University of Nairobi

School of Mathematics

ISSN: 2410-1397

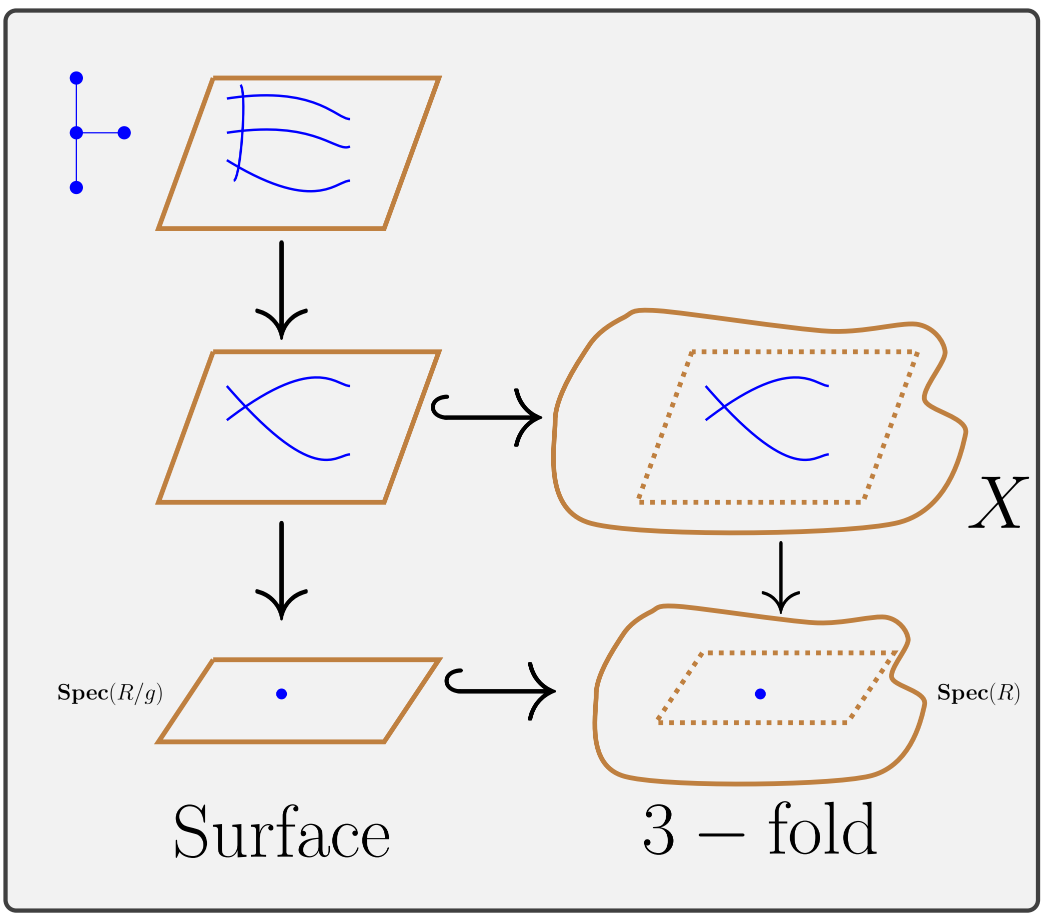
Dissertation in Pure Mathematics

Deformation and Resolution of Surface Singularities

Research Report in Mathematics, Number 26, 2020

Stephen Ochieng Mboya

September 2020



Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

Dissertation in Pure Mathematics **University of Nairobi**

September 2020

Deformation and Resolution of Surface Singularities

Research Report in Mathematics, Number 26, 2020

Stephen Ochieng Mboya

School of Mathematics
College of Biological and Physical sciences
Chiromo, off Riverside Drive
30197-00100 Nairobi, Kenya

Master Thesis

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

In this dissertation, we study ADE surface singularities in terms of Dynkin diagram obtained by deforming and resolving the singularity. Using classic invariant theory, we describe how these surface emerge as quotient of $\mathbb{C}^{\mathbf{2}}/\mathbf{\Gamma}$, where $\mathbf{\Gamma} \subseteq \boldsymbol{SL_2}(\mathbb{C})\boldsymbol{,}$ is a finite subgroup of the group of $\mathbf{2} \times \mathbf{2}$ matrix of determinant $\mathbf{1}$ over $\mathbb{C}$. We further describe how these hypersurface embed in $\mathbb{C}^{\mathbf{3}}$ as an affine varieties. We deform $\boldsymbol{A_n}$ type singularity and show its relation to McKay-quivers.

Finally, we investigate the the exceptional locus of the resolution of the those isolated singularities using sequence of blowup and from this we obtain the corresponding Dynkin diagram of ADE type.

Master Thesis in Mathematics at the University of Nairobi, Kenya.
ISSN 2410-1397: Research Report in Mathematics

DISTRIBUTOR: School of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

STEPHEN OCHIENG MBOYA
Reg No. I56/16759/2018

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

Dr. Jared Ongaro
School of Mathematics,
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: ongaro@uonbi.ac.ke

Dedication

I dedicate this dissertation to Gedion Mboya, Philgona Mboya, Lameck Mboya and Salome Apiyo.

Contents

Acknowledgments

On the very outset of this dissertation, I would like to extend my sincere and heartfelt obligation towards all the personage who have helped me in this endeavor. I am ineffably indebted to my supervisor Jared Ongaro for conscientious guidance, friendship, mentorship and encouragement to accomplish this project. Furthermore, his teaching introduced me to new areas of mathematics and he took the time to explain them to me in detail. I also express my profound gratitude to Geoffrey Mboya for his selfless and valuable guidance and support for completion of this project.

I am grateful to the University of Nairobi for offering me a scholarship to pursue my master degree in pure mathematics. Many thanks to the former director, school of mathematics Patrick Weke. I appreciate my lecturers in the department of Mathematics for equipping me with knowledge to enable me face this work with ease. I am especially thankful to Dr. Katende, Prof. Nzimbi and Dr. Luketero, for their advice, friendship and always being there for me. I am also grateful to EAUMP for allowing me to attend summer school on algebraic topology in Kampala. It was a motivation towards my project in algebraic geometry.

I am extremely thankful and pay my gratitude to Lameck Mboya for his selfless and support for completion of this project. I am also thankful to my friend Laurence Muthama for reviewing the preliminary draft of my thesis and providing many constructive comments. To my classmates Serryann, Jacob, Evance, Lokaran, Chelulei and Njeru; our many discussions and your kind and healthy suggestions improved this dissertation.

I also acknowledge with a deep sense of reverence, my gratitude towards my parents, sisters; Everline,Lilian, Salome and Elvince who has always supported me morally and economically.

STEPHEN OCHIENG MBOYA

Nairobi, 2020.

Introduction

The problem of resolution of singularities for a given algebraic variety X entails construction of a proper birational morphism; $\pi : Y \to X$ such that Y is regular. The resolution process has an impressive history. For curves the existence of resolution is known, see as presented in [**Kar00**]. For the surface case, Zariski (1935) gave an algebraic proof and existence of resolution in characteristic zero fields. Hironaka (1964) came up with famous result of resolution for an arbitrary dimensional variety X over field of characteristic zero. The geometric version of Hironaka theorem states that,for every real or complex algebraic varieties, no matter how badly singularity is, can be dominated by smooth algebraic variety isomorphic to it at each of its smooth points, see in [**Kar00**].

In this project we adopt Hironaka method and theorem on resolution of singularities which comprise of an idea of sequence of blowup of singular locus along regular subvariety that lies on singular locus of the proceeding variety. The output of the whole resolution process consist of tree of charts, where the root nodes is isomorphic to $\mathbb{C}^n$; The ambient space of the given **hypersurface**. The sub branches, that is the children of each node are obtained from the parent nodes by blowing up and finally the proper transform of the given **hypersurface** in the leaf node are regular and the total transform is normal crossing divisor, see in [**Bier97**].

Firstly, we classify simple surface singularities as the quotient of $\mathbb{C}^2$ by finite subgroup $\Gamma \subseteq \mathbf{SL_2}(\mathbb{C})$. The elements of $\mathbf{SL_2}(\mathbb{C})$ acts on the ring $\mathbb{C}[u, v]$ such that for each subgroup $\Gamma \subset \mathbf{SL_2}(\mathbb{C})$, there exist certain polynomial in $\mathbb{C}[u, v]$ that are invariant with respect to this action and satisfying an algebraic relation as depicted from [**Igor07**]. In our case we take $f \in \mathbb{C}[x, y, z]$ to be a polynomial describing this relation and consider f as surface defined by the association of orbit of Γ resulting to a complex **hypersurface** in $\mathbb{C}^3$. We find that f has an isolated singularity at the origin which turn to be isomorphic to $\mathbb{C}^2/\Gamma$. These singularities have been competently called, the **Kleinian Singularities** which appears throughout the classification of surface and other known areas of geometry.

Du val showed that, resolving the **Kleinian** singularities by blowing up yields an exceptional divisor that can be converted to **Dynkin diagram** of type **ADE** , see in [**Dav 97**].

The outline of the thesis is as follows:

Chapter 1: We use this chapter to introduce the standard notation and concept of singularities which form a basis for resolution of surface singularities. Morphism , sheaf of regular function on algebraic varieties and classification of finite subgroup of $\mathbf{SL_2}(\mathbb{C})$ are also introduced owing to its usefulness in understanding the sequel in other sections.

Chapter 2: Here, an introduction to classification of simple surface singularities as quotient of $\mathbb{C}^2$ by finite subgroup Γ of $\mathbf{SL_2}(\mathbb{C})$ using classic invariant theory is presented. We delve into finding invariant polynomials under this G action on $\mathbb{C}^2 \setminus \{0\}$ to achieve **ADE**- type classification of surface singularities. We also present the space of universal deformation of cyclic quotient singularities. Furthermore, we give the correspondence between the McKay quiver and deformation of A_n- type singularities.

Chapter 3: We present the concept of blowup as an important surgical tool in resolution process. We discuss blowup of a point in an affine varieties, $p \in \mathbb{A}^n$, blowup of a point in a subvariety, $p \in X \subseteq \mathbb{A}^n$ and blowup along subavriety, $X \subset Y \subseteq \mathbb{A}^n$. We extend to resolve the surface singularities of **ADE**-type and further explore the correspondence of these (minimal) resolutions with dual graphs called **dynkin diagrams**.

1 Preliminaries

The content of this chapter consists of classical Algebraic geometry material whose details can be found in [**Harts77**, **Dav97**, **Wil06**] and [**Mac69**]. This is intended to fix notation used in the rest of this project.

1.1 Affine Spaces and Affine Varieties

We work over the field $\mathbb{C}$ of complex numbers.

Definition 1.1.1. *An* ***n*-dimensional affine space** *is defined by* $\mathbb{A}^n := \mathbb{C}^n$ *to be the set of all* ***n*-tuples** *of elements of* $\mathbb{C}$.

Definition 1.1.2. *For a positive integer* n, *let* $R = \mathbb{C}[x_1, \dots, x_n]$ *be a polynomial ring. An* **affine variety** *defined on* $S \subset R = \mathbb{C}[x_1, \dots, x_n]$ *is the vanishing set*

$$\mathbb{V}(S) := \Big\{p \in \mathbb{C}^n \;\Big|\; f(p) = 0, \text{ for } f \in S\Big\}.$$

Definition 1.1.3. *Let* $V \subset \mathbb{C}^n$ *be affine variety. The* **vanishing ideal** *of* V

$$\mathbb{I}(V) = \Big\{f \in \mathbb{C}[x_0, \dots, x_n] : f(p) = 0, \text{ for all } p \in V\Big\}$$

is the ideal of all homogeneous polynomials vanishing on V. *The* **coordinate ring** $\mathbb{C}[V]$ *of* V *is the quotient ring*

$$\mathbb{C}[V] = \frac{\mathbb{C}[x_0, \dots, x_n]}{\mathbb{I}(V)}.$$

Further, we call a subset $V \subset \mathbb{P}^n$ *an* **algebraic set** *if there exists a homogeneous ideal* I *for which* $V = \mathbb{V}(I)$. *We say* V *is* **reducible** *if* $V = \mathbb{V}(I) = \mathbb{V}(I_1) \cup \mathbb{V}(I_2)$ *for some proper ideals* $I_1, I_2 \in \mathbb{C}[x_0, \dots, x_n]$, *otherwise it is* **irreducible**.

Theorem 1.1.4 (Hilbert's Nullstellensatz). *Let* I *be an ideal of* R. *Then*

(i) $I \subsetneq R$ *if and only if* $\mathbb{V}(I) \neq \varnothing$

(ii) $\mathbb{I}(V(I)) = \sqrt{I} = \{f \in R = \mathbb{C}[x_0, \dots, x_n] : f^m \in I \text{ for some } m\}$ *In particular, if* I *is radical, [i.e.* $\sqrt{I} = I$ *or equivalently* R/I *is a reduced ring (has no nilpotent elements) e.g.* $I = (x) \subset \mathbb{C}[x]$ *] then* $\mathbb{I}(\mathbb{V}(I)) = I$.

Corollary 1.1.5. *There are order-reversing bijections*

$$\begin{aligned}
\{\textit{varieties }\} &\longleftrightarrow \{\textit{radical ideals }\}\\
\{\textit{irreducible varieties}\} &\longleftrightarrow \{\textit{prime ideals}\} = \mathbf{Spec}(R)\\
\{\textit{points}\} &\longleftrightarrow \{\textit{maximal ideals}\} = \mathbf{Spec}\, m(R)\\
X &\longleftrightarrow \mathbb{I}(X)\\
V(I) &\longleftrightarrow I.
\end{aligned}$$

Lemma 1.1.6. *A collection of Algebraic sets has the following properties*

i) Empty space and the whole projective space are algebraic sets.

ii) Arbitrary intersection of algebraic sets is an algebraic set.

iii) Finite union of algebraic sets is an algebraic set.

Definition 1.1.7 (Zariski topology)**.** *Zariski topology on $\mathbb{P}^n$ is the topology on algebraic varieties where the open sets are complements of algebraic sets which satisfy Lemma 1.1.6.*

1.2 Affine Hypersurfaces

For $R := \mathbb{C}[x_1, \dots, x_n]$, let $p = (a_1, \dots, a_n) \in \mathbb{C}^n$ be a point and $f \in R$ be an irreducible polynomial. We define a hypersurface $X \subset \mathbb{C}^n$ to be a vanishing set

$$\mathbb{V}(f) := \left\{p \in \mathbb{C}^n \middle| \, f(p) = 0\right\}.$$

Definition 1.2.1. *An algebraic hypersurface in an affine $n-$ space is an algebraic variety defined by a single implicit equation of the form $f(x_1, \dots, x_n) = 0$.*

Example 1.2.2. *An affine hyperplane defined by a linear polynomial of the form $X := \mathbb{V}(x_1 + \dots + x_n)$ is an hypersurface.*

Definition 1.2.3. *Let $X \subset \mathbb{C}^n$ be an hypersurface defined by $f \in R$*

i. A point $p \in \mathbb{C}^n$ is a singular point (**Sing**) *or critical* (**Crit**) *point of f if ;*

$$\frac{\partial f}{\partial x_i}(p) = 0 \quad \textit{for all } i = 1, \dots, n.$$

Otherwise, we say that the point $p \in X$ is regular point or smooth point.

A singular hypersurface $X = \mathbb{V}(f)$ is defined as;

$$\mathbf{Sing}(X) := \left\{p \in X \;\middle|\; f(p) = \frac{\partial f}{\partial x_1}(p) = \dots = \frac{\partial f}{\partial x_n}(p) = 0\right\}.$$

Example 1.2.4. *We consider a variety* $X = \mathbb{V}(x^2 + zy^2 + z) \subset \mathbb{A}^3$. *The partial derivatives are*

$$\begin{cases} \frac{\partial f}{\partial x} = 2x \\ \frac{\partial f}{\partial y} = 2yz \\ \frac{\partial f}{\partial z} = y^2 + 1 \end{cases}.$$

Thus the singular points are $(0, i, 0)$ *and* $(0, -i, 0)$.

ii. *The tangent space to the hypersurface* X *at point* p *is defined as*

$$\mathrm{T_p}X = \Big\{(x_1, \ldots, x_n) \in \mathbb{C}^n \Big| \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x - a_i) = 0 \text{ for all } i = 1, \ldots, n\Big\}.$$

iii. *For a hypersurface* X *we define a dimension of* X *by*

$$\dim\ X := \min\Big\{\dim\ \mathrm{T_p}X \ \Big|\ p \in X\Big\}.$$

From definition (ii)*and* (iii), *we have that* $\dim\ \mathrm{T_p}X \leq \dim\ \mathrm{T_p}X$ *for all* $p \in X$. *If* $\dim\ \mathrm{T_p}X = \dim\ X$, *then we say that* X *is nonsingular at a point* p.

iv. *The tangent cone of* X *at* p *is defined as;* $\mathrm{C_p}(V) = \mathbb{V}(f_{p,\min} \mid f \in \mathbb{I}(V))$, *where*

$$f_{p,\ \min} = f_{p,\ j} := \min\Big\{j \in \mathbb{Z} \Big| f_{p,j} \neq 0\Big\}.$$

Example 1.2.5. *Let* $V \subseteq \mathbb{C}^n$ *with* $I_V = \langle f \rangle$. *Then* $\mathrm{T_p}(V)$ *is defined by the single equation* $d_p f = 0$. *Thus,* $\mathrm{T_p}(V) = \mathbb{C}^n \iff \frac{\partial f}{\partial x_i} = 0$ *for all* i. *On the other hand, if some* $\frac{\partial f}{\partial x_i} \neq 0$, *then* $\dim_{\ \mathbb{C}} \mathrm{T_p}V = n - 1$.

Definition 1.2.6. *Let* $U \subset \mathbb{C}^n$ *be an open subset,* $f : U \to \mathbb{C}$ *be a holomorphic function and* $X := \mathbb{V}(f) = f^{-1}(0)$ *be an* **hypersurface** *define by* f *in* U. *A point* $x \in U \subset \mathbb{C}^n$ *is called isolated critical point of* f *if there exist a neighborhood* V *of* x *such that,*

$$\mathrm{Sing}(f) \cap V \setminus \{x\} = \varnothing.$$

Definition 1.2.7. *The characterization of singularities by the vanishing of the partial derivatives is called the* ***Jacobian*** *criterion for smoothness.*

1.3 Morphisms and Regular Functions on Affine Varieties

Definition 1.3.1. *Let* $Y \subseteq \mathbb{C}^n$ *be a variety. A function* $f : Y \to \mathbb{C}$ *is* **regular** *at a point* $P \in Y$ *if there is an open neighborhood* U *with* $P \in U \subseteq Y$, *and homogenous polynomials* $g, h \in R = \mathbb{C}[x_1, \ldots, x_n]$, *of the same degree, such that* h *is nowhere zero on* U, *and* $f = g/h$ *on* U. f *is regular on* Y *if it is regular at every point of* Y. *The set of all regular function will be denoted by* $\mathcal{O}_X(U)$.

Definition 1.3.2. *For an affine variety* X. *Let* $f \in \mathbb{C}[x_1, \ldots, x_n]$ *on* X. *Then we call*

$$\mathrm{D(f)} := X \setminus \mathbb{V}(f) = \left\{ x \in X : \ f(x) \neq 0 \right\}$$

the distinguished open subset of f *in* X.

Example 1.3.3. *We consider a* ***3-dimensional*** *affine variety* $X = \mathbb{V}(x_1x_4 - x_2x_3) \subseteq \mathbb{A}^4$ *and the open subset*

$$U = X \setminus \mathbb{V}(x_2, x_4) = \left\{ (x_1, x_2, x_3, x_4) \in X \ \middle| \ x_2, x_4 \neq 0 \right\} \subset X.$$

Then

$$\begin{aligned} \varphi : & U \to \mathbb{C} \\ & (x_1, x_2, x_3, x_4) \mapsto \begin{cases} \frac{x_1}{x_2} & \text{if } x_2 \neq 0 \\ \frac{x_3}{x_4} & \text{if } x_4 \neq 0 \end{cases} \end{aligned}$$

is a regular function on U.

Definition 1.3.4. *Let* $V \subseteq \mathbb{C}^n$ *and* $W \subseteq \mathbb{A}^m$ *be affine varieties. A map* $f : V \to W$ *is a* ***morphism*** *of affine varieties if it is the restriction of a polynomial map on the affine space* $\mathbb{C}^m$ *to affine space* $\mathbb{C}^n$. *A morphism* $f : V \to W$ *is an* ***isomorphism*** *if there exist a morphism* $g : W \to V$ *such that* $f \circ g = \mathrm{Id}_W$ *and* $g \circ f = \mathrm{Id}_V$.

Example 1.3.5. *Consider the polyomial map*

$$\begin{aligned} & F : \mathbb{C}^n \to \mathbb{C}^m \\ & F(x_1, \ldots, x_n) \mapsto \big(f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_m) \big) \\ & \text{where } f_i \in \mathbb{C}[x_1, \ldots, x_n] \text{ for all } 1 \leq i \leq n \end{aligned}$$

More specifically if we consider an affine variety $\mathbb{V}(xy - z^2) \subseteq \mathbb{C}^3$ *and a restriction map*

$$\begin{aligned} F : & \mathbb{C}^3 \to \mathbb{C}^2 \\ & (x, y, z) \mapsto (xy, xz) \end{aligned}$$

Then the variety $\mathbb{V}(xy - z^2) \subseteq \mathbb{C}^3 \cong \mathbb{V}(zY - yZ) \subseteq \mathbb{C}^2$ *through the mapping* F, *where* $Y = xy$ *and* $Z = xz$.

Example 1.3.6. *Consider two affine varieties* $X, Y \subset \mathbb{C}^3$ *with* $X = \mathbb{V}(x^2 + y^2 + z^2)$ *and* $Y = \mathbb{V}(x + y + z)$. *The mapping*

$$\varphi(x, y, z) = x^2 + y^2 + z^2$$

is a morphism from Y *to* X. *We also note that the affine variety* X *is isomorphic to* Y *via the coordinate change* $\varphi(x, y, z) = (x, y, iz)$.

Definition 1.3.7. *A **rational map** $\varphi : X \to Y$ is an equivalence class of pairs $\langle U, \varphi_U \rangle$ where U is a nonempty open subset of X, φ_U is a morphism of U to Y, and where $\langle U, \varphi_U \rangle$ and $\langle V, \varphi_V \rangle$ are equivalent if φ_U and φ_V agree on $U \cap V$. The family of rational map φ denoted by $K(X)$ is called function.*

Example 1.3.8. *We consider $X := \mathbb{V}(xy - z^2) \subseteq \mathbb{C}^3$, then $\frac{x}{z}$ is a rational function. Moreover, $\frac{z}{y}$ is the same rational function, that is $\frac{x}{z} \sim \frac{z}{y}$ because $xy = z^2$ on X.*

1.4 Finite Groups action on Affine Varieties

In this section we present the concept of quotient singularity. Let X be an affine variety and G be a finite group acting on X. Then we are interested on a singular point of the quotient X/G.

Theorem 1.4.1 (Cartan). *Any complex quotient singularity $(X/G, x)$ is isomorphic to $(\mathbb{C}^n/G, 0)$, where $G \subset \mathrm{GL}(n, \mathbb{C})$ is a finite subgroup.*

Definition 1.4.2. *Let $p \in X$ be a point in a variety X. We define an algebraic group as a variety X with regular maps satisfying the usual multiplication and inverse in a group,*

$$g : X \times X \to X$$
$$\mathrm{Id} : X \mapsto X.$$

Definition 1.4.3. *A **morphism** of algebraic groups is defined as a map $\varphi : G \to H$, that is both a regular map and a group **homomorphism**.*

Example 1.4.4 (The General Linear Group $\mathrm{GL}(n, \mathbb{C})$. *The set of invertible $n \times n$ matrices, whose* **hypersurface** *is given by determinant, is just a distinguished open subset of $\mathbb{A}^{2n}$, and thus it is an affine variety.*

The regular map $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ exhibit, by definition, an inverse map. The subgroup $\mathrm{SL}(n, \mathbb{C}) \subset \mathrm{GL}(n, \mathbb{C})$ is a subvariety.

Definition 1.4.5. *An algebraic group G is said to act on algebraic variety X, if the there exist a regular map: $\varphi : G \times X \to X$ such that $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ and $\varphi(e, x) = x$, for all $g, h \in G$ and $x \in X$.*

1.5 Classification of finite subgroup of $\mathrm{SL}_2(\mathbb{C})$

A homomorphism on $\mathrm{SU}_2(\mathbb{C})$

We consider subgroup of general linear group of a vector space V over a field $\mathbb{C}$ given by $\mathrm{GL}(2,\mathbb{C}) := \Big\{ f : V \to V$ such that f is linear invertible$\Big\}$.

We have that $\mathrm{SU} = \Big\{ A \in \mathrm{GL}(2,\mathbb{C}) \ |AA^* = A^*A = I, \ \det \ A = 1 \Big\}$.
Suppose we let

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

with $UU^* = I$, we have that the determinant $|a|^2 + |c|^2 = 1, |b|^2 + |d|^2 = 1$, then we have that $\bar{a}b + \bar{c}d = 0$ and thus we only consider the values of a, b, c, d in a unit disk $\mathbb{C}$.

We define the parametrization

$$a = \exp^{i\alpha}\cos(x), \ b = \exp^{i\beta}\sin(x), \ c = -\exp^{i\gamma}\sin(x) \text{ and } d = \exp^{i\eta}\cos(x)$$

from which we get

$$\exp^{i(\alpha+\zeta)}\cos^2(x) + \exp^{i(\beta+\gamma)}\sin^2(x) = \exp^{i(\alpha+\zeta)} = 1. \tag{1}$$

From equation (1), we obtain

$$\alpha = -\zeta \ \text{ and } \beta = -\gamma \implies a = \bar{d} \text{ and } b = -\bar{c}.$$

Therefore we get an element of the form $U \in \mathrm{SU}_2(\mathbb{C})$,

$$U = \begin{bmatrix} x & y \\ -\bar{y} & \bar{x} \end{bmatrix} = \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}.$$

Lemma 1.5.1. *There exist a natural surjective homomorphism between the subgroups* $\mathrm{SU}_2(\mathbb{C})$ *and real orthogonal subgroup,* $\mathrm{SO}_3(\mathbb{R})$ *with kernel* $\Big\{ I, -I \Big\}$ *and in fact we have that* $\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{C})/\mathbb{C}_2$ *defined by*

$$\varphi : \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R})$$

$$\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} \mapsto \begin{bmatrix} a^2-b^2-c^2+d^2 & 2ab+2cd & -2ac+2bd \\ -2ab+2cd & a^2-b^2-d^2 & 2ad+2bc \\ 2ac+2bd & -2ad+2bc & a^2-b^2-c^2-d^2 \end{bmatrix}.$$

Lemma 1.5.2. *Every finite subgroup of* $\mathrm{SL}_n(\mathbb{C})$ *is conjugate to a subgroup of* $\mathrm{SU}_n(\mathbb{C})$.

Theorem 1.5.3. *There is an exact sequence of group homomorphism*

$$\mathbf{1} \to \mathbb{Z}_2 \to \mathbf{SU_2}(\mathbb{C}) \xrightarrow{\pi} \mathbf{SO_3}(\mathbb{R}) \to \mathbf{1}.$$

More precisely

$$\ker(\pi) = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Topologically π *is the map* $S^3 \to \mathbb{RP}^3$.

Remark 1.5.4. *Every finite subgroup* $G \subset \mathbf{GL_2}(\mathbb{C})$ *can be embedded into* $\mathbf{SL_3}(\mathbb{C})$ *by group homomorphism*

$$\begin{aligned} \mathbf{GL_2}(\mathbb{C}) &\to \mathbf{SL_3}(\mathbb{C}) \\ g &\mapsto \begin{bmatrix} g & 0 \\ 0 & \frac{1}{\det(g)} \end{bmatrix}. \end{aligned}$$

From the above theorem, it follows that there is an isomorphism between subgroups of $\mathbf{SU_2}(\mathbb{C})$ *and* $\mathbf{SO_3}(\mathbb{R})$*. The classification of finite isometry groups of* $\mathbb{R}^3$ *is a classical results in Felix Klein of the finite subgroups of* $\mathbf{SO_3}(\mathbb{R})$.

Theorem 1.5.5 (Classification of finite subgroup of $\mathbf{SL_2}(\mathbb{C})$)**.** *The classification of the non-trivial finite subgroup of* $\mathbf{SL_2}(\mathbb{C})$ *upto conjugation are precisely the binary polyhedral groups given below. Here we let* $\zeta_m = \exp(\frac{2\pi i}{m})$.

(i) A_n*: For* $n \geq 1$*, the* **cyclic group**, $G \cong \mathbb{Z}_m$ *where* $m = n + 1$*, of order* m *generated by:*

$$\begin{bmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{bmatrix}.$$

(ii) D_n*: For* $n \geq 4$*, the* **binary dihedral group** D_m, $|D_n| = 4m$*, where* $m = n - 2$. *To find the generators of this subgroup, we consider the mapping*

$$\pi : \mathbf{SU_2}(\mathbb{C}) \to \mathbf{SO_3}(\mathbb{C})$$

such that $\mathbb{D}_n = \langle A, B \rangle$ *with the relation*

$$\begin{cases} A^m = B^2 \\ B^4 = Id \\ BAB^{-1} = A^{-1}. \end{cases}$$

For $m = n-2$*, order* $4m$*, which is generated by* A, B

$$A = \begin{bmatrix} \zeta_{2m} & 0 \\ 0 & \zeta_{2m}^{-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

(iii) E_6*: The* **binary tetrahedral group** T*, of order* 24*, and is generated by:*

$$\delta = \begin{bmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & i \\ i & o \end{bmatrix}, \quad \mu = \frac{1}{1-i} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}.$$

(iv) E_7*: The* **binary octahedral group** $\mathbb{O}$*, of order,* 48*, and is generated by :*

$$\kappa = \begin{bmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \mu = \frac{1}{1-i} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}.$$

(v) E_8*:The binary icosahedral group, denoted by* $\mathbb{I}$*, of order* 120*, generated by:*

$$\gamma = \begin{bmatrix} \zeta_{10} & 0 \\ 0 & \zeta_{10}^{-1} \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \Omega = \frac{1}{\sqrt{5}} \begin{bmatrix} \zeta_5 - \zeta_5^4 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_3^5 & -\zeta_5 + \zeta_5^4 \end{bmatrix}.$$

The above theorem is an example of ADE classification. In the next part, we show that indeed the generators of above groups yield the same dual graph of resolution.

Example 1.5.6. *Let* $G \subseteq \mathrm{GL}_2(\mathbb{C})$ *finite subgroup, acting on one dimensional space* $\mathbb{P}^1$ *with projective coordinates* $[t_0 : t_1]$ *dual to the standard basis* $(e_1, e_2) \in \mathbb{C}$*, for* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$ *and* $x = [\alpha : \beta] \in \mathbb{P}^1$*;*

$$\begin{aligned} G \times \mathbb{P}^1 &\to \mathbb{P}^1 \\ g \cdot x = a &\mapsto [\alpha + b\beta : c\alpha + d\beta]. \end{aligned}$$

2 Surface Singularities and Deformation

Here we present simple surface singularities as the quotient of affine space $\mathbb{C}^2$ by linear action of finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$.

For a subgroup Γ acting on an affine variety X by automorphism, we have that Γ also act on the algebra $\mathbb{C}[X]$ of regular function on X such that for $g \in \Gamma,\ f \in \mathbb{C}[X]$and $x \in X$ we have that, $g \circ f(x) := f\Big(g^{-1}(x)\Big)$. The $\Gamma-$invariant elements in $\mathbb{C}[X]$ form a finitely generated subalgebra denoted by $\mathbb{C}[X]^{\Gamma}$ with the corresponding algebraic variety, denoted by X/Γ.

The inclusion $\mathbb{C}[X]^{\Gamma} \subset \mathbb{C}[X]$ give rise to a finite morphism $\pi : X \to X/\Gamma$ of algebraic varities. This quotient turn out to be the spectrum of the ring of invariants of the action

$$\mathbb{C}^n/G := \mathrm{Spec}(\mathbb{C}[V]^G).$$

2.1 ADE-Surface Singularities

2.1.1 G-invariant subrings

Definition 2.1.1 (Relative invariant)**.** *A homogeneous polynomial F is a* ***relative invariant*** *if and only if any $g \in G$, then $g^*(F) = \mathbb{V}(F)$.*

Definition 2.1.2 (Grundformen)**.** *A Grundformen is a relative invariant F with divisor of zeroes equal to an exceptional orbit (i.e. an orbit with a non-trivial stabilizer).*

Definition 2.1.3. *Let $\Gamma \in \mathrm{SU}_2(\mathbb{C})$ be a finite subgroup and $f \in \mathbb{C}[u,v]$. The* ***Reynolds operator****, R_Γ of Γ acting on $f \in \mathbb{C}[u,v]$ is defined as;*

$$R_\Gamma(f) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g \circ f.$$

Theorem 2.1.4 (Noether)**.** *The ring of invariant of Γ action on $\mathbb{C}[u,v]$ is generated algebraically by;*

$$\langle R_\Gamma(u^\alpha v^\beta) : \alpha + \beta \leq |\Gamma| \rangle.$$

We note that, whenever $-I \in \Gamma$, then we have for every $g \in \Gamma \implies -g \in \Gamma$. Furthermore, if $\alpha+\beta = 2n+1$ for $n \in \mathbb{Z}^+$, then $(-g)\cdot u^\alpha v^\beta = -(g \cdot u^\alpha v^\beta) \implies I \in \Gamma$,$\alpha + \beta$ *is odd*, $R_\Gamma(u^\alpha v^\beta) = 0$. More precisely, if Γ is cyclic of order $2n$,

then $-I \in \Gamma$, and thus if $\alpha+\beta$ is odd, then $R_\Gamma(u^\alpha v^\beta)=0$.
Similarly, if $\alpha+\beta=2n,\ \forall\ n \in \mathbb{Z} \implies (-g)\cdot u^\alpha v^\beta = g\cdot g\cdot u^\alpha v^\beta$.
Thus, if Γ is cyclic of order $2n$ and $\alpha+\beta$ is even, then;

$$\begin{aligned} R_\Gamma(u^\alpha v^\beta) &= \frac{1}{2n}\sum_{k=i}^{2n} g^k\cdot u^\alpha v^\beta \\ &= \frac{1}{2n}\sum_{k=i}^{n}(g^k\cdot u^\alpha v^\beta + g^{n+k}\cdot u^\alpha v^\beta) \\ &= \frac{1}{2n}\sum_{k=i}^{n}(g^k\cdot u^\alpha v^\beta + (-g^k\cdot u^\alpha v^\beta) \\ &= \frac{1}{2n}\sum_{k=i}^{n}2(g^k\cdot u^\alpha v^\beta) \\ &= \frac{1}{n}\sum_{k=i}^{n}g^k\cdot u^\alpha v^\beta. \end{aligned}$$

Thus, for a cyclic group of order $2n$, the ring of invariant of Γ is generated algebraically by;

$$\langle\Big(\tfrac{1}{n}\sum_{k=2}^{n} g^k\cdot u^\alpha v^\beta) : 2|(\alpha+\beta) \leq 2n|\rangle = \langle(\sum_{k=2}^{n} g^k\cdot u^\alpha v^\beta) : 2|(\alpha+\beta\Big) \leq 2n|\rangle.$$

In our case, the systematic application of Noether's theorem to cyclic group $G \cong \mathbb{Z}_m$, where $m=n+1$, of order m, the binary dihedral group D_m, where $m = n-2$ of order $4m$, binary tetrahedral group, $\mathbb{T}$ of order 24, Binary octahedral group $\mathbb{O}$ of order 48 and binary icosahedral group, $\mathbb{I}$, of order 120 will give the result as shown below.

Group	Fundamental Invariants
Cyclic ($\mathbb{Z}_m$)	uv, u^m, v^m
Binary diheral (D_n)	u^2v^2, $u^{2n}+(-1)^n v^{2n}$, $uv(u^{2n}+(-1)^{n-1}v^{2n})$
Binary Tetrahedral ($\mathbb{I}$)	$u^8+14u^4v^4+v^8$, $u^{10}v^2-2u^6v^6+u^2v^{10}$, $u^{17}v-34u^{13}v^5+34u^5v^{13}-uv^{17}$

Table 1. Fundamental invariants

2.1.2 Cyclic Quotient Singularities

Theorem 2.1.5. *The ring of polynomial invariants of linear actions of a finite group on an affine space is finitely generated.*

Let $\mathbb{C}[u,v]$ be ring of polynomial and G be a fine subgroup of $\mathrm{SL}_2(\mathbb{C})$ acting on $\mathbb{C}[u,v]$, we intend to find set of polynomials invaraint under the action of G, that is

$$\mathbb{C}[u,v]^G := \Big\{ f \in \mathbb{C}[u,v] \mid gf = f \ \ \forall \ \ g \in G \Big\}.$$

Suppose we consider a finite subgroup $\Gamma \subset \mathrm{SU}_2(\mathbb{C})$, we seek invariant polynomial $f(u,v) \in \mathbb{C}[u,v]$ under all the elements of Γ. For $G \cong \mathbb{Z}/m \cong \mathbb{C}_m$ a cyclic group of order $m = n+1$. We consider $\mathbb{C}^m_{u,v}/G$. Let G act on $\mathbb{C}^m \setminus \Big\{0\Big\}$ by:

$$\phi : G \times \mathbb{C}^m \setminus \Big\{0\Big\} \to \mathbb{C}^m$$
$$(u,v) \mapsto \begin{bmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \zeta_m u, \zeta_m^{-1} v$$

Under this action, $u^m \to u^m$, $v^m \to v^m$, $uv \to uv$. So that $\mathbb{C}[u,v]^G = \mathbb{C}[u^m, v^m, uv]$

We let $f_1 = u^m$, $f_2 = v^m, f_3 = uv$ generates the algebra of invariant with the relation

$$f_3^m = f_1 f_2 \implies \mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/(xy + z^m).$$

Thus we have that cyclic quotient singularities,

$$A_n : \mathbb{V}(xy - z^{n+1}) \cong \mathbb{C}[x,y,z]/\langle xy - z^{n+1} \rangle.$$

2.1.3 Binary Dihedral and Binary Tetrahedral Quotient Singularities

Theorem 2.1.6. *Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup. Let $F(x,y,z) = 0$ be homogeneous polynomial and let f_1, f_2, f_3 generates the ring of invariant with $\mathbb{C}[u,v]^G = \mathbb{C}[f_1, f_2, f_3]$, then $\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/\langle F(x,y,z) \rangle$.*

Remark 2.1.7. *A monomial $u^a v^b$ is invariant if and only if for the matrix of generators, we have that, $\zeta_n^{a-b} = 1$, since $u^a v^b = (uv)^a v^{b-a}$ for all $a \leq b$.*

Proof. We consider the remark 2.1.7 to prove the following cases:

(a) Case 1: $G \cong D_n$ is a binary dihedral group of order $4m$. With respect to generators defined above, the **relative invariants (Grundformen)** are given as

$$\Phi_1 = u^n + v^n,\ \Phi_2 = u^n - v^n,\ \Phi_2 = uv.$$

We have that

$$f_1 = uv(u^{2m} - 2u^m v^m + v^{2m}),\ with\ (m = n-2).$$

When n is odd,then we have that,

$$f_2 = u^{2m} - v^{2m},\ f_3 = u^2 v^2, \text{ with relation}$$
$$\tilde{f}_1^2 = f_3 f_2^2,\ where \tilde{f}_1 = f_1 + 2 f_3^{\frac{m+1}{2}}.$$

When n is even, we have that,

$$f_2 = u^{2m} - 2u^m v^m + v^{2m},\quad f_3 = uv(u^{2m} - v^{2m}), \text{ with the relation}$$
$$f_3^2 = f_1 f_2^2 \implies (f_3 + 2 f_1^{\frac{n+1}{2}})^2 - 4 f_2^{n+1} - f_1 f_2^2 = 0.$$

By substitution and scaling the generators, we get

$$\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/(x^2 + y^2 z + z^{n+1}).$$

In case n is even, we use the relation

$$f_1 = \Phi_3^2,\ f_2 = \Phi_2^2,\ f_3 = \Phi_1 \Phi_2 \Phi_3,$$

With the same relation, and we obtain the algebra of invariants isomorphic to the same ring in the case of odd n.

(b) Case 2: $G \cong \mathbb{T}$ is a binary tetrahedral group of order 24, with the Grundformen given as

$$\Phi_1 = uv(u^4 - v^4),\ \Phi_2, \Phi_3 = u^4 \pm 2\sqrt{-3u^2} v^2 + v^4$$

with the relation that

$$f_1 = \Phi_1,\ f_2 = \Phi_2 \Phi_3,\ f_3 = \Phi_2^3 + \Phi_3^3.$$

We obtain that;

$$f_2^2 = u^2 v^2 (u^4 + v^4)^2 - 4u^4 v^4 = (12\sqrt{-3})^{-1}(\Phi_2^3 - \Phi_3^3).$$

This show that Φ_2^3 and Φ_3^3 can be expressed in terms of f_1, f_2, f_3. By substitution we obtain that,

$$f_3^2 = f_1^4 + 4f_2^3.$$

And thus we have that,

$$\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/(x^2+y^3+z^4).$$

(c) Case 3: $G \cong \mathbb{O}$ is a binary octahedral group of order 48, with Grundformen given as;

$\Phi_1 = uv(u^4 - v^4),$
$\Phi_2 = u^8 + 14u^4v^4 + v^8 = (u^4 + 3\sqrt{-3u^2}v^2 + v^4)(u^4 - 2\sqrt{-3u^2}v^2 + v^4),$
$\Phi_3 = (u^4+v^4)((u^4+v^4)^2 - 36u^4v^4).$

We have that, suppose we identify our invariant polynomials with Grundformen such that:

$$f_1 = \Phi_1^2,\ f_2 = \Phi_2,\ f_3 = \Phi_3\Phi_1.$$

We obtain the relation,

$$\Phi_2^3 - \Phi_3^2 = 108\Phi_1^4.$$

Thus we can express the invariant Φ_3^2 in terms of f_1, f_2, f_3, by so doing this, we obtain the relation,

$$f_3^2 = \Phi_1^2\Phi_3^2 = f_1(f_2^3 - 108f_1^2).$$

By substitution, we obtain that

$$\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/(x^2 + z(y^3 + z^2)).$$

(d) Case 4: $G \cong \mathbb{I}$ is a binary icosahedral group of order 120, with respect to generators above, we have that, the Grundformen is given by;

$$\Phi_3 = uv(u^{10} + 11u^5v^5 - v^{10}).$$
$$\Phi_2 = -(u^{20}+v^{20}) + 228(u^{15}v^{15} - u^5v^{15}) - 494u^{10}v^{10}.$$
$$\Phi_3 = u^{30} + v^{30} + 522(u^{25}v^5 - u^5v^{25}) - 10005(u^{20}v^{10} + u^{10}v^{20}).$$

In this case, the Grundformen are invariants, therefore any invariant is a polynomial in Grundformen Φ_1, Φ_2, Φ_3. We obtain the relation;

$$\Phi_1^2 + \Phi_2^3 = 1728\Phi_3^5.$$

Hence we have that:

$$\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/(x^2+y^3+z^5).$$

$\square$

2.1.4 General results Obtained from Classification

Let $\Gamma \subset SU_2(\mathbb{C}) \subset SL_2(\mathbb{C})$ be a finite subgroup. Up to conjugacy class we obtain five classes of such groups, as depicted in [Bries96].

Group	Order	Classification ($\underline{\Delta}$)	R(x,y,z)=0
Cyclic ($\mathbb{Z}_n$)	$n,\ n \geq 2$	A_n	$xy - z^{n+1}$
Binary Dihedral (D_n)	$4n,\ n \geq 2$	D_{n+2}	$x^2 + zy^2 + z^{n+1}$
Binary Tetrahedral ($\mathbb{T}_{24}$)	24	E_6	$x^2 + y^3 + z^4$
Binary Octahedral ($\mathbb{O}_{48}$)	48	E_7	$x^2 + y^3 + yz^3$
Binary Isocahedral ($\mathbb{I}_{120}$)	120	E_8	$x^2 + y^3 + z^5$

Table 2. Classification of ADE-Singularities

The Klenian attached to Γ is the quotient singularity $S = \mathbb{C}^2/\Gamma$, which we explain by the invariant theory of Γ, as a hypersurface in $\mathbb{C}^3$.

Suppose R is a relation between three fundamental generator of the invariant ring $\mathbb{C}[u,v]^\Gamma$ of $\mathbb{C}^2$, then we describe this as

$$S = \mathbb{C}^2/\Gamma := \Big\{(x,y,z) \in \mathbb{C}^3 \ | R(x,y,z) = 0\Big\}$$

with isolated singularity at the origin $(0,0,0)$.

2.2 Deformation of Singularities

Definition 2.2.1. *Let (X,x) and (S,s) be complex space germs. A deformation of (X,x) over (S,s) consist of a flat morphism (i.e. all fibers have the same dimension) $\phi : \big(\phi^{-1}(s), x\big) \to (S,s)$ of complex germs together with an isomorphism $\pi : (X,x) \xrightarrow{\cong} \big(\phi^{-1}(s), x\big)$. Where $\big(\phi^{-1}(s), x\big)$ is called the total space and (S,s) the base space.*

Definition 2.2.2. *Let* $\phi : \left(\phi^{-1}(s), x\right) \to (S, s)$ *be a deformation of* (X, x) *and let* $f : (T, t) \to (S, s)$ *be a holomorphic map. The induced deformation is the flat map* $f^*(\Phi) : (X_{\times S} T, t) \to (T, t)$.

Definition 2.2.3. *A deformation* $\phi : \left(\phi^{-1}(s), x\right) \to (S, s)$ *of* (X, x) *is miniversal or semi-universal if every deformation* $\rho : \left(\rho^{-1}(t), x\right) \to (T, t)$ *of* (X, x) *is isomorphic to a deformation* $f^*(\phi)$*, for some map* $f : (T, t) \to (S, s)$, *such that the differential* df *of* f *is uniquely determined.*

Theorem 2.2.4. *Let* $X = \mathbb{V}(f) \in \mathbb{C}^n$ *be a hypersurface with an isolated singularity at the origin. Then a semi-universal deformation of* X *is given by*

$$X = \left\{(x, s) \in \mathbb{C}^n \times \mathbb{C}^r \;\middle|\; f(x) = \sum_{i=o}^{r} s_i b_i(x)\right\}$$

$$\downarrow \;\; \phi$$

$$(\mathbb{C}^r, 0)$$

$$\pi : S \hookrightarrow X$$

$$x \mapsto (x, 0)$$

Where $b_1, b_2, \ldots, b_r$ *are basis of space* $\mathbb{C}[x_1, \ldots, x_n]/\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\rangle$.

Example 2.2.5. *Consider* $X = \mathbb{V}(x^2 + y^2 + z^{n+1}) \subseteq \mathbb{C}^3$ *be the singularity of type* A_n. *Then* $\mathbb{C}[x, y, z]/\langle x, y, z^n\rangle$ *has the basis* $\left\{1, z, z^2, \ldots, z^{n-1}\right\}$. *Thus the semi-universal deformation of* X *is given by*

$$X = \left\{(x, y, z, s_1, s_2, \ldots, s_n) \in \mathbb{C}^{3+n} \middle| \; z^{n+1} + \sum_{i=1}^{n} s_i z^{n-i} + x^2 + y^2 = 0\right\}$$

$$\downarrow \;\; \phi$$

$$(\mathbb{C}^n, 0)$$

Definition 2.2.6. *Let* $X = \mathbb{V}(f) \subseteq \mathbb{C}^n$ *be an affine variety defined by an hypersurface* $f = 0$. *Then the space defined by;*

$$T_f^1 = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle f, \frac{\partial f}{x_1}, \ldots, \frac{\partial f}{\partial x_n}\rangle}$$

is called **Zariski tangent space**.

Theorem 2.2.7. *Let* $X \subset \mathbb{C}^n$ *be an affine variety with isolated singularity and* $g_1, \ldots, g_r$ *be regular function such that their image in* T_f^1 *form a basis. Then the deformation given by* $f_t = f + \sum_i s_i g_i$ *where* t_i *are coordinates on the germ* $(\mathbb{C}^n, 0)$ *is a veresal deformation.*

Definition 2.2.8 (Infinitesimal deformation). *An infinitesimal deformation of an affine variety X, is a flat morphism together with $\mathbb{C}$-**isomorphism** $X_T \times \text{Spec } \mathbb{C} \xrightarrow{\sim} X$ such that it induces a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d] & \widetilde{X} \arrow[d, "\phi \quad flat"] \\ \text{Spec } \mathbb{C} \arrow[r] & \text{Spec } R. \end{tikzcd}$$

Example 2.2.9. *For $n \geq 1$, we consider deformation of A_n surface singularities defined by the equation $f = x^2 + y^2 + z^{n+1}$. The derivatives are $(n+1)z^n, 2x, 2y$ so that we take $g_i = z^i$ for $i = 0, \ldots, n-1$. The versal deformation space has dimension n and is given by;*

$$y^2 + x^2 + z^{n+1} + s_0 + s_1 z + s_2 z^2 + \ldots + s_{n-1} z^{n-1}$$

Furthermore we extend this to infinitesimal deformation given by;

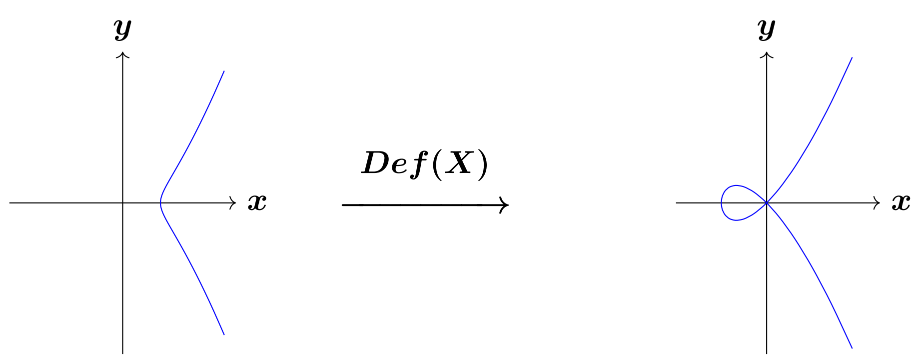
$$\begin{tikzcd} \frac{\mathbb{C}\left\{x,y,z\right\}}{(x^2+y^2+z^{n+1})} & \frac{\mathbb{C}\left\{x,y,z,s\right\}}{(x^2+y^2+z^{n+1}+s)} \arrow[l] \\ \mathbb{C} \arrow[u] & \mathbb{C}\left\{s\right\}. \arrow[l] \arrow[u] \end{tikzcd}$$

Definition 2.2.10 (First order deformation). *First order deformation is the deformations of an affine scheme X over $A = \mathbb{C}[s]/(s^2) = \mathbb{C} \oplus \mathbb{C}[s]$ with $s^2 = 0$. Suppose I_A is an ideal of $A[x_1, \ldots, x_n]$ then*

$$I_A = \Big((f_1 + g_1 s), \ldots, (f_{n-d} + g_{n-d} s)\Big).$$

Where $g_i \in \mathbb{C}[x_1, \ldots, x_n]$.

Example 2.2.11. *We consider the nodal curve $X = \mathbb{V}(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$ so that $\frac{\partial f}{\partial x} = -3x^2 - 2x$ and $\frac{\partial f}{\partial y} = 2y$*



$\mathbb{V}(t_1 + t_2 x + -x^2 - x^3 + y^2)$ $\qquad$ $X = \mathbb{V}(y^2 - x^3 - x^2)$

Figure 1. Deformation of nodal curve

With $T_f^1 = \frac{\mathbb{C}[x,y]}{\langle x,y\rangle}$ *with* $g_i = x^i$. *Thus the versal deformation space of the nodal curve has dimension two and is given by;* $f_t = t_1 + t_2 x + x^2 - x^3 + y^2$

Example 2.2.12. *We consider a* **hypersurface** $X = \mathbb{V}(f_i) \subseteq \mathbb{C}^n$. *Then,*

$$\operatorname{Spec} A = \mathbb{C}[X]/\Big(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big).$$

Specifically, if we consider a cone $X = \mathbb{V}(z^2 - xy) \subseteq \mathbb{C}^3$, *then we have that,*

$$\operatorname{Spec} A = \mathbb{C}[x,y,z]/(f, -y, -x, 2z)$$

with $f = z^2 - xy + t$. *This is a deformation of* A_1 *singularity.*

Theorem 2.2.13 (Artin, Lipman,Wah). *Let* X *be a rational singularity. Then* $Def(\tilde{X})$ *maps finitely to one to a component of* $Def(X)$, *is called Artin component ;*

$$(Def(X))_{art} \cong Def(\tilde{X})/\prod W_j$$

where W_j *are the rational double point configuration supported by the exceptional set* $\widetilde{X}$.

2.2.1 Root System from Deformation

Let V be a finite dimensional Euclidean vector space with standard Euclidean inner product denoted by $\langle \cdot, \cdot \rangle$:

1. **A reflection S on V is an orthogonal transformation**

$$S : V \to V \text{ such that for every } \alpha \in V,\ S(\alpha) = -\alpha$$

and it fixes point wise the hyperplane

$$H_\alpha = \Big\{\beta \in V \ \Big|\ \langle \beta, \alpha \rangle = 0\Big\}$$

of V. We describe this reflection by the formula $S_\alpha(\beta) = \beta - 2\frac{\langle \beta,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$.

2. **A subset Φ of V of finite set of non zero vectors is called a root system if:**
   (a) **The roots span V.**
   (b) **For every $\alpha \in \Phi$ the only multiple of V in Φ are $\pm\alpha$.**
   (c) **Φ is crystallographic, that is, for every $\alpha, \beta \in \Phi$ we have that $\zeta_{\beta\alpha} = 2\frac{\langle \beta,\alpha\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$.**

(d) For every $\beta \in \Phi$, there exist a unique reflection S_β such that

$$S_\beta(\Phi) = \Phi.$$

Definition 2.2.14. *Finite group of isometries of V generated by reflection, $S_\alpha(\beta)$ through hyperplane associated to the roots of Φ is called the **Weyl group**.*

Definition 2.2.15. *The rank of a root system Φ is the dimension of V.*

Example 2.2.16. *We note that, there is only one root system of rank 1 consisting of non-zero vectors $\left\{\alpha, -\alpha\right\}$. This root is called A_1.*
In rank two there are four non zero vectors corresponding to the reflection

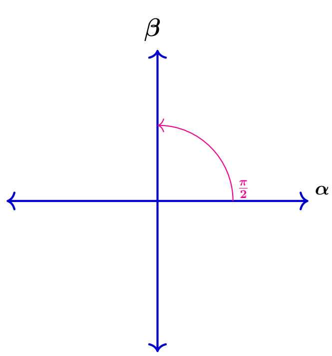
$$S_\alpha(\beta) = \left\{\beta + n\alpha \;\middle|\; n = 0, 1, 2, 3\right\}$$

(a) The simplest root system corresponds to $\theta = \frac{\pi}{2}$ and is called $A_1 \times A_1$ which is a direct sum of rank 1 root system A_1.

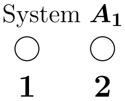
(b) When the of reflection $\theta = \frac{\pi}{3}$, the root system comprise of 6 vectors that corresponds to the vertices of a regular hexagon.

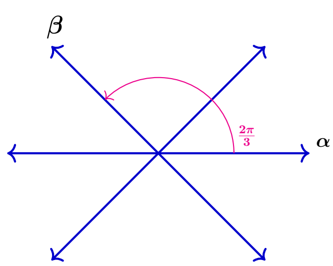
(c) If $\theta = \frac{\pi}{4}$ the root system comprise of 8 vectors corresponding to the midpoint of the edges of a regular square. This root system is called B_2.

(d) If $\theta = \frac{\pi}{6}$ the root system consist of 12 vectors corresponding to the vertices of two regular hexagon of different sizes and rotated away from each other by an angle of $\frac{\pi}{6}$.

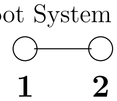


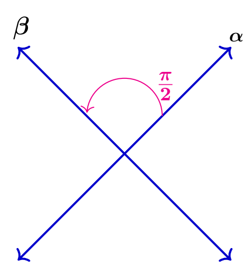
Root System $A_1 \times A_1$



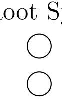


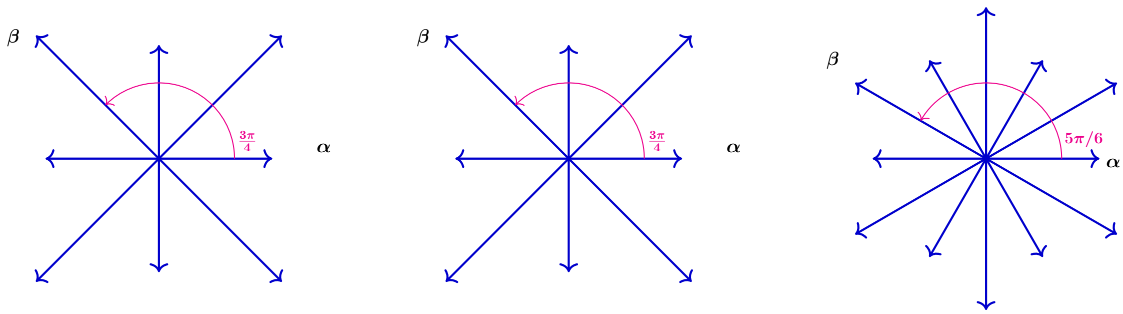
Root System A_2





Root System D_2





Root System B_2 — $1 \longleftarrow 2$

Root System C_2 — $1 \longrightarrow 2$

Root System G_2 — $1 \longleftarrow 2$

Definition 2.2.17. *A subset $\Phi^+ \subseteq \Phi$ is called a positive root if the following conditions hold:*

(a) For each root $\alpha \in \Phi$ only one of $\{\alpha, -\alpha\}$ is contained in Φ^+

(b) For any distinct $\alpha, \beta \in \Phi^+$ with $\alpha + \beta$ a root,then $\alpha + \beta \in \Phi$.

Lemma 2.2.18. *For every set of positive roots Φ^+, the elements of the form $-\Phi$ are called negative roots.*

Definition 2.2.19. *An element of Φ^+ is called a simple root if it cannot be expressed as a sum of two elements of Φ^+.*

Example 2.2.20. *We consider the set of positives for the G_2 root system with simple roots α_1 and α_2.*

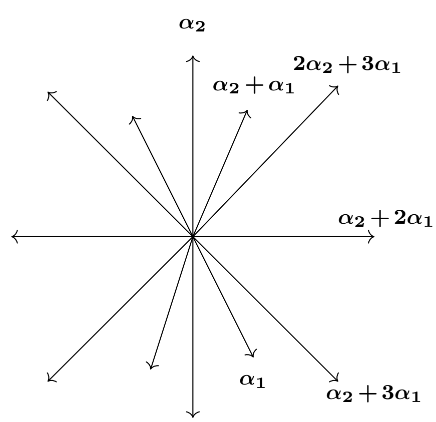


Figure 2. Positive and Simple Root System

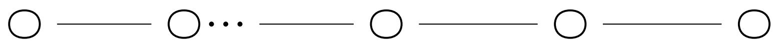
2.2.2 Classification of Root System

Definition 2.2.21. *The coxeter graph of a root system Φ is a multigraph that has a vertex for each simple root of a root system Φ and every pair α, β of distinct vertex is connected by $\zeta_{\alpha\beta} \cdot \zeta_{\beta\alpha} = 4\cos^2\theta \in \{0, 1, 2, 3\}$ edges.*

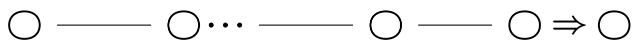
Definition 2.2.22. *A Dynkin diagram of a root system is its coxeter graph with arrows attached to the double and triple edges that point to the shortest root.*

Theorem 2.2.23. *The Dynkin diagram of an irreducible root system is one of the diagram shown below.*

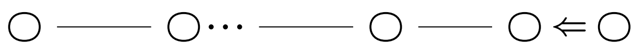
$\boldsymbol{A_n, n \geq 1}$**:**



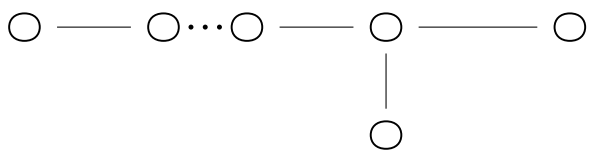
$\boldsymbol{B_n, n \geq 2}$**:**



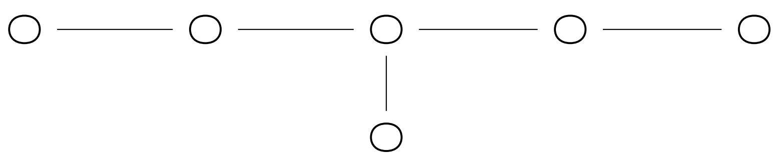
$\boldsymbol{C_n, n \geq 3}$**:**



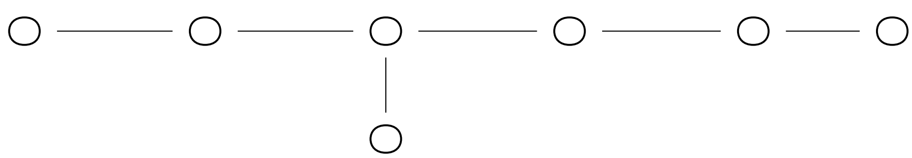
$\boldsymbol{D_n, n \geq 4}$**:**



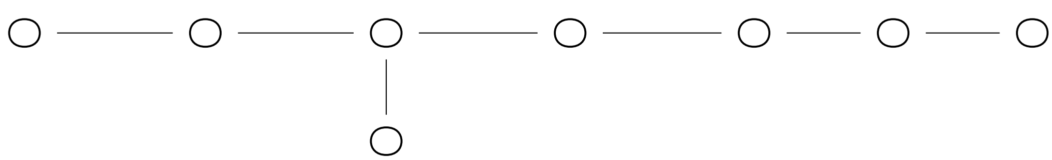
$\boldsymbol{E_6}$**:**



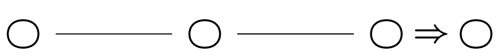
$\boldsymbol{E_7}$**:**



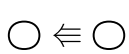
$\boldsymbol{E_8}$**:**



$\boldsymbol{F_4,}$**:**



$\boldsymbol{G_2,}$**:**



2.3 McKay quivers and Deformation of Cyclic quotient Singularities

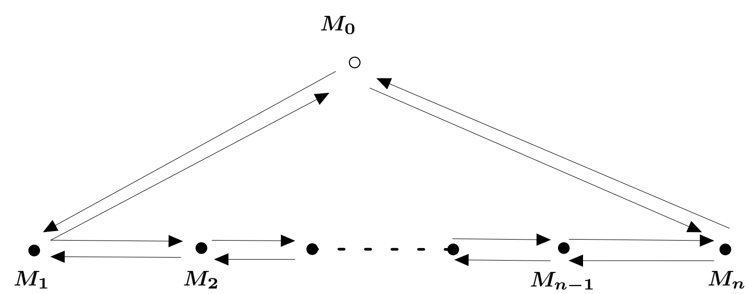
As discussed in [RO98], A_n singularities are cyclic quotient singularities $\mathbb{C}^n/\mathbb{Z}/(n+1)\mathbb{Z}$ with the action on $\mathbb{C}[u,v]$ by :

$$(u,v) \mapsto (\zeta_{n+1}u\ \zeta_{n+1}^{-1}v) \tag{2}$$

with ζ_{n+1} is a primitive $(n+1)$ of root of unity. We obtain the generating invariants $x_0 = u^{n+1},\ x_1 = uv,\ x_2 = v^{n+1}$, and the generating equation

$$x_o x_2 = (uv)^{n+1} = x_1^{n+1}.$$

The MacKay- quiver is of the form,



with $n+1$ vertices are one dimensional representations define by tensoring ;

$$M_j : u \to \zeta_{n+1}^j u,\ \ u \in \mathbb{C},\ \ j = 0, \ldots, n$$

This representation [2] yields : $M_j \otimes \mathbb{C}^2 = M_{j-1} \oplus M_{j+1},\ \ j \in \mathbb{Z}_{n+1}$,
We fix a basis for each M_j with a number u_j and

$$U = \begin{pmatrix} 0 & 0\cdots & \cdots 0 & 0 & u_n \\ u_o & 0\cdots & \cdots 0 & 0 & 0 \\ 0 & u_1\cdots & \cdots 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0\cdots & \cdots & u_{n-1} & 0 \end{pmatrix}$$

Such that, we can identify $M_j \to M_{j+1}$ as a linear homomorphism by describing the endomorphism $U \in End\ M$, with expression $M = M_o \oplus \ldots \oplus M_n$

Similarly, we have an endomorphism $V \in End\ M$ with

$$V = \begin{pmatrix} 0 & v_1 & 0 & \cdots & 0 \\ 0 & 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & v_n \\ v_0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

These endomorphism have diagonal: $U^{n+1} = u_0 \cdots u_n E_{n+1}$,

$$UV = diag(v_o u_n \cdot v_1 u_0 \cdots v_n u_{n-1}), \quad V^{n+1} = v_0 \cdots v_n E_{n+1}.$$

Taking the entries of this map; $u_o \cdots u_n, v_0 u_n, v_1 u_0, \ldots, v_n u_{n-1}, v_o \ldots, u_n$ as generators of subalgebra of $\mathbb{C}[u_o, \ldots, u_n v_0 \ldots, v_n]$, presented by

$$\mathbb{C}[x_0, x_1^{(0)}, \ldots, x_1^{(n)}, x_2] / \langle x_0 x_2 = x_1^{(0)} \ldots x_1^{(n)} \rangle$$

with $x_o = u_o \ldots u_n, \;\; x_2 = v_o \ldots v_n, \;\; x_1^{(j)} = v_j u_{j-1}, \quad j \in \mathbb{Z}_{n+1}$. We express the corresponding deformation in the form :

$$[U, V] = UV - VU = diag(\lambda, \ldots, \lambda_n), \qquad \sum_{j=0}^{n} \lambda_j = 0.$$

The following theorem [RO98] says that in order to understand the deformation theory of cyclic quotient surface singularities we need to consider underlying quiver seriously as MCKAY- quiver.

Theorem 2.3.1. *For a given (n, q) take in $\mathbb{C}[u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}]$ the subalgebra generated by those elements of diagonal matrices*

$$U^{i_\rho} V^{j_\rho}, \quad \rho = 0, \ldots r + 1$$

belonging to a special representation. then this algebra is canonically isomorphic to the algebra of the total space of the Artin components of the $A_{n,q}$ singularity up to smooth factor.

3 Resolution of Singularities and Dual graph

In this section we resolve Kleinian singularity. These singularity have an excellent characterization of absolute isolated double point. The resolution process involve successive blowup where each blowup is at an isolated double point, therefore whenever we blowup a singularity of this type, the result is either non-singular or the singularities are also Du Val type.

Moreover, there is a correspondence between Du Val singularity and Dykin diagram in a way that; Suppose we have a resolution $\pi : \tilde{X} \to X$, then the preimage of the origin will be a tree of projective lines and any two lines either intersect transversely in a singular point or are disjoint. On the other hand, any point of intersection contains only two lines. We therefore draw a graph whose nodes are the projective lines and connect two nodes with an edge if the corresponding lines intersect. This graph will be a Dykin diagram of type ADE.

3.1 Projective Spaces and Projective Varieties

Definition 3.1.1. *An* ***n-dimensional projective space*** *is defined by $\mathbb{P}^n := \mathbb{C}^n/\mathbb{C}^*$ where $\mathbb{C}^*$ acts by $\lambda(x_i) = (\lambda x_i)$. A point $p = [x_0 : \ldots : x_n] \in \mathbb{P}^n$ is the equivalence class of (x_i), with x_i the homogeneous coordinates of p. Further, we have that*

$$\mathbb{P}^n = \cup_{i=0}^n U_i \cong \cup_{i=0}^n \mathbb{C}^n_{\left\langle \frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i} \right\rangle} = \mathbb{C}^n \sqcup \mathbb{P}^{n-1} = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \ldots \cup \mathbb{C}^1 \cup \{\infty\}$$

is covered by $n+1$ affine $n-$spaces.

Example 3.1.2. $\mathbb{P}^2 = \mathbb{C}^2 \sqcup \mathbb{P}^1 = \mathbb{C}^2 \cup \mathbb{C} \cup \left\{\infty\right\}$ *with the disjoint pieces $\mathbb{C}^2$ and $\mathbb{P}^1$ identified through the map*

$$[x_0 : x_1 : x_2] \mapsto \begin{cases} (\frac{x_1}{x_0}, \frac{x_2}{x_0}) \in \mathbb{C}^2 \ for \ x_0 \neq 0 \\ [x_1 : x_2] \in \mathbb{P}^1 \ for \ x_0 = 0. \end{cases}$$

Let $u_1 = \frac{x_1}{x_0}, u_2 = \frac{x_2}{x_0}; v_1 = \frac{x_0}{x_1}, v_2 = \frac{x_2}{x_1}$ and $w_1 = \frac{x_0}{x_2}, w_2 = \frac{x_1}{x_2}$ be the affine coordinates in respective three charts $U_i = \{x_i \neq 0\}$.

The gluing data on $U_0 \cap U_1 = \{x_0 \neq 0, x_1 \neq 0\}, U_0 \cap U_2 = \{x_0 \neq 0, x_2 \neq 0\}$ and $U_1 \cap U_2 = \{x_1 \neq 0, x_2 \neq 0\}$ is given by

$$\begin{bmatrix} u_1^{-2} & 0 \\ 0 & u_1^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} 0 & u_2^{-2} \\ u_2^{-1} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \ \textit{and} \ \begin{bmatrix} v_2^{-1} & 0 \\ 0 & v_2^{-2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

respectively.

Definition 3.1.3. *A* ***codimension*** m ***projective variety***

$$X := \mathbb{V}(f_{d_1}, \dots, f_{d_m}) = \cup_{i=0}^{n} X \cap U_i \subset \mathbb{P}^n$$

is the zero set of homogeneous polynomials $f_{d_j} \in \mathbb{C}[x_0, \dots, x_n]$, *that is* $f_{d_j}(\lambda p) = \lambda^{d_j} f_{d_j}(p)$, *with* $p \in \mathbb{P}^n$ *and the dimension* $\dim X = n - m$.

Further, X *is covered by* $n+1$ *affine varieties*

$$X \cap U_i = \mathbb{V}(\widetilde{f_{d_1}}, \dots, \widetilde{f_{d_m}}) \subset U_i = \{x_i \neq 0\} \cong \mathbb{C}^n$$

with

$$\widetilde{f_{d_j}} = f_{d_j}(x_0, \dots, x_j = 1, \dots, x_n)$$

the dehomogenisation of f_{d_j}. *The projective variety* X *is a degree* $d = d_1$ *hypersurface if* $m = 1$.

Example 3.1.4. *The conic* $V = \mathbb{V}(x_0^2 + x_1^2 - x_2^2) \subset \mathbb{P}^2 = \mathbb{C}^2 \sqcup \mathbb{P}^1$ *is covered by* 3 *affine varieties* $V \cap U_i \subset U_i = \{x_i \neq 0\} \cong \mathbb{C}^2$. *We then have, from Example 3.1.2 that;*

$$\begin{aligned} V &= (V \cap U_0) \cup (V \cap U_1) \cup (V \cap U_2) \\ &= \mathbb{V}(1 + u_1^2 - u_2^2) \cup \mathbb{V}(v_1^2 + 1 - v_2^2) \cup \mathbb{V}(w_1^2 + w_2^2 - 1). \end{aligned}$$

Let $(V, \mathcal{O}_V)$ be a projective variety furnished by it sheaf of regular functions. The following example is a demonstration of how to compute $\mathcal{O}_V$.

Example 3.1.5. *In this example, we find the sheaf* $\mathcal{O}_{\mathbb{P}^2_{[x_i]}}(U)$ *of regular functions on* $U = \mathbb{P}^2 \setminus \mathbb{V}(x_0^2 + x_1^2 - x_2^2)$. *From Example [3.1.4] , we know that the basic open sets* $D_f = \{f \neq 0\}$ *of* U *are*

$$\begin{aligned} U \cap U_0 &= \left\{(u_1, u_2) \in \mathbb{C}^2 : 1 + u_1^2 - u_2^2 \neq 0\right\} \\ &= \mathbb{C}^2 \setminus \mathbb{V}(1 + u_1^2 - u_2^2), \end{aligned}$$

$$\begin{aligned} U \cap U_1 &= \left\{(v_1, v_2) \in \mathbb{C}^2 : v_1^2 + 1 - v_2^2 \neq 0\right\} \\ &= \mathbb{C}^2 \setminus \mathbb{V}(v_1^2 + 1 - v_2^2), \end{aligned}$$

and

$$\begin{aligned} U \cap U_2 &= \left\{ (w_1, w_2) \in \mathbb{C}^2 : w_1^2 + w_2^2 - 1 \neq 0 \right\} \\ &= \mathbb{C}^2 \setminus \mathbb{V}(w_1^2 + w_2^2 - 1). \end{aligned}$$

Now, denote $X = \mathbb{P}^2_{[x_i]}$ *then;*

$$\mathcal{O}_X(U) \xrightarrow{x_0 \neq 0} \mathcal{O}_{\mathbb{P}^2}(U \cap U_0) = \mathcal{O}_{\mathbb{A}^2}(D_{1+u_1^2-u_2^2}).$$

$$\mathcal{O}_X(U) \xrightarrow{x_1 \neq 0} \mathcal{O}_{\mathbb{P}^2}(U \cap U_1) = \mathcal{O}_{\mathbb{A}^2}(D_{v_1^2+1-v_2^2}).$$

$$\mathcal{O}_X(U) \xrightarrow{x_2 \neq 0} \mathcal{O}_{\mathbb{P}^2}(U \cap U_2) = \mathcal{O}_{\mathbb{A}^2}(D_{w_1^2+w_2^2-1}).$$

3.2 Blow Up

Blowups are the basic device for fabricating resolution of singularities. They constitute a certain type of transformation of regular schemes X to yield a relatively new regular scheme $\tilde{X}$ above X, the blowup of X, together with defined projection $\pi : \tilde{X} \to X$ called the blowup map.
The blowup plays an important role in a way that, it untie the singularities of a given vanishing set $Y \subseteq X$ by emphasizing on its inverse image E in $\tilde{Y} \subseteq \tilde{X}$.
Blowing up a variety at a point means that we replace the point with projectivisation of the tangent space at that point.

Definition 3.2.1 (Resolution of singularities)**.** *Let $\tilde{Y}$ be regular and irreducible variety, a resolution of a singularities of an irreducible varieties X is a* ***proper birational morphism***

$$\pi : \tilde{Y} \to X,$$

that is, π restricts to an isomorphism $\pi^{-1}(\tilde{Y}) \xrightarrow{\sim} X^{reg}$.

Theorem 3.2.2 (Hironaka)**.** *Let X be any variety over a field of characteristic zero. Then there exists a variety Y and a regular map $\pi : \tilde{Y} \to X$ that is a birational equivalence.*

3.2.1 The blowup of $\mathbb{C}^n$ at the origin

We define the blowup at a point in affine $n-$space $\mathbb{C}^n$, as a closed subvariety of $\mathbb{C}^n \times \mathbb{P}^{n-1}$.

$$\begin{aligned} \mathrm{Bl}_0\, \mathbb{C}^n =& \left\{ (x, \ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} | x \in \ell \right\} \subseteq \mathbb{C}^n \times \mathbb{P}^n \\ =& \left\{ \big((x_1, \dots, x_n); [y_1 : \dots : y_n] \in \mathbb{C}^n \times \mathbb{P}^{n-1} \;\; | \;\; x_i y_j = x_j y_i \big) \right\}, \end{aligned}$$

with the line ℓ uniquely determined by $(x_1,\ldots,x_n)$. We have that, the projective space $\mathbb{P}^n$ is covered by affine charts, with at least $y_i \neq 0$ with coordinates $[y_1:\ldots:y_n]$, then the coordinates in $x_1,\ldots,x_{n-1}$ are redundant, we obtain an affine space with coordinate $(y_1,\ldots,y_{n-1},x_n)$ and therefore our definition translates to,

$$\mathrm{Bl}_0\,\mathbb{C}^n = \Big\{\Big((x_1,\ldots,x_n);(y_1,\ldots,y_{n-1})\Big)\in\mathbb{C}^n\times\mathbb{P}^{n-1}|x_j=x_ny_j,\ 1\leq j\leq n-1\Big\}.$$

Remark 3.2.3. *Suppose a point p is not the origin, we can still blow up at that point by shifting origin to k by a transformation, $\phi_p : x \mapsto x-k$ and define the blowup as;*

$$Bl_p\mathbb{C}^n = \mathbb{V}\Big((x_i-k_i)y_j-(x_j-k_j)y_i : 1\leq i,j\leq n\Big)\subseteq\mathbb{C}^n\times\mathbb{P}^{n-1}.$$

Since this map is not defined at the origin, we intend to eliminate this indeterminacy of this map by constructing a graph Γ of π to be open subset of the quasi projective variety $\mathbb{C}^n\times\mathbb{P}^{n-1}$.

Example 3.2.4. *We consider a variety $V=\mathbb{V}(x^2+zy^2+z)\subseteq\mathbb{C}^3$. By Jacobian criterion, we obtain two singularities at the point $p=(0,\pm i,0)$.*

Suppose we need to blowup the singularity at the point $p_1=(0,-i,0)$, we adopt the coordinate change, $x\to X,\quad y\to Y+i,\quad z\to Z$.

This implies that, $x^2+zy^2+z=0$ translates to $X^2+ZY^2+2iYZ=0.$

We now blowup the the final variety defined by $\mathbb{V}(X^2+ZY^2+2iYZ)\subseteq\mathbb{C}^3$ to obtain resolution $\pi:\tilde{V}\to V.$

Definition 3.2.5. *The blowup at a point $p\in\mathbb{C}^n$ is a birational morphism,*

$$\pi:\mathrm{Bl}_p\mathbb{C}^n\to\mathbb{C}^n$$

and the fibres of π are:

$$\pi^{-1}(x)=\begin{cases}(x,[x]) & for\ x\neq 0\\ \Big\{0\Big\}\times\mathbb{P}^{n-1} & if\ x=0.\end{cases}$$

So π is an isomorphism between $\mathrm{Bl}_p\,\mathbb{C}^n\setminus\pi^{-1}(0)$ and $\mathbb{C}^n\setminus\Big\{0\Big\}$, but the origin is replaced with $\mathbb{P}^{n-1}$.

Definition 3.2.6. *The set $E=\pi^{-1}(0)\subseteq\mathrm{Bl}_0\,\mathbb{C}^n$ is called the exceptional divisor. The points on E are in bijection with lines through the* **origin***, $(0,0,\ldots,0)\in\mathbb{C}^n$.*

Proposition 3.2.7. *The blowup* $\mathbf{Bl_0}\,\mathbb{C}^n$ *is a smooth irreducible variety of dimension* n*.*

Proof. For a blowup $\mathbf{Bl} \subseteq \mathbb{C}^n \times \mathbb{P}^{n-1}$. Let $U_i \subseteq \mathbb{C}^{n-1}$ be a standard affine chart of $\mathbb{C}^{n-1}$. It suffices to check that each $B_pl \cap (\mathbb{C}^n \times U_i)$ is smooth. For simplicity, we let $i = n$. Then we have that;

$$\mathbf{Bl}_p \cap (\mathbb{C}^n \times \mathbb{P}^{n-1}) \cong \mathbb{C}^n.$$

This implies that:

$$\begin{aligned}\mathbf{Bl}\cap(\mathbb{C}^n \times \mathbb{P}^{n-1}) =&\{(x_1,\dots x_n);[y_1:\dots:y_n] \;\Big|\; x_iy_j - x_jy_i = 0\} \\ =&\{(x_1,\dots x_n);[\frac{y_1}{y_n}:\dots:\frac{y_{n-1}}{y_n}:1] \;\Big|\; x_j = x_n(\frac{y_j}{y_n}), y_n \neq 0\}.\end{aligned}$$

We obtain an isomorphism; $\Big((x_1,\dots x_n);[\frac{y_1}{y_n}:\dots:\frac{y_{n-1}}{y_n}] \mapsto (\frac{y_1}{y_n},\dots,\frac{y_{n-1}}{y_n},y_n)\Big)$. That is,

$$\begin{aligned}\mathbb{A}^n &\to \mathbf{Bl}\cap U_n \\ \Big(t_1,\dots t_{n-1},t_n\Big) &\mapsto \Big((t_nt_1,\dots t_nt_{n-1},t_n);[t_1:\dots:t_{n-1}:1]\Big).\end{aligned}$$

□

Example 3.2.8. *We consider the blowup* $\pi : X \to \mathbb{C}^3$ *of a point* $p = (0,0,0)$ *in an affine three space* $\mathbb{C}^3$*. Let* (x,y,z) *be the coordinates of the afiine* $3-space$ *and* $[u,v,w]$ *be the coordinates of the projective space* $\mathbb{P}^2$*. Then by definition, the blowup map is given by;*

$$\begin{aligned}Bl_{(0,0,0)}\mathbb{C}^3 =&\{(x,\ell) \in \mathbb{C}^3 \times \mathbb{P}^2 : x \in \ell\} \subseteq \mathbb{C}^3 \times \mathbb{P}^2 \\ &= \{\Big((x,y,z),[u:v:w]\Big) \in \mathbb{C}^3 \times \mathbb{P}^2 : x_iu_j - x_ju_i = 0\}.\end{aligned}$$

For the projective space $\mathbb{P}^2$ *is covered by its three standard charts,*

$$\mathbb{P}^2 = \mathbb{A}^2_x \cup \mathbb{A}^2_y \cup \mathbb{A}^2_w.$$

The blowup is also covered by three charts. The intersection with each chart of the preimage of the variety $X \subset \mathbb{C}^3$ *is determined as:*

$$\mathbb{C}^3 \xleftarrow{\pi} \mathbf{Bl}_{(0,0,0)} \subset \mathbb{C}^3 \times \mathbb{P}^2 \cong (\mathbb{C}^3 \times \mathbb{A}^2_x) \cup (\mathbb{C}^3 \times \mathbb{A}^2_y) \cup (\mathbb{C}^3 \times \mathbb{A}^2_w).$$

For the exceptional divisor $E=\pi^{-1}(p)\cong\mathbb{P}^2$. *We consider a closed point* $q\in\pi^{-1}(p)$ *and define a regular parameter* (u,v,w) *satisfying the relations,*

$$xv=yu, xw=zu, yw=zv.$$

For the chart $u\neq 0$, *the above relation translates to;* $x=x, y=xv, z=xw.$ *We see that* $x=0$ *is local equation of the exceptional divisor* $E.$

Similarly, for the chart $v\neq 0$ *we have the relation* $x=yu, y=y, z=yw.$ *Thus we obtain a local equation,* $y=0$ *of the exceptional divisor* E.

Lastly, for the chart $w\neq 0$ *we have the blowup relation:* $x=zu, y=zv, z=z.$ *We obtain* $z=0$ *as a local equation of the exceptional locus* E. *For the chart* $w\neq 0$ *we obtain;*

$$\begin{aligned}X =&\pi^{-1}\Big(\mathbb{V}(X)\Big)\cap(\mathbb{C}^3\times\mathbb{A}^2_z)\\ =&\Big\{(x,y,z),[u:v:w]\;\Big|\;\mathbb{V}(X)\Big\}\\ =&\Big\{(x,yzu,v)\mid xv=yu, xw=zu, yw=zv, \mathbb{V}(f)\Big\}\subset\mathbb{A}^5.\end{aligned}$$

We repeat the same steps to blow up the along the charts $v\neq 0$ *and* $u\neq 0.$ *Projecting* $\mathbb{A}^5$ *into 3-dimensional space* $\mathbb{C}^3$ *with coordinates* (u,v,z) *the preimage set of blowup surface becomes* $\Big\{(u,v,z)\;\Big|\;\mathbb{V}(f)\Big\}.$

This is to say, if $U=\mathrm{Spec}(R)\in\mathbb{C}^3$ *is an affine neighborhood of* $p=(0,0,0)$, *with* p *having an ideal* $m_p=(x,y,z)\in R$. *Then*

$$\pi^{-1}(U)=\mathrm{Proj}\Big(\oplus m_p^n\Big)=\quad\mathrm{Spec}\Big(R[\frac{y}{x},\frac{z}{x}]\cup\mathrm{Spec}\Big(R[\frac{x}{y},\frac{z}{y}]\cup\mathrm{Spec}\Big(R[\frac{x}{z},\frac{y}{z}]\Big).$$

3.2.2 The blowup of $X\subseteq\mathbb{C}^n$ at $p\in X$

Definition 3.2.9. *We define the strict transform of a variety* X *as*

$$\mathrm{Bl}_p X=\overline{\pi^{-1}(X\setminus\{p\})}\subset\mathrm{Bl}_p\mathbb{C}^n$$

the closure of $\pi^{-1}(X\setminus\{p\})\subset\mathrm{Bl}_p\mathbb{C}^n.$

The blowup of X at p is the projection map $\pi|_{\mathrm{Bl}_p X}:\mathrm{Bl}_p X\to X.$ The exceptional divisor is $E=\pi^{-1}(p)\subset\mathrm{Bl}_p X.$

Example 3.2.10. *We consider a variety* $V=\mathbb{V}(xy-z^3)\subset\mathbb{C}^3.$ *By Jacobian criterion this variety has an isolated singularity at the origin,* $p=(0,0,0)$. *We blowup this* *point by defining equation;*

$$\begin{aligned}&\mathrm{Bl}_p\,\mathbb{C}^3 = \{(x,\ell)\in\mathbb{C}^3\times\mathbb{P}^2 : x\in\ell\}\subseteq\mathbb{C}^3\times\mathbb{P}^2\\ &= \{\big((x,y,z),[u:v:w]\big)\in\mathbb{C}^3\times\mathbb{P}2 : xv-yu=0, yw-zv=0, xw-zu=0\}.\end{aligned}$$

Blowing up along the three charts, we obtain the exceptional divisors, $x^2=y^2=z^2=0$ *and corresponding smooth strict transforms respectively:*

$$\begin{aligned}\tilde{Y}_{u\neq 0} &: y-xz^3=0,\\ \tilde{Y}_{v\neq 0} &: x-yz^3=0,\\ \tilde{Y}_{w\neq 0} &: xy-z=0.\end{aligned}$$

Remark 3.2.11. *We note that* $\pi:\mathrm{Bl}_p X\to X$ *defines an isomorphism between* $B_pX\setminus E$ *and* $X\setminus\{p\}$, *and therefore* π *is a birational morphism. The total preimage in* $\mathrm{Bl}_p\,\mathbb{C}^n$, $\pi^{-1}(X)=E\cup\mathrm{Bl}_p X\subset\mathrm{Bl}_p\,\mathbb{C}^n$ *is called the* **the total transform of** X. *It is the union of the exceptional divisor* $E=\pi^{-1}(p)$ *in* $\mathrm{Bl}_p\,\mathbb{C}^n$ *and the strict transform* $\mathrm{Bl}_p X$.

3.2.3 Blowup along Subvariety

We consider the blowup of an $\mathbb{C}^n$ **along** $Y\subset X\subseteq\mathbb{C}^n$**, with ideal** $I_Y=\langle f_1,\dots,f_m\rangle$**. We define the blowup as the closure in** $\mathbb{C}^n\times\mathbb{P}^{n-1}$ **of the image of the map,**

$$\begin{aligned}\pi : X\setminus Y &\to \mathbb{C}^n\times\mathbb{P}^{n-1}\\ (x_1,\dots,x_n) &\mapsto (x_1,\dots,x_n, f_1(x_1,\dots,x_n)\dots,f_m(x_1,\dots,x_n)).\end{aligned}$$

Definition 3.2.12. *Let* X *and* Y *be quasi projective variety and* $\pi:X\to Y$ *be a regular map. The graph is set defined as:*

$$\begin{aligned}\Gamma_\pi : X\to Y\\ \{(x,y)\mid\pi(x)=y\}\subseteq X\times Y.\end{aligned}$$

We refers to the closure of the graph $\overline{\Gamma}$ *as the blow up,* $\mathrm{Bl}_I Y$ *along an ideal* I.

Example 3.2.13. *Consider the blowup of* $\mathbb{C}^3$ *along the* $z-axis$. *Suppose we take the* ***projective coordinates*** $((x,y,z);[u:v:w])\in\mathbb{C}^3\times\mathbb{P}^2$.

Blowing along the **z-axis** *implies the change coordinates,* $xv-yu=0$ *and we consider the kernel of blowup map,*

$$((x,y,z);[u:v])\mapsto[x,y,z;x:y].$$

Therefore we have that $\pi:\mathbb{C}^3\times\mathbb{P}^1\to\mathbb{C}^3$.

Thus the image of the blowup is $\mathbb{C}^3\times\mathbb{P}^1$ *with* $\pi^{-1}(x,y,z)$ *is a point unless,* $x=y=0$ *when it becomes a projective space,* $\mathbb{P}^1$.

Remark 3.2.14. *In case we require successive blowup to resolve a given variety X, each blowup is completely specified by its center which is defined to be regular closed subscheme Z chosen in relation to zero sets of the variety X. Therefore at the center of the blowup, the mapping fails to be* **isomorphic** *.*

3.3 Resolution of ADE Singularities and Dynkin Diagrams

3.3.1 Resolving Singularities of A_n -types

Here we resolve the A_n singularities with generating polynomial

$$Y := \mathbb{V}(x^2 + y^2 + z^{n+1}) \subseteq \mathbb{C}^3$$

and construct the corresponding dual graph. We resolve this type of singularity by blowing up the only singular point $p = (0,0,0)$.

This surface is covered by three open affine chart. If we consider the chart $w \neq 0$ we have that $y = z\frac{v}{w}$, $x = z\frac{u}{w}$, $z = z$. So our original equation becomes;

$$z^2\Big((\frac{u}{w})^2 + (\frac{v}{w})^2 + z^{n-1}\Big) = 0.$$

This has two irreducible components, one where $z^2 = 0$ corresponding to the $\pi^{-1}(0,0,0)$ and the other $(\frac{u}{w})^2 + (\frac{v}{w})^2 + z^{n-1}) \cong u^2 + v^2 + z^{n-1} = 0$, defines the strict transform (i.e. the closure of $\pi^{-1}(X \setminus \{0\})$ inside this open set).
The exceptional divisor with open set defined by $z \neq 0$ is the set of point,

$$\Big((0,0,0),[a : \pm ia : 1]\Big) \text{ with } a \in \mathbb{C}.$$

For the chart defined by $u \neq 0$, we obtain equation

$$x^2(1 + (\frac{v}{u})^2) + x^{n-1}(\frac{w}{u})^{n+1} = 0 \cong x^2(1 + v^2 + x^{n-1}w^{n+1}) = 0.$$

We consider the second irreducible part, $1 + v^2 + x^{n-1}w^{n+1} = 0$ and the intersection with $\pi^{-1}(0)$ is an affine line $\Big((0,0,0),[1 : \pm i : b]\Big)$ with $b \in \mathbb{C}$. By symmetry, we obtain the same on the chart $v \neq 0$.

Remark 3.3.1. *We note that* $\Big((0,0,0);[a : \pm ia : 1]\Big) = \Big((0,0,0);[1 : \pm i : b]\Big)$ *when* $a, b \in \mathbb{C}^*$ *and* $b = a^{-1}$. *This can be seen by checking on the limits* $b \to \infty$, *on the point* $\Big((0,0,0);[b^{-1} : \pm ib^{-1} : 1]\Big) \implies b^{-1} = 0$ *so that we have the representation of this limits point as* $\Big((0,0,0);[0 : 0 : 1]\Big)$ *which corresponds to the case* $a = 0$.

We conclude that the exceptional divisor E consist of two projective lines intersecting at the point $\big((0,0,0);[0:0:1]\big)$

When $n>2$, this point of intersection is singular point of the resulting blowup surface $\tilde{Y}_{i=w,v}$ and the the singularity obtained here is of type A_{n-2}. (i.e. the the variety defined by an equation $u^2+v^2+z^{n-1}=0$, has the singularity of A_{n-2}-type).
Again we can blowup and obtain the exceptional divisor which is either single $\mathbb{P}^1$ or two copies Γ_o and Γ_1 intersecting at a point, say Q. In the first case , the inverse image of E and E' intersect this copy in different points and in the second case we will have E intersect Γ_o in a point distinct from Q and E' intersect Γ_1 in a point distinct from Q.

By this successive blowup of singular point until we obtain a nonsingular variety. Ultimately, we will get projective lines, say $L_1,\dots,L_n$ above the origin such that these projective lines L_i intersect L_{i+1} at distinct points.
The figure below represent the exceptional divisor of A_{n-1} type singularity and resolution dual graph of A_n type singularity.

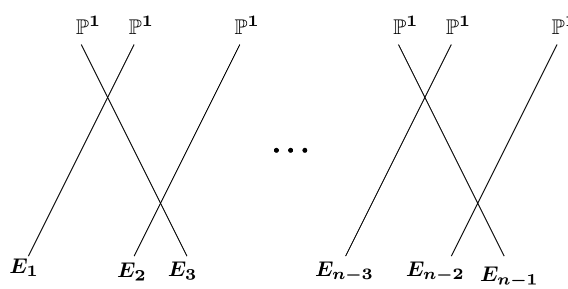


Figure 3. The exceptional divisor of A_{n-1} type singularity.

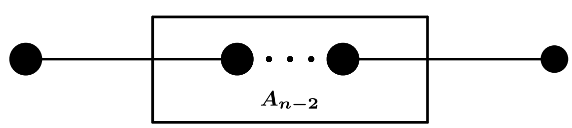


Figure 4. Resolution Graph of A_n.

Example 3.3.2 (The ordinary double point). *We start by an ordinary quadratic cone in* $\mathbf{3}-\boldsymbol{space}$, *with an* **hypersurface** *given by;*

$$p=(0,0,0)\in X:(xy=z^2)\subset\mathbb{C}^3$$

The ordinary double point occurs throughout the theory of algebraic surface and can be used to illustrate the entire hierarchy of argument.

Since this surface is a cone with vertex at p ***(the only singular point)*** *and base the plane conic* $X = xy - z^2 \subset \mathbb{A}^3$. *This singularity has the standard "cylinder" resolution:* $\pi : \tilde{Y} \to X$.

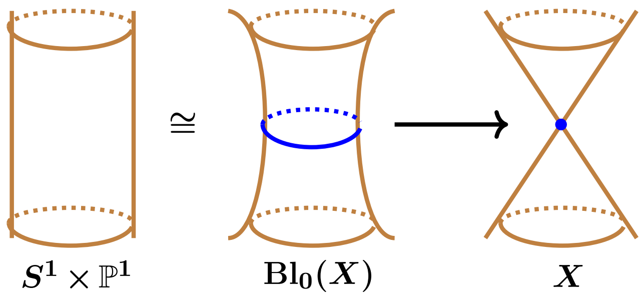


Figure 5. Resolution of A_1 surface singularity

The cone consist of union of generating lines ℓ, *through* p, *and* $\tilde{Y}$ *is the disjoint union of these generating lines,* ℓ. *(That is to say,* $\tilde{Y}$ *is the correspondence between the cone and its generating lines* ℓ*):*

$$\tilde{Y} = \Big\{(x,\ell) : \mathbb{C}^3 \times \mathbb{P}^2 \ : \ x \in \ell\Big\}.$$

We obtain $\tilde{Y}$ *by blowing up point the origin,* $p = (0,0,0)$. *The exceptional curve* $\pi^{-1}(p) = \Gamma$ *of the resolution is a* $-2-curve$ *with* $\Gamma \cong \mathbb{P}^1$.

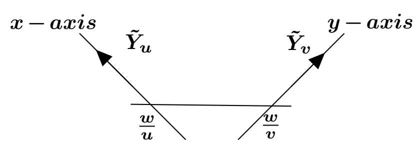


Figure 6. The exceptional curves of the blowup of A_1 singularities.

Since we have only one $\mathbb{P}^1$ *our dual graph will just be represented by a single node.*

Figure 7. Dual graph of A_1-type singularity.

3.3.2 Resolving Singularities of D_n -types

We consider a variety defined by

$$Y := \mathbb{V}(x^2 + y^2 z + z^{n-1}) \subseteq \mathbb{C}^3 \ \text{ where } \ n \geq 4.$$

The exceptional divisor E of the blowup $\tilde{Y}$ is a projective line. For the chart $u \neq 0$ does not intersect the exceptional divisor E, while the intersection of $\tilde{Y}$ with blowup along the chart $v \neq 0$ is defined by equation,

$$(\frac{u}{v})^2 + y(\frac{w}{v}) + y^{n-3}(\frac{w}{v})^{n-1} \cong u^2 + yw + y^{n-3}w^{n-1} = 0.$$

This singularity is analytically isomorphic to A_1 type singularity, since the ideal generated by this equation and its partial derivatives is the maximal ideal, $\langle u, y, w \rangle$.

Now we consider the intersection of $\tilde{Y}$ with chart $w \neq 0$ with defining equation

$$(\frac{u}{w})^2 + z(\frac{v}{w})^2 + z^{n-3} \cong u^2 + zv^2 + z^{n-3} = 0.$$

If $n = 4$ it contains two singular points corresponding to the chart $u = 0, v = \pm i, z = 0$ both of A_1-type singularity. For the case of $v = -i$, we translate the coordinates by $v \to v + i$ to obtain $u^2 + v^2 z + 2izv$ and its partial derivatives are $2u, 2z(v+i), v(v+2i)$. Since we are working on the power series ring, we have that, $v + i$ and $v + 2i$ are units, so the partial derivatives generates the maximal ideal. Furthermore, for $n > 4$ it contain one singular point, $u = 0, v = 0, z = 0$, of type D_{n-2} at the origin.

Resolving this new singularity requires successive blowup until we obtain non-singular variety. This singular point generates a tree of projective line which intersect in the pattern of D_{n-2}.

The preimage of the exceptional divisor E, will intersect in a single point,and also contains another projective line from A_1 singularity from the previous chart.

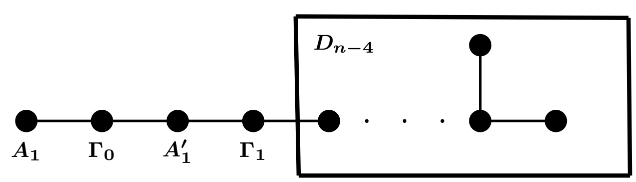


Figure 8. Dual graph of D_n type singularity.

Example 3.3.3. *We consider D_4 and its resolution*
The singularity is at $p = (0,0,0) \in X$ with $Y := \mathbb{V}(x^2 + y^3 + z^3) \subseteq \mathbb{C}^3$. Let the blowup at this singular point p be defined by $\pi : \tilde{Y} \to Y$. It is covered by three **affine pieces***. Let $[u : v : w] \in \mathbb{P}^2$ be projective coordinates and $(x, y, z) \in \mathbb{C}^3$ be the affine coordinates as before. The morphism*

$$\pi : B_{p=(0,0,0)}\ell \to \mathbb{C}^3$$

is defined by $x = uz, y = zv, z = z$ (i.e. the blowup along the chart $w \neq 0$). The inverse image of morphism π is defined by;

$$f(uz, vz, z) = z^2(u^2 + v^3 z + z).$$

Here the factor z^2 vanishes on the exceptional $uv - plane$,

$$\mathbb{A}^2 = \pi^{-1}(p) \mid (z=0) \subset B_p\ell,$$

and the other irreducible component $\tilde{Y}_1 : (u^2 + v^3 z + z = 0) \subset B_p\ell$ is the **birational transform** *of Y. It is clear that the inverse image of p under morphism $\pi : \tilde{Y} \to Y$ is the $v - axis$ and the strict transform $\tilde{Y}_1 : u^2 + z(v^3+1) = 0$ has an ordinary double point at the three points where $u = z = o$ and $v^3 + 1 = 0$.*

The resolution is obtained by blowing up these three points.

It follows that π^{-1} consist of four **-2-curves** *, $\Gamma_o, \Gamma_1, \Gamma_2, \Gamma_3$ intersecting to yield the configuration of the Dykin diagram D_4.*

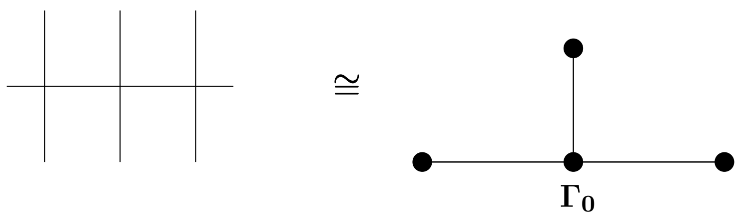


Figure 9. Dual graph of D_4

3.3.3 Resolving Singularity of E_6-type

In this case, we consider the singularity of E_6-type and compute its minimal resolution as well as its dual graph. Let the projective coordinates be given by $[u : v : w]$. This E_6 type singularities is defined by the vanishing of a variety X;

$$X := \mathbb{V}(x^2 + y^3 + z^4) \subseteq \mathbb{C}^3$$

Suppose $\pi : \tilde{Y} \to X$ is a minimal resolution, then the exceptional fiber $E = \cup E_i = \pi^{-1}(0)$. The blowup equations are given by;

$$\tilde{Y} = \mathbb{V}((xv - yu, yw - zv, xw - zu, x^2 + y^3 + z^4)$$

We blowup the origin, the equation of total transform of X on the first coordinate patch is given by;

$$x^2 + (x\frac{v}{u})^3 + (x\frac{w}{u})^4 \cong x^2(1 + v^3 x + w^3 + x^2 w^4) = 0.$$

So the equation of the strict transform $\tilde{Y}$ is given by; $\tilde{Y} := \mathbb{V}(1 + v^3 x + w^3 + x^2 w^4)$. This is smooth and do not intersect E.

The equation of the total transform of X on the second coordinate patch is given by;

$$y^2\Big((\frac{u}{v})^2+y+y^2(\frac{w}{v})^4\Big)\cong y^2(u^2+y+y^2w^4)=0.$$

So that the equation of the strict transform of X is given by; $\tilde{Y}:=\mathbb{V}(s^2+y+u^4+y^2)$. This is a smooth hypersurface and intersect E whenever $u=0$.

The equation of the total transform on the third coordinate patch is given by

$$z^2\Big((\frac{u}{w})^2+z(\frac{v}{w})^3+z^2\Big)\cong z^2(u^2+zw^3+z^2)=0.$$

Here the equation of the strict transform is given by ;

$$u^2+zw^3+z^2=u^2+(z+\frac{w^3}{2})^2-\frac{w^6}{4}=0.$$

This equation represent an equation of A_5-type singularity. The exceptional locus C is the (double) line $u=0$.

If we blowup one more time we get two more exceptional curves, copies of $\mathbb{P}^1$, and the strict transform of C passes through the unique singular point, which an a A_3-singularity. Blowing up one more time, we get two copies of $\mathbb{P}^1$ and again the strict transform of C passes through unique singular point of A_1-type singularity. Blowing up one more introduces one more copy of $\mathbb{P}^1$ and the strict transform of C intersect the middle curve of the A_5 chain. We obtain the dual graph of resolution of E_6 type singularity.

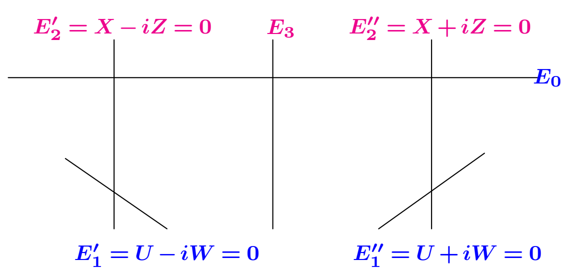


Figure 10. Exceptional divisor of E_6 after third blowups

.

We therefore obtain the resolution dual graph of E_6

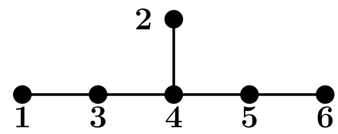


Figure 11. Resolution dual graph of E_6 type singularities.

3.4 Resolution Problems.

3.4.1 Choosing the Centers of Blowup

Our primary step in resolution of singular varieties consist of getting an overview of a possible a path which could results to a solution of problem. Under this consideration, the center of blowup constitutes to primary object of interest and therefore by making a correct choice, we are bound to obtain a resolution $\pi : \tilde{W} \to W$

Given a variety X embedded in a regular ambient scheme W with the defining ideal J, of structure sheaf $\mathcal{O}_W$. Essentially, resolving J, will implies, resolving X.

According to the Philosophy of resolution of singularities, the worst point of a variety X, is attacked first. Therefore, an appropriate center of blowup Z, should lie inside the locus of point where the order of the ideal J is maximal.

Definition 3.4.1 (Top locus of J). *A closed subscheme $top(J) \subset W$ defined as: $top(J) = \left\{a \in W \;\middle|\; ord_a J \text{ is maximal}\right\}$ is the top locus of an ideal $J \subset X$*

Suppose $top(J)$ consist of two transverse lines and one of them has to be chosen as the center, in this case, our center is not unique and therefore, our variety X has asymmetry obtained by interchanging two variables to yield a permutation of two lines, that is, We obtain permutation group S_2 acting on variety X.

In making a choice for the center Z, we intend to keep this symmetry and therefore none of these lines will make a good center of blowup. Instead, we prefer to choose the intersection of these two lines which appears to be $S_2 - invaiant$. However, this center may be too small. This is the paradox for the resolution process.

Example 3.4.2. *We consider $X = \mathbb{V}(x^3 - y^2z^2) \subset \mathbb{C}^3$. The top locus consist of one point, the origin where $ord(J) = 3$. Thus $Z = \left\{0\right\}$ as our center of blowup. By blowing up the origin, we obtain the total transforms in each three charts as follows:*

$x - chart$	$y - chart$	$z - chart$
$x^3(1 - xy^2z^2)$	$y^3(x^3 - yz^2)$	$z^3(x^2 - y^2z)$

From the equation defining the strict transforms, we observe that there is a symmetry in y-chart and z-chart, thus J is invariant under interchange of y and z

Remark 3.4.3. *As a general rule, larger centers of blowups improves the singularities faster than relatively small small centers. By this remark, We require Z to be a regular closed subvariety of $top(J)$ of possible maximum dimension, see [**Hau98**]*

3.4.2 Equiconstant Point

Definition 3.4.4. *Let $J \subset W$, $Z \subset top(J)$ and $\pi : \tilde{W} \to W$ be a blowup with center Z. For a point $a \in Z$, we set $q = ord_a J = ord_Z J$. A point a' in the exceptional locus $\tilde{X}$, is called an **equicnstant** point if:*

$$ord_{a'} J^{\gamma} = ord_a J, \quad \textit{here } J^{\gamma} \textit{ denotes the weak transform.}$$

Example 3.4.5. *We consider blowing up the origin of Whitney's umbrella define by,*

$$X = \mathbb{V}(x^2 - y^2 z) \subset \mathbb{C}^3.$$

Blowing up along the **z-patch** *produces the same singularity $x^2 - y^2 z = 0$ with total transform generated by; $z^2(x^{-}y^2 z) = 0$. Hence, the origin of $z - chart$ is an equiconstant point.*

We observe that the order of $f(x, y, z) := x^2 + y^2 z$ is 2 at every point of the $z - axis$ and 1 at any other point of X, thus the $top(J)$ is the $z - axis$.

We blowup X along the $z - axis$ to obtain the resolutions with regular strict transform given by; $(1 - y^2 z)$ and $(x^2 - z)$ along the $u - patch$ and $v - patch$ respectively.

Example 3.4.6. *Consider $X = \mathbb{V}(x^3 - y^3 z^3) \in \mathbb{C}^3$ In this case, the top locus will consist of two transverse lines $y - axis$ and $z - axis$, and thus J is $S_2 - invariant$ obtained by interchanging y **and** z, this implies that, the symmetric choice of the center of blowup $Z = \Big\{(0, 0, 0)\Big\}$. We obtain the total transform $J^* = x^3(1 - x^3 y^3 z^3)$ regular along the $x - chart$, and $y^3(x^3 - y^3 z^3)$ in the $y - chart$(symmetric to $z - chart$). Along these two charts, the singularity remained the same,even though our natural choice of center $Z = \Big\{0\Big\}$ was the best!.*

3.4.3 General resolution of $X = \mathbb{V}(x^q + y^r z^s) \in \mathbb{C}^3$ singularities

We finally consider varieties $X := \mathbb{V}(x^q + y^r z^s) \in \mathbb{C}^3$ such that $r + s \geq q$. We determine the top locus, top(J) which depends on the values, q, r *and* s. Suppose r, s $\geq q$, then we have that, $top(J)$ is the union of the y-axis *and* z-axis, thus we can have the three choices for the center, $Z = \Big\{0\Big\}$ *or* $Z = \Big\{y - axis\Big\}$, *or* $Z =$

$\left\{z - axis\right\}$. In all the three choices of the center Z, the tangent cone of the ideal J consist of the monomial x^q, except when $r + s = q$. If this the case, then the tangent cone of J is $x^q + y^r z^s$.

In obtaining the equiconstant point, we consider $V = \left\{x = 0\right\}$, because x appears in the tangent cone. We know that, the strict transform $V' \subset W'$ contains all the equiconstant points of $\tilde{X}$. However, outside $\tilde{X}$, all other points will be an equiconstant with no relevant information.

Suppose we choose $Z = \left\{(0,0,0)\right\} \in \mathbb{C}^3$ the origin, with $r, s < q$. We have that $V' \cap \tilde{X}$ lies entirely in the component of the $x - chart$, we omit this chart from our consideration as explained above. We consider the two symmetric charts: y-chart and z- chart and suppose we take the $y - chart$, we obtain the total transform;

$$J^* = x^q y^q + y^{r+s} z^s = y^q (x^q + y^{r+s-q} z^s).$$

We note that, the origin of this $y - chart$ is an equiconstant point if and only if

$$r + 2s - q \geq q \implies r + 2s \geq 2q.$$

Furthermore, we have that $s < q$, this implies that the $y - exponent$ has decreased from r to $\left\{r - (q - s)\right\}$.
Suppose that $s \geq q$ we could have considered $y - axis$ or $z - axis$ respectively, as the center of blowups because it gives a larger center which in turn improves the singularities faster as compared to smaller center, $Z = \left\{0\right\}$.
We conclude that, for any ideal $J \subset W$ and any point $p \in W$, there exist locally at point p, a regular hypersurface $V \subset W$, whose strict transform $\tilde{V}$ contains all equiconstant points $p' \in \tilde{W}$ above p.

Bibliography

[Harts77] R. Hartshorne, ***Algebraic geometry Graduate Text in Mathematics.*** **Springer, New York, 1977.**

[Cut04] STEVEN DALE CUTKOSKY, ***Resolution of Singularities, Graduate Studies in Mathematics*** **Vol. 63, 2004.**

[Art86] ARTIN M. ***Lipman's proof of resolution of singularities in Arithmetic Geometry (eds.G. Cornell, J. H. Silverman)*** **Springer, New York, 1986.**

[Hau98] HAUSER H, ***Seventeen obstacles for resolution of singularities,*** **The Brieskorn Anniversary volume (ed.: V.I. Arnold et al.). Birkhauser, 1998.**

[Wil06] Andrew Wilson, ***Birational Maps and Blowing Things Up.*** **University of Edinburgh, 2006.**

[Bur83] D. Burns, ***On the Geometry of Elliptic Modular Surfaces and Representations of Finite Groups.*** **University of Michigan, Ann Arbor, 1983.**

[Dav97] D. Cox J. Little and D. 0' Shea, ***Ideal,Varieties and Algorithms, second edition, Springer-Verlag, New York***, **1997.**

[Abra17] Dan Abramovich , ***Resolution of singularities of complex algebraic varieties and their families*** **, 2017.**

[Hau03] Hauser H., ***The Hironaka Theorem on resolution of singularities,*** **Bulletin American Mathematical Society, $40(2003), 323-403$.**

[Igor07] Igor D., ***Mc kay Correspondence, http://www.math.lsa.umich.edu/ idolga/ Mc Kay book.pdf.***

[Bier97] Bierstone E., Milman, P., ***Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant.*** **Invent Math.128, $(1997)-302$.**

[Vill08] O. Villamayor, Jurgen H. and Holger B., ***Three Lectures on Commutative Algebra, American Mathematical Society, Vol. 42, (2008).***

[R098] Riemenshneider Oswal, ***Cyclic quotient Singularities: Constructing The Artin Components via the Mc Kay-Quiver*** **Singularity and complex Analytic Geometry, 1998.**