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Research Report in Mathematics, Number 30, 2020

Jacob Abuor Ogola

December 2020

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

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Master Thesis

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Abstract

This thesis mainly focuses on the Mathematical and Numerical aspects of the *Fredholm* integral equations of the Second Kind. Due to its wide range of physical applications, we are going to deal with three types of equations namely: differential equations, integral equations and integro-differential equations.

Some of the applications of integral equations are heat conducting radiation, elasticity, potential theory and electrostatics.

Generally, we define an integral equation where an unknown function occurs under an integral sign. Also integral equations can be classified according to three different dichotomies:

1. Nature of the limits of integration
2. Placement of the unknown function
3. Nature of the known function

After the classification of these integral equations we will have to investigate some analytical and numerical methods for solving the *Fredholm* integral equations of the Second Kind. **Analytical methods include: degenerate kernel methods, Adomain decomposition methods and Successive approximation methods** and **Numerical Methods include: Degenerate kernel methods, Projection methods, Nystrom methods and Spectral methods.**

The main objective of the thesis is to study Fredholm integral equations of the Second Kind. In chapter 4 we have given the approximate methods (**Spectral Methods**) to solve these equations **Using The Classical Orthogonal polynomials** which is the main idea of this thesis where we apply the Spectral Approximation Methods for approximating the *Fredholm* integral equations of the Second Kind.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

JACOB ABUOR OGOLA
Reg No. I56/13777/2018

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

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Dedication

I dedicate this thesis to my family and friends.

A special feeling of gratitude to my beloved Mum, Grace Adhiambo who has been the source of inspiration and strength to me, may God continue blessing you, my sisters, Caren, Gladys and brother, Clement whose words of wisdom and encouragement ring in my ears.

I also dedicate this thesis to my friends who have supported me throughout this process. I will always appreciate all you have done to me, especially Evans and Gideon for helping me understand those mathematics concepts during the group discussions.

Last but not least, I dedicate this thesis to one beloved person who has meant and continue to mean so much to me. Although he is no longer of this world, his memories continue to regulate my life, my late Dad Benjamin. Thank you so much "JAPUONY", I will never forget you.

List of Abbreviations

BC	Boundary Condition
IC	Initial Condition
IVPs	Initial Value Problems
BVPs	Boundary Value Problems
ODEs	Ordinary Differential Equations
PDE	Partial Differential Equations
FIE	Fredholm Integral Equation
VIE	Volterra Integral Equation

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Last but not least, I would like to thank my classmates and my family members for their support throughout the whole process. Thank you.

Jacob Abuor Ogola

Nairobi, 2020.

1 Introduction

Integral equations occur naturally in many fields of science and engineering. A computational approach to solve integral equations is an essential work scientific research.

Integral equations were first encountered in the theory of Fourier Integral. In 1826, another integral equation was discovered by Abel. Actual development of the theory of integral equations began with the works of the Italian Mathematician V. Volterra (1896) and Swedish Mathematician I. Fredholm (1900).

Integral equations are encountered in a variety of applications in many fields including; queuing theory, mathematical problems of radiative equilibrium, medicine, the particle transport problem of astrophysics and reactor theory, acoustics, fluid mechanics, steady state heat conduction, fructure mechanics and radiative heat transfer problems. Fredholm is one of the most important integral equations.

Most initial value problems $(IVPs)$ and boundary value problems $(BVPs)$ which are also associated with ordinary differential equations $(ODEs)$ and partial differential equations $(PDEs)$ can be evaluated more easily by the use of integral equation methods. Generally, integral equations is one of the useful tools in many branches of pure analysis famously known as the theories of functional analysis.

Integral equations can be viewed as equations which are results of transformation of points in a given vector spaces of integrable functions by the use of certain specific integral operators to points in the same space. If, in particular, one is concerned with function spaces spanned by polynomials for which the kernel of the corresponding transforming integral operator is separable being comprised of polynomial functions only,then several approximate methods of solution of integral equations can be developed.

1.1 HISTORICAL BACKGROUND OF THE INTEGRAL EQUATIONS

An integral equation is an equation in which an unknown function under one or more integral signs. There is a close connection between differential equations and integral equations and some problems may be formulated either way. Its basic form is given by

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt, \quad x \in [a,b], \quad \lambda \neq 0 \tag{1}$$

K is known as the kernel of the integral equation. Since the kernel K is integrable then it satisfies all the conditions of $Fredholm$ theorems. For $g(x) \neq 0$, we have λ which is a nonzero real or complex parameter and g given, and we seek f, this is the non homogeneous problem. For $y = 0$, equation (1) becomes an eigenvalue problem, and we seek both the eigenvalue λ and the eigen function f. The integral Equation (1) is therefore given by;

$$(I - \lambda K)f = y \tag{2}$$

with K being the integral operator on a Banach space X

Toward the end of 19^{th} century, interest on integral equations increased, majorly because of their connection with some of the differential equations **(be it Ordinary or Partial)** of mathematical physics.

In this section we present the works of $Ivar\ Fredholm$ $[1903]$, $David\ Hilbert$ $[1904]$ and $Erhard\ Schmidt$ $[1907]$ on the theory of $Fredholm$ Integral equations of the Second Kind. Although the original copies of $Fredholm's$ and $Hilbert's$ papers had been anticipated in special cases-especially by $Carl\ Neumann$ and $Henry\ Poincare'$. They were actually the first to treat the problem in full generality, that is; independent of special applications. But $Schmidt$ derives and extends the results of $Fredholm$ and $Hilbert$, which is entirely a different of view.

We have seen that a $Fredholm$ integral equation of the Second Kind has the general form as:

$$y(x) = f(x) - \lambda \int_a^b K(x,t)y(t)dt, \quad x \in [a,b] \tag{3}$$

Where $y(x)$ is the unknown function, $f(x)$ and $K(x,t)$ are known. $K(x,t)$ is called the kernel of the integral equation, and λ is a parameter.

Our authors assume that $y(x)$ and $K(x,t)$ satisfy certain regularity conditions on the interval. The authors were also quick to point out their results apply to more general regions of integration in higher dimensional spaces.

Both $Fredholm$ and $Hilbert$ start from the corresponding linear system below:

$$y = (I - \lambda K)f \quad \textbf{(4)}$$

Where K is a square matrix and y and f are vectors. But $Fredholm$, implicitly takes $\lambda = -1$ and is also concerned with how to solve (4) in such away that the process can be generalized to (3). He does not justify his generalization but simply writes down formulas and then shows that they actually work. In the process, he treats the right hand side of (3) as an operator on functions, thus ensuring his place among the founders of $Functional$ $Analysis$. The crowning glory of his ($Fredholm's$) paper is an elegant theory of what happens when (3) is 'singular' i.e when -1 is an eigenvalue of an arbitrary multiplicity of $K(x,t)$

On the other hand, $Hilbert$ takes K to be symmetric and is concerned with generalizing the finite dimensional concept of eigenvalue and eigenvector in such away that functions can be expanded in terms of $eigenfunctions$ of the kernel $K(x,t)$. (It was $Hilbert$ who introduced the terms $Eigenwert$ and $Eigenfunktion$). Unlike $Fredholm$ who first develops a complete theory for linear systems and $eigensystems$ and then by a limiting process generalizes the theory to (3). He is then forced to assume that his eigenvalues are not multiple(although he relaxes this assumption toward the end of his paper). There,s no significant use of operators.

$Schmidt$ covers the territory mapped by $Fredholm$ and $Hilbert$ (and then some), but with an important difference. Instead of starting with the finite dimensional problem, he works directly with the integral equations. In addition, he goes ahead and introduces what we would now call the Singular Value Decomposition for $unsymmetric$ kernels and proves an important approximation theorem associated with the decomposition. He achieved this by introducing a finite collection of functions $\varphi_1, \varphi_2...$ that are orthonormal in the sense that

$$\int_a^b \varphi_i(s)\varphi_j(s)ds = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad \textbf{(5)}$$

There are many analytical methods which are developed for solving Fredholm Integral Equations. Such methods as the degenerate kernel methods, **converting Fredholm integral equations to ordinary differential equations, the Adomain decomposition , altered decomposition method, series solutions**. Numerical methods for solving Fredholm integral equations can be subdivided into the following categories, **Degener-**

ate kernel approximation methods, projection methods, Nyström methods and Spectral methods.

All of these methods have iterative invariants. There are other numerical methods,but the above methods have their invariants the most popular. For more interesting history, see [15].

1.2 Problem Statement

This involves the study of the Fredholm integral equations of the Second Kind including their properties, analytical solutions and numerical solutions.

1.3 Objectives

- To study and understand the Historical Background of Fredholm integral equation of the Second Kind.
- To study and understand various analytical and numerical methods for solving Fredholm integral equation of the Second Kind.
- To study and understand the application of spectral methods in Fredholm integral equation of the Second Kind.

1.4 Literature Review

We have already defined an integral equation as an equation in which an unknown function to be determined appears in the integral sign.

Over the years, we have seen an increasing interest in integral equations mainly because of their connection with some of the differential equations of mathematical physics. This also explains why in engineering and scientific applications of a natural phenomenon we usually find ourselves in front of one equation of three, **differential equation, integral equation or integro-differential equation.** In fact, the conversation of a scientific phenomenon to integral equations is the easy way to obtain numerical solutions, enable to prove the fundamental results on the existence and uniqueness of the solution.

At a time in the early 1960's, researchers were interested majorly in one-dimensional case. It was for a kernel function K that was at least continuous: and the it was assumed that $K(x,t)$ was seventh times continuously differentiable. This was the type of equation studied by Ivar Fredholm, and in his honor such equation is called Fredholm Integral Equation of the Second Kind. Today, the work involves multi-dimensional in which the equations are completely continuous and the region of integration is commonly a surface of R^3 and the kernel function K is often singular. The Fredholm theory is still valid for such equations and the theory is critical as well for the convergence and stability analysis of associated numerical methods.

In the recent past, many methods of solving Fredholm integral equations of the second kind have been developed by researchers, such as quadrature method, collocation method and Galerkin method, expansion method, product-integration method, deferred correction method, graded mesh method and Petrov-Galerkin method. Furthermore, the iterated kernel method is a traditional method for solving the integral equation. However, it also requires a huge size of data of calculations.

Some of the recent researchers include: Iman Malmir [30] who in his research he used the numerical solution method based on Chebyshev and Legendre polynomials to solve the Fredholm integral equation of the Second Kind. Also, Also in their research Salih Y. and T. Akkaya [16] used a matrix method for approximately solving linear Fredholm integral equations of the Second Kind. The solution involves the truncated Legendre series approximation.

Their method is based on first taking the truncated Legendre series expansions of the functions in equation and then substituting their matrix forms into the equation reducing it to a matrix equation corresponding to a linear system of algebraic equations with unknown Legendre coefficients among many other researchers [13].

He used their expansions because of their convergence and recurrence properties. First, he tries to expand the unknown function in the integral equation based on the related formulas, then develop kernel of integral equation by determining a function which can be represented as the solution of linear differential equation then substitute into the integral equation to find the coefficients of the function.

In this thesis, we are going to use the Legendre Polynomials and Chebyshev Polynomials methods to approximate the solution of the equation (3). Usually, the main advantage of spectral methods lies to their rapid convergence and their relatively simpler numerical implementation than other methods especially the Legendre Series approximation method.

1.5 Outline

In this study we discuss the following points:

The first chapter is about introduction and definition of integral equations, historical background and theory of integral equations. In the second chapter we expose the mathematical preliminary of the integral equations which aims to familiarize the reader with the concept of integral equation including their applications and relations with ordinary differential equations and partial differential equations. The third chapter is devoted primarily to presenting various methods of analytical and numerical resolution of Fredholm integral equations of the Second Kind, especially to exhibit approximation methods, such as the methods of Adomain decomposition, variational iteration, projection, collocation, Galerkin, Nystrom, successive aproximation$\cdots$ and to illustrate the validation of these methods by instructive examples. The fourth chapter contains the introduction to spectral methods together with classical orthogonal polynomials and applications of some of these polynomials to approximate the solution of Fredholm integral equation of the Second Kind. These methods seek the solution as a linear combination of polynomials of degree N. Indeed, it is known that the orthogonal polynomials have interesting properties for approximating the Fredholm integral equations of the Second Kind. These orthogonal polynomials include: D'Olinde Rodrigues', Laguerre, Hermitian, Legendre, Chebyshev and Jacobi. In this work, we are only going to study into details the Chebyshev and Legendre polynomials.

(a) David Hilbert

(b) Erhard Schmidt

(c) Ivar Fredholm

2 Mathematical Preliminaries

Definition An integral equation is an equation in which the unknown function $y(x)$ to be determined appears under the integral sign. A general form of an integral equation in $y(x)$ is of the form:

$$u(x) = v(x) + \int_{\omega(x)}^{\mu(x)} K(x,s)u(s)ds \tag{6}$$

$K(x,s)$ is the kernel of the integral equation. $\omega(x)$ and $\mu(x)$ are the limits of integration. The limits of integration $\omega(x)$ and $\mu(x)$ are either constants, variables or mixed and they can also be in uni-dimension or multi-dimension.

For example; for $a \leq x \leq b$; $a \leq s \leq b$ the equations

$$y(x) = \int_a^b K(x,s)y(s)ds \tag{7a}$$

$$y(x) = u(x) + \int_a^b K(x,s)y(s)ds \tag{7b}$$

$$y(x) = \int_a^b K(x,s)[y(s)]^2 ds \tag{7c}$$

In equation (1), the unknown function $y(x)$ is easily observed to appear both inside and outside the integral sign as we stated earlier in the definition. It is important to note that the kernel $K(x,t)$ and the function $f(x)$ are always given in advance. Therefore, the main idea is to determine $y(x)$ such that it it agrees with equation (1).

Integral equations arise naturally in Physics, Chemistry and Biology and Engineering applications modeled by initial value problems for a finite interval $[a,b]$. They also arise as representation formulas for the solutions of differential equations. Therefore, a $differential$ equation can be replaced by an integral $equation$ that incorporates its boundary conditions. An $integral \quad equation$ can also be replaced by differential equations that incorporates the limits of integration. For that reason, each solution of an integral equation automatically satisfies the boundary conditions. $Integral$ equations also form one of the most important tool in many branches of pure analysis, such as functional analysis and stochastic processes.

2.1 Classificaion of Integral Equations

2.1.1 Class of Integral Equations

There are two major types of integral equations and other related types. The following is the list of the class of integral equations:

1. $Fredholm$ types
2. $Volterra$ types
3. Integro-diferential Equations
4. Singular integral Equations
5. $Volterra-Fredholm$ type
6. $Volterra-Fredholm\ Integro-differential$ Equations

2.1.2 Fredholm Integral Equations

The standard form of the Fredholm Integral Equations take the form;

$$\varphi(x)u(x) = v(x) + \lambda \int_a^b K(x,s)y(s)ds, a \leq x, s \leq b \tag{8}$$

The kernel function of the integral equation is $K(x,s)$ and the non-homogeneous function is $v(x)$ which are normally given in advance and the parameter λ. The value $\varphi(x)$ will give the following kind of Fredholm Integral equations;

1. When the value $\varphi(x) = 0$, the equation (8) becomes;

   $$0 = v(x) + \lambda \int_a^b K(x,s)y(s)ds \tag{9}$$

   and the Fredholm Integral Equation is called Fredholm Integral Equation of the First Kind.

2. When the value $\varphi(x) = 1$, the equation (8) can be written as;

   $$u(x) = v(x) + \lambda \int_a^b F(x,s)y(s)ds \tag{10}$$

and the integral equaton is then called the Fredholm Integral Equation of The Second Kind

Without loss of generality, we can always obtain equation (9) from (8) through dividing (8) by $\varphi(x)$ on condition that $\varphi(x) \neq 0$.

2.1.3 Volterra Integral Equations

Its general form is given by

$$\varphi(x)u(x) = v(x) + \lambda \int_a^x K(x,s)u(s)ds \tag{11}$$

with its interval of integration being the function of x

We realize that equation (10) is a special case of the Fredholm integral equation when the kernel $K(x,s)$ vanishes for $s > x$, x is in the range of integration $[a,b]$. As in Fredholm integral equations, Volterra integral equations fall under two kinds.

1. For $\varphi(x) = 0$, then (10) is

$$0 = v(x) + \lambda \int_a^x K(x,s)y(s)ds \tag{12}$$

   and in this case the integral equation is called the Volterra integral equation of the First Kind.

2. For $\varphi(x) = 1$, then (10) is

$$y(x) = v(x) + \lambda \int_a^x K(x,s)y(s)ds \tag{13}$$

   and the integral equation is then called the Volterra integral equation of the Second Kind.

In a bit of a summary, the Volterra intgral equation is of the First Kind if the unknown function $y(x)$ appears only under the integral sign. However, the Volterra integral equation is of the 2^{nd} type if $y(x)$ apears inside and outside the integral sign.

Linearity and Homogeinity Property of Fredholm and Volterra Integral Equations

2.1.4 Classification of Linear Integral equations

There are two main classes of integral equations.

1. Class of $Fredholm$
2. Class of $Volterra$

Fredholm Linear Integral Equations

The standard form of Fredholm Linear Integral equation is given by;

$$y(x) = v(x) + \lambda \int_a^b K(x,s)y(s)ds, a \leq x, b \leq s \quad \textbf{(14)}$$

a and b are constants limits of integration, $F(x,s)$ the kernel of the integral equation and λ is a parameter

The equation (13) is called Linear because the unknown function $y(x)$ under the integral sign occurs linearly, i.e the power of $y(x)$ is one.

2.1.5 Volterra Linear Integral equations

The standard form of a Linear Volterra Integral Equation is given by;

$$y(x) = v(x) + \lambda \int_a^s K(x,s)y(s)ds \quad \textbf{(15)}$$

where the limits of integration are a constant a and a variable x and the unknown function $y(x)$ occurs linearly under the integral sign. The kernel of the integral equation is given by $K(x,s)$.

Remark 2.1.1. ***The Structure of Fredholm and Volterra Integral Equations.*** *The unknown function $y(x)$ appears linearly under the integral sign in Linear Fredholm and Volterra Integral Equations of the First Kind. Nevertheless, the unknown function $y(x)$ appear linearly inside as well as outside the integral sign of the Second Kind of both the Fredholm and Volterra Integral Equations*

The limits of Integration. *In Fredholm Integral Equations, the integral is taken over a finite interval with fixed limits of integration. However, jn Volterra integral equations, at least one limit of integration is a variable, and is usually the upper limit.*

2.1.6 Linearity Property

As we have seen earlier, the function $y(x)$ in Linear Fredholm and Volterra Integral Equations (9) and (12) must occurs in the first powers of 1 whenever it exists.

However, non-linear ones come about when the unknown function $y(x)$ is substituted by a non-linear function $K(y(x))$ such as $y^2(x)$, $\sin(y(x))$, $e^y(x)$ and so on. The following are examples of non-linear integral equations;

$$y(x) = v(x) + \lambda \int_a^x K(x,s)y^2(s)ds \tag{16a}$$

$$y(x) = v(x) + \lambda \int_a^x K(x,s)e^y(s)ds \tag{16b}$$

$$y(x) = v(x) + \lambda \int_a^x K(x,s)\sin(y(s))ds \tag{16c}$$

2.1.7 Homogeinity Property of Integral Equations

When $v(x) = 0$ in Fredholm and Volterra Integral equations of the Second Kind given by (9) and (12), the resulting integral equation is called homogeneous integral equation, otherwise it is non-homogeneous integral equation.

2.1.8 Integro-Differential Equations

In this type of integral equations, the unknown function $v(x)$ occurs on one side as an ordinary derivative and appears on the other side under the integral sign.

Examples of Integro-differential Integral Equations

1.
$$y''(x) = 2x + \int_0^s xsy(s)ds, \quad y(0) = 0, y' = 1 \tag{17}$$

2.
$$y''(x) = \sin x + \int_0^{sx} y(s)ds, \quad y(0) = 1 \tag{18}$$

Equations (17) and (18) are Volterra-Integro differential equations while equation (**??**) is Fredholm Inegro-differential equation.

2.1.9 Infinite-Integral Equations

These types of integral equations arise when one or all the limits of integration become infinite. Also when the kernel $K(x,s)$ becomes infinite. Some of the examples are:

1.
$$y(x) = v(x) + \lambda \int_{-\infty}^{\infty} e^{|s^2 - t^2|} y(t)dt \tag{19}$$

2.
$$y(x) = \int_0^x [\frac{1}{(x-s)^{\alpha}}] y(s)ds, \quad 0 < \alpha < 1 \tag{20}$$

Where the singular behaviour in the equation 20 results from the kernel being infinite.

$$K(x,s)_{x \to s} = \infty$$

Remark 2.1.2. *The Abel′s type of integral equation is normally given by the equation* (20).

Remark 2.1.3.
$$y(x) = \frac{1}{3} - \sqrt{x} - \int_0^x \frac{1}{\sqrt{(x-s)}} y(s)ds \quad \textit{(21)}$$

Is an example of a weak − singular integral equation. Where the singular behaviour in this type arise from the kernel again becoming infinite as $x \to s$.

2.1.10 Volterra-Fredholm Integral Equations

The Volterra-Fredholm Integral Equation, which is a combination of disjoint Volterra and Fredholm Integral Equations.These equations arise from boundary value problems especially, when converting a boundary value problem to integral equation whose basic form is:

$$y(x) = f(x) + \int_0^x K_1(x,s)y(s)ds + \int_a^b K_2(x,s)y(s)ds \quad (22)$$

Where $K_1(x,t)$ and $K_2(x,s)$ are the kernels of the integral equation.

Numerical examples are given below,

$$y(x) = x^2 - \int_0^x (x+s)y(s)ds + \int_0^{\frac{\pi}{3}} x^3 y(s)ds \quad (23)$$

$$y(x) = \tan x - \cot x - \int_0^x y(s)ds + \int_0^{\frac{\pi}{4}} y(s)ds \quad (24)$$

Remark 2.1.4. *The unknown function* $y(x)$ *appears inside the Volterra and Fredholmintegrals and outside those integrals.*

2.1.11 Volterra-FredholmIntegro-Differential Equations

This is a combination of disjoint Volterra and Fredholm integrals and a Differential Operator. This type of integral equation arise from many physical and chemical applications similar to Volterra-Fredholm integral equations. The standard form is given by;

$$y^n(x) = f(x) + \int_0^x K_1(x,s)y(s)ds + \int_a^b K_2(x,s)y(s) \quad (25)$$

Where $K_1(x,s)$ and $K_2(x,s)$ are the kernels of the integral equation and n is the order of the ordinary derivative of $y(x)$. Because this kind of equation contains ordinary derivatives, we need the to prescribe the initial and boundary conditions depending on the order of the derivative involved. Other examples include:

$$y'(x) = 2x + \int_0^x (x^2 - t^2)y(s)ds + \int_0^2 x^2 s^2 y(s)ds, y(0) = 2 \tag{26}$$

and

$$y''(x) = -2x - \frac{1}{24}x^4 + \int_0^x y(s)ds + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} xsy(s)ds, \quad y(0) = 1, \quad y'(0) = 2 \tag{27}$$

2.2 Relations Between Ordinary Differential Equations and Integral Equations

In order to convert differential equations to I.E we will first have to state and prove the $Cauchy's \quad Integral \quad Formula$

Cauchy's Integral Formula

Theorem 2.2.1. *For f a continuous real valued function, the n^{th} repeated integrals of f based at a is assumed to be.*

$$\int_a^x \int_a^{x_1} ... \int_a^{x_{n-1}} f(x_n)dx_n...dx_2dx_1 = \frac{1}{(n-1)!}\int_a^x (x-t)^(n-1)f(t)dt$$

a single integral.

Proof. Applying Mathematical Induction. Since f is continuous, it follows from the Fundamental Theorem of Calculus: Let

$$I_n(x) = \int_a^x \int_a^{x_1} ... \int_a^{x_{n-1}} f(x_n)dx_n...dx_2dx_1$$

Therefore,

$$\frac{d}{dx}[I_1(x)] = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Where

$$I_1(a) = \int_a^a f(t)dt = 0$$

If the result above is true for n then we can also prove for $n+1$ by applying the induction hypothesis as well as changing the order of integration.

$$I_{n+1}(x) = \int_a^x \int_a^{x_1} ... \int_a^{x_{n-1}} f(x_{n+1}) dx_{n+1} ... dx_2 dx_1 = \frac{1}{(n-1)!} \int_a^x \int_t^{x_1} (x_1 - t)^{n-1} f(t) dx_1 dt$$

On changing the order of integration, we obtain

$$I_{n+1}(x) = \frac{1}{n!} \int_a^x (x-t)^n f(t) dt$$

□

Example 2.2.2. ***Let us have a look at the following general conversion of an IVP to integral equation***

Consider the Initial Value Problem:

$$y^n(x) + a_1(x) y^{n-1}(x) + ... + a_n(x) y(x) = F(x)$$

With the initial conditions;

$$y(x_0) = C_0, \quad y'(x_0) = C_1, \quad y^{n-1}(x_0) = C_{n-1}$$

Where the functions $a_i(x)$ $(i = 1, 2, ..., n)$ and $F(x)$ are real valued functions and continuous on the interval $a \leq x \leq b$ Let

$$y^n(x) = u(x)$$

Integrating the above equation with respect to x over x_0 to x, we obtain;

$$y^{n-1}(x) - y^{n-1}(x_0) = \int_{x_0}^x u(x) dx$$

OR

$$y^{n-1}(x) - C_{n-1} = \int_{x_0}^x u(x) dx$$

Where

$$y^{n-1}(x_0) = C_{n-1}$$

OR

$$y^{n-1}(x) = \int_{x_0}^{x} u(x)dx = C_{n-1}$$

Since t **is a dummy variable, we can instead use** x *Thus, when we integrate again, we obtain;*

$$y^{n-2}(x) - y^{n-2}(x_0) = \int_{x_0}^{x} u(x)dx^2 + C_{n-1}\int_{x_0}^{x} dx$$

OR

$$y^{n-2}(x) - C_{n-2} = \int_{x_0}^{x} u(x)dx^2 + C_{n-1}(x - x_0)$$

Where

$$C_{n-2} = y^{n-2}(x_0)$$

OR

$$y^{n-2}(x) = \int_{x_0}^{x} u(x)dx^2 + C_{n-1}(x - x_0) + C_{n-2}$$

But

$$\int_{x_0}^{x} u(x)dx^2 = \int_{x_0}^{x}\int_{x_0}^{x} u(x)dxdx$$

$$= \int_{x_0}^{x} (x - x_0)u(x)dx$$

I.e by Cauchy's Integral Formulae, we obtain the above integral
Integrating again with respect to x *over* x_0 *to*x*, we obtain;*

$$y^{n-3}(x) = \int_{x_0}^{x} u(x)dx^3 + C_{n-1}\frac{(x - x_0)^2}{2!} + C_{n-2}(x - x_0) + C_{n-3}$$

Where

$$C_{n-3} = y^{n-3}(x_0)$$

And after n *times integration, we have:*

$$y(x) = \int_{x_0}^{x} u(x)dx^n + C_{n-1}\frac{(x - x_0)^{n-1}}{(n-1)!} + C_{n-2}\frac{(x - x_0)^{n-2}}{(n-2)!} + C_{n-3}\frac{(x - x_0)^{n-3}}{(n-3)!} + \cdots + C_1(x - x_0) + C_0$$

And by Cauchy's integral formulae, we get;

$$y(x) = \int_{x_0}^{x} \frac{(x-t)^{n-1}}{(n-1)!} u(t)dt + C_{n-1}\frac{(x-x_0)^{n-1}}{(n-1)!} + C_{n-2}\frac{(x-x_0)^{n-2}}{(n-2)!} + C_{n-3}\frac{(x-x_0)^{n-3}}{(n-3)!} + \cdots + C_1(x-x_0) + C_0$$

Now, putting these values of y and its derivatives in the given initial value problem in the differential equation, then we shall have:

$u(x) =$ *an expression for n^th derivative of y with respect to x.*

$$\begin{aligned} u(x) + a_1(x)(\int_{x_0}^{x} u(t)dt + C_{n-1}) + a_2(x)(\int_{x_0}^{x}(x-t)u(t)dt + C_{n-1}(x-x_0) + C_{n-2}) \\ + a_3(x)(\int_{x_0}^{x} \frac{(x-t)^2}{2!}u(t)dt + \frac{(x-x_0)^2}{2!}C_{n-1} + (x-x_0)C_{n-2} + C_{n-3}) + \cdots \\ + a_n(x)(\int_{x_o}^{x} \frac{(x-t)^{n-1}}{(n-1)!}u(t)dt + \frac{(x-x_0)^{n-1}}{(n-1)!}C_{n-1} + \frac{(x-x_0)^{n-2}}{(n-2)!}C_{n-2} + \cdots \\ + C_1(x-x_0) + C_0) = F(x) \end{aligned}$$

By letting

$$F(x) = u(x) + \phi(x) - \int_{x_0}^{x} K(x,t)u(t)dt$$

Where

$$\phi(x) = C_{n-1}a_1(x) + [C_{n-2} + (x-x_0)C_{n-1}]a_2(x) + [C_0 + (x-x_0)C_1 + ... + C_{n-1}\frac{(x-x_0)^{n-1}}{(n-1)!}]a_n(x)$$

and

$$K(x,t) = -[a_1(x) + (x-t)a_2(x) + ... + \frac{(x-t)^{n-1}}{(n-1)!}a_n(x)]$$

Hence we have;

$$u(x) = f(x) + \int_{x_0}^{x} K(x,t)u(t)dt$$

Which is a Volterra integral equation of the Second Kind.

Usually we convert the IVPs and BVPs to equivalent Volterra and Fredholm integral equations respectively.

Example 2.2.3. *Convert the following IVP to V. I. E.*

$$u''(s) - u'(s)\cos s + y(s)\sin s = e^s \tag{28}$$

and the initial conditions are;

$$y(0) = 1, y\prime(o) = -1 \tag{29}$$

We begin by letting $u = y$ *and* $s = x$ *Let us put* $y''(x) = u(x)$
Integrating between 0 *and* x*, gives*

$$\int_0^x y''(x)dx = \int_0^x u(t)dt \tag{30}$$

t *being a dummy variable.*
Using the limits of integration and the initial conditions we have:

$$y'(x) - y(0) = \int_0^x u(t)dt \tag{31}$$

But $y'(0) = -1$ *thus, we have;*

$$y'(x) = \int_0^x u(t)dt - 1 \tag{32}$$

Integrating again between 0 *and* x*, we get*

$$y'(x)dx = \int_0^x \int_0^x u(t)dt^2 - \int_0^x dx \tag{33}$$

Again using the limits of integration and applying the initial conditions, we get

$$y(x) - y(0) = \int_0^x \int_0^x u(t)dtdt - x \tag{34}$$

By Cauchy,s Integral Formulae, we have

$$y(x) = \int_0^x (x-t)u(t)dt - x + 1 \quad y(0) = 1 \tag{35}$$

$$y(x) = \int_0^x (x-t)u(t)dt - x + 1 \tag{36}$$

Substituting the values of $y''(x)$*,* $y'(x)$ *and* $y(x)$ *in the given differential equation, we will obtain:*

$$u(x) - (\int_0^x u(t)dt - 1)\cos x + \sin x(\int_0^x (x-t)u(t)dt - x + 1) = e^x$$

Rearranging the terms;

$$u(x)-\int_0^x \cos xu(t)dt+\cos x+\int_0^x \sin x(x-t)u(t)dt-x\sin x+sinx=e^x \qquad (37)$$

Rearranging the terms again, we obtain;

$$u(x)=e^x-cosx+sinx(x-1)+\int_0^x [cosx-sinx(x-t)]u(t)dt \qquad (38)$$

Equation (36) *can also be written as:*

$$u(x)=f(x)+\int_0^x K(x,t)u(t)dt \qquad (39)$$

With

$$f(x)=e^x-cosx+sinx(x-1) \quad and \quad K(x,t)=cosx-sinx(x-t) \qquad (40)$$

Is a Volterra Integral Equation of the Second Kind given by

$$u(x)=e^x-\cos x+\sin(x-1)=\int_0^x (\cos x-\sin(x-t))u(t)dt \qquad (41)$$

2.2.1 Transformation of (BVPs) Into Integral Equations

When an ordinary differential equation is to be solved under conditions involving dependent variable or derivatives at two different values of the independent variables, then the problem under consideration is called a boundary value problem (BVP).

Example 2.2.4. *Let us consider the following boundary value problem*

$$y''(x)+\lambda y(x)=e^x, y(0)=0, y(1)=1 \qquad (42)$$

We construct a Fredholm integral equation associated with the boundary value problem above: We integrate both sides of (42) *with respect to x over* 0 *to x in the following manner*

$$\int_0^x y''(x)dx+\lambda\int_0^x y(x)dx=\int_0^x e^x dx[y'(x)]_0^x=[e^x]_0^x-\lambda\int_0^x y(x)dx$$

$$y'(x)-y'(0)=e^x-1-\lambda\int_0^x y(x)dx \qquad (43)$$

Let $y'(0)=C$, then we can rewrite equation (43) *as*

$$y'(x)=e^x-1-\lambda\int_0^x y(x)dx+C$$

Again integrating with respect to x over 0 to x and using the Cauchy's Integral Formula, we obtain:

$$\int_0^x y'(x)dx = \int_0^x e^x dx - \int_0^x dx - \lambda \int_0^x y(x)dxdx + \int_0^x C(x)dx$$

Using the formula below;

$$\int_0^x ... \int_0^x y(x)dxdx... = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} y(t)dt$$

We get

$$y(x) - y(0) = e^x - 1 - x - \lambda \int_0^x (x-t)y(t)dt + Cx \quad (44)$$

But $y(0) = 0$, then (44) becomes

$$y(x) = Cx - x + e^x - 1 - \lambda \int_0^x (x-t)y(t)dt \quad (45)$$

C is unknown We therefore, make C the subject of the formula in equation (44). Since $y(1) = 1$ then C can be expressed as

$$1 = C - 1 + e - 1 - \lambda \int_0^1 (1-t)y(t)dt$$
$$\Longrightarrow C = 3 - e + \lambda \int_0^1 (1-t)y(t)dt$$

We substitute the value of C in the equation to get

$$y(x) = x[3 - e + \lambda \int_0^1 (1-t)y(t)dt] - x + e^x - 1 - \lambda \int_0^x (x-t)y(t)dt$$

Rearranging to get

$$y(x) = 2x - 1 - xe + e^x + \lambda \int_0^1 x(1-t)y(t)dt - \lambda \int_0^x (x-t)y(t)dt$$
$$y(x) = 2x - 1 - xe + e^x - \lambda \int_0^x (x-t)y(t)dt + \lambda \int_0^1 x(1-t)y(t)dt$$

We now break the interval from 0 to x from 0 to x and the from x to 1 thus we have;

$$y(x) = 2x - 1 - xe + e^x + \lambda \int_0^x (x-t)y(t)dt + \lambda[\int_0^x x(1-t)y(t) + \int_x^1 x(1-t)y(t)dt]$$

We combine the two integrals where the limit of integration at 0 to x so we write it as;

$$y(x) = 2x - 1 - xe + e^x + \lambda[\int_0^x (x(1-t) - (x-t))y(t)dt + \int_x^1 x(1-t)y(tdt)]$$

$$y(x) = 2x - xe - 1 + e^x + \lambda[\int_0^x t(1-x)y(t)dt + \int_x^1 x(1-t)y(t)dt] \qquad (46)$$

OR

$$y(x) = f(x) + \lambda \int_0^1 K(x,t)y(t)dt \qquad (47)$$

Which is a Fredholm Integral equation of the Second Kind. Where

$$f(x) = 2x - xe - 1 + e^x$$

$$K(x,t) = \begin{cases} t(1-x), & 0 \le t \le x \\ x(1-t), & x \le t \le 1 \end{cases}$$

Example 2.2.5. ***Transform the BVP into its equivalent integral equaotion***

$$y''(x) = \sin x - xy(x)$$
$$y(0) = y(1) = 0$$

Integrating the differential equation given with respect x over the interval 0 to x as shown below

$$\int_o^x y''(x)dx = \int_0^x \sin x dx - \int_0^x xy(x)dx$$
$$[y'(x)]_0^x = [-\cos x]_0^x - \int_0^x xy(x)dx$$
$$y'(x) - y'(0) = -\cos x + 1 - \int_0^x xy(x)dx$$

Let $y'(0) = B$, then we have from above as

$$y'(x) = -\cos x + b + 1 - \int_o^x xy(x)dx \qquad (48)$$

Integrating again the equation (48), we obtain

$$\int_0^x y'(x)dx = -\int_0^x \cos x dx + \int_0^x B(x)dx + \int_o^x dx - \int_0^x \int_0^x xy(x)dxdx$$

Taking into consideration the Cauchy's Integral Formula; we then have

$$[y(x)]_0^x = -[\sin x]_0^x + [b(x)]_0^x + x - \int_0^x (x-t)ty(t)dt$$

From above, we apply the limits of integration and we obtain

$$y(x) - y(0) = -\sin x + Bx + x - \int_0^x (x-t)ty(t)dt$$

Taking $y(0) = 0$, *we get*

$$y(x) = -\sin x + Bx + 1 - \int_0^x (x-t)ty(t)dt \tag{49}$$

Since B *is unknown and given the condition that* $y(1) = 0$ *we then have from equation* (49) *as*

$$\begin{aligned} y(1) &= -\sin 1 + B + 1 - \int_0^1 (1-t)ty(t)dt \\ 0 &= -0.01745 + B + 1 - \int_0^1 (1-t)ty(t)dt \\ \implies B &= -0.98255 + \int_0^1 (1-t)ty(t)dt \end{aligned}$$

Substituting back the value of B *in equation* (49) *above, we get*

$$y(x) = -\sin x + x(-0.98255 + \int_0^1 (1-t)ty(t)dt) - \int_0^x (x-t)ty(t)dt$$

Rearranging to get

$$\begin{aligned} y(x) &= -\sin x + 0.01745x + x\int_0^1 (1-t)ty(t)dt - \int_0^x (x-t)ty(t)dt \\ y(x) &= -\sin x + 0.01745x - \int_0^x (x-t)ty(t)dt + x\int_0^1 (1-t)ty(t)dt \end{aligned}$$

We now break the interval of integration from 0 *to* 1 *from* 0 *to* x *then* x *to* 1 *as shown below*

$$y(x) = -\sin x + 0.01745x - \int_0^x (x-t)ty(t)dt + x[\int_0^x (1-t)ty(t)dt + \int_x^1 (1-t)ty(t)dt] \tag{50}$$

We combine the two integrals with the limits of integration running from 0 *to* x *which suffices to write* (50) *as*

$$\begin{aligned} y(x) &= -\sin x + 0.01745x - (\int_0^x x[(1-x)t - (x-t)t])y(t)dt + \int_x^1 x(1-t)ty(t)dt \\ y(x) &= -\sin x + 0.01745x - \int_0^x x(1-x)ty(t)dt + \int_x^1 x(1-t)ty(t)dt \end{aligned}$$

We then have

$$y(x) = -\sin x + 0.01745x - \int_0^x t^2(1-x)y(t)dt + \int_x^1 xt(1-t)y(t)dt \tag{51}$$

Which can be written in the form

$$y(x) = f(x) + \lambda \int_0^1 K(x,t)y(t)dt$$

Where

$$f(x) = 0.01745x + sinx$$

$$K(x,t) = \begin{cases} t^2(1-x), & 0 \leq t < x \\ xt(1-t), & x \leq t \leq 1 \end{cases}$$

2.2.2 Relations Between Integral Equations and Partial Differential Equations (Green's Functions)

As we have seen above, here also we consider two cases which are initial value problem and boundary value problem. We want to see how we can come up with an Integral Equation given a differential operator L

Let L be defined by:

$$Lu(x) = [A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx}C(x)]u(x), \quad a < x < b \tag{52}$$

Where $A(x)$ is continuously differentiable positive function. Let its adjoint operator M be defined by:

$$Mv(x) = \frac{d^2}{dx^2}[A(x)v(x)] - \frac{d}{dx}[B(x)v(x)] + C(x)v(x), \quad a < x < b \tag{53}$$

We integrate (53) by integration by parts to obtain:

$$\int_a^b (v(x)Lu(x) - u(x)Mv(x))dx = [A(X)(v(x)u'(x) - u(x)v'(x)) + u(x)v(x)(B(x) - A'(x))]_a^b \tag{54}$$

The above equation is known as the $Green's$ Formula for the Operator L

And we use the relation below to prove the theorem

$$A(X)y''(x) + B(x)y'(x) + C(x)y(x) = F(x) \tag{55}$$

Which proves our theorem can still be written in the form:

$$\frac{d}{dx}[p(x)\frac{dy(x)}{dx}]+q(x)y(x)=F(x) \tag{56}$$

Which is clearly self-adjoint.

The function $p(x)$ is again continuously differentiable and positive and $q(x)$ and $F(x)$ are continuous in a given interval (a,b).
The equation(54) takes the simple form:

$$\int_a^b (v(x)Lu(x)-u(x)Lv(x))dx=[p(x)v(x)u'(x)-u(x)v'(x)] \tag{57}$$

Let us consider the following homogeneous second order equation:

$$\frac{d}{dx}(p(x)\frac{dy}{dx})+q(x)y(x) \tag{58}$$

Also let $u_1(x)$ and $u_2(x)$ be two linearly independent solutions of the homogeneous equation (58) which are also twice continuously differentiable in the interval $a<x<b$.

Any solution of this equation is only a linear combination of $u_1(x)$ and $u_2(x)$. I.e;

$$y(x)=C_1u_1(x)+C_2u_2(x)$$

Where C_1 and C_2 are constants.

Case one: Initial Value Problem

Consider the IVP below:

$$\frac{d}{dx}(p(x)\frac{dy}{dx})+q(x)y(x)=F(x) \tag{59}$$

$$y(a)=0,y'(a)=0 \tag{60}$$

To form an integral equation from the problem above, we consider a function:

$$w(x) = u_1(x)\int_a^x u_2(t)F(t)dt - u_2(x)\int_a^x u_1(t)F(t)dt, \tag{61}$$

Where $u_1(x)$ and $u_2(x)$ are solutions of the homogeneous equation (58).
Differentiating the equation (61), we obtain:

$$w'(x) = u_1'(x)\int_a^x u_2(t)F(t)dt - u_2(x)\int_a^x u_1(t)F(t)dt + u_1(x)u_2(x)F(x) - u_1(x)u_2(x)F(x)$$

$$= u_1'(x)\int_a^x u_2(t)F(t)dt - u_2'(x)\int_a^x u_1(t)F(t)dt$$

Hence $w(a) = w\prime(a) = 0$ and

$$\frac{d}{dx}[p(x)\frac{dw}{dx}] = \frac{d}{dx}[p(x)\frac{du_1}{dx}]\int_a^x u_2(t)F(t)dt - \frac{d}{dx}[p(x)\frac{du_2}{dx}] \times \int_a^x u_1(t)F(t)dt + p(x)[u_1\prime(x) - u_2'(x)(x)u_1(x)]F(\tag{62}$$

And the $u_1(x)$ and $u_2(x)$ satisfy the homogeneous equation (58)

Intead of x, let's now use p, u_1 and u_2 to obtain:

$$\frac{d}{dx}[p(u_1'u_2 - u_2'u_1] = \frac{d}{dx}(pu_1')u_2 - \frac{d}{dx}(pu_2')u_1 + pu_1'u_2 - pu_2'u_1 = 0$$

And since because u_1 and u_2 satisfy (58), we have:

$$p(x)[u_1'(x)u_2' - u_2'(x)u_1'(x)] = A \tag{63}$$

Where A is a constant. The negative of the expressions in the brackets in the above relation is called the $Wronskian\ W\ W(u_1, u_2; x)$ of the solution u_1 and u_2. I.e

$$W(u_1, u_2; x) = u_1(x)u_2'(x) - u_2(x)u_1'(x) \tag{64}$$

From (62) and (64), it follows that the function (61) satisfies the system:

$$\frac{d}{dx}(p\frac{dW}{dx}) + qW = AF(x) \tag{65}$$

$$W(a) = 0, \quad W' = 0 \tag{66}$$

Dividing equation (65) by the constant A then comparing it with (56), we derive the required relation for $y(x)$ given by:

$$y(x) = \int_a^x R(x,t)F(t)dt$$

Where

$$R(x,t) = \frac{1}{A}(u_1(x)u_2(t) - u_2(x)u_1(t)) \tag{67}$$

Also $R(x,t) = -R(t,x)$

We can use the $delta\ diac$ function property to verify that, for a fixed value of t, the function $R(x,t)$ is completely the solution of the initial value problem:

$$LR = \frac{d}{dx}[p(x)\frac{dR}{dx}] + q(x)R = \delta(x-t)$$

$$R|_{x=t} = 0, \frac{dR}{dx}|_{x=t} = \frac{1}{p(t)} \tag{68}$$

The equation (68) describes the effects on the value of y at x due to a concentrated disturbance at t. Also known as the $InfluenceEquation$.

The function $G(x;t)$ given by:

$$G(x;t) = \begin{cases} 0, & \text{if} \quad x < t \\ R(x;t), & \text{if} \quad x > t \end{cases}$$

Which is called the $Causal\ Green's$ Function.

Case 2: Boundary Value Problem

Consider the BVP:

$$\frac{d}{dx}(p(x)\frac{dy}{dx})+q(x)y(x)=F(x), \quad a\leq x\leq b \tag{69}$$

and the boundary conditions:

$$y(a)=0, \quad y(b)=0 \tag{70}$$

The general solution of (69) is of the form:

$$y(x)=\int_a^x R(x;t)F(t)dt+C_!u_1(x)+C_2u_2(x) \tag{71}$$

Where $u_1(x)$ and $u_2(x)$ are two linearly independent solutions of the homogeneous equation (58). When we substitute the boundary conditions (70) in the boundary value problem (69), we obtain

$$C_1u_1(a)+C_2u_2(a)=0 \tag{72}$$

$$C_1u_1(b)+C_2u_2(b)=-\int_a^x R(b,t)F(t)dt$$

Which will determine a unique pair of constants C_1 and C_2 provided the following holds for the determinant D given by;

$$D=u_1(a)u_2(b)-u_2(a)u_1(b)\neq 0 \tag{73}$$

Assuming that (73) holds; then

$$C_1=[\frac{u_2(a)}{D}]\int_a^x R(b,t)F(t)dt+[\frac{u_2(a)}{D}]\int_a^b R(b,t)F(t)dt \tag{74}$$

And C_2 is given by

$$C_2=-[\frac{u_1(a)}{D}]\int_a^x R(b,t)F(t)dt-[\frac{u_1(a)}{D}]\int_a^b R(b,t)F(t)dt \tag{75}$$

Putting the values of C_1 and C_2 in (76), then we have the solution of $y(x)$ as:

$$y(x)=\int_a^x (R(x,t)+[\frac{u_2(a)u_1(x)-u_1(a)u_2(x)}{D}]R(b,t))F(t)dt+\int_a^b [\frac{u_2(a)u_1(x)-u_1(a)u_2(x)}{D}]R(b,t)F(t)dt \tag{76}$$

Now, using (63) and (67) and carrying out some forms of algebraic manipulations, we get:

$$R(x,t)+[\frac{u_2(a)u_1(x)-u_1(a)u_2(x)}{D}]R(b,t)=\frac{[u_1(a)u_2(t)-u_2(a)u_1(t)][u_1(x)u_2(b)-u_2(x)u_1(b)]}{AD} \quad \textbf{(77)}$$

Which we finally have as

$$G(x;t)=\begin{cases}\frac{[u_1(x)u_2(a)-u_2(x)u_1(a)][u_1(t)u_2(b)-u_2(t)u_1(b)]}{AD}, & \text{if} \quad x<t\\ \frac{[u_1(t)u_2(a)-u_2(t)u_1(a)][u_1(x)u_2(b)-u_2(x)u_1(b)]}{AD}, & \text{if} \quad x>t\end{cases}$$

Then the solution of $y(x)$ by the value of C_1 takes the form:

$$y(x)=-\int_a^x G(x;t)F(t)dt \quad \textbf{(78)}$$

Which is called the $Green's$ Function. It is clearly symmetric:

$$G(x;t)=G(t;x) \quad \textbf{(79)}$$

Which satisfies the auxiliary problem for all values of t

$$LG=\frac{d}{dx}[p(x)\frac{dG}{dx}]+q(x)G=\delta(x-t) \quad \textbf{(80)}$$

$$G|_{x=a}=G|_{x=b}=0 \quad \textbf{(81)}$$

$$G|_{x=t^+}-G|_{x=t^-}=0 \quad \textbf{(82)}$$

$$\frac{dG}{dx}|_{t^+}-\frac{dG}{dx}|_{t^-}=-\frac{1}{p(t)} \quad \textbf{(83)}$$

The condition (82) means that the $Green's$ function is continuous at $x=t$. Also, the condition (83) states that $\frac{dG}{dx}$ has a $jump discontinuity$ of magnitude $-\frac{1}{p(t)}$ at $x=t$ Generally, the condition (83) is as a result of (79) and (81), and indeed the value of the jump in $G(x;t)$ can be obtained by integrating (81) over a small interval say $(t-\epsilon,x)$. But the indefinite integral of $\delta(x-t)$ is the $Heaviside function$ $H(x-t)$, thus we have;

$$p(t)\frac{dG(x;t)}{dx}+\int_{t-\epsilon}^x q(x)G(x;t)dx=p(t-\epsilon)\frac{dG(t-\epsilon;t)}{dx}-H(x-t)$$

When x traverses the source point t, then on the R.H.S the $Heavyside$ function has a unit jump discontinuity. But since other terms are continuous functions of x, then it follows that $\frac{dG}{dx}$ has at t a jump discontinuity.

Example 2.2.6. ***Determine the*** $Green's$ ***function for the boundary value problem (E):***$x^2y"(x)+xy'(x)+(\lambda^2x^2-n^2)y(x)=f(x)$**,** $0\leq x\leq a$ **and** n **is any natural number** *The boundary conditions* $y(0)$ *is finite and* $y(a)=0$

We observe that the associated homogeneous equation is the Bessel's Equation written as

$$x^2y''(x)+xy'(x)+(\lambda^2x^2-n^2)y(x)=0$$

written in the parametric form. This means that the ordinary differential equation can be written as a Sturm – Liouville equation if we divide both sides by x i.e

$$xy''(x)+y'(x)+(\lambda^2x-\frac{n^2}{x})y(x)-\frac{f(x)}{x}$$

Or

$$\frac{d}{dx}(xy'(x))+(\lambda^2x-\frac{n^2}{x})y(x)=\frac{f(x)}{x}$$

The Green's function $G_n(x,\zeta)$ for the given problem must satisfy the equation:

$$\frac{d}{dx}(x\frac{dG_n(x,\zeta)}{dx})+(\lambda^2x-\frac{n^2}{x})G_n(x,\zeta)=\delta(x-\zeta)$$

But the Bessel's equations in parametric form has solutions $J_n(\lambda x)$ and $N_n(\lambda x)$ which are linearly independent. Note that $J_n(x)$ is the Bessel function of order n and $N_n(x)$ is the Neumann function of order n given by:

$$N_n(x)=\frac{J_n(x)cosn\pi-J_{-n}(x)}{sinn\pi}$$

A Bessel's equation whose solution is:

$$y(x)=AJ_n(\lambda x)+BN_n(\lambda x) \tag{84}$$

We now want to construct the Green's function. Since the Neumann function $N_n(\lambda x)$ is singular at the origin, so, the only solution that satisfy the boundary condition at the origin is $J_n(\lambda x)$ and so we set

$$y_1(x)=J_n(\lambda x)$$

Now, to determine the function $y_2(x)$, the solution of the Bessel equation which satisfies the boundary condition at $x=a$. We first observe that

$$y(a)=AJ_n(\lambda a)+BN_n(\lambda a)=0 \implies thatB=-\frac{J_n(\lambda a)}{N_n(\lambda a)}A$$

So, if we put $A+1$, we have a solution

$$y_2(x)=J_n(\lambda x)-\frac{J_n(\lambda a)}{N_n(\lambda a)}N_n(\lambda x)$$

which satisfies the second boundary condition $y_2(a) = 0$ Thus we write

$$y_2(x) = \frac{J_n(\lambda x)N_n(\lambda a) - J_n(\lambda a)N_n(\lambda x)}{N_n(\lambda a)}$$

We now use the method of variation of parameters to solve the Boundary Value Problem. We let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the Boundary Value Problem. Then we write

$$y(x) = C_1(x)y_1(x) + C_2(x)y_2(x)$$

$$\Longrightarrow y'(x) = C_1'(x)y_1(x) + C_2'(x)y_2(x) + C_1(x)y_1'(x) + C_2(x)y_2'(x)$$

And we now assume that

$$C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \tag{85}$$

so we remain with

$$y'(x) = C_1(x)y_1'(x) + C_2(x)y_2'(x)$$

$$y''(x) = C_1'(x)y_1'(x) + C_2'(x)y_2'(x) + C_1(x)y_1''(x) + C_2(x)y_2''(x)$$

Using the values of $y'(x)$ and $y''(x)$ in the equation

$$xy''(x) + y'(x) + (\lambda^2 x - \frac{n^2}{x})y(x) = \frac{f(x)}{x}$$

We observe that

$$x[C_1(x)y_1''(x) + C_2(x)y_2''(x) + C_1'(x)y_1'(x) + C_2'(x)y_2'(x)] + [C_1(x)y_1'(x) + C_2(x)y_2'(x)] + (\lambda^2 - \frac{n^2}{x})(C_1(\tag{86}$$

Which is equivalent to

$$= C_1(x)[xy_1''(x) + y_1'(x) + (\lambda^2)x - \frac{n^2}{x}y_1(x)] + C_2(x)[xy_2''(x) + y_2'(x) + (\lambda^2 x - \frac{n^2}{x})y_2(x)]x[C_1'(x)y_1'(x) + \tag{87}$$

Since $y_1(x)$ and $y_2(x)$ are solutions of the associated homogeneous equation it implies that

$$C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = \frac{f(x)}{x} \tag{88}$$

The equations (85) *and* (88) *permits to solve the system for* $C_1'(x)$ *and* $C_2'(x)$ *if the determinant* $W(x)$

$$W(x) = W[y_1(x), y_2(x)]$$

$$= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

If we let

$$\Delta_1 = \begin{vmatrix} 0 & y_2(x) \\ \frac{f(x)}{x^2} & y_2'(x) \end{vmatrix}$$

$$= \frac{-y_2(x)f(x)}{x^2}$$

and

$$\Delta_2 = \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & \frac{f(x)}{x^2} \end{vmatrix}$$

$$= \frac{y_1(x)f(x)}{x^2}$$

The $Cramer's$ *Formula gives*

$$C_1'(x) = \frac{\Delta_1}{\Delta} = \frac{-y_(x)f(x)}{x^2W(x)}$$

$$C_2'(x) = \frac{\Delta_2}{\Delta} = \frac{y_1(x)f(x)}{x^2W(x)}$$

Now, to determine the values of $C_1(x)$ *and* $C_2(x)$, *we use the* $Lagrange'sIdentity$ *If* u *and* v *are solutions of the self-adjoint equation*

$$L[u] = \frac{d}{dx}(p(x)\frac{du}{dx}) + q(x)u(x) = 0$$

then

$$0 = vL[u] - uL^*[v] = \frac{d}{dx}[p(x)(u'v - uv')]$$

Therefore,

$$p(x)(u'v - uv') = C$$

i.e

$$W[u,v] = \frac{C}{p(x)}$$

But

$$p(x) = x$$

and

$$W[y_1(x), y_2(x)] = \frac{C}{x}$$

Let

$$C_1'(x) = \frac{\frac{-y_2(x)f(x)}{x}}{C}$$

$$C_2'(x) = \frac{\frac{y_1(x)f(x)}{x}}{C}$$

We now determine the value of the constant C *so that* $C_1(x)$ *and* $C_2(x)$ *are completely determined. For this, we calculate the* $Wronskian$ $W[y_1(x), y_2(x)]$. *Here, we use the values of*

$$y_1(x) = J_n(\lambda x)$$

and

$$y_2(x) = \frac{J_n(\lambda x)N_n(\lambda a) - J_n(\lambda a)N_n(\lambda x)}{N_n(\lambda a)}$$

We now show that $\lim_{x\to 0} J_n(x) = \frac{x^n}{2^n n!}$

$$\lim_{x\to 0} N_n(x) = \frac{-(n-1)!}{\pi}(\frac{2}{\zeta})^n, n \neq 0$$

Here,

$$J_n(\lambda x) = \frac{1}{n!}(\frac{\lambda x}{2})^n$$

$$N_n(\lambda x) = -\frac{(n-1)!}{\pi}(\frac{2}{\lambda x})^n$$

as $x \to 0$ *Now,*

$$W[y_1, y_2] = W[J_n(\lambda x), J_n(\lambda x) - \frac{J_n(\lambda a)}{N_n(\lambda a)} N_n(\lambda x)]$$

$$-\frac{J_n(\lambda a)}{N_n(\lambda a)} W[J_n(\lambda x), N_n(\lambda x)]$$

By using the properties of the determinants hence we have

$$\lim_{x \to 0} W[J_n(x), N_n(x)] = \frac{2}{\pi x}$$

Where $J_n(x)$ *and* $N_n(x)$ *are replaced by their values as* $x \to 0$ *Similarly,*

$$\lim_{x \to 0} W[J_n(\lambda x), N_n(\lambda x)] = \frac{2}{\pi x}$$

i.e

$$W[y_1, y_2] = (\frac{J_n(\lambda a)}{N_n(\lambda a)}) \frac{2}{\pi x}$$

$$\Longrightarrow C = xW[y_1, y_2] = -\frac{2}{\pi} \frac{J_n(\lambda a)}{N_n(\lambda a)}$$

This implies that:

$$C_1(x) = \int_x^a \frac{\frac{y_2(t) f(t)}{t}}{C} dt$$

$$C_2(x) = \int_0^x \frac{\frac{y_1(t) f(t)}{t}}{C} dt$$

And

$$y(x) = C_1(x) y_1(x) + C_2(x) y_2(x)$$

We then have from above:

$$G(x, \zeta) = \begin{cases} \frac{\pi}{2}[\frac{J_n(\lambda\zeta)N_n(\lambda a) - N_n(\lambda\zeta)J_n(\lambda a)}{J_n(\lambda a)}] J_n(\lambda x) & \text{if} \quad 0 \leq x \leq \zeta \\ \frac{\pi}{2}[\frac{J_n(\lambda\zeta)N_n(\lambda a) - N_n(\lambda x)J_n(\lambda a)}{J_n(\lambda a)}] J_n(\lambda\zeta) & \text{if} \quad \zeta < x \leq a \end{cases}$$

Which is an integral equation

2.3 Application of Integral Equations

Due to the advancement of science has led to formulation of many physical problems, which are governed by integral equations. The following are some of the areas where integral equations have been applied:

2.3.1 Ocean Waves

Ocean waves are actually energy passing through the water, causing it to move in a circular manner. From this point of view, the waves interact which causes the energy transfer among different wave components. In order to determine the approximate solution of the resultant nonlinear transfer action function, we consider a set of four progressive waves traveling with different numbers and frequencies. This interaction results in energy exchange.

This can be determined by numerical integration method using a fourth order $Runge-kutta$ scheme in computing the non-linear transfer action function.

2.3.2 Seismic Response of Dams

The dynamic behavior of a dam with a full reservoir is known to be different from that of a dam with an empty reservoir.

The developed hydrodynamic pressures affect the motion of the dam, while the dam response influences in turn the dynamic response of the reservoir. This phenomenon is termed **Dynamic Dam-Reservoir Interaction (DRI)** and it could be catastrophic in cases of resonance, that is when the two domains (dam and reservoir) are vibrating in two phases.

In order to analyze the safety and stability of an earth dam during an earthquake, we need to know the response of the dam to earthquake ground motion so that the inertia forces that will be generated in the dam by the earthquake can be derived.

Once the inertia forces are known, the safety and stability of the structure can be determined.

Some approximations are made to create a mathematical model in the form of integral equations, that is;

$$y = \int_0^t \ddot{a}(\tau) g[(t-\tau), z] d\tau$$

Where z is the level, y is the displacement, $\ddot{a}$ is the base acceleration and τ is the shear stress.

2.3.3 Flow of Heat in a Metal Bar

Here, we consider a classical problem of heat flow in a metal bar to demonstrate the aplication of integral equations. The unsteady flow of heat in one-dimensional medium should be well defined. The ruling partial differential equation with its boundary are given as follows:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} - qT$$

The boundary conditions are;

$$T(0,t) = T(l,t), \quad \frac{\partial T}{\partial T}(0,t) = \frac{\partial T}{\partial c}(l,t)$$

and the initial condition is;

$$T(x,0) = g(x)$$

Where $T(x,t)$ is the temperature at any time and at any position on the bar, α is the thermal diffusivity, q is the heat loss through the bar, $x = 0$ is the left-hand side of the bar and $x = l$ is the right-hand side of the bar, and l is the length of the bar and t is the time.

2.3.4 Calculation of the Flow Characteristics around a Submerged Sphere in Regular Wave.

Here, we consider a combination of two classical problems in order to find the hydrodynamic characteristic of motion of a body in time harmonic waves.
These are radiation and diffraction problem. In **radiation problem,** the body undergoes prescribed oscillatory motions in calm fluid while **diffraction problem**, the body is held fixed in the incident wave field and determines the influence of it over the incident wave. These boundary value problems can be formulated as two different types of integral equations called **direct boundary integral formulation function** as a superposition of a single layer and double-layer potentials and **indirect boundary integral formulation** representing the unknown function with the aid of a source distribution of Green's function with fictitious singularities of adjustable strength.

2.4 Types of Kernels

The main special types of Kernels of an Integral Equation are:

1. Symmetric Kernels
2. Separable / Degenerate Kernels
3. Resolvent Kernels
4. Iterated Kernels

So, we discuss the various types of Kernels in Integral Equations since Kernels play a big role when in finding the solution of the given system of integral equations.

2.4.1 Symmetric Kernel

A Kernel $K(x,t)$ is symmetric (or complex symmetric or $Hermitian$) if:

$$K(x,t) = K^*(t,x) \tag{89}$$

Where $*$ denotes the complex conjugate. If the Kernel is real, then we have;

$$K(x,t) = K(t,x) \tag{90}$$

For example,the following are some examples of symmetric Kernels

1. $K(x,t) = \cos(x+t)$
2. $K(x,t) = \log xt$
3. $K(x,t) = x^2t^2 + xt$

But again, $K(x,t) = \sin(2x+3t)$ and $K(x,t) = xt^2 + 2$ are not symmetric. Also, $K(x,t) = i(x-t)$ is symmetric Kernel since in this case; If

$$K(x,t) = i(x-t)$$

then

$$K(t,x) = i(t-x)$$

and so,

$$\bar{K}(t,x) = -i(t-x) = i(x-t) = K(x,t)$$

But, $K(x,t) = i(x+t)$ is NOT a symmetric kernel, since in this case if $K(x,t) = i(x+t)$ then

$$\bar{K}(t,x) = i(t \bar{+} x) = -i(x+t) = -K(x,t) \neq \bar{K}(x,t)$$

Usually, Integral Equations with $Symmetric$ kernels are of frequent occurrence in the formulation of physically motivated problems

2.4.2 Separable Kernels

A kernel $K(x,t)$ is said to be separable or degenerate if if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x and t only i.e

$$K(x,t) = \sum_{i=1}^{n} a_i(x)b_i(t) \tag{91}$$

Where $a_i(x)$ and $b_i(t)$ are assumed to be linearly independent, otherwise the number of terms in relation (91) can be reduced by linear independence if $c_1a_1 + c_2a_2 + \ldots + c_na_n = 0$, where c_i are arbitrary constants i.e $c_1 = c_2 = \ldots = c_n = 0$

2.4.3 Resolvent Kernel

Let the solutions of the integral equations below be given

$$u(x) = \phi(x) + \lambda \int_a^b G(x,t)u(t)dt \tag{92}$$

and

$$v(x) = \phi(x) + \lambda \int_a^x G(x,t)v(t) \tag{93}$$

respectively by

$$u(x) = \phi(x) + \lambda \int_a^b R(x,t;\lambda)u(t)dt \tag{94}$$

and

$$v(x) = \phi(x) + \lambda \int_a^x \Gamma(x,t;\lambda)v(t)dt \tag{95}$$

Then $R(x,t;\lambda)$ and $\Gamma(x,t;\lambda)$ are called the $Resolvent$ or $Reciprocal$ kernels of the given integral equations.

2.4.4 Iterated Kernels

Here we consider the general form of $Fredholm$ integral equation of the Second Kind.

$$\phi(x) = \varphi(x) + \lambda \int_a^b G(x,t)\phi(t)dt \tag{96}$$

Then the iterated kernels $G_n(x,t)$ for $n = 1, 2, ...$ are defined as shown below:

$$G_1(x,t) = G(x,t)$$

$$G_n(x,t) = \int_a^b G(x,z)G_{n-1}(x,t)dt, n = 2, 3, ... \tag{97}$$

Where $G(x,t)$ is the $Kernel$ of the integral equation

2.4.5 Eigenvalues and Eigenfunctions

Consider the homogeneous Fredholm Integral Equation:

$$y(x) = \lambda \int_a^b G(x,t)y(t)dt \tag{98}$$

Usually, the equation (98) has two types of solutions $y(x) = 0$ and $y(x) \neq 0$ For $y(x) = 0$ is called the $trivial$ solution of the integral equation For $y(x) \neq 0$ is called the nontrivial solution of the integral equation (98) and the parameter λ is the eigenvalue of (98) while every nonzero solution of (98) is called the $eigenfunction$ corresponding to eigenvalue λ.

Some Fundamental Properties of Eigenvalues and Eigenfunctions

Theorem 2.4.1. *If a kernel is symmetric, then all its iterated kernels are also symmetric*

Proof. Let $G(x,t)$ represent a symmetric kernel, then

$$G(x,t) = \bar{G}(t,x) \tag{99}$$

By definition, the iterated kernels are defined as follows:

$$G_1(x,t) = G(x,t) \tag{100}$$

$$G_n(x,t) = \int_a^b G(x,y)G_{n-1}(y,t)dy, n = 2, 3, ..., n \tag{101}$$

Let $G_n(x,t)$ be symmetric for $n=m$, then by definition we have;

$$G_m(x,t) = \bar{G_m}(t,x) \tag{102}$$

Now we prove that G_{m+1} is also symmetric i.e

$$G_{m+1}(x,t) = \bar{G}_{m+1}(t,x)$$

using (101) We have;

$$G_{m+1}(x,t) = \int_a^b \bar{G}(x,y)G_m(y,t)dy$$

using (99)and (102)

$$= \int_a^b \bar{G}(y,x)\bar{G}_m(t,y)dy,$$

$$= \int_a^b \bar{G}_m(t,y)\bar{G}(y,x)dy = \bar{G}_{m+1}(t,x)$$

Thus by Mathematical induction, $G_n(x,t)$ is symmetric for $n = 1,2,3,...$ □

Theorem 2.4.2. *(HILBERT THEOREM) Every symmetric kernel with a non zero norm has at least one eigenvalue.*

Theorem 2.4.3. *The eigenfunctions corresponding to distinct eigenvalues are orthogonal to each other.*

Proof. Let $\varphi_1(s)$ and $\varphi_2(s)$ be the eigenfunctions to the two distinct eigenvalues λ_1 and λ_2 of the homogeneous Fredholm Integral Equation:

$$\varphi(s) = \lambda \int_a^b G(s,t)\varphi(t) \tag{103}$$

and suppose that the Kernel $G(s,t)$ is symmetric, then $\lambda = 0$ is not an eigenvalue of (103) since it will result to a trivial solution i.e $\varphi(s) = 0$. The functions φ_1 and φ_2 satisfies the equation (103) in the sense that;

$$\varphi_1(s) = \lambda_1 \int_a^b G(s,t)\varphi_1(t)dt \tag{104}$$

and

$$\varphi_2(s) = \lambda_2 \int_a^b G(s,t)f_2(t)dt \tag{105}$$

We multiply (104) by $\varphi_2(s)$ then integrate it with respect to s over (a,b) to obtain;

$$\int_a^b \varphi_1(s)\varphi_2(s)ds = \lambda_1 \int_a^b \varphi_2(s)[\int_a^b G(s,t)\varphi_1(t)dt]ds$$

Changing the order of integration we get:

$$\int_a^b \varphi_1(s)\varphi_2(s)ds = \lambda_1 \int_a^b \varphi_1)(t)[\int_a^b G(s,t)\varphi_2(s)ds]dt$$

Applying symmetry of the kernel $G(s,t)$, we obtain;

$$\int_a^b \varphi_1(s)\varphi_2(s)ds = \lambda_1 \int_a^b \varphi_1(t)[\int_a^b G(t,s)\varphi_2(s)ds]dt$$

Using (105),we obtain;

$$\int_a^b \varphi_1(s)\varphi_2(s)ds = \lambda \int_a^b \varphi_1(t)\frac{\varphi_2(t)}{\lambda_2}dt = \frac{\lambda_1}{\lambda_2}\int_a^b \varphi_1(t)\varphi_2(t)dt$$

which gives

$$(\lambda_1 - \lambda_2)\int_a^b \varphi_1(s)\varphi_2(s)ds = 0 \quad \textbf{(106)}$$

since $\lambda_1 \neq \lambda_2$ we then conclude that:

$$\int_a^b \varphi_1(s)\varphi_2(s)ds = 0$$

Hence eigenfunctions corresponding to distinct eigenvalues are orthogonal to each other. □

Theorem 2.4.4. *The eigenvalues of a Fredholm integral equation with a real symmetric kernel are real.*

Proof. let λ_1 be an imaginary eigenvalue corresponding to a complex eigenfunction $f_1(x)$. Then the complex conjugate $\bar{\lambda_1}$ will be an eigenvalue corresponding to the eigenfunction $f_1\bar{(}x)$, which is the complex conjugate of $f_1(x)$ Hence using (106),we obtain:

$$(\lambda_1 - \bar{\lambda_2})\int_a^b f_1(x)\bar{f_2}(x)dx = 0 \quad \textbf{(107)}$$

if $\lambda_1 = \alpha_1 + i\beta_1$ and $f_1(x) = a_1(x) + ib_1(x)$ then (106) gives:

$$2i\beta_1 \int_a^b (a_1^2 + b_1^2)dx = 0$$

Since $f_1(x) \neq 0$, the integral cannot be zero unless the imaginary part of λ_1 i.e β_1 must vanish. Hence we conclude that the eigenvalues of a Fredholm integral equation with a real symmetric kernel are real. □

3 Methods of Solutions of Fredholm Integral Equations of the Second Type

In this chapter, we will present some analytical and numerical methods to solve $Fredholm$ integral equation of the second kind, but we begin by stating some theorems on existence, uniqueness of solutions.

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \tag{108}$$

a and b are given constants and the kernel function is $K(x,t)$. Also, in $Fredholm$ integral equations of the Second Kind, the unknown function $y(x)$ appears both inside and outside the integral sign.
In equation (108),$K(x,t)$ and the function $f(x)$ are real valued functions and λ is a parameter. When $f(x) = 0$ then the $Fredholm$ integral equation is said to be $homogeneous$

3.0.1 Existence and Uniqueness of Solutions of Fredholm integral equations of the Second Kind

In this section, we give some conditions which ensure the existence of the unique solution of Fredholm integral equations.

Existence and Uniqueness

Theorem 3.0.1. *Let the following equation*

$$y(x) - \lambda \int_a^b K(x,t)y(t)dt = f(x) \tag{109}$$

If the kernel $K(x,t)$ *is continuous over* $[a,b] \times [a,b]$, $f \in L^2([a,b])$ *and* $|\lambda| B < 1$, *where*

$$B = \sqrt{\int_a^b \int_a^b |K(x,t)|^2 dxdt}$$

Then the equation (109) *admits a unique solution for* $y \in L^2([a,b])$

PROOF. We consider the equation

$$T(y)(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \tag{110}$$

Since

$$f \in L^2([a,b]), \quad Ty \in L^2([a,b]) \quad and \quad \int_a^b K(x,t)y(t)dt \in L^2([a,b])$$

Using the equality of $Shwartz$, so

$$|\int_a^b K(x,t)y(t)dt| \leq \int_a^b |K(x,t)y(t)|dt \leq \left(\int_a^b |K(x,t)|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b |y(t)|^2 dt\right)^{\frac{1}{2}}$$

so

$$\left|\int_a^b K(x,t)y(t)dt\right|^2 \leq \left(\int_a^b \left|K(x,t)\right|^2 dt\right) \left(\int_a^b \left|y(t)\right|^2 dt\right)$$

Also

$$\int_a^b \left|\int_a^b K(x,t)y(t)dt\right|^2 dx \leq \int_a^b \left(\int_a^b \left|K(x,t)\right|^2 dt\right) \left(\int_a^b \left|y(t)dt\right|^2\right) dx$$

Which is

$$\leq \int_a^b \int_a^b \left|K(x,t)\right|^2 dtdx \int_a^b \left|y(t)\right|^2 dt$$

Since

$$\int_a^b \int_a^b \left|K(x,t)\right|^2 dtdx < \infty$$

and then

$$\int_a^b \left|y(t)\right|^2 dt < \infty$$

Then equation (110) is satisfactory and T of $L^2([a,b])$ in itself.
We notice that the demonstration below is also that the operator defined by

$$(Ay)(x) = \int_a^b K(x,t)y(t)dt$$

is bounded and therefore by theorem above the equation

$$Ty = y$$

admits a unique solution for $|\lambda|B < 1$. □

Lemma 3.0.2. *Let U and V be two compact spaces. The set of continuous functions from U to V with the uniform norm is complete.*

Theorem 3.0.3. *Let $K:[a,b]\times[a,b]\to R$ and $f:[a,b]\to R$ be two continuous functions. So, if λ is small enough, then the equation*

$$y(x) = f(x)+\lambda\int_a^b K(x,t)y(t)dt$$

admits a unique solution that needs to be extended further to $[a,b]$

Proof. We consider the set F of continuous functions on $[a,b]\to[a,b]$, endowed with the uniform norm, so by the lemma above, it implies that F is complete. We then consider the application

$$\Phi : F \to F$$

given by

$$\Phi(y)(x) = f(x)+\lambda\int_a^b K(x,t)y(t)dt$$

We show that Φ^p is a contraction application for a certain p. $\forall x\in[a,b]$, we have;

$$\left|\left|\Phi(y)-\Phi(y')\right|\right|_{\infty} \leq \left|\left|\Phi(y)(x)-\Phi(y')(x)\right|\right|$$

$$= \left|\lambda\right|\left|\left|\int_a^b K(x,t)(y(t)-y'(t))dt\right|\right| \leq \left|\lambda\right|\left|\left|K\right|\right|_{\infty}\left|b-a\right|\left|\left|y-y'\right|\right|_{\infty}$$

If it is small enough and if p is large enough, then the application Φ^p is a contraction, so it has a unique fixed point. This point is the solution of the Fredholm integral equation of the Second Kind. □

Here, we will majorly use the method of degenerate kernels.

And such a kernel can be expressed as:

$$K(x,t) = \sum_{i=1}^{n} g_i(x)h_i(t) \tag{111}$$

some of the examples of degenerate kernels are

$$K(x,t) = x-t$$
$$K(x,t) = (x-t)^2$$

In what follows we state without proof the *Fredholm Alternative Theorem*.

Theorem 3.0.4 (*Fredholm Alternative Theorem*)**.** *If the homogeneous Fredholm Integral Equation*

$$y(x) = \lambda \int_a^b K(x,t)y(t)dt \quad (112)$$

has only the trivial or zero solution $y(x) = 0$*, then the corresponding non-homogeneous Fredholm integral equation:*

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \quad (113)$$

has always a unique solution.
We consider this theorem for a Kernel which is separable.

Theorem 3.0.5 (Uniqueness Solution)**.** *If the Kernel* $K(x,t)$ *in the Fredholm integral equation* (108) *is continuous, real valued function, bounded in the square* $a \leq x \leq b$ *and* $a \leq t \leq b$ *and if* $f(x)$ *is a continuous real valued function, then a necessary condition for existence of a unique solution for Fredholm integral equation* (108) *is given by*

$$|\lambda| M(b-a) < 1 \quad (114)$$

where;

$$|K(x,t)| \leq M \in R \quad (115)$$

However, if the necessary condition does not hold, then a continuous solution may exist for Fredholm integral equation.

A number of analytical and numerical methods have been developed to solve these equations. The analytical methods are also known as *traditional* methods.

3.1 Analytical Methods for Solving Fredholm Integral Equations of the Second Kind

Some integral equations have solutions and others have no solutions or have innumerable solutions.

Remark 3.1.1. *It is important to say that we will discuss analytical methods in space* $\left(C([a,b]),||.||_\infty\right)$

Some of the analytical methods include:

1. The degenerate kernel method
2. $The\ Adomain$ Decomposition Method
3. The Modified Decomposition Method
4. The Noise Terms Phenomenon
5. The Variational Iteration method
6. The Direct Computation Method
7. Successive Approximations ($Neumann$ Series and $Picard's$ method)
8. Convertion of Integral equations to BVPs

3.1.1 The Degenerate or Separable kernel Method

We defined a degenerate or separable kernel as:

$$K(x,t) = \sum_{i=1}^{n} g_i(x)h_i(t) \tag{116}$$

The mappings $g_1(x)$,$g_2(x)$,...,$g_n(x)$ and $h_1(t)$,$h_2(t)$,...$h_n(t)$ are linearly independent. Given the integral equation below

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \tag{117}$$

Since

$$K(x,t) = \sum_{i=1}^{n} g_i(x)h_i(t)$$

Then (117) becomes:

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} g_i(x) \int_a^b h_i(t)y(t)dt \tag{118}$$

The technique of solving the equation (118) is essentially dependent on the choice of the complex parameter λ and therefore we define:

$$\beta_i = \int_a^b h_i(t)y(t)dt \tag{119}$$

Where the quantities β_i are constants. Using (119) in (118) we obtain:

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} \beta_i g_i(x) \tag{120}$$

And we now can find the values of β_i Using (120)in (118) we have;

$$\sum_{i=1}^{n} g_i(x)[\beta_i - \int_a^b h_i(t)[f(t) + \lambda \sum_{k=1}^{n} \beta_k g_k(t)]dt] = 0 \tag{121}$$

From linear independence of $g_i(x)$, it then suffices to have

$$\beta_i - \int_a^b h_i(t)[f(t) + \lambda \sum_{k=1}^{n} \beta_k g_k(t)]dt = 0 \tag{122}$$

Now, using the simplified notation;

$$\int_a^b h_i(t)f(t) = f_i \tag{123}$$

And

$$\int_a^b h_i(t)g_k(t)dt = G_{ik} \tag{124}$$

Then (122) becomes;

$$\beta_i - \lambda \sum_{k=1}^{n} \beta_k G_{ik} = f_i, i = 1, 2, ..., n \tag{125}$$

Which is a system of $Algebraic$ equations for the unknown β_i The system (125) has the determinant:

$$D(\lambda) = \begin{vmatrix} 1-\lambda G_{11} & -\lambda G_{12} & \cdots & -\lambda G_{1n} \\ -\lambda G_{21} & 1-\lambda G_{22} & \cdots & -\lambda G_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ -\lambda G_{n1} & -\lambda G_{n2} & \cdots & 1-\lambda G_{nn} \end{vmatrix} \tag{126}$$

Which is a polynomial in λ of degree n.

1. $\forall\lambda$ for which $D(\lambda)\neq 0$, the *Algebraic* system (126) and thereby the integral equation (117) has a unique solution.

2. On the other hand, $\forall\lambda$ for which $D(\lambda)=0$, the *Algebraic* system (125), and with it the integral equation (117) , either is insoluble or has an infinite number of solutions.

Example 3.1.2. *We want to show that the equation*

$$u(x)=v(x)+\frac{1}{\pi}\int_0^{2\pi}\sin(x+t)u(t)dt \tag{127}$$

have no solution if the non-homogeneous part $v(x)=x$ *and has a solution for* $v(x)=1$. *We rewrite* $G(x,t)=\sin(x+t)$ *as*

$$K(x,t)=\sin x\cos t+\cos x\sin t \tag{128}$$

$=$

$$h_1(x)g_1(t)+h_2(x)g_2(t)$$

and

$$G_{ik}=\int_0^{2\pi}h_i(t)g_k(t)dt,\quad i,k=1,2 \tag{129}$$

Here

$$G_{11}=\int_0^{2\pi}h_1(t)g_1(t)dt=\int_0^{2\pi}\cos t\sin tdt=0$$

$$G_{12}=\int_0^{2\pi}h_1(t)g_2(t)dt=\int_0^{2\pi}\cos t\cos tdt=\pi$$

$$G_{21}=\int_0^{2\pi}h_2(t)g_1(t)dt=\int_0^{2\pi}\sin t\sin tdt=\pi$$

$$G_{22}=\int_0^{2\pi}h_2(t)g_2(t)dt=\int_0^{2\pi}\sin t\cos tdt=0$$

Therefore,

$$D(\lambda)=\begin{vmatrix}1-\lambda G_{11} & -\lambda G_{12}\\ -\lambda G_{21} & 1-\lambda G_{22}\end{vmatrix} \tag{130}$$

$$D(\lambda)=\begin{vmatrix}1 & -\lambda\pi\\ -\lambda\pi & 1\end{vmatrix} \tag{131}$$

$$= 1 - \lambda^2\pi^2 \tag{132}$$

If $D(\lambda) \neq 0$ then we have a unique solution.
Here the eigenvalues are given by equating (132) to 0:

$$1 - \lambda^2\pi^2 = 0$$

where we obtain:

$$\lambda_1 = \frac{1}{\pi} \quad and \quad \lambda_2 = -\frac{1}{\pi} \tag{133}$$

The algebraic system corresponding to (127) therefore given by:

$$\begin{pmatrix} 1-\lambda G_{11} & -\lambda G_{12} \\ -\lambda G_{21} & 1-\lambda G_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{134}$$

Now, using (131) we obtain:

$$A - \lambda\pi B = 0$$
$$-\lambda\pi A + B = 0$$

$K(x,t) = \sin(x+t)$ *being symmetric: When $\lambda = \frac{1}{\pi}$, yields $A = B$. Then the corresponding eigenfunction $u_1(x)$ is therefore given by:*

$$u_1(x) = \lambda \sum_{i=1}^{n} ABg_i(x) = \frac{1}{\pi}(Ag_1(x) + Bg_2(x))$$
$$u_1(x) = \frac{1}{\pi}(A\sin x + A\cos x) = \frac{A}{\pi}(\sin x + \cos x)$$

When $\lambda = -\frac{1}{\pi}$, we obtain:$A = -B$, then the corresponding eigenfunction is given by:

$$u_2(x) = \frac{1}{\pi}(A\mu - A\omega) = \frac{A}{\pi}(\mu - \omega), \quad \mu = \sin x, \quad \omega = \cos x$$

When $v(x) = x$, we have:

$$\int_0^{2\pi} v(x)u_1(x)dx = \frac{A}{\pi}\int_0^{2\pi} x(\mu+\omega)dx \neq 0$$
$$\int_0^{2\pi} v(x)u_2(x)dx = \frac{A}{\pi}\int_0^{2\pi} x(\mu-\omega)dx \neq 0$$

which gives that $v(x)$ is not orthogonal to $u_1(x)$ and $u_2(x)$ and so (127) will have no solution.
When $v(x) = 1$, we obtain the following:

$$\int_0^{2\pi} v(x)u_1(x)dx = \frac{A}{\pi}(mu + \omega)dx = 0$$
$$\int_0^{2\pi} v(x)u_2(x)dx = \frac{A}{\pi}\int_0^{2\pi} (\mu - \omega)dx = 0$$

Then (127) *will have infinitely many solutions of the type:*

$$u(x) = v(x) + \frac{A}{\pi}u_1(x) + \frac{A}{\pi}u_2(x)$$

$$u(x) = 1 + C\frac{A}{\pi}(\mu + \omega) + D\frac{A}{\pi}(\mu - \omega)$$

$$u(x) = 1 + Ecosx + Fsinx$$

Where E *and* F *are arbitrary constants.*

3.1.2 The Adomain Decomposition Method

It was introduced by $George\ Adomain$, and it involves of decomposition of of the unknown function $y(x)$.

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \tag{135}$$

into a sum of infinite number of components defined by the decomposition series:

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{136}$$

or equivalently

$$y(x) = y_0(x) + y_1(x) + y_2(x) + ... + y_n(x) \tag{137}$$

Where the components $y_n(x)$, $n \geq 0$ will be determined recurrently. This is achieved by substituting (136) into (135) to obtain.

$$\sum_{n=0}^{\infty} y_n(x) = f(x) + \lambda \int_a^b K(x,t)(\sum_{n=0}^{\infty} y_n(t))dt \tag{138}$$

or equivalently,

$$y_0(x) + y_1(x) + y_2(x) + ... = f(x) + \lambda \int_a^b K(x,t)[y_0(t) + y_1(t) + y_2(t) + ...]dt \tag{139}$$

The component $y_0(x)$ is known as the $zeroth$ which is identified by all the terms outside the integral sign.

$$y_0(x) = f(x) \tag{140}$$

and

$$y_{n+1}(x) = \lambda \int_a^b K(x,t)y_n(t)dt, \quad n \geq 0 \tag{141}$$

Therefore $y(x)$ can be easily obtained in the form of a series solution with the help of the assumption we made in (136) and (137)

Remark 3.1.3. *The components of $y_0(x), y_1(x) \cdots$ are completely determined. As a result, the solution $y(x)$ is easily obtained as a series using the series assumption in (136). The decomposition method converts the integral equation into an elegant determination of the computable components, if an exact solution exists for the problem, then the series obtained converges very quickly to this exact solution. However, for concrete problems, when a closed form solution is not obtained, a truncated number of terms is usually used at numeric ends. The more components we use the more precision we get.*

Example 3.1.4. *Let us solve the integral of the second type below*

$$y(x) = 2x + \frac{x}{2}\int_{-1}^{1} ty(t)dt \tag{142}$$

We know that from (142) *we have*

$$\sum_{n=0}^{\infty} y_n(x) = 2x + \frac{x}{2}\int_{-1}^{1} t\sum_{n=0}^{\infty} y_n(t)dt \tag{143}$$

Or equivalently as

$$y_0(x)+y_1(x)+y_2(x)+\cdots = 2x + \frac{x}{2}\int_{-1}^{1} t(y_0(t)+y_1(t)+y_2(t)+\cdots)dt \tag{144}$$

We therefore identify the zeroth component by all terms that are not identified under the integral sign. Therefore, we obtain the following recurrence relation

$$y_0(x) = 2x \tag{145}$$

$$y_{n+1}(x) = \frac{x}{2}\int_{-1}^{1} ty_n(t)dt \tag{146}$$

It follows from (145) *that*

$$\begin{aligned}
y_1(x) &= \frac{x}{2}\int_{-1}^{1} ty_0(t)dt = \frac{x}{2}\int_{-1}^{1} 2t^2 dt = \frac{2x}{3} \\
y_2(x) &= \frac{x}{2}\int_{-1}^{1} ty_1(t)dt = \frac{x}{2}\int_{-1}^{1} \frac{2t^2}{3} dt = \frac{2x}{9} \\
y_3(x) &= \frac{x}{2}\int_{-1}^{1} ty_2(t)dt = \frac{x}{2}\int_{-1}^{1} \frac{2t^2}{9} dt = \frac{2x}{27}
\end{aligned}$$

Hence

$$y(x) = 2x + x\left(\tfrac{2}{3}+\tfrac{2}{9}+\tfrac{2}{27}+\tfrac{2}{81}+\cdots\right) \tag{147}$$

We notice that the equation (147) *represents an infinite geometric series at the right hand side and has*

$$a = \frac{2}{3} \quad r = \frac{1}{3}$$

but we know that

$$S = \frac{a}{1-r} = \frac{\frac{2}{3}}{1-\frac{1}{3}} = \frac{2}{3}\times\frac{3}{2} = 1$$

Therefore, the exact solution is given by

$$y(x) = 2x + x \tag{148}$$

$$y(x) = 3x \tag{149}$$

3.1.3 The Modified Decomposition Method

The method introduces a slight variation to the recurrence relation

$$u_0(s) = v(s) \quad (150)$$

$$u_{k+1}(s) = \lambda \int_a^b K(s,t)u(t)dt \quad k \geq 0 \quad (151)$$

whose infinite sum is given by

$$u(s) = \sum_{n=0}^{\infty} u_n(x) \quad (152)$$

that lead to finding the components of $u(s)$ in the manner that is easier and faster. In many events the function $v(s)$ can be picked up as the sum of two partial functions that is to say $v_1(s)$ and $v_2(s)$ or as

$$v(s) = v_1(s) + v_2(s) \quad (153)$$

In view of (153), we introduce a change in the structure of the occurrence relation (150). In order to minimize the magnitude of calculation we identify the zeroth component $u_0(s)$ by one part of $v(s)$ that is $v_2(s)$ and $v_2(s)$. While the other part can be added to the component $u_1(s)$ as well as other terms.
To put it in another way, the method introduces modified occurrence relation

$$u_0(s) = v_1(s) \quad (154)$$

$$u_1(s) = v_2(s) + \lambda \int_a^b K(s,t)u_0(t)dt \quad (155)$$

$$u_{k+1}(s) = \lambda \int_a^b K(s,t)u_k(t)dt \quad k \geq 1 \quad (156)$$

Which displays the disparity between the standard occurrence relation (150) and the altered relation (154) rests only on the categorization of the first two components $u_0(s)$ and $u_1(s)$ while the rest of the components $u_i(s) \quad i \geq 2$ remainunchanged in the two occurrence relations. Even though this variation in the formation of $u_0(s)$ and $u_1(s)$ is small, nonetheless it takes part in the accelerating the convergence of the solution and in minimizing the magnitude of computational work.Furthermore, decreasing the number of terms in $v_1(s)$ influences the components of $u_1(x)$ and other components as well.

Remark 3.1.5. *Proper selection of $v_1(s)$ and $v_2(s)$, the exact solution $u(s)$ can be acquired by using very few iterations and sometimes by computing only two components. The outcome of this alteration depends only on the correct choice of $v_1(s)$ and $v_2(s)$, and this can be achieved through tests only. Upto to now there is no rule guiding on* *the correct selection of* $v_1(s)$ *and* $v_2(s)$.

If $v(s)$ *contains one term only, the standard decomposition can be used in this case.*

Example 3.1.6. *Consider;*

$$y(x) = sinx - x + x\int_0^{\frac{\pi}{2}} ty(t)dt \tag{157}$$

First, we separate $f(x) = sinx + x$ *into two parts namely,*

$$g(x) = sinx$$

$$h(x) = -x$$

Now, using the Modified Recurrence formula (154) *we obtain;*

$$y_0(x) = g(x) = sinx \tag{158}$$

$$y_1(x) = h(x) + x\int_0^{\frac{\pi}{2}} ty_0(t)dt = -x + x\int_0^{\frac{\pi}{2}} tsintdt = 0 \tag{159}$$

Consequently,

$$y_{i+1}(x) = \int_0^{\frac{\pi}{2}} K(x,t)y_i(t)dt = 0 \qquad i \geq 1 \tag{160}$$

Also, using (160), *the the structure of* $y_i, \quad i \geq 1$ *is equivalent to zero. Hence*

$$y(x) = sinx$$

3.1.4 The Noise Term Phenomenon

The noise terms refers to the undistinguishable termswith mutually exclusive sign that spring up in the structure of $y_0(s)$ and $y_1(s)$ as well as other terms in the answer to the problem of $Fredholm \quad integral \quad equation$ of the second type. These terms came into sight only for unique type of non-homogeneous integral equation.

Remark 3.1.7. *If the noise terms exist in the $y_0(s)$ and $y_1(s)$ components then without loss of generality a closed structure of slution can be obtained after two consecutive iterations.*
If we cancel the noise terms between $y_0(s)$ and $y_1(s)$ while $y_1(s)$ having more terms, the terms which have not been canceled in $y_0(s)$ sometimes yield the correct answer to the problem. So, it is inevitable to verify the non-canceled terms of $y_0(s)$ fulfill the integral equation. However, we will be required to carry out many iterations of $y(s)$ in order to obtain the answer in a series form.
In many scenarios, not all nonhomogeneous equations have noise terms phenomenon.

Example 3.1.8. *Find the exact answer to the problem below using noise terms phenomenon method:*

$$y(s) = s(\pi - 2 + \sin s + \cos s) - \int_0^{\pi} sy(t)dt \tag{161}$$

Following the Standard Adomain Decomposition Method, we use the recurrence formula:

$$y_0(s) = s(\pi - 2 + \sin s + \cos s)$$

$$y_{i+1}(s) = \int_0^{\pi} sy_0(t)dt$$

The above gives:

$$y_0(s) = s(\pi - 2 + \sin s + \cos s) \tag{162}$$

$$y_1(s) = \int_0^{\pi} sy_0(t)dt \tag{163}$$

$$y_1(s) = s\int_0^{\pi} t(\pi - 2 + \sin s + \cos s)dt \tag{164}$$

On using Integration by Parts, we obtain:

$$y_1(s) = s\int_0^{\pi} (\pi t - 2t + t\sin t + t\cos t)dt = s(\frac{\pi^3}{2} - \pi^2 + \pi - 2) \tag{165}$$

It follows from (162) *and* (165) *that the noise terms $\pm 2s$ and $\pm\pi$ appears in $y_0(s)$ and $y_1(s)$ so, canceling these terms from the zeroth component $y_0(s)$ yields the exact solution;*

$$y(s) = x(\sin s + \cos s)$$

3.1.5 The Variational Iterative Method

In this section, we will apply the variational iterative method to handle $Fredholm$ integral equations. This technique only works best for a degenerate kernel such that

$$G(x,t) = u(x)v(t), \quad u = g \quad and \quad v = h. \tag{166}$$

From (166) it implies that we differentiate both sides of the Fredholm integral equation to convert to its identical $fredholm - Integro\ differential$ equation which needs a defined initial condition. For this reason, we confine ourselves with the study of $u(x) = x^n, \quad n \geq 1.$

The standard Fredholm integral equation is of the form:

$$y(x) = f(x) + \int_a^b G(x,t)y(t)dt \tag{167}$$

Or equivalently from (166)

$$y(x) = f(x) + u(x)\int_a^b v(t)y(t)dt \tag{168}$$

Since the value under the integral sign i.e the integrand is constant. So, differentiating the equation (168) with respect to x yields;

$$y'(x) = f'(x) + g'(x)\int_a^b h(t)y(t)dt \tag{169}$$

Which is called the Integro-differential equation and its correction functional is given by/

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\zeta)(y_n'(x) - f'(\zeta) - u'(\zeta)\int_a^b v(s)\bar{y_n}(s)ds)d\zeta \tag{170}$$

The variational iteration method entails two essential steps:

1. We use integration by parts to optimally obtain the $Lafrange's$ multipliers.
2. We substitute the value λ obtained into (170) where the restriction was excluded which yields:

$$y_{n+1}(x) = y_n - \int_0^x [y_n'(\zeta) - f'(\zeta) - g'(\zeta)\int_a^b h(r)y_n(s)ds]d\zeta \tag{171}$$

and it is used for the computation of consecutive (successive) approximations $y_{n+1}(x) \quad n \geq 0$ of the solution $y(x)$ From (171), the zeroth approximate $y_0(x)$ can be any selective function, and making use of the first value y_0 is used for the selective zeroth term approximation and eventually we obtain;

$$y(x) = \lim_{n\to\infty} y_n(x) \tag{172}$$

Example 3.1.9. *Using the method of* ***Variational Iteration Method*** *solve the following Fredholm integral equation:*

$$y(x) = \sin x - x + x\int_0^{\pi} y(t)dt \tag{173}$$

The solution of the above Fredholm integral equation by Variational Iteration Method requires us to first differentiate the given integral equation (173) *with respect to* x *keeping t fixed and we obtain:*

$$y'(x) = \cos x - 1 + \int_0^{\pi} y(t)dt \tag{174}$$

From (174) *the corresponding correction functional is therefore given by: That is after using the formula for Integro-differential equation* (171)

$$y_{n+1}(x) = y_n(x) - \int_0^{x} [y_n'(\zeta) - \cos\zeta + 1 - \int_0^{\pi} y_n(s)ds]d\zeta \tag{175}$$

We realize that λ *is taken to be* -1 *for the first order integro-differential equation and the initial condition* $y(0) = 0$ *is reached after substituting* $x = 0$ *in equation* (173). *Subsequently, we will be using this initial condition to select* $y_0(x) = y(0) = 0$ *and finally applying the same selection into the correction functional gives the following successive approximations.*

$$y_0(x) = 0 \tag{176}$$

$$y_1(x) = y_0(x) - \int_0^{x} [y_0'(\zeta) - \cos\zeta + 1 - \int_0^{\pi} y_0(s)ds]d\zeta \tag{177}$$

but $y_0'(\zeta)$ *is given by;*

$$y_0'(\zeta) = \cos\zeta - 1 + \int_0^{\pi} y_0(s)ds = 0$$

applying the above in the equation (177) *we have*

$$y_1(x) = -\int_0^{x} [cos\zeta + 1 -]d\zeta = sinx - x$$

thus we obtain

$$y_1(x) = \sin x - x \tag{178}$$

$$y_x(x) - y_1(x) - \int_0^{x} [y_1'(\zeta) - cos\zeta + 1 - \int_0^{\pi} y_1(s)ds]d\zeta \tag{179}$$

Here

$$y_1'(\zeta) = \cos\zeta - 1 = 0 \quad at\zeta = 0$$

And using the value of $y_1'(\zeta) = 0$ *in* (179) *we have;*

$$y_2(x) = \sin x - x - \int_0^x [-\cos\zeta + 1 - \int_0^\pi (\sin s - s)ds]d\zeta$$

$$= \sin x - x - \int_0^x [-\cos\zeta - 1 + \frac{\pi^2}{2}]d\zeta$$

$$= \sin x - x - [\sin x - x + \frac{\pi^2}{2}x]$$

$\pm\sin x$ *is one of the noise terms and we therefore cancel it to obtain:*

$$y_2(x) = (\sin x - x) + (x - \frac{\pi^2}{2}) \qquad \textbf{(180)}$$

Using (180), *we have*

$$y_3(x) - y_2(x) - \int_0^x [y_2'(\zeta) - \cos\zeta + 1 - \int_0^\pi y_2(s)ds]d\zeta \qquad \textbf{(181)}$$

Using the same method as in the case of $y_0(x)$, $y_1(x)$ *and* $y_2(x)$ *in* $y_3(x)$ *we obtain,*

$$y_3(x) = (\sin x - x) + (x - \frac{\pi^2}{2}x) + (x - \frac{\pi^4}{4}x) \qquad \textbf{(182)}$$

$$\Longrightarrow y_4(x) = (\sin x - x) + (x - \frac{\pi^2}{2}x) + (x - \frac{\pi^4}{4}x) + (x - \frac{\pi^{16}}{16}x)$$

and the general solution therefore becomes:

$$y_n(x) = (\sin x - x) + (x - \frac{\pi^2}{2}x) + (x - \frac{\pi^4}{4}x) + (x - \frac{\pi^{16}}{16}x) + (x - \frac{\pi^{256}}{256}x) + ... + (x - \frac{\pi^n}{n}x) \qquad \textbf{(183)}$$

Hence, from (183) *the noise terms are* $\pm x$ *and* $\pm\frac{\pi^n}{n}$ *which we now cancel out to obtain the answer to the problem as*

$$y(x) = \sin x \qquad \textbf{(184)}$$

3.1.6 The Direct Computation Method

The Direct Computation Method normally approaches the Fredholm integral equations of the Second Kind in a more direct manner and the resultant solution is usually given in an exact manner but not in series form as we have seen in other methods of solution.

Also, a point to note is that this method is only applicable to degenerate or separable kernels of the form:

$$K(x,t) = \sum_{i=1}^{n} u_i(x)v_i(t) \tag{185}$$

which is a kernel of the Fredholm integral equation of the Second Kind;

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \tag{186}$$

Procedures to be followed when dealing with the Direct Method

1. Substitute (185) into (186)

2. On substitution, we obtain:

$$y(x) = f(x) + u_1(x)\int_a^b v_1(t)y(t)dt + u_2(x)\int_a^b v_2(t)y(t)dt + ... + u_n(x)\int_a^b v_n(t)y(t)dt \tag{187}$$

   Here we assumed $\lambda = 1$

3. Each and every integral in the right hand side depends only on a variable t and the limits of integration for t being constant which also implies that each integral is equal to a constant. Based on the explanations above, the equation (187) becomes:

$$y(x) = f(x) + \lambda\beta_1 u_1(x) + \lambda\beta_2 u_2(x) + \lambda\beta_3 u_3(x) + ... + \lambda\beta_n u_n(x) \tag{188}$$

   Where

$$\beta_i = \int_a^b v_i(t)y(t)dt, \qquad 1 \leq i \leq n \tag{189}$$

4. If we substitute (189) into (188), we obtain a system of n algebraic equations that can be solved to find the value of the constants $\beta_i \quad 1 \leq i \leq n$. Thus using the obtained numerical values of β_i into (188), the solution of the Fredholm integral equation (186) is readily obtained.
   The direct method will be best demonstrated by the following example.

Example 3.1.10. *Solve the homogeneous Fredholm integral equation using the direct computation method*

$$y(x) = \lambda \int_0^{\pi} \sin(x+t) y(t) dt \tag{190}$$

We notice that the Kernel $K(x,t) = \sin(x+t) = \sin x \cos t + \cos x \sin t$ *is separable. Then equation* (190) *can be written as*

$$y(x) = \omega \lambda \sin x + \gamma \lambda \cos x \tag{191}$$

Where

$$\omega = \int_0^{\pi} \cos t y(t) dt, \quad \gamma = \int_0^{\pi} \sin t y(t) dt \tag{192}$$

Substituting (191) *into* (192) *yields*

$$\omega = \int_0^{\pi} \cos t(\omega \lambda \sin t + \gamma \lambda \cos t) dt \tag{193}$$

$$\gamma = \int_0^{\pi} \sin t(\omega \lambda \sin t + \gamma \lambda \cos t) dt \tag{194}$$

Solve the algebraic system above gives

$$\omega = \frac{1}{2} \gamma \lambda \pi \quad \gamma = \frac{1}{2} \omega \lambda \pi \tag{195}$$

For $\omega \neq 0, \quad \gamma \neq 0$, *we find that the eigenvalue* λ *is given by*

$$\lambda = \pm \frac{2}{\pi}, \quad \omega = \gamma \tag{196}$$

Which inturn gives an eigenfunction $y(x)$ *by*

$$y(x) = \pm \frac{C}{\pi} (\sin x + \cos x), \quad C = 2\omega \tag{197}$$

3.1.7 Successive Approximations Method

This technique gives a systematic plan that can be used to find a solution to IVPs an integral equations. Normally, we begin by first assuming $f_0(x)$ which will be used to find other approximations. $f_0(x)$ is also known as the $zeroth$ recurrence function approximations. Taking into account the $Fredholm$ of the 2^{nd} type

$$f(x) = g(x) + \lambda \int_{-1}^{1} K(x,t) f(t) dt \tag{198}$$

Where $y(x)$ is the unknown function to be determined, $K(x,t)$ the kernel and λ a parameter. This method takes the form:

$$f_0(x) = zeroth \quad term \tag{199}$$

$$f_{n+1}(x) = g(x) + \lambda \int_{-1}^{1} K(x,t) f_n(t) dt \tag{200}$$

Just like the Adomain Decomposition method,in Successive Approximations method, we also have that;

$$f_0(x) = any \quad selective \quad real-valued \quad function \quad not \quad inside \quad the \quad integral \quad sign \tag{201}$$

$$f_1(x) = \lambda \int_{-1}^{1} K(x,t) f_0(t) dt \tag{202}$$

$$f_2(x) = \lambda \int_{-1}^{1} K(x,t) f_1(t) dt \tag{203}$$

$$f_{n+1}(x) = f_n(x) + \lambda \int_{-1}^{1} K(x,t) f_n(t) dt \tag{204}$$

One of the methods associated with Successive Approximations Method depending on the choice of $f_0(x)$ is the $Neumann \quad Series$ approximation method.

Neumann Series Method

The Neumann series is obtained when $f_0(x) = g(x)$ in other terms which are not inside the integral sign of (198). Given:

$$f_0(x) = g(x) \tag{205}$$

$$f_1(x) = g(x) + \lambda \int_{-1}^{1} K(x,t) f_0(t) dt = g(x) + \lambda \int_{-1}^{1} K(x,t) g(t) dt \tag{206}$$

$$= f_1(x) = g(x) + \lambda \Phi_1(x) \tag{207}$$

Where

$$\Phi_1(x) = \int_{-1}^{1} K(x,t)g(t)dt \quad \textbf{(208)}$$

The second approximation is therefore given by:

$$f_2(x) = g(x) + \lambda \int_{-1}^{b} K(x,t)f_1(t)dt \quad \textbf{(209)}$$

$$= g(x) + \lambda \int_{-1}^{1} K(x,t)[g(t) + \lambda\Phi_1(t)]dt \quad \textbf{(210)}$$

$$= g(x) + \lambda\Phi_1(x) + \lambda^2\Phi_2(x) \quad \textbf{(211)}$$

Where

$$\Phi_2(x) = \int_{-1}^{b} K(x,t)\Phi_1(t)dt \quad \textbf{(212)}$$

From (212), the third approximation can be obtained as:

$$f_3(x) = g(x) + \lambda\Phi_1(x) + \lambda^2\Phi_2(x) + \lambda^3\Phi_3(x) \quad \textbf{(213)}$$

And the n^{th} approximation as:

$$f_{n+1}(x) = g(x) + \lambda\Phi_1(x) + \lambda^2\Phi_2(x) + \lambda^3\Phi_3(x) + ... + \lambda^n\Phi_n(x) + \lambda^{n+1}\Phi_{n+1}(x) \quad \textbf{(214)}$$

$$= g(x) + \sum_{n=1}^{\infty} \lambda^n\Phi_{n+1}(x), \quad for \quad n \geq 0 \quad \textbf{(215)}$$

Where

$$\Phi_{n+1}(x) = \int_{-1}^{1} K(x,t)\Phi_n(t)dt \quad \textbf{(216)}$$

The series (216) is known as under the name of the $Neumann$ series. this series is absolutely and uniformly convergent. Thus, the solution is given by

$$f(x) = g(x) + \lim_{n\to\infty} \sum_{i=1}^{n} \lambda^i\Phi_i(x) \quad \textbf{(217)}$$

Example 3.1.11. *Determine the solution of Fredholm integral equation of the Second Kind using Successive Approximations Method;*

$$y(x) = e^x + \int_{-1}^{1} xty(t)dt \quad \textbf{\textit{(218)}}$$

For the zeroth approximation approximation $y_0(x)$, we let;

$$y_0(x) = e^x \tag{219}$$

The method of successive approximations admits the use of the iteration formula:

$$y_{n+1}(x) = e^x + \int_{-1}^{1} xty_n(t)dt \quad n \geq 0 \tag{220}$$

Using (219) *and* (220) *we obtain:*

$$y_1(x) = e^x + x\int_{-1}^{1} ty_0(t)dt = e^x + x\int_{-1}^{1} te^t dt = e^x + x\frac{2}{e}$$

$$y_2(x) = e^x + x\int_{-1}^{1} ty_1(t)dt = e^x + x\int_{-1}^{1} (te^t + \frac{2}{e}t^2)dt = e^x + x\frac{2}{e} + x\frac{2}{e} \times \frac{2}{3}$$

$$y_3(x) = e^x + x\int_{-1}^{1} ty_2(t)dt = e^x + x\int_{-1}^{1} (te^t + \frac{2}{e}t^2 + \frac{2}{e} \times \frac{2}{3}t^2)dt = e^x + x\frac{2}{e} + x\frac{2}{e} + x\frac{2}{e} \times \frac{2}{3} + x\frac{2}{e} \times (\frac{2}{3})^2$$

Therefore,

$$y_4(x) = e^x + x\frac{2}{e} + x\frac{2}{e} \times \frac{2}{3} + x\frac{2}{e} \times (\frac{2}{3})^2 + x\frac{2}{e} \times (\frac{2}{3})^3 \tag{221}$$

From (221), *we obtain:*

$$y_{n+1}(x) = e^x + x\frac{2}{e}(1 + \frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3 + (\frac{2}{3})^n) \tag{222}$$

We notice that the infinite geometric series on the right hand side has $a = 1$ and the common ratio as $r = \frac{2}{3}$.
Therefore, the sum of the infinite geometric series in (222) *is given by:*

$$S = \frac{a}{a - r} = \frac{1}{1 - \frac{2}{3}} = 3 \tag{223}$$

Using (223), *the series solution in* (222) *therefore becomes*

$$y(x) = e^x + 6xe^{-1}$$

Example 3.1.12. *Let the integral equation of the Second Kind be*

$$u(x) = 1 + \int_{0}^{1} xu(t)dt$$

Solve using the successive approximation (method of Neumann)
For the solution, consider the zeroth approximation as $u_0(x) = 1$, and then the first approximation can be calculated as

$$\begin{aligned} u_1(x) &= 1 + \int_0^1 x u_0(t) dt \\ &= 1 + \int_0^1 x dt = 1 + x \end{aligned}$$

In this way, we find

$$\begin{aligned} u_2(x) &= 1 + \int_0^1 x u_1(t) dt \\ u_2(x) &= 1 + \int_0^1 x(1+t) dt \\ u_2(x) &= 1 + \frac{1}{2}x \end{aligned}$$

So we get

$$u_n(x) = 1 + x\left[1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots + \tfrac{1}{2^{n-1}}\right]$$

And so

$$\begin{aligned} u_(x) &= \lim_{n\to\infty} u_n(x) \\ u(x) &= \lim_{n\to\infty} x \sum_{i=0}^{n} \frac{1}{2^i} \\ u(x) &= 1 + \lim_{n\to\infty} x.2(1 - \frac{1}{2^i}) \\ u(x) &= 1 + 2x \end{aligned}$$

which is the solution.

3.1.8 Picard's Method

The Picard's method is obtained when we let $y_0(x) = 0, \quad 1, \quad x$, or any other real-valued function.Given the Fredholm integral equation of the Second Kind:

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) dt \quad a \le x \le b \tag{224}$$

From (224), we have:

$$y_0(x) = Any \quad real - valued \quad function \tag{225}$$

Therefore, the first approximation $y_1(x)$ is defined by:

$$y_1(x) = f(x) + \lambda \int_a^b K(x,t) y_0(t) dt = f \tag{226}$$

The second approximation is given by:

$$y_2(x) = f(x) + \lambda \int_a^b K(x,t) y_1(t) dt \tag{227}$$

Continuing in the same way, we obtain the n^{th} approximation by:

$$y_{n+1}(x) = f(x) + \lambda \int_a^b K(x,t) y_n(t) dt \tag{228}$$

Hence, the final solution is then given by;

$$\lim_{n \to \infty} y_{n+1}(x) = y(x) \tag{229}$$

Example 3.1.13. *Determine the solution of the Fredholm integral equation below by using Picard's Method*

$$y(x) = \sin x + \cos x + \int_0^{\frac{\pi}{2}} \sin xt y(t) dt \tag{230}$$

Here, we let the zeroth approximation $y_0(x)$ to be

$$y_0(x) = 0 \tag{231}$$

The first approximation is then given by:

$$y_1(x) = \sin x + \cos x + \sin x \int_0^{\frac{\pi}{2}} t y_0(t) dt = \sin x + \cos x \tag{232}$$

The second approximation is also given by:

$$y_2(x) = \sin x + \cos x + \sin x \int_0^{\frac{\pi}{2}} t y_2(t) dt = \tag{233}$$

$$y_2(x) = \sin x + \cos x = \sin x \int_0^{\frac{\pi}{2}} (t \sin t + t \cos t) dt = \sin x + \cos x + \frac{\pi}{2} \sin x$$

The third approximation is given by:

$$y_3(x) = \sin x + \cos x + \sin x \int_0^{\frac{\pi}{2}} t y_2(t) dt = \sin x + \cos x + \pi \sin x \tag{234}$$

Using the same approach we have the fourth approximation as:

$$y_4(x) = \sin x + \cos x + \sin x \int_0^{\frac{\pi}{2}} t y_3(t) dt = \sin x + \cos x + \frac{3\pi}{2} \sin x \qquad (235)$$

and

$$y_5(x) = \sin x + \cos x + \sin x \int_0^{\frac{\pi}{2}} t y_4(t) dt = \sin x + \cos x + 2\pi \sin x \qquad (236)$$

From (236), *we cancel the noise terms in* $\pm\pi$ *such that the exact solution is*

$$y(x) = \sin x + \cos x$$

Convergence of Successive Approximation Method

Theorem 3.1.14. *Let* $A : X \to X$ *an operator continuous on a Banach space* X *whose spectral radius verifies* $r(A) < 1$. *Then the successive approximation method defined by* $\forall n \in X$, $u_{n+1} = Au_0 + f$ *converges for all* $u_0 \in X$ *and* $f \in X$ *to a unique solution of* $u - Au = f$.

Remark 3.1.15. *This result is very natural in the finite dimensional space where the spectral radius can be interpreted as the greatest value of* $N(A)$ *for the set of algebraic norms* N.

3.1.9 Conversion of Fredholm Integral Equations to ODE

This method involves converting F. I. Es to their equivalent BVPs. This is obtained by integrating both sides of the given Fredholm integral equation w.r.t s in order to do away with the integral sign on the right hand side which then yields the ODE.

Leibnitz's **Theorem**

Theorem 3.1.16. *[7] To convert the differential equations to integral equations, we use the Leibnitz'sRule of Differentiating Under the Integral Sign, i.e*

If $G(s,t)$ *and* $\frac{\partial G(s,t)}{\partial s}$ *are continuous functions of* s *and* t *in the domain* $a \leq s \leq b$, $t_0 \leq t \leq t_1$, $a = g(s)$, $\quad b = h(s)$, *then*

$$\frac{d}{ds} \int_{g(s)}^{h(s)} G(s,t) dt = \int_{g(s)}^{h(s)} \frac{\partial G(s,t)}{\partial s} dt + \frac{dh(s)}{ds} G(s, h(s)) - \frac{dg(s)}{ds} G(s, g(s)) \qquad (237)$$

provided $g(s)$ and $h(s)$ defined functions with continuous derivatives in the intervals defined above. We may also use this rule to convert integral equations to equivalent ordinary differential equations. For example, we have
(a). For Volterra integral equation:

$$\frac{d}{ds}[\int_a^s G(s,t)u(t)dt] = \int_a^s \frac{\partial G}{\partial s}dt + G(s,t)u(s) \tag{238}$$

(b). For Fredholm Integral Equation:

$$\frac{d}{ds}[\int_a^b G(s,t)v(t)dt] = \int_a^b \frac{\partial G}{\partial s}v(t)dt \tag{239}$$

Where $u(t)$ and $v(t)$ are independent of x and hence taking partial derivatives with respect to x, $u(t)$ and $v(t)$ are treated as constants.
The boundary conditions can be obtained by substituting $s = a$, and $s = b$ into $y(s)$ and the resultant boundary value problem can be solved easily by using the Ordinary Differential Equations methods e.g Taylor series, Runge-Kutta and multi-step methods.

There are two types of problems associated with this method of solutions. From the general point of view
Given the Fredholm Integral Equation of the Second Kind:

$$y(s) = f(s) + \lambda \int_a^b G(s,t)y(t)dt, \quad a \le s,t \le b \tag{240}$$

Where $f(s)$ is given and the kernel $G(s,t)$ can be of two types:

1. $$G(s,t) = \begin{cases} t(1-s)h(s), & \text{for} \quad a \le t \le s \\ s(1-t)h(s), & \text{for} \quad s \le t \le b \end{cases}$$

2. $$G(s,t) = \begin{cases} th(s), & \text{for} \quad a \le t \le s \\ sh(s), & \text{for} \quad s \le t \le b \end{cases}$$

So, in a simpler way, $h(s) \equiv \lambda$, a constant. From the first expression, the Fredholm Integral equation 240 takes the form:

$$y(s) = f(s) + \lambda \int_a^s t(1-s)y(t)dt + \lambda \int_s^b s(1-t)y(t)dt \tag{241}$$

and from the second expression, we have:

$$y(s) = f(s) + \lambda \int_a^s ty(t)dt + \lambda s \int_s^b y(t)dt \tag{242}$$

The R.H.S of equations (241) and $(3.1.9)$ are a representation of a product of two functions of s, so when we differentiate them with respect to s both sides of(241) and $(3.1.9)$ using Leibnitz rule to obtain for equation (241) we have:

$$y'(s) = f'(s) + \lambda s(1-s)y(s) - \lambda \int_a^s ty(t)dt - \lambda s(1-s)y(s) + \lambda \int_s^b (1-t)y(t)dt$$

$$y'(s) = f'(s) - \lambda \int_a^s ty(t)dt + \lambda \int_s^b (1-t)y(t)dt \tag{243}$$

we differentiate both sides again with respect to s in order to do away with the integral sign in (243),

$$y''(s) = f''(s) - \lambda sy(s) - \lambda(1-s)y(s) \tag{244}$$

wich then give rise to the ODE:

$$y''(s) + \lambda sy(s) = f''(s) \tag{245}$$

With boundary conditions as:

$$y(a) = f(a), \quad yb = f(b) \tag{246}$$

And for equation $(3.1.9)$ given by:

$$y(s) = f(s) + \lambda \int_a^s ty(t)dt + \lambda s \int_s^b y(t)dt$$

We differentiate both sides of $(3.1.9)$ with respect to x using product rule of differentiation and using Leibnitz rule, we obtain:

$$y'(s) = f'(s) + \lambda \int_s^b y(t)dt \tag{247}$$

We again differentiate (247) with respect to x so that we remove the integral sign to get

$$y''(s) = f''(s) - \lambda y(s) \tag{248}$$

Therefore, the resulting ordinary differential equation is given by:

$$y''(s) + \lambda y(s) = f''(s) \tag{249}$$

Here, the boundary conditions are:

$$y(a) = f(a), \quad y'(b = f'(b)) \tag{250}$$

Combining (249) and (250) yields the boundary value problem equivalent to the Fredholm integral equation (240).

However, if $h(s)$ is not a constant, we can then proceed in similar manner to the above discussion above to obtain the boundary value problem for the two different types of kernel. The method is best demonstrated in the following examples:

Example 3.1.17. *Convert the following equation to its equivalent boundary value problem:*

$$y(x) = \sin x + \int_0^{\frac{\pi}{2}} K(x,t)y(t)dt \tag{251}$$

Where

$$K(x,t) = \begin{cases} t(\frac{\pi}{2} - x), & \textit{for} \quad 0 \le t \le x \\ x(\frac{\pi}{2} - t), & \textit{for} \quad x \le t \le \frac{\pi}{2} \end{cases}$$

and therefore, l equation (251) *becomes;*

$$y(x) = \sin x + (\frac{\pi}{2} - x)\int_0^x ty(t)dt + x\int_x^{\frac{\pi}{2}} (\frac{\pi}{2} - t)y(t)dt \tag{252}$$

Differentiating (252) *twice with respect to* x *using the product rule of differentiation and using Leibnitz Rule of differentiation, we obtain:*

$$y'(x) = \cos x - \int_o^x ty(t)dt + \int_x^{\frac{\pi}{2}} (\frac{\pi}{2} - t)y(t)dt \tag{253}$$

$$y''(x) = -\sin x - [ty(t)]_0^x + [(\frac{\pi}{2} - t)y(t)]_x^{\frac{\pi}{2}}$$

$$y''(x) = -\sin x - xy(x) - \frac{\pi}{2}y(x) + xy(x)$$

$$y''(x) = -\sin x - \frac{\pi}{2}y(x) \tag{254}$$

From (254), *the resulting ordinary differential equation is then given by:*

$$y''(x) + \frac{\pi}{2} = -\sin x \tag{255}$$

With the BCs as:

$$y(0) = f(0) = 0, \quad y(\frac{\pi}{2}) = f(\frac{\pi}{2}) = 1 \tag{256}$$

3.2 Numerical Methods For Solving Fredholm Integral Equations of the Second Kind

There are many numerical methods for solving Fredholm integral equations of the Second Kind. These include:

1. The Degenerate Kernel Method
2. The Collocation Method
3. The Galerkin Method
4. The Nystrom (Quadrature) Method

3.2.1 The Degenerate Kernel Method

We discussed about the Degenerate Kernel Method as one of the Analytical Methods for solving Fredholm Integral equations of the Second Kind. However, we can also use it to numerically solve the Fredholm integral equation below.

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt, \quad a \leq x \leq b \tag{257}$$

A kernel $K(x,t)$ is degenerate if it can be expressed as the sum of a finite number of terms, each of which is a product of functions of x and t only such that:

$$K(x,t) = \sum_{i=1}^{n} g_i(x)h_i(t) \tag{258}$$

which we now seek to approximate them by degenerate kernels. From equation (257), $K(x,t)$ is the function to be approximated by a sequence of degenerate kernel functions;

$$K_n(x,t) = \sum_{i=1}^{n} g_{i,n}(x)h_{i,n}(t), \quad n \geq 1 \tag{259}$$

Where K_n is an integral operator associated with the integral equation (257) which can be rewritten as:

$$K_n y(x) = \lambda \int_a^b K_n(x,t)y(t)dt, \quad a \leq x \leq b, \quad n \geq 1 \tag{260}$$

Thus, the integral equation (257) can be simplified as:

$$(I - \lambda K)y = f \tag{261}$$

Using (260), (261) can be written as;

$$(I-\lambda K_n)y_n = f \tag{262}$$

Where y_n is the solution of the approximating equation.
Using the formula (259) for $K_n(x,t)$, the integral equation (262) becomes

$$y_n(x) = f(x) + \lambda \sum_{i=1}^{n} g_{i,n}(x) \int_a^b h_{i,n}(t) y_n(t) dt$$

As we have already seen in section 3.1.1, we then have:

$$y_n(x) = f(x) + \lambda \sum_{i=1}^{n} \beta_i g_i(x) \tag{263}$$

Where

$$\beta_i - \lambda \sum_{k=1}^{n} G_{ik}\beta_k = q_i \quad i = 1,2,...,n \tag{264}$$

represents a system of n algebraic equations with β_i the unknowns such that

$$q_i = \int_a^b h_i(t) g_i(t) dt \tag{265}$$

and

$$G_{ik} = \int_a^b h_i(t) g_k(t) dt \tag{266}$$

From (264), (265) and (266) we have:

$$D(\lambda) = \begin{vmatrix} 1-\lambda G_{11} & -\lambda G_{12} & \cdots & -\lambda G_{1n} \\ -\lambda G_{21} & 1-\lambda G_{22} & \cdots & -\lambda G_{2n} \\ . & & & \\ . & & & \\ . & & & \\ -\lambda G_{n1} & -\lambda G_{n2} & \cdots & 1-\lambda G_{nn} \end{vmatrix} \tag{267}$$

which is a polynomial in λ of degree n.

Analysis of the solution $y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) dt$ by the degenerate kernel

When one of the q_i or all of them are nonzero, then we have the following:

1. If $D(\lambda) \neq 0$, then a unique nonzero solution of the system (264) exists, which implies that (257) has a unique solution given by (263).

2. If $D(\lambda) = 0$, then the system (264) has either no solution or possesses infinitely many solutions. Thus, (257) has either no solution or infinite solution.

When $f(x) = 0$, then from (265) it implies that $q_i = 0, \quad i = 1, 2, ..., n$ which suffices the algebraic system (264) to a system of homogeneous linear equation hence the following are possible depending on the value of $D(\lambda)$

1. If $D(\lambda) \neq 0$, then (264) has a unique zero solution for $\beta_i = 0$ which also implies that (257) has a unique zero solution i.e $y_n(x) = 0$.

2. If $D(\lambda) = 0$, then (264) has infinite nonzero solutions so that (257) has infinite nonzero solutions as well where the values of λ for which $D(\lambda) = 0$ are the eigenvalues and any corresponding nonzero solutions of $y(x) = \lambda \int_a^b K(x,t)y(t)dt$ are called the eigenfunctions of the integral equation.

When $f(x) = 0$, but

$$\int_a^b f(t)h_i(t)dt = 0, \quad i = 1, 2, ..., n \tag{268}$$

which simply means that the function $f(x)$ is orthogonal to the functions $h_i(t)$ thus the q_i reduce to zeros for $i = 1, 2, ..., n$. From (268), the system (265) reduces to a system of homogeneous linear equations and the possible case are:

1. If $D(\lambda) \neq 0$, then (257) has a unique zero solution for $\beta_i = 0, \quad i = 1, 2, ..., n.$ i.e $y_n(x) = 0$

2. If $D(\lambda) = 0$, then the system (264) possesses infinite nonzero solutions which implies that (257) has infinite nonzero solutions.

For a kernel which is not degenerate, we use the Taylor Series Approximation Method.

Taylor Series Approximation

Consider the Fredholm Integral Equation of the Second Kind;

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \quad a \leq x \leq b \tag{269}$$

Where oftenly, $K(x,t)$ can be written as a power series in t,

$$K(x,t) = \sum_{i=0}^{\infty} P_i(x)(t-a)^i \tag{270}$$

Or in x,

$$K(x,t) = \sum_{i=0}^{\infty} P_i(t)(x-a)^i \tag{271}$$

Now, suppose that K_n is the partial sum of the first n terms on the right hand side of (270) which can be rewritten as:

$$K_n(x,t) = \sum_{i=1}^{n-1} P_i(x)(t-a)^i \tag{272}$$

And using the notation;

$$K_n(x,t) = \sum_{i=1}^{n} g_{i,n}(x)h_{i,n}(t), \quad n \geq 1$$

Where K_n is a degenerate kernel with

$$g_i(x) = P_{i-1}(x), \quad h_i(t) = (t-a)^{i-1}, \quad i = 1,...,n \tag{273}$$

And the linear system (264) can now be written as:

$$G_i - \lambda \sum_{j=1}^{n} G_j \int_a^b (t-a)^{i-1} P_{j-1}(t)dt = \int_a^b f(t)(t-a)^{i-1}dt \quad i = 1,...,n \tag{274}$$

and the approximate solution is therefore given by:

$$y_n(x) = f(x) + \lambda \sum_{i=0}^{n-1} G_{i+1}P_i(x) \tag{275}$$

Where the integrals in equations (274) are evaluated numerically.

Example 3.2.1. *Solve the Fredholm integral equation*

$$y(x) = e^x + \int_0^1 \sin xty(t)dt, \quad 0 \leq x,t \leq 1. \tag{276}$$

Let us first approximate the Kernel $K(x,t)$ by the sum of the first three terms in its Taylor Series:

$$K(x,t) = \sin xt \tag{277}$$

Expanding (277) *using the Taylor expansion, we have*

$$\sum_{k=0}^{\infty}\frac{f^k(0)}{k!}(x-0)^k \approxeq f(0)+\frac{f'(o)}{1!}(xt)+\frac{f''(0)}{2!}(xt)^2+\frac{f'''(0)}{3!}(xt)^3+\frac{f''''(0)}{4!}(xt)^4+\cdots \tag{278}$$

From (278) *we obtain*

$$K(x,t)=xt-\frac{(xt)^3}{3!}+\frac{(xt)^5}{5!}$$

which is a separable kernel.
Then the given Fredholm integral equation takes the form:

$$y(x)=e^x+\int_0^1 (xt-\frac{(xt)^3}{3!}+\frac{(xt)^5}{5!})y(t)dt \tag{279}$$

$$y(x)=e^x+x\int_0^1 ty(t)dt-\frac{x^3}{3!}\int_0^1 t^3y(t)dt+\frac{x^5}{5!}\int_0^1 t^5y(t)dt \tag{280}$$

Let

$$A_1=\int_0^1 ty(t)dt \tag{281}$$

$$A_2=-\frac{1}{6}\int_0^1 t^3y(t)dt \tag{282}$$

$$A_3=\frac{1}{120}\int_0^1 t^5y(t)dt \tag{283}$$

Then (279) *yields*

$$y(x)=e^x+A_1x-A_2x^3+A_3x^5 \tag{284}$$

$$y(t)=e^t+A_1t-A_2t^3+A_3t^5 \tag{285}$$

Substituting the value of $y(t)$ *given by* (285) *in* (281), *we obtain*

$$A_1=\int_0^1 t(e^t+A_1t-A_2t^3+A_3t^5)dt$$
$$A_1=\Big[te^t-et\Big]_0^1+\frac{A_1}{3}-\frac{A_2}{5}+\frac{A_3}{7}$$

Therefore,

$$\frac{2}{3}A_1 + \frac{A_2}{5} - \frac{A_3}{7} = 1 \tag{286}$$

Substituting the value of $y(t)$ *given by* (285) *in* (282), *we get*

$$A_2 = -\frac{1}{6}\int_0^1 t^3(e^t + A_1 t - A_2 t^3 + A_3 t^5)dt$$
$$A_2 = -\frac{1}{6}\left[t^3e^t - 3t^2e^t + 6te^t - 6e^t\right]_0^1 - \frac{A_1}{30} + \frac{A_2}{42}A_2 - \frac{A_3}{54}$$

$$\frac{A_1}{5} + \frac{41}{7}A_2 + \frac{A_3}{9} = 2e - 6 \tag{287}$$

Also, substituting the value of $y(t)$ *given by* (285) *in* (283), *we obtain*

$$A_3 = \frac{1}{120}\int_0^1 t^5(e^t + A_1 t - A_2 t^3 + A_3 t^5)dt$$
$$A_3 = \frac{1}{120}\left[t^5e^t - 5t^4e^t + 20t^3e^t - 60t^2e^t + 120te^t - 120e^t\right]_0^1 + \frac{A_1}{840} - \frac{A_2}{1080} + \frac{A_3}{1320}$$

$$A_3 = \frac{A_1}{840} - \frac{A_2}{1080} + \frac{A_3}{1320} + 1 - \frac{11}{30}e$$
$$-\frac{A_1}{840} + \frac{A_2}{1080} + \frac{1319}{1320}A_3 = 1 - \frac{11}{30}e$$

$$-\frac{A_1}{7} + \frac{A_2}{9} + \frac{1319}{11}A_3 = 120 - 44e \tag{288}$$

Solving the algebraic systems (286), (287) *and* (288) *gives*

$$A_1 = 1.5459$$
$$A_2 = -0.1491$$
$$A_3 = 0.005279$$

So, using these values in (284) *gives the solution of* (279) *as below*

$$y(x) = e^x + 1.5459x - 0.1491x^3 + 0.005279x^5 \tag{289}$$

Which is the required approximate solution.

Example 3.2.2.

$$y(x) = \frac{3x}{6} + \frac{1}{2}\int_0^1 (e^{xt} - 1)y(t)dt$$

We approximate the kernel by the sum of the first three terms in its Taylor series.

$$K(x,t) \approx \left[1 + \tfrac{xt}{1!} + \tfrac{(xt)^2}{2!} + \tfrac{(xt)^3}{3!} - 1\right]$$

which is a degenerate kernel.
Then the given integral equation can be written in the form

$$y(x) = \frac{3x}{6} + \frac{1}{2}\int_0^1 \left(xt + \tfrac{x^2t^2}{2} + \tfrac{x^3t^3}{6}\right) y(t)dt \tag{290}$$

$$y(x) = \frac{3x}{6} + \frac{1}{2}x\int_0^1 ty(t)dt + \frac{1}{4}x^2\int_0^1 t^2y(t)dt + \frac{1}{12}x^3\int_0^1 t^3y(t)dt \tag{291}$$

We let

$$A_1 = \frac{1}{2}\int_0^1 ty(t)dt \tag{292}$$

$$A_2 = \frac{1}{4}\int_0^1 t^2y(t)dt \tag{293}$$

$$A_3 = \frac{1}{12}\int_0^1 t^3y(t)dt \tag{294}$$

Then (290)*gives*

$$y(x) = \frac{3x}{6} + A_1x + A_2x^2 + A_3x^3 \tag{295}$$

$$y(t) = \frac{3t}{6} + A_1t + A_2t^2 + A_3t^3 \tag{296}$$

Applying the same procedure as in the above example, we obtain the algebraic system below

$$\frac{5}{3}A_1 - \frac{A_2}{4} - \frac{A_3}{5} = \frac{1}{6} \tag{297}$$

$$-\frac{A_1}{4} + \frac{19}{5}A_2 - \frac{A_3}{6} = \frac{1}{8} \tag{298}$$

$$-\frac{A_1}{5} - \frac{A_2}{6} + \frac{83}{7}A_3 = \frac{1}{12} \tag{299}$$

Solving the system, we obtain

$$A_1 = 0.1072$$
$$A_2 = 0.04036$$
$$A_3 = 0.009403$$

Hence, from equation (295) *the required approximate solution is given by*

$$y(x) = \frac{x}{2} + 0.1072x + 0.04036x^2 + 0.009403x^3$$

3.2.2 Collocation Method

Collocation method is one of the numerical methods for solving ordinary differential equations, partial differential equations and integral equations.

The main idea in collocation method is to select a finite-dimensional space of polynomials up to a certain degree say degree n. The method also involves the selection of a number of points in the domain as well a solution which satisfies the given equation at the collocation points. Therefore, the fundamental principle of collocation method is to choose parameters and basis functions such that the residual is zero at the collocation points. Now, given the Fredholm integral equation of the Second Kind:

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt, \quad a \leq x,t \leq b \tag{300}$$

The equation (300) can be written in the form:

$$\varepsilon[y(x)] \equiv f(x) - \lambda \int_a^b K(x,t)y(t) - F(x) \tag{301}$$

We seek an approximate solution of equation (301) in the form such that:

$$Y_n(x) = \Phi(x, A_1, ..., A_n) \tag{302}$$

Where the parameters $A_1, ..., A_n$ are known as the indeterminate coefficients.
When we substitute (302) into (301), we obtain:

$$\varepsilon[Y_n(x)] = Y_n(x) - \lambda \int_a^b K(x,t)Y_n(t)dt - F(x) \tag{303}$$

If $f(x)$ is an exact solution, then clearly the residual $\varepsilon[f(x)] = 0$, which allows us to choose the parameters $A_1, ..., A_n$ such that the error $\varepsilon[F_n(x)]$ is small.

Again, we have that $Y_n(x)$ are linearly dependent on the parameters $A_1,...,A_n$, hence from (302), if

$$\lim_{n\to\infty} Y_n(x) = y(x),$$

then we can find te solution $f(x)$ arbitrarily by taking sufficiently large number of the parameters $A_1,...,A_n$.

Determination of an Approximate Solution $F_n(x)$

Let's set

$$Y_n(x) = \Phi_0(x) + \sum_{i=1}^{n} A_i \Phi_i(x) \tag{304}$$

Assuming the functions $\Phi_i(x) \quad i = 1,...,n$ are linearly independent.
If we take $\Phi_0(x) = Y(x)$ or $\Phi_0(x) = 0$ and then substitute into (304), it gives;

$$\varepsilon[Y_n(x)] = \Phi_0(x) + \sum_{i=1}^{n} A_i \Phi_i(x) - y(x) - \lambda \int_a^b K(x,t)[\Phi_0(t) + \sum_{i=1}^{n} A_i \Phi_i(t)]dt$$

or simply as:

$$\varepsilon[Y_n(x)] = h_0(x,\lambda) + \sum_{i=1}^{n} A_i h_i(x,\lambda) \tag{305}$$

Where

$$h_0(x,\lambda) = \Phi_0(x) - Y(x) - \lambda \int_a^b K(x,t)\Phi_0(t)dt$$
$$h_i(x,\lambda) = \Phi_i(x) - \lambda \int_a^b K(x,t)\Phi_i(t)dt$$

We require that $\varepsilon[Y_n(x)] = 0$ of the given system at the collocation points $x_1,...,x_n$ on the interval $[a,b]$ i.e;

$$\varepsilon[Y_n(x_j)] = 0 \quad j = 1,2,...,n \quad a \le x_1 < x_2 < ... < x_n \le b$$

Usually, $x_1 = a$ and $x_n = b \implies$ the algebraic system (305) becomes:

$$\sum_{i=1}^{n} A_i h_i(x_j,\lambda) = -h_0(x_j,\lambda) \quad j = 1,...,n \tag{306}$$

Suppose that the determinant of the algebraic system (306) is nonzero i,e

$$det[h_i(x_j,\lambda)] = \begin{vmatrix} h_1(x_1,\lambda) & h_1(x_2,\lambda) & \cdots & h_1(x_n,\lambda) \\ h_2(x_1,\lambda) & h_2(x_2,\lambda) & \cdots & h_2(x_n,\lambda) \\ . & & & \\ . & & & \\ . & & & \\ h_n(x_1,\lambda) & h_n(x_2,\lambda) & \cdots & h_n(x_n,\lambda) \end{vmatrix} \neq 0 \tag{307}$$

then (306) uniquely determines the numbers $A_1,...,A_n$ which makes it possible to find the approximate solution $Y_n(x)$ using the formula (304)

Example 3.2.3. *Solve the following Fredholm integral equation of the Second Kind using Collocation Method;*

$$y(x) = 2x + \frac{1}{2}\int_{-1}^{1} xty(t)dt \tag{308}$$

Set $n = 3$ *such that the linearly independent functions are given by;*

$$\Phi_1(x) = 1, \quad \Phi_2(x) = x \quad and \quad \Phi_3(x) = x^2 \tag{309}$$

Therefore, the approximate solution of (308) *is;*

$$Y_n(x) = \sum_{i=1}^{3} A_i\Phi_i(x) = A_1 + A_2x + A_3x^2 \tag{310}$$

Substituting (310) *into* (308) *we obtain:*

$$Y_3(x) = 2x + \frac{1}{2}\int_{-1}^{1} xt(A_1 + A_2t + A_3t^2)dt + \varepsilon(x,A_1,A_2,A_3)$$

Where

$$\varepsilon(x,A_1,A_2,A_3)$$

is the error term or the residual term.

$$Y_3(x) = 2x + \frac{1}{2}x\int_{-1}^{1} (A_1t + A_2t^2 + A_3t^3)dt + \varepsilon(x,A_1,A_2,A_3)$$

Integrating with respect to t *over* $-1 \leq t \leq 1$ *we obtain:*

$$Y_3(x) = 2x + \frac{x}{2}(\frac{2}{3}A_2) + \varepsilon(x,A_1,A_2,A_3) \tag{311}$$

Now, from (311) *we need to form three algebraic equations that will enable us find the values* $A_1, \quad A_2 \quad and \quad A_3$.
Asserting that the error term is zero at the collocation points, arbitrarily chosen to be $x_1 = -1$, $x_2 = 0$ *and* $x_3 = 1$. *Then we have:*

$$for \quad x = -1$$
$$A_1 - A_2 + A_3 = -2 - \frac{1}{3}A_2$$
$$A_1 - \frac{2}{3}A_2 + A_3 = -2$$

which yields;

$$-A_1 + \frac{2}{3}A_2 - A_3 = 2 \tag{312}$$
$$x = 0, \quad A_1 = 0 \tag{313}$$

$$x = 1$$
$$A_1 + A_2 + A_3 = 2 + \frac{1}{3}A_2$$

resulting:

$$\frac{2}{3}A_2 + A_3 = 2 \tag{314}$$

Using (313) *we have;*

$$\frac{2}{3}A_2 - A_3 = 2$$
$$\frac{2}{3}A_2 + A_3 = 2$$

Adding the two equations and solving for $A_2 \quad and A_3$ *we obtain:*

$$A_2 = 3 \quad and \quad A_3 = 0$$

Therefore, the approximate solution of (308) *is given by;*

$$y(x) = 3x$$

Which is the same as an exact solution (148) obtained by the use of Adomain decomposition method.

3.2.3 Galerkin Approximation Method

The solution of the fredholm integral equation of the Second Kind

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt, \quad a \leq x,t \leq b \tag{315}$$

$y(x)$ can be approximated by the partial sum:

$$Y_n(x) = \sum_{i=1}^{n} A_i \Phi_i(x) \tag{316}$$

of n linearly independent functions $\Phi_1(x), \Phi_2(x), ..., \Phi_n(x)$ on the interval (a,b). Here the associated error $\varepsilon(x, A_1, A_2, ..., A_n)$ depends on x and the choice of the coefficients $A_1, A_2, ..., A_n$. Thus, when we substitute the approximate solution for $y(x)$, the equation (315) becomes;

$$Y_n(x) = f(x) + \lambda \int_a^b K(x,t)Y_n(t)dt + \varepsilon(x, A_1, A_2, ..., A_n) \tag{317}$$

Where the n conditions must be found to give the n equations that will determine the coefficients $A_1, A_2, ..., A_n$.
For Galerkin Method, we assume that the error term $\varepsilon(x, A_1, A_2, ..., A_n)$ is orthogonal to n linearly independent functions $\Phi_1(x), \Phi_2(x), ..., \Phi_n(x)$ on the interval $[a,b]$ where the n conditions are given by;

$$\int_a^b \Phi_k(x)(\varepsilon, x, A_1, A_2, .., A_n)dx \quad k = 1, ..., n$$

$$\int_a^b \Phi_k(x)[Y_n(x) - f(x) - \int_a^b K(x,t)Y_n(t)dt]dx = 0, \quad k = 1, ..., n$$

Where $A_1, ..., A_n$ are intermediate coefficients and asserting that $\lambda = 0$.
Which implies that the approximate solution can be written in the form:

$$\int_a^b \Phi_k(x)[Y_n(x) - \int_a^b K(x,t)Y_n(t)dt]dx = \int_a^b \Phi_(x)f(x)dx \quad k = 1, ..., n \tag{318}$$

OR

$$\int_a^b \Phi_k(x)(\sum_{k=1}^{n} A_k\Phi_k(x) - \int_a^b K(x,t)[\sum_{k=1}^{n} A_k\Phi_k(t)]dt)dx = \int_a^b \Phi_k(x)f(x), \quad k = 1, ..., n \tag{319}$$

which is the general formula after substituting for the partial sum (316).
"Without loss of generality, the first set of linearly independent functions $\Phi_k(x)$ are different from the second set $\Phi_k(x)$, but some functions may be used for convenience."
The scheme is best demonstrated in the example below,

Example 3.2.4. *Approximate the solution of the following,*

$$y(x) = 2x + 0.5\int_{-1}^{1} xty(t)dt, \quad -1 \leq x,t \leq 1 \tag{320}$$

Choosing $\Phi_1(x) = 1$, $\Phi_2(x) = x$ *and* $\Phi_3(x) = x^2$ *as the linearly independent functions to give the approximate solution:*

$$Y_3(x) = A_1 + A_2x + A_3x^2 \tag{321}$$

Substituting (320) *in* (317), *we obtain;*

$$\varepsilon(x, A_1, A_2, A_3) = A_1 + A_2x + A_3x - 2x - \frac{1}{2}\int_{-1}^{1} xt(A_1 + A_2t + A_3t^2)dt \tag{322}$$

Where the error/residual term $\varepsilon(x, A_1, A_2, A_3)$ *must be orthogonal to the three linearly independent functions which we chose to be;*

$$\Phi_1(x) = 1$$
$$\Phi_2(x) = x$$
$$\Phi_3(x) = x^2$$

For $\Phi_1(x) = 1$, *we have,*

$$\frac{1}{2}\int_{-1}^{1} 1[A_1 + A_2x + A_3x^2 - \frac{1}{2}\int_{-1}^{1} xt(A_1 + A_2t + A_3t^2)dt]dx = \int_{-1}^{1} xdx$$

Integrating the second integral on the left hand side with respect to t *and the integral on the right hand side with respect to* x *over* $-1 \leq x,t \leq 1$, *we obtain:*

$$\frac{1}{2}\int_{-1}^{1}(A_1 + \frac{2}{3}A_2x + A_3x^2)dx = 0$$

which gives;

$$A_1 + \frac{1}{3}A_3 = 0 \tag{323}$$

For $\Phi_2(x) = x$, *we have,*

$$\frac{1}{2}\int_{-1}^{1} x[A_1 + A_2x + A_3x^2 - \frac{1}{2}\int_{-1}^{1} xt(A_1 + A_2t + A_3t^2)dt]dx = \frac{1}{2}\int_{-1}^{1} x^2dx$$

Canceling $\frac{1}{2}$ *throughout and integrating the second integral on the left hand side with respect to* t *and the integral on the right hand side with respect to* x, *we obtain:*

$$\int_{-1}^{1}(A_1x + \frac{1}{3}A_2x^2 + A_3x^3)dx = \frac{2}{3}$$

which gives

$$A_2 = 3 \qquad \textbf{(324)}$$

For $\Phi_3(x) = x^2$, *we have;*

$$\frac{1}{2}\int_{-1}^{1} x^2[A_1 + A_2x + A_3x^2 - \frac{1}{2}\int_{-1}^{1} xt(A_1 + A_2t + A_3t^2)dt]dx = \frac{1}{2}\int_{-1}^{1} x^3 dx$$

Canceling out $\frac{1}{2}$ *as in the above and integrating the second integral on the left hand side with respect to* t *and the integral on the right hand side with respect to* x *over* $-1 \le x, t \le 1$. *We obtain:*

$$\int_{-1}^{1} (A_1x^2 + \frac{1}{3}A_2x^3 + A_3x^4)dx = 0$$

Giving rise to'

$$\frac{2}{3}A_1 + \frac{2}{5}A_3 = 0 \qquad \textbf{(325)}$$

Using (323), (324) *and* (325) *gives* $A_1 = A_3 = 0$ *and* $A_2 = 0$ *which gives the approximate solution as*

$$y_3(x) = 3x$$

which is also the same to the exact solution given by (148). *and we realize that the same result is obtained if we use the Collocation method.*

3.2.4 Quadrature Method (Nystrom Method) of Approximation

The Nystrom method of approximation of Fredholm integral equations of the Second Kind is based on numerical integration of the integral operator in the equation:

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt, \quad a \le x, t \le b \qquad \textbf{(326)}$$

Where the interval $[a,b]$ is subdivided into n equal sub-intervals known as node points of width $\Delta t = \frac{b-a}{n}$.

Normally the quadrature formula takes the general form:

$$\int_a^b \phi(x)dx = \sum_{i=1}^{n} A_i\phi(x_i) + \varepsilon[\phi] \qquad \textbf{(327)}$$

Where the x_i are the node points of the integral equation, $A_i \quad (i = 0,1,2,...,n)$ are numerical coefficients of the function $\phi(x)$ and $\varepsilon[\phi]$ is the error term.
There are quite a number of quadrature methods, these include;

1. Rectangle Rule
2. Simpson Rule
3. Trapezoidal Rule

In this section we are going to discuss the Trapezoidal rule due to its wide range of applications and accuracy to approximate the numerical solution of the Fredholm integral equations of the Second Kind.

Trapezoidal Rule

Here, the numerical solution to the general Fredholm integral equation of the Second Kind is given by a finite sum and solving the simultaneous equations which is the end result. The general form of the integral equation is given by;

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) dt, \quad a \leq x, t \leq b \tag{328}$$

of which the interval [a,b] can be subdivided into n equal sub-intervals known as node points with $h = \frac{b-a}{n}$.

If

$$t_0 = a, \quad t_j = a + jh = t_0 + ih, \quad j = 0, 1, ..., n$$

But since t and x are dummy variables, we have;

$$\begin{aligned} x_0 &= t_0 = a \\ x_n &= t_n = b \\ x_i = x_o + ih \quad (i.e \quad x_i &= t_i) \end{aligned}$$

Let $h = \Delta t$ Therefore, from equation (328) we can rewrite;

$$\begin{aligned} y(x) &= y(x_i) \\ f(x) &= f(x_i) \\ K(x,t) &= K(x_i, t_i) \end{aligned}$$

which further yields,

$$y(x_i) = y_i$$
$$f(x_i) = f_i$$
$$K(x_i, t_j) = K_{ij}$$

Now applying the Trapezoidal rule on equation (328) then we obtain,

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \approxeq f(x) + \lambda\Delta t[\frac{1}{2}K(x,t_0)y(t_0) + K(x,t_1)y(t_1) + ... + \frac{1}{2}K(x,t_n)y(t_n)]$$

Which can be simplified as:

$$y(x) \approxeq f(x) + \lambda\Delta t[\frac{1}{2}K(x,t_0)y_0 + K(x,t_1)y_1 + ... + \frac{1}{2}K(x,t_n)y_n] \quad \textbf{(329)}$$

Since there are $n+1$ values of y_i for $(i = 0,1,2,...,n)$ then equation (329) becomes a system of $n+1$ equations in y_i for instance,

$$y_i = f_i + \lambda\Delta t[\frac{1}{2}K_{i0}y_0 + K_{i1}y_1 + K_{i(n-1)}y_{n-1} + \frac{1}{2}K_{in}y_n], \quad i = 0,1,2,...,n. \quad \textbf{(330)}$$

The equation (330) gives the approximate solution of (327) at $x = x_i$, where the terms in y are shifted to the left of the equations resulting into $n+1$ equations in y_0, y_1, y_2, ..., y_n

Since $h = \Delta t$ therefore from (330) the system of simultaneous equations is given by,

$$i = 0, \quad (1 - \lambda\frac{h}{2}K_{00})y_0 - \lambda h K_{01}y_1 - \lambda h K_{02}y_2 - ... - \lambda\frac{h}{2}K_{0n}y_n = f_0$$
$$i = 1, \quad -\lambda\frac{h}{2}K_{10}y_0 - (1 - \lambda h K_{11})y_1 - \lambda h K_{12} - ... - \lambda\frac{h}{2}K_{1n}y_n = f_1$$
$$i = n, \quad -\lambda\frac{h}{2}K_{n0}y_0 - \lambda h K_{n1}y_1 - ... + (1 - \lambda\frac{h}{2}K_{nn})y_n = f_n$$

and its general form is written as,

$$\lambda KY = F$$

where K is a matrix of coefficients which is given by:

$$K = \begin{bmatrix} 1-\lambda\frac{h}{2}K_{00} & -\lambda h K_{01} & \cdots & -\lambda\frac{h}{2}K_{0n} \\ -\lambda\frac{h}{2}K_{i0} & 1-\lambda h K_{11} & \cdots & -\lambda\frac{h}{2}K_{1n} \\ . & & & \\ . & & & \\ . & & & \\ -\lambda\frac{h}{2}K_{n0} & -\lambda h K_{n1} & \cdots & 1-\lambda\frac{h}{2}K_{nn} \end{bmatrix}$$

Y is the matrix of solution given by;

$$Y = \begin{bmatrix} y_0 \\ y_1 \\ . \\ . \\ . \\ y_n \end{bmatrix}$$

and F is the matrix of the non-homogeneous part and is given by;

$$F = \begin{bmatrix} f_0 \\ f_1 \\ . \\ . \\ . \\ f_n \end{bmatrix}$$

Which will give a unique solution of a system of linear equations where $|K| \neq o$ and either infinite or zero solutions when $|K| = 0$.

Example 3.2.5. *Use the trapezoidal to approximate the solution of the following Fredholm integral equation of the Second Kind;*

$$y(x) = \sin x = \int_0^1 (1 - x\cos xt)y(t)dt$$

at $x = 0, \frac{1}{2}$ *and* 1. *Taking* $n = 2$ *and* $\Delta t = \frac{1-0}{2} = \frac{1}{2}$, *so that* $t_i = i\Delta t = \frac{i}{2} = x_i$. *Using*

$$y_i = g_i + \frac{1}{2}(\frac{1}{2}K_{i0}y_0 + K_{i1}y_1 + \frac{1}{2}K_{i2}y_2), \quad i = 0, 1, 2$$

which can also be given in the matrix form;

$$\begin{bmatrix} 1 - \frac{1}{4}K_{00} & -\frac{1}{2}K_{01} & -\frac{1}{4}K_{02} \\ -\frac{1}{4}K_{10} & 1 - \frac{1}{2}K_{11} & -\frac{1}{4}K_{12} \\ -\frac{1}{4}K_{21} & -\frac{1}{2}K_{21} & 1 - \frac{1}{4}K_{22} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \sin 0 \\ \sin \frac{1}{2} \\ \sin 1 \end{bmatrix}$$

substituting for $g_i = g(x_i) = \sin\frac{i}{2}$ *and* $K_{ij} = K(x_i, t_i) = 1 - \frac{i}{2}\cos\frac{ij}{4}$ *into the matrix, we obtain;*

$$\begin{bmatrix} 1-\frac{1}{4}(1-0) & -\frac{1}{2}(1-0) & -\frac{1}{4}(1-0) \\ -\frac{1}{4}(1-\frac{1}{2}) & 1-\frac{1}{2}(1-\frac{1}{2}\cos\frac{1}{4}) & -\frac{1}{4}(1-\frac{1}{2}\cos\frac{1}{2}) \\ -\frac{1}{4}(1-1) & -\frac{1}{2}(1-\cos\frac{1}{2}) & 1-\frac{1}{4}(1-\cos 1) \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \sin 0 \\ \sin\frac{1}{2} \\ \sin 1 \end{bmatrix}$$

Which yields the system below

$$\begin{bmatrix} 0.75 & -0.5 & -0.25 \\ -0.125 & 0.7422 & -0.1403 \\ 0 & -0.0612 & 0.8851 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4794 \\ 0.8415 \end{bmatrix}$$

Using Gaussian Elimination method we obtain; $y_0 \approxeq 1.0132, \quad y_1 \approxeq 1.0095$ *and* $y_2 \approxeq 1.0205$ *Since the exact solution is given by* $y(x) = 1$ *which compares exactly as the approximate values.*

4 Spectral Approximation Methods

In this chapter we focus on how to use the spectral methods for solving the $Fredholm$ integral equation of the Second Kind. This technique is based on the approximation of the unknown in the form of a series.

4.1 Orthogonal Polynomials

Here we consider a range I of R, bounded or not and a weight function $w(x) > 0$.

Any polynomial is therefore integrable for the measurement of weight function $w(x)$ with respect to the $Lebesgue$ measure on I. We then apply the $Gram - Schmidt$ orthogonalization process to construct the natural base $1, X, ..., X^n$ of P_n an orthogonal family of polynomials P_n which can be assumed to be unitary. If we want an orthonormal basis, it suffices to divide the P_n by their norm. We then obtain a family $\bar{P_n} = \frac{P_n}{||P_n||}$ where $||\dot{a}||$ is the norm associated with the dot product.

From these families of polynomials we can check $||P_n||$ the recurrence relations which can be written;

$$P_n(x) = (X - \lambda_n)P_{n-1}(X) - \mu_n P_{n-2}(X), \quad (n \geq 1)$$

Where

$$\lambda_n = \frac{< XP_{n-1}, P_{n-1} >}{||P_{n-1}||}$$

and

$$\mu_n = \frac{||P_{n-1}||^2}{||P_{n-2}||^2}$$

In particular, $\mu_n > 0$ for all $n \geq 2$ in the case of unitary polynomials $P_n(x) = x^n + ...$ by setting $P_{-1} = 0$.

In general, we show that any orthonormal family P_n of orthogonal polynomials of degree n verifies a recurrence relation of the same type:

$$xP_{n-1} = \alpha_n P_n + \beta_n P_{n-1} + \gamma_n P_{n-2}$$

The expression above is important because we are sometimes required to use other orthogonal families other than those normalized by their norm or their higher degree. This is in particular for the case of $Chebyshev$ polynomials T_n, occurring naturally in the form;

$$T_n(x) = \cos(n \arccos(x))$$

Here, P_n are normalized by their higher degree of coefficients if $\alpha_n = 0$ for all n provided they have taken $P_0 = 1$ and by their $Euclidean$ norm if $\gamma_n = \alpha_{n-1}$ for all n.

Moreover, the roots of the P_n polynomials are for al $n \geq 1$, real, within the interval I and distinct.

Definition and Properties

Usually, we denote the interval I by $I = [a,b]$, $-\infty \leq a \leq b \leq \infty$ an interval of R and by $w(x)$ the real and positive measure such that;

$$\int_a^b x^n w(x)dx < +\infty, \quad n \in N$$

Given that $w(x)$ is the weight function and $C[a,b]$ the span of continuous function on the interval I.

Remark 4.1.1. *We can also work with a complex value function instead of a real valued function by the introduction of $Hermitian$ scalar product*

Example 4.1.2. *Let $f, \quad g \in C(I)$ then the dot product is defined by;*

$$< f,g >= \int_a^b f(x)g(\bar{x})w(x)dx$$

Definition 4.1.3. *The family $(P_i)i \geq 0$ of polynomials is said to be a family of orthogonal polynomials if P_i, P_j of $C(I)$ on a*

$$< f,g >= \int_a^b P_i(x)\bar{P_j}(x)w(x)dx = 0$$

In general, the family $(P_i)i \geq 0$ is referred to as a family of orthogonal polynomials on the interval I at the weight function $w(x)$ if the degree of P_i is i for all integers i and

$$< P_i, P_j >= 0, \quad i \neq j$$

4.1.1 $Gram-Schmidt$ Orthogonalization Procedure

Let X be a $Hilbert$ space, we consider a sequence of free vectors $V_1, V_2, ..., V_n$ of X then an orthogonal basis $V_1', V_2', ..., V_n'$ of vectors of X defined by;

$$\forall i = 1, ..., n; \quad V_i' \in V_1, V_2, ..., V_n$$

on $V_1' = V_1$ which is being constructed: we look for V_2' such that;

$$V_2 + \alpha_{1,2} V_1' \quad and \quad < V_1', V_2' >= 0$$

is realized if and only if,

$$\alpha_{1,2} = -\frac{< V_1', V_1' >}{||V_1'||^2}$$

Also from the vectors $V_1', V_2', ..., V_n'$ being constructed, we look for V_k' in the form:

$$V_k + \alpha_{1,k} V_1' + ... + \alpha_{k-1,k} V_{k-1} \quad and \quad < V_i', V_k >= 0, \quad i = 1, 2, ..., k-1$$

and can only be achieved only if:

$$\alpha_{1,k} = \frac{< V_i', V_k >}{||V_i'||^2}$$

4.2 Classical Orthogonal Polynomials

The classical orthogonal polynomials are the most widely used since they constitute most important class of orthogonal polynomials and are defined according to the values of the interval of integration I and according to the expression of the measure of the weight function $w(x)$.These orthogonal polynomials include:

1. D'Olinde Rodrigues' formula
2. Laguerre polynomials
3. Hermitian polynomials
4. Legendre polynomial
5. Chebyshev's polynomials
6. Jacobi polynomials

These polynomials have many important applications in areas such as mathematical physics (in particular, the theory of random matrices), approximation theory, numerical analysis and many more.

To come up with the above class of orthogonal polynomials, we use the orthogonalization process of $Gram - Schmidt$ which leads to respective class of orthogonal polynomials.

4.2.1 D'Olinde Rodrigues' polynomial

It is a simple formula which allows us to calculate a series of orthogonal polynomials.
Let $(P_n(x))n \in N$ be a sequence of orthogonal polynomials and satisfies the orthogonality condition

$$\int_a^b P_m(x)P_n(x)w(x)dx = K_{m,n}\delta_{m,n}$$

where $w(x)$ is the weight function, $K_{m,n}$ are constants and $\delta_{m,n}$ is the $Kronecker$ delta.
It can also be shown that $P_n(x)$ satisfies the recurrence relation of the form:

$$P_n(x) = \frac{1}{w(x)}\frac{d^n}{dx^n}(Q^n(x)w(x))$$

Where $Q^n(x)$ is a polynomial with degree at most n.

Generative function and Classical Orthogonal Polynomials

We call the generating function of the system of functions $\phi(x)$, a function $K(x,t)$ whose development in series according to the powers of t in a certain domain D is as follows:

$$K(x,t) = \sum_{n=0}^{+\infty} \phi_n(x)t^n$$

The assumption is that the functions $\phi_n(x)$ are polynomials $P_n(x)$ of degree n.

Theorem 4.2.1. *A necessary and sufficient condition for the $P_n(x)$ polynomials to be defined by the development $K(x,t) = \sum_{n=0}^{+\infty} P_n(x)t^n$ are orthogonal over the interval $[a,b]$ relative to the weight function at $w(x)$ is that*

$$I = \int_a^b K(x,t)K(x,t')w(x)dx$$

depends only on the product tt'

PROOF. In effect to

$$I = \int_a^b \sum_{n,m=0}^{+\infty} P_n(x)P_m(x)t^n T'^m w(x)dx = \sum_{n,m=0}^{+\infty} t^n t'^m < P_n, P_m >$$

so if $n = m$ then $< P_n, P_m >= 0$ and we have that

$$I = \sum_{n=0}^{+\infty} t^n t'^m < P_n, P_m >$$

□

4.2.2 Laguerre Polynomials

The family of $Laguerre$ polynomials is normally characterized by the following presentations:

1.The sequence of $Laguerre$ generalized polynomial $L_n^\alpha(x)$ is given by the generating function;

$$K_\alpha(x,t) = \frac{1}{(x-t)^{\alpha+1}} \exp(\frac{-xt}{1-t})$$

with

$$L_n^\alpha(x) = \sum_{j=0}^{n} \frac{(-1)^j(\alpha+n)!}{j!(n-j)!(\alpha+j)!} x^j$$

if $\alpha = 0$ we have $L_n^o(x) = L_n(x)$ with

$$L_n(x) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} C_n^j X^j$$

which gives an orthonormal basis in $L^2([0,+\infty])$ which permits to have:

$$< L_n, L_m >= 0, \quad ||l_n||_{L^2([0,+\infty])=1}$$

2. The polynomial $L_n^\alpha(x)$ can also be defined by the formula of $d'Olinde\ Rodrigues$

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!} \frac{d^n}{d^n(x)}(x^{n+\alpha}e^{-x})$$

3. Also the sequence of generalized $Laguerre$ polynomials verifies the recurrence relation;

$$L_{n+1}^\alpha(x) - (2n+\alpha+1-x)L_n^\alpha(x) + n(n+\alpha)L_{n-1}^\alpha(x) = 0$$

4.2.3 The $Hermitian$ Polynomials

They are characterized by of the following three presentations:
1. The $Hermitian$ polynomials $H_n(x)$ are given by the generating function;

$$K(x,t) = e^{-2tx-t^2} - \sum_{n=0}^{+\infty} H_n(x)t^n$$

2. $H_n(x)$ polynomials are defined by the formula of $d'Olinde$ Rodrigues' i.e

$$H_n(x) = (1-)^n e^{x^2} \frac{d^n}{d^n(x)}(e^{-x^2})$$

3. The series of the $Hermitian$ polynomials $(H_n)_{n\geq 1}$ satisfies the recurrence relation:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

which is orthogonal in $L^2(R, e^{-x^2}dx)$
4. The family

$$\bar{H_n}(x) = \frac{H_n(x)}{2^n n! \sqrt{\pi}}$$

which is an orthonormal base in $L^2(R, e^{-x^2}dx)$

4.2.4 $Legendre$ Polynomials

The family of $Legendre$ polynomials is characterized by the following presentations:
1. By the generating function the $Legendre's$ polynomials $L_n(x)$ is given by;

$$K(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{+\infty} L_n(x)t^n$$

2. From the $d'Olinde$ Rodrigues' formula the $Legendre's$ polynomials $L_n(x)$ take the form:

$$L_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}((x^2-1)^n)$$

3. The of $Legendre's$ polynomials $(L_n(x))_{n\geq 0}$ satisfies the recurrence relation;

$$L_{n+1}(x) = \frac{2n+1}{n+1}xL_n(x) - \frac{n}{n+1}L_{n-1}(x)$$

where $(L_n(x) \geq 0)$ is a family of orthogonal polynomials in $L^2([-1,1])$

4.2.5 $Chebyshev's$ Polynomials

They are characterized by one of the following three presentations;
1. The $Chebyshev's$ polynomials $T_n(x)$ are defined by the generating function:

$$K(x,t) = \frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{+\infty} T_n(x)t^n$$

2. $T_n(x)$ polynomials are defined by the formula of $d'Olinde$ Rodrigues':

$$T_n(x) = \frac{(-1)^n 2^{2n-1}(n-1)!}{(2n)!}\sqrt{1-x^2}\frac{d^n}{d^n(x)}((1-x^2)^{n-\frac{1}{2}})$$

3. The sequence $(T_n(x)_{n\geq 0})$ satisfies the recurrence relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

which is also an orthonormal basis in $L^2([-1,1],(1-x^2)^{-\frac{1}{2}}dx)$, and satisfies the relation;

$$T_n(x) = \cos(n\arccos(x))$$

4.2.6 Jacobi Polynomials

From the fundamental interval $[-1,1]$ and measure

$$d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}dx, \quad \alpha,\beta > -1$$

with Standard Normalization (if $\alpha = 0$)

$$P_n^{(\alpha,\beta)} = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)}$$

square of the standard of the normalized polynomial

$$\frac{\pi 2^{(\alpha+\beta+1)}\Gamma(n+1+\alpha)\Gamma(n+1+\beta)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}$$

4.3 Spectral Approximation Methods

Introduction

Spectral methods were introduced by $D.\ Gottlieb$ and $S.\ Orszag$ [18] and they have been widely used for the approximation of solutions of equations with partial derivatives.

Initially they were introduced by the use of truncated $Fourier$ series which were mainly used for the approximation of the problems with periodic boundary conditions. Also by the use of high-degree polynomials that form tensorized bases of approximation spaces which is a fundamental property of the construction of spectral methods.

Generally, the spectral approximation methods use polynomials from the family of $Jacobi$ polynomials.

$$P_n^{(\alpha,\beta)} = \frac{1}{2^n}\sum_{j=0}^{k}\begin{pmatrix} k+\alpha \\ l \end{pmatrix}\begin{pmatrix} k+\beta \\ k-l \end{pmatrix}(x-1)^l(x+1)^{k-1}, \quad \forall x \in [-1,1]$$

compared to the weight function

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}$$

Normally the nature and choice of the approximation depends on the domain on which we are looking for the solution.

For example, for the $Legendre's$ polynomials we take (α=0 and β=0) and for the $Tchebyshev's$ polynomials of the first kind (α=-$\frac{1}{2}$ and β=-$\frac{1}{2}$) a multiplicative factor close and basically depends on the degree k if we place ourselves on the interval $I = [-1,1]$.

Numerically, spectral method call for the use of $Tchebyshev's$ polynomials which are sometimes difficult to implement and analyze numerically, specifically, when dealing with spectral methods.

We notice that the definition and numerical analysis of spectral methods are fully based on the properties of orthogonal polynomials.

Consider the $Fredholm$ integral equation of the Second Kind;

$$y(x) = f(x) + \int_a^b K(x,t)y(t)dt \quad C([a,b]) = [-1,1] \tag{331}$$

such that $f(x)$ is continuous for $x \geq 0$ and the Kernel $K(x,t)$ a function defined on

$$I = ((x,t) \geq -1, \quad t \leq 1)$$

is a function to be determined.

4.4 Numerical Resolution of the Integral Border Points

Proposition 4.4.1. *Consider the following function system;*

$$l_0(x), l_1(x), ..., l_n(x)$$

where $l_0(X) = 1, \quad l_1(x) = x$ *and*

$$l_n(x) = \frac{1}{2^n} \sum_{j=0}^{[\frac{n}{2}]} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \begin{pmatrix} 2k-2j \\ k \end{pmatrix} x^{k-2j}$$

this system forms an orthonormal basis in $L^2([-1,1])$

Proof. We check that

$$< l_m(x), l_n(x) >= \int_{-1}^{1} l_m(x) l_n(x) dx = 0, \quad m \neq n$$

with the properties:

$$l_k(\pm 1) = (\pm 1)^k$$
$$||l_k(x)'|| \leq \frac{1}{2} k(k+1), \quad -1 \leq x \leq 1$$
$$\int_{-1}^{1} l_k^2(x) dx = (k + \frac{1}{2})$$

where $[\frac{k}{2}]$ denotes the integral part of k and the $Legendre's$ polynomials take the recurrence relation:

$$l_{n+1}(x) = \frac{2n+1}{n+1} x l_n(x) - \frac{n}{n+1} l_{n-1}$$

□

Now, in order to solve the integral equation (4.3) we use the projection by the approximation of $y_n(x)$ which is a linear combination of orthogonal polynomials $l_n(x)$ as well as the solution of the integral equation (4.3)

Given

$$y_n(x) = f(x) + \int_{-1}^{1} K(x,t) y_n(t) dt \qquad \textbf{(332)}$$

and by taking the linear combination of the $legendre's$ polynomials;

$$y_n(x) = \sum_{j=0}^{n} C_j l_j(x) \qquad \textbf{(333)}$$

when we replace (333) in (332) we obtain:

$$\sum_{j=0}^{n} C_j l_j(x) = f(x) + \int_{-1}^{1} K(x,t) \sum_{j=0}^{n} C_j l_j(t) dt \tag{334}$$

Taking

$$b_j(x) = \int_{-1}^{1} K(x,t) l_j(t) dt$$

We write

$$\sum_{j=o}^{n} C_j (l_j(x) - b_j(x)) = f(x) \tag{335}$$

Also taking the inner product of (335) by $l_i(x)$, we obtain:

$$\sum_{j=0}^{n} C_j < l_j(x) - b_j(x), l_i(x) > = < f(x), l_i(x) > \tag{336}$$

If we use the orthogonality condition in (336), we obtain a system;

$$C_i - \sum_{j=0}^{n} C_j < b_j(x), l_i(x) > = < f(x), l_i(x), \quad i = 0, 1, 2, ..., n \tag{337}$$

whose determinant is given by:

$$D(\lambda) = \begin{vmatrix} 1 - \lambda < b_0, l - 0 > -\lambda < b_1, l_0 > \cdots - \lambda < b_n, l_0 > \\ -\lambda < b_0, l_1 > 1 - \lambda < b_1, l_1 > \cdots - \lambda < b_n, l_1 > \\ . \\ . \\ . \\ -\lambda < b_0, l_n > -\lambda < b_1, l_n > \cdots 1 - \lambda < b_n, l_n > \end{vmatrix}$$

If $D(\lambda) \neq 0$, then the sustem (337) has a unique solution.

4.4.1 Numerical Techniques

[17][19] Here, the main idea is to approximate the Fredholm integral equation equation of the Second Kind using:

- Legendre Series Solution Method
- Chebyshev Polynomials Solution Method

4.4.2 Mapped Chebyshev Spectral Method

For the last three decades researchers like Grosch and Orszag [18] showed that we can easily solve the differential equations on a semi-infinite interval through mapping it into $[-1,1]$ using the algebraic function for the same map.Their technique was later developed by BOYD [20] by generalizing this technique which is provided in [19] an incredible extensive review of general properties and practical implementation for many of these approaches.

The maps which have used more frequently are logarithmic, algebraic and exponential given as below:

- Algebraic map given by;

$$y=\frac{sx}{\sqrt{1-x^2}}, \quad x=\frac{y}{\sqrt{y^2-s^2}}$$

- Exponential map given by;

$$y=\sinh sx=s^{-1}(\ln x+\sqrt{x^2+1})$$

- Logarithmic map given;

$$y=s\tanh^{-1}x=\frac{s}{2}\ln\frac{1+x}{1-x}, \quad x=\tanh^{-1}(s^{-1}y)$$

Where $y\in(-\infty,\infty)$. Here, we can choose the maps by how y quickly increases with $x\to\pm1$

Theoretically, we have already seen that the Chebyshev polynomials of the first kind of degree n are defined by:

$$T_n(x)=\cos(n\arccos x), \quad n=0,1,2,... \tag{338}$$

(338) is defined by a recurrence relation;

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x), \quad n\geq 1 \tag{339}$$

(339) is an orthornormal basis on $L^2([-1,1],(1-x^2)^{-\frac{1}{2}}dx)$ where $(1-x^2)^{-\frac{1}{2}}$ is the weight function which implies that Chebyshev polynomials are a system of of orthogonal polynomials given by:

$$\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{(1-x^2)}}=\frac{\alpha_n\pi}{2}\delta_{m,n} \tag{340}$$

on the interval $[-1,1]$ with respect to the weight function Where $\alpha_0 = 2$ and $\alpha_n = 1$ for $n \geq 1$
Such that the Chebyshev expansion of a function $u \in L_w^2(-1,1)$ is:

$$u(x) = \sum_{n=0}^{N} \hat{u}_n T_n(x), \quad \hat{u}_n = \frac{2}{\pi\alpha_n}\int_{-1}^{1} u(x)T_n(x)w(x)dx$$

Like always, the Chebyshev polynomial $T_{n+1}(x)$ of degree $n+1$ has different simple zeros or roots in $[-1,1]$ at

$$x_i = \cos(\frac{2n+1}{2n+2}\pi), \quad n = 0,1,2,...,N \tag{341}$$

Theorem 4.4.2. *For $g \in C^{2n}[-1,1]$ and $x_0, x_1, ..., x_n$ are the zeros of $T_{n+1}(x)$, then*

$$\int_{-1}^{1} \frac{g(x)}{\sqrt{1-x^2}}dx \approx \frac{\pi}{n+1}\sum_{k=0}^{n} g(x_k) \tag{342}$$

Procedure For The Solution

We consider the Fredholm Integral Equation of the Second Kind:

$$\phi(x) = g(x) + \int_{-1}^{1} K(x,t)\phi(t)dt \tag{343}$$

Where the functions $K(x,t)$ and $g(x)$ are given while $\phi(x)$ is the unknown function to be determined. Using the logarithmic map, then equation (343) becomes

$$\phi(x) = g(x) + \int_{-1}^{1} K(x, s\tanh^{-1}t)\phi(s\tanh^{-1}t)\frac{s}{1-t^2}dt \tag{344}$$

Suppose, $x = s\tanh^{-1}t$, then

$$\phi(s\tanh^{-1}t) = g(s\tanh^{-1}t) + \int_{-1}^{1} K(s\tanh^{-1}t, s\tanh^{-1}t)\phi(s\tanh^{-1}t)\frac{s}{1-t^2}dt \tag{345}$$

So, by letting $u(z) = \phi(s\tanh^{-1}z)$, then from (345) we obtain

$$y(z) = g(s\tanh^{-1}z) + \int_{-1}^{1} K(s\tanh^{-1}z, s\tanh^{-1}t)\frac{s}{1-t^2}dt \tag{346}$$

Now, we set

$$M(z,t)=\frac{K(s\tanh^{-1}z,s\tanh^{-1}t)}{\sqrt{1-t^2}}dt$$

and

$$G(z)=g(s\tanh^{-1}z)$$

then (346) becomes:

$$y(z)=G(z)+\int_{-1}^{1}M(z,t)y(t)w(t)dt \tag{347}$$

Also, using the same method as above in algebraic map, we obtain the integral equation of the form:

$$M(z,t)=\frac{K(sz(1-z^2)^{-\frac{1}{2}},st(1-t^2)^{-\frac{1}{2}})}{1-t^2}$$

and

$$G(z)=g(sz(1-z)^{-\frac{1}{2}})$$

From the two cases, it suffices to solve the approximate solution of the mapped integral equation in the form:

$$(I-sK)u=G \tag{348}$$

In order to solve (348), we assume the operator K is compact on the space $L^2_w(-1,1)$ with P_n the $n+1-$dimensional subspace spanned by the Chebyshev polynomials $T_0,...,T_n$ which motivates us to approximate the integral equation (347) by trying to solve;

$$(I-sP_nK)u_n=P_nG, \quad U_n\in P_n \tag{349}$$

Using the Gauss-Chebyshev formula given in the theorem $(4.4.2)$ to approximate the integral part of (347) we obtain:

$$\hat{u}(z)=G(z)+\frac{s\pi}{n+1}\sum_{i=0}^{n}M(z,t)\hat{u}(t_i) \tag{350}$$

which is a semi-discrete equation that is almost exact in the manner that the residual;

$$r_n(x)=\hat{u}(z)-\frac{s\pi}{n+1}\sum_{i=0}^{n}M(z,t)\hat{u}(t_i)-G(z) \tag{351}$$

The residual $r_n(z)=0$ at the collocation points $z_k, \quad k=0,...,n$. This leads to the following system of linear equations:

$$\hat{u}(z_k)-\frac{s\pi}{n+1}\sum_{i=0}^{n}M(z_k,t_i)\hat{u}(t_i)=G(z_j) \tag{352}$$

Thus, the Chebyshev polynomial approximations is given by

$$P_n u(z) = \sum_{k=0}^{n} c_k T_i(z) \tag{353}$$

Where

$$c_0 \approx \frac{1}{n+1}\sum_{i=0}^{n} \hat{u}(z_k) \tag{354}$$

and

$$c_k \approx \frac{2}{n+1}\sum_{k=0}^{n} \hat{u}(z_i)T_i(z_k), \quad i = 0,...,n \tag{355}$$

Eventually, the approximate solution of equation (343) from the real line is given by

- Logarithmic map

$$\phi_n(x) = \sum_{k=0}^{n} c_k T_k(\tanh(s^{-1}x)) \tag{356}$$

- Algebraic map

$$\phi_n(x) = \sum_{k=0}^{n} c_k T_i(\frac{x}{\sqrt{x^2+s^2}}) \tag{357}$$

For more details and illustrative examples, see [11],[30], [31], [22]

4.4.3 Legendre Series Solution Method

Here, the solution involves the use of truncated Legendre series approximation.
Normally, we first take the truncated Legendre series expansions of the functions in the Fredholm integral equation and then substituting their matrix forms into the equation, which corresponds to a linear system of algebraic equations with unknown Legendre coefficients.
For more detailed information and illustrative examples, see [16], [31] and [17].

Consider the Fredholm integral equation of the Second Kind:

$$y(x) = g(x) + \lambda \int_{-1}^{1} K(x,t)y(t)dt, \quad -1 \leq x,t \leq 1 \tag{358}$$

Where $y(x)$ is the function to be determined λ is aconstant,while the kernel $K(x,t)$ and $g(x)$ are given continuous functions in $-1 \leq x,t \leq 1$ or just as $[-1,1]$.

The solution of equation (358) is therefore expressed as a truncated Legendre Series, i.e:

$$y(x) = \sum_{n=0}^{\infty} a_n P_n(x) \tag{359}$$

Where $P_n(x)$ is the Legendre polynomial of degree n.
The expression (359) can be expressed in its equivalent matrix form as:

$$\boldsymbol{y(x) = P_x A} \tag{360}$$

Where

$$P_x = P_0(x)P_1(x)P_2(x)...P_n(x),$$

$$A = [a_0 a_1 a_2 ... a_n]^T$$

and

$$a_n, \quad n = 0, 1, 2, ..., \infty$$

are coefficients to be determined.
We choose n such that equation (359) becomes;

$$y(x) = \sum_{n=0}^{N} a_n(x) P_n(x)$$

Procedure For the Determination of $y(x)$

In order for one to obtain $y(x)$ for equation (358) in the form of equation (359), we will first have to deduce some of the Matrix approximations corresponding to the Legendre polynomials, that is letting the known function $g(x)$ to be approximated by the truncated Legendre series as given below,

$$g(x) = \sum_{n=0}^{N} g_n P_n(x) \tag{361}$$

with its equivalent matrix form is given by;

$$\boldsymbol{[g(x)] = P_x G} \tag{362}$$

Where

$$G = [g_0 g_1 g_2 ... g_n]$$

If we then consider the Kernel $K(x,t)$ which we can approximate by double Legendre Series of degree N (because of the presence of the two variables x and t) as:

$$K(x,t) = \sum_{n=0}^{N}\sum_{n=0}^{N} K_{m,n}P_m(x)P_n(t) \tag{363}$$

Which we can also in its equivalent matrix form as :

$$[\boldsymbol{K(x,t)}] = \boldsymbol{P_x K P_t^T} \tag{364}$$

Where

$$P_t = [p_0(t)p_1(t)p_2(t)...p_n(t)]$$

and

$$K = \begin{pmatrix} K_{00} & K_{01} & \cdots & K_{0N} \\ K_{10} & K_{11} & \cdots & K_{1N} \\ . & & & \\ . & & & \\ . & & & \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{pmatrix}$$

Conversely, the unknown function $y(t)$ in the integrand, we write it from the expression (359) and (360);

$$[\boldsymbol{y(t)}] = \boldsymbol{P_t A} \tag{365}$$

When we substitute the matrix forms (360), (362), (364) and (365) which are congruent to the functions $y(x)$, $g(x)$, $K(x,t)$ and $y(t)$ respectively into equation (358) and simplify the resultant equation, we obtain the matrix equation;

$$A = G + \lambda K(\int_{-1}^{1} P_t^T P_t dt)A \tag{366}$$

OR simply as:

$$(I - \lambda KQ)A = G \tag{367}$$

Where

$$Q = \int_{-1}^{1} P_t^T P_t dt = [q_{mn}], \quad m,n = 0,1,,2,...,N \tag{368}$$

and I is the identity matrix.

$$q_{mn} = \int_{-1}^{1} P_n(t)P_s(t)dt = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

In equation (368), when

$$D(\lambda) = |I - \lambda KQ| \neq 0$$

we then obtain;

$$A = (I - \lambda KQ)^{-1}G, \quad \lambda \neq 0 \tag{369}$$

which implies that the unknown coefficients $a_n,\ n = 0,1,2,...,N$ can be determined uniquely by the equation (369) thus the integral equation (358) has a unique solution as we have already shown in the previous sections. The solution is normally given by the truncated Legendre Series.

Exactness of the Solution

We can always check if this method is exact or accurate. Because equation (359) represents the approximate solution of equation (358) which must be satisfied.
For $x_i \in [-1,1]$, then

$$\epsilon(x_i) = |y(x_i) - g(x_i) - \lambda \int_{-1}^{1} K(x,t)y(t)dt| \approxeq 0$$

or

$$\epsilon(x_i) \leq 10^{-k}, \tag{370}$$

k_i is any positive integer.

suppose $\epsilon(x_i) \leq 0, \quad k$ is any positive integer, is specified, then the truncation limit N is increased until the difference $\epsilon(x_i)$ at each of the points x_i smaller and smaller then the specified 10^{-k}.

Conversely, the error function can be determined by

$$\epsilon(x_i) = y(x) - g(x) - \lambda \int_{-1}^{1} K(x,t)y(t)dt$$

Example 4.4.3. *Approximate the following Fredholm integral equation*

$$y(x) = 3x - 5x^3 + \int_{-1}^{1} (1 - xt)y(t)dt \tag{371}$$

whose exact solution is given by;

$$y(x) = 3x - 5x^3 \tag{372}$$

We seek a solution of $y(x)$ in the Legendre Series;

$$y(x) = \sum_{n=0}^{N} a_n P_n(x) \tag{373}$$

So that:

$$\begin{aligned} f(x) &= 3x - 5x^3 \\ K(x,t) &= (1 - xt) \\ \lambda &= 1 \\ N &= 3 \end{aligned}$$

By using the expressions for the powers of x^n in terms of the Legendre Polynomials $p_n(x)$ given by;

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \tag{374}$$

Where

$$P_0(x) = 1 \implies 1 = P_0(x) \tag{375}$$

$$P_1(x) = x \implies x = P_1(x) \tag{376}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \implies x^2 = \frac{2P_2(x)}{3} + \frac{1}{3} P_0(x) \tag{377}$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \implies x^3 = \frac{2P_3(x)}{5} + \frac{3}{5} P_1(x) \tag{378}$$

Using the substitutions above in (371) we obtain

$$\begin{aligned} f(x) = 3x - 5x^3 = 3P_1(x) - 5(\frac{2P_3(x)}{5} + \frac{3}{5} P_1(x)) &= -2P_3(x) \\ K(x,t) = (1 - xt) &= P_0(x)P_0(t) - P_1(x)P_1(t) \end{aligned}$$

Therefore, from the expressions (362) and (364) we obtain the matrices:

$$F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} \tag{379}$$

and

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{380}$$

Which is given by the coefficients of

$$K = \begin{bmatrix} P_0(x)P_0(t) & P_0(x)P_1(t) & P_0(x)P_2(t) & P_0(x)P_3(t) \\ P_1(x)P_0(t) & P_1(x)P_1(t) & P_1(x)P_2(t) & P_1(x)P_3(t) \\ P_2(x)P_0(t) & P_2(x)P_1(t) & P_2(x)P_2(t) & P_2(x)P_3(t) \\ P_3(x)P_0(t) & P_3(x)P_1(t) & P_3(x)P_2(t) & P_3(x)P_3(t) \end{bmatrix} \tag{381}$$

Now, using the expression (368) *where*

$$Q = \int_{-1}^{1} P_t^T P_t dt = \begin{bmatrix} q_{mn} \end{bmatrix} \quad m, n = 0, 1, 2, 3$$

Given by

$$\begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} \begin{bmatrix} P_0(t) & P_1(t) & P_2(t) & P_3(t) \end{bmatrix} \tag{382}$$

Whose recurrence relation is given by:

$$P_n(x) = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

Yielding a diagonalized matrix Q*,*

$$Q = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & \frac{2}{7} \end{bmatrix} \tag{383}$$

From (367) *and* (369) *we have*

$$\left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}-\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}2&0&0&0\\0&\frac{2}{3}&0&0\\0&0&\frac{2}{5}&0\\0&0&0&\frac{2}{7}\end{bmatrix}\right)\begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\-2\end{bmatrix} \tag{384}$$

Or equivalently as

$$\begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix}=\left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}-\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}2&0&0&0\\0&\frac{2}{3}&0&0\\0&0&\frac{2}{5}&0\\0&0&0&\frac{2}{7}\end{bmatrix}\right)^{-1}\begin{bmatrix}0\\0\\0\\-2\end{bmatrix} \tag{385}$$

And from the expression (369) *we obtain*

$$\begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}-1&0&0&0\\0&-6&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\0\\-2\end{bmatrix}$$

and clearly,

$$\begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\-2\end{bmatrix} \tag{386}$$

Thus, using (359) *we have*

$$y(x)=a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+a_3P_3(x) \tag{387}$$

and from (386), (387) *becomes*

$$y(x)=-2P_3(x) \tag{388}$$

where $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Giving

$$y(x) = -2\left(\tfrac{1}{2}(5x^3 - 3x)\right) \tag{389}$$

Hence,

$$y(x) = 3x - 5x^3$$

which is the same as the exact solution obtained analytically.

Example 4.4.4. *We want to approximate the Fredholm integral equation below using Legendre Series Solution Method,*

$$y(x) = 2x + \frac{1}{2}\int_{-1}^{1} xty(t)dt \tag{390}$$

We seek a solution $y(x)$ *in the form of Legendre Series*

$$y(x) = \sum_{n=0}^{N} a_n P_n(x)$$

so that

$$f(x) = 2x, \quad K(x,t) = xt, \quad \lambda = \frac{1}{2} \quad \text{and} \quad N = 3$$

By using the expression for the powers of x^n *in terms of the Legendre Polynomials* $P_n(x)$ *given by*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

$$\begin{aligned}
P_0(x) = 1 &\implies 1 = P_0(x) \\
P_1(x) = x &\implies x = P_1(x) \\
P_2(x) = \frac{1}{2}(3x^2 - 1) &\implies x^2 = \frac{2P_2(x)}{3} + \frac{1}{3}P_0(x) \\
P_3(x) = \frac{1}{2}(5x^3 - 3x) &\implies x^3 = \frac{2P_3(x)}{5} + \frac{3}{5}P_1(x)
\end{aligned}$$

From the expressions (362) *and* (363), *the matrices become*

$$F = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad and \quad K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, using the expression (367)

$$Q = \int_{-1}^{1} P_t^T P_t dt = \left[q_{mn}\right], \quad m,n = 0,1,2,3$$

$$= \begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} \begin{bmatrix} P_0(t) & P_1(t) & P_2(t) & P_3(t) \end{bmatrix}$$

From

$$P_n(x) = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

$$Q = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & \frac{2}{7} \end{bmatrix}$$

But we know that

$$A = \left(I - \lambda KQ\right)^{-1} F$$

Therefore

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \\ 0 \\ 0 \end{bmatrix}$$

$$a_0 = 0$$
$$a_1 = \frac{3}{2}$$
$$a_2 = 0$$
$$a_3 = 0$$

Applying

$$y(x) = \sum_{n=0}^{3} a_n P_n(x)$$

gives

$$y(x) = \frac{3}{2} P_1(x)$$

$$y(x) = \frac{3}{2} x$$

Which is not the same as the exact solution as in the case of the standard decomposition method where we found the exact solution as

$$y(x) = 3x$$

Example 4.4.5. *Let us consider the problem*

$$y(x) = e^{2x} - \frac{3x}{4e^2} - \frac{1}{4} e^2 x \int_{-1}^{1} xty(t)dt \tag{391}$$

Using the same same procedures from the two examples above, we want to analyze the errors for $N = 9, 10$

From figure 2, we can easily conclude that the larger the N *the more the accuracy hence in order to obtain the most approximate solution then* N *should be sufficiently large since the margin of error is minimal.*

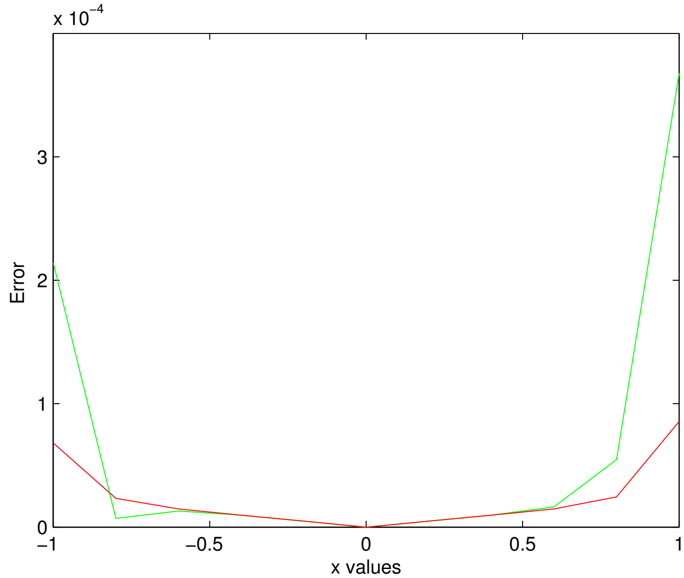

Figure 2. Numerical results for N=9,10

5 Conclusion

In this thesis we have presented some analytical and numerical methods for solving the Fredholm integral equation of the Second Kind. The analytical methods presented here are: degenerate kernel methods, converting Fredholm integral equation to Ordinary Differential Equations, the Adomain decomposition method, the modified decomposition method, variational iteration method, noise terms methods, direct computation method and successive approximation methods. Furthermore, we have used numerical methods: projection methods including collocation method and Galerkin method, degenerate kernel approximation methods and Nystr$\ddot{o}$m methods for solving Fredholm integral equation of the second kind. Additionally, we have also approximated the Fredholm integral equation of the Second Kind with the numerical solution using mapped Chebyshev polynomials and Legendre Series methods. The importance of the Legendre series method presented for the approximate solution of Fredholm integral equation has been demonstrated. The advantage of the method is that the solution is expressed as a truncated Legendre series which means that after calculation of the Legendre coefficients, the solution $y(x)$ can be easily evaluated for arbitrary values of x at low computation effort. If the functions $f(x)$ and $K(x,t)$ can be expanded to the Legendre series in $-1 \leq x,t \leq 1$, then there exists the solution $y(x)$; else, the method cannot be used in. On the other hand, it would appear that the method shows to best advantage when the known function $f(x)$ and $K(x,t)$ have Taylor series especially when they converge slowly. For computational efficiency the truncation limit N must be chosen sufficiently larger.

5.1 Future Research

Due to time constraints, there are many areas in which we intended to extend this work. One of the areas include:

1. Applications of Integral Equations in Aerodynamics and Geophysics.

Matlab Codes

Listing 5.1. Matlab codes

```
clc,clear all,close all
E_9=[2.14066E-04,7.043E-06,1.2992E-05,9.6585E-06,4.843E-06,0,4.843E-06,9.717E-06,1.6442E-05,5.4748E-05,0.000367792];
E_10=[6.81207E-05,2.32566E-05,1.46982E-05,9.6881E-06,4.8431E-06,0,4.843E-06,9.688E-06,1.4736E-05,2.4448E-05,8.5605E-05];
x=[-1,-0.8,-0.6,-0.4,-0.2,0,0.2,0.4,0.6,0.8,1];
plot(x,E_9,'g')
hold on
plot(x,E_10,'r')
xlabel('x values')
ylabel ('Error ')
```

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