic ordering,Random variate generation.MLE method was used to estimate its three parameters.The distribution exhibited decreasing,increasing and bathtub hazard rate depending on its parameters.Numerical examples to show flexibility of the model was explored and derivation of bi-variate version of the proposed distribution studied.It was noted that it is a better model compared to Exponential and Gamma distribution on real time data.Another version of three paremeter Lindley distribution,G3L(2) was proposed by Elbatal et al,2013.This distribution generalizes Lindley distributions including;Power Lindley,Sushila,Lindley Pareto,Lindley half Logistic and Classical Lindley distribution.The distribution was applied to two real lifetime data set and proved its superiority over Exponential Power Lindley,Power Lindley,Exponential Lindley Geometric,Classical one parameter Lindley and Lindley exponential distribution in modeling of lifetime data set.

2.5 Four Parameter Lindley Distribution.

A four parameter generalized Lindley distribution was introduced by AL-Babtain et al(2015) as a special case of Five parameter Lindley distribution.This distribution(G4L) can be constructed and two versions obtained by finite mixture of gamma(α, θ) and gamma(β, θ).The first version G4L(1) is obtained when the other parameter k is from the weight$\left(w_1 = \frac{\theta k}{\theta k+1}\right)$. while parameters($\alpha, \theta, \beta$) from the two gamma distributions.The second version G4L(2) is obtained by finite mixture of the two gamma distributions mentioned above contributing parameter α, θ and β and the fourth parameter n from weight $\left(w_1 = \frac{\theta}{\theta+n}\right)$.Its statistical properties such as moment generating function and related measures were explained.Method of moments and Maximum Likelihood Estimation method were used to estimate its parameters.

2.6 Five Parameter Lindley Distribution.

Five Parameter Lindley Distribution(G5L) is a proposed generalization of Lindley distribution (AL-Babtain A et al,2015).This distribution was obtained from a concept of finite mixture.Gamma(α, θ) and Gamma(β, θ) provided three parameters (α, θ, β) and addition of two other parameters namely k and n from the weights of finite mixture,$\left(w_1 = \frac{\theta k}{\theta k+n}\right)$. Structural properties investigated include;Statistical measures and Reliability analysis.In addition, Lorenz, Bonferroni curves and Renyi entropy as measure for uncertainity are derived.Maximum Likelihood Estimation method was used to estimate its five parameters.This distribution was fit on real data collected by Nichols and Padgett of stress on carbon fiber(Gba).It proved to be a better model compared to LD,G2L,G3L and G4L.

2.7 Other Generalizations of Lindley Distribution.

Nadarajah et al(2011),proposed two parameter distribution as an alternative to Gamma,Lognormal,Weibull and exponentiated Exponential distribution for modeling lifetime data.Its cdf was constructed by considering the cdf of One parameter Lindley distribution where each of the identical random variables of Lindley are independently distributed till size α.It was proven to have a better Hazard rate properties than Wei-bull,Gamma and Log-Normal distribution since it allows monotonically decreasing,increasing and bathtub shaped hazard rate functions but not for constant hazard rate functions.Presentation was also done on its mathematical properties such as Quantile function,Shapes,Statistical orders,Moments,Conditional moments,L moments,Mean deviation,Bonferroni,Lorenz curves,Entropies,Order statistics, Extreme values and Reliability.Method of Moments and Maximum Likelihood Estimation approach was used to estimate the parameters.A simulation concept was also explained and application done on real data.The new model provides more accurate estimates as well as better fits when compared to Gamma,Log normal and Wei-bull distribution.

Bakouch et al(2012) extended Lindley distribution by exponentiating it so as offered more flexible model for lifetime data,namely reliability in terms of its failure rate shapes.This distribution accommodates both decreasing and increasing failure rates as well as bathtub and unimodal shaped failure rates.Statistical properties discussed include;(Reversed)Failure rate, (Reversed)Mean Residual lifetime,Moments,Order statistics,Bonferroni and Lorenz curves.They also derived observed information matrix.The estimation of parameters was done using Maximum Likelihood Estimation method.A real data was used in modeling and it provided better fit than Lindley distribution.

A two parameter Power Lindley and Associated Inference(M.E Ghitany et al,2012) was introduced.It is a two component mixture of Weibull distribution(with shape α and scale θ) and Generalized Gamma distribution(with shape parameter $2, \alpha$ and scale θ) with mixing proportion $\left(w_1 = \frac{\theta}{\theta+1}\right)$.The statistical properties discussed include;Shape of the density and Hazard rate function,Moments,Quantile function,Skewness and Kurtosis and limiting distributions of Order Statistics.Ghitany et al proposed three Algorithms for generation of random data of the said distribution.Maximum Likelihood method of estimating parameters was derived and asymptotic Standard errors obtained.An application of the model using real data set proved that Power Lindley distribution provided better fit than Lindley and other two well known parameter distributions such as Gompertz,Weibull and Gamma.

A three parameter distribution which generalizes Lindley,Gamma and Exponential was introduced by Elbatal et al(2013).This model is obtained from a mixture of gamma($\alpha.\theta$) and gamma(β, θ) with a weight,$\left(w_1 = \frac{\theta}{\theta+1}\right)$. Statistical properties examined include; moment,generating function and inequality measures.Distribution of order statistics,regression based method of least squares and weighted least squares estimates were also discussed.The estimation of parameters by MLE was done and Likelihood Ratio Statistics was used to compare the model with its baseline.Application of the distribution to real data shows that it is effective in providing better fit than Lindley since it provides larger flexibility in modeling real data.

Exponentiated Power-Lindley distribution with its applications was introduced by Gayan Warahena-Liyanage and Mavis Pararani(2014).This is a generalization of Power Lindely distribution initially introduced by Ghitany et al(2013).The mathematical properties studied include;Hazard and Reverse hazard functions,Moments,Conditional moments,Entropies,Mean deviations,Lorenz and Bonferroni curves.Fisher information and MLE of parameters were also explained.The distribution was finally fitted on real data and proved to be better model compared to Power Lindley.

Hassan,2014 proposed a formula from sums of independent and identical Lindley with different parameters which are based on elementary compaction previously applied by Akkouch(2005,2008).Another formula presented by Zakerzedah and Dolati(2009) for instance a case of π being equal to a number;$p > 0$ by use of Moment generating function.Section 2 of Marwa's research paper explained Convolution of Lindley distribution with different parameters.Section 3;Convolution of Lindley distribution with same parameters.Section 4;Convolution of order statistics having Lindley distribution.

M.R Irshad and D.S Shibu,2016 proposed a new version of three parameter distribution which is a generalization of Lindley(1958),Three parameter Lindley distribution(Zakerzadadeh and Dolati,2009) and Generalized Lindley distribution by Elbatal et al(2013).Its Mathematical and Statistical techniques have been discussed.A real life data was used to apply the this distribution so as to compare it with other distributions such as; using MOM and MLE.This distribution provided a best fit compared to Lindley,Three parameter Lindley distribution and Generalized Lindley distribution by Elbatal et al(2013).

2.8 Use of Lindley Distribution.

2.8.1 Lindley Distribution as Mixture distribution.

Bhati et al,2014 proposed Lindley Exponential(L-E) distribution generated by Lindley distribution.Important properties such as moments,asymptotic distribution of sample,maximum, minimum and entropy were obtained.MLE method was used to estimate parameters and also give asymptotic confidence interval.Application of the Lindley exponential distribution was also shown using two real data sets and by comparing Lindley-exponential with other models such as three parameter distribution(Elbatal et al 2013),Power Lindley(Ghitany et al 2013),Lindley and Exponential distribution.Lindley Exponential distribution proved to be a better model fit.

Lindley Pareto distribution was introduced by Lazri et al(2016) for modeling lifetime data.Statistical properties discussed include;Method of moments,Quantile function,MLE, Entropy and also limiting distribution of Extreme order statistics.Pareto distribution (V.Pareto,1896) on the other hand was introduced to explore unequal wealth distribution.It is mainly used in actuarial science(Re-Insurance)due to its heavy tail properties.A simulation study on Lindley-Pareto was done to examine the Quantiles,Mean,Median and Mode of the distribution.

2.8.2 Lindley Distribution as a conditional distribution.

Poisson-Lindley distribution which is a discrete random variable was introduced by M.Sankaran(1970) to model count data closely related to the Lindley distribution.This is a mixture distribution obtained from Poisson distribution assuming that the parameter of Poisson by itself is a random variable with Lindley distribution.To explore this Ghitany and Al-Mutairi(2009),Borah and Begum(2002) and Bora and Deka Nath(2001) investigated some statistical properties of this distribution including other related probability distributions.To be precise,Ghitany and Al-Mutairi developed Algorithm for generating random data from Poisson-Lindley distribution basing on the fact that the random variable considered is a mixed distribution.Ghitany et al(2008) were instrumental in molecular biology by considering zero-truncated Poisson-Lindley distribution and analogous Algorithm in order to generate samples which are random from the mentioned truncated distribution.

Jodra(2010) later discussed procedures to generate random variables with Lindley,Poisson-Lindley or zero truncated Poisson-Lindley distribution as simple alternatives to existing Algorithms.The basis of his argument was on the fact that the quantile functions of the said probability distributions can be expressed in closed form and also in terms of the Lambert W function.This made it easier for the extreme order statistics from the distribution to be made computer generated in a straight forward manner.

Negative Binomial-Lindley(NB-L) was constructed by Zamani and Ismail(2010)and applied to an insurance count claim data reported by Klugman et al(2008) having been collected by Dropkin between 1956-1958.This continuous mixture model provided a better fit compared to Poisson and Negative Binomial distribution.Basic properties such as Moments,Mean and Variance were studied.Maximum likelihood estimation method was used to approximate parameter values.The Insurance count data had large value at zero and also with thick tail.It was analyzed and the model proved to be a better fit for prediction purposes though not very close to the actual claims as was expected.

J.Sarguta(2012) constructed mixed poison distributions which are significant for modeling non-homogeneous populations.Among other mixing distributions,Lindley(1958) was considered and the mixing done explicitly to obtain its pdf.This was done with an aim of improving frequency modeling for thick and long tailed data from Insurance claims.

M.Wakoli(2015) proposed Hazard functions of exponential mixtures and their link with mixed poison distribution.The pdf of Poison-Lindley distribution was obtained,its survival and hazard rate function derived.The hazard rate function of Poison-Lindley,proved to be the difference of two hazard functions of Pareto ii (Lomax).

2.9 Summary of Literature Review.

One-Parameter Lindley distribution has been generalized,modified,extended and mixed with other distributions (Discrete and Continuous) in various ways;

1. a.)Generalization of Lindley has been done by finite mixture of two gamma distributions with certain weights.The table below shows specific values of parameters used to generate sub-models of G5L.

Model	θ	α	β	k	n	Author.
G5L						Al-Babtain et al(2015).
G4L(1)				1		Al-Babtain et al(2015).
G4L(2)					1	Al-Babtain et al(2015).
G3L(1)			$\alpha+1$	1		Zakerzadeh and Dolati(2009).
G3L(2)				1	1	Elbatal et al(2013).
G2L(1)		1	2	1		Shanker et al(2012).
QLD		1	2		θ	Shanker and Mishra(2013).
G2L(2)		1	2		1	Al-Babtain et al(2015).
LD		1	2	1	1	Lindley(1958).

   b.)Other forms of generalization of Lindley include Exponentiation of One parameter Lindley distribution as initially explained so as to provide more flexibility for lifetime data.

   c.)Their exists several modifications and extensions of Lindley such as discrete version of Lindley and Power Lindley.

2. Lindley as a mixture distribution.
   Some of these type of distribution include; Lindley Exponential(L-E) distribution and Lindley Pareto (LP) distribution for modeling lifetime data.

3. Lindley as a conditional distribution.
   Lindley mixture of Poisson distribution where Lindley was the mixing distribution (Discrete Poison-Lindley(P-L)distribution) and a continuous mixture of Negative Binomial and Lindley to form Negative Binomial-Lindley(NB-L) distribution.

4. Related distributions to Lindley consists of;

   a.) Sums of independent and identically distributed Lindley random variables which involved convolution of Lindley with same parameters and also with different parameters.

   b.) Construction of excess loss (Residual lifetime) and mean residual lifetime (MRL) of various Lindley distributions.

The Generalizations,extensions and modifications of Lindley mentioned above have been constructed in terms of their Probability density functions,Cumulative distribution function,Survival(Reliability analysis) function and Hazard Rate function.

Some statistical properties have been discussed;Shapes,Reverse Hazard function,Cumulative Hazard function,Residual Lifetime(Excess loss function);Mean Residual Lifetime(Mean excess loss function),Moment Generating Functions and related measures,Stochastic ordering,Distributions of Order Statistics,Measure of Inequality and Uncertainty(Lorenz curves,Bonferroni curves,Bonferroni index,Gini index and Renyi Entropy).

The parameter values have been investigated by Method of Moments(MoM) and Maximum Likelihood Estimation(MLE).

Some of the methods which have been used to a certain goodness of fit include;Negative Log-Likelihood(-LL),Akaike Information Criterion(AIC),Corrected Akaike Information Criterion(AICc),Bayesian Information Criterion(BIC),Kolgomorov-Smirnov test(K-S) and the p-value.

3 CONSTRUCTION AND PROPERTIES OF A G5L DISTRIBUTION.

3.1 Introduction.

The objective of this chapter is to construct and obtain properties of a five-parameter Lindley distribution,G5L.
The construction is based on finite mixture of two gamma distributions and is expressed in terms of Probability density function,Survival function and Hazard function.
Properties considered are:Moment generating function,Mean,Variance,Skewness,Kurtosis and Mode.
We shall first re-examine construction and properties of One-parameter,Two-parameter,Three-parameter and Four-parameter Lindley distribution.

3.2 One parameter Lindley distribution.

This is a finite mixture of an exponential distribution and gamma distribution given as follows;
Let,

$f_1(x) = \theta e^{-\theta x}, x > 0; \theta > 0$ which is exponential with parameter θ. **(1)**

$f_2(x) = \frac{\theta^2}{\Gamma(2)} e^{-\theta x} x^{2-1} = \theta^2 x e^{-\theta x}$ which is a gamma distribution with parameter 2 and θ. **(2)**

Therefore, we have the following results,when the weighing probability is

$$w = \frac{\theta}{\theta+1} \tag{3}$$

Proposition 3.2.1. *The Probability density function f(x),the Cumulative density function F(x),the Survival function S(x) and Hazard function h(x) of a one-parameter Lindley distribution are;*

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \text{ ;} x > 0, \theta > 0. \tag{4}$$

$$F(x) = 1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x} \text{ ; } x > 0, \theta > 0. \tag{5}$$

$$S(x) = \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x} \text{ ; } x > 0, \theta > 0. \tag{6}$$

$$h(x) = \frac{\theta^2(1+x)}{\theta+\theta x-1} \text{ ; } x > 0, \theta > 0. \tag{7}$$

Proofs:

Probability density function.

$$f(x) = p_1 * Gamma(1,\theta) + p_2 * Gamma(2,\theta)$$

$$f(x) = \frac{\theta}{\theta+1}(\theta e^{-\theta x}) + \frac{1}{\theta+1}\left(\theta^2 e^{-\theta x} x\right)$$
$$f(x) = \frac{\theta^2}{\theta+1}\left[1+\frac{\theta(\theta x)}{\theta^2}\right]e^{-\theta x}$$
$$f(x) = \frac{\theta^2}{\theta+1}\left[1+\frac{(\theta x)}{\theta}\right]e^{-\theta x}$$
$$f(x) = \frac{\theta^2}{\theta+1}\left[1+x\right]e^{-\theta x} \text{ ,where } \theta, (1+\theta) > 0$$

Cumulative distribution function.

$$F(x) = \int_0^\infty \frac{\theta^2}{\theta+1} f(x)dx = \int_0^\infty \frac{\theta^2}{\theta+1}\left[1+x\right]e^{-\theta x}dx$$

$$
\begin{aligned}
F(x) &= \frac{\theta^2}{\theta+1}\int_0^\infty \left[[1+x]\, e^{-\theta x} dx\right] \\
&= \frac{\theta^2}{\theta+1} * I_1 \\
I_1 &= \int_0^\infty (x+1)e^{-\theta x} dx \\
u &= x+1 \Rightarrow \frac{du}{dx} = 1 \\
dv &= e^{-\theta x} \\
v &= \int_0^\infty e^{-\theta x} dx = \frac{-e^{-\theta x}}{\theta} \\
I_1 &= \frac{-(x+1)e^{-\theta x}}{\theta} - \int_0^\infty \frac{-e^{-\theta x}}{\theta} dx \\
I_2 &= \int_0^\infty \frac{-e^{-\theta x}}{\theta} dx \\
u &= -\theta x \Rightarrow \frac{du}{dx} = -\theta \\
& \frac{1}{\theta^2}\int_0^\infty e^u du \\
& \frac{1}{\theta^2}\int_0^\infty e^u du = \frac{e^u}{\theta^2} = \frac{e^{-\theta x}}{\theta^2}
\end{aligned}
$$

combining I_1 and I_2

$$
\begin{aligned}
& \frac{\theta^2}{\theta+1}\int_0^\infty (x+1)e^{-\theta x} dx \\
&= \frac{\theta^2}{\theta+1}\left[\frac{-(x+1)e^{-\theta x}}{\theta} - \frac{e^{-\theta x}}{\theta^2}\right] \\
&= \frac{-\theta(x+1)e^{-\theta x}}{\theta+1} - \frac{e^{-\theta x}}{\theta+1} \\
&= 1 - \frac{(\theta x+\theta+1)e^{-\theta x}}{\theta+1} \\
F(x) &= 1 - \left[1 + \frac{\theta x}{\theta+1}\right] e^{-\theta x}
\end{aligned}
$$

Survival function.

$$S(x) = 1 - F(x)$$

$$S(x) = 1 - \left[1 - \frac{(\theta x + \theta + 1)e^{-\theta x}}{\theta + 1}\right]$$
$$S(x) = \frac{(\theta x + \theta + 1)e^{-\theta x}}{\theta + 1}$$

Hazard function.

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\frac{\theta^2}{\theta+1}\left[1 + x\right]e^{-\theta x}}{\frac{(\theta x+\theta+1)e^{-\theta x}}{\theta+1}}$$
$$h(x) = \frac{\theta^2(1 + x)}{\theta x + (\theta + 1)}$$

This completes the proofs.

3.3 Two-Parameter Lindley Type 1

Proposition 3.3.1. *The Probability density function f(x),the Cumulative density function F(x),the Survival function S(x) and Hazard function h(x) of a Two-parameter Lindley distribution version 1 are;*

$$f(x) = \frac{\theta^2}{\theta + n}\left[1 + nx\right]e^{-\theta x} \qquad \textit{where } \theta, n \textit{ and } (n + \theta) > 0 \tag{8}$$
$$F(x) = 1 - \left[1 + \frac{\theta n x}{\theta + n}\right]e^{-\theta x} \tag{9}$$
$$S(x) = \frac{(\theta n x + \theta + n)e^{-\theta x}}{\theta + n} \quad ,\textit{where } \theta, n > 0 \tag{10}$$
$$h(x) = \frac{\theta^2(1 + nx)}{\theta n x + (\theta + n)} \tag{11}$$

Proofs:

Probability density function.

$$f(x) = p_1 * Gamma(1,\theta) + p_2 * Gamma(2,\theta)$$

$$f(x) = \frac{\theta}{\theta+n}(\theta e^{-\theta x}) + \frac{n}{\theta+n}\left(\theta^2 e^{-\theta x} x\right)$$
$$f(x) = \frac{\theta^2}{\theta+n}\left[1+\frac{n\theta(\theta x)}{\theta^2}\right]e^{-\theta x}$$
$$f(x) = \frac{\theta^2}{\theta+n}\left[1+\frac{n(\theta x)}{\theta}\right]e^{-\theta x}$$
$$f(x) = \frac{\theta^2}{\theta+n}[1+nx]e^{-\theta x} \text{ where } \theta, n \text{ and } (n+\theta) > 0$$

The Cumulative distribution function.

$$F(x) = \int_0^\infty \frac{\theta^2}{\theta+n} f(x)dx = \int_0^\infty \frac{\theta^2}{\theta+n}[1+nx]e^{-\theta x}dx$$
$$F(x) = \frac{\theta^2}{\theta+n}\int_0^\infty [1+nx]e^{-\theta x}dx$$
$$= \frac{\theta^2}{\theta+n} * I_1$$
$$I_1 = \int_0^\infty (nx+1)e^{-\theta x}dx$$
$$u = nx+1 \Rightarrow \frac{du}{dx} = n$$
$$dv = e^{-\theta x} \Rightarrow v = \int_0^\infty e^{-\theta x}dx = \frac{-e^{-\theta x}}{\theta}$$
$$I_1 = \left[\frac{-(nx+1)e^{-\theta x}}{\theta}\right]_0^\infty - \int_0^\infty \frac{-ne^{-\theta x}}{\theta}dx$$
$$I_1 = \left[0 - \frac{-(1)}{\theta}\right] - \int_0^\infty \frac{-ne^{-\theta x}}{\theta}dx$$

$$I_2 = \int_0^\infty \frac{-ne^{-\theta x}}{\theta} dx$$
$$u = -\theta x \Rightarrow \frac{du}{dx} = -\theta$$
$$I_2 = \frac{1}{\theta^2} \int_0^\infty e^u du$$
$$I_2 = \frac{n}{\theta^2} \int_0^\infty e^u du = \frac{ne^u}{\theta^2} = \frac{ne^{-\theta x}}{\theta^2}$$
$$I_1 = \frac{-(nx+1)e^{-\theta x}}{\theta} - \frac{ne^{-\theta x}}{\theta^2}$$
$$F(x) = \frac{\theta^2}{\theta+n} \int_0^\infty (nx+1)e^{-\theta x}$$
$$F(x) = 1 + \frac{-\theta(nx+1)e^{-\theta x}}{\theta+n} - \frac{ne^{-\theta x}}{\theta+n}$$
$$F(x) = 1 - \frac{(\theta nx+\theta+n)e^{-\theta x}}{\theta+n}$$
$$F(x) = 1 - \left[1 + \frac{\theta nx}{\theta+n}\right] e^{-\theta x}$$

Survival function.

$$S(x) = Prob[X > x] = 1 - Prob[X \leq x] = 1 - F(x)$$

$$S(x) = 1 - \left[1 - \frac{(\theta nx+\theta+n)e^{-\theta x}}{\theta+n}\right]$$
$$S(x) = \frac{(\theta nx+\theta+n)e^{-\theta x}}{\theta+n} \quad \text{,where } \theta, n > 0$$

Hazard function.

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\frac{\theta^2}{\theta+n}[1+nx]e^{-\theta x}}{\frac{(\theta nx+\theta+n)e^{-\theta x}}{\theta+n}}$$
$$h(x) = \frac{\theta^2(1+nx)}{\theta nx + (\theta+n)}$$

This completes the proofs.

3.4 Two- Parameter Lindley Type 2.

Proposition 3.4.1. *The Probability density function f(x),the Cumulative density function F(x),the Survival function S(x) and Hazard function h(x) of a Two-parameter Lindley distribution version 2 are;*

$$f(x) = \frac{\theta^2}{k\theta+1}[k+x]e^{-\theta x} \quad \textit{where } \theta, k \textit{ and } (1+k\theta) > 0 \tag{12}$$

$$F(x) = 1 - \left[1 + \frac{\theta x}{\theta k+1}\right] e^{-\theta x} \tag{13}$$

$$S(x) = \frac{(\theta x + \theta k + 1)e^{-\theta x}}{\theta k+1} \tag{14}$$

$$h(x) = \frac{\theta^2(k+x)}{\theta x + (\theta k+1)} \tag{15}$$

Proofs:

Probability density function.

$$f(x) = p_1 * Gamma(1,\theta) + p_2 * Gamma(2,\theta)$$

$$f(x) = \frac{k\theta}{k\theta+1}(\theta e^{-\theta x}) + \frac{1}{k\theta+1}\left(\frac{\theta^2 e^{-\theta x} x}{1}\right)$$

$$f(x) = \frac{\theta^2}{k\theta+1}\left[k + \frac{\theta(\theta x)}{\theta^2}\right] e^{-\theta x}$$

$$f(x) = \frac{\theta^2}{k\theta+1}\left[k + \frac{(\theta x)}{\theta}\right] e^{-\theta x}$$

$$f(x) = \frac{\theta^2}{k\theta+1}[k+x]e^{-\theta x} \quad \text{where } \theta, k \text{ and } (1+k\theta) > 0$$

Cumulative distribution function.

$$F(x) = \int_0^\infty \frac{\theta^2}{k\theta+n} f(x)dx = \int_0^\infty \frac{\theta^2}{k\theta+1}[k+x]e^{-\theta x}dx$$

$$F(x) = \frac{\theta^2}{k\theta+1}\int_0^\infty \left[[k+x]e^{-\theta x}dx\right]$$

$$F(x) = \frac{\theta^2}{k\theta+1} * I_1$$
$$I_1 = \int_0^\infty (x+k)e^{-\theta x}dx$$
$$u = x+k \Rightarrow u' = 1$$
$$dv = e^{-\theta x} \Rightarrow v = \int_0^\infty e^{-\theta x}dx = \frac{-e^{-\theta x}}{\theta}$$
$$I_1 = \left[\frac{-(x+k)e^{-\theta x}}{\theta}\right]_0^\infty - \int_0^\infty \frac{-e^{-\theta x}}{\theta}dx$$
$$I_1 = \left[0 - \frac{-(x+k)}{\theta}\right] - \int_0^\infty \frac{-e^{-\theta x}}{\theta}dx$$
$$I_1 = \left[\frac{(x+k)}{\theta}\right] - \int_0^\infty \frac{-e^{-\theta x}}{\theta}dx$$
$$I_2 = \int_0^\infty \frac{-e^{-\theta x}}{\theta}dx$$
$$u = -\theta x \Rightarrow \frac{du}{dx} = -\theta$$
$$I_2 = \frac{1}{\theta^2}\int_0^\infty e^u du$$
but, $\int_0^\infty e^u du = e^u$

therefore, $I_2 = \frac{1}{\theta^2}\int_0^\infty e^u du = \frac{e^u}{\theta^2} = \frac{e^{-\theta x}}{\theta^2}$
$$I_1 = \frac{(x+k)}{\theta}\frac{-(x+k)e^{-\theta x}}{\theta} - \frac{e^{-\theta x}}{\theta^2}$$
$$I_1 = \frac{(x+k)(1-e^{-\theta x})}{\theta} - \frac{e^{-\theta x}}{\theta^2}$$
$$I_1 = \frac{(x+k)(1-e^{-\theta x})}{\theta} - \frac{e^{-\theta x}}{\theta^2}$$
$$F(x) = \frac{\theta^2}{\theta k+1}\int_0^\infty (x+k)e^{-\theta x}dx$$
$$F(x) = 1 + \frac{-\theta(x+k)e^{-\theta x}}{\theta k+1} - \frac{e^{-\theta x}}{\theta k+1}$$
$$F(x) = 1 - \frac{(\theta x+\theta k+1)e^{-\theta x}}{\theta k+1}$$
$$F(x) = 1 - \left[1 + \frac{\theta x}{\theta k+1}\right]e^{-\theta x}$$

Survival function.

$$S(x) = 1 - F(x)$$

$$S(x) = 1 - \left[1 - \frac{(\theta x + \theta k + 1)e^{-\theta x}}{\theta k + 1}\right]$$
$$S(x) = \frac{(\theta x + \theta k + 1)e^{-\theta x}}{\theta k + 1}$$

Hazard function.

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\frac{\theta^2}{k\theta+1}\left[k + x\right] e^{-\theta x}}{\frac{(\theta x+\theta k+1)e^{-\theta x}}{\theta k+1}}$$
$$h(x) = \frac{\theta^2(k+x)}{\theta x + (\theta k + 1)}$$

3.5 Three Parameter Lindley distribution.

Proposition 3.5.1. *The Probability density function f(x),the Cumulative density function F(x),the Survival function S(x) and Hazard function h(x) of a Three-parameter Lindley distribution are stated as;*

$$f(x) = \frac{\theta^2}{k\theta + n}\left[k + nx\right] e^{-\theta x} \text{ ,where } \theta, k, n > 0. \tag{16}$$
$$F(x) = 1 - \left[1 + \frac{\theta n x}{\theta k + n}\right] e^{-\theta x} \tag{17}$$
$$S(x) = \frac{(\theta n x + \theta k + n)e^{-\theta x}}{\theta k + n} \tag{18}$$
$$h(x) = \frac{\theta^2(k + nx)}{\theta n x + (\theta k + n)} \tag{19}$$

Proofs:

Probability density function

$$f(x) = p_1 * Gamma(1,\theta) + p_2 * Gamma(2,\theta)$$

$$f(x) = \frac{k\theta}{k\theta+n}(\theta e^{-\theta x}) + \frac{n}{k\theta+n}\left(\theta^{\beta} e^{-\theta x} x\right)$$

$$f(x) = \frac{\theta^2}{k\theta+n}\left[k + \frac{n\theta(\theta x)^{2-1}}{\theta^2}\right] e^{-\theta x}$$

$$f(x) = \frac{\theta^2}{k\theta+n}\left[k + \frac{n(\theta x)}{\theta}\right] e^{-\theta x}$$

$$f(x) = \frac{\theta^2}{k\theta+n}\left[k + nx\right] e^{-\theta x}$$

Cumulative distribution function

$$F(x) = \int_0^{\infty} \frac{\theta^2}{k\theta+n} f(x)dx = \int_0^{\infty} \frac{\theta^2}{k\theta+n}\left[k+nx\right] e^{-\theta x} dx$$

$$F(x) = \frac{\theta^2}{k\theta+n}\int_0^{\infty}\left[k+nx\right] e^{-\theta x} dx$$

$$= \frac{\theta^2}{k\theta+n} * I_1$$

$$I_1 = \int_0^{\infty} (nx+k)e^{-\theta x} dx$$

$$u = nx+k \Rightarrow \frac{du}{dx} = n$$

$$dv = e^{-\theta x} \Rightarrow v = \int_0^{\infty} e^{-\theta x} dx = \frac{-e^{-\theta x}}{\theta}$$

$$I_1 = \left[\frac{-(nx+k)e^{-\theta x}}{\theta}\right]_0^{\infty} - \int_0^{\infty} \frac{-ne^{-\theta x}}{\theta} dx$$

$$I_2 = \int_0^{\infty} \frac{-ne^{-\theta x}}{\theta} dx$$

$$u = -\theta x \Rightarrow \frac{du}{dx} = -\theta$$

$$
\begin{aligned}
&= \frac{k}{\theta^2}\int_0^\infty e^u du \quad \text{but} \int_0^\infty e^u du = e^u \\
I_2 &= \frac{n}{\theta^2}\int_0^\infty e^u du = \frac{ne^u}{\theta^2} = \frac{ne^{-\theta x}}{\theta^2} \\
I_1 &= \frac{-(nx+k)e^{-\theta x}}{\theta} - \frac{ne^{-\theta x}}{\theta^2} \\
F(x) &= \frac{\theta^2}{\theta k+n}\int_0^\infty (nx+k)e^{-\theta x} \\
&= \frac{-\theta(nx+k)e^{-\theta x}}{\theta k+n} - \frac{ne^{-\theta x}}{\theta k+n} \\
&= 1 - \frac{(\theta nx+\theta k+n)e^{-\theta x}}{\theta k+n} \\
F(x) &= 1 - \left[1+\frac{\theta nx}{\theta k+n}\right]e^{-\theta x}
\end{aligned}
$$

Survival function

$$S(x) = 1 - F(x)$$

$$
\begin{aligned}
S(x) &= 1 - \left[1 - \frac{(\theta nx+\theta k+n)e^{-\theta x}}{\theta k+n}\right] \\
S(x) &= \frac{(\theta nx+\theta k+n)e^{-\theta x}}{\theta k+n}
\end{aligned}
$$

Hazard function

$$h(x) = \frac{f(x)}{S(x)}$$

$$
\begin{aligned}
h(x) &= \frac{\frac{\theta^2}{k\theta+n}\left[k+nx\right]e^{-\theta x}}{\frac{(\theta nx+\theta k+n)e^{-\theta x}}{\theta k+n}} \\
h(x) &= \frac{\theta^2(k+nx)}{\theta nx+(\theta k+n)}
\end{aligned}
$$

3.6 Four- Parameter Lindley Type 1.

Proposition 3.6.1. *The Probability density function f(x),the Cumulative density function F(x),the Survival function S(x) and Hazard function h(x) of a Four-parameter Lindley distribution version 1 are stated as;*

$$f(x) = \frac{\theta^2}{\theta+n}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x} \quad \text{,where } \theta,\alpha,n,\beta > 0 \tag{20}$$

$$F(x) = \frac{1}{\theta+n}[\theta\gamma_{\alpha(\theta x)} + n\gamma_{\beta(\theta x)}] \tag{21}$$

$$S(x) = \frac{n+\theta-[\theta\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]}{\theta+n} \tag{22}$$

$$h(x) = \frac{\theta^2\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}}{n+\theta k-[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]} \tag{23}$$

Proofs:

Probability density function.

$$f(x) = p_1 * Gamma(\alpha,\theta) + p_2 * Gamma(\beta,\theta)$$

$$f(x) = \frac{\theta}{\theta+n}\left(\frac{\theta^\alpha e^{-\theta x}x^{\alpha-1}}{\Gamma(\alpha)}\right) + \frac{n}{\theta+n}\left(\frac{\theta^\beta e^{-\theta x}x^{\beta-1}}{\Gamma(\beta)}\right)$$

$$f(x) = \frac{\theta^2}{\theta+n}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n\theta(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}$$

thus the pdf is $f(x) = \frac{\theta^2}{\theta+n}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}$,where $\theta,\alpha,n,\beta > 0$

Cumulative distribution function.

$$F(x) = \int_0^\infty \frac{\theta^2}{\theta+n}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}dx$$

$$F(x) = \frac{\theta^2}{\theta+n}\int_0^\infty \left[\frac{(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}dx$$

$$F(x) = \frac{\theta^2}{\theta+n}\left[\int_0^{\infty}\frac{(\theta x)^{\alpha-1}}{\Gamma(\alpha)}dx + \int_0^{\infty}\frac{n(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}dx\right]e^{-\theta x}$$

$$F(x) = \frac{1}{\theta+n}\left[\frac{\theta}{\Gamma(\alpha)}\int_0^{\infty}\frac{(\theta x)^{\alpha-1}}{1}e^{-\theta x}dx + \frac{n}{\Gamma\beta}\int_0^{\infty}\frac{(\theta x)^{\beta-1}}{1}e^{-\theta x}dx\right]$$

$$F(x) = \frac{1}{\theta+n}[\theta\gamma_{\alpha(\theta x)} + n\gamma_{\beta(\theta x)}]$$

$$\text{where, } \gamma_{\alpha}(\theta x) = \frac{1}{\Gamma(\alpha)}\int_0^{\theta x}(\theta x)^{\alpha-1}e^{-\theta x}dx = \frac{\gamma(\alpha,\theta x)}{\Gamma(\alpha)}$$

$$\text{and } \gamma_{\beta}(\theta x) = \frac{1}{\Gamma(\beta)}\int_0^{\theta x}(\theta x)^{\beta-1}e^{-\theta x}dx = \frac{\gamma(\beta,\theta x)}{\Gamma(\beta)}$$

Survival function.

$$S(x) = 1 - F(x)$$

$$S(x) = 1 - \frac{1}{\theta+n}[\theta\gamma_{\alpha}(\theta x) + n\gamma_{\beta}(\theta x)]$$

$$S(x) = \frac{n+\theta-[\theta\gamma_{\alpha}(\theta x) + n\gamma_{\beta}(\theta x)]}{\theta+n}$$

Hazard function

$$h(x) = \lim_{\Delta x}\frac{F(x+\Delta x) - F(x)}{\Delta x S(x)} = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\frac{\theta^2}{n+\theta}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\theta n(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}}{1 - \frac{1}{n+\theta}[\theta\gamma_{\alpha}(\theta x) + n\gamma_{\beta}(\theta x)]}$$

$$h(x) = \frac{\frac{\theta^2}{n+\theta}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\theta n(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}}{\frac{n+\theta-[\theta\gamma_{\alpha}(\theta x)+n\gamma_{\beta}(\theta x)]}{n+\theta}}$$

$$h(x) = \frac{\theta^2\left[\frac{(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\theta n(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}}{n+\theta-[\theta\gamma_{\alpha}(\theta x) + n\gamma_{\beta}(\theta x)]}$$

$$h(x) = \frac{\theta^2\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}}{n+\theta k-[\theta k\gamma_{\alpha}(\theta x) + n\gamma_{\beta}(\theta x)]}$$

This completes the proof.

3.7 Four- Parameter Lindley Type 2.

Proposition 3.7.1. *The Probability density function f(x),the Cumulative density function F(x),the Survival function S(x) and Hazard function h(x) of a Four-parameter Lindley distribution version 2 are stated as;*

$$f(x) = \frac{\theta^2}{k\theta+1}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x} \text{ where } \theta,k,\alpha,\beta > 0 \tag{24}$$

$$F(x) = \frac{1}{\theta k+1}[\theta k\gamma_{\alpha(\theta x)} + \gamma_{\beta(\theta x)}] \tag{25}$$

$$S(x) = \frac{1+\theta k - [\theta k\gamma_\alpha(\theta x) + \gamma_\beta(\theta x)]}{\theta k+1} \tag{26}$$

$$h(x) = \frac{\theta^2\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}}{1+\theta k - [\theta k\gamma_\alpha(\theta x) + \gamma_\beta(\theta x)]} \tag{27}$$

Proofs:

Probability density function.

$$f(x) = p_1 * Gamma(\alpha,\theta) + p_2 * Gamma(\beta,\theta)$$

$$f(x) = \frac{k\theta}{k\theta+1}\left(\frac{\theta^\alpha e^{-\theta x}x^{\alpha-1}}{\Gamma(\alpha)}\right) + \frac{1}{k\theta+1}\left(\frac{\theta^\beta e^{-\theta x}x^{\beta-1}}{\Gamma(\beta)}\right)$$

$$f(x) = \frac{\theta^2}{k\theta+1}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\theta(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}$$

$$f(x) = \frac{\theta^2}{k\theta+1}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x} \qquad \text{where } \theta,k,\alpha,\beta > 0$$

Cumulative distribution function.

$$F(x) = \int_0^\infty \frac{\theta^2}{k\theta+1}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}dx$$

$$F(x) = \frac{\theta^2}{k\theta+1}\int_0^\infty \left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}dx$$

$$F(x)=\frac{\theta^2}{k\theta+1}\left[\int_0^{\infty}\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}dx+\int_0^{\infty}\frac{(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}dx\right]e^{-\theta x}$$

$$F(x)=\frac{1}{k\theta+1}\left[\frac{\theta k}{\Gamma(\alpha)}\int_0^{\infty}\frac{(\theta x)^{\alpha-1}}{1}e^{-\theta x}dx+\frac{1}{\Gamma\beta}\int_0^{\infty}\frac{(\theta x)^{\beta-1}}{1}e^{-\theta x}dx\right]$$

$$F(x)=\frac{1}{\theta k+1}[\theta k\gamma_{\alpha(\theta x)}+\gamma_{\beta(\theta x)}]$$

$$\text{where;}\gamma_\alpha(\theta x)=\frac{1}{\Gamma(\alpha)}\int_0^{\theta x}(\theta x)^{\alpha-1}e^{-\theta x}dx=\frac{\gamma(\alpha,\theta x)}{\Gamma(\alpha)}$$

$$\gamma_\beta(\theta x)=\frac{1}{\Gamma(\beta)}\int_0^{\theta x}(\theta x)^{\beta-1}e^{-\theta x}dx=\frac{\gamma(\beta,\theta x)}{\Gamma(\beta)}$$

Survival function.

$$S(x)=1-F(x)$$

$$S(x)=1-\frac{1}{\theta k+1}[\theta k\gamma_\alpha(\theta x)+\gamma_\beta(\theta x)]$$

$$S(x)=\frac{1+\theta k-[\theta k\gamma_\alpha(\theta x)+\gamma_\beta(\theta x)]}{\theta k+1}$$

Hazard function.

$$h(x)=\frac{f(x)}{S(x)}$$

$$h(x)=\frac{\frac{\theta^2}{1+\theta k}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\theta(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}}{1-\frac{1}{1+\theta k}[\theta k\gamma_\alpha(\theta x)+\gamma_\beta(\theta x)]}$$

$$h(x)=\frac{\frac{\theta^2}{1+\theta k}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\theta(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}}{\frac{1+\theta k-[\theta k\gamma_\alpha(\theta x)+\gamma_\beta(\theta x)]}{1+\theta k}}$$

$$h(x)=\frac{\theta^2\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\theta(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}}{1+\theta k-[\theta k\gamma_\alpha(\theta x)+\gamma_\beta(\theta x)]}$$

$$h(x)=\frac{\theta^2\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}}{1+\theta k-[\theta k\gamma_\alpha(\theta x)+\gamma_\beta(\theta x)]}$$

3.8 Construction of Five Parameter Lindley Distribution,G5L.

3.8.1 Probability density function of G5L.

Proposition 3.8.1. *The pdf of G5L is given as;*

$$f(x;\theta,\alpha,\beta,k,n) = \frac{1}{k\theta+n}\left[\frac{k\theta^{\alpha+1}x^{\alpha-1}}{\Gamma(\alpha)} + \frac{n\theta^{\beta}x^{\beta-1}}{\Gamma(\beta)}\right]e^{-\theta x} \tag{28}$$

for, $x>0,\theta>0,\alpha>0,\beta>0,k\geq,n\geq 0,$ *where k and n are not simultaneously zeros.*

Proof:
Let $f_1(x)$ be a p.d.f of a gamma distribution with parameters (θ,α) and $f_2(x)$ be a p.d.f of another gamma distribution with parameters (θ,β).

$$\text{therefore, } f_1(x;\alpha,\theta) = \frac{\theta^{\alpha}e^{-\theta x}\lambda^{\alpha-1}}{\Gamma(\alpha)};x>0,\theta,\alpha>0$$

$$\text{and } f_2(x;\beta,\theta) = \frac{\theta^{\beta}e^{-\theta x}\lambda^{\beta-1}}{\Gamma(\beta)};x>0,\theta,\beta>0$$

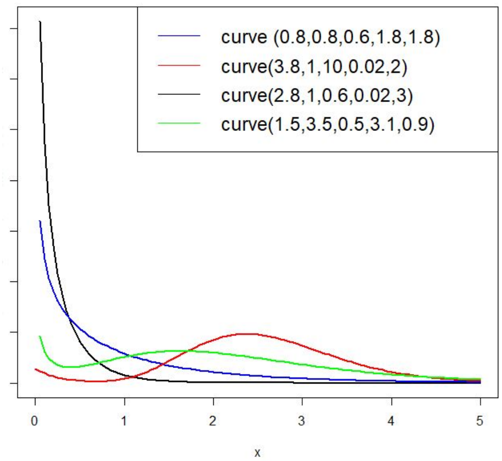
Let $w_1 = \frac{k\theta}{k\theta+n} \Rightarrow 1-w_1 = \frac{n}{k\theta+n}$.

We can now obtain the p.d.f of G5L by finite mixture as follows;

$$f(x;\theta,\alpha,\beta,k,n) = w_1 f_1(x;\alpha,\theta) + (1-w_1)f_2(x;\beta,\theta)$$
$$f(x;\theta,\alpha,\beta,k,n) = \frac{k\theta}{k\theta+n}\left(\frac{\theta^{\alpha}e^{-\theta x}x^{\alpha-1}}{\Gamma(\alpha)}\right) + \frac{n}{k\theta+n}\left(\frac{\theta^{\beta}e^{-\theta x}x^{\beta-1}}{\Gamma(\beta)}\right)$$
$$f(x;\theta,\alpha,\beta,k,n) = \frac{\theta^2}{k\theta+n}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n\theta(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}$$
$$f(x;\theta,\alpha,\beta,k,n) = \frac{\theta^2}{k\theta+n}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}$$

$$f(x;\theta,\alpha,\beta,k,n) = \frac{1}{k\theta+n}\left[\frac{k\theta^{\alpha+1}x^{\alpha-1}}{\Gamma(\alpha)} + \frac{n\theta^{\beta}x^{\beta-1}}{\Gamma(\beta)}\right]e^{-\theta x}$$

Figure 1. Pdf Shapes from four sets of Parameter values.



3.8.2 Cumulative distribution function.

Proposition 3.8.2. *Cdf of is stated as;*

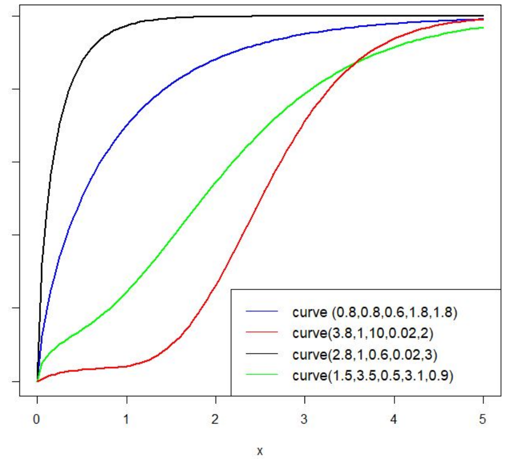
$$F(x;\theta,\alpha,\beta,k,n) = \frac{1}{\theta k+n}[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)] \tag{29}$$

Proof.

$$F(x;\theta,\alpha,\beta,k,n) = \frac{1}{k\theta+n}\int_0^\infty\left[\frac{k\theta^{\alpha+1}x^{\alpha-1}}{\Gamma(\alpha)}+\frac{n\theta^\beta x^{\beta-1}}{\Gamma(\beta)}\right]e^{-\theta x}dx$$
$$F(x;\theta,\alpha,\beta,k,n) = \frac{\theta^2}{k\theta+n}\int_0^\infty\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}+\frac{n(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}dx$$
$$F(x;\theta,\alpha,\beta,k,n) = \frac{1}{k\theta+n}\left[\frac{\theta^2 k}{\Gamma(\alpha)}\int_0^\infty(\theta x)^{\alpha-1}e^{-\theta x}dx+\frac{\theta^2 n}{\theta\Gamma\beta}\int_0^\infty(\theta x)^{\beta-1}e^{-\theta x}dx\right]$$
$$F(x;\theta,\alpha,\beta,k,n) = \frac{1}{k\theta+n}\left[\frac{\theta k}{\Gamma(\alpha)}\int_0^x\theta(\theta x)^{\alpha-1}e^{-\theta x}dx+\frac{n}{\Gamma\beta}\int_0^x\theta(\theta x)^{\beta-1}e^{-\theta x}dx\right]$$
$$F(x;\theta,\alpha,\beta,k,n) = \frac{1}{k\theta+n}\left[\frac{\theta k}{\Gamma(\alpha)}\int_0^x\theta(\theta x)^{\alpha-1}e^{-\theta x}dx+\frac{n}{\Gamma\beta}\int_0^x\theta(\theta x)^{\beta-1}e^{-\theta x}dx\right]$$

$$F(x;\theta,\alpha,\beta,k,n) = \frac{1}{\theta k+n}[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)] \text{ where, } ;x>0,\theta>0,\alpha>0,\beta>0,k\geq,n\geq 0$$

Figure 2. Cdf shapes from four sets of Parameter values.



3.8.3 Survival function.

Proposition 3.8.3. *The survival function of G5L is stated as;*

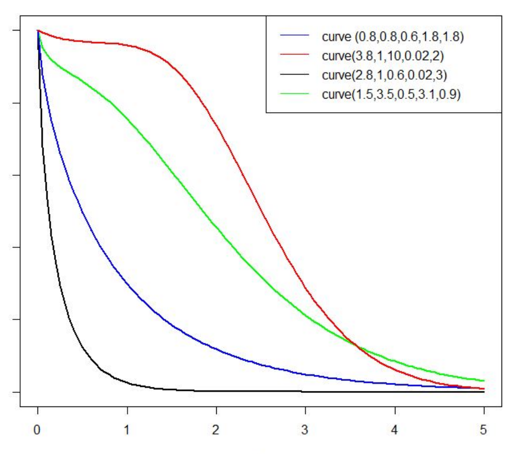
$$S(x) = \frac{n + \theta k - [\theta k \gamma_\alpha(\theta x) + n\gamma_\beta(\theta x)]}{\theta k + n} \tag{30}$$

Proof

$$S(x) = 1 - F(x)$$
$$S(x) = 1 - \frac{1}{\theta k + n}[\theta k \gamma_\alpha(\theta x) + n\gamma_\beta(\theta x)]$$

$$S(x) = \frac{n + \theta k - [\theta k \gamma_\alpha(\theta x) + n\gamma_\beta(\theta x)]}{\theta k + n} \quad ; x > 0, \theta > 0, \alpha > 0, \beta > 0, k \geq 0, n \geq 0$$

Figure 3. Shapes of survival function with four sets of Parameter values.



3.8.4 Hazard function.

Proposition 3.8.4. *The hazard function of G5L is stated as;*

$$h(x)=\frac{\theta^2\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}+\frac{n(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}}{n+\theta k-[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]} \tag{31}$$

Proof

$$h(x)=\frac{f(x)}{S(x)}$$

$$h(x)=\frac{\frac{\theta^2}{n+\theta k}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\theta n(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}}{1-\frac{1}{n+\theta k}[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]}$$

$$h(x)=\frac{\frac{\theta^2}{n+\theta k}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\theta n(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}}{\frac{n+\theta k-[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]}{n+\theta k}}$$

$$h(x)=\frac{\theta^2\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\theta n(\theta x)^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}}{n+\theta k-[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]}$$

$$h(x)=\frac{\theta^2\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}+\frac{n(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}}{n+\theta k-[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]}\quad ;x>0,\alpha,\beta,\theta>0 \text{ and also } k,n\geq 0$$

3.9 Properties of Five Parameter Lindley Distribution.

Moment generating function $M_x(t)$,the Mean $E(x)$,Variance $Var(x)$,Coefficient of variation ρ,Skewness ρ_1,Kurtosis ρ_2 and Mode x_0 are given by;

Proposition 3.9.1.

$$M_X(t)=\frac{1}{\theta k+n}\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{\theta^r}\left(\theta k\binom{-\alpha}{r}+n\binom{-\beta}{r}\right)t^r\right] \tag{32}$$

$$\textit{Mean(X)},\ \mu_1'=\frac{\alpha\theta k+\beta n}{\theta(\theta k+n)} \tag{33}$$

$$Var(X)=\frac{\alpha^2\theta^2k^2+4\alpha\theta kn+\beta n^2}{\theta^2(\theta k+n)^2} \tag{34}$$

$$\textit{Coefficient of variation},\ \rho=\frac{\sqrt{\mu_2'-\mu_1^2}}{\mu_1}=\frac{(\mu_2'-\mu_1^2)^{\frac{1}{2}}}{\mu_1} \tag{35}$$

$$\text{Measure of skewness, } \rho_1 = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{[\mu_2' - \mu_1'^2]^{\frac{3}{2}}} \tag{36}$$

$$\text{Measure of kurtosis, } \rho_2 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^2}{[\mu_2' - \mu_1'^2]^2} \tag{37}$$

$$x_0 = \frac{n\Gamma(\alpha)(\beta-1)\theta^{\beta-2} - k\Gamma(\beta)(\theta-\alpha+1)\theta^{\alpha-1}}{n\Gamma(\alpha)\theta^{\beta-1}} \tag{38}$$

Proofs.

3.9.1 Moment generating function.

$$M_X(t) = E[e^{Xt}] = \int_0^\infty e^{tX} f(x)dx$$

$$M_X(t) = \int_0^\infty e^{tx} * \frac{\theta^2}{\theta k + n}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n}{\theta}\frac{(\theta x)^{\beta-1}}{\Gamma(\beta)}\right] e^{-\theta x} dx$$

$$M_X(t) = \frac{\theta^2}{\theta k + n}\left[\frac{k(\theta^{\alpha-1})}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-(\theta-t)x}dx + \frac{n(\theta)^{\beta-2}}{\Gamma(\beta)}\int_0^\infty x^{\beta-1}e^{-(\theta-t)x}dx\right]$$

$$M_X(t) = \frac{1}{\theta k + n}\left[\frac{k(\theta^{\alpha+1})}{\Gamma(\alpha)} + \frac{n\theta^\beta}{(\theta-t)^\beta}\right]$$

$$M_X(t) = \frac{1}{\theta k + n}\left[\theta k\left(\frac{\theta}{\theta-t}\right)^\alpha + n\left(\frac{\theta}{\theta-t}\right)^\beta\right]$$

$$M_X(t) = \frac{1}{\theta k + n}\left[\theta k\left(\frac{\theta-t}{\theta}\right)^{-\alpha} + n\left(\frac{\theta-t}{\theta}\right)^{-\beta}\right]$$

$$M_X(t) = \frac{1}{\theta k + n}\left[\theta k\left(1-\frac{t}{\theta}\right)^{-\alpha} + n\left(1-\frac{t}{\theta}\right)^{-\beta}\right]$$

$$\text{using identity, } (1-x)^{-d} = \sum_{r=0}^{\infty}(-1)^r\binom{-d}{r}x^r$$

$$\text{then, } \left(1-\frac{t}{\theta}\right)^{-\alpha} = \sum_{r=0}^{\infty}(-1)^r\binom{-\alpha}{r}\left(\frac{t}{\theta}\right)^r$$

$$\text{and simmilarly, } \left(1-\frac{t}{\theta}\right)^{-\beta} = \sum_{r=0}^{\infty}(-1)^r\binom{-\beta}{r}\left(\frac{t}{\theta}\right)^r$$

$$\text{therefore, } M_X(t) = \frac{1}{\theta k + n}\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{\theta^r}\left(\theta k\binom{-\alpha}{r} + n\binom{-\beta}{r}\right)t^r\right] \text{,where } \theta > t.$$

r^{th} **non-central moments.**

Using equation 13 we apply first and second order derivative and then substitute t=0 so as to obtain the mean and variance respectively.

$$M_X(t) = \frac{1}{\theta k+n}\left[\theta k\left(1-\frac{t}{\theta}\right)^{-\alpha}+n\left(1-\frac{t}{\theta}\right)^{-\beta}\right]$$

$$M_X'(t) = \frac{1}{\theta k+n}\left[-\alpha\theta k\left(1-\frac{t}{\theta}\right)^{-\alpha-1}.\frac{-1}{\theta}+-\beta n\left(1-\frac{t}{\theta}\right)^{-\beta-1}.\frac{-1}{\theta}\right]$$

$$M_X'(t) = \frac{1}{\theta k+n}\left[\frac{\alpha\theta k}{\theta}\left(1-\frac{t}{\theta}\right)^{-\alpha-1}+\frac{\beta n}{\theta}\left(1-\frac{t}{\theta}\right)^{-\beta-1}\right]$$

$$M_X'(t) = \frac{1}{\theta(\theta k+n)}\left[\alpha\theta k\left(1-\frac{t}{\theta}\right)^{-(\alpha+1)}+\beta n\left(1-\frac{t}{\theta}\right)^{-(\beta+1)}\right]$$

$$\text{when t=0, } M_\Lambda'(0) = \frac{1}{\theta(\theta k+n)}[\alpha\theta k+\beta n]$$

$$M_X'(0) = \frac{\alpha\theta k+\beta n}{\theta(\theta k+n)} = \mu_1'$$

$$M_X''(t) = \frac{1}{\theta^2(\theta k+n)}\left[\alpha\theta k(\alpha+1)\left(1-\frac{t}{\theta}\right)^{-(\alpha+1)-1}+\beta n(\beta+1)\left(1-\frac{t}{\theta}\right)^{-(\beta+1)-1}\right]$$

$$\text{when t=0, } M_X''(0) = \frac{1}{\theta^2(\theta k+n)}[\alpha\theta k(\alpha+1)+\beta n(\beta+1)]$$

$$M_X''(0) = \frac{\alpha\theta k(\alpha+1)+\beta n(\beta+1)}{\theta^2(\theta k+n)} = \mu_2'$$

$$\text{when t=0, } M_X'''(0) = \frac{1}{\theta^3(\theta k+n)}[\alpha\theta k(\alpha+1)(\alpha+2)+\beta n(\beta+1)(\beta+2)]$$

$$M_X'''(0) = \frac{\alpha\theta k(\alpha+1)(\alpha+2)+\beta n(\beta+1)(\alpha+2)}{\theta^3(\theta k+n)} = \mu_3'$$

$$\text{simmilarly, } M_\Lambda^{iv}(0) = \frac{\alpha\theta k(\alpha+1)(\alpha+2)(\alpha+3)+\beta n(\beta+1)(\alpha+2)(\beta+3)}{\theta^4(\theta k+n)} = \mu_4'$$

3.9.2 Mean and Variance.

$$\text{Mean (X), } \mu_1' = \frac{\alpha\theta k+\beta n}{\theta(\theta k+n)}$$

$$Var(X) = \frac{\alpha\theta k(\alpha+1)+\beta n(\beta+1)}{\theta^2(\theta k+n)} - \left(\frac{\alpha\theta k+\beta n}{\theta(\theta k+n)}\right)^2$$

$$Var(X) = \frac{\alpha^2\theta^2k^2+4\alpha\theta kn+\beta n^2}{\theta^2(\theta k+n)^2}$$

i^{th} central moments.

The i^{th} central moments μ_i can be obtained from the r^{th} non-central moments as follows; $\mu_i = \frac{\sum f(x-\bar{X})^i}{N}$ where;
x is a random variable.
$\bar{X}$ is the mean.
N is the total number of random variables.
i is the central moment. Now let $x = a + cu \Rightarrow \bar{X} = a + c\bar{u}$ where a is an assumed mean and c is a constant.

$$\begin{aligned}
\mu_i &= \frac{\sum f(x-\bar{X})^i}{N} = \frac{\sum f(a+cu-a+c\bar{u})^i}{N} \\
&= \frac{\sum f(cu-c\bar{u})^i}{N} \\
&= \frac{c^i \sum f(u^i - iu^{i-r}\bar{u} + iu\bar{u}^{i-r} - \bar{u}^i)}{N} \\
\text{therefore, } \mu_i &= c^i \left[\frac{\sum fu^i}{N} - \frac{i\bar{u}\sum fu^{i-r}}{N} + (i-r)\bar{u}^i \right] \\
\text{let } c &= 1,\ a = 0 \text{ and } i = x. \\
\mu_i &= \left[\frac{\sum fx^i}{N} - \frac{i\bar{x}\sum fx^{i-r}}{N} + (i-r)\bar{x}^i \right] \\
\mu_i &= \mu_i' - i\mu_r'\mu_{i-r}' + (i-r)\left(\mu_r'\right)^i \\
\text{hence, } \mu_i &= \sum_{r=0}^{i} \binom{i}{r} (-1)^{i-r} (\mu)^{i-r} \mu_r' \\
\text{when } i = r = 1,\ \mu_1 &= \sum_{r=1}^{1} \binom{1}{1} (-1)^{1-1} (\mu)^{1-1} \mu_1' \\
\text{but, } \mu_1 = \mu_1' &= \frac{\alpha\theta k + n\beta}{\theta(n+\theta k)} \\
\Rightarrow \mu_i &= \sum_{r=0}^{i} \binom{i}{r} (-1)^{i-r} \left(\frac{\alpha\theta k + n\beta}{\theta(n+\theta k)} \right)^{i-r} \mu_r' \\
\text{for } i = 2\ ,\ r = 1,2\ \mu_2 &= \sum_{r=1}^{2} \binom{2}{r} (-1)^{2-r} (\mu)^{2-r} \mu_r' \\
\mu_2 &= \binom{2}{1} (-1)^{2-1} \left(\frac{\alpha\theta k + n\beta}{\theta(n+\theta k)} \right)^{2-1} \mu_1' + \binom{2}{2} (-1)^{2-2} \left(\frac{\alpha\theta k + n\beta}{\theta(n+\theta k)} \right)^{2-2} \mu_2' \\
\mu_2 &= \mu_2' - \left(\frac{\alpha\theta k + \beta n}{\theta(\theta k + n)} \right) \mu_1' \\
\text{therefore, } \mu_2 &= \mu_2' - \left(\mu_1'\right)^2
\end{aligned}$$

This implies that $\mu_2 = Variance(X)$.

simmilarly for $i=3$; $r=1,2,3$. $\mu_3 = \sum_{r=1}^{3}\binom{3}{r}(-1)^{3-r}(\mu)^{3-r}\mu_r'$

$$\mu_3 = \binom{3}{1}(-1)^{3-1}\left(\mu_1'\right)^{3-1}\mu_1' + \binom{3}{2}(-1)^{3-2}\left(\mu_1'\right)^{3-2}\mu_2' + \binom{3}{3}(-1)^{3-3}\left(\mu_1'\right)^{3-3}\mu_3'$$

$$\mu_3 = 2\left(\mu_1'\right)^2\mu_1' - 3\left(\mu_1'\right)\mu_2' + \mu_3'$$

Re-arrange, $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\left(\mu_1'\right)^3$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_1'\mu_2' - 3\left(\mu_1'\right)^2$$

Other Moment related measures.

3.9.3 Coefficient of variation.

$$\rho = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\mu_2' - \mu_1^2}}{\mu_1} = \frac{(\mu_2' - \mu_1^2)^{\frac{1}{2}}}{\mu_1}$$

3.9.4 Measure of skewness.

$$\rho_1 = \frac{E[\lambda - \mu]^3}{\sigma^3}$$

$$\rho_1 = \frac{\mu_3}{(\sigma^2)^{\frac{3}{2}}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{[\mu_2' - \mu_1'^2]^{\frac{3}{2}}}$$

3.9.5 Measure of kurtosis.

$$\rho_2 = \frac{E[x - \mu]^4}{\sigma^4} = \frac{E[x - \mu]^4}{[\sigma^2]^2}$$

$$\rho_2 = \frac{\mu_4}{\mu_2^2} = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^2}{[\mu_2' - \mu_1'^2]^2}$$

3.9.6 Mode of G5L.

$$f(x)=\frac{\theta^2}{\theta k+n}\left[\frac{k\theta^{\alpha-1}x^{\alpha-1}}{\Gamma(\alpha)}+\frac{n\theta^{\beta}\lambda^{\beta-1}}{\theta^2\Gamma(\beta)}\right]e^{-\theta x}$$

$$f'(x)=\frac{\theta^2}{\theta k+n}\left[\frac{d}{dx}\left(\frac{k\theta^{\alpha-1}x^{\alpha-1}}{\Gamma(\alpha)}\right)e^{-\theta\lambda}+\frac{d}{dx}\left(\frac{n\theta^{\beta}x^{\beta-1}}{\theta^2\Gamma(\alpha)}\right)e^{-\theta x}\right]$$

$$f'(x)=\frac{\theta^2}{\theta k+n}\left[\frac{k\theta^{\alpha-1}}{\Gamma(\alpha)}*\frac{d}{dx}x^{\alpha-1}e^{-\theta x}+\frac{n\theta^{\beta}}{\theta^2\Gamma(\beta)}*\frac{d}{dx}x^{\beta-1}e^{-\theta x}\right]$$

let $\lambda^{\alpha-1}=u$ and $v=e^{\theta\lambda}\Rightarrow u'=(\alpha-1)x^{\alpha-2}$

$$\text{hence, }\frac{d}{dx}\left(\frac{x^{\alpha-1}}{e^{\theta x}}\right)=\frac{(\alpha-1)x^{\alpha-2}e^{\theta x}-x^{\alpha-1}\theta e^{\theta x}}{(e^{\theta x})^2}$$

$$=\frac{((\alpha-1)x^{\alpha-2}-\theta x^{\alpha-1})e^{\theta x}}{e^{2\theta x}}$$

$$\frac{\theta^2}{\theta k+n}*\frac{k\theta^{\alpha-1}}{\Gamma(\alpha)}(((\alpha-1)x^{\alpha-2}-\theta x^{\alpha-1})e^{-\theta x})=0$$

$$\frac{\theta^2}{\theta k+n}*\frac{k\theta^{\alpha-1}}{\Gamma(\alpha)}(((\alpha-1)x^{\alpha-2}-\theta x^{\alpha-1})e^{-\theta x})+$$

$$\text{simmilarly, }\frac{\theta^2}{\theta k+n}*\frac{n\theta^{\beta}}{\theta^2\Gamma(\beta)}(((\beta-1)x^{\beta-2}-\theta x^{\beta-1})e^{-\theta x})=0$$

$$\frac{\theta^2}{\theta k+n}\left(\frac{k(\alpha-1)x^{\alpha-2}\theta^{\alpha-1}-k\theta^{\alpha}x^{\alpha-1}}{\Gamma(\alpha)}+\frac{n(\beta-1)x^{\beta-2}\theta^{\beta-2}-n\theta^{\beta-1}x^{\beta-1}}{\Gamma(\beta)}\right)e^{-\theta x}=0$$

$$\left(\frac{k(\alpha-1)x^{\alpha-2}\theta^{\alpha-1}}{\Gamma(\alpha)}+\frac{n(\beta-1)x^{\beta-2}\theta^{\beta-2}}{\Gamma(\beta)}\right)e^{-\theta x}=\left(\frac{kx^{\alpha}x^{\alpha-1}}{\Gamma(\alpha)}+\frac{n\theta^{\beta-1}x^{\beta-1}}{\Gamma(\beta)}\right)e^{-\theta x}$$

$$\frac{k\Gamma(\beta)(\alpha-1)x^{\alpha-2}\theta^{\alpha-1}+n\Gamma(\alpha)(\beta-1)x^{\beta-2}\theta^{\beta-2}}{\Gamma(\alpha)\Gamma(\beta)}=\frac{k\Gamma(\beta)x^{\alpha-1}\theta^{\alpha}+n\Gamma(\alpha)x^{\beta-1}\theta^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$$

$$k\Gamma(\beta)(\alpha-1)x^{\alpha-2}\theta^{\alpha-1}+n\Gamma(\alpha)(\beta-1)x^{\beta-2}\theta^{\beta-2}=k\Gamma(\beta)x^{\alpha-1}\theta^{\alpha}+n\Gamma(\alpha)x^{\beta-1}\theta^{\beta-1}$$

$$n\Gamma(\alpha)(\beta-1)x^{\beta-2}\theta^{\beta-2}=k\Gamma(\beta)x^{\alpha-2}\theta^{\alpha-1}(\theta x-(\alpha-1))+n\Gamma(\alpha)x^{\beta-1}\theta^{\beta-1}$$

$$n\Gamma(\alpha)x^{\beta-1}\theta^{\beta-1}+k\Gamma(\beta)x^{\alpha-2}\theta^{\alpha-1}(\theta x-\alpha+1))-n\Gamma(\alpha)(\beta-1)x^{\beta-2}\theta^{\beta-2}=0$$

$$x_0=\frac{2\left[n\Gamma(\alpha)(\beta-1)\theta^{\beta-2}-k\Gamma(\beta)(\theta-\alpha+1)\theta^{\alpha-1}\right]}{2n\Gamma(\alpha)\theta^{\beta-1}}$$

$$x_0=\frac{n\Gamma(\alpha)(\beta-1)\theta^{\beta-2}-k\Gamma(\beta)(\theta-\alpha+1)\theta^{\alpha-1}}{n\Gamma(\alpha)\theta^{\beta-1}}\qquad;x>0,\alpha,\beta,\theta>0,k,n\geq 0$$

4 EXCESS LOSS AND EQUILIBRIUM DISTRIBUTION OF G5L.

4.1 Introduction.

In this chapter we shall construct pdf of Excess loss,Mean excess loss(MRL),Equilibrium distribution,Survival function and Hazard function of equilibrium of G5L.Thereafter,we prove the relationship between MRL and Hazard function of equilibrium distribution.

4.2 Excess loss function.

4.2.1 Probability distribution function.

Proposition 4.2.1. *The pdf of excess loss function is given as;*

$$g(x) = \frac{\left[\frac{k\theta^{\alpha-1}}{\Gamma(\alpha)}\left(\theta x^{\alpha-1}+(\alpha-1)x^{\alpha-2}\right)+\frac{n\theta^{\beta-2}}{\Gamma(\beta)}\left(\theta x^{\beta-1}+(\beta-1)x^{\beta-2}\right)\right]e^{-\theta x}}{n+\theta k-[\theta k\gamma_{\alpha}(\theta x)+n\gamma_{\beta}(\theta x)]} \tag{39}$$

,where $\theta,\alpha,\beta,k,n>0.$

Proof.
Excess loss function also known as residual lifetime is obtained by;
$g(x)=\frac{\int_x^\infty f(t)dt}{S(x)}$, where $\int_x^\infty f(t)dt$ is the infinite sums of excess random variables given that an item has survived beyond a given value within specified time and $S(x)$ is our survival function.

$$S(x) = \frac{n+\theta k-[\theta k\gamma_{\alpha}(\theta x)+n\gamma_{\beta}(\theta x)]}{\theta k+n}$$

$$\int_x^\infty f(t)dt = \int_x^\infty \frac{\theta^2}{\theta k+n}\left[\frac{k(\theta t)^{\alpha-1}}{\Gamma(\alpha)}+\frac{n(\theta t)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta t}dt$$

$$g(x) = \frac{\frac{\theta^2}{n+\theta k}\int_x^\infty\left[\frac{k(\theta t)^{\alpha-1}}{\Gamma(\alpha)}+\frac{n(\theta t)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta t}dt}{1-\frac{1}{\theta k+n}\left[\theta k\gamma_{\alpha}(\theta x)+n\gamma_{\beta}(\theta x)\right]}$$

$$g(x) = \frac{\frac{\theta^2}{n+\theta k}\int_x^\infty\left[\frac{k(\theta t)^{\alpha-1}}{\Gamma(\alpha)}+\frac{n(\theta t)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta t}dt}{\frac{n+\theta k-[\theta k\gamma_{\alpha}(\theta x)+n\gamma_{\beta}(\theta x)]}{\theta k+n}}$$

$$g(x) = \frac{\theta^2\int_x^\infty\left[\frac{k(\theta t)^{\alpha-1}}{\Gamma(\alpha)}+\frac{n(\theta t)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta t}dt}{n+\theta k-[\theta k\gamma_{\alpha}(\theta x)+n\gamma_{\beta}(\theta x)]}$$

$$\text{Consider the integral;}\quad \theta^2\int_x^\infty \frac{k(\theta t)^{\alpha-1}}{\Gamma(\alpha)}e^{-\theta t}dt$$

$$\begin{aligned}
&= \frac{k\theta^{\alpha+1}}{\Gamma(\alpha)}\int_x^\infty t^{\alpha-1}e^{-\theta t}dt \\
&= \frac{k\theta^{\alpha+1}}{\Gamma(\alpha)}\left(\frac{\theta x^{\alpha-1}+(\alpha-1)x^{\alpha-2}}{\theta^2}\right)e^{-\theta x} \\
&= \frac{k\theta^{\alpha-1}}{\Gamma(\alpha)}\left(\theta x^{\alpha-1}+(\alpha-1)x^{\alpha-2}\right)e^{-\theta x} \\
&= \frac{n\theta^{\beta}}{\Gamma(\beta)}\int_x^\infty t^{\beta-1}e^{-\theta t}dt \\
&= \frac{n\theta^{\beta}}{\Gamma(\beta)}\left(\frac{\theta x^{\beta-1}+(\beta-1)x^{\beta-2}}{\theta^2}\right)e^{-\theta x} \\
&= \frac{n\theta^{\beta-2}}{\Gamma(\beta)}\left(\theta x^{\beta-1}+(\beta-1)x^{\beta-2}\right)e^{-\theta x}
\end{aligned}$$

$$g(x) = \frac{\frac{k\theta^{\alpha-1}}{\Gamma(\alpha)}\left(\theta x^{\alpha-1}+(\alpha-1)x^{\alpha-2}\right)e^{-\theta x}+\frac{n\theta^{\beta-2}}{\Gamma(\beta)}\left(\theta x^{\beta-1}+(\beta-1)x^{\beta-2}\right)e^{-\theta x}}{n+\theta k-[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]}$$

$$g(x) = \frac{\left[\frac{k\theta^{\alpha-1}}{\Gamma(\alpha)}\left(\theta x^{\alpha-1}+(\alpha-1)x^{\alpha-2}\right)+\frac{n\theta^{\beta-2}}{\Gamma(\beta)}\left(\theta x^{\beta-1}+(\beta-1)x^{\beta-2}\right)\right]e^{-\theta x}}{n+\theta k-[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]}$$

4.2.2 Mean Residual lifetime.

The Mean Residual Lifetime of G5L is stated as;

Proposition 4.2.2.

$$m(x) = \frac{\alpha\theta k\Gamma(\alpha+1)(\theta x)+\beta n\Gamma(\beta+1)(\theta x)}{\theta(n+\theta k-[\theta k\gamma_\alpha(\theta\lambda)+n\gamma_\beta(\theta x)])}-x \tag{40}$$

,where $\theta,\alpha,\beta,k,n>0.$

Proof

$$m(x) = \frac{\int_x^\infty t f(t)}{R(x)} dt - xa \qquad ; x > 0$$

$$m(x) = \frac{\frac{\theta^2}{n+\theta k} \int_x^\infty t \left[\frac{k(\theta t)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n(\theta t)^{\beta-1}}{\theta\Gamma(\beta)} \right] e^{-\theta x} dt}{\frac{n+\theta k - [\theta k \gamma_\alpha(\theta x) + n\gamma_\beta(\theta x)]}{\theta k + n}} - x$$

$$m(x) = \frac{\theta^2 \int_x^\infty t \left[\frac{k(\theta t)^{\alpha-1}}{\Gamma(\alpha)} + \frac{n(\theta t)^{\beta-1}}{\theta\Gamma(\beta)} \right] e^{-\theta x} dt}{n+\theta k - [\theta k \gamma_\alpha(\theta x) + n\gamma_\beta(\theta x)]} - x$$

$$m(x) = \frac{\theta k \int_x^\infty \frac{\theta \theta^{\alpha-1} t^\alpha e^{-\theta t}}{\Gamma(\alpha)} dt + \frac{n\theta \int_x^\infty \theta^{\beta-1} t^\beta}{\Gamma(\beta)} e^{-\theta t} dt}{n+\theta k - [\theta k \gamma_\alpha(\theta x) + n\gamma_\beta(\theta \lambda)]} - x$$

$$m(x) = \frac{\alpha\theta^2 k \Gamma(\alpha+1)(\theta x) + \beta n \theta \Gamma(\beta+1)(\theta x)}{(n+\theta k - [\theta k \gamma_\alpha(\theta x) + n\gamma_\beta(\theta x)])} - x$$

$$m(x) = \frac{\alpha\theta k \Gamma(\alpha+1)(\theta x) + \beta n \Gamma(\beta+1)(\theta x)}{\theta(n+\theta k - [\theta k \gamma_\alpha(\theta \lambda) + n\gamma_\beta(\theta x)])} - x$$

MRL satisfies the following conditions;

1. $m(x) \geq 0$
2. $\frac{d}{dx} m(x) \geq$-1
3. $\int_0^\infty [m(x)]^{-1} dx = \infty$
4. $m(0) = \mu_1' = \frac{\alpha\theta k + \beta n}{\theta(n+\theta k)}$

4.3 Equilibrium distribution of G5L.

The equilibrium distribution $f_e(x)$,Survival function of equilibrium distribution $S_e(x)$ and Hazard function of equilibrium distribution $h_e(x)$ is given as;

Proposition 4.3.1.

$$f_e(x) = \frac{n+\theta k - [\theta k \gamma_\alpha(\theta x) + n\gamma_\beta(\theta x)]}{\alpha\theta k + \beta n} \tag{41}$$

$$S_e(x) = \frac{\alpha\theta^2 k \Gamma(\alpha+1)(\theta x) + \theta n \beta \Gamma(\beta+1)(\theta x)}{\alpha\theta k + \beta n} - x \tag{42}$$

$$h_e(x) = \frac{n+\theta k - [\theta k \gamma_\alpha(\theta x) + n\gamma_\beta(\theta x)]}{\alpha\theta^2 k \Gamma(\alpha+1)(\theta x) + \theta n \beta \Gamma(\beta+1)(\theta x) - x} \tag{43}$$

,where $\theta, \alpha, \beta, k, n > 0$.

Proofs.

$$f_e(x) = \frac{S(x)}{E(X)} = \frac{\frac{(n+\theta k-[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)])}{n+\theta k}}{\frac{\alpha\theta k+\beta n}{\theta(\theta k+n)}}$$

$$\text{,thus } f_e(x) = \frac{n+\theta k-[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]}{\alpha\theta k+\beta n}$$

$$S_e(x) = \frac{\int_x^\infty S(t)dt}{E(X)}$$

$$\begin{aligned}
\text{but, } \int_x^\infty S(t;\theta,\alpha,\beta,k,n)dt &= \int_x^\infty 1dt - \left[\frac{1}{\theta k+n}(\theta k\gamma_\alpha(\theta t)+n\gamma_\beta(\theta t))\right]dt \\
&= \int_x^\infty 1dt - \int_x^\infty \left[\frac{1}{\theta k+n}(\theta k\gamma_\alpha(\theta t)+n\gamma_\beta(\theta t))\right]dt \\
&= -x + \int_x^\infty \frac{\theta k}{\theta k+n} - \gamma_\alpha(\theta t)dt + \int_x^\infty \frac{n}{\theta k+n} - \gamma_\beta(\theta t)dt \\
&= -x + \frac{\theta k}{\theta k+n}\alpha\Gamma(\alpha+1)(\theta x) + \frac{n}{\theta k+n}\beta\Gamma(\beta+1)(\theta x) \\
&= -x + \frac{1}{\theta k+n}[\alpha\theta k\Gamma(\alpha+1)(\theta x)+n\beta\Gamma(\beta+1)(\theta x)]
\end{aligned}$$

$$\text{now, } S_e(x) = \frac{\frac{1}{\theta k+n}[\alpha\theta k\Gamma(\alpha+1)(\theta x)+n\beta\Gamma(\beta+1)(\theta x)]}{\frac{\alpha\theta k+\beta n}{\theta(\theta k+n)}} - x$$

$$S_e(x) = \frac{\theta[\alpha\theta k\Gamma(\alpha+1)(\theta x)+n\beta\Gamma(\beta+1)(\theta x)]}{\alpha\theta k+\beta n} - x$$

$$S_e(x) = \frac{\alpha\theta^2 k\Gamma(\alpha+1)(\theta x)+\theta n\beta\Gamma(\beta+1)(\theta x)}{\alpha\theta k+\beta n} - x$$

$$\text{,and } h_e(x) = \frac{f_e(x)}{S_e(x)} = \frac{\frac{n+\theta k-[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]}{\alpha\theta k+\beta n}}{\frac{\alpha\theta^2 k\Gamma(\alpha+1)(\theta x)+\theta n\beta\Gamma(\beta+1)(\theta x)}{\alpha\theta k+\beta n} - x}$$

$$h_e(x) = \frac{n+\theta k-[\theta k\gamma_\alpha(\theta x)+n\gamma_\beta(\theta x)]}{\alpha\theta^2 k\Gamma(\alpha+1)(\theta x)+\theta n\beta\Gamma(\beta+1)(\theta x)-x}$$

4.3.1 Relationship between equilibrium of Survival function and equilibrium of Hazard function.

$S_e(t) = 1 - F_e(t)$,where $F_e(t)$ represents the Cdf of equilibrium distribution.

$$\frac{d}{dt}S_e(t) = -f_e(t)$$
$$-\frac{d}{dt}S_e(t) = f_e(t)$$
$$h_e(t) = \frac{f_e(t)}{S_e(t)} = \frac{1}{S_e(t)} * -\frac{d}{dt}S_e(t)$$
$$h_e(t) = \frac{f_e(t)}{S_e(t)} = -\frac{1}{S_e(t)}\frac{d}{dt}S_e(t)$$

work reversely by introducing logarithm; $\Rightarrow h_e(t) = -\frac{d}{dt}\log S_e(t)$

$$h_e(t)dt = -d\log S_e(t)$$

Integrate both sides of equation; $\int_0^x h_e(t)dt = -\int_0^x d\log S_e(t)dt$

$$\int_0^x h_e(t)dt = -\int_0^x \log S_e(t)dt$$
$$\int_0^x h_e(t)dt = -[\log S_e(t) - \log S_e(0)]$$
$$\int_0^x h_e(t)dt = -[\log S_e(t) - \log 1]$$
$$\int_0^x h_e(t)dt = -\log S_e(t)$$
$$-\int_0^x h_e(t)dt = \log S_e(t)$$

Introducing exponent on both sides of equation; $e^{-\int_0^x h_e(t)dt} = e^{\log S_e(t)}$

$$e^{-\int_0^x h_e(t)dt} = S_e(t)$$
$$\Rightarrow S_e(t) = e^{-\int_0^x h_e(t)dt}$$

similarly; $\Rightarrow S_e(t) = S_e(0)e^{-\int_0^x h_e(t)dt}$

4.3.2 Relationship between equilibrium of Survival function and equilibrium of Mean Residual Lifetime.

$$\Rightarrow S_e(t) = e^{-\int_0^x h_e(t)dt}$$

Expectation of equilibrium distribution=Equilibrium of Mean Residual Lifetime.
i.e.$E_e(X) = (m_e(x))$

$$m_e(x) = \int_x^{\infty} \frac{1 - F_e(y)}{1 - F_e(x)} dy$$

when $x > 0$; equilibrium mean residual lifetime $m_e(0)$ becomes;

$$m_e(0) = \int_0^{\infty} [1 - F_e(y)]dy = \int_0^{\infty} S_e(y)dy \Rightarrow \int_0^{\infty} S_e(x)dx = E_e(x)$$

Therefore;

$$\frac{S_e(x)}{E_e(X)} = \frac{e^{-\int_0^x h_e(t)dt}}{m_e(x)}$$

but from $h_e(t) = \frac{f_e(t)}{S_e(t)}$ and also $h_e(t) = \frac{1}{m_e(t)}$

$$S_e(x) = \frac{E_e(X)}{m_e(x)} e^{-\int_0^x \frac{dt}{m_e(t)}}$$

$$S_e(x) = \frac{m_e(0)}{m_e(x)} e^{-\int_0^x \frac{dt}{m_e(t)}} \tag{44}$$

5 ESTIMATION AND APPLICATION.

5.1 Introduction.

G5L has five parameters to be estimated; $(\theta,\alpha,\beta,k,n)$.In this chapter we shall estimate these parameters by MoM and MLE methods.

5.1.1 Estimation.

Method of moment.

With reference to the first two moments about origin we have;

$$p=\frac{\mu_2'}{\mu_1'^2}=\frac{\alpha\theta k(\alpha+1)+\beta n(\beta+1)}{\theta^2(n+\theta k)}*\left(\frac{\theta(n+\theta k)}{\alpha\theta k+\beta n}\right)^2$$

$$p=\frac{[\alpha\theta k(\alpha+1)+\beta n(\beta+1)](\theta k+n)}{(\alpha\theta k+\beta n)^2}$$

$$\Rightarrow \alpha\theta k(\alpha+1)+\beta n(\beta+1)](\theta k+n)=p(\alpha\theta k+\beta n)^2$$

$$\textbf{hence, } \alpha(\alpha+1)n(\theta k)+\alpha(\alpha+1)(\theta k)^2+\beta n^2(\beta+1)+\beta n(\beta+1)\theta k=$$

$$=p\alpha^2(\theta k)^2+2p\alpha\theta k\beta n+p(\beta n)^2$$

$$\left\{\alpha^2+\alpha-p\alpha^2\right\}(\theta k)^2+\left\{(\alpha^2+\alpha)n+\beta n(\beta+1)-2p\alpha\beta n\right\}\theta k+\beta n^2((\beta+1)-p\beta)=0$$

let $\theta k=b$

$$\left\{\alpha^2+\alpha-p\alpha^2\right\}b^2+\left\{(\alpha^2+\alpha)n+\beta n(\beta+1\right\}-2p\alpha\beta n)b+\beta n^2((\beta+1)-p\beta)=0$$

The value of b is obtained by solving the quadratic equation.
Replace population mean with sample mean and parameters with their estimates;

$$\mu_1' = \frac{\alpha\theta k + \beta n}{\theta(\theta k + n)}$$
$$\bar{x} = \frac{\hat{\alpha}\hat{\theta}\hat{k} + \hat{\beta}\hat{n}}{\hat{\theta}(\hat{\theta}\hat{k} + \hat{n})}$$
$$\bar{x} = \frac{\hat{\theta}\hat{k}}{\hat{\theta}\hat{k} + \hat{n}}\bar{x_1} + \frac{\hat{n}}{\hat{\theta}\hat{k} + \hat{n}}\bar{x_2}$$

where,
$\bar{x}$ is sample mean of G5L.
$\bar{x_1}$ is sample mean of Gamma(α,θ).
$\bar{x_2}$ is sample mean of Gamma(β,θ).

$$\bar{x} = \frac{\hat{\theta}\hat{k}}{\hat{\theta}\hat{k} + \hat{n}}\left(\frac{\hat{\alpha}}{\hat{\theta}}\right) + \frac{\hat{n}}{\hat{\theta}\hat{k} + \hat{n}}\left(\frac{\hat{\beta}}{\hat{\theta}}\right)$$
$$\Rightarrow \frac{\hat{\alpha}\hat{\theta}\hat{k}}{\hat{\theta}} + \frac{\hat{n}\hat{\beta}}{\hat{\theta}} = \hat{\theta}\hat{k}\bar{x} + \hat{n}\bar{x}$$
$$\hat{n}\left(\frac{\hat{\beta}}{\hat{\theta}} - \bar{x}\right) = \left(\bar{x} - \frac{\hat{\alpha}}{\hat{\theta}}\right)\hat{\theta}\hat{k}$$

since $b = \hat{\theta}\hat{k}$, then

$$\hat{n} = \left(\frac{\hat{\theta}\bar{x} - \hat{\alpha}}{\hat{\beta} - \hat{\theta}\bar{x}}\right)\hat{\theta}\hat{k}$$
$$\hat{n} = \left(\frac{\hat{\theta}\bar{x} - \hat{\alpha}}{\hat{\beta} - \hat{\theta}\bar{x}}\right)b$$

similarly, $\hat{k} = \dfrac{\hat{n}(\hat{\beta} - \hat{\theta}\bar{x})}{\hat{\theta}(\hat{\theta}\bar{x} - \hat{\alpha})}$

$$\hat{\beta} = \frac{(\hat{\theta}\bar{x} - \hat{\alpha})b + \hat{n}\hat{\theta}\bar{x}}{\hat{n}}$$
$$\hat{\theta} = \frac{b\hat{\theta}(\hat{\theta}\bar{x} - \hat{\alpha})}{\hat{n}(\hat{\beta} - \hat{\theta}\bar{x})}$$
$$\hat{\alpha} = \frac{\bar{x}\hat{\theta}(\hat{n} + b) - \hat{n}\hat{\beta}}{b}$$

Alternatively,

$$\bar{x} = \frac{\hat{\alpha}b + \hat{n}\hat{\beta}}{\hat{\theta}(\hat{n}+b)} \tag{45}$$

$$\hat{\theta} = \left(\frac{\hat{\alpha}b + \hat{n}\hat{\beta}}{\hat{n}+b}\right) * \frac{1}{\bar{x}} \tag{46}$$

$$\hat{k} = \frac{b}{\hat{\theta}} = \frac{b\bar{x}(\hat{n}+b)}{\hat{\alpha}b + \hat{n}\hat{\beta}} \tag{47}$$

$$\hat{\alpha} = \frac{b\hat{n}\bar{x} + b^2\bar{x} - \hat{n}\hat{\beta}\hat{k}}{\hat{k}b} \tag{48}$$

$$\hat{n} = \frac{\bar{x}\hat{\theta}(\hat{n}+b) - \hat{\alpha}b}{\hat{\beta}} \tag{49}$$

$$\hat{\beta} = \frac{\bar{x}\hat{\theta}(\hat{n}+b) - \hat{\alpha}b}{\hat{n}} \tag{50}$$

Maximum Likelihood Method.

In this section,we consider the method of likelihood to estimate five parameters and use them to come up with confidence intervals for the unknown parameters.Let $X_1, X_2 \cdots, X_m$ be of a sample size m from FPLD.The Likelihood function is given as;

$$\text{Likelihood function; } l = \prod_{j=1}^{m} f(\theta, \alpha, \beta, k, n/x_j)$$

$$= \left(\frac{1}{\theta k + n}\right)^m \prod_{j=1}^{m} \left[\frac{k\theta^{\alpha+1}x_j^{\alpha-1}}{\Gamma(\alpha)} + \frac{n\theta^{\beta}x_j^{\beta-1}}{\Gamma(\beta)}\right] e^{-\theta x_j}$$

$$= \left(\frac{1}{\theta k + n}\right)^m e^{-\theta\sum_{j=1}^{m}x_j} [(\Gamma(\alpha)\Gamma(\beta)]^{-m} \prod_{j=1}^{m} \left[k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right]$$

The log-likelihood function;$L = lnl$

$$\text{hence, } L = -mln(\theta k + n) - \theta\sum_{j=1}^{m} x_j - mln\Gamma(\alpha) - mln\Gamma(\beta)$$

$$+ \sum_{j=1}^{m} [k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}]$$

$$\frac{dL}{d\theta} = -\frac{mk}{n+\theta k} - \sum_{j=1}^{m} x_j + \sum_{j=1}^{m}\left[\frac{k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1} + n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right] = 0 \tag{51}$$

now we divide each term by sample size,m and solve as follows;

$$\frac{1}{m}\sum_{j=1}^{m}x_j = -\frac{k}{n+\theta k}+\frac{1}{m}\sum_{j=1}^{m}\left[\frac{k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]$$

$$\bar{x} = \frac{1}{m}\sum_{j=1}^{m}\left[\frac{k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]-\frac{k}{n+\theta k}$$

where$\bar{x} = \frac{\alpha\theta k+n\beta}{\theta(\theta k+n)}$.

$$n+\theta k = \frac{mk}{\left[\frac{k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]+m\bar{x}}$$

$$\hat{\theta} = \frac{1}{k}\left[\frac{mk}{\left[\frac{k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]+m\bar{x}}-n\right]$$

and with reference to method of moments, a quadratic equation was obtained where $b=\theta k \Rightarrow \theta = \frac{b}{k}$.

$$\hat{\theta} = \frac{1}{k}\left[\frac{mk}{\left[\frac{k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]+m\bar{x}}-n\right]$$

$$\Rightarrow b = \frac{mk}{\left[\frac{k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]+m\bar{x}}-n$$

hence, $E(\hat{\theta}) = E\left(\frac{\theta k}{k}\right) = \theta$,therefore $\hat{\theta}$ is **Unbiased Estimator** of θ.

$$\frac{dL}{d\alpha} = \frac{-m\ln\Gamma'(\alpha)}{\ln\Gamma(\alpha)}+\sum_{j=1}^{m}\left[\frac{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\log(\theta x_j)+n\Gamma'(\alpha)\theta^{\beta}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma'(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]=0 \quad (52)$$

$$= -m\psi(\alpha)+\sum_{j=1}^{m}\left[\frac{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\log(\theta x_j)+n\Gamma'(\alpha)\theta^{\beta}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]=0$$

$$\frac{dL}{d\beta} = -m\psi(\beta)+\sum_{j=1}^{m}\left[\frac{k\Gamma'(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\log(\theta x_j)}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]=0 \quad (53)$$

$$\frac{dL}{dk} = \frac{-m\theta}{n+\theta k} + \sum_{j=1}^{m} \left[\frac{\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}} \right] = 0 \tag{54}$$

$$\frac{dL}{dn} = \frac{-m}{n+\theta k} + \sum_{j=1}^{m} \left[\frac{\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}} \right] = 0 \tag{55}$$

$$\psi(a) = \frac{d}{da} ln\Gamma(a) = \frac{\Gamma'(a)}{\Gamma(a)}$$, $\psi(a)$ is a dia-gamma function.

The above five equations are solved using non-linear optimization algorithms such as quasi-Newton algorithms to numerically maximize log-likelihood functions.

To estimate parameter intervals and test the Hypothesis we require Fisher Information matrix (I) to solve the above five equations iteratively. The standard errors and asymptotic confidence intervals are obtained by use of large sample approximation ($n \to \infty$) ,where ML estimates are taken to approximate tri-variate normal with zero means and variance-co variance matrix as I^{-1}.

Fisher Information matrix based on a single observation is stated as;

$$I(\theta) = \left[I_{ij}(\theta)\right] = E\left[\frac{-\partial^2}{\partial\theta_i \partial\theta_j} \ln f(X,\theta)\right].$$

Therefore,

$$I = E \begin{pmatrix} \frac{-\partial^2 L}{\partial\theta^2} & \frac{-\partial^2 L}{\partial\theta\partial\alpha} & \frac{-\partial^2 L}{\partial\theta\partial\beta} & \frac{-\partial^2 L}{\partial\theta\partial k} & \frac{\partial^2 L}{\partial\theta\partial n} \\ \frac{-\partial^2 L}{\partial\theta\partial\alpha} & \frac{-\partial^2 L}{\partial\alpha^2} & \frac{-\partial^2 L}{\partial\alpha\partial\beta} & \frac{-\partial^2 L}{\partial\alpha\partial k} & \frac{\partial^2 L}{\partial\alpha\partial n} \\ \frac{-\partial^2 L}{\partial\theta\partial\beta} & \frac{-\partial^2 L}{\partial\alpha\partial\beta} & \frac{-\partial^2 L}{\partial\beta^2} & \frac{-\partial^2 L}{\partial\beta\partial k} & \frac{\partial^2 L}{\partial\beta\partial n} \\ \frac{-\partial^2 L}{\partial\theta\partial k} & \frac{-\partial^2 L}{\partial\alpha\partial k} & \frac{-\partial^2 L}{\partial\beta\partial k} & \frac{-\partial^2 L}{\partial k^2} & \frac{-\partial^2 L}{\partial k\partial n} \\ \frac{-\partial^2 L}{\partial\theta\partial n} & \frac{-\partial^2 L}{\partial\alpha\partial n} & \frac{-\partial^2 L}{\partial\beta\partial n} & \frac{-\partial^2 L}{\partial n\partial k} & \frac{-\partial^2 L}{\partial n^2} \end{pmatrix}$$

Hence as $m \to \infty$,the asymptotic distribution of the MLE is given by,

$$\begin{bmatrix} \hat{\theta} \\ \hat{\alpha} \\ \hat{\beta} \\ \hat{k} \\ \hat{n} \end{bmatrix} \approx N \left[\begin{bmatrix} \theta \\ \alpha \\ \beta \\ k \\ n \end{bmatrix} \begin{pmatrix} \hat{I_{11}} & \hat{I_{12}} & \cdots & \hat{I_{15}} \\ \hat{I_{21}} & \hat{I_{22}} & \cdots & \hat{I_{25}} \\ \hat{I_{31}} & \hat{I_{32}} & \cdots & \hat{I_{35}} \\ \hat{I_{41}} & \hat{I_{42}} & \cdots & \hat{I_{45}} \\ \hat{I_{51}} & \hat{I_{52}} & \cdots & \hat{I_{55}} \end{pmatrix} \right] \text{,where } (\hat{I}_{ij} = I_{ij}/\theta = \hat{\theta})$$

and also, $$\begin{pmatrix} I_{11} & I_{12} & \cdots & I_{15} \\ I_{21} & I_{22} & \cdots & I_{25} \\ I_{31} & I_{32} & \cdots & I_{35} \\ I_{41} & I_{42} & \cdots & I_{45} \\ I_{51} & I_{52} & \cdots & I_{55} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 L}{\partial\theta^2} & \frac{\partial^2 L}{\partial\theta\partial\alpha} & \frac{\partial^2 L}{\partial\theta\partial\beta} & \frac{\partial^2 L}{\partial\theta\partial k} & \frac{\partial^2 L}{\partial\theta\partial n} \\ \frac{\partial^2 L}{\partial\theta\partial\alpha} & \frac{\partial^2 L}{\partial\alpha^2} & \frac{\partial^2 L}{\partial\alpha\partial\beta} & \frac{\partial^2 L}{\partial\alpha\partial k} & \frac{\partial^2 L}{\partial\alpha\partial n} \\ \frac{\partial^2 L}{\partial\theta\partial\beta} & \frac{\partial^2 L}{\partial\alpha\partial\beta} & \frac{\partial^2 L}{\partial\beta^2} & \frac{\partial^2 L}{\partial\beta\partial k} & \frac{\partial^2 L}{\partial\beta\partial n} \\ \frac{\partial^2 L}{\partial\theta\partial k} & \frac{\partial^2 L}{\partial\alpha\partial k} & \frac{\partial^2 L}{\partial\beta\partial k} & \frac{\partial^2 L}{\partial k^2} & \frac{\partial^2 L}{\partial k\partial n} \\ \frac{\partial^2 L}{\partial\theta\partial n} & \frac{\partial^2 L}{\partial\alpha\partial n} & \frac{\partial^2 L}{\partial\beta\partial n} & \frac{\partial^2 L}{\partial n\partial k} & \frac{\partial^2 L}{\partial n^2} \end{pmatrix}$$

where,

$$\frac{\partial^2 L}{\partial k^2} = \frac{\partial}{\partial k}\left(\frac{\partial L}{\partial k}\right)$$

$$\frac{\partial^2 L}{\partial k\partial n} = \frac{\partial}{\partial k}\left(\frac{\partial L}{\partial n}\right) = \frac{\partial}{\partial n}\left(\frac{\partial L}{\partial k}\right)$$

$$\text{with, } I^{-1} = -E\left[\begin{pmatrix} I_{11} & I_{12} & \cdots & I_{15} \\ I_{21} & I_{22} & \cdots & I_{25} \\ I_{31} & I_{32} & \cdots & I_{35} \\ I_{41} & I_{42} & \cdots & I_{45} \\ I_{51} & I_{52} & \cdots & I_{55} \end{pmatrix}\right]$$

$$I_{11} = \frac{\partial^2 L}{\partial\theta^2} = -\frac{mk^2}{(n+\theta k)^2} + \sum_{j=1}^{m}\left[\frac{\alpha(\alpha+1)k\Gamma(\beta)\theta^{\alpha-1}\lambda_j^{\alpha-1} + \beta(\beta-1)n\Gamma(\alpha)\theta^{\beta-2}\lambda_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}\lambda_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}\lambda_j^{\beta-1}}\right]$$
$$-\sum_{j=1}^{m}\left[\frac{\left((\alpha+1)k\Gamma(\beta)\theta^{\alpha-1}x_j^{\alpha-1} + n\beta\Gamma(\alpha)\theta^{\beta-2}x_j^{\beta-1}\right)^2}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$\frac{\partial^2 L}{\partial\theta^2} = -\frac{m}{(n+\theta k)^2} + \sum_{j=1}^{m}\left[\frac{\alpha(\alpha+1)k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + \beta(\beta-1)n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}{\theta^2\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}\right]$$

$$-\sum_{j=1}^{m}\left[\frac{\left((\alpha+1)k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\beta\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}{\theta^2\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \quad \textbf{(56)}$$

$$I_{22} = \frac{\partial^2 L}{\partial\alpha^2} = -m\psi'(\alpha) + \sum_{j=1}^{m}(M_j + N_j),\text{where} \quad \textbf{(57)}$$

$$M_j = \frac{k\theta^{\alpha+1}\Gamma(\beta)(\ln\theta)^2 x_j^{\alpha-1} + 2k\theta^{\alpha+1}\Gamma(\beta)\ln(\theta)x_j^{\alpha-1}\ln(x_j) + k\Gamma(\beta)\theta^{\alpha+1}(\ln x_j)^2 x_j^{\alpha-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}$$
$$+ \frac{n\psi'(\alpha)\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1} + n(\psi'(\alpha))^2\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}$$
$$N_j = \frac{\left(k\theta^{\alpha+1}\Gamma(\beta)(\ln\theta)x_j^{\alpha-1} + k\theta^{\alpha+1}\Gamma(\beta)x_j^{\alpha-1}\ln(x_j) + n\Gamma(\alpha)\theta^{\beta}\psi(\alpha)x_j^{\beta-1}\right)^2}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}$$

$$I_{33} = \frac{\partial^2 L}{\partial\beta^2} = -m\psi'(\beta) + \sum_{j=1}^{m}\frac{k\psi'(\beta)\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n(\psi'(\beta))^2\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}$$
$$+ \frac{k\Gamma(\alpha)\theta^{\beta}(\ln\theta)^2 x_j^{\alpha-1} + 2n\Gamma(\alpha)\theta^{\beta}x_j^{\alpha-1}\ln(\lambda_j)\ln(\theta)) + n(\ln x_j)^2\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}$$
$$- \frac{\left(k\psi(\beta)\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\ln(\theta x_j)\right)^2}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2} \quad (58)$$

$$I_{44} = I_{kk} = \frac{\partial}{\partial k}\left(\frac{\partial L}{\partial k}\right) = \frac{\partial}{\partial k}\left(\frac{-m\theta}{\theta k + n} + \sum_{j=1}^{m}\frac{\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}}\right)$$

using the quotient rule, $\frac{u'v - v'u}{v^2}$ let $u = m\theta$ and $v = \theta k + n$ for the first term of equation to be partially differentiated in the second order.

$$\frac{\partial}{\partial k}\left(\frac{-m\theta}{\theta k + n}\right) = -\frac{\partial}{\partial k}\left(\frac{m\theta}{\theta k + n}\right) = -\left(\frac{0(\theta k + n) - \theta(m\theta)}{(\theta k + n)^2}\right) = \frac{m\theta^2}{(\theta k + n)^2}$$
$$\frac{\partial}{\partial k}\left(\sum_{j=1}^{m}\left[\frac{\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]\right)$$
$$= \sum_{j=1}^{m}\left[\frac{0(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}) - \left(\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\right)^2}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$
$$= \sum_{j=1}^{m}\left[\frac{-\left(\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\right)^2}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$I_{44} = \frac{m\theta^2}{(\theta k+n)^2} - \sum_{j=1}^{m}\left[\frac{\left(\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\right)^2}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \quad (59)$$

$$I_{55} = I_{nn} = \frac{\partial}{\partial n}\left(\frac{\partial L}{\partial n}\right) = \frac{\partial}{\partial n}\left(\frac{-m}{\theta k+n} + \sum_{j=1}^{m}\frac{\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}}\right)$$

$$\frac{\partial}{\partial n}\left(\frac{-m}{\theta k+n}\right) = -\frac{\partial}{\partial n}\left(\frac{m}{\theta k+n}\right) = -\left(\frac{0(\theta k+n)-(m)}{(\theta k+n)^2}\right) = \frac{m}{(\theta k+n)^2}$$

$$\frac{\partial}{\partial n}\left(\sum_{j=1}^{m}\left[\frac{\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]\right)$$

$$= \sum_{j=1}^{m}\left[\frac{0(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}) - \left(\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)\left(\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$= \sum_{j=1}^{m}\left[\frac{-\left(\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$I_{55} = \frac{m}{(\theta k+n)^2} - \sum_{j=1}^{m}\left[\frac{\left(\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \quad (60)$$

$$I_{12} = I_{\theta\alpha} = \sum_{j=1}^{m}\left[\frac{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}(P_j+Q_j)}{\theta(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1})^2}\right] \quad (61)$$

where,

$$P_j = n\ln(\theta)\alpha\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1} + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\ln(\theta) + k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}$$

and

$$Q_j = n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1} + n\alpha\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\ln(x_j) + n\ln(x_j)\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1} + \beta\psi(\alpha)\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}$$
$$- n\theta^{\beta}\beta\Gamma(\alpha)\ln(\theta)x_j^{\beta-1} - n\beta\Gamma(\alpha)\ln(x_j\theta^{\beta}x_j^{\beta-1} - n\theta^{\beta}\psi(\alpha)\alpha\Gamma(\alpha)x_j^{\beta-1} - n\psi(\alpha)\Gamma(\alpha)\theta^{\beta}x_j.$$

$$I_{13} = \sum_{j=1}^{m}\frac{k\psi(\beta)\Gamma(\beta)\theta^{\alpha+1}(\alpha+1)x_j^{\alpha-1} + n\Gamma(\alpha)\beta\theta^{\beta}x_j^{\beta-1}\ln(\theta x_j) + n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}{\theta(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1})}$$
$$-\sum_{j=1}^{m}\frac{\left(k(\alpha+1)\theta^{\alpha+1}\Gamma(\beta)x_j^{\alpha-1}+n\theta^{\beta}\beta\Gamma(\alpha)x_j^{\beta-1}\right)\left(k\theta^{\alpha+1}\psi(\beta)\Gamma(\beta)x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)\ln(\theta x_j)x_j^{\beta-1}\right)}{\theta\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2} \quad (62)$$

$$I_{23}=\sum_{j=1}^{m}\frac{k\psi(\beta)\Gamma(\beta)\theta^{\alpha+1}\ln(\theta x_j)x_j^{\alpha-1}+n\theta^{\beta}\psi(\alpha)\Gamma(\alpha)x_j^{\beta-1}\ln(\theta x_j)}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}}$$

$$-\sum_{j=1}^{m}\frac{\left(k\theta^{\alpha+1}\Gamma(\beta)x_j^{\alpha-1}\ln(\theta x_j+n\theta^{\beta}\psi(\alpha)\Gamma(\alpha)x_j^{\beta-1}\right)\left(k\theta^{\alpha+1}\psi(\beta)\Gamma(\beta)x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)\ln(\theta x_j)x_j^{\beta-1}\right)}{\left(k\theta^{\alpha+1}\Gamma(\beta)x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}\right)^2} \quad \textbf{(63)}$$

$$I_{14}=I_{\theta k}=\frac{\partial}{\partial k}\left(\frac{\partial L}{\partial\theta}\right)$$

$$=\frac{\partial}{\partial k}\left(\frac{-mk}{\theta k+n}-\sum_{j=1}^{m}x_j+\sum_{j=1}^{m}\frac{k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}}\right)$$

$$-\frac{\partial}{\partial k}\left(\frac{mk}{\theta k+n}\right)=-\left(\frac{m(\theta k+n)-mk(\theta)}{(\theta k+n)^2}\right)=-\frac{mn}{(\theta k+n)^2}$$

$$\frac{\partial}{\partial k}\left(\sum_{j=1}^{m}\left[\frac{k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]\right)$$

$$=\sum_{j=1}^{m}\left[\frac{\left((\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}\right)\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$-\sum_{j=1}^{m}\left[\frac{\left(\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\right)\left(k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$I_{14}=-\frac{mn}{(\theta k+n)^2}+\sum_{j=1}^{m}\left[\frac{\left((\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}\right)\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$-\sum_{j=1}^{m}\left[\frac{\left(\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\right)\left(k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \quad \textbf{(64)}$$

$$I_{15}=I_{\theta n}=\frac{\partial}{\partial n}\left(\frac{\partial L}{\partial\theta}\right)$$

$$=\frac{\partial}{\partial n}\left(\frac{-mk}{\theta k+n}-\sum_{j=1}^{m}x_j+\sum_{j=1}^{m}\frac{k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}}\right)$$

$$-\frac{\partial}{\partial n}\left(\frac{mk}{\theta k+n}\right)=-\left(\frac{0(\theta k+n)-mk(1)}{(\theta k+n)^2}\right)=\frac{mk}{(\theta k+n)^2}$$

$$\frac{\partial}{\partial n}\left(\sum_{j=1}^{m}\left[\frac{k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]\right)$$

$$=\sum_{j=1}^{m}\left[\frac{\left(\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}\right)\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$-\sum_{j=1}^{m}\left[\frac{\left(\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)\left(k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$I_{15}=\frac{mk}{(\theta k+n)^2}+\sum_{j=1}^{m}\left[\frac{\left(\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}\right)\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$-\sum_{j=1}^{m}\left[\frac{\left(\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)\left(k(\alpha+1)\Gamma(\beta)\theta^{\alpha}x_j^{\alpha-1}+n\beta\Gamma(\alpha)\theta^{\beta-1}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \quad (65)$$

$$I_{45}=I_{kn}=\frac{\partial}{\partial n}\left(\frac{\partial L}{\partial k}\right)=\frac{\partial}{\partial n}\left(\frac{-m\theta}{\theta k+n}+\sum_{j=1}^{m}\frac{\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}}\right)$$

$$\frac{\partial}{\partial n}\left(\frac{-m\theta}{\theta k+n}\right)=-\frac{\partial}{\partial n}\left(\frac{m\theta}{\theta k+n}\right)=-\left(\frac{0(\theta k+n)-1(m\theta)}{(\theta k+n)^2}\right)=\frac{m\theta}{(\theta k+n)^2}$$

$$\frac{\partial}{\partial n}\left(\sum_{j=1}^{m}\left[\frac{\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}}\right]\right)$$

$$=\sum_{j=1}^{m}\left[\frac{0(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1})-\left(\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)\left(\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$=-\sum_{j=1}^{m}\left[\frac{\left(\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)\left(\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right]$$

$$I_{45} = \frac{m\theta}{(\theta k+n)^2} - \sum_{j=1}^{m}\left[\frac{\left(\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)\left(\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \tag{66}$$

$$I_{24} = \frac{\partial}{\partial k}\left(-m\psi(\alpha)+\sum_{j=1}^{m}\frac{k\theta^{\alpha+1}\Gamma(\beta)\ln(\theta x_j)x_j^{\alpha-1}+n\Gamma'(\alpha)\theta^{\beta}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}}\right)$$

$$\begin{aligned} I_{24} &= \sum_{j=1}^{m}\left[\frac{\left(\theta^{\alpha+1}\Gamma(\beta)\ln(\theta x_j)x_j\right)\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \\ &= -\sum_{j=1}^{m}\left[\frac{\left(\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\right)\left(k\Gamma(\beta)\theta^{\alpha+1}\ln(\theta x_j)x_j^{\alpha-1}+n\Gamma'(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \end{aligned} \tag{67}$$

$$I_{34} = \frac{\partial}{\partial k}\left(\frac{\partial L}{\partial \beta}\right) = \frac{\partial}{\partial k}\left(-m\psi(\beta)+\sum_{j=1}^{m}\frac{k\theta^{\alpha+1}\Gamma'(\beta)x_j^{\alpha-1}+n\theta^{\beta}x_j^{\beta-1}\Gamma(\alpha)\ln(\theta x_j)}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}}\right)$$

$$\begin{aligned} I_{34} &= \sum_{j=1}^{m}\left[\frac{\left(\theta^{\alpha+1}\Gamma'(\beta)x_j^{\alpha-1}\right)\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \\ &= -\sum_{j=1}^{m}\left[\frac{\left(\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}\right)\left(k\theta^{\alpha+1}\Gamma'(\beta)x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}\ln(\theta x_j)x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \end{aligned} \tag{68}$$

$$I_{25} = \frac{\partial}{\partial n}\left(-m\psi(\alpha)+\sum_{j=1}^{m}\frac{k\theta^{\alpha+1}\Gamma(\beta)\ln(\theta x_j)x_j^{\alpha-1}+n\Gamma'(\alpha)\theta^{\beta}x_j^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}}\right)$$

$$\begin{aligned} I_{25} &= \sum_{j=1}^{m}\left[\frac{\left(\theta^{\beta}\Gamma'(\alpha)x_j^{\beta-1}\right)\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \\ &= -\sum_{j=1}^{m}\left[\frac{\left(\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)\left(k\Gamma(\beta)\theta^{\alpha+1}\ln(\theta x_j)x_j^{\alpha-1}+n\Gamma'(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2}\right] \end{aligned} \tag{69}$$

$$I_{35} = \frac{\partial}{\partial n}\left(\frac{\partial L}{\partial \beta}\right) = \frac{\partial}{\partial n}\left(-m\psi(\beta)+\sum_{j=1}^{m}\frac{k\theta^{\alpha+1}\Gamma'(\beta)x_j^{\alpha-1}+n\theta^{\beta}x_j^{\beta-1}\Gamma(\alpha)\ln(\theta x_j)}{k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}}\right)$$

$$I_{35} = \sum_{j=1}^{m} \left[\frac{\left(\theta^{\beta}\Gamma(\alpha)x_j^{\beta-1}\ln(\theta x_j)\right)\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2} \right]$$

$$= -\sum_{j=1}^{m} \left[\frac{\left(\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)\left(k\theta^{\alpha+1}\Gamma'(\beta)x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}\ln(\theta x_j)x_j^{\beta-1}\right)}{\left(k\Gamma(\beta)\theta^{\alpha+1}x_j^{\alpha-1}+n\Gamma(\alpha)\theta^{\beta}x_j^{\beta-1}\right)^2} \right] \quad (70)$$

The solutions of inverse dispersion matrix will yield asymptotic variance-covariance of M.L.E parameters.
The $100(1-\aleph)\%$ confidence interval is obtained where $z_{\frac{\aleph}{2}}$ is the upper $\aleph^{th}$ percentile of standardized normal.

$$\hat{\theta} \pm z_{\frac{\aleph}{2}}\sqrt{\hat{I}_{11}}$$

$$\hat{\alpha} \pm z_{\frac{\aleph}{2}}\sqrt{\hat{I}_{22}}$$

$$\hat{\beta} \pm z_{\frac{\aleph}{2}}\sqrt{\hat{I}_{33}}$$

$$\hat{k} \pm z_{\frac{\aleph}{2}}\sqrt{\hat{I}_{44}}$$

$$\hat{n} \pm z_{\frac{\aleph}{2}}\sqrt{\hat{I}_{55}}$$

5.2 Application.

In this section we shall use a real data set to show that G5L distribution can be a better model than the Lindley distribution.The data set given in table 1 consist of uncensored data corresponding to the breaking stress of carbon fibers(in Gba) of random sample size 66 initially reported by Nichols and Padgett.

Table1:Uncensored data of Stress on Carbon fibers(in Gba).

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11	3.56
4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.90	1.57
2.67	2.93	3.22	3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85
1.25	4.38	1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03
1.89	2.88	2.82	2.05	3.65	3.75	2.43	2.95	2.97	3.39	2.96
2.35	2.55	2.59	2.03	1.61	2.12	3.15	1.08	2.56	1.80	2.53

Table2:Summary of data-set.

Units	Min	1st Qu	Median	Mean	3rd Qu	Max	Var
66	0.390	2.178	2.835	2.760	3.278	4.900	0.794

AIC,AICC,BIC and K-S Tests.

In order to compare the two distribution models,we shall consider criteria like:-2LL,AIC(Akaike Information Criterion),AICC(Corrected Akaike Information Criterion),BIC(Bayesian Information Criterion) and K-S(Kolmogorov-Smirnov test) for the data.The best distribution corresponds to *smallest* AIC,-2LL, AICC,BIC and K-S.AIC act as a guard against overfitting;the more the number of parameters you fit the more the penalty is imposed.

$$AIC = 2q - (-2LL) = 2q + 2LL$$

$$AICC = AIC + \frac{2q(q+1)}{m-q-1}$$

$$BIC = q\log(m) + 2LL$$

$$K-S = \sup_x |F_m(x) - F(x)|$$

where $F_m(x) = \frac{1}{m}\sum_{t=1}^{m} I_{x_{t \leq x}}$ is empirical distribution function,$F(x)$ is cumulative distribution function,q is the number of parameters in the statistical model,m is sample size and LL is the maximized value of the Log-likelihood function under considered model.

Table3:MLEs,measure of -LL and AIC under considered models based on our data.

Mode	Parameter Estimated	-LL	AIC
Lindley	$\hat{\theta} = 0.590188$	122.387	246.774
G2L(version 1)	$\hat{\theta} = 0.722999$ $\hat{n} = 98.555751$	112.1951	228.3902
G3L(version 2)	$\hat{\theta} = 2.788$ $\hat{\alpha} = 7.412$ $\hat{\beta} = 8.485$	91.107	188.215
G5L	$\hat{\theta} = 5.2241$ $\hat{\alpha} = 6.1834$ $\hat{\beta} = 15.2660$ $\hat{k} = 0.0000425$ $\hat{n} = 0.021$	85.864	181.393

Table4:MLEs,measure of -2LL, AIC,AICC,BIC and K-S of models based on data.

Mode	-2LL	AIC	AICC	BIC	K-S
Lindley	244.774	246.768	246.830	248.957	0.282
G3L(version 2)	182.214	188.215	188.602	194.784	0.117
G5L	171.728	181.393	182.393	192.341	0.061

Log-Likelihood Ratio Test (LR).

Using R-software we can compute the maximum unrestricted and restricted Log-likelihood functions to construct Likelihood ratio(LR) test statistics for testing and comparing sub-models with G5L.
For instance,we can use the LR test statistics to check if G5L for a sampled data set is statistically superior to Lindley distribution.

Let our LR test statistic for null (H_0) against alternative(H_1) hypothesis be;

$\varepsilon = 2(l(\hat{\phi};\lambda) - l(\hat{\phi}_0;\lambda))$,where $\hat{\phi}$ and $\hat{\phi}_0$ are MLEs under H_1 and H_0 respectively.

The ε is asymptotically (as $m \to \infty$) distributed as χ^2_p,where p is the length of parameter vector ϕ of interest.

We reject H_0 iff $\varepsilon > \chi^2_{p,\aleph}$,where $\chi^2_{p,\aleph}$ denotes the upper 100% quantile of the χ^2_p distribution.
The LR test statistic for testing the hypothesis:

1. $H_0 : \alpha = k = n = 1, \beta = 2$ (α, β, k, n are not significant in the model.)
   $H_1 : \alpha \neq 1, k \neq 1, n \neq 1, \beta \neq 2$ (α, β, k, n are significant in the model.)
2. Level of significance $\aleph = 0.05$ i.e. the 95^{th} percentile of χ_{q-1} variable.
3. Test statistic, $C = (-2LL_{G5L}) - (-2LL_{LD})$
   $\Rightarrow 244.774 - 171.728 = 73.04$.
4. Tabulated value,$T = \chi^2_{q-1,\aleph} = \chi^2_{5,0.05} = 9.487729$.
5. Result;$73.04 > 9.487729 \Rightarrow C > T$ at $\aleph$-level of significance.
6. Decision;We reject H_0 (Null hypothesis) and conclude that the additional parameters α, β, k and n are significant to the model.

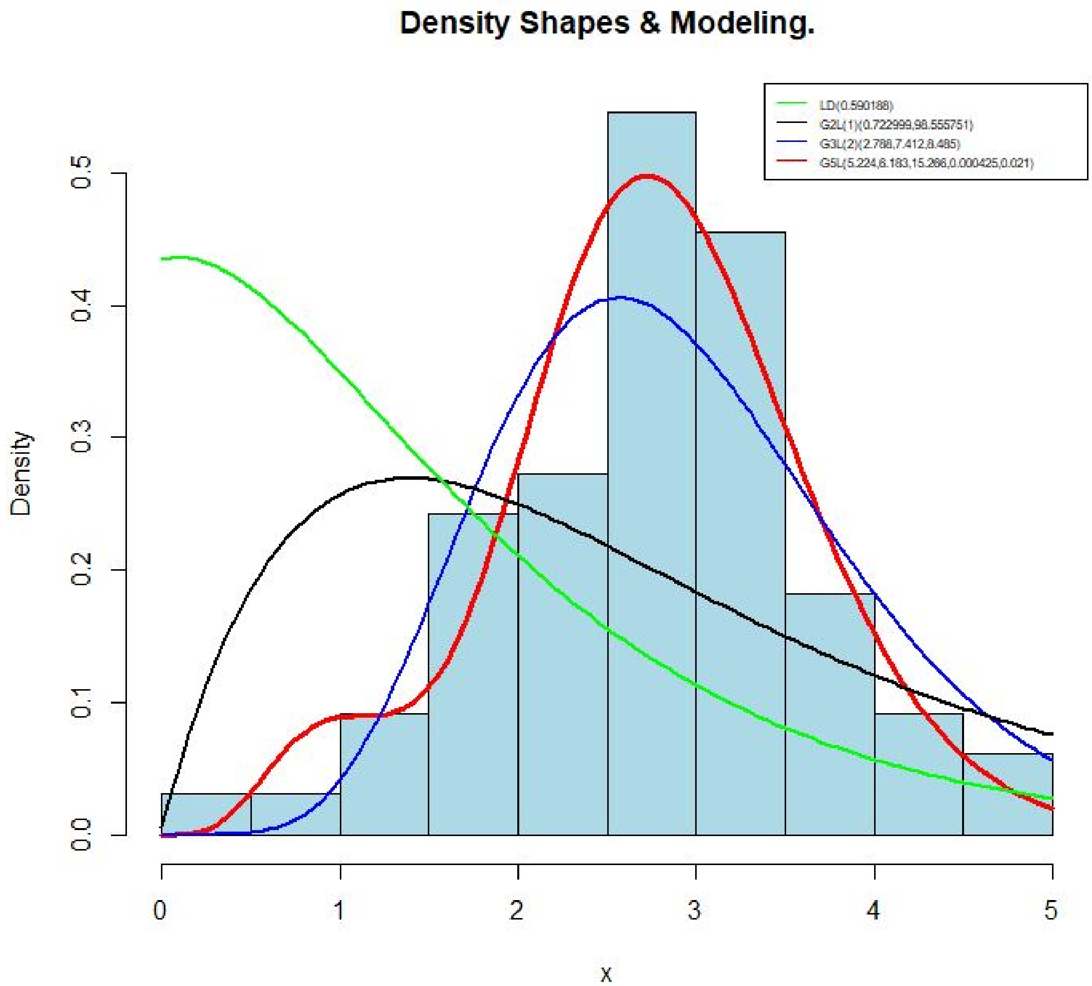


Figure 4. Density shapes fitted on Lifetime data.

Discussion of Results.

Results from table 4 ,proves that G5L is a strong competitor to LD in fitting our data. Fig 4. presents Pdf of G5L,G3L,G2L and LD with best parameter values and also distribution of our data in terms of Histogram.

The fitted density for G5L(**red**) model is closest to the empirical histogram (of our data) than the shapes of G3L (**blue curve**),G2L (**black**) and Lindley (**green**).G5L proves to be the most extensive and flexible model compared to other generalizations of Lindley.

6 CONCLUSION.

We have constructed G5L by finite mixture of two gamma distributions.It provided larger flexibility in modeling our lifetime data.Mathematical and Statistical properties studied included; pdf,cdf and hazard rate function.Other properties included;moments and moment generating functions.Parameters were estimated by method of moments,maximum likelihood and best shapes drawn(Fig:4).G5L could accommodate our data compared to G3L,G2L and Lindley.

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