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Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

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Master Thesis

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Abstract

For any linear operator T acting on a Hilbert space H, its Aluthge transform $\tilde{T}$ where $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is another linear operator on H. It is known that $\tilde{T}$ preserves the spectral properties of T and more importantly that T has a non trivial closed invariant subspace if and only if T has. In this project Aluthge transforms of different classes of operators in Hilbert spaces were studied. In addition, generalized Aluthge transforms, as well as powers of Aluthge transformations were sort and looked at. Lastly, the numerical range of T was discussed but for some classes of operators.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

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In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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Dedication

This project is dedicated to my dear siblings, my dear friends and myself.

Contents

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Mutuku Serryanne Wavinya

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1 Introduction

1.1 DEFINITIONS AND NOTATIONS.

In this chapter terminologies which featured in this write up were defined, different notations were explained, series of inclusions of different classes of operators in Hilbert spaces were outlined and a brief historical development of Hilbert spaces was visited.
We start off by discussing the following terminologies.

Definition 1.1.1. *A set is called Banach space if it is a complete normed vector space with respect to the norm.*

Definition 1.1.2. *A norm is a function denoted $f \mapsto ||f||$, that maps vectors to non negative scalars and has the following properties:*

(i.) *If $f \neq 0$ then $||f|| \neq 0$;*

(ii.) *Given a scalar k, $||kf|| = |k|.||f||$, where $|k|$ is the absolute value of k;*

(iii.) *Given two vectors f,g; $||f+g|| \leq ||f|| + ||g||$.*

Definition 1.1.3. *$(X, ||\cdot||)$ is called a normed linear space if for X a linear space and $||\cdot||$ a norm on X, the map $||\cdot|| : X \to \mathbb{R}$ satisfies:*

(i.) *$||x|| \geq 0$, with equality iff $x = 0$.*

(ii.) *$||\alpha x|| = |\alpha|||x||$.*

(iii.) *$||x+y|| \leq ||x|| + ||y||$.*

Remark 1.1.4. *Normed linear spaces are denoted by $n.l.s$.*

Definition 1.1.5. *Let H be a vector space over a field $\mathbb{C}$. A mapping $\langle \, , \rangle : H \times H \to K$(where K is $\mathbb{C}$ or $\mathbb{R}$) which associates with every ordered pair $(x,y) \in H \times H$, a scalar denoted by $\langle x,x \rangle$ is called an inner product on $H \times H$ if it satisfies the following properties:*

(i.) *$\langle x,x \rangle \geq 0$ for all $x \in H$.*

(ii.) *$\langle x,x \rangle = 0$, if $x = \overline{0}$, for all $x \in H$.*

(iii.) *$\langle x,y \rangle = \overline{\langle y,x \rangle}$for all $x,y \in H$ (where bar denotes complex conjugate).*

(iv.) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ *for all* $x, y \in H$ *and* $\alpha \in \mathbb{C}$.

(v.) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ *for all* $x, y, z \in H$.

Definition 1.1.6.

Remark 1.1.7. *Inner product on a Hilbert space* H *is denoted by* $\langle , \rangle$.
Thus the vector space H *together with the inner product mapping* $\langle , \rangle$ *is known as an inner product space over* K.
If $K = \mathbb{R}$, *then the inner product space is called a real inner product space. If* $K = \mathbb{C}$, *then the inner product space is called a complex inner product space.*

Remark 1.1.8. *Inner product spaces are denoted by* $i.p.s$.

Definition 1.1.9. *An inner product space* H *is called a Hilbert space if* $(H, ||.||)$ *is a Banach space with* $||.||$, *the norm induced by the inner product function.*

Remark 1.1.10. *Hilbert spaces are denoted by* H.

Definition 1.1.11. *A subset* M *of a vector space* X *is a set which is itself a vector space with respect to vector space axioms.*

Definition 1.1.12. *Let* V *and* W *be vector spaces over the same field* H. *A function* $f : V \to W$ *is said to be a linear transformation or a linear map if for any two vectors* $u, v \in V$ *and any scalar* $c \in H$ *the following conditions are satisfied:*

(i.) $f(u+v) = f(u) + f(v)$

(ii.) $f(cu) = cf(u)$.

That is the addition and scalar multiplication operations are preserved.

Remark 1.1.13. *A linear transformation is a mapping* $V \to W$ *between two vector spaces that preserves the operations of addition and scalar multiplication.*

Definition 1.1.14. *Let* X *and* Y *be two sets. A correspondence which assigns a uniquely defined element* $A(x) \in Y$ *to every element* x *of a subset* $D \subset X$ *is called an operator* A *from* X *into* Y. *Written as* $A : D \to Y$ *for* $D \subset X$.

Remark 1.1.15. *An operator is a mapping or a function that acts on elements of a space to produce elements of another space.*

Definition 1.1.16. *Let* U *and* V *be vector spaces over a field* K. *A mapping* $A : U \to V$ *is called linear operator if* $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$, $\forall x, y \in U$ *and* $\alpha, \beta \in K$.

Remark 1.1.17. *It can easily be noted that a linear operator is a linear transformation.*

Definition 1.1.18. *Let U and V be vector spaces over the same ordered field K which are equipped with norms. Then linear operator from U to V is called bounded if there exists $C > 0$ such that $||Ax||_V \leq C||x||_U, \forall\ x \in V$.*

Definition 1.1.19. *The spectrum of an operator T is the set $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T\}$ is not invertible.*

Definition 1.1.20. *The point spectrum of an operator T is the set $\sigma_p(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) \neq 0\}$*

Remark 1.1.21. *Point spectrum is the set of eigenvalues of T.*

Definition 1.1.22. *The continuous spectrum of an operator T is the set $\sigma_c(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = 0, \overline{(\lambda I - T)} = H$ and $Ran(\lambda I - T) \neq H\}$*

Definition 1.1.23. *The numerical range of an operator T is the set $W(T) = \{\langle Tx, x\rangle : ||x|| = 1, x \in H\}$*

Definition 1.1.24. *Let T be an operator on H. Then $W_q(T) = \{\langle Tx, x\rangle : x, y \in \mathbb{C}^n, ||x|| = ||y|| = 1, \langle x, y\rangle = q\}$ is called the q-numerical range of T.*

Definition 1.1.25. *The numerical radius of an operator T is the set $\omega(T) = \{sup|\lambda| : \lambda \in W(T)\}$.*

Definition 1.1.26. *The spectral radius of T is the set $r(T) = \{sup|\lambda| : \lambda \in \sigma(T)\}$*

Definition 1.1.27. *The residual spectrum of an operator T is the set $\sigma_r(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = 0$ and $\overline{Ran(\lambda I - T)} \neq H\}$*

Proposition 1.1.28. *Let $T \in B(H)$, then $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$ holds, where $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are mutually disjoint parts of $\sigma(T)$.*

Proposition 1.1.29. *Let $T \in B(H)$, then $\sigma(T) = \sigma_{ap}(T) \cup \sigma_{cp}(T)$ holds, where $\sigma_{ap}(T)$ and $\sigma_{cp}(T)$ are not necessary disjoint parts of $\sigma(T)$. Also, $\sigma(T) = \sigma_r(T) \cup \sigma_{ap}(T)$ holds.*

Definition 1.1.30. *Let H be a Hilbert space and $T \in B(H)$. Then there exists $T^* \in B(H)$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$. T^* is called the adjoint of the operator T.*

Definition 1.1.31. *An operator T in $L(H)$ is compact if it is the limit of a sequence of finite rank operators.*

Remark 1.1.32. *We denote the set of compact operators by $\mathbb{K}$.*

Definition 1.1.33. *An operator $T \in B(H)$ is said to be self adjoint if $T^* = T$.*

Definition 1.1.34. *An operator $T \in B(H)$ is called an involution if $T^2 = I$.*

Definition 1.1.35. *An operator $T \in B(H)$ is said to be unitary if $T^*T = TT^* = I$.*

Definition 1.1.36. *An operator $T \in B(H)$ is said to be normal if $T^*T = TT^*$.*

Definition 1.1.37. *An operator $T \in B(H)$ is said to be binormal if T^*T and TT^* commute. That is $[T^*T, TT^*] = 0$.*

Definition 1.1.38. *An operator $T \in B(H)$ is said to be subnormal if it has a normal extension. That is, if there exists a normal operator N on a Hilbert space K such that H is a subspace of K and the subspace H is invariant under N and the restriction of N to H coincides with T.*

Definition 1.1.39. *An operator $T \in B(H)$ is said to be hyponormal if $T^*T \geq TT^*$.*

Remark 1.1.40. *Every subnormal operator is hyponormal.*

Definition 1.1.41. *An operator $T \in B(H)$ is said to be seminormal if either T or T^* is hyponormal.*

Definition 1.1.42. *An operator T is said to be paranormal if $||Tx||^2 \leq ||T^2x||\ ||x||$.*

Definition 1.1.43. *Let q be a positive number with $q \neq 1$. Let T be a closed, densely defined operator in H. If T satisfies $TT^* = qT^*T$, then T is called a deformed normal operator with deformation parameter q.*

Definition 1.1.44. *An operator A is said to be a square-normal operator if $A^2(A^*)^2 = (A^*)^2A^2$.*

Proposition 1.1.45. *If A is a normal operator then A is a square-normal operator.*

Proof. If A is a normal operator then $A^2(A^*)^2 = AAA^*A^* = AA^*AA^* = A^*AA^*A = A^*A^*AA = (A^*)^2A^2$. So A is a square-normal operator. □

Remark 1.1.46. *The converse for proposition* 1.1.45 *is not true though. This is proved by Example* 4.1.12 *given in the Examples section.*

Proposition 1.1.47. *A is a square-normal operator if and only if A^2 is normal.*

Proof. Let A be a square-normal operator, so
$A^2(A^*)^2 = (A^*)^2A^2$
$\Leftrightarrow A^2(A^2)^* = (A^2)^*A^2$
$\Leftrightarrow A^2$ is normal. □

Theorem 1.1.48. *If A is a square-normal operator and $0 \notin W(A)$ then A is normal.*

Definition 1.1.49. *An operator $T \in B(H)$ is said to be an isometry if $T^*T = I$.*

Definition 1.1.50. *An operator $T \in B(H)$ is said to be a co-isometry if $TT^* = I$.*

Definition 1.1.51. *An operator $T \in B(H)$ is said to be a partial isometry if $T = TT^*I$.*

Definition 1.1.52. *An operator $T \in B(H)$ is said to be quasinormal if $T(T^*T) = (T^*T)T$, or equivalently $[T^*T, T] = 0$.*

Remark 1.1.53. *Every quasinormal operator is subnormal.*

Definition 1.1.54. *Let $T \in B(H)$ be an operator with polar decomposition $T = U|T|$ where U is a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$; we have $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ as its Aluthge transform denoted by ΔT.*

Definition 1.1.55. *Let $T = U|T|$ be the polar decomposition of T and let $s,t > 0$. Then generalized Aluthge transformation of T is defined as $\tilde{T}_{s,t} = |T|^sU|T|^t$. Also, we define $\tilde{T}^*_{s,t} = (\tilde{T}_{s,t})^* = |T|^tU^*|T|^s$.*

Definition 1.1.56. *Let q be a positive number with $q \neq 1$. Let T be a closed, densely defined operator in H with polar decomposition $T = U|T|$. If T satisfies the relation $U|T| \subset \sqrt{q}|T|U$, then T is called a deformed quasinormal operator with deformation parameter q.*

Remark 1.1.57. *For a deformed normal (respectively deformed quasinormal) operator T with deformation parameter q, we simply say T is q-normal (respectively q-quasinormal).*

Proposition 1.1.58. *Let T be a closed, densely defined operator in a Hilbert space H. Then the following statements hold:*

(i.) If T is q-normal, then T is q-quasinormal.

(ii.) T is q-normal if and only if $D(T) = D(T^*)$, and $||T^*\eta|| = \sqrt{q}||T\eta||$, $\eta \in D(T)$ where $D(T)$ is the domain of T.

(iii.) If T is q-quasinormal, then $D(T) \subseteq D(T^*)$, and $||T^*\eta|| \leq \sqrt{q}||T\eta||$, $\eta \in D(T)$.

Definition 1.1.59. *Let q be a positive number with $q \neq 1$. A densely defined operator T is called q-hyponormal (or a deformed hyponormal operator with deformation parameter q) if it satisfies $D(T) \subseteq D(T^*)$ and $||T^*\eta||\sqrt{q}||T\eta||$ for all $\eta \in D(T)$. If a q-hyponormal operator T satisfies $||T^*\eta|| = \sqrt{q}||T\eta||$ for all $\eta \in D(T)$, then T is said to be q-formally normal.*

Remark 1.1.60. *Every q-quasinormal operator is q-hyponormal. One can also check that a q-hyponormal operator T is closable and that its closure $\tilde{T}$ is also q-hyponormal. Such an operator with deformation parameter q is sometimes called a q-deformed operator as a generic term.*

Lemma 1.1.61. *Let T be a densely defined operator in H. Then T is q-hyponormal if and only if there is a contraction K such that $T^* \supset \sqrt{q}KT$. In this case, the contraction K is taken such that $R(K^*) \subseteq \overline{R(T)}$, or equivalently $kerK \supseteq kerT^*$. Moreover, K is uniquely determined under this condition.*

Proof. Suppose that T is q-hyponormal. Define an operator K_0 from $R(T)$ to $R(T^*)$ by $K_0T\eta = \frac{1}{\sqrt{q}}T^*\eta$ for $\eta 2 \in D(T)$. Then K_0 is a contraction on $R(T)$, so that K_0 continuously extends $\tilde{K_0}$ on $\overline{R(T)}$. Put $K = \tilde{K_0}$ on $\overline{R(T)}$ and $K = 0$ on $R(T)^{\perp}$. Then K is a contraction such that $T^* \supset \sqrt{q}KT$ and $(K^*) \subseteq \overline{R(T)}$. The converse and the uniqueness of K can easily be shown. □

Corollary 1.1.62. *Let T be a q-hyponormal operator in H. Then $R(T) \subseteq R(T^*)$. In particular, if T is q-normal, then $R(T) = R(T^*)$.*

Proof. The first relation follows from $T \subset \sqrt{q}T^*K^*$. If T is q-normal, then T^* is q^{-1}-normal. Therefore, $R(T) = R(T^*)$. $\square$

Definition 1.1.63. *For each q-hyponormal operator T, we denote by K_T the contraction K in previous Lemma uniquely determined in the lemma. K_T is called the attached contraction to T.*

Proposition 1.1.64. *Let T be a q-hyponormal operator in H. Then T is a q-formally normal if and only if K_T is a partial isometry with initial domain $\overline{R(T)}$.*

Proof. If T is q-formally normal, $||K_T T\eta|| = \frac{1}{\sqrt{q}}||T^*\eta|| = ||T\eta||$, for $\eta \in D(T)$. Hence, by K_T is a partial isometry with initial domain $\overline{R(T)}$. The converse is clear. $\square$

Theorem 1.1.65. *Let T be a closed q-hyponormal operator in H and let $T = U|T|$ be the polar decomposition. Then T is a q-quasinormal if and only if $K_T = (U^*)^2$.*

Proof. Suppose that T is q-quasinormal. Since U^*U is the orthogonal projection onto $\overline{R(|T|)}$, we have $T^* = |T|U^* \supset \sqrt{q}U^*|T| = \sqrt{q}(U^*)^2T$. Since UU^* is the orthogonal projection onto $R(T)$, it follows that $U^* = 0$ on $R(T)^\perp$. Hence, by the uniqueness of K_T , $K_T = (U^*)^2$. Assume, conversely $K_T = (U^*)^2$. By the definition and the same way as mentioned above, we have $|T|U^* \supset \sqrt{q}U^*|T|$. Hence, $U|T| \subset \sqrt{q}|T|U$. Thus T is q-quasinormal. $\square$

Proposition 1.1.66. *The following statements hold:*

(i.) A unilateral weighted shift S_u in H with weights w_n is q-quasinormal if and only if $|w_n| = (\frac{1}{\sqrt{q}})^n|w_0|$ for all $n \geq 0$. In particular, a unilateral weighted shift cannot be q-normal.

(ii.) A bilateral weighted shift S_b in H with weights w_n is q-normal if and only if the equation in (i.) is valid for all $n \in \mathbb{Z}$.

(iii.) A weighted shift S_u (respectively S_b) is q-hyponormal if and only if $|w_{n+1}| \geq \frac{1}{\sqrt{q}}|w_n|$ for all $n \geq 0$ (respectively $n \in \mathbb{Z}$).

Proof. Let S_u be a unilateral shift with the weights w_n and let $S_u = U|S_u|$ be the polar decomposition of S_u. If S_u is q-quasinormal, then $U|S_u|e_n = \sqrt{q}|S_u|Ue_n$, and so $\sqrt{q}|w_{n+1}| = |w_n|$ for all $n > 0$. Hence $|w_n| = q^{-\frac{n}{2}}|w_0|$, $n \geq 0$. Conversely, suppose that the equality in (i.) is valid for $n > 0$. It is clear $U|S_u|e_n = \sqrt{q}|S_u|Ue_n$ for each n, and so $U|S_u| = \sqrt{q}|S_u|U$ on De_n. For each $\eta \in D(|S_u|) = D(S_u)$, there is a sequence η_n in De_n such that $\eta_n \to \eta$ and $S_u\eta_n \to S_u\eta$, as $n \to \infty$. Since $S_u\eta_n = U|S_u|\eta_n = \sqrt{q}|S_u|U\eta_n$, we have $\sqrt{q}|S_u|U\eta_n \to S_u\eta$ and $U\eta_n \to U\eta$, as $n \to \infty$. Since $|S_u|$ is closed, $U\eta \in D(|S_u|)$ and $\sqrt{q}|S_u|U\eta = S_u\eta$. Hence $U|S_u| \subset \sqrt{q}|S_u|U$. If S_u is $q-$normal, then we have $qS_u^*S_ue_0 = S_uS_u^*e_0 = 0$. Hence $w_n = w_0 = 0$ for all n. Thus statement (i) holds. Next, let S_b be a bilateral weighted shift with the weights w_n. Then $S_bS_b^*e_n = |w_{n-1}|^2e_n$ and $S_b^*S_be_n = |w_n|^2e_n$ for all $n \in \mathbb{Z}$. Hence, if S_b is q-normal, we have $|w_n| = q^{-\frac{n}{2}}|w_0|$, for all $n \in \mathbb{Z}$. Conversely, assume that the equality in (i.) is valid for all $n \in \mathbb{Z}$. Then it is easily seen that $D(S_b) = D(S_b^*)$. By our assumption, $S_bS_b^*e_n = qS_b^*S_be_n$ for all $n \in \mathbb{Z}$. Hence, $||S_b^*\xi|| = \sqrt{q}||S_b\xi||$ for all $\xi \in De_n$. For each $\eta \in D(S_b)$, there is a sequence ξ_n in De_n such that $\xi_n \to \eta$ and $S_b\xi_n \to S_b\eta$, as $n \to \infty$. Since S_b^* is closed, it follows that the sequence $S_b^*\xi_n$ converges to $S_b^*\eta$.Hence, $||S_b^*\eta|| = \sqrt{q}||S_b\eta||$. Finally, we have to prove statement (iii.). It is verified by the same way as in statement (ii). □

Corollary 1.1.67. *Let T be a (unilateral or bilateral) weighted shift with weights w_n, with respect to a basis e_n. If T satisfies $T^*T - qTT^* = 1$, $q > 0$, $q \neq 1$. on De_n, then T is q^{-1}-hyponormal with $D(T^*) = D(T)$.*

Proof. The relation implies that $|w_{n+1}|^2 - q|w_n|^2 = 1$. It follows that $D(T^*) = DD(T)$. Clearly, $|w_{n+1} > \sqrt{q}|w_n|$ for all n. Hence, from (ii.) in Proposition 1.1.66 T is q^{-1}-hyponormal. □

Definition 1.1.68. *Let T be a bounded linear operator on a complex Hilbert space H and let $T = U|T|$ be the polar decomposition of T. Then T is called class p-$wA(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$ and $(|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp}$ where $0 < s,t$ and $0 < p \leq 1$.*

Definition 1.1.69. *Let T be a bounded linear operator on a complex Hilbert space is called class p-$A(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$ where $0 < s,t$ and $0 < p \leq 1$.*

Remark 1.1.70. *If $p = s = t = 1$, then class p-$A(s,t)$ coincides with class A operators.*

Definition 1.1.71. *Let $T = U|T|$ be the polar decomposition of T and $0 < p \leq 1$, $0 < s,t$. T is called class p-A if $|T^2|^p \geq |T|^{2p}$.*

Definition 1.1.72. *An operator T is said to be w-hyponormal if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$.*

Definition 1.1.73. *An operator $T \in B(H)$ is said to be a convexoid if $\overline{W(T)} = conv(\sigma(T))$.*

Definition 1.1.74. *An operator $T \in B(H)$ is said to be a normaloid if $r(T) = ||T||$.*

Definition 1.1.75. *An operator $T \in B(H)$ is said to be a spectraloid if $\omega(T) = r(T)$.*

Definition 1.1.76. *An operator $T \in B(H)$ is said to be a scalar if it is a scalar multiple of the identity operator. That is if $T = \alpha I$, where $\alpha \in \mathbb{C}$.*

Definition 1.1.77. *An operator $T \in B(H)$ is said to be invertible if there exists an operator S such that $ST = TS = I$ (where I is the identity).*

Remark 1.1.78. *The class of invertible linear operators $T : H \to K$ is denoted by $G(H,K)$.*

Definition 1.1.79. *The commutator of two operators A and B denoted by $[A,B]$ is defined by $[A,B] = AB - BA$.*

Definition 1.1.80. *The self commutator of an operator A is defined by $[A^*,A] = A^*A - AA^*$.*

Definition 1.1.81. *Two operators $T \in B(H)$ and $S \in B(K)$ are said to be similar(denoted $T \approx S$) if there exists an invertible operator $X \in G(H,K)$ such that $XT = SX$.*

Definition 1.1.82. *Two operators $T \in B(H)$ and $S \in B(K)$ are said to be equivalent (denoted $T \equiv S$) if there exists an unitary operator $U \in G(H,K)$ such that $UT = SU$.*

Definition 1.1.83. *An operator $X \in B(H,K)$ is called quasiaffinity or a quasi-invertible if it is injective and with dense range.*

Definition 1.1.84. *An operator $T \in B(H)$ is quasiaffine transform of $S \in B(K)$ if there exists a quasiaffinity $X \in B(H,K)$ such that $XT = SX$.*

Definition 1.1.85. *Two operators $T \in B(H)$ and $S \in B(K)$ are said to be quasisimilar(denoted by $T \sim S$) if they are quasiaffine transform of each other. That is, if there exists quasiaffinites $X \in B(H,K)$ and $Y \in B(K,H)$ such that $XT = SX$ and $YS = TY$.*

Definition 1.1.86. *An operator is said to be Hermitian if it is equal to its own transpose conjugate.*

Remark 1.1.87. *Hermitian operators are self adjoint.*

Definition 1.1.88. *An operator $T \in B(H)$ is idempotent if $T^2 = T$.*

Definition 1.1.89. *An operator $T \in B(H)$ is said to be an isoloid if any isolated point of $\delta(T)$ is an eigenvalue of T.*

Definition 1.1.90. *An operator $T \in B(H)$ is said to be p-hyponormal if $(T^*T)^p \geq (TT*)^p$ for $0 < p \leq 1$.*

Definition 1.1.91. *An operator $T \in L(H)$ is posinormal if there exists a positive $P \in L(H)$ such that $TT^* = T^*PT$. Here, P is called an interrupter of T.*

Definition 1.1.92. *An operator $A \in B(H)$ is coposinormal if A^* is posinormal.*

Corollary 1.1.93. *Every hyponormal operator is posinormal.*

Corollary 1.1.94. *In order for a cohyponormal operator A to be posinormal it is necessary that $KerA = KerA^*$.*

Theorem 1.1.95. *Every invertible operator is posinormal.*

Proof. If A is invertible, then $A^* = A^*(A^{-1}A) = (A^*A^{-1})A$, so $A^* \in [A]$. □

Corollary 1.1.96. *Every invertible operator is coposinormal.*

Corollary 1.1.97. *Assume $A \in B(H)$ and $\lambda \notin \sigma(A)$, the spectrum of A. Then $A-\lambda$ is posinormal.*

Definition 1.1.98. *For an operator $A \in B(H)$, the posispectrum of A which is denoted $p(A)$ is the set $\{\lambda : A-\lambda$ is not posinormal$\}$.*

Remark 1.1.99. *By Corollary 1.1.98, it is clear that $p(A)$ is a subset of $\sigma(A)$.*

Proposition 1.1.100. *If A is hyponormal, then $p(A) = \emptyset$.*

Proof. Since translates of a hyponormal operator are hyponormal, $A-\lambda$ is hyponormal and hence posinormal for every λ. □

Proposition 1.1.101. *If U is the unilateral shift, then $p(U) = \emptyset$ and $p(U^*) = \sigma(U^*)$ $= \{\lambda : |\lambda| \leq 1\}$.*

Corollary 1.1.102. *Assume $A-\lambda$ is posinormal for three distinct real values of λ and that the same positive operator P functions as an interrupter for $A-\lambda$ for each of those three values. Then A is normal.*

Theorem 1.1.103. *Assume $A-\lambda$ is posinormal for two distinct values of λ, and assume that the same operator B functions as a multiplier for $A-\lambda$ for both of those values. Then A is normal.*

Proof. Assume $(A-\lambda_1)^* = B(A-\lambda_1)$ and $(A-\lambda_2)^* = B(A-\lambda_2)$ where $\lambda_1 \neq \lambda_2$. Then $(\overline{\lambda}_1 - \overline{\lambda}_2)I = (\lambda_1 - \lambda_2)B$, so $B = ((\overline{\lambda}_1 - \overline{\lambda}_2)/(\lambda_1 - \lambda_2))I$. Therefore $P = B^*B = I$ serves as an interrupter for $A-\lambda$ when $\lambda = \lambda_1, \lambda_2$; it then follows that A is normal. □

Definition 1.1.104. *Let $p > 0$. An operator $T \in L(H)$ is said to be p-posinormal if $(TT^*)^p \leq \mu(T^*T)^p$ for some $\mu > 1$.*

Remark 1.1.105. *It is clear that 1-hyponormal and 1-posinormal are hyponormal and posinormal, respectively. If T is hyponormal then it is isoloid.*

Definition 1.1.106. *An operator $T \in B(H)$ is said to be semi-hyponormal if $(T^*T)^{\frac{1}{2}} \geq (TT^*)_{\frac{1}{2}}$.*

Definition 1.1.107. *An operator $T \in B(H)$ is (r,t) weakly-hyponormal if $|\check{T}_{r,t}| \geq |T| \geq |d\check{T}_{r,t}|$.*

Definition 1.1.108. *An operator $T \in B(H)$ is said to be log-hyponomal if T is invertible and $logT^*T \geq logTT^*$.*

Definition 1.1.109. *A bounded operator T on a Hilbert space is said to be a quasi-isometry if and only if $T^{*2}T^2 = T^*T$. More generally, given a positive integer q, greater than 1, T is said to be a q-quasi-isometry if and only if $T^{*q}T^q = T^*T$.*

Definition 1.1.110. *An operator $T \in B(H)$ is said to be complex symmetric if there exists a conjugate linear involution.*

Definition 1.1.111. *An operator $T \in B(H)$ is said to be Fredholm if the null space of T and T^* are finite dimensional and the range of T is closed.*

Definition 1.1.112. *An operator $T \in B(H)$ is said to be nilpotent if $T^n = \{0\}$ for some positive integer n.*

Definition 1.1.113. *An operator $T \in B(H)$ is said to be quasinilpotent if $||T^n||^{\frac{1}{n}} \to 0$ for some n.*

Definition 1.1.114. *An operator $T \in B(H)$ is said to be centred if the sequence $T^2(T^2)^*, TT^*, T^*T, (T^2)^*T^2, \ldots$ is commutative.*

Definition 1.1.115. *An operator $T \in B(H)$ is said to be a contraction if $||T|| \leq 1$.*

Definition 1.1.116. *An operator $T \in B(H)$ is reducible if it has a non trivial reducible subspace, otherwise it is said to be irreducible.*

Definition 1.1.117. *An operator $T \in L(H)$ is said to be polaroid if every isolated point of the spectrum $\sigma(T)$ is a pole of the resolvent of T.*

Definition 1.1.118. *Let $T \in B(H)$, then we have:*

(i.) $KerT = N(T) = \{x \in H : Tx = 0\}$, the kernel of T which is a subspace of H that contains all elements mapped to identity by T.

(ii.) $R(T)$, the range or image of T.

Definition 1.1.119. *A bounded operator T on a Hilbert space is called a generalized projection if and only if $T^2 = T^*$. More generally, given an integer $q > 1$, T is a generalized q-projection if and only if $T^q = T^*$.*

Definition 1.1.120. *Let $T = U|T|$ be the polar decomposition of an operator T. Then Aluthge transformation of T is defined as $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ and its adjoint is $(\tilde{T})^* = |T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}$.*

Remark 1.1.121. *I. Jung, Ko and C. Pearcy in [IEC00] defined n-th Aluthge transformation of an operator.*

Definition 1.1.122. *For every $T \in B(H)$, the sequence $\{\tilde{T}\}^{(n)}$ of Aluthge iterates of T is defined by $\{\tilde{T}\}^{(0)} = T$ and $\{\tilde{T}\}^{(n+1)} = \widetilde{T^{(n)}}$ for every positive integer n.*

Definition 1.1.123. *An operator T is said to be a k-quasi-class A operator if $T^{*k}(|T^2| - |T|^2)T^k \geq 0$, where k is a positive integer number.*

Definition 1.1.124. *An operator $T \in B(H)$ is powers of N-class A_k if $|T|^{2p} \leq N|T^{k+1}|^{\frac{2p}{k+1}}$ for a fixed $N > 0$ and $0 < p \leq 1$.*

Definition 1.1.125. *An integral operator is a linear operator that associates with every function f another function g by means of an integral equation.*

Definition 1.1.126. *A subspace M of H is invariant under $T \in B(H)$ or T-invariant if $Tx \in M$ whenever $x \in M$. That is, $TM \subseteq M$, where $TM = \{Tx : x \in M\}$*

Remark 1.1.127. *The collection of all subspaces of H invariant under T is denoted by $LatT$. For collections of bounded operators $L \subseteq B(H)$, we define $Lat(L) = \cap_{T\in L} LatT$. For $T \in B(H)$, $LatT$ is closed under intersections and spans and is complete in the sense that both ϕ and H are in $LatT$.*
ϕ and H are called the trivial invariant subspaces. Some linear operators have only the trivial invariant subspaces. A good example is the rotation operator in two dimensional vector space. A space V is said to be simple if it has no non trivial invariant subspaces. V is said to be semisimple if it is a direct sum of simple invariant subspaces. V is said to be diago- nalizable if there is a basis $e_{i i\in I}$ such that for all $i \in I$, $Te_i \in \langle e_i \rangle$ equivalently, V is a direct sum of one-dimensional invariant subspaces. Thus diagonalizable implies semisimple.

Theorem 1.1.128. *The following statements are equivalent:*

(i.) V is semisimple.

(ii.) If $W \subset V$ is an invariant subspace, it has an invariant complement: i.e., there is an invariant subspace W' such that $V = W \oplus W'$.

(iii.) V is spanned by its simple invariant subspaces.

Proof. Three times in the following argument we assert the existence of invariant subspaces of V which are maximal with respect to a certain property. When V is finite-dimensional it doesn't matter what this property is: one cannot have an infinite, strictly ascending chain of subspaces of a finite-dimensional vector space. In the general case the claimed maximality follows from Zorn's Lemma; (i) $\Rightarrow$ (ii): Suppose $V = \oplus_{i\in I} S_i$, with each S_i a simple invariant. For each $J \subset I$, put $V_J = \oplus_{i\in J} S_i$. Now let W be an invariant subspace of V. There is a maximal subset $J \subset I$ such that $W \cap V_J = 0$. For $i \notin J$ we have $(V_J \oplus S_i) \cap W \neq 0$, so choose $0 \neq x = y + z$, $x \in W$, $y \in V_J$, $z \in S_i$. Then $z = x - y \in (V_j + W) \cap S_i$, and if $z = 0$, then $x = y \in W \cap V_j = 0$, contradiction. So $(V_J \oplus W) \cap S_i \neq 0$. Since S_i is simple, this forces $S_i \subset V_J \oplus W$. It follows that $V = V_J \oplus W$. (ii) $\Rightarrow$ (i): The hypothesis on V passes to all invariant subspaces of V. We claim that every non zero invariant subspace $C \subset V$ contains a simple invariant subspace. Proof of claim: Choose $0 \neq c \in C$, and let D be an invariant subspace of C that is maximal with respect to not containing c. By the observation of the previous paragraph, we may write $C = D \oplus E$. Then E is simple. Indeed, suppose not and let $0 \subsetneq F \subsetneq E$. Then $E = F \oplus G$ so $C = D \oplus F \oplus G$. If both $D \oplus F$ and $D \oplus G$ contained c, then $c \in (D \oplus F) \cap (D \oplus G) = D$, contradiction. So either $D \oplus F$ or $D \oplus G$ is a strictly larger invariant subspace of C than D which does not contain c, contradiction. So E is simple, establishing our claim. Now let $W \subset V$ be maximal with respect to being a direct sum of simple invariant subspaces, and write $V = W \oplus C$. If $C \neq 0$, then by the claim C contains a non zero simple sub module, contradicting the maximality of W. Thus $C = 0$ and V is a direct sum of simple invariant subspaces. (i) $\Rightarrow$ (iii) is immediate. (iii) $\Rightarrow$ (i): There is an invariant subspace W of V that is maximal with respect to being a direct sum of simple invariant subspaces. We must show $W = V$. If not, since V is assumed to be generated by its simple invariant subspaces, there exists a simple invariant subspace $S \subset V$ that is not contained in W. Since S is simple we have $S \cap W = 0$ and thus $W + S = W \oplus S$ is a strictly larger direct sum of simple invariant subspaces than W hence a contradiction. □

Theorem 1.1.129. *If $T \in B(H)$ and $P = P|_M$ is the projection onto M, then $M \in LatT$ if and only if $TP = PTP$.*

Proof. Suppose M is invariant under T. Let $x \in H$. Then $Px \in M$. Therefore $P(TPx) = TPx \forall x \in H$. That is $PTP = TP$. Conversely, let $PTP = TP$. Let $x \in M$. Then $(PTP)x = TPx = T(Px)$. But $(PTP)x = P(TPx) \in M$ since $M = Ran(P)$. Therefore $T(Px) \in M$. That is, $Tx \in M$ since $Px = x$. This means that $Tx \in M$ whenever $x \in M$. That is $TM \subseteq M$. Therefore M is invariant under T. □

Theorem 1.1.130. *For any $T \in B(H)$, $LatT^* = \{M : M^\perp \in LatT\}$.*

Remark 1.1.131. *I.Jung, E.Ko and C.Pearcy in [IEC00] proved that an operator T has a non trivial invariant subspace if and only if $\tilde{T}$ does.*

Definition 1.1.132. *For real numbers α and β with $0 \leq \alpha \leq 1 \leq \beta$, an operator T acting on a Hilbert space H is called (α, β)-normal if $\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T$.*

Definition 1.1.133. *$T \in B(H)$ is called an n-normal operator if $T^nT^* = T^*T^n$.*

Remark 1.1.134. *All non zero nilpotent operators are n-normal operators, for $n \leq k$ where k is the index of nilpotence, but they are not normal. Example 4.1.19 in the examples section is used to illustrate this.*

Proposition 1.1.135. *Let $T \in B(H)$. Then T is n-normal if and only if T^n is normal where $n \in \mathbb{N}$.*

Proof. If T is n-normal, then $T^nT^* = T^*T^n$. Therefore $T^n(T^*)^n = T^*T^n(T^*)^{n-1} = T^*(T^nT^*)(T^*)^{n-2} = (T^*)^2T^n(T^*)^{n-2} = (T^*)^nT^n$. Then T^n is normal. Now, let T^n is normal. Since $T^nT = TT^n$, by Fuglede theorem, $T^*T^n = T^nT^*$. Therefore T is n-normal. □

Proposition 1.1.136. *Let $T \in B(H)$ be n-normal. Then*

(1.) T^* is n-normal.

(2.) If T^{-1} exists, then (T^{-1}) is n-normal.

(3.) If $S \in B(H)$ is unitary equivalent to T, then S is n-normal.

(4.) If M is a closed subspace of H such that M reduces T, then $S = T/M$ is an n-normal operator.

Proof. (1.) Since T is n-normal, T^n is normal. So $(T^n)^* = (T^*)^n$ is normal, T^* is an $n-$normal operator.

(2.) Since T is $n-$normal, T^n is normal. Since $(T^n)^{-1} = (T^{-1})^n$ is normal, T^{-1} is an n-normal operator.

(3.) Let T be an n-normal operator and S be unitary equivalent of T. Then there exists unitary operator U such that $S = UTU^*$ so $S^n = UT^nU^*$. Since T^n is normal, S^n is normal. Therefore S is n-normal.

(4.) Since T is n-normal, T^n is normal. So $T^n = M$ is normal. And since M is invariant under T, $T^n/M = (T/M)^n$. Thus $(T/M)^n$ is normal. So T/M is n-normal.

□

Definition 1.1.137. *Let $T \in L(H)$, and let $T = U|T| = |T^*|U$ be the polar decomposition of T. Then, for every $\lambda \in [0,1]$ the λ-Aluthge transform is defined by $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$.*

Definition 1.1.138. *Let T be a bounded linear operator on a complex Hilbert space with the polar decomposition $T = U|T|$. Let $T(t) = |T|^t U|T|^{1-t}$ for $0 < t < 1$, and $T(O) = U^*UU|T|$ and $T(1) = |T|U$. $T(t)$ is called the generalized Aluthge transform of T.*

Definition 1.1.139. *An operator $T \in B(H)$. Let $T^* = U^*|T^*|$ be the polar decomposition of T^*. Then $*$-Aluthge transformation is defined as $\tilde{T}^{(*)} = (\widetilde{T^*})^* = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$ and its adjoint is $(\tilde{T}^{(*)})^* = |T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}$.*

Definition 1.1.140. *Let M be a subspace of a Hilbert space H. Let $H = M \oplus M^\perp$, then the map $P_M : H \to M$ defined by $P_M x = x'$ where $x = x' + x''$, for $x' \in M$ and $x'' \in M^\perp$ is called an orthogonal projection of H onto M. P_M is self-adjoint ($P_M^* = P_M$), idempotent ($P_M^2 = P_M$) and $Ker(P) \perp Ran(P)$.*

Definition 1.1.141. *An operator $T \in B(H)$ is called paranormal if $||Tx||^2 \leq ||T^2x||||x||$ for all $x \in H$.*

Remark 1.1.142. *It is well known that a power of a paranormal operator is paranormal. This is also true for $p(k)$ operators.*

Definition 1.1.143. *An operator T is called k-quasi-paranormal or $p(k)$ operator if it satisfies the following inequality: $||T^{k+1}x||^2 \leq ||T^{k+2}x||||T^k x||$ for all $x \in H$ and k a positive integer.*

Remark 1.1.144. *It is obvious that every paranormal operator is a $p(k)$ operator.*

Theorem 1.1.145. *Let T be an algebraically k-quasi-paranormal operator. Then T is polaroid.*

Corollary 1.1.146. *A k-quasi-paranormal operator is isoloid.*

Definition 1.1.147. *An operator T is called k-quasi $*$- paranormal if it satisfies the inequality: $||T^*T^k x||^2 \leq ||T^{k+2}x||||T^k x||$ for all unit vector $x \in H$ where k is a natural number.*

Definition 1.1.148. *An operator $T \in B(H)$ is called $n-$power normal if T^n commutes with T^*, that is, $T^n T^* = T^* T^n$.It is denoted by $[nN]$.*

Definition 1.1.149. *An operator T in $B(H)$ is said to be skew symmetric if there exists conjugation C on H such that $CTC = -T^*$. T is said to be complex symmetric if $CTC = T^*$ for some conjugation C on H.*

Definition 1.1.150. *An operator $T \in B(H)$ is called transaloid if $T - \lambda I$ is normaloid for all $\lambda \in \mathbb{C}$.*

Definition 1.1.151. *An operator $T \in B(H)$ is said to be reguloid if for every isolated point λ of $\sigma(T)$, $\lambda I - T$ is relatively regular, that is there exists $S_\lambda \in B(H)$ such that $(\lambda I - T)S_\lambda(\lambda I - T) = \lambda I - T$.*

Definition 1.1.152. *A bounded linear operator T on H, an arbitrary complex Hilbert space, is called quasihyponormal if $T^*(T^*T - TT^*)T \geq 0$ or equivalently $||T^*Tx|| \leq ||TTx||$ for all $x \in H$.*

Definition 1.1.153. *A Hilbert space operator $T \in B(H)$ is said to be p-quasihyponormal for some $0 < p \leq 1$, $T \in p - QH$, if $T^*(|T|^{2p} - |T^*|^{2p})T \geq 0$.*

Definition 1.1.154. *An operator T is called (p,k)-quasihyponormal if $T^{*(k)}(|T|^{(2p)} - |T^*|^{(2p)})T^k \geq 0$, ($0 < pleq1$; k is an element of $\mathbb{Z}^+$), which is a common generalization of p-quasihyponormality and k-quasihyponormality.*

Remark 1.1.155. *Hyoun Kim [Kim04] introduced (p,k)-quasihyponormal operators and proved many interesting properties of (p,k)-quasihyponormal operators.*

Definition 1.1.156. *An operator $T \in B(H)$ is dominant if $Ran(T - \lambda) \subseteq Ran(T - \lambda)^*$ for all $\lambda \in \sigma(T)$.*

Remark 1.1.157. *Stampfli and Wadhwa in this publication [JB77] studied dominant operators.*

Definition 1.1.158. *An operator $T \in B(H)$ is M-hyponormal if there exists a real number M such that $||(T-\lambda)^* f|| \leq M||(T-\lambda)f||$ for all $f \in H$ and all complex numbers λ.*

Definition 1.1.159. *Let $T \in B(T)$ and $\mu \in \sigma(T)$, where μ is the Lebesque measure. Then we denote :*

(i.) $m(T,\mu)$*, the algebraic multiplicity of eigenvalue* μ *for* T*.*

(ii.) $m_0(T,\mu) - DimKer(T-\mu I)$*, the algebraic multiplicity of* μ*.*

1.2 OTHER NOTATIONS USED ARE:

1. $T = U(T)$; The polar decomposition of T where U is unitary.
2. $\tilde{T} = \Delta(T) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$; Aluthge transform of T.
3. $\mathbb{C}$; Space of complex numbers.
4. $\delta\_cp(T) = \{\lambda \in \mathbb{C} : \overline{R(T-\lambda)} \subsetneq H\}$; Compressive spectrum of T.
5. $\varphi(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is invertible $\}$; Resolvent set of T.
6. $\langle . \rangle$; Inner product function.
7. $M_r(\mathbb{C})$; Algebra of complex $r \times r$ matrices.
8. $GL_r(\mathbb{C})$; Group of all invertible elements in $M_r(\mathbb{C})$.
9. $U(r)$; Group of unitary operators.
10. $L(H)$; The algebra of all bounded linear operators on H.
11. $\{\Delta^n(T)\}_{n=0}^{\infty}$; The Aluthge sequence.
12. $\Delta_\lambda(T) = |T|^{1-\lambda} U |T|^{\lambda}$; Generalized Aluthge transform of T.
13. $W(T) = \{\langle Tx, x\rangle : ||x|| = 1, x \in H\}$; Numerical range of T.
14. $\omega(T) = Sup\{|\lambda|; \lambda \in W(T)\}$; Numerical radius of T.
15. $W_q(T) = \{\langle Tx, x\rangle : x, y \in \mathbb{C}^n, ||x|| = ||y|| = 1, \langle x, y\rangle = q\}$; q-numerical range of T.
16. $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is not invertible $\}$; Spectrum of T.
17. $r(T) = Sup\{|\lambda| : \lambda \in \sigma(T)\}$; Spectral radius of T.
18. $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is not bounded below $\}$; Approximate point spectrum of T.

19. $||\cdot||$; Norm.

20. $Lat(T)$; The invariant subspace lattice of an arbitrary operator $T \in B(H)$.

21. $R(\Delta) = \{\tilde{T} : T \in B(H)\}$; Range of Δ, where Δ is a map defined on $B(H)$.

1.3 SERIES OF INCLUSIONS OF CLASSES OF OPERATORS.

In this section, some classes of operators and other higher classes were discussed, more so those which contain all normal operators, but they are non-normal themselves. Hyponormal operators introduced by Stampfli [Sta62], 1962, generalizes normal operators. Also, every normal operator is hyponormal. This is seen when the inequality is relaxed to equality in definition of a hyponormal operator. k-quasihyponormal operators were later introduced by Campbell and Gupta [CG79], 1978, as the extension of the class of hyponormal operators. He showed that very hyponormal operator is k-quasihyponormal if we let $k = 1$. Hyponormal operators were generalized by Aluthge,[AAL], 1990, by introducing the p-hyponormal operators. Letting $p = 1$ in the definition of a p-hyponormal operator one gets a hyponormal operator.Thus p-hyponormal operators generalizes hyponormal operators.p-quasihyponormal operators contains all quasihyponormal operators. This is obtained by letting $p = 1$ in the definition of p-quasihyponormal operator to get a quasihyponormal operator. Furuta, 1997, introduced a larger class called class A operators which contains the class of p-hyponormal operators. Tanahashi [TAN99], 1999 introduced another class, log-hyponormal operators which contains all invertible hyponormal operators. He further showed that invertible p-hyponormal operators are log-hyponormal and log-hyponormal operators are class A operators. later, Fujii, et al. 2000 generalized class A operators into class $A(s,t)$ and in the same year, the latter was generalized into the class of absolute-(s,t) paranormal by Yanagida,2000. Fujii [FJLLN], et al.2000, introduced another class which contains all invertible operators of class $A(s,t)$, the class $AI(s,t)$, which contains all invertible operators of class $A(s,t)$. Aluthge [AD00], et al.2000 generalized both log- and p-hyponormal operators into w-hyponormal operators using Aluthge transformations. Every $semi$-hyponormal operator is w-hyponormal from the definition of w-hyponormality. For every $p \geq \frac{1}{2}$ $semi$-hyponormal operators contain all p-hyponormal operators. Therefore w-hyponormal operators contain all p-hyponormal operators. Ito, 2001 introduced class $wA(s,t)$as a generalization of w-hyponormality. Putting $s = \frac{1}{2}$ and $t = \frac{1}{2}$ in the definition of $wA(s,t)$ operator, one gets w-hyponormal operator. The class $wA(1,1)$ is called wA. This means that T is a member of class wA, if and only if $|T^2| \geq |T|^2$ and $|T^*|^2 \geq |T^{2*}|$. Since T belongs to class A if $|T^2| \geq |T|^2$, from the definition of class A, then it is clear that class A contains class wA but from analysing class $wA(s,t)$ operators, it follows that $wA(s_1,t_1)$ is contained in $wA(s_2,t_2)$ where $s_2 \geq s_1$ and $t_2 \geq t_1$. This shows that $wA(\frac{1}{2},\frac{1}{2}) \subseteq wA(1,1)$. Class A is a generalization of w-hyponormal operators since $wA(\frac{1}{2},\frac{1}{2})$corresponds to w-hyponormal operators while $wA(1,1)$ corresponds to class A. Yanagida [Yan03], 2003, introduced class $A(k)$ as a generalization of class A. All class A

operators are $A(k)$, letting $k=1$ in definition of class $A(k)$ one gets $|T^2| \geq |T|^2$. From the definition of class $A(s,t)$ operators, class $A(k,1)$ equals class $A(k)$, class $A(k)$ is a subclass of $A(s,t)$. p-quasihyponormal and q-quasihyponormal operators are generalized by (p,k)-quasihyponormal operators which was introduced by Hyoun, 2003. Putting $p=1$ and $k=$ in the definition of a (p,k)-quasihyponormal operator one gets a k-quasihyponormal and a p-quasihyponormal operator respectively. q-quasihyponormal operator is a (p,k)-quasihyponormal operator since $0<q<p$ and the (p,k)-quasihyponormal operators contain p-hyponormal operators. Jibril [AAJ1], 2007, introduced the class of $2-$Power normal operators which is an extension of normal operators. Later [AAJ2] in 2008 he generalized the class of 2-Power normal operators into n-Power normal, where n is a positive integer. Ahmed [Ahm11], 2011 generalized the results by Jibril into the class of $n-$Power quasinormal operators. He also showed that every n-Power normal operator is n-Power quasinormal. Panayan [Pan12], 2012, introduced an extension of all normal operators and called it the n-Power Class(Q) operators.

1.3.1 Series of inclusions of Hilbert space operators.

The following inclusions hold and are known to be proper among the classes of operators discussed above;

(i.) self-adjoint $\subset$ normal $\subset$ hyponormal $\subset$ p-hyponormal $\subset$ p-quasihyponormal $\subset$ (p,k)-quasihyponormal.

(ii.) hyponormal $\subset$ quasihyponormal $\subset$ k-quasihyponormal $\subset$ (p,k)-quasihyponormal.

(iii.) p-hyponormal $\subset$ *semi*-hyponormal $\subset$ w-hyponormal $\subset$ wA $\subset$ class A $\subset$ class $A(k)$ $\subset$ class $A(s,t)$.

(iv.) p-hyponormal $\subset$ *semi*-hyponormal $\subset$ w-hyponormal $\subset$ wA $\subset$ class $A(s,t)$ $\subset$ class $A(s,t)$.

(v.) log-hyponormal $\subset$ w-hyponormal $\subset$ class $AI(s,t)$ $\subset$ class $wA(s,t)$ $\subset$ class $A(s,t)$.

(vi.) ∞-hyponormal $\subset$ k-hyponormal $\subset$ p-hyponormal $\subset$ w-hyponormal $\subset$ class $AI(s,t)$ $\subset$ class $wA(s,t)$ $\subset$ class $A(s,t)$.

(vii.) subnormal $\subset$ hyponormal $\subset$ quasihyponormal $\subset$ class(A) $\subset$ paranormal.

(viii.) hyponormal $\subset$ transaloid $\subset$ convexoid.

(ix.) ∞-hyponormal $\subset$ normal $\subset$ quasinormal $\subset$ n-Power-quasinormal.

(x.) spectroid $\subset$ *hen*-spectroid $\subset$ numeroid $\subset$ transaloid $\subset$ normaloid $\subset$ spectraloid.

(xi.) spectroid $\subset$ *hen*-spectroid $\subset$ numeroid $\subset$ transaloid $\subset$ convexoid.

(xii.) normal $\subset$ hyponormal $\subset$ p-hyponormal $\subset$ normaloid $\subset$ HN.

(xiii.) $CTHN \subset THN \subset HN$.

(xiv.) ∞-hyponormal $\subset$ normal $\subset$ n-powernormal $\subset$ n-Power-quasinormal.

(xv.) ∞-hyponormal $\subset$ normal $\subset$ quasinormal $\subset$ quasihyponormal $\subset$ p-quasihyponormal $\subset$ (p,k)-quasihyponormal.

1.4 BACKGROUND OF THE STUDY.

Hilbert space was introduced in the $20th$ century. One of the developments that led to its introduction was an observation which arose during David Hilbert and Erhard Schmidt's study of integral equations, that two square-integrable real valued functions f and g on an interval $[a,b]$ have an inner product which has many familiar properties of the Euclidean dot product. John Von Neumann coined the term abstract Hilbert space in his work on unbounded Hermitian operators. Mathematicians Hermann Weyl and Norbert Wiener had already studied particular Hilbert spaces but Von Neumann the first complete and axiomatic treatment of them. It is a generalization of Euclidean space. Prior to the development of Hilbert spaces, other generalizations of Euclidean spaces were known to mathematicians and physicists. Particularly, the idea of an abstract linear space(vector space) had gained some traction towards the end of $19th$ century. Hilbert space is a vector space with structure of an inner product whereby length and angle can be measured. Hilbert spaces were earliest studied in the first decade of the $20th$ century by David Hilbert, Erhard Schmidt and Frigyes Riesz. They are crucial tools in the theories of partial differential equations, quantum mechanics, fourier analysis (including applications to signal processing and heat transfer) and ergodic theory. The success of Hilbert space methods plays an important role in functional analysis. Ariyadasa Aluthge [AAL] introduced Aluthge transforms to study p-hyponormal linear operators.

2 Chapter 2

The aim of this chapter was to showcase the class of $\tilde{T}$ after identifying that of T. This chapter was the longest chapter in this project. We started by classifying Aluthge transforms of some classes of operators. Also generalized Aluthge transforms of different classes were investigated herein. In addition, different results touching on iterated Aluthge transforms and those about powers of Aluthge transformations were borrowed and discussed.

2.1 ALUTHGE TRANSFORMS OF DIFFERENT CLASSES OF OPERATORS.

Remark 2.1.1. *Recall that the class of w-hyponormal operators contain all normal operators but there are some w-hyponormal operators which are not normal operators. The following result by [AD00], [SJG11], [SJG13], [AF12] and [CHK01] says $\tilde{T}$ is normal if T is w-hyponormal.*

Corollary 2.1.2. *Let $T = U|T|$ be a w-hyponormal operator. If $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is normal then T is also a normal operator.*

Proof. Since T is w-hyponormal, it is also quasinormal. Therefore $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = U|T|$ and $\tilde{T}^* = |T|U^*$. Hence $|T|^2 = |\tilde{T}|^2 = |T^*|^2$. This shows that $|T| = |T^*|$ so T is normal. $\square$

Proposition 2.1.3. *Let $T \in B(H)$. Then :*

(i.) $\widetilde{(cT)} = c(\tilde{T})$ *for all* $c \in \mathbb{C}$.

(ii.) $\widetilde{(VTV^*)} = V(\tilde{T})V^*$ *for some* V *being unitary operator.*

(iii.) *If* $T = T_1 + T_2$, *then* $(\tilde{T}) = (\tilde{T_1}) + (\tilde{T_2})$.

(iv.) $||\tilde{T}||_2 \leq ||T||_2$.

(v.) T *and* $\tilde{T}$ *have the same characteristic polynomial in particular* $\sigma(\tilde{T}) = \sigma(T)$.

Proposition 2.1.4. *Let $T \in B(T)$,*

(i.) *If* $0 \in \sigma(T)$, *then there exist* $n \in \mathbb{N}$ *such that* $m(T,0) = m_0(\tilde{T}^n)$, $m_0(T,\mu) \leq m_0(\tilde{T},\mu)$.

(ii.) *For every $\mu \in \sigma(T)$, $m_0(T,\mu) \leq m_0(\tilde{T},\mu)$. This implies that if T is diagonalizable. That is $m_0(T,\mu)) = m(T,\mu)$ for every μ. Then also $\tilde{T}$ is diagonalizable.*

Remark 2.1.5. *The Aluthge transform of an operator $T = U|T|$ does not depend on the partial isometry U of the polar decomposition of that operator.*

Theorem 2.1.6. *Let $T = U|T|$ be the polar decomposition of $T \in B(H)$ for a Hilbert space H. Let $\tilde{T}$ denote the Aluthge transform of T. Then the following assertions hold:*

(i.) *The spectrum of T, $\sigma(T) = \sigma(\tilde{T})$*

(ii.) *The point spectrum of T, $\sigma_p(T) = \sigma_p(\tilde{T})$*

(iii.) *The approximate point spectrum of T, $\sigma_{ap}(T) = \sigma_{ap}(\tilde{T})$*

(iv.) *The essential spectrum of T, $\sigma_e(T) = \sigma_e(\tilde{T})$*

(v.) *The left essential spectrum of T, $\sigma_{le}(T) = \sigma_{le}(\tilde{T})$*

(vi.) *The right essential spectrum of T, $\sigma_{re}(T) = \sigma_{re}(\tilde{T})$*

(vii.) $||\tilde{T}|| \leq ||T^{\frac{1}{2}}|| \leq ||T||$.

Remark 2.1.7. *T is quasiaffinity, that is T is one-to-one and has a dense range if and only if $|T|$ is a quasiaffinity. U is a unitary operator, and therefore $\tilde{T}$ is quasiaffinity if T is. Moreover, in this case, T and $\tilde{T}$ are quasisimilar. Furthermore, T is invertible if and only if $\tilde{T}$ is and in this case T and $\tilde{T}$ are similar.*

Theorem 2.1.8. *Let $T = U|T|$ be an arbitrary quasiaffinity for $T \in L(H)$ and $Lat(T)$ the invariant subspace lattice of an arbitrary operator $T \in B(H)$. Then the mapping $\phi : N \longrightarrow (|T|^{\frac{1}{2}}N)^-$, for $N \in Lat(T)$, maps $Lat(T)$ into $Lat(\tilde{T})$, and moreover if $\{\overline{0}\} \neq N = H$, then $\{\overline{0}\} \neq \phi(N) = (|T|^{\frac{1}{2}}N)^- \neq H$. Moreover, the mapping $\varphi : M \longrightarrow (U|T|^{\frac{1}{2}}M)^-$, $M \in Lat(T)$, maps $Lat(\tilde{T})$ into $Lat(T)$ and if $\{\overline{0}\} \neq M \neq N$, then $\{\overline{0}\} \neq \varphi(M) = (U|T|^{\frac{1}{2}}M)^- \neq H$. Consequently, $Lat(T)$ is a non trivial if and only if $Lat(\tilde{T})$ is non trivial.*

Remark 2.1.9. *The following theorem shows that Aluthge transform preserves complex symmetry. An article [Ram08] by Stephan Ramon Garcia explains this.*

Theorem 2.1.10. *The Aluthge transform of a complex symmetric operator is complex symmetric. In other words, if $T = CT^*C$ for some conjugation C, then there exists a conjugation J such that $\tilde{T} = J(\tilde{T})^*J$.*

Proof. We may write $T = CJ|T|$ where J is a conjugation on H which commutes with $|T|$. Since $\tilde{T} = |T|^{\frac{1}{2}}CJ|T|^{\frac{1}{2}}$ and $(CJ)^* = JC$, it follows that
$J(\tilde{T})^*J = J|T|^{\frac{1}{2}}JC|T|^{\frac{1}{2}}J$
$= |T|^{\frac{1}{2}}CJ|T|^{\frac{1}{2}}\ \tilde{T}$ $\square$

Lemma 2.1.11. *If T is a complex symmetric operator, then the following are equivalent:*

(i.) *T is quasinormal,*

(ii.) *C and $|T|$ commute. That is $|T|$ is also C-symmetric,*

(iii.) *T is normal.*

Theorem 2.1.12. *If T complex symmetric then $\tilde{T} = T$ if and only if T is normal.*

Proof. Since $\tilde{T} = T$ if and only if T is quasinormal, this follows from Lemma 2.1.11 and the fact that all normal operators are complex symmetric. □

Theorem 2.1.13. *$\tilde{T} = 0$ if and only if T is nilpotent of order two. That is $T^2 = 0$.*

Proof. Let $T = U|T|$ denote the polar decomposition of T. If $\tilde{T} = 0$, then
$T^2 = U|T|U|T| = U|T|^{\frac{1}{2}}\tilde{T}|T|^{\frac{1}{2}} = 0$
so that T is nilpotent of order two.
Conversely, if $T^2 = 0$, then $U|T|U|T| = 0$ whence $|T|U|T| = 0$ since U^*U is the orthogonal projection onto $cl(ran|T|)$. This implies that $|T|^{\frac{1}{2}}\tilde{T}|T|^{\frac{1}{2}} = 0$. Since $\tilde{T}$ vanishes on $ker|T|$, it suffices to show that $\tilde{T}$ also vanishes on $cl(ran|T|)$. Suppose that $y \in ran|T|$ but that $z = \tilde{T}y \neq 0$. Writing $y = |T|^{\frac{1}{2}}x$ it follows that
$0 = |T|^{\frac{1}{2}}\tilde{T}|T|^{\frac{1}{2}}x = |T|^{\frac{1}{2}}\tilde{T}y = |T|^{\frac{1}{2}}z \neq 0$
since z is a non zero vector in $ran|T|$. This is a contradiction that shows $\tilde{T}$ vanishes identically on $ran|T|$ and hence on $cl(ran|T|)$ as well. Thus $\tilde{T} = 0$. □

Corollary 2.1.14. *Let $n \in \mathbb{N}$ and $T \in B(H)$. Then $\tilde{T}$ is centered if and only if for each positive integer n, $W_{n-1}U = (W_0U)^n$. In particular, if T is a centered operator, then $\tilde{T}$ is centered if and only if $W_{n-1}U = U^n$.*

Proof. By the definition $\tilde{T}$ is centered if and only if for each positive integer n, $W_{n-1}U = (W_0U)^n$. If T is centered then T is binormal, and so $W_0 = I$. Thus we conclude that $\tilde{T}$ is centered if and only if $W_{n-1}U = U^n$. □

Corollary 2.1.15. *If $T \in B(H)$ is quasinormal, then $\tilde{T}$ is centered.*

Remark 2.1.16. *An operator A is said to be w-hyponormal if $|\tilde{A}| \geq |A| \geq |\tilde{A}^*|$. The class of w-hyponormal operators is more general than that of both semi-hyponormal and hyponormal operator classes. However, if an operator A is w-hyponormal then its first and second Aluthge transforms are semi-hyonormal and hyponormal respectively as shown in the following example by A. Aluthge and D. Wang in [AD00].*

Lemma 2.1.17. *If A is w-hyponormal, then $\tilde{A}$ is semi-hyponormal and $\tilde{\tilde{A}}$ is hyponormal. Where $\tilde{\tilde{A}}$ is the second Aluthge of operator A.*

Remark 2.1.18. *It is known that if A is a normal operator then the kernel condition holds, that is, $kerA = kerA^*$. However, if A is w-hyponormal, the kernel condition does not hold. By imposing an additional requirement on w-hyponormal operators A. Aluthge and D. Wang [AD00] stated and proved the following result.*

Lemma 2.1.19. *Let A be w-hyponormal with $kerA \subset kerA^*$. If $\tilde{A}$ is normal, then $A = \tilde{A}$.*

Remark 2.1.20. *A p-hyponormal operator for $p > 1$ is also hypornormal. More interest has been put on p-hyponormal operators for $0 < p \leq 1$. For more information, Aluthge has discussed this on [AAL].*

Lemma 2.1.21. *Suppose that $T = U|T|$ (polar decomposition) is an arbitrary p-hyponormal operator in $L(H)$ for some $p \in [\frac{1}{2}, 1]$. Then its Aluthge transform $\tilde{T}$ is a hyponormal operator.*

Remark 2.1.22. *We earlier stated on a remark that Aluthge transform of an operator $T = U|T|$ does not depend on the partial isometry U. This applies to any other partial isometry, say for instance V for polar decomposition $T = V|T|$.*

Lemma 2.1.23. *Let $T = U|T|$ be the polar decomposition of T. If there exists another decomposition $T = V|T|$, then $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}$.*

`Proof.` Let $H = N(|T|^{\frac{1}{2}}) \oplus N(|T|^{\frac{1}{2}})^{\perp}$. In case $x \in N(|T|^{\frac{1}{2}}) \cdot \tilde{T}x = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0$ $= |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}x$. In case $x \in N(|T|^{\frac{1}{2}})^{\perp} = \overline{R(|T|^{\frac{1}{2}})}$. There exists $y \in H$ such that $x = |T|^{\frac{1}{2}}y$. Then we have $\tilde{T}x = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = |T|^{\frac{1}{2}}U|T|y = |T|^{\frac{1}{2}}Ty = |T|^{\frac{1}{2}}V|T|y = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}x$. Hence we have $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}$ on $H = N(|T|^{\frac{1}{2}}) \oplus N(|T|\frac{1}{2})^{\perp}$. □

Remark 2.1.24. *A. Aluthge in [AAL] proved that for a p-hyponormal operator $T = U|T|$ in $L(H)$ with $0 < p < \frac{1}{2}$, then the aluthge transform $\tilde{T}$ of T is a $(p + \frac{1}{2})$-hyponormal operator and the second aluthge transform $\tilde{\tilde{T}}$ of T is a hyponormal operator.*

Theorem 2.1.25. *Let $T \in L(H)$ be p-hyponormal. Then*
- *If $p \geq \frac{1}{2}$, then $\tilde{T}$ is hyponormal,*
- *If $p < \frac{1}{2}$, then $\tilde{T}$ is $(p + \frac{1}{2})$-hyponormal,*
- *It holds that $\tilde{\tilde{T}}$ is hyponormal.*

Remark 2.1.26. *The following theorem implies that if $\tilde{T}$ has a non trivial invariant subspace then T does. It is then better when looking for invariant subspaces of T to find invariant subspaces of $\tilde{T}$ in their place.*

Theorem 2.1.27. *If $Lat(T)$ denotes the lattice of invariant subspaces of a given operator $T \in L(H)$, then $Lat(T) \simeq Lat(\tilde{T})$.*

Theorem 2.1.28. *If T is a complex symmetric operator, then $\tilde{T} = C(\tilde{T^*})C$.*

`Proof.` Since T is complex symmetric, it follows that $C(TT^*)C = T^*T$ and hence $C(TT^*)^pC = (T^*T)^p$ for all $p \geq 0$.
In particular, we note that
$T^* = CTC = C(CJ\sqrt{T^*T}C = J\sqrt{T^*T}C = JC\sqrt{TT^*}$
whence
$C(\tilde{T^*})C = C[(TT^*)(TT^*)\frac{1}{4}JC(TT^*)\frac{1}{4}]C = (T^*T)\frac{1}{4}CJ(T^*T)\frac{1}{4} = \tilde{T}.$ □

Theorem 2.1.29. *Let $T = U|T| \in B(H)$ be powers of N-class A^k and U an isometry operator then $\tilde{T}$ is powers of N-class A_K operator.*

`Proof.` From the definition of powers of N-class A_k operator $|T|^{2p} \leq T|T^{k+1}|^{\frac{2p}{k+1}}$
$(T^*T)^p \leq N((T^*T)^{k+1})^{\frac{p}{k+1}}$
$(U^*|T^*|U|T|)^p$
$\leq N((U^*|T^*|U|T|)^{k+1})^{\frac{p}{k+1}}$
$(U^*|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}})^p$
$\leq N((U^*|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}})^{k+1})^{\frac{p}{k+1}}$
$U((\tilde{T}^*)^*\tilde{T}^*)^pU^*$
$\leq NU(((\tilde{T}^*)^*\tilde{T}^*)^{k+1})^{\frac{p}{k+1}}U^*$
$U|\tilde{T}^*|^{2p}U^* \leq NU|(\tilde{T}^*)^{2(k+1)}|^{\frac{p}{k+1}}U^*$
$|\tilde{T}|^{2p} \leq N|(\tilde{T})^{K+1}|^{\frac{2p}{k+1}2p}$
Therefore $\tilde{T}$ is powers of N-class A_k operator. □

Lemma 2.1.30. *The Aluthge transform map $T \to \tilde{T}$ is $(||.||, ||.||)$-continuous on $B(H)$.*

Remark 2.1.31. *We know that w-hyponormal operators are normal operators. We also know that the class of normal operators is contained in the class of quasinormal operators. Jung, Ko and Pearcy proved in [IEC00] that $\tilde{T} = T$ if and only if T is a quasinormal operator.*

Theorem 2.1.32. *Let T be a w-hyponormal operator with the polar decomposition $T = U|T|$. If $\tilde{T}$ is quasinormal, then T is also quasinormal. Hence T coincides with its Aluthge transform $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.*

`Proof.` Since T is a w-hyponormal operator, $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$. Then Douglass theorem implies that $\mathbb{R}(\tilde{T}) = \mathbb{R}(\tilde{T}^*) \subset \mathbb{R}(|T|) = \mathbb{R}|\tilde{T}|$ where M denotes the norm closure of M. Let $\tilde{T} = W|\tilde{T}|$ be the polar decomposition of $\tilde{T}$. Then $E := W^*W = U^*U \geq WW^* =: F$.
Put

$$|\tilde{T}^*| = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$$

$$W = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}$$

on $H=\overline{\mathbb{R}(\tilde{T})}\oplus ker(\tilde{T}^*)$.
Then X is injective and has a dense range. Since $\tilde{T}$ is quasinormal, W commutes with $|\tilde{T}|$ and
$|\tilde{T}|=W^*W|\tilde{T}|=W^*|\tilde{T}|W\geq W^*|T|W\geq W^*|\tilde{T}|W=|\tilde{T}|$.
Hence $|\tilde{T}|=W^*|\tilde{T}|W=W^*|T|W$.
and

$$|\tilde{T}^*|=W|\tilde{T}|W^*=WW^*,|\tilde{T}|WW^*=WW^*|T|WW^*=\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$$

Since

$$WW^*=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

implying that $|\tilde{T}|$ and T are of the forms

$$|\tilde{T}|=\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\geq |T|=\begin{pmatrix} X & 0 \\ 0 & Z \end{pmatrix}$$

, where $\overline{\mathbb{R}(Y)}=\mathbb{R}(|T|)\ominus\overline{\mathbb{R}(\tilde{T})}$
$=ker(\tilde{T}^*)\ominus ker(T)$.
Since W commutes with $|\tilde{T}|$,

$$\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}=\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}$$

So $W_1X=XW_1$ and $W_2Y=XW_2$, and hence $\overline{\mathbb{R}(W_1)}$ and $\overline{\mathbb{R}(W_2)}$ are reducing subspaces of X. Since $W^*W|T|=|T|$, we have $W_1^*W_1=1$ and
$X^k=X^k=W_1^*W_1X^k=W_1^*X^kW_1$,
$Y^k=W_2^*W_2Y^k=W_2^*X^kW_2$. Put

$$U=\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

Then $\tilde{T}=|T|^{\frac{1}{2}}U|T|_{\frac{1}{2}}=W|\tilde{T}|$ implies

$$\begin{pmatrix} X^{\frac{1}{2}} & 0 \\ 0 & Z_{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} X^{\frac{1}{2}} & 0 \\ 0 & Z^{\frac{1}{2}} \end{pmatrix}=\begin{pmatrix} W1 & W2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

Hence $X^{\frac{1}{2}}U_{11}X^{\frac{1}{2}}=W_1X=X^{\frac{1}{2}}W_1X^{\frac{1}{2}}$,
$X^{\frac{1}{2}}U_{12}Z^{\frac{1}{2}}=W_2Y=XW_2$

and
$X^{\frac{1}{2}}(U_{11}-W_1)X^{\frac{1}{2}}=0$,
$X^{\frac{1}{2}}(U_{12}Z^{\frac{1}{2}}-X^{\frac{1}{2}}W_2)=0$ Since X is injective and has a dense range, $U_{11}=W_1$ is isometry and $U_{12}Z^{\frac{1}{2}}=X^{\frac{1}{2}}W_2$. Then

$$U^*U=\begin{pmatrix} U_{11}^*U_{11}+U_{21}^*U_{21} & U_{11}^*U_{12}+U_{21}^*U_{22} \\ U_{12}^*U_{11}+U_{22}^*U_{21} & U_{12}^*U_{12}+U_{22}^*U_{22} \end{pmatrix}$$

on $H=\overline{\mathbb{R}(\tilde{T})}\oplus ker(\tilde{T}^*)$ is the orthogonal projection onto $\overline{\mathbb{R}(|T|)}\supset\overline{\mathbb{R}(\tilde{T})}$ and

$$U^*U=\begin{pmatrix} 1 & 0 \\ 0 & U_{12}^*U_{12}+U_{22}^*U_{22} \end{pmatrix}.$$

Since $U_{12}Z^{\frac{1}{2}}=X^{\frac{1}{2}}W_2$, we have $Z\geq Z^{\frac{1}{2}}U_{12}^*U_{12}Z^{\frac{1}{2}}=W_2^*XW_2=Y$, and $Z\geq Z^{\frac{1}{2}}U_{12}^*U_{12}Z^{\frac{1}{2}}=W_2^*XW_2=Y\geq Z$. Hence $Z^{\frac{1}{2}}U_{12}^*U_{12}Z^{\frac{1}{2}}=Z=Y$, so $Z=Y$ and $|\tilde{T}|=|T|$. Since $Z=Z^{\frac{1}{2}}U_{12}^*U_{12}Z^{\frac{1}{2}}\leq Z_{12}^{\frac{1}{2}}U_{12}Z^{\frac{1}{2}}+Z^{\frac{1}{2}}U_{22}^*U_{22}Z^{\frac{1}{2}}\leq ZZ^{\frac{1}{2}}U_{22}^*U_{22}Z^{\frac{1}{2}}=0$
and
$U_{22}Z^{\frac{1}{2}}=0$. This implies $\mathbb{R}(U_{22}^*)\subset ker(Z)$. Since $\mathbb{R}(U_{12}^*U_{12}+U_{22}^*U_{22})\subset\mathbb{R}(Z)$ and $U^*U_{22}\leq U_{12}^*U_{12}+U_{22}^*U_{22}$, we have $\mathbb{R}(U_{22}^*)\subset\mathbb{R}(Z)$.
Hence $U_{22}=0$,

$$U=\begin{pmatrix} W_1 & U_{12} \\ 0 & 0 \end{pmatrix}$$

and $\mathbb{R}(U)\subset\mathbb{R}(\tilde{T})\subset\mathbb{R}|T|)=\mathbb{R}(E)$. Since W commutes with $|\tilde{T}|=|T|$, W commutes with $|T|$ and $|T|^{\frac{1}{2}}(W-U)|T|^{\frac{1}{2}}=W|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}-|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}=W|\tilde{T}|-\tilde{T}=0$. Hence $E(W-U)E=0$ and $U=UE=EUE=EWE=WE=W$. Thus $U=W$ commutes with $|T|$ and T is quasinormal. □

Remark 2.1.33. *Since T and $\tilde{T}$ have the same spectrum, they are both singular or non singular. Since $(\widetilde{T^{-1}})=(\tilde{T})^{-1}$ is true in the case where T is invertible.*

Lemma 2.1.34. *The matrix $T\in\mathbb{C}^{n\times n}$ is invertible if and only if $\tilde{T}$ is invertible, and in this case T and $\tilde{T}$ are similar.*

Theorem 2.1.35. *Let $T=U|T|$ be a contraction such that $ker(T)=Ker(T^2)$. If $\tilde{T}$ is a partial isometry, then $T=\tilde{T}=U$ and T is a quasinormal partial isometry.*

`Proof.` We first note that
$ker(\tilde{T})=ker(T^2)=ker(T)=ker(U)$,
so $\mathbb{R}(\tilde{T}^*)=\overline{\mathbb{R}(T^*)}=\overline{\mathbb{R}(|T|)}$.
Since $\tilde{T}=U$ on $ran(\tilde{T}^*)=\overline{\mathbb{R}(|T|)}$ and
$ker(\tilde{T})=ker(U)=\mathbb{N}(T)$,

$\tilde{T} = U$ because $H = \overline{\mathbb{R}(|T|)} \oplus ker(T)$. This shows that
$\mathbb{R}(U) = \mathbb{R}(\tilde{T}) \subset \overline{\mathbb{R}(|T|)} = \mathbb{R}(U^*U)$.
Thus $U = UU^*U = U^*UU$. Let

$$|T| = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, U^*U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

on $H = \overline{\mathbb{R}|T|} \oplus ker(T)$.
Since T is a contraction, we have $U^*|T|U \leq 1$ and $0 \leq X \leq 1$.
Then
$U^*U = \tilde{T}^*\tilde{T} = |T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}} \leq |T| \leq U^*U$.
Hence $|T| = U^*U$ and $T = U|T| = UU^*U = U = \tilde{T}$.
Thus T is a quasinormal partial isometry. □

Corollary 2.1.36. *Let $T = U|T|$ be w-hyponormal operator. If $\tilde{T}$ is a partial isometry, then $\tilde{T} = T$ and T is a quasinormal partial isometry.*

`Proof.` Since $|\tilde{T}|$ is a contraction and $|\tilde{T}| \geq |T|$, it follows that T is a contraction and $ker(T) = ker(\tilde{T}) = ker(T^2)$.
Now the result follows from Theorem 2.1.35. □

Corollary 2.1.37. *Every bounded, q-quasinormal operator T is quasinilpotent, so that $\sigma(T) = \{0\}$.*

`Proof.` Let T be a non-zero bounded, q-quasinormal operator. Then, we have $q > 1$. By the equation, $||T^n|| = |||T^n||| \leq (\frac{1}{\sqrt{q}})^{\frac{n(n-1)}{2}}||T||^n$. Since $q > 1$, $||T^n||^{\frac{1}{n}} \leq (\frac{1}{\sqrt{q}})^{\frac{n-1}{2}}||T|| \to 0$ as $n \to \infty$. Hence, T is quasinilpotent and hence $\sigma(T) = \{0\}$. □

Lemma 2.1.38. *Let $T = U|T|$ be the polar decomposition of T. If there exists another decomposition $T = V|T|$, then $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}$.*

`Proof.` Let $H = N(|T|^{\frac{1}{2}}) \oplus N(|T|^{\frac{1}{2}})^{\perp}$. In case $x \in N(|T|^{\frac{1}{2}})$. $\tilde{T}x = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0 = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}$. In case $x \in N(|T|^{\frac{1}{2}})^{\perp} = \overline{R(|T|^{\frac{1}{2}})}$. There exists $y \in H$ such that $x = |T|^{\frac{1}{2}}y$. Then we have $\tilde{T}x = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = |T|^{\frac{1}{2}}U|T|y = |T|^{\frac{1}{2}}Ty = |T|^{\frac{1}{2}}V|T|y = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}x$. Hence we have $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}$on $H = N(|T|^{\frac{1}{2}}) \oplus N(|T|^{\frac{1}{2}})^{\perp}$. □

Theorem 2.1.39. *Let $T \in B(H)$. Then the following assertions are equivalent:*

(i.) *T is normaloid.*

(ii.) *$||T|| = ||\tilde{T}_n||$ for all natural number n.*

Theorem 2.1.40. *Let $T \in B(H)$. Then $lim_{n\to\infty}||\tilde{T_n}|| = r(T)$.*

Theorem 2.1.41. *Let $T = U|T|$ be the polar decomposition of powers of N-class $A(k)$ operator for $N > 0$ then $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is $(p+\frac{1}{2})$ powers of N-class $A(k)$ operators for $0 < p \leq 1$.*

Proof. Let T be the powers of N-class $A(k)$ operator then
$|T|^{2p} \leq N(T^*|T|^{2k}T)^{\frac{p}{k+1}}$
$|T|^{2p} \leq (T^*T)^{p(p+\frac{1}{2})}$
$= (U^*|T^*|U|T|)^{p(p+\frac{1}{2})}$
$= (U^*|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}})^{p(p+\frac{1}{2})}$
$= (\tilde{T}^*\tilde{T})^{p(p+\frac{1}{2})}$
$= |\tilde{T}|^{2p(p+\frac{1}{2})}$
$[N(T^*|T|^{2k}T)^{\frac{p}{k+1}}]^{(p+\frac{1}{2})} = [N(T^*T^{*k}T^kT)^{\frac{p}{k+1}}]^{(p+\frac{1}{2})}$
$= [N(U^*|T^*|U^{*k}|T^*|^kU^k|T|^kU|T|)^{\frac{p}{k+1}}]^{(p+\frac{1}{2})}$
$= [N\tilde{T}^*|T|^{\frac{k}{2}}U^{*k}|T|^{\frac{k}{2}}|T|^{\frac{k}{2}}U^k|T|^{\frac{k}{2}}\tilde{T})^{\frac{p}{k+1}}]^{(p+\frac{1}{2})}$
$= [N(\tilde{T}^*|T|^{2k}\tilde{T})^{\frac{p}{k+1}}]^{(p+\frac{1}{2})}$
Hence $|\tilde{T}|^{2p(p+\frac{1}{2})} \leq [N(\tilde{T^*}|T|^{2k}\tilde{T})^{\frac{p}{k+1}}]^{(p+\frac{1}{2})}$ □

Theorem 2.1.42. *Let $T \in B(H)$, then $\tilde{T}^{(*)}$ is p-hyponormal.*

Proof. Let $T \in B(H)$ then
$|T|^{2p} \leq N(T^*|T|^{2k}T)^{\frac{p}{k+1}}p$
$|T|^{2p} = (U^*|T^*|U|T|)^p$
$= (|T|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})^p$
$= (\tilde{T^{(*)}}(\tilde{T^{(*)}})^*)^p$
$N(T^*|T|^{2k}T)^{\frac{p}{k+1}} = N(U^*|T^*|U^{*k}|T^*|^kU^k|T|^kU|T|)^{\frac{p}{k+1}}$
$= N(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}|T^*|^{\frac{k}{2}}U^{*k}|T^*|^{\frac{k}{2}}|T^*|^{\frac{k}{2}}U^k|T^*|^{\frac{k}{2}}|T^*|^{\frac{k}{2}}U|T^*|^{\frac{k}{2}})^{\frac{p}{k+1}}$
$= N(\tilde{T}^{(*)})^*[(\tilde{T}^{(*)})^*]^k(\tilde{T}^{(*)})^k\tilde{T}^{(*)\frac{p}{k+1}}$
$= N[(\tilde{T}^{(*)})^*\tilde{T}^{(*)}]^p$
Hence $N((\tilde{T}^{(*)})^*\tilde{T}^{(*)})^p \geq (\tilde{T}^{(*)}(\tilde{T}^{(*)})^*)^p$. □

Theorem 2.1.43. *If $T \in B(H)$ is powers of N-class $A(k)$ operator then $(\tilde{T}^{(*)})^*$ is p-hyponormal.*

Theorem 2.1.44. *If $T \in B(H)$ is powers of N-class $A(k)$ operator then $\tilde{T}$ is p-hyponormal.*

Theorem 2.1.45. *If $T \in B(H)$ is powers of N-class $A(k)$ operator then $\widetilde{T^*}$ is p-hyponormal.*

Theorem 2.1.46. *Let $T \in B(H)$, then $\tilde{T}$ is p-hyponormal $\Leftrightarrow$ $\tilde{T}^{(*)}$ is p-hyponormal.*

Theorem 2.1.47. *Let $T \in B(H)$, then $\widetilde{T^*}$ is p-hyponormal $\Leftrightarrow$ $(\tilde{T}^{(*)})^*$ is p-hyponormal.*

Corollary 2.1.48. *Let $T = U|T|$ be p-posinormal operator for $0 < p < 1$. Then*

(1.) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is* $(p+\frac{1}{2})$*-posinormal for* $0 < p < \frac{1}{2}$.

(2.) $\tilde{T}$ *is posinormal for* $\frac{1}{2} \leq p < 1$.

Theorem 2.1.49. *Let* $T = U|T|$ *be the polar decomposition of* T. *Then* T *is class* $p - wA(s,t)$ *if and only if* $|\tilde{T}_{s,t}|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$ *and* $|T|^{2sp} \geq |\tilde{T}^*_{s,t}|^{\frac{2sp}{s+t}}$.

Proof. $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$
$\Leftrightarrow (U|T|^tU^*|T|^{2s}U|T|^tU^*)^{\frac{tp}{s+t}} \geq U|T|^{2tp}U^*$
$\Leftrightarrow U(|T|^tU^*|T|^{2s}U|T|^t)^{\frac{tp}{s+t}}U^* \geq U|T|^{2tp}U^*$
$\Leftrightarrow (|T|^tU^*|T|^{2s}U|T|^t)^{\frac{tp}{s+t}} \geq |T|^{2tp}$
$\Leftrightarrow |\tilde{T}_{s,t}|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$.
Also, $(|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp}$
$\Leftrightarrow (|T|^sU|T|^{2t}U^*|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp} \Leftrightarrow |\tilde{T}^*_{s,t}|^{\frac{2sp}{s+t}} \leq |T|^{2sp}$. □

Corollary 2.1.50. *If* T *is class* $p - wA(s,t)$, *then* $\tilde{T}_{s,t}$ *is* $\frac{min\{sp,tp\}}{s+t}$*-hyponormal.*

Theorem 2.1.51. *Let* $T = U|T|$ *be the polar decomposition. Then* $\tilde{T} = U|\tilde{T}| \Leftrightarrow T$ *is binormal.*

Proof. We now prove the theorem.
$\Rightarrow$ Assume that $\tilde{T} = U|\tilde{T}|$, we have $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U|T^{|}\frac{1}{2}U^* = \tilde{T}U^* = U|\tilde{T}T|U^* \geq 0$, then $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = |T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}}$, that is, T is binormal.
$\Leftarrow$ If T is binormal, then $0 \leq |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = ||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}|$. Then $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U$ $= ||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}|U = UU^*(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}}U = U(|T|^{\frac{1}{2}}U * |T|U|T|^{\frac{1}{2}})^{\frac{1}{2}} = U|\tilde{T}|$. Hence the proof is complete. □

Remark 2.1.52. *Note,* $\tilde{T} = U|\tilde{T}|$ *in above Theorem is not the polar decomposition since* $(U) = N(\tilde{T})$ *does not hold. One might expect that* $\tilde{T}$ *is also binormal if* T *is binormal. But there is a counterexample for this expectation as follows:*

Example 2.1.53. *There exists a binormal operator* T *such that* $\tilde{T}$ *is not binormal. Let*

$$T = \begin{pmatrix} 0 & 0 & 5 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \end{pmatrix}$$

and $T = U|T|$ *be the polar decomposition. Then* T *is binormal since*

$$T^*T \cdot TT^* = TT^* \cdot T^*T = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 25 \end{pmatrix},$$

and also

$$|T| = (T^*T)^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

so that

$$U = T|T|^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \end{pmatrix}.$$

Therefore

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 5 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{15}}{2} & \frac{-\sqrt{5}}{2} & 0 \end{pmatrix}$$

We get that

$$(\tilde{T})^*\tilde{T} \cdot \tilde{T}(\tilde{T})^* = \begin{pmatrix} 20 & -\sqrt{3} & 0 \\ -5\sqrt{3} & 2 & 0 \\ 0 & 0 & 25 \end{pmatrix}$$

and

$$\tilde{T}(\tilde{T})^* \cdot (\tilde{T})^*\tilde{T} = \begin{pmatrix} 20 & -5\sqrt{3} & 0 \\ -\sqrt{3} & 2 & 0 \\ 0 & 0 & 25 \end{pmatrix}$$

Hence $\tilde{T}$ is not binormal.

Theorem 2.1.54. *Let $T = U|T|$ be the polar decomposition of a binormal operator T. Then the following assertions are equivalent:*

(i.) $\tilde{T}$ *is binormal;*

(ii.) $[U^2|T|(U^2)^*, |T|] = 0$.

`Proof.` We first note that T is binormal if and only if $[U|T|U^*, |T|] = 0$. Then we obtain
$[U|T|U^*, U^2|T|(U^2)^*] = 0$
since $U^2|T|(U^2)^* \cdot U|T|U^* = U \cdot U|T|U^* \cdot |T| \cdot U^*$
$= U \cdot |T| \cdot U|T|U^* \cdot U_*$ by
$= U|T|U^*\dot{U}^2|T|(U^2)^*$. Therefore we have $|\tilde{T}|^2|\tilde{T}^*T^*|^*$
$= |T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}} \cdot |T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}}$
$= U^* \cdot U|T|^{\frac{1}{2}}U^* \cdot |T| \cdot U|T|U^*\dot{U}^2|T|(U^2)^* \cdot U|T|\frac{1}{2}U^* \cdot U$
$= U^*|T \cdot U^2|T|(U^2)^*U|T|^2U^*U$

$= U^*|T| \cdot U^2|T|(U^2)^*U|T|^2$
and
$|\tilde{T}^*|^2|\tilde{T}|^2 = |T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}} \cdot |T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}$
$= U^* \cdot U|T|^{\frac{1}{2}}U^* \cdot U^2|T|(U^2)^* \cdot U|T|U^* \cdot |T| \cdot U|T|^{\frac{1}{2}}U^* \cdot U$
$= U^*U^2|T|(U^2) * \dot{|}T|U|T|^2U^*U$
$= U^*U^2|T|(U^2)^* \cdot |T|U|T|^2.$
Hence proof of (ii.) $\Rightarrow$ (i).
Proof of (i.) $\Rightarrow$ (ii.). Since $\tilde{T}$ is binormal, we have
$U^2|T|(U^2)^* \cdot |T|U|T|^2 = UU^*U^2|T|(U^2)^* \cdot |T|U|T|^2$
$= UU^*|T|\dot{U}^2|T|(U^2)^*U|T|^2$
$= |T|UU^* \cdot U^2|T|(U^2)^*U|T|^2$
$= |T| \cdot U^2|T|(U^2)^*U|T|^2$, that is, $U^2|T|(U^2)^* \cdot |T| = |T| \cdot U^2|T|(U^2)^*$ on $\overline{R(U|T|^2)} = N(|T|^2U^*)^\perp = N(UU^*)^\perp = R(UU^*)$. In other words, $(U^2|T|(U^2)^* \cdot |T| \cdot UU^* = |T| \cdot U^2|T|(U^2)^* \cdot UU^*$ holds. Hence, we have $U^2|T|(U^2)^* \cdot |T| = U^2|T|(U^2)^* \cdot UU^* \cdot |T|$
$= U^2|T|(U^2)^* \cdot |T| \cdot UU^*$
$= |T| \cdot U^2|T|(U^2)^* \cdot UU^*$
$= |T| \cdot U^2|T|(U^2)^*$. Therefore the proof is complete. ☐

Theorem 2.1.55. *Let $T = U|T|$ be the polar decomposition. Then for each non-negative integer n, the following assertions are equivalent:*

(i.) $\tilde{T}_k$ *is binormal for all* $k = 0, 1, 2, \cdots, n$;

(ii.) $[U^k|T|(U^k)^*, |T|] = 0$ *for all* $k = 1, 2, \cdots, n+1$.

Proposition 2.1.56. *Let $T = U|T|$ be the polar decomposition of a binormal operator T. Then $\tilde{T} = U^*UU|\tilde{T}|$ is also the polar decomposition of $\tilde{T}$.*

Proof. Since $|T|^{\frac{1}{2}} = U^*U|T|^{\frac{1}{2}}$ and $|T^*|^{\frac{1}{2}} = UU^*|T^*|^{\frac{1}{2}}$ are the polar decompositions of $|T|^{\frac{1}{2}}$ and $|T^*|^{\frac{1}{2}}$ respectively, then $|T^{\frac{1}{2}}||T^*|^{\frac{1}{2}} = U^*UUU^*|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}$ is the polar decomposition of $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}$. Therefore we have that $\tilde{T} = U^*UUU^* \cdot U|\tilde{T}| = U^*UU|\tilde{T}|$ is also the polar decomposition of $\tilde{T}$. ☐

Theorem 2.1.57. *Let $T = U|T|$ be the polar decomposition of p-hyponormal for $1 \geq p > 0$. Then$\tilde{T} = |T|^qU|T|^q$ is hyponormal for any q such that $p \geq q > 0$.*

Proof. As T is p-hyponormal for $p > 0$, T is q-hyponormal for q such that $p \geq q > 0$ by the Löwner-Heinz theorem, then we have $U^*|T|^{2q}U \geq |T|^{2q} \geq U|T|^{2q}U^*$ for any q such that $p \geq q > 0$, this implies $(\tilde{T}^*\tilde{T}) - (\tilde{T}\tilde{T}^*) = |T|^q(U^*|T|^{2q}U - U|T|^{2q}U^*)|T|^q \geq 0$ for any q such that $p \geq q > 0$, that is, $\tilde{T}$ is hyponormal, so the proof is complete. ☐

Theorem 2.1.58. *Let H be a Hilbert space and $T \in L(H)$.*

(i.) *For* $0 < p < \frac{1}{2}$*, if* T *is* p*-hyponormal, then* $\tilde{T}$ *is* $p + \frac{1}{2}$*-hyponormal.*

(ii.) *For* $\frac{1}{2} \leq p \leq 1$*, if* T *is* p*-hyponormal, then* $\tilde{T}$ *is* 1*-hyponormal.*

Theorem 2.1.59. *Let* H *be a Hilbert space,* $T \in L(H)$*, and* T *be invertible. If* T *is log-hyponormal, then* $\tilde{T}$ *is* $\frac{1}{2}$*-hyponormal.*

Theorem 2.1.60. *Let* $T \in L(H)$*.*

(i.) $||\tilde{T}|| \leq ||T||$*.*

(ii.) T *is quasinormal if and only if* $T = \tilde{T}$*.*

Proof. Let $T = U|T|$ be the polar decomposition of T. One can note that (if $U \neq 0$) $||U|| = 1$. Further, one can see that $||T|| = |||T|^2||^{\frac{1}{2}} = |||T||| = |||T|^{\frac{1}{2}}||^2$ and hence $|||T|^{\frac{1}{2}}|| = ||T||^{\frac{1}{2}}$. Now $||\tilde{T}|| = |||T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}||$
$\leq |||T|^{\frac{1}{2}}|| \cdot ||U|| \cdot |||T|^{\frac{1}{2}}||$
$= |||T|^{\frac{1}{2}}||^2$
$= ||T||$
Further, $T = \tilde{T} \Rightarrow T = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \Rightarrow T|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U|T| \Rightarrow T|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}T \Rightarrow |T|^{\frac{1}{2}}$ commutes with $T \Rightarrow |T|$ commutes with $T \Rightarrow T^*T$ commutes with $T \Rightarrow T$ is quasinormal. Also, T is quasinormal $\Leftrightarrow U$ and $|T|$ commute thus $\tilde{T} = T$. □

Proposition 2.1.61. *Let* T *be a partial isometry of* $B(H)$*. Then its Aluthge transform is given by* $\tilde{T} = T^*T^2$*.*

Proof. Since T is a partial isometry, the set $R(T^*)$ is closed and $|T|^2 = T^*T = pr_{R(T^*)}$. Thus $|T|^{\frac{1}{2}} = pr_{R(T^*)}$. Let $x = x_1 + x_2$ be the decomposition of an arbitrary vector of H with respect to the (orthogonal) direct sum $H = N(T) \oplus R(T^*)$. We get $Tx = Tx_2 = T\,pr_{R(T^*)}x$, so $U\,pr_{R(T^*)} = T\,pr_{R(T^*)}$ and $N(U) = N(T)$. Finally we obtain $U = T$ and $\tilde{T} = pr_{R(T^*)}T\,pr_{R(T^*)} = pr_{R(T^*)}T = T^*T^2$, because $R(T^*) = [N(T)]^{\perp}$. □

Theorem 2.1.62. *Suppose that* $B(H)$ *is a Banach algebra with respect to the unitary invariant norm* $|||\cdot|||$ *and* $|||I||| = 1$*. Let* $T \in B(H)$ *be invertible and* $0 < \lambda < 1$*. Then* $lim_{n\to\infty}|||\tilde{T}_{\lambda}^{n}||| = r(T)$*.*

Proof. For each $k \in \mathbb{N}$, the sequence $\{|||(T_n)^k|||^{\frac{1}{k}}\}_{n\in\mathbb{N}}$ is non-increasing and converges to $s = lim_{n\to\infty}|||T_n|||$. So for all $n, k \in \mathbb{N}$, $s \leq |||(T_n)^k|||^{\frac{1}{k}}$. Suppose that $r(T) < s$, that is $r(T_n) < s$ for all n. Then for a fixed $n \in \mathbb{N}$, and sufficiently large k, we have $|||(T_n)^k|||^{\frac{1}{k}} < s$; a contradiction. So $r(T) = s$. □

Corollary 2.1.63. *Let* $\lambda \in (0,1)$*. If the sequence* $\tilde{S}_{\lambda}^{n}$ *converges for every invertible matrix* $S \in \mathbb{M}_r(\mathbb{C})$ *and every* $r \in \mathbb{N}$*, then the sequence* $\tilde{T}_{\lambda}^{n}$ *converges for all* $T \in \mathbb{M}_r(\mathbb{C})$ *and every* $r \in \mathbb{N}$*.*

Proof. Let $T \in \mathbb{M}_r(\mathbb{C})$. We can assume that $m(T,0) = m_0(T,0)$. We note that in this case, $N(\tilde{T}_\lambda) = N(T)$, since $N(T) \subseteq N(\tilde{T}_\lambda)$ and $m(\tilde{T}_\lambda, 0) = m(T,0)$. On the other hand $R(\tilde{T}_\lambda) \subseteq R(|T|)$ so that $R(\tilde{T}_\lambda)$ and $N(\tilde{T}_\lambda)$ are orthogonal subspaces. Thus, there exists a unitary matrix U such that

$$U\tilde{T}_\lambda U^* = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$$

where $S \in \mathbb{M}_s(\mathbb{C})$ is invertible $(s = r - m(T,0))$. Since for every $n \geq 2$,

$$\tilde{T}_\lambda^n = U^* \begin{pmatrix} \tilde{S}_\lambda^{n-1} & 0 \\ 0 & 0 \end{pmatrix} U,$$

the sequence $\tilde{T}_\lambda^n$ converges because the sequence $\tilde{S}_\lambda^{n-1}$ converges by hypothesis. □

Theorem 2.1.64. *Let $T = U|T|$ be the polar decomposition of T. If T is binormal then $\tilde{T} = U^*UU|\tilde{T}|$ is the polar decomposition of $\tilde{T}$.*

Theorem 2.1.65. *Let T be invertible and suppose that $T = U|T|$ is the polar decomposition of T. If T is binormal, then $\tilde{T} = U|\tilde{T}|$ is the polar decomposition of $\tilde{T}$.*

Proof. Since T is invertible and U is unitary, then by above Theorem the proof follows. □

Theorem 2.1.66. *Let T be an operator on $B(H)$. Then $\tilde{T}$ is binormal for all $n \geq 0$ iff T is centered.*

Theorem 2.1.67. *If U is a partial isometry, then the following assertions are mutually equivalent:*

(i.) U is binormal.

(ii.) $\tilde{U}$ is a partial isometry.

(iii.) U^2 is a partial isometry.

Corollary 2.1.68. *Let $T = U|T|$ be paranormal and $\tilde{T} = U|\tilde{T}|$. Then T is binormal and hyponormal.*

Proof. $\tilde{T} = U|\tilde{T}|$ and T is binormal by Theorem 2.1.67. Since T is binormal and paranormal, therefore T is hyponormal. □

2.2 GENERALIZED ALUTHGE TRANSFORMS.

Generalized Aluthge transform is very useful in studying p-hyponormal operators which was studied and discussed by Ariyadasa Aluthge in [AAL].

Proposition 2.2.1. *Let $T \in L(H)$ and $\lambda \in [0,1]$. Then:*

1. $\widetilde{cT}_{\lambda} = c\tilde{T}_{\lambda}$ *for every* $c \in \mathbb{C}$.
2. $(\widetilde{VTV^*})_{\lambda} = V\tilde{T}_{\lambda}V^*$ *for every* $V \in U(H)$.
3. $||\tilde{T}_{\lambda}|| \leq ||T||$.
4. $\sigma(\tilde{T}_{\lambda}) = \sigma(T)$.
5. *If* $dimH < \infty$, *then* T *and* $\tilde{T}_{\lambda}$ *have the same characteristic polynomial.*

Remark 2.2.2. *If $dimH = n < \infty$, equality holds for $k = n$. Indeed, if $T = U|T|$ is the polar decomposition of T, then $det|\tilde{T}_{\lambda}| = [det(|T|^{\lambda}U|T|^{1-\lambda})^*(|T|^{\lambda}U|T|^{1-\lambda})]^{\frac{1}{2}}$
$= (det|T|^2)^{\frac{1}{2}} = det|T|$.*

Theorem 2.2.3. *Let $\lambda \in (0,1)$, $1 \leq p < \infty$ and $T \in L^p(H)$. Then $||\tilde{T}_{\lambda}||_p = ||T||_p$ if and only if T is normal.*

Remark 2.2.4. *Theorem 2.2.3 fails for $\lambda = 1$. Take, for example, $T \in L^2(H)$ with polar decomposition $T = U|T|$, where $U \in U(H)$. In this case, $||\tilde{T}_1||_2 = ||T||_2$. The following example shows that the Theorem may be false for other unitarily invariant norms. In particular, for the spectral norm. Let*

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Then,

$$\tilde{T}_{\lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for every $\lambda \in (0,1)$, and therefore $1 = ||\tilde{T}_{\lambda}||_p < ||T||_p = 2^{\frac{1}{p}}$ but $\tilde{T}_{\lambda}|| = ||T|| = 1$.

Corollary 2.2.5. *Let $T \in \mathbb{M}_n(\mathbb{C})$ and $\lambda \in (0,1)$. Then, $\rho(T) = \lim_{n\to\infty} ||\tilde{T}_{\lambda}^n||$.*

`Proof.` Take a subsequence $\tilde{T}_{\lambda}^{nk}$ that converges to a limit point L. Since L is normal and $\sigma(L) = \sigma(T)$, it holds that $||L|| = \rho(L) = \rho(T)$. Hence $\lim_{k\to\infty} ||\tilde{T}_{\lambda}^{nk}|| = ||L|| = \rho(L) = \rho(T)$. Finally, since the whole sequence $||\tilde{T}_{\lambda}^n||$ converges because it is non-increasing, thus obtain the desired result. □

Remark 2.2.6. *If $dim H = n < \infty$ equality holds for $k = n$. Indeed if $T = U|T|$ is the polar decomposition of T, then $det|\tilde{T}_\lambda| = [det|T|^\lambda U|T|^{1-\lambda})^*(|T|^\lambda U|T|^{1-\lambda})]^{\frac{1}{2}} = (det|T|^2)^{\frac{1}{2}} = det|T|$.*

Corollary 2.2.7. *Let $p \geq 1$, $T \in L^p(H)$ and $\lambda \in [0,1]$. Then, $\tilde{T}_\lambda \in L^p(H)$ and $||\tilde{T}_\lambda||_p \leq ||T||_p$.*

Proof. We have
$\sum_{i=1}^n S_i(\tilde{T}_\lambda)^p \leq \sum_{i=1}^k S_i(T)^p$, $k \in \mathbb{N}$. By taking limit, we obtain $tr|\tilde{T}_\lambda|^p = \sum_{i=1}^\infty S_i(\tilde{T}_\lambda)^p \leq \sum_{i=1}^\infty S_i(T)^p = tr|T^p$ □

Theorem 2.2.8. *Let $\lambda \in (0,1)$, $1 \leq p < 1\infty$ and $T \in L^p(H)$. Then $||\tilde{T}_\lambda||_p = ||T||_p$ if and only if T is normal.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Fix $1 \leq p < \infty$. Then with $A = |T|^\lambda$ and $B^* = U|T|^{1-\lambda}$, we get $tr|\tilde{T}_\lambda|^p \leq tr||T|^\lambda T^*|^{1-\lambda}|^p$. With $A = |T|^\lambda$ and $B = |T^*|^{1-\lambda}$, we get $tr||T|^\lambda T^*|^{1-\lambda}|^p \leq tr(T|^{p\lambda}|T^*|^{p(1-\lambda}$.
Then, for the conjugate numbers λ^{-1} and $(1-\lambda)^{-1}$, $tr|\tilde{T}_\lambda|^p \leq tr(T|^{p\lambda}|T^*|^{p(1-\lambda)})$
$\leq (1-\lambda)tr|T|^p + \lambda tr|T^*||^p = tr|T|^p$. Therefore, if $\tilde{T}_\lambda||_p = ||T||_p$, then equality holds in Young's inequality. We now conclude that $|T|^p = |T^*|^p$. Hence T is normal. □

Proposition 2.2.9. *Let $T \in \mathbb{M}_n(\mathbb{C})$. Then, the limit points of the sequence $\tilde{T}_\lambda^n$, $n \in \mathbb{N}$ are normal. Moreover, if L is a limit point, then $\sigma(L) = \sigma(T)$ with the same algebraic multiplicity.*

Proof. Let $\tilde{T}_\lambda^{n_k}$, $k \in \mathbb{N}$ be a subsequence which converges in norm to a limit point L. By the continuity of Aluthge transforms, $\tilde{T}_\lambda^{n_k+1} \longrightarrow_{k\to\infty} \tilde{L}_\lambda$. Then $||\tilde{L}_\lambda||_2 = lim_{k\to\infty}||\tilde{T}_\lambda^{n_k+1}||_2 = lim_{n\to\infty}||\tilde{T}_\lambda^n||_2 = lim_{k\to\infty}||\tilde{T}_\lambda^{n_k}||_2 = ||L||_2$. Hence, by a previous Proposition L is normal. It only remains to prove that $\sigma(L) = \sigma(T)$ with the same algebraic multiplicity, or equivalently, that $tr(T^m) = tr(L^m)$ for every $m \in \mathbb{N}$. Indeed, $trL^m = lim_{k\to\infty} tr(\tilde{T}_\lambda^{n_k})^m = trT^m$, $m \in \mathbb{N}$, because, for each $k \in \mathbb{N}$, $\sigma(\tilde{T}_\lambda^{n_k}) = \sigma(T)$ (with algebraic multiplicity), and therefore $tr(\tilde{T}_\lambda^{n_k})^m = trT^m$. □

Theorem 2.2.10. *Let $T \in \mathbb{M}_2(\mathbb{C})$ and $\lambda \in (0,1)$. Then, the sequence $\tilde{T}_\lambda^n$, $n \in \mathbb{N}$ converges.*

Proof. Suppose that $\sigma(T) = \{\mu_1, \mu_2\}$. Since we have proved that the limit points of the sequence $\tilde{T}_\lambda^n$ are normal, if $\mu_1 = \mu_2 = c$, then $\tilde{T}_\lambda^n \longrightarrow_{n\to\infty} cI$. From now on we shall consider the case in which
$\mu_1 \neq \mu_2$.
We denote $T_n = \tilde{T}_\lambda^n$. Fix $n \geq 0$. If $T_n = U_n|T_n|$ is the polar decomposition of T_n, then $|T_n^*|^t = U|T_n|^t U^*$, for every $t > 0$. Therefore we obtain $(T_{n+1} - T_n)U_n^* = |T_n|^\lambda U_n|T_n|^{1-\lambda}U_n^* - U_n|T_n|U_n^*$

$= |T_n|^{\lambda}|T_n^*|^{1-\lambda} - |T_n^*|$
$= (|T_n|^{\lambda} - |T_n^*|^{\lambda})|T_n^*|^{1-\lambda}$, and then
$||T_{n+1} - T_n||_2 \leq |||T_n|^{\lambda} - |T_n^*|^{\lambda}||_2|||T_n^*|^{1-\lambda}||_2$
$\leq 2|||T_n|^{\lambda} - |T_n^*|^{\lambda}||_2|||T_n^*|^{1-\lambda}||_{sp}$
$= 2|||T_n|^{\lambda} - |T_n^*|^{\lambda}||_2||T_n||^{1-\lambda}$
$\leq |||T_n|^{\lambda} - |T_n^*|^{\lambda}||_2||T||^{1-\lambda}$. Applying previous Proposition to $A = T_n^*T_n$, $B = T_nT_n^*$ and $r = \frac{\lambda}{2}$, since $||I_2||_2^{1-\frac{\lambda}{2}} \leq 2$, we get
$||T_{n+1} - T_n||_2 \leq 2|||T_n|^{\lambda} - |T_n^*|^{\lambda}||_2||T||^{1-\lambda} \leq (4||T||^{1-\lambda})||T_n^*T_n - T_nT_n^*||_2^{\frac{\lambda}{2}}$. If $\gamma(T,\lambda) \in (0,1)$ is the constant of a previous Lemma, and $a = \gamma(T,\lambda)^{\frac{\lambda}{2}} < 1$, then
$||T_{n+1} - T_n||_2 \leq (4||T||^{1-\lambda})||T_n^*T_n - T_nT_n^*||_2^{\frac{\lambda}{2}}$
$\leq a^n(4||T||^{1-\lambda}||T^*T - TT^*||_2^{\frac{\lambda}{2}})$. Denote $N(T,\lambda) = 4||T||^{1-\lambda}||T^*T - TT^*||_2^{\frac{\lambda}{2}})$. Then, if $n,m \in \mathbb{N}$, with $n < m$, $||T_m - T_n||_2 \leq \sum_{k=n}^{m-1} ||T_{k+1} - T_k||_2$
$\leq N(T,\lambda)\sum_{k=n}^{m-1} a^k \longrightarrow_{n,m\to\infty} 0$, which shows that the $lim_{n\to\infty}T_n = lim_{n\to\infty}\tilde{T}_{\lambda}^n$ exists. □

Corollary 2.2.11. *Let $\lambda \in (0,1)$. If the sequence $\tilde{S}_{\lambda}^m$ converges for every invertible matrix $S \in \mathbb{M}_n(\mathbb{C})$ and every $n \in \mathbb{N}$, then the sequence $\tilde{T}_{\lambda}^m$ converges for all $T \in \mathbb{M}_n(\mathbb{C})$ and every $n \in \mathbb{N}$.*

Proof. Let $T \in \mathbb{M}_n(\mathbb{C})$. By Proposition 2.1.4, we can assume that $m(T,0) = m_0(T,0)$. Note that, in this case, $N(\tilde{T}_{\lambda}) = N(T)$ since $N(T) \subseteq N(\tilde{T}_{\lambda})$ and $m(\tilde{T}_{\lambda},0) = m(T,0)$. On the other hand, $R(\tilde{T}_{\lambda}) \subseteq R(|T|)$ so that $R(\tilde{T}_{\lambda})$ and $N(\tilde{T}_{\lambda})$ are orthogonal subspaces. Thus, there exists a unitary matrix U such that

$$U\tilde{T}_{\lambda}U^* = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$$

where $S \in \mathbb{M}_s(\mathbb{C})$ is invertible $(s = n - m(T,0))$. Since for every $m \geq 2$

$$\tilde{T}_{\lambda}^m = U^* \begin{pmatrix} \tilde{S}_{\lambda}^{m-1} & 0 \\ 0 & 0 \end{pmatrix} U,$$

the sequence $\tilde{T}_{\lambda}^m$ converges, because the sequence $\tilde{S}_{\lambda}^{m-1}$ converges by hypothesis. □

Remark 2.2.12. *If $T \in \mathbb{M}_n(\mathbb{C})$ is invertible, then $|T|^{\lambda}$ is invertible, for every $\lambda \in (0,1)$, and $\tilde{T}^{\lambda}(T) = |T|^{\lambda}T|T|^{-\lambda}$.*

Theorem 2.2.13. *Let $T = U|T|$ be the polar decomposition of a p-posinormal operator for $0 < p \leq 1$. Then the following assertions hold:*

(1.) $\tilde{T}_{s,t} = |T|_sU|T|_t$ is $\frac{p+min\{s,t\}}{s+t}$-posinormal for $s,t > 0$ such that $max\{s,t\} \geq p$.

(2.) $\tilde{T}_{s,t}$ *is posinormal for* $0 < s,t \leq p$.

Proof. Suppose that $|T^*|^{2p} \leq \mu |T|^{2p}$ for some $\mu > 1$.

(1.) Let $A = \mu|T|^{2p}$ and $B = |T^*|^{2p}$. Then $(\tilde{T}_{s,t}^* \tilde{T}_{s,t})^{\frac{p+min\{s,t\}}{s+t}} = (|T|^t U^* |T|^{2s} U |T|^t)^{\frac{p+min\{s,t\}}{s+t}}$
$= U^*(|T^*|^t |T|^{2s} |T^*|^t)^{\frac{p+min\{s,t\}}{s+t}} U$
$= \mu^{-\frac{s}{p}\frac{p+min\{s,t\}}{s+t}} U^*(B^{\frac{t}{2p}} A^{\frac{s}{p}} B^{\frac{t}{2p}})^{\frac{p+min\{s,t\}}{s+t}} U \geq \mu^{-\frac{s}{p}\frac{p+min\{s,t\}}{s+t}} U^* B^{\frac{p+mins,t}{p}} U$ by Furuta Inequality $= \mu^{-\frac{s}{p}\frac{p+min\{s,t\}}{s+t}} |T|^{2(p+min\{s,t\})}$
since $\frac{s+t}{p+min\{s,t\}} \geq 1$ and $(1+\frac{t}{p})\frac{s+t}{p+mins,t} \geq \frac{s}{p} + \frac{t}{p}$.
And $(\tilde{T}_{s,t} \tilde{T}_{s,t}^*)^{\frac{p+min\{s,t\}}{s+t}}$
$= (|T|^s U |T|^{2t} U^* |T|^s)^{\frac{p+min\{s,t\}}{s+t}}$
$= (|T|^s |T^*|^{2t} |T|^s)^{\frac{p+min\{s,t\}}{s+t}}$
$= \mu^{-\frac{s}{p}\frac{p+min\{s,t\}}{s+t}} (A^{\frac{s}{2p}} B^{\frac{t}{p}} A^{\frac{s}{2p}})^{\frac{p+min\{s,t\}}{s+t}}$
$\leq \mu^{\frac{t-s}{p}\frac{p+min\{s,t\}}{s+t}} A^{\frac{p+min\{s,t\}}{p}}$ by Furuta Inequality $= \mu^{\frac{t-s}{p}\frac{p+min\{s,t\}}{s+t}} |T|^{2(p+min\{s,t\})}$
since $\frac{s+t}{p+mins,t} \geq 1$ and $(1+\frac{s}{p})\frac{s+t}{p+min\{s,t\}} \geq \frac{t}{p} + \frac{s}{p}$.
We have
$(\tilde{T}_{s,t} \tilde{T}_{s,t}^*)^{\frac{p+min\{s,t\}}{s+t}}$
$\leq \mu^{\frac{t}{p}\frac{p+min\{s,t\})}{s+t}} (\tilde{T}_{s,t} \tilde{T}_{s,t}^*)^{\frac{p+min\{s,t\}}{s+t}}$, that is, $\tilde{T}_{s,t}$ is $\frac{p+min\{s,t\}}{s+t}$-posinormal for $s,t > 0$ such that $max\{s,t\} \geq p$.

(2.) Applying Löwner-Heinz Inequality to the first equation, $|T^*|^{2s} \leq \mu^{\frac{s}{p}} |T|^{2s}$ and $|T^*|^{2t} \leq \mu^{\frac{t}{p}} |T|^{2t}$ hold for any $0 < s, t \leq p$.
We then have, $\tilde{T}_{s,t}^* \tilde{T}_{s,t}$
$= |T|^t U^* |T|^{2s} U |T|^t$
$\geq \mu^{-\frac{s}{p}} |T|^t U^* |T^*|^{2s} U |T|^t$
$= \mu^{-\frac{s}{p}} |T|^{2(s+t)}$ and $\tilde{T}_{s,t} \tilde{T}_{s,t}^*$
$= |T|^s U |T|^{2t} U^* |T|^s \leq |T|^s \mu^{\frac{t}{p}} |T|^{2t} |T|^s$
$= \mu^{\frac{t}{p}} |T|^{2(s+t)}$.
So $\tilde{T}_{s,t} \tilde{T}_{s,t}^* \leq \mu^{\frac{s+t}{p}} \tilde{T}_{s,t}^* \tilde{T}_{s,t}$ and hence $\tilde{T}_{s,t}$ is posinormal.

□

Remark 2.2.14. *We note that Theorem* 2.2.13 *leads the next result by putting* $s = t = \frac{1}{2}$.

Proposition 2.2.15. *Let* V *be a co-isometry of* $B(H)$. *Then we have* $\tilde{V}^n = V^{*n} V^{n+1}$ *for any non negative integer* n.

Proof. We proceed by induction. It is obvious for $n = 0$. Let $n \in \mathbb{N}$ and assume that $\tilde{V}^n = V^{*n} V^{n+1}$. The equality $|\tilde{V}^n|^2 = V^{*n+1} V^n V^{*n} V^{n+1} = V^{*n+1} V^{n+1}$ shows that $|\tilde{V}^n|^2$ is a projection, denoted by P_{n+1}. Since $N(V^{n+1}) = N(P_{n+1})$, the polar decomposition of $\tilde{V}^n$ is $\tilde{V}^n = \tilde{V}^n P_{n+1}$. It yields to the equality $\tilde{V}^{n+1} = P_{n+1}\tilde{V}P_{n+1} = V^{*n+1}V^{n+2}$ and the result is proved. □

Lemma 2.2.16. *Let $T(t) = |T|^t U|T|^{1-t}(0 \le t \le 1)$ be the generalized Aluthge transform of T. Then $\sigma_a(T) = \sigma_a(T(t))$ for $0 \le t < 1$. In general, $\sigma_a(T) \ne \sigma_a(T(1))$.*

Theorem 2.2.17. *Let $T = U|T|$ be the polar decomposition of a binormal operator $T \in B(H)$ with$N(T) = N(T^*)$. Then the generalized Aluthge transformation $\tilde{T} = |T|^q U|T|^q$ accepts the polar decomposition.*

Proof. Note that $[|T|, |T^*|] = 0$ implies that $[|T|^q, |T^*|^q] = 0$, for all $q > 0$. By induction, the equality $[|T|^{\frac{n}{m}}, |T^*|^{\frac{n}{m}}] = 0$ holds for all positive integers n, m. Hence $[|T|^q, |T^*|^q] = 0$, as $\frac{n}{m} \to q$. The proof follows from a previous theorem and the fact that $N(|T|) = N(|T|^q)$, for all $q > 0$. □

Corollary 2.2.18. *Let $T = U|T|$ be the polar decomposition of an operator $T \in B(\chi)$. If T and $\tilde{T}$ are binormal operators with $N(T) = N(T^*)$, then $\tilde{T}$ and $(\tilde{\tilde{T}}) = |\tilde{T}|^{\frac{1}{2}}\tilde{U}|\tilde{T}|^{\frac{1}{2}}$ accept the polar decompositions.*

Proof. We first prove that $N(\tilde{T}) = N((\tilde{T})^*)$. Since $N(T) = N(|T|)$, the definition of $\tilde{T}$ implies that $N(T) \subseteq N(\tilde{T})$. Then results we have that $||\tilde{T}x||^2\chi = ||\langle |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x, |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x\rangle||A$
$= ||\langle x, |T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}x\rangle||A = ||\langle x, U^*|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}}Ux\rangle||A$
$= ||\langle x, U^*||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}|^2Ux\rangle||A$
$= ||\langle x, U^*(|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}})^2Ux\rangle||A$
$= ||\langle x, U^*(|T||T^*|)Ux\rangle||A$
$= ||\langle x, U^*|T^*||T|Ux\rangle||A$. Now let $x \in N(\tilde{T})$. By the above equality $U^*|T^*||T|Ux = 0$, so $UU^*|T^*||T|Ux = 0$. Projecting UU^* on $R(|T^*|)$ and binormality of T imply that $|T^*||T|Ux = |T||T^*|Ux = 0$, that is $|T^*|Ux \in N(|T|)$. Since $N(|T|) = N(T) \subseteq N(T^*) = N(|T^*|)$, hence $|T^*|^2Ux = 0$, so $Ux \in N(|T^*|^2) = N(|T^*|) = N(U^*)$, whence $U^*Ux = |U|^2x = 0$. Therefore $N(\tilde{T}) = N(T)$. Obviously $N(T^*) \subseteq N((\tilde{T})^*)$, by $N(T) = N(T^*)$. $||(\tilde{T})^*x||^2\chi = ||\langle |T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}x,$
$|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}x\rangle||A$
$= ||\langle x, |T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}}x\rangle||A = ||\langle x, |T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}}x\rangle||A$
$= ||\langle x, ||T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}}|^2x\rangle||A = ||\langle x, (|T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}})^2x\rangle||A$
$= ||\langle x, |T||T^*|x\rangle||A$. Suppose that $x \in N((\tilde{T})^*)$. By the assumption and above equality, we reach that $|T^*|x \in N(|T|) = N(T) \subseteq N(T^*) = N(|T^*|)$, hence $x \in N(|T^*|^2) = N(|T^*|) = N(T^*)$, therefore $N(T^*) = N((\tilde{T})^*)$. Consequently $N((\tilde{T})^*) = N(\tilde{T})$. This means that $\tilde{T}$ satisfies assumption that $\tilde{T} = U^*UU|\tilde{T}|$ for a binormal operator T, hence the second Aluthge transformation $(\tilde{\tilde{T}})$ possesses the polar decomposition. □

Remark 2.2.19. *In the above corollary A denotes a C^*-algebra and χ denotes Hilbert A-modulus.*

2.3 ITERATED ALUTHGE TRANSFORMS.

Remark 2.3.1. *For every $T \in L(H)$, the sequence $||((\widetilde{T^{(n)}})||_{n=0}^{\infty}$ is decreasing such that $r(T) \leq \widetilde{T^{(n)}} \leq ||T||$.*

Proof. Since $\sigma(T) = \sigma(\tilde{T}) = \sigma(\widetilde{T^{(n)}})$ for all $n \in \mathbb{N}$, we have, $r(T) = r(\widetilde{T^{(n)}}) \leq ||\widetilde{T^{(n)}}||$ for all $n \in \mathbb{N}$. The fact $||\widetilde{T^{(n)}}|| \leq ||T||$ follows easily. Hence $||(\widetilde{T^{(n)}})||_{n=0}^{\infty}$ is a convergent sequence. □

Remark 2.3.2. *Moreover, T. Yamazaki [Yam] established the following interesting formula for the spectral radius $lim_{n\to\infty}||\tilde{T}_{\lambda}^{n}|| = r(T)$ where $\tilde{T}_{\lambda}^{n}$ is the n-th iterate of $\tilde{T}_{\lambda}$, that is $\tilde{T}_{\lambda}^{n+1} = \tilde{T}_{\lambda}(\tilde{T}_{\lambda}^{n})$, $\tilde{T}_{\lambda}^{0} = T$.*

Lemma 2.3.3. *The n-th Aluthge iterate of T is given by*

$$\tilde{T}^{(n)} = U|\tilde{T}^{(n)}| \tag{1}$$

where

$$|\tilde{T}^{(n)}| = \prod_{k=0}^{n} \tilde{\alpha}^{k}\left(|T|^{\frac{\binom{n}{k}}{2^{n}}}\right) \tag{2}$$

Proof. By induction over n, (1.) and (2.) hold for $n = 0$. Let $N \geq 1$, and assume that (2.) holds for $n = N - 1$. Then for the N-th Aluthge iterate we have:

$$\tilde{T}^{(N)} = [\textstyle\prod_{k=0}^{N-1} \tilde{\alpha}^{k}(|T|^{\frac{\binom{N-1}{k}}{2^{N-1}}})]^{\frac{1}{2}} U [\prod_{l}^{N-1} \tilde{\alpha}^{l}(|T|^{\frac{\binom{N-1}{l}}{2^{N-1}}})]^{\frac{1}{2}}$$

$$= [\textstyle\prod_{k=0}^{N-1} \tilde{\alpha}^{k}(|T|^{\frac{\binom{N-1}{k}}{2^{N}}})] U [\prod_{l=0}^{N-1} \tilde{\alpha}^{l}(|T|^{\frac{\binom{N-1}{l}}{2^{N}}})]$$

$$= [\textstyle\prod_{k=0}^{N-1} \tilde{\alpha}^{k+1}(|T|^{\frac{\binom{N-1}{k}}{2^{N}}})] U [\prod_{l=0}^{N-1} \tilde{\alpha}^{l}(|T|^{\frac{\binom{N-1}{l}}{2^{N}}})]$$

$$= [\textstyle\prod_{k=1}^{N-1} \tilde{\alpha}^{k}(|T|^{\frac{\binom{N-1}{k-1}}{2^{N}}})] U [\prod_{l=0}^{N-1} \tilde{\alpha}^{l}(|T|^{\frac{\binom{N-1}{l}}{2^{N}}})]$$

$$= U\tilde{\alpha}^N(|T|^{\frac{\binom{N-1}{N-1}}{2^N}})U[\textstyle\prod_{k=1}^{N-1}\tilde{\alpha}^k(|T|^{\frac{\binom{N-1}{k-1}}{2^N}})] + \left(\begin{matrix} N-1 \\ k \end{matrix}\right)/2^N)]|T|^{\frac{1}{2^N}}$$

$= U\prod_{k=0}^{N}\tilde{\alpha}^k(|T|^{\frac{\binom{N}{k}}{2^N}})$, and this shows that (1.) and (2.) as well hold for $n = N$. □

2.4 POWERS OF ALUTHGE TRANSFORMS.

Corollary 2.4.1. *Let $T = U|T|$ be the polar decomposition of an invertible p-hyponormal operator for $1 \geq p > 0$, where U is a unitary operator and $|T| > 0$. Let q and r be any positive numbers such that $q \geq p$ and $\frac{1}{2}(1+\frac{p}{q}) \geq r$. Also let $\tilde{T} = \tilde{U}|\tilde{T}|$ be the polar decomposition of an operator $\tilde{T} = |T|^q U|T|^q$. Then $\tilde{\tilde{T}} = |\tilde{T}|^r\tilde{U}|\tilde{T}|^r$ is hyponormal.*

Proof. As T is invertible p-hyponormal, $1 \geq p > 0$, T can be decomposed into $T = U|T|$, where U is unitary and $|T| > 0$, so that $\tilde{T} = |T|^q U|T|^q$ is $\frac{1}{2}(1+\frac{p}{q})$-hyponormal for q such that $q \geq p$. Then we obtain $\tilde{\tilde{T}} = |\tilde{T}|^r\tilde{U}|T\tilde{T}|^r$ is hyponormal for any r such that $\frac{1}{2}(1+\frac{p}{q}) \geq r$, so the proof is complete. □

Remark 2.4.2. *The following theorem is proved in [Ima]. The equality holds since T is invertible and therefore all Aluthge transforms of T are same.*

Theorem 2.4.3. *If T is w-hyponormal operator which is invertible, then $\tilde{T^k}_n$ and $\tilde{T}_n^k$ are also w-hyponormal operators which are invertible for $k, n \in J^+$. Also $\tilde{T^k}_n = \tilde{T}_n^k$.*

3 Chapter 3

In this chapter spectral properties of different classes of operators were discussed. Further, numerical ranges including that of Aluthge transform of different operators were looked at.

3.1 SPECTRAL PROPERTIES OF DIFFERENT CLASSES OF OPERATORS.

I.Jung, E.Ko and C.Pearcy [IEC00] proved that spectral properties of an operator are preserved by the Aluthge transform.

Lemma 3.1.1. *Let S be a posinormal operator. If $z \in \sigma_p(S)$ for $0 < p < \frac{1}{2}$, then $\bar{z} \in \sigma_p(S^*)$.*

Proof. Suppose 0 is in point spectrum of S. Then there exists a non zero vector $x \in H$ such that $Sx = 0$. Since $|S|^2 x = S^*Sx = 0$ and $|S| \geq 0$, we have $(S^*S)^{\frac{1}{2}k} x = 0 (k = 1,2,\cdots)$. For $m \in \mathbb{N}$ such that $\frac{1}{m} < p$, we have $(S^*S)^{\frac{1}{2}m} x = 0$. It then follows that, $(S^*S)^p x = 0$. Clearly $(SS^*)^p x = 0$ since S is posinormal. Therefore $S^*x = 0$. Next assume that $z \in \sigma_p(S)$ for non zero $z \in \mathbb{C}$. Then there exists a non zero vector $y \in H$ such that $Sy = zy$. Let $S = U|S|$ be a polar decomposition of S with unitary operator U. Since $U|S|y = zy$, it follows that $|S|^{\frac{1}{2}} U |S|^{\frac{1}{2}} |S|^{\frac{1}{2}} y = z|S|^{\frac{1}{2}} y$. We know that $\tilde{S} = |S|^{\frac{1}{2}} U |S|^{\frac{1}{2}}$.
Hence, we have $\tilde{S^*} = |S|^{\frac{1}{2}} U^* |S|^{\frac{1}{2}} y = \bar{z} |S|^{\frac{1}{2}} y$.
Thus $S^*(|S|y) = \bar{z}|S|y$. Since $|S|y \neq 0$, then $\bar{z} \in \sigma_p(S^*)$. □

Lemma 3.1.2. *Let $S = (S_1, S_2, \cdots, S_n)$ be doubly commuting n-tuple of posinormal operators on H. If $z = (z_1, z_2, \cdots, z_n) \in \sigma_p(S)$, then $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n) \in \sigma_p(S^*)$, where $S^* = S_1^*, S_2^*, \cdots, S_n^*$*

Proof. There exists a non-zero vector $x \in H$ such that $S_1 x = z_1 x (i = 1, \cdots, n)$. We assume that $z_1, \cdots, z_k$ are non-zero and $z_{k+1} = \cdots = z_n = 0$. therefore we obtain $S_{k+1}^* = \cdots = S^* x = 0$. Also $S_i^*(S_i|x) = \bar{z}_i |S_i| x$, where S_{S_i} is the positive operator in a polar decomposition $Si = U_i |S_i|$ where $i = 1, \cdots, k$. Suppose $|S_1| \cdots |S_k| x = 0$. Since $(S_1 \cdots S_k)$ is doubly commuting k-tuple of a posinormal operator, then U_i and $|S_i|$ commute with U_j and $|S_j|$ for every $i \neq j$. Thus we have $S_1 \cdot S_2 \cdots S_k x = 0$. It follows that $z_1, \cdots, z_k = 0$. Since every $z_i \neq 0 (i = 1, \cdots, k)$. Therefore we have $|S_1| \cdots |S_k| x \neq 0$. For $i (i = 1, \cdots, k)$, we have $S_i^*(|S_1| \cdots |S_k| x) = |S_1| \cdots |S_{i-1}| \cdot |S_{1+1}| \cdots |S_k| \cdot S_i^* |S_i| x$
$= \bar{z}_i (|S_i| \cdots |S_k| x)$. Since also S_i commutes with $|S_1 \cdots |S_k|$, we have $S_i^*(|S_1| \cdots |S_k| x) = 0 (i = k+1, \cdots, n)$ Therefore it follows that $\bar{z} = (\bar{z}_1, \cdots, \bar{z}_n) \in \sigma_p(S^*)$. □

Remark 3.1.3. *The following lemma says that both the spectrum and the point spectrum of an operator T is preserved by generalized aluthge transform of T.*

Lemma 3.1.4. *Let $T(t) = |T|^t U|T|^{1-t}(0 \leq t \leq 1)$ be the generalized Aluthge transform of T. Then $\sigma(T) = \sigma(T(t))$ and $\sigma_p(T) = \sigma_p(T(t))$ for $0 \leq t \leq 1$.*

Remark 3.1.5. *The following theorem shows that spectral properties of an operator T are preserved under the aluthge transform for $T = U|T|$. Jung, Ko and Pearcy proved this in [IEC00].*

Theorem 3.1.6. *For every $T = U|T|$ (polar decomposition) in $L(H)$; $\sigma(T) = \sigma(\tilde{T}), \sigma_{ap}(T) = \sigma_{ap}(\tilde{T}), \sigma_p(T) = \sigma_p(\tilde{T}), \sigma_{ap}(T^*)/(0) = \sigma_{ap}((\tilde{T})^*)/(0)$, and $\sigma_p(T^*)/(0) = \sigma_p((\tilde{T})^*)/(0)$.*

Corollary 3.1.7. *Let $T \in B(H)$. Then the following assertions are equivalent:*

(i.) *T is a spectraloid.*

(ii.) *$\omega(T) = \omega(\tilde{T}_n)$ for a natural number n.*

Proof. Since $||T|| \geq \omega(T) \geq r(T)$, we have $\lim_{narrow\infty} \omega(\tilde{T}_n) = r(T)$. We therefore obtain the following inequalities: $\omega(T) \geq \omega(\tilde{T}) \geq \cdots \geq \omega(\tilde{T}_n)$. Hence the proof is complete. □

Lemma 3.1.8. *If T is an algebraic operator, then $\sigma(T) = \sigma_p(T)$ (point spectrum of T).*

Proof. It is well known that an operator T is algebraic if and only if its spectrum consists of poles only. But a pole of an operator is always an eigenvalue. Hence for an algebraic operator the spectrum and the point spectrum coincide. □

Proposition 3.1.9. *For any $T \in B(H)$, $\sigma_c(T) \subset \sigma_{ap}(T)$.*

Proof. If $\lambda \in \sigma(T)$ but $\lambda \notin \sigma_{ap}(T)$, then there exists $c > 0$ such that $||Tx - \lambda x|| \cdot c||x||$ for all $x \in H$. But this implies $R(T - \lambda I)$ is closed, so $\lambda \notin \sigma_c(T)$. □

Proposition 3.1.10. *If T is a bounded normal operator, then $\sigma_r(T) = \phi$.*

Proof. If $\lambda \in \sigma_r(T)$, then $ker(T* - \overline{\lambda} I) = R(T - \lambda I)^{\perp} \cdot 0$. Since T is normal, so is $T - \lambda I$, and thus for any x, $||(T - \lambda I)x|| = ||(T* - \overline{\lambda} I)x||$, so $ker(T - \lambda I) = ker(T* - \overline{\lambda} I) \cdot 0$. But this implies $\lambda \in \sigma_p(T)$, which is a contradiction. □

Proposition 3.1.11. *Let $T \in B(H)$ be a normal operator. Then*

(a.) *T is self-adjoint if and only if $\sigma(T) \subset \mathbb{R}$.*

(b.) T is a projection if and only if $\sigma(T) \subset \{0,1\}$.

(c.) T is unitary if and only if $\sigma(T) \subset \{z \in \mathbb{C} : |z| = 1\}$.

Theorem 3.1.12. *Let T be a normal operator in $B(H)$. Then*

(a.) T is self-adjoint iff $\sigma(T) \subset \mathbb{R}$,

(b.) T is unitary iff $\sigma(T) \subset T = \{\lambda : |\lambda| = 1\}$.

Theorem 3.1.13. *If T is a normal operator then $\sigma(T) = \sigma_{app}(T)$.*

Proof. Taking into account that $T - \lambda$ is normal, for any
$x \in H$, $||(T-\lambda)x|| = ||(T^* - \overline{\lambda})x||$
so $\sigma_p(T) = \overline{\sigma_p(T^*)}$.
Also,
$\lambda \in \sigma_p(T^*)$,
$\Longrightarrow Ker(T^* - \lambda) \neq (0)$,
$\Longrightarrow Ran(T^* - \lambda)^*$ is not dense,
$\Longrightarrow Ran(T - \overline{\lambda})$ is not dense,
$\Longrightarrow \lambda \in \sigma_{comp}(T)$.
Conclusion: $\sigma_{comp}(T) \subset \sigma_p(T) \subset \sigma_{app}(T)$.
Since $\sigma(T) = \sigma_{app}(T) \cup \sigma_{comp}(T)$ the result follows. □

Theorem 3.1.14. *Let T be a compact operator, let λ be a non-zero complex number, and suppose that $T - \lambda$ is not bounded below. Then $\lambda \in \sigma_p(T)$.*

Proof. Let x_n be a sequence of unit vectors such that $||(T-\lambda)x_n|| \to 0$, $n \to \infty$. Since B_1 is weakly compact, x_n has a weakly convergent subsequence x_{n_k}, so the compactness of T implies that Tx_{n_k} is a convergent sequence. Let $x = lim_k Tx_{n_k}$. Notice that $||x|| \geq ||\lambda x_{n_k}|| - ||(T-\lambda)x_{n_k}|| \to |\lambda|$ so x is a non-zero vector. Moreover, $||(T-\lambda)x|| \leq ||(T-\lambda)(Tx_{n_k} - x)|| + ||(T-\lambda)Tx_{n_k}|| \to 0$ so $\lambda \in \sigma_p(T)$. □

Remark 3.1.15. *Since T^* is also compact, by above theorem we can conclude that $\overline{\lambda} \in \sigma_p(T^*)$.*

Corollary 3.1.16. *The spectrum of a compact operator consists of 0 and its eigenvalues.*

Remark 3.1.17. *An operator T is quasinilpotent if $\sigma(T) = \{0\}$. More properties of the spectrum of T are discussed in the following theorem.*

Theorem 3.1.18. *(a.) If T is a unitary operator then $\sigma(T)$ is a subset of the unit circle.*

(b.) If T is a self-adjoint operator then $\sigma(T)$ is a subset of the real axis.

(c.) If T is a positive operator then $\sigma(T)$ is a subset of the non-negative real axis.

(d.) If T is a non-trivial projection then $\sigma(T) = \{0,1\}$.

Proof. All operators listed in the theorem are normal, it suffices to prove assertions $(a.)$ to $(d.)$ with $\sigma_{app}(T)$ instead of $\sigma(T)$. To that end, we will prove that, if λ does not belong to the appropriate set, then $T-\lambda$ is bounded below.

(a.) If T is unitary and $|\lambda| \neq 1$, then $||Tx-\lambda x|| \geq |||Tx|| - ||\lambda x||| = |(1-\lambda)|||x||$ so T is bounded below.

(b.) Let $\lambda = \alpha+\beta$.Then $||Tx-\lambda x||^2$
$= ||Tx-\alpha x||^2 - 2Re\langle Tx-\alpha x, i\beta x\rangle + ||i\beta x||^2$.

(c.) $||Tx-\lambda x||^2 = ||Tx-\alpha x||^2 - 2Re\langle Tx-\alpha x, i\beta x\rangle + ||i\beta x||^2$. If α, β are real numbers and $T = T^*$ we have that $\langle T-\alpha x, x\rangle \in \mathbb{R}$, it follows that $Re\langle Tx-\alpha x, i\beta x\rangle = 0$ Therefore, $||Tx-\lambda x||^2 \geq |\beta|^2||x||^2$, so $\beta \neq 0$ implies that $T-\lambda$ is bounded below.

(d.) If $T \geq 0$ then T is self-adjoint, so $\sigma(T) \subset \mathbb{R}$. Notice that $||Tx-\lambda x||^2 = ||Tx||^2 - 2Re\langle Tx, \lambda x\rangle + ||\lambda x||^2$. If $\lambda < 0$ then $\langle Tx, \lambda x\rangle < 0$ (by definition of a positive operator) so $||Tx-\lambda x||^2 \geq |\lambda|^2||x||^2$ and $T-\lambda$ is bounded below.

(e.) If T is a non-trivial projection then neither T nor $I-T$ (the projection on the orthogonal complement of the range of T) can be invertible, so $\{0,1\} \subset \sigma(T)$. If $\lambda \notin \{0,1\}$, a calculation shows that $\frac{1}{\lambda(1-\lambda)}T - \frac{1}{\lambda}$is the inverse of T.

□

Theorem 3.1.19. *Let $T \in B(H)$ be k-quasi-paranormal operator, the range of T^k be not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $H = [ranT^k] \oplus kerT^{*k}$. Then T_1 is paranormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Theorem 3.1.20. *If T is k-quasi-paranormal, then $r(T) \geq \frac{||T^n||}{||T^{n-1}||}$ for every positive integer $n \geq k+1$.*

Proof. Since $\frac{||T^{k+n}||}{T^{k+n-1}||} \geq \frac{||T^{k+n-1}||}{||T^{k+n-2}||} \geq \cdots \geq \frac{||T^{k+1}||}{||T^k||}$, $\frac{||T^{k+n}||}{||T^k||} \geq (\frac{||T^{k+1}||}{||T^k||})^n$.
Thus, $||T^n|| \geq (\frac{||T^{k+1}||}{||T^k||})^n$ or $||T^n||^{\frac{1}{n}} \geq \frac{||T^{k+1}||}{||T^k||}$.
Letting $n \to \infty$, we get $r(T) \geq \frac{||T^{k+1}||}{||T^k||}$.

Similarly $r(T) \geq \frac{||T^{k+2}||}{T^{k+1}||}$.
In general, $r(T) \geq \frac{||T^{n}||}{||T^{n-1}||}$ for every positive integer $n \geq k+1$. $\square$

Remark 3.1.21. *Yamazaki in [?] proved that if T and T^* are class A operators, then T is normal. However, corresponding result is not true for k-quasi-paranormals. In case, the adjoint of a k-quasi-paranormal operator T is hyponormal, then T turns out to be normal. To see this we note that if T^* is hyponormal, then $kerT^* \subseteq kerT$ which implies T is paranormal. Hence we have the following supposition that: If the adjoint of a k-quasi-paranormal operator T is paranormal, then the operator T is normal.*

Theorem 3.1.22. *σ is continuous on the set of all hyponormal operators.*

3.2 NUMERICAL RANGES OF ALUTHGE TRANSFORMS.

Remark 3.2.1. *T. Yamazaki in [Yam] proved the following theorem that the numerical range of $\tilde{T}$ is contained in the numerical range of T.*

Theorem 3.2.2. *Let $T \in B(H)$. Then $w(T) \geq w(\tilde{T})$.*

Corollary 3.2.3. *Let $T \in L(H)$ and $\lambda \in [0,1]$. Then, for every complex analytic function f defined in a neighbourhood of $\sigma(T)$, $\overline{W(f(\tilde{T}_\lambda))} \subseteq \overline{W(f(T))}$. In particular $\overline{W(\tilde{T}_\lambda)} \subseteq \overline{W(T)}$.*

Proof. For every $\mu \in \mathbb{C}$ it holds that $||f(\tilde{T}_\lambda) - \mu I|| \leq ||f(T) - \mu I||$. So, using the well known formula $\overline{W(T)} = \cap_{\mu \in \mathbb{C}} z : |z - \lambda| \leq ||T - \lambda I||$.
We have that $\overline{(f(\tilde{T}_\lambda))} = \cap_{\mu \in \mathbb{C}} z : |z - \mu \leq ||f(\tilde{T}_\lambda) - \mu I||$
$\subseteq \cap_{\mu \in \mathbb{C}} |z : |z - \mu| \leq ||f(T) - \mu I|| = \overline{Wf(T)})$. $\square$

Corollary 3.2.4. *Let $T \in B(H)$. Then the following assertions are equivalent:*

(i.) *T is spectraloid.*

(ii.) *$w(T) = w(\tilde{T}_n)$ for all natural number n.*

Proof. Since $||T|| \geq w(T) \geq r(T)$, we have $\lim_{n\to\infty} w(\tilde{T}_n) = r(T)$ By previous theorem we obtain the following inequalities: $w(T) \geq w(\tilde{T}) \geq \cdots \geq w(\tilde{T}_n)$. Hence the proof is complete. $\square$

Theorem 3.2.5. *Let $T = U|T|$ be a decomposition. If we can choose U as isometry, then $\overline{W(T)} \supset \overline{W(\tilde{T})}$.*

To prove this theorem, we first cite the following result:

Theorem 3.2.6. *Let $T \in B(H)$. Then $W(T) = \cap_{\mu \in \mathbb{C}} \lambda : |\lambda - \mu| \leq w(T - \mu I)$.*

The proof is as follows:

Proof. First, we shall show the following assertion: If $S = V|S|$ is a decomposition such that V is isometry, then for each $\lambda \in \mathbb{C}$, $w(S - \lambda I) \leq 1 \rightarrow w(\tilde{S} - \lambda I) \leq 1$. We have the following inequalities: $||\tilde{S} - zI|| \leq |||S|V - zI||^{\frac{1}{2}}||S - zI||^{\frac{1}{2}}$
$= ||V^*(S - zI)V||^{\frac{1}{2}}||S - zI||^{\frac{1}{2}}$ (by $V^*V = I$)
$\leq ||S - zI||$ for all $z \in \mathbb{C}$. Assume that $w(S - \lambda I) \leq 1$. Then, we have $||\tilde{S} - \lambda I - zI|| \leq ||S - \lambda I - zI|| \leq 1 + 1 + |z|^{2\frac{1}{2}}$ for all $z \in \mathbb{C}$. Hence we obtain $w(\tilde{S} - \lambda I) \leq 1$ by a previous theorem. Next, for each $\mu \in \mathbb{C}$, put $S = \frac{T}{w(T - \mu I)}$ and $\lambda = \frac{\mu}{w(T - \mu I)}$. Then $|S| = \frac{|T|}{w(T - \mu I)}$ holds, and $S = U\frac{|T|}{w(T - \mu I)}$ is a decomposition such that U is isometry, and also $\tilde{S} = \frac{\tilde{T}}{w(T - \mu I)}$. Moreover, $w(S - \lambda I) \leq 1$. Then, we obtain $w(\tilde{S} - \lambda I) = \frac{w(\tilde{T} - \mu I)}{w(T - \mu I)} \leq 1$. It is equivalent to $w(\tilde{T} - \mu I) \leq w(T - \mu I)$ for all $\mu \in \mathbb{C}$. Hence the proof is complete by a previous theorem. □

Corollary 3.2.7. *If T is an $n \times n$ matrix, then $W(T) \supset W(\tilde{T})$.*

Proof. Since T is an$n \times n$ matrix, we can choose a unitary matrix U such that $T = U|T|$. Since it is in the finite-dimensional case, $W(T)$ and $W(\tilde{T})$ are both closed, and the proof is complete by a previous theorem. □

Corollary 3.2.8. *Let $T \in B(H)$ with $N(T) \subset N(T^*)$. Then $\overline{W(T)} \supset \overline{W(\tilde{T})} \supset \overline{W(\tilde{T_2})} \supset \ldots \supset \overline{W(\tilde{T_n})}$ hold for all natural number n.*

Proof. Since $N(T) \subset N(T^*)$, we can choose an isometry U such that $T = U|T|$. Then we have $W(T) \supset W(\tilde{T})$ by a previous theorem. So we have only to prove $N(\tilde{T}) \subset N(\tilde{T}^*)$ if $N(T) \subset N(T^*)$. By the definition of Aluthge transformation, $N(T) \subset N(\tilde{T})$ and $N(T) \subset N(\tilde{T}^*)$ hold easily. So, we shall show $N(T) \supset N(\tilde{T})$. Let $x \in N(\tilde{T})$. Then by $N(|T|^{\frac{1}{2}}) = N(T) \subset N(T^*) = N(|T^*|^{\frac{1}{2}})$, we
$\tilde{T}x = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0 \rightarrow Tx = |T^*|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0$.
Hence we obtain $N(\tilde{T}) \subset N(T)$, and $N(\tilde{T}) = N(T) \subset N(\tilde{T}^*)$. So the proof is complete by Theorem 3.2.5. □

Remark 3.2.9. *For each set X, we write coX for the convex hull of X, especially we write $co\sigma(T)$ for the convex hull of the spectrum of T. Recall that an operator T is said to be hyponormal and convexoid if $T^*T \geq TT^*$ and $W(T) = co\sigma(T)$, respectively. It is well known that every hyponormal operator is convexoid and normaloid, and every convexoid operator is spectraloid.*

Corollary 3.2.10. *Let T be a hyponormal operator. Then for each natural number n, we have $\overline{W(T)} = \overline{W(\tilde{T})} = \cdots = \overline{W(\tilde{T_n})} = co\sigma(T)$.*

Proof. Since T is hyponormal, $N(T) \subset N(T^*)$ holds and T is convexoid. Then we obtain the following inclusion relations:
$co\sigma(T) = \overline{W(T)} \supset \overline{W(\tilde{T})} \supset \overline{W(\tilde{T_2})} \supset \cdots \supset \overline{W(\tilde{T_n})} \supset co\sigma(\tilde{T_n}) = co\sigma(T)$. Hence the proof is complete. □

Theorem 3.2.11. $\overline{W(\tilde{A})} \subseteq \overline{W(A)}$ *for any operator A.*

Remark 3.2.12. *To prove this theorem, we need another dual notion of the Aluthge transform defined by T. Yamazaki. Let $A = V|A|$ be any polar decomposition of A. The $*$-Aluthge transform $A^{\tilde{(*)}}$ of A is the operator $|A^*|^{\frac{1}{2}}V|A^*|^{\frac{1}{2}}$. It is easily seen that $A\tilde{(*)}$ is again independent of the choice of V in A. Yamazaki showed in a Theorem in [Yam] that the numerical radii of $\tilde{A}$ and $A^{\tilde{(*)}}$ are equal. Recall that the numerical radius $w(A)$ of operator A is the quantity $sup|z| : z \in W(A)$. The next theorem says that more is true.*

Theorem 3.2.13. $\overline{W(\tilde{A})} = \overline{W(A^{\tilde{(*)}})}$ *for any operator A.*

To prove this, we need the following two lemmas.

Lemma 3.2.14. *Let $A = V|A|$ be any polar decomposition of A. Then*

(a.) $A^* = V^*|A^*|$ *is a polar decomposition of A^*,*

(b.) $(\tilde{A^*})^* = A^{\tilde{(*)}}$*, and*

(c.) $A^{\tilde{(*)}} = V\tilde{A}V^*$.

Remark 3.2.15. *The assertions of the Lemma* 3.2.14 *can be proved by delving into the construction of the polar decomposition and using the properties of V, $|A|$ and $|A^*|$.*

Lemma 3.2.16. *If A and B are operators such that $A = X^*BX$ for some contraction X, then $W(A) \subseteq (W(B) \cup \{0\})^\wedge$. If, in addition, X is a co-isometry ($XX^* = 1$), then we also have $W(B) \subseteq W(A)$.*

Proof. If x is a unit vector with $Xx = 0$, then $\langle Ax, x\rangle = \langle X^*BXx, x\rangle = 0$, which is in $W(B) \cup \{0\})^\wedge$. On the other hand, if $Xx \neq 0$, then $\langle Ax, x\rangle = \langle BXx, Xx\rangle = ||Xx||^2 \cdot \langle B(\frac{Xx}{||Xx||}), \frac{Xx}{||Xx||}\rangle + (1 - ||Xx||^2) \cdot 0$, which shows that $\langle Ax, x\rangle$ is again in $(W(B) \cup \{0\})^\wedge$. Hence $W(A) \subseteq (W(B) \cup 0)^\wedge$. If in addition, X is a co-isometry, then from $A = X^*BX$ we obtain $B = XAX^*$. For any unit vector x, we have
$\langle Bx, x\rangle = \langle XAX^*x, x\rangle = \langle AX^*x, X^*x\rangle$. Since X^*x is also a unit vector, this shows that $\langle Bx, x\rangle$ is in $W(A)$. Hence $W(B) \subseteq W(A)$ as asserted. □

We are now ready for the proof of the Theorem 3.2.13.

Proof. Two cases are considered separately here.

(i.) $dimkerA \leq dimkerA^*$. In this case, the partial isometry V in the polar decomposition $A = V|A|$ of A can be taken to be an isometry. Since $\widetilde{A^{(*)}} = V\tilde{A}V^*$ by previous lemma part (c), we may apply the preceding lemma to obtain $W(\tilde{A}) \subseteq W(\widetilde{A^{(*)}}) \subseteq (W(\tilde{A}) \cup \{0\})^\wedge$. It follows that $\overline{W(\tilde{A})} \subseteq \overline{W(\widetilde{A^{(*)}})} \subseteq \overline{(W(\tilde{A})} \cup \{0\})^\wedge$. If 0 is in $\overline{W(\tilde{A})}$, then these containments imply that $\overline{W(\tilde{A})} = \overline{W(\widetilde{A^{(*)}})}$. On the other hand, if 0 is not in $\overline{W(\tilde{A})}$, then 0 cannot be in $\sigma(\tilde{A})$. Hence $\tilde{A} = |A|^{\frac{1}{2}}V|A|^{\frac{1}{2}}$ is invertible. This implies the invertibility of $|A|^{\frac{1}{2}}$ and V. Thus V is a unitary operator, and $\tilde{A}$ and $\tilde{A^*}$ are unitarily equivalent. Hence we obviously have
$\overline{W(\tilde{A})} = \overline{W(\widetilde{A^{(*)}})}$.

(ii.) $dimkerA^* \leq dimkerA$. For this case, we apply (i) to A^* to obtain $\overline{W(\tilde{A^*})} = \overline{W(\widetilde{A^{*(*)}})}$. By Lemma 3.2.14 part (b), we have $(\tilde{A^*})^* = \widetilde{A^{(*)}}$ and $(\widetilde{A^{*(*)}})* = \tilde{A}$. Thus $\overline{W(\widetilde{A^{(*)}})} = \overline{W(\tilde{A})}$ as required.

□

Finally, we come to prove first Theorem 3.2.11.

Proof. Again, we consider two cases separately.

(i.) $dimkerA \leq dimkerA^*$. Here, for completeness, we give a simplified sketch. As before, we can choose the partial isometry V in $A = V|A|$ to be an isometry. Then

$$||\tilde{A} - zI|| \leq |||A|V - zI||^{\frac{1}{2}}||A - zI||^{\frac{1}{2}} = ||V^*(A - zI)V||^{\frac{1}{2}}||A - zI||^{\frac{1}{2}} \leq ||A - zI|| \tag{3}$$

for any $z \in \mathbb{C}$, where inequality (3.) is a consequence of Heinz inequality $(||A^{\frac{1}{2}}XB^{\frac{1}{2}}|| \leq ||AXB||^{\frac{1}{2}}||X||^{\frac{1}{2}}$ for positive operators A and B and an arbitrary operator X. This implies $\overline{W(\tilde{A})} \subseteq \overline{W(A)}$ since $\overline{W(A)} = \cap_{\lambda \in \mathbb{C}} z \in \mathbb{C} : |z - \lambda| \leq ||A - \lambda I||$
and similarly for $\overline{W(\tilde{A})}$.

(ii.) $dimkerA^* \leq dimkerA$. For this case, we apply (i) to A^* to obtain $\overline{W(\tilde{A^*})} \subseteq \overline{W(A^*)}$. Therefore,
$\overline{W(\tilde{A})} = \overline{W(\widetilde{A^{(*)}})} = \overline{W((\tilde{A^*})^*)} \subseteq \overline{W(A)}$

by Theorem 3.2.13 and Lemma 3.2.14 part (b) and by taking the adjoints. This completes the proof. □

Remark 3.2.17. *It remains to be seen whether* $W(\tilde{A}) = W(\widetilde{A^{(*)}})$ *and* $W(\tilde{A}) \subseteq W(A)$ *(without the closures) hold for an arbitrary operator* A. *The former is indeed true when* $1 \leq dimkerA \leq dimkerA^*$ *or* $1 \leq dimkerA^* \leq dimkerA$. *This is because, assuming that* $1 \leq dimkerA \leq dimkerA^*$, *we have* 0 *as an eigenvalue of* A *and hence of* $\tilde{A}$, *which implies that* 0 *is in* $W(\tilde{A})$ *and hence we obtain* $W(\widetilde{A^{(*)}}) \subseteq W(\tilde{A})$ *from* $W(\widetilde{A^{(*)}}) \subseteq (W(\tilde{A}) \cup \{0\})^{\wedge}$ *and the equality* $W(\tilde{A}) = W(\widetilde{A^{(*)}})$ *as in the proof of case (i) of Theorem* 3.2.13.

Corollary 3.2.18. *If* T *is an* $n \times n$ *matrix, then* $W(T) \supset W(\tilde{T})$.

Proof. Since T is an $n \times n$ matrix, there exists a unitary matrix U such that $T = U|T|$. Since it is in the finite dimensional case, $W(T)$ and $W(\tilde{T})$ are both closed, and the proof is complete. □

Theorem 3.2.19. *Suppose that* T *is an* $n \times n$ *matrix and* $f(z)$ *is a polynomial in* z, *then the inclusion* $W_q(f(\tilde{T})) \subset W_q(f(T))$, *holds for every complex number* q *with* $|q| \leq 1$.

Theorem 3.2.20. *{Toeplitz-Hausdorff theorem} The numerical range of every bounded linear operator* $T \in B(H)$ *is convex, that is,* $W(T)$ *is convex for all* $T \in B(H)$.

Theorem 3.2.21. *{The Ellipse theorem} If* T *is a linear transformation on* $\mathbb{C}^2$, *then* $W(T)$ *is an elliptical disc.*

Theorem 3.2.22. *Let* $T \in B(H)$ *be a normal operator. Then* $\overline{W(T)} = Conv(\sigma(T))$.

Theorem 3.2.23. *Let* $T \in B(H)$. *Then* $\sigma_e(T) \subseteq W_e(T)$.

Theorem 3.2.24. *Let* $T \in B(H)$ *be a normal operator on an infinite dimensional Hilbert space* H. *Then* $W_e(T) = Conv(\sigma_e(T)$.

Lemma 3.2.25. *Let* H *be a complex Hilbert space and* $B(H)$ *the algebra of all bounded linear operators on* H. *Let* $S \in B(H)$ *be posinormal then* $W(S)$ *is an ellipse whose foci are the eigenvalues of* S.

Proof. Let

$$S = \begin{pmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{pmatrix}$$

where λ_1 and λ_2 are eigenvalues of S. Now if $\lambda_1 = \lambda_2 = \lambda$, we have

$$S - \lambda I = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

Let $x=(x_1,x_2)$, then

$$(S-\lambda I)x = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_2 \\ 0 \end{pmatrix} = a\begin{pmatrix} x_2 \\ 0 \end{pmatrix}$$

Therefore, $||S-\lambda I|| = sup\{||a(x_2,0)|| : |x_1|^2+|x_2|^2=1\} = |a|$. Hence the radius is $\frac{1}{2}|a|$. Therefore the numerical range $W(S)=\{z : |z| \leq \frac{|a|}{2}\}$. It thus follows that $W(S)$ is a circle with center at λ and radius $\frac{|a|}{2}$. Now if $\lambda_1 \neq \lambda_2$ and $a=0$ we have

$$S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

If $x=(x_1,x_2)$, then

$$Sx = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{pmatrix}.$$

Therefore taking the inner product $\langle Sx, x\rangle$ we get

$$\langle Sx, x\rangle = \begin{pmatrix} \lambda_1 x_1 \lambda_2 x_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \overline{x_1} + \lambda_2 x_2 \overline{x_2} \end{pmatrix} = \begin{pmatrix} \lambda_1 |x_1|^2 + \lambda_2 |x_2|^2 \end{pmatrix}.$$

So $\langle Sx, x\rangle = \lambda_1|x_1|^2+\lambda_2|x_2|^2$. Now letting $t=|x_1|^2$, we write the above equation as follows $\langle Sx, x\rangle = t\lambda_1 + (1-t)\lambda_2$, since $|x_1|^2+|x_2|^2=1$. So $W(S)$ is the set of convex combinations of λ_1 and λ_2 and is the segment joining them. If $\lambda_1 \neq \lambda_2$ and $a \neq 0$ we choose λ such that it lies between λ_1 and λ_2. We therefore have

$$S - \frac{\lambda_1+\lambda_2}{2}I = \begin{pmatrix} \frac{\lambda_1+\lambda_2}{2} & a \\ 0 & \frac{\lambda_2-\lambda_1}{2} \end{pmatrix}.$$

In this case, we let $z=re^{-i\theta}$,
$\frac{\lambda_1-\lambda_2}{2} = re^{-i\theta}$ and $\frac{\lambda_2-\lambda_1}{2} = -re^{-i\theta}$
so,

$$e^{-i\theta}(S - \frac{\lambda_1+\lambda_2}{2}) = \begin{pmatrix} \frac{\lambda_1-\lambda_2}{2} & a \\ 0 & \frac{\lambda_2-\lambda_1}{2} \end{pmatrix} = S'.$$

Here we see that $W(S')$ is an ellipse with the center at $(0,0)$ and the minor axis $|a|$, and foci at $(r,0)$ and $(-r,0)$. Thus, $W(S)$ is an ellipse with foci at λ_1 and λ_2 and the major axis has an inclination of θ with the real axis. $\square$

Lemma 3.2.26. *If $T \in B(H)$ is posinormal, where H is a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Then $W(T)$ is nonempty.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal sequence of vectors in H. For $\{x_n\}_{n=1}^{\infty}$ to exist in H then $lim_{n\to\infty}\langle Tx_n, x_n\rangle = a$. The sequence $\{\langle Tx_n, x_n\rangle\}_{n=1}^{\infty}$ is bounded and $||x||$. Now, using $T = T^*$ (since all posinormal operators are self adjoint because they are all positive operators) we have $\langle Tx_n - ||T||x_n, Tx_n - ||T||x_n\rangle = \langle Tx_n, Tx_n\rangle - \langle Tx_n, ||T||x_n\rangle - \langle ||T||x_n, Tx_n\rangle +$
$\langle ||T||x_n, ||T||x_n\rangle$
$= ||Tx_n||^2 - 2||T||\langle Tx_n, x_n\rangle + ||T||^2||x_n||^2$
$\leq 2||T||^2||x_n||^2 - 2||T||\langle Tx_n, x_n\rangle$
$= 2||T||^2x_n||^2 - 2||T||\langle Tx_n, x_n\rangle$
$\Rightarrow 2||T||^2||x_n||^2 - 2||T||^2||x_n||^2 = 0$ Therefore, as $n \to \infty$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to 0 in H such that $lim_{n\to\infty}\langle Tx_n, x_n\rangle = a$. Thus x is an eigenvector for eigenvalue $||T||$. This implies that $W(T)$ is non empty. □

Theorem 3.2.27. *Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Let $S \in B(H)$ be posinormal then $||S|| = \overline{W(S)}$.*

Corollary 3.2.28. *Let $S \in B(H)$ be posinormal then $0 \in W(S)$.*

Proof. Since S is bounded, then every eigenvalue of S that lies on the boundary of $W(S)$ is a normal eigenvalue. An eigenvalue λ is said to be normal for an operator $S \in B(H)$ if $Ker(S - \lambda I) = Ker(S^* - \overline{\lambda} I)$. Assume without loss of generality that $\lambda = 0$. Suppose there is a unit vector f for which $Sf = 0$ but $S^*f \neq 0$. Let $g = \frac{S^*f}{||S^*f||}$. Because $\langle f, S^*f\rangle = \langle Sf, f\rangle = \langle 0, f\rangle = 0$ the pair (f, g) is orthonormal in H, and therefore spans a two dimensional subspace M. It follows that $W(S)$ contains the numerical range of the compression S_M of S to M. It is enough to show that 0 is in the interior of $W(S_M)$. Now the matrix of S_M with respect to the orthonormal basis (f, g) of M is of the form

$$\begin{pmatrix} 0 & a \\ 0 & * \end{pmatrix},$$

where $a = <S_M g, f>$. We need to show that $a \neq 0$, this will establish $W(S_M)$ as a non degenerate elliptical disk with one focus at 0, and therefore complete the proof. Now, $a = \langle S_M g, f\rangle = \langle PSg, f\rangle = \langle Sg, f\rangle = \langle g, S^*\rangle>$, where the term on the right, upon recalling that $g = \frac{S^*f}{||S^*f||}$, is just $\frac{\langle S^*f, S^*f\rangle}{||S^*f||} = ||S^*f|| \neq 0$. Hence the proof. □

Theorem 3.2.29. *Let $S \in B(H)$ on a complex Hilbert space H be posinormal. Then $\sigma_p(S) \subseteq \overline{W_p(S)}$.*

Proof. If λ is not a member of $\overline{W_p(S)}$, then $d = dist(\lambda, \overline{W_p(S)}) > 0$, (where $dist$ is the distance function derived from the modulus in $\mathbb{C}$) then $\lambda I - S$ has an inverse and

$||\lambda I - S)^{-1}|| < \frac{1}{d}$. By definition of distance d we have
$d \leq |\langle Sx, x\rangle - \lambda|, \forall x \in H||x|| = 1$ This implies, $d||x||^2 \leq |\langle (S-\lambda I)x, x\rangle|, \forall x \in H$ and using the cauchy-Schwatz inequality, we see that $(||S-\lambda I)x|| \geq d||x||$. Since $(<S-\lambda I)$ is bounded below, $(\lambda I - S)^{-1}$ exists on $R_{(S-\lambda I)}$ and is bounded. Moreover $||(S-\lambda I)^{-1}y|| \geq d^{-1}||y||, \forall \lambda \in R_{(S-\lambda I)}$ Hence, there are only two possibilities, that is $\lambda \in \rho(S)$ or $\lambda \in R\sigma(S)$. Suppose $\lambda \in R\sigma(S)$. Since, $\overline{R_{(S-\lambda I)}}^{\perp} = \{R_{S-\lambda I}\}^{\perp}$
$= ker(S^* - \overline{\lambda} I)$. If $\lambda \in R\sigma(S)$, then $\overline{R_{(S-\lambda I)}}^{\perp} \neq \{0\}$, that is, $ker(S^* - \overline{\lambda} I) \neq \{0\}$. Hence $\overline{\lambda}$ is an eigenvalue of S^*. If $x \in H$, $||x|| = 1$ and is such that $S^*x = \overline{\lambda}x$, then $Sx = \lambda x$ for $x \neq 0$
$\langle Sx, x\rangle = \langle x, S^*x\rangle = \langle x, \lambda x\rangle = \lambda \langle x, x\rangle = \lambda ||x||^2 = \lambda$ This implies that $\lambda \in W_p(S)$, which is a contradiction. Hence, if λ is not a member of $\overline{W_p(S)}$ then λ is not a member of $\sigma_p(S)$, this shows that $\sigma_p(S) \subseteq \overline{W_p(S)}$. Alternatively, $\lambda \in W_p(S)$ implies that there exists a sequence $\{x_n\}$ of unit vectors in H such that since for such x_n
$|\lambda - \langle Sx_n, x_n\rangle| = |\langle (\lambda I - S)x_n, x_n\rangle| \leq ||(\lambda I - S)x_n||||x_n|| \leq ||(\lambda I - S)x_n|| \to 0$ as $n \to \infty$
Therefore $\lambda = lim_{n\to\infty}\langle Sx_n, x_n\rangle$. It therefore follows that $\lambda \in W_p(S)$.
Since $|\lambda| = ||S|| = w(S) = sup|\lambda| : \lambda \in \sigma_p(S)$ So $||S|| \in \sigma_p(S)$ implies that $||S|| \in \overline{W^p(S)}$, hence $\sigma_p(S) \subseteq \overline{W_p(S)}$. □

Theorem 3.2.30. *Let S be posinormal, then $\overline{W_e(S)} = conv(\sigma_e(S))$ if and only if $\forall \lambda \in conv(\sigma_e(S))$, $||R_\lambda(S)|| \leq (d(\lambda, conv(\sigma_e(S)))$, where $d = dist(\lambda, \overline{W_e(S)}) > 0$, (dist is the distance function derived from the modulus in $\mathbb{C}$.)*

Proof. We apply the transformation $S \to \alpha S + \beta$ and suppose that $[\lambda < 0, 0 \in conv\sigma_e(S) \subset \{z \in \mathbb{C} : Rez0\}]$, $\forall \lambda < 0$. Let $\overline{W_e(S)} = conv(\sigma_e(S))$. Now for all $x \in H$, we have $||(S-\lambda)x||^2 = ||Sx||^2 - \lambda(Sx, x) + (x, Sx)] + \lambda^2||x||^2 \geq \lambda^2||x||^2$ Since $(S-\lambda)$ is invertible, we have $||x||^2 \geq \lambda^2||(S-\lambda)^{-1}x||^2, \forall x \in H$. Hence $|\lambda|^{-1} \geq ||(S-\lambda)^{-1}x||$, or $|\lambda| = d(\lambda, conv\sigma(S))$. Conversely, suppose that $||R_\lambda(S)|| \leq (d(\lambda, Conv\sigma_e(S)))$. We need to prove that $\overline{W_e(S)} = conv(\sigma_e(S))$. It suffices to show that if λ is not in the convex hull of $\sigma_e(S)$, then also λ is not in $\overline{W_e(S)}$. By applying the transformation $S \to \alpha S + \beta$ we can assume that $[\lambda < 0$, $0 \in conv\sigma_e(S) \subset \{z \in \mathbb{C} : Rez \geq 0\}]$, $\forall \lambda < 0$. The estimate $dist(c, Conv\sigma_e(S)) \geq |c|$ implies $||(S-c)^{-1}|| \leq |c|^{-1}$, so $c^2||x||^2 \leq ((S-c)x|(S-c)x)$. Let c tend to infinity, therefore $(Sx|x) + (x|Sx) \geq 0$. Hence, $\overline{W_e(S)} \subset z \in \mathbb{C} : Rez \geq 0$, that is, λ is not in $\overline{W_e(S)}$ as desired. □

Theorem 3.2.31. *Let S be a posinormal operator on H. Then $\sigma_e(S) \subseteq W_e(S)$.*

Proof. Let $\sigma_e(S) \in W_e(S)$ and let $B = \lambda I_H - S$, $\forall S \in B(H)$. Here we consider three cases: the range of B is not closed, the kernel of B is infinite dimensional, or the kernel of B^* is infinite dimensional. If the range of B is not closed, then B is not bounded below on the orthogonal complement of $ker(B)$. Let $X = ker(B)^\perp$. Then there exists a unit vector $x_1 \in X$ such that $||Bx_1|| \leq 1$. Then, since B is not bounded below, there must exist a unit vector $x_2 \in X$ orthogonal to x_1 such that $||Bx_2|| \leq \frac{1}{2}$. Repeating this process gives us an orthonormal sequence $\{x_n\}_{n\geq 1}$ such that $lim_{n\to\infty}||Bx_n|| = 0$. Thus $\lambda \in \sigma_e(S)$. If the kernel of B is infinite dimensional, we can easily construct an orthonormal sequence $\{x_n\}_{n\geq 1}$ such that $\langle Bx_n, x_n\rangle = 0$ for all n. In the same way if the kernel of B^* is infinite dimensional then $\overline{\lambda} \in W_e(S^*)$. We know that $W_e(S^*) = \overline{W_e A}$ therefore $\overline{\lambda} \in \overline{W_e S}$ hence $\lambda \in W_e(S)$ and thus $\sigma_e(S) \subseteq W_e(S)$. $\square$

4 Chapter 4

In this chapter, general examples touching on linear operators and Aluthge transforms were discussed. Some applications of Aluthge transforms and linear operators in general were looked at. Lastly, conclusions about this project were outlined.

4.1 EXAMPLES ON LINEAR OPERATORS.

Example 4.1.1. *Let*

$$A = \begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix}$$

Then

$$A^{-1} = \begin{pmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{pmatrix}$$

Example 4.1.2. *The following is an example of a normal operator.*
Let

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Clearly $TT^* = T^{*T}$.

Example 4.1.3. *Let*

$$T = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Then

$$T^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$$

Example 4.1.4. *If the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ maps each vector on the x-axis, we can construct a projection operator or a linear transformation P as follows: $P(U) = W = (w_1, w_2) = (x + 0y, 0x + 0y) = (x, 0)$*

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

P maps every vector in $\mathbb{R}^2$ to its orthogonal projection in the x-axis.

Example 4.1.5. *An example of a quasinilpotent operator which is not nilpotent.*
$T : l^2 \to l^2$
$T(x_1, x_2, x_3, \cdots) \mapsto (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots).$

Example 4.1.6. *If the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ maps each vector on the xy-plane, we can construct a projection operator or a linear transformation P as follows: $P(U) = W = (w_1, w_2, w_3) = (x + 0y + 0z, 0x + y + 0z, 0x + 0y + 0z) = (x, y, 0)$*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

P maps every vector in $\mathbb{R}^3$ to its orthogonal projection in the xy-plane.

Example 4.1.7.

$$A = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

A is unitary since $A^* = A^{-1}$.

Example 4.1.8.

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

A is unitary since $A^* = A^{-1}$.

Example 4.1.9. *Let $T \in L(H)$, $T = U|T|$ a polar decomposition of T, and $\lambda \in (0,1)$. Then:*

1. *$\tilde{T}_\lambda = T$ if and only if $U|T| = |T|U$.*
2. *If $T^2 = T$ then $\tilde{T}_\lambda$ is the orthogonal projection onto $R(T^*)$.*

Example 4.1.10. *Consider the operator d/dx. This operator is linear since:*
$(d/dx)[f(x) + g(x)] = (d/dx)f(x) + (d/dx)g(x)$
$(d/dx)[cf(x)] = c(d/dx)f(x)$ $\forall f(x), g(x)$ *in some space H and some scalar c. Generally, given a function $f = x^2 + x$ then $df/dx = d/dx(x^2 + x) = 2x$.*

Example 4.1.11. *Consider the operator $T = \int_0^1 f(x)$ where $f(x) = x^2 + 2 - 1$. Then $T(x^2 + 2 - 1) = \int_0^1 (x^2 + 2 - 1), 0 \le x \le 1$ is a linear operator $T : V \to V$.*

The following is an example of a square-normal operator which is not normal:

Example 4.1.12.

$$A = \begin{pmatrix} i & 0 \\ i & -i \end{pmatrix}, A^* = \begin{pmatrix} -i & -i \\ 0 & i \end{pmatrix}$$

Since

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$(A^*)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

,

$$A^2(A^*)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$(A^*)^2A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

So A is a square-normal operator. But

$$AA^* = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and

$$A^*A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

So A is not normal.

Example 4.1.13.

For vectors $x, y \in H$, let $x \otimes y$ denote the operator defined as $z \in H$, $x \otimes y(z) = \langle z, y \rangle x$. An elementary calculation gives: $\Delta_\lambda(x \otimes y) = \Delta(x \otimes y) = \frac{\langle x,y \rangle}{||y||^2} y \otimes y$, for $y \neq 0$.

Example 4.1.14.

$$T = \begin{pmatrix} 1 & 2 \\ -4 & -8 \end{pmatrix}, \tilde{T} = \begin{pmatrix} -1.4 & -2.8 \\ -2.8 & -5.6 \end{pmatrix}$$

The following are examples of self adjoint operators.

Example 4.1.15.

$$T = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = T^*$$

Example 4.1.16.

$$T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 3 \end{pmatrix} = T^*$$

Example 4.1.17.

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

in $B(\mathbb{C})$ *with* $\alpha = \sqrt{(3-\sqrt{5})/2}$ *and* $\beta = \sqrt{(3+\sqrt{5})/2}$. *is an example of* (α,β)*-normal operator which is neither normal nor hyponormal.*

Example 4.1.18.

$$T^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

is also (α,β)*-normal.*

Example 4.1.19.

$$T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$$

is 2*-normal but not normal.*

Example 4.1.20. *Let* U *be unilateral shift on* $\mathbf{L}^2$. *That is* $U(\alpha_0,\alpha_1,\cdots) = (0,\alpha_0,\alpha_1,\cdots)$. *Then* U *is subnormal but for any* $n \in \mathbb{N}$, U^n *is not normal.*

Example 4.1.21. *Let*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We use the standard basis in $\mathbb{R}^2$. *Let*

$$M_1 = span \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then $AM_1 \subseteq M_1$. Therefore M_1 is invariant under A. Let

$$M_2 = span\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then $AM_2 \subseteq M_2$ and therefore M_2 is also invariant under A.

Example 4.1.22. *Let*

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We use the standard basis in $\mathbb{R}^3$. Let

$$M_1 = span\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$M_2 = span\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$M_3 = span\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $TM_1 \subseteq M_1$, $TM_2 \subseteq M_2$, and $TM_3 \subseteq M_3$. Therefore M_1,M_2 and M_3 are T-invariant or invariant under T.

Example 4.1.23. *Let*

$$T = \begin{pmatrix} 1 & 2 \\ -4 & -8 \end{pmatrix}$$

Then

$$\tilde{T} = \begin{pmatrix} -1.4 & -2.8 \\ -2.8 & -5.6 \end{pmatrix}$$

Example 4.1.24. *Let*

$$A = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}$$

Then $\sigma(A) = \{-\sqrt{3i}, \sqrt{3i}\}$ and $r(A) = \sqrt{3}$

Example 4.1.25. *Let*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Then $\sigma(T) = \{-1, 1\}$

Example 4.1.26. *Let*

$$T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}.$$

We want to show by direct decomposition that

$$T^2T^* = \begin{pmatrix} i & 0 \\ -2 & -i \end{pmatrix} = T^*T^2$$

hence T *is* 2*-normal operators. Again we show that;*

$$(T^*)^2T^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (T^*T^2)^2$$

so T *is* 2*- power class (Q) operator.*

Example 4.1.27. *Consider the operator*

$$T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}.$$

So

$$T^* = \begin{pmatrix} -i & 0 \\ 2 & i \end{pmatrix}.$$

We see that

$$(T^*)^2T^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (T^*T^2)^2$$

holds and hence T *is* 2*- power class (Q) operator. Again, we see that*

$$T^2T^*T^2 = \begin{pmatrix} -i & 0 \\ 2 & i \end{pmatrix} = T^*T^4$$

and therefore T *is a* 2*- power quasi* 2*- normal operator.*

Example 4.1.28. *Let T be the unilateral shift on l_2 of square summable sequences. For any $x \in l_2$, $x = (x_1, x_2, x_3, \cdots)$, with $||x|| = 1$ and $\Sigma_{i=1}^{\infty} |x_i|^2 < \infty$, the unilateral right shift operator $T : l_1 \to l_2$ is given by*
$Tx = (0, x_1, x_2, x_3, \cdots)$. Now,

$$\langle Tx, x\rangle = \langle \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rangle = 0\overline{(x_1)} + x_1\overline{x_2} + x_2\overline{x_3} + \cdots = x_1\overline{x_2} + x_2\overline{x_3} + \cdots$$

Thus, $(|x_1| - |x_2|)^2 \geq 0$ by the arithmetic-geometric mean inequality implies that $|x_1|^2 + |x_2|^2 \geq 2(|x_1||x_2|)$.Similarly, $|x_2|^2 + |x_3|^2 \geq 2(|x_2||x_3|)$, also $|x_3|^2 + |x_4|^2 \geq 2(|x_3||x_4|)$ and so on. Therefore adding all the terms on the left and similarly on the right of the above equations, we obtain $|x_1|^2 + 2|x_2|^2 + 2|x_3|^2 + 2|x_4|^2 + \cdots \geq 2|x_1||x_2| + 2|x_2||x_3| + 2|x_3||x_4| + \cdots$
We therefore have $|\langle Tx, x\rangle| \leq |x_1x_2| + |x_2x_3| + \cdots = |x_1||x_2| + |x_2||x_3| + \cdots$
$= |x_1||x_2| + |x_2||x_3| + \cdots$
$= \frac{1}{2}(2|x_1||x_2| + 2|x_2||x_3| + \cdots)$
Now since $||x|| = |x_1|^2 + |x_2|^2 + \cdots = 1$,
we have $|\langle Tx, x\rangle| = \frac{1}{2}[|x_1|^2 + 2|x_2|^2 + 2|x_3|^2 + \cdots]$
$= \frac{1}{2}[(|x_1^2| + |x_2|^2 + |x_3|^2 + \cdots) + (|x_2|^2 + |x_3|^2 + \cdots)]$
$= \frac{1}{2}[(1 + |x_2|^2 + |x_3|^2 + \cdots)$
$= \frac{1}{2}[1 + (1 - |x_1|^2)]$
$= \frac{1}{2}[2 - |x_1|^2]$
If $|x_1| \neq 0$ we see that $|\langle Tx, x\rangle| \leq 1$. For if $|x_1| = 0$ and x contains a finite number of non zero entries, we have $|\langle Tx, x\rangle| = 1$ if we consider a minimum natural number n such that $x_n \neq 0$. Therefore, $W(T)$ is an open disc of radius < 1.

4.2 EXAMPLES ON ALUTHGE TRANSFORMS.

Remark 4.2.1. *Ken Dykema and Hanne Schultz proved that the Aluthge transformation map $T \longrightarrow \tilde{T}$ is continuous on $L(H)$. This has been proved in the following result.*

Theorem 4.2.2. *The Aluthge transform map $T \to \tilde{T}$ is $(||.||, ||.||)-$continuous on $B(H)$*

Proof. Let $T \in B(H)$ and take $\varepsilon \in (0, 1]$. Let $R = ||T|| + 1$ and let p and q be polynomials for these values of R and ε. Let $\delta \in (0, 1]$ be such that $||T - S|| < \delta$ implies that $||p(T^*T)Tq(T^*T) - p(S^*S)Sq(S^*S)|| < \varepsilon$. Then $||T - S|| < \delta$ implies that $||\tilde{T} - \tilde{S}|| < \tilde{T} - p(T^*T)Tq(T^*T)|| + ||\tilde{S} - p(S^*S)Sq(S^*S)|| + \varepsilon < 3\varepsilon$, where the last inequality is by choice of p and q. Hence the proof. □

Example 4.2.3. *Examples of Aluthge transformation Δ:*

(i.) Let T be mapped to $\tilde{T}$ by $\Delta(T) \to \tilde{T}$ for T and $\tilde{T}$ as follows:

$$T = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} and \tilde{T} = \begin{pmatrix} 2.7 & 1.8487 \\ -0.0487 & 3.3 \end{pmatrix}$$

The map Δ is given by the matrix.

$$\Delta = \begin{pmatrix} 0.9 & 0.0162 \\ -0.0162 & 1.1108 \end{pmatrix}$$

(ii.) Let $\Delta(A) \to \tilde{A}$ for A and $\tilde{A}$ as follows:

$$A = \begin{pmatrix} 1 & 2 \\ -4 & -8 \end{pmatrix}, \tilde{A} = \begin{pmatrix} -1.4 & -2.8 \\ -2.8 & -5.6 \end{pmatrix}$$

The map Δ is given by the matrix.

$$\Delta = \begin{pmatrix} -1.4 & 0 \\ 0 & 0.7 \end{pmatrix}$$

Here the map Δ is not a unique solution since A has no inverse.

Example 4.2.4. *Here we show how to compute Aluthge transform of a 2 by 2 matrix. Let*

$$T = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, T^* = \begin{pmatrix} 3 & 0 \\ 2 & 3 \end{pmatrix}$$

so we have

$$T^*T = \begin{pmatrix} 9 & 6 \\ 6 & 13 \end{pmatrix}$$

From definition of Aluthge transform $\tilde{T}$, $|T| = (T^*T)^{\frac{1}{2}}$. To obtain $|T|$ we first obtain the eigenvalues of T^*T. The eigenvalues are: $\lambda_1 = 17.3246$ and $\lambda_2 = 4.6755$. Two equations with two unknowns are obtained α and β.

$$(\lambda_1)^{\frac{1}{2}} = \alpha(\lambda_1) + \beta = 4.1623 \tag{4}$$

$$(\lambda_2)^{\frac{1}{2}} = \alpha(\lambda_2) + \beta = 2.1623 \tag{5}$$

α and β are solved from the following equations:

$$4.1623 = 17.3246\alpha + \beta \tag{6}$$

$$2.1623 = 4.6755\alpha + \beta \tag{7}$$

$\alpha = 0.1518$ *and* $\beta = 1.4231$. *So*

$$(T * T)^{\frac{1}{2}} = \alpha T^*T + \beta I = \begin{pmatrix} 2.8460 & 0.9486 \\ 0.9486 & 3.4784 \end{pmatrix} = |T|$$

We can find U for $T = U|T|$ since we have both T and $|T|$.

$$U = \begin{pmatrix} 0.9486 & 0.3162 \\ -0.3162 & 0.9486 \end{pmatrix}.$$

U is a unitary operator. It can be seen that $U^*U = U^{-1}U = I$.
It then remains to show the Aluthge transform which is given by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Here $|T|^{\frac{1}{2}}$ is solved same way $(T^*T)^{\frac{1}{2}}$ was solved above.
The eigenvalues of $|T|$ are $\lambda_1 = 4.1621$ and $\lambda_2 = 2.1623$.

$$(\lambda_1)^{\frac{1}{2}} = \alpha(\lambda_1) + \beta = 2.0401 \tag{8}$$

$$(\lambda_2)^{\frac{1}{2}} = \alpha(\lambda_2) + \beta = 1.4705 \tag{9}$$

α and β are obtained from the two equations above. $\alpha = 0.2848$ *and* $\beta = 0.8546$
Now,

$$|T|^{\frac{1}{2}} = \alpha|T| + \beta I = \begin{pmatrix} 1.6651 & 0.2702 \\ 0.2702 & 1.8452 \end{pmatrix}$$

Therefore

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \begin{pmatrix} 2.6995 & 1.8482 \\ -0.0486 & 3.2990 \end{pmatrix}$$

Example 4.2.5. *Let*

$$T = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, T^* = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

so we have

$$T^*T = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}$$

It is known that $|T| = (T^*T)^{\frac{1}{2}}$. *To obtain* $|T|$ *the eigenvalues of* T^*T *are first obtained. The eigenvalues are:* $\lambda_1 = 8.5312$ *and* $\lambda_2 = 0.4689$. *There are two equations with two unknowns* α *and* β *to be solved.*

$$(\lambda_1)^{\frac{1}{2}} = \alpha(\lambda_1) + \beta = 2.9208 \tag{10}$$

$$(\lambda_2)^{\frac{1}{2}} = \alpha(\lambda_2) + \beta = 0.6848 \tag{11}$$

α *and* β *are obtained from the following equations.*

$$2.9208 = 8.5312\alpha + \beta \tag{12}$$

$$0.6848 = 0.4689\alpha + \beta \tag{13}$$

$\alpha = 0.2773$ *and* $\beta = 0.5548$. *So*

$$(T * T)^{\frac{1}{2}} = \alpha T^*T + \beta I = \begin{pmatrix} 0.8321 & 0.5546 \\ 0.5546 & 2.7732 \end{pmatrix} = |T|$$

U *can be solved for* $T = U|T|$ *since both* T *and* $|T|$ *are known.*

$$U = \begin{pmatrix} 0.8321 & 0.5546 \\ -0.5546 & 0.8321 \end{pmatrix}.$$

U *is a unitary operator. It can be seen that* $U^*U = U^{-1}U = I$.
It then remains to show the Aluthge transform which is given by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. $|T|^{\frac{1}{2}}$ *is solved the same way* $(T^*T)^{\frac{1}{2}}$ *was solved above.*
The eigenvalues of $|T|$ *are* $\lambda_1 = 2.7732$ *and* $\lambda_2 = 0.8321$.

$$(\lambda_1)^{\frac{1}{2}} = \alpha(\lambda_1) + \beta = 1.6653 \tag{14}$$

$$(\lambda_2)^{\frac{1}{2}} = \alpha(\lambda_2) + \beta = 0.9122 \tag{15}$$

α and β are obtained from the two equations above. $\alpha = 0.3880$ and $\beta = 0.5893$
Now

$$|T|^{\frac{1}{2}} = \alpha|T| + \beta I = \begin{pmatrix} 0.9122 & 0.2152 \\ 0.2152 & 1.6653 \end{pmatrix}$$

Therefore

$$\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} = \begin{pmatrix} 0.7309 & 1.2784 \\ -0.3553 & 2.3461 \end{pmatrix}$$

Example 4.2.6. *A 3 by 3 case can be done as in the case for a 2 by 2 matrix in Example* 4.2.4. *Let*

$$T = \begin{pmatrix} 0 & 0 & 5 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \end{pmatrix}, T^* = \begin{pmatrix} 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ 5 & 0 & 0 \end{pmatrix}$$

so we have

$$T^*T = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 25 \end{pmatrix}$$

It is known that $|T| = (T^*T)^{\frac{1}{2}}$. To obtain $|T|$ the eigenvalues of T^*T are first obtained. The eigenvalues are: $\lambda_1 = 8.5312$ and $\lambda_2 = 0.4689$. Three equations are obtained with two unknowns α and β.

$$(\lambda_1)^{\frac{1}{2}} = \alpha(\lambda_1) + \beta = 5 \tag{16}$$

$$(\lambda_2)^{\frac{1}{2}} = \alpha(\lambda_2) + \beta = 5 \tag{17}$$

$$(\lambda_3)^{\frac{1}{2}} = \alpha(\lambda_3) + \beta = 1 \tag{18}$$

α and β can then be solved for from the following equations.

$$5 = 25\alpha + \beta \tag{19}$$

$$5 = 25\alpha + \beta \tag{20}$$

$$1 = \alpha + \beta \tag{21}$$

$\alpha = 6$ and $\beta = -5$.
So

$$(T*T)^{\frac{1}{2}} = \alpha T^*T + \beta I = \begin{pmatrix} 145 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 145 \end{pmatrix} = |T|$$

It can then be solved for U from $T = U|T|$ since both T and $|T|$ are known.

$$U = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \end{pmatrix}.$$

U is a unitary operator. It can be seen that $U^*U = U^{-1}U = I$.
It then remains to show the Aluthge transform given by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. $|T|^{\frac{1}{2}}$ can then be solved for as it was solved for $(T^*T)^{\frac{1}{2}}$ above.
The eigenvalues of $|T|$ are $\lambda_1 = 1$, $\lambda_2 = 5$ and $\lambda_3 = 1$.

$$(\lambda_1)^{\frac{1}{2}} = \alpha(\lambda_1) + \beta = 1 \tag{22}$$

$$(\lambda_2)^{\frac{1}{2}} = \alpha(\lambda_2) + \beta = \sqrt{5} \tag{23}$$

$$(\lambda_3)^{\frac{1}{2}} = \alpha(\lambda_3) + \beta = 1 \tag{24}$$

α and β are solved for from the two equations above. $\alpha = 0.3090$ and $\beta = 0.6910$
Now

$$|T|^{\frac{1}{2}} = \alpha|T| + \beta I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix}$$

Therefore

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & \sqrt{5} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{15}}{2} & \frac{-\sqrt{5}}{2} & 0 \end{pmatrix}$$

4.3 SOME APPLICATIONS OF ALUTHGE TRANSFORMS AND LINEAR OPERATORS.

Hilbert spaces are fine, that is every operator in them has a unique adjoint. This makes them a preference in this project compared to other spaces. Any linear operator T acting on an Hilbert space H its Aluthge transform $\tilde{T}$ is another linear operator on H. It is known that $\tilde{T}$ preserves the spectral properties of T. $\tilde{T}$ has a non trivial closed invariant subspace if and only if T has. For T normal, the spectral radius of $\tilde{T}$ and T are equal.

4.4 CONCLUSIONS.

In Chapter 1 we have managed to define various classes of operators; for instance: normal, hyponormal and w-hyponormal operators. See definitions 1.1.36, 1.1.40 and 1.1.72 respectively. We have also captured notations used in the entire write in Chapter 1; see 1.2. In Chapter 1 we have also managed to capture series of inclusions of different classes of operators notably 1.3.1.

In Chapter 2 we have managed to discuss Aluthge transform of different classes of operators. Corollary 2.1.2 gives a proof that if an operator $\tilde{T}$ is normal then a w-hyponormal operator T is also normal. We have also discussed generalized aluthge transform in Chapter 2. In Lemma 2.2.16 we have stated that the approximate spectrum of T is equal to that of generalized aluthge transform of $T(t) = |T|^t U|T|^{1-t}$ of T, that is $\sigma_a(T) = \sigma_a(T(t))$ for $0 \leq t < 1$. However, this does not hold for $t = 1$. We have also discussed iterated aluthge transform in the same chapter. Remark 2.3.2 shows an interesting formula established by T. Yamazaki [Yam] for the spectral radius $lim_{n\to\infty}||\tilde{T}_\lambda^n|| = r(T)$ where $\tilde{T}_\lambda^n$ is the n-th iterate of $\tilde{T}_\lambda$. We have also discussed powers of aluthge transform as well in Chapter 3. We have a result 2.4.3 that says if T is w-hyponormal operator which is invertible, then $\tilde{T^k}_n$ and $\tilde{T}_n^k$ are also w-hyponormal operators which are invertible and they are equal from the fact that T is an invertible operator.

In Chapter 3 we have managed to discuss spectral properties of different classes of operators and numerical ranges of aluthge transforms. Lemma 3.1.4 and theorem 3.1.6 show that the spectral properties of an operator are preserved by the generalized aluthge transform and aluthge transform respectively.

In this Chapter we have managed to give examples on linear operators and Aluthge transforms. We have also managed to give some applications of Aluthge transforms and linear operators in general. Examples 4.2.4 and 4.2.5 show how to compute aluthge transforms of a 2 by 2 and a 3 by 3 matrix respectively.

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