University of Nairobi

School of Mathematics

ISSN: 2410-1397

Master Project in Applied Mathematics

Numerical Solutions for Partial Differential Equations of Conservation Laws

Research Report in Mathematics, 06 , 2021

James Wamwenge Wangui

June 2021

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

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Master Thesis

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

The goal of this project, is to come up with a sufficient background so that we can approach the current literature of research possessing the necessary tools and detailed understanding. We will concern ourselves with analyzing the discontinuous galerkin method (DGM) by looking at its background and formulation. We will deal with the theory of mathematical outlook of these equations first and then solutions. This will cause us to emphasize more on the tools of mathematics that are very important in the of development, analyzing and successful utilization of the finite difference method for the non-linear systems of conservation laws, in particular for problems involving Shallow Water Equations. The derivation of these equations will be provided. Also the shallow water equations will be given in both conservative and non-conservative form. The main type of method used in the approximation of differential equations of this kind will be given i.e the finite difference method. We will later formulate the solutions to the shallow water equation in MATLAB.

Master Thesis in Mathematics at the University of Nairobi, Kenya.
ISSN 2410-1397: Research Report in Mathematics

DISTRIBUTOR: School of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

19/7/2021

Signature Date

WAMWENGE JAMES WANGUI
Reg No. I56/34291/2019

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

19/07/2021

Signature Date

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Dedication

This project is dedicated to my father Peter Kinyanjui Chege and mother Lydia Wangui Chege, for emotional and spiritual support in the time of dire need. To my aunt Elizabeth Chege for standing with me during tough financial times. To my brothers Simon mwangi (sr), Samwel Chege, Joel Njuguna, Simon Mwangi(jr), Noah Wangai and Joseph Ngugi. And finally to my fiancee Victoria Wangari for reminding me of all the beautiful things in life and been there during the development of this project. May God bless you.

Contents

Acknowledgments

First and foremost I would like to appreciate the Lord God almighty for granting me the serenity to be, for providence, health and all His immeasurable kindness. Secondly I thank my supervisor Dr Victor Ogesa Juma for assisting me initiate this project, investing his time and resources to provide guidance all through the project in the midst of the Covid-19 pandemic.

I also thank my friend Abuor for providing insight on Latex, my colleges for moral support and ideas. Special thanks to the great University of Nairobi through the School of Mathematics for granting me a scholarship to do my Masters degree without which my dreams would have been shattered "AHSANTE".

James Wamwenge Wangui

Nairobi, 2021.

1 Introduction

The world we live in is in nature a very complicated one. Quit often we can not explain what is happening or perhaps how it happened. But by the help of mathematics we can predict the future occurrences, one is because mathematics is cheaper since instead of running experiments everyday we use mathematical equations to predict the future. Secondly this problems are so complex in nature such that when we relate them to partial difference equations (PDEs) we more often than not have non-linearities or complications coming from the pdes themselves or the from the boundaries that is when we have complicated boundaries [1]. For that reason its is evident we can not solve them analytically and that's why we need numerical solutions, that means we try to get approximate solutions as accurate as possible.

Figure 1. An image showing different world problems that can be related to pdes.

In Figure (1), the first image shows the Solai tragedy that occurred in Kenya. The second one shows a traffic situation in an intersection. The third one simulates a bullet in motion, while the last shows a linear accelerator used the to treat cancer.

1.1 Numerical methods

Numerical methods are procedures or techniques that are used to approximate mathematical processes (an example of a mathematical process is an integral). They are schemes that are applied to find an approximation when analytical answers are either absent or impossible to find. Numerical methods for all partial differential equations (PDEs) is made up of two parts: First a space discretisation that transforms the system of PDEs into a system of ODEs and the second is a time discretisation that transforms the system of ODEs into a series of algebraic equations that can be solved using techniques of linear algebra [2].

When modeling the physical world, conservation becomes a fundamental principle to be applied which stretches from quantum mechanics, continuum mechanics to gravitational physics, having prominent examples that include the famous Navier's equations of elasticity, Euler equations of gas dynamics, Einstein's equations of gravitation and Maxwell's equations of electromagnetics.

It therefore does not come as a surprise why there is deep research in the area of mathematical analysis of conservation laws whose stretches back to the Leibniz. The process that involve developing accurate and efficient computational methods used to solve such problems have taken the center stage of scientific pioneers including von Neumann, Lax, and Lions. However, even if a problem may appear very simple, a detail investigation reveals serious hurdles when we attempt to understand the most basic conservation law together with the nature of the solutions.

1.1.1 Why are numerical methods necessary?

When we find an exact solution to a problem and in a shorter time than forever, then we say that the problem is "analytically solvable". For instance, "Victor has 2 mangoes while James has 3 mangoes, how many mangoes do they have together?" is solvable analytically. It's 5. $Exactly$ 5.

Precisely solving such problems is what we often think that mathematicians are busy doing, however more often than not scientists stumble on problems that can't be solved analytically. In real sense "mathematics" widely involves finding **an answer** rather than **the answer**. While it could be hard to find the exact answer, we can try to find answers which are "arbitrary precise". This is like saying, you can determine an approximate solution and the more computer time you are willing to invest, the closer that approximate answer will be to the correct answer. This trick is called the "numerical method". Numerical methods are rather strange: they determine solutions close to the exact answer without ever knowing what that answer is in the first place. So that there is actually an

answer: we require numerical methods because majority of problems are not solvable analytically and we know that these methods work because each one of them method comes with a proof that it works.

The choice of a particular numerical method to use in solving a certain problem, depends on the ease in applying it to that particular problem, the ability of the method to produce accurate results when compared to other numerical methods and the consistency of the numerical methods.

1.2 Conservation laws

Systems of hyperbolic conservation consists of nonlinear and time dependent systems of pdes with simple structure. In one-dimensional space the equations are of the form

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}fu(x,t)=0. \tag{1}$$

Here u is a vector of m-dimension and represent quantities which are conserved state variables for example momentum, energy dynamics and mass problem. Even more precisely, u_j represents the state variables density function at position j. The explanation is that at time t the integral $\int_{x_1}^{x_2} u_j(x,t)dx$ equals to all the quantity in the interval (x_1,x_2) of state variable.

When we say that we are conserving the state variables we are trying to say that the integral $\int_{-\infty}^{\infty} u_j(x,t)dx$ is supposed to be fixed respecting t.These functions u_j themselves at time t which stand for how the state variables are distributed changes with time. We are assuming that for equation (1) when we know $u(x,t)$ at a certain points and time enables determination of the flux, of every state variable (x,t).

The functions of the flux are typically functions of u, which are nonlinear that leads to systems (nonlinear) of PDEs. Generally it's impossible to determine the actual solutions to the equations, therefore the necessity to formulate and discuss numerical schemes for getting those solutions which are approximate.

As a example of equations we are looking at in our entire investigations, we look at a conservation law, typically written as

$$\partial u/\partial t+\partial f(u)/\partial x=0, \quad (0,1)\times(0,T) \tag{2}$$

together wit initial condition

$$u(x,0) = u_0(x) \tag{3}$$

subject to some periodic boundary conditions. Suppose that the partition of $(0,1)$ is given by $\{I_j\}_{j=1}^{N}$, with

$$\triangle_j = x_{j+1/2} - x_{j-1/2},$$

to find the element's center we use

$$x_j = \frac{(x_{j-1/2} + x_{j+1/2}}{2},$$

if (2) and (15) are multiplied by an arbitrary function say $v(x)$ then integrating by parts over I_j, we obtain the weak problem statement, that is

$$\int_{I_j} \partial_t uv dx = \int_{I_j} f(u)v' dx + f[u(x^+_{j-\frac{1}{2}},t)]v(x^+_{j-\frac{1}{2}},t) - f[u(x^-_{j+\frac{1}{2}},t)]v(x^-_{j+\frac{1}{2}},t) \tag{4}$$

$$\int_{I_j} u(x,0)v(x)dx = \int_{I_j} u_0(xv(x)dx \tag{5}$$

where $x^+_{j-\frac{1}{2}}$ and $x^-_{j-\frac{1}{2}}$ represents the limit from the right and the left respectively.

The problem statement is that we find a solution u_h to u for every $t \in (0,T)$, with $v(x)$ and $u_h(x,t)$ belonging to semi-finite dimesion space

$$V_h^K = \{v \in L^1(0,1) : v|_{I_j} \in P^K(I_j), \quad j = 1,2,...,N\} \tag{6}$$

This means that, in the space dimension V_h^K, u_h and v are polynomials of degree K. Since u_h and v are both discontinuos at $x^+_{j-\frac{1}{2}}$ and $x^-_{j-\frac{1}{2}}$, then the ambiguity in the terms of (4) which involves the non-linear fuxes $f[u(x_{j+\frac{1}{2}},t)]$ and $f[u(x_{j-\frac{1}{2}},t)]$ should be substituted by numerical fuxes which entirely depends on u_h at $x_{j+\frac{1}{2}}$ and $x_{j-\frac{1}{2}}$ I.e

$$\bar{f}_{j+\frac{1}{2}} = \bar{f}(u_h|^-_{j+\frac{1}{2}}, u_h|^+_{j+\frac{1}{2}}), \quad \bar{f}_{j-\frac{1}{2}} = \bar{f}(u_h|^-_{j-\frac{1}{2}}, u_h|^+_{j-\frac{1}{2}}) \tag{7}$$

which is yet to be chosen. Therefore if we use Legendre's polynomials P_m to expand the approximation u_h in terms of basis functions

$$u_h|I_j = \sum_{m=0}^{K} u_{mj}(t) P_m(\frac{2(x-x_j)}{\triangle_j}), \quad \forall j = 1,2,...,N, \tag{8}$$

Then the test functions v_h are considered to be equal to the functions of the basis, that is, $v(x) = \{P_m\}_{m=0}^K$,[3], wheres at the same time we invoke the orthogonality property of Legendre's polynomials

$$\int_{-1}^{1} P_m(\psi) P_l(\psi) d\psi = \frac{2}{2l+1} \delta_{ml} \tag{9}$$

where

$$\psi = \frac{2(x-x_j)}{\triangle_j}$$

δ_{ml}= $\begin{cases} 1, & \text{m=l} \\ 0, & \text{otherwise} \end{cases}$

the weak form becomes

$$\frac{du_{lj}(t)}{dt} = \frac{2l+1}{\triangle_j} \int_{-1}^{1} f[u_h(\psi,t)] P_l'(\psi) d\psi + \frac{2l+1}{\triangle_j} (-1)^l f^+_{j-1/2} - f^-_{j+1/2}$$

$$u_{lj}(0) = \frac{2l+1}{2} \int_{-1}^{1} u_0(\psi) P_l(\psi) d\psi, \quad \forall l and j$$

Notice that we apply the property

$$P_l(-1) = (-1)^l, \quad P_l(1) = 1$$

1.2.1 Why study Conservation laws

There are multiple reasons as to why we consider this particular group of equations alone:

1. Majority of problems in engineering and science involve conservation and eventually ends to partial differential equations of this group.
2. There exists technicalities that come with finding solutions to these systems which don't exist any other place and has to looked at with precision when coming up with numerical schemes. Methods concerned with basic FD approximations might be okay with finer solutions however can produce horribly wrong outcomes when there is presence of discontinuities.
3. Even though only a handful of exact solutions are known, there in so much information about the mathematical nature of these problems including their answers. Exploiting this facts helps come up with better methods that beat some of the numerical difficulties experienced with more basic approaches [4].

1.3 Early Numerical methods for Conservation laws

One of the earliest development in an attempt to solve conservation laws was the Discontinuous Galerkin (DG) method. DG methods were originally developed around 1970s [5] as a way of solving partial differential equations numerically [4] . In 1973 Reed and Hill [6] discovered a DG method to solve the NEUTRON TRANSPORT EQUATION (NTE) equation.

$$\sigma u + \nabla(au) = f \tag{10}$$

where $\sigma, a(x) \in \mathbb{R}$ and u is to be found

This equation has roots going back more than a century ago to the Boltzamann's equation [7].

$$\frac{N_j}{N} = \frac{\bar{w}_j e^{\frac{-E_j}{kT}}}{\sum \bar{w}_j e^{\frac{-E_i}{kT}}}$$

which was formulated initially to facilitate the study of kinetic theory of gases. Further study of radiation transport in atmosphere lead to several analytical Solutions the transport problems in early 1930s. The physics surrounding these problems, however, caused a

Figure 2. WH Reed

Figure 3. TR Hill

confinement in interest to one dimensional semi-infinite medium geometries. During the advent of nuclear chain reactors in 1940s there became interest involving neutral particle transport problems in a wider range of configurations in geometry which were found in the applications of nuclear reactor and radiation shielding. A number of good analytical methods to solve the transport problems who pursued since the 1940s. The Weiner-Hopf method [8], singular eigenfunction expansions among other analytical methods provided a great deal of input into the general nature of the transport processes through the study of greatly idealized configurations, which included the Milne problem.

At the same time there was an increase in sophisticated numerical methods which began being developed, concurrent with the rapidly increasing use of digital computers as a

computational power. Among these numerical methods the discontinuous galerkin was conceived.

Until their most recent development these method made it's way into the heart of computational fluid dynamics where they found use in a wide variety of applications including but has not been limited to nonlinear conservation laws, the compressible Navier-stokes equation and Hamilton Jacobi-like equations.

This quest, however, is far from trivial simply because of two main reasons

1. The first is that the exact solution of (nonlinear) purely convective problems develops discontinuities in finite time;
2. the second is that these solutions might display a very rich and sophisticated structure near such discontinuities.

Thus,when constructing numerical methods for these problems, it must be guaranteed that the discontinuities of the approximate solution are those which are physically relevant. In addition, it must be ensured that the appearance of a discontinuity in the approximated solution does not bring about sparious oscillations that tamper with the quality of the approximation;on the other hand, when ensuring this, the method must remain accurate enough near that discontinuity in order to capture the possibly rich structure of the exact solution.

Owing to their finite element nature, the DG methods have the following main advantages over other classical finite volume and finite difference methods:

1. The real order of accuracy of Discontinuous Galerkin methods entirely depends on the exact solution; DG methods of arbitrarily high formal order of accuracy can be obtained by choosing a suitable degree of approximating polynomials.
2. DG methods are highly parallelizable. since the elements are discontinuous,the mass matrix is block diagonal and since the blocks and the number of degrees of freedom are equal inside the corresponding elements, the blocks can be inverted (by using a symbolic manipulation or by hand) once and for all.
3. DG handles adaptive techniques as polishing of the grid can be achieved without taking considering the continuity restriction typical of conforming finite element methods. Moreover, the degree of approximately polynomial can be easily Transformed

from one element to the other without losing generality. Adaptivity is of much importance in hyperbolic problems considering the complexity of the structure of the discontinuities.

1.3.1 The original DG method for the neutron transport problem

The original finite element method was introduced in 1973 by Reed and Hill for solving the neutron transport equation (10).The usefullness of the method was recognised by LeSaint and Raviart who in 1974 published its first mathematical analysis. To display the method, we multiply the equation by a test function v and integrate over an arbitrary subset of Ω say,K. After a formal integration by parts, we get

$$\sigma(u,v)_K - (u, a.\nabla v)_K + (a.n_k u, v)_{dK} = (f,v)_K$$

where n_k denotes the outward unit normal of dK, and

$$(u,v)_K = \int_K uvdx$$

,

$$(w,v)_{dK} = \int_{dK} wvds.$$

Next, we construct a triangulation $\tau_h = K$ of Ω, and take our approximate solution u_h to be a polynomial of degree at most k on each elements of the triangulation. The approximate solution u_h is then determined as the unique solution of the following weak formulation:

$$\forall k \in \tau_h :$$

$$\sigma(u_h,v)_K - (u_h, a.\nabla v)_K) + (h,v)_{dK} = (f,v)_K,\ \forall v \in P^k(K),$$

where $P^k(K)$ denotes the space of polynomials of degree at most k on the element Ka and h on the numerical flux given by

$$h(x) = a.n_K(x) \lim_{s\to 0} u_h(x - sa).$$

Note that the value $\lim_{s\to 0}(x - sa)$ is nothing but the value of u_h upstream the characteristic direction a. As a consequence the degree of freedom of the approximate solution u_h in the element K can be computed in terms of the values of u_h upstream the characteristics hitting dK. In other words, the approximate solution u_h can be computed element by element when the elements are suitably ordered according to the characteristic direction a.

1.3.2 The DG method for ODEs

The first analysis of the DG method as applied to ordinary differential equations, was performed in 1974 by LeSaint and Raviart [9] who showed that the method is strongly stable of order $2k+1$ at mesh points, and that the Gauss-Radau discretization of the DG method is also of order $2k+1$ when polynomials (piecewise) of degree k are applied. It is rather interesting that about a year later before the introduction of the DG method by Reed and Hill, another mathematician, Hulme had studied a method for ordinary differential equations which word is same week formulation as the DG method but employed a continuous approximate solution u_h; this method is, however,of order k at mesh points.

A study of global error control for ordinary differential equations for DG method was done in 1994 by Estep and French. Another work on DG methods for ODEs was carried out in 1981 by Delfour, Hager and Trochu; they introduced a class of DG methods which are proven to give an order of accuracy up to $2k+2$ at the mesh points. Recently, Schotzau and Schwab have obtained a new estimate on the size of time step needed to solve the implicit system of equations determined by the DG method by way of a simple fixed point iteration method.

In 1988,Johnson [10] gave an analysis of error control for the DG method for harder ODEs. And lastly, in 1996, Bottcher and Rannacher introduced a new global error control method for ODEs by using the DG method.

With a variety of well-tested and successfully applied methods, one cannot help but ask why there is any need to consider another method. To embrace this, let us take off by trying to understand the strengths and weaknesses of the standard methods. we will consider how one-dimensional conservation law for the solution $u(x,t)$

$$\frac{\partial u}{\partial t}+\frac{\partial f}{\partial x}=h, \quad x\in\Omega \tag{11}$$

corresponding to a set of boundary and initial condition on, $\partial\Omega$. Where f = flux and $h(x,t)$ is a function which has been prescribed. To come up with any numerical method that solves a pde calls for consideration of two factors

1. How to represent u using the approximation u_h?

2. In which manner does this approximation get to fit the Partial Differential Equation (PDE)?

This draws the line between separate methods giving different properties.

Consider a very simple method used over the years and perhaps for the oldest time, called the FDM. Here, x^k, is given in space and using difference methods we approximate derivatives; i.e,

$$\frac{du_g(x^k,t)}{dt} + \frac{f_g(x^{k+1},t) - f_g(x^{k-1},t)}{g^k + g^{k-1}} = h(x^k,t), \tag{12}$$

where u_g is solution and f_g is flux numerically approximated, wheres

$$g^k = x^{k+1} - x^k,$$

represents size of grid. To construct a FDM we need, within the surrounding of every point x^k, the flux and its soution are therefore considered as approximated numerically using polynomials.

$$x \in [k-1, k+1] : u_g(x,t) = \sum_{i=0}^{2} a_i(t)(x-x^k)^i, f_g(x,t) = \sum_{i=0}^{2} b_i(t)(x-x^k)^i,$$

wheres $a_i(t)$ and $b_i(t)$ are calculated ensuring the approximated function exists at points , x^k. putting this approximation in equation (11), gives the residual

$$x \in [x^{K-1}, x^{k+1}] : R_h(x,t) = \frac{\partial u_h}{\partial t} + \frac{\partial f_h}{\partial x} - g(x,t).$$

Clearly,

$$R_h(x,t) \neq 0,$$

In the table below we give the summary of methods. By a look at it one should remember that this comparison only look at some basic problems and that many of those problems which have been addressed and restrictions can be rectified and overcome in a variety of ways. In addition, the comparison gives an insight of which shortcoming one should struggle to resolve when trying to come up with a new method.

Method	FDM	FVM	FEM	DG-FEM
Complex Geometries	×	✓	✓	✓
h_p-adaptivity and High-order accuracy	✓	×	✓	✓
Explicit semi-discrete form	✓	✓	×	✓
Conservation laws	✓	✓	(✓)	✓
Elliptic problems	✓	(✓)	✓	✓

Table 1. Generic properties of methods used in discretizing partial differential equations

On the table a ✓ shows success, while × shows that there is a shortcoming in that method. finally, (✓) shows that the method, when subjected to modifications, can be used to solve search problems but it should not the recommended for as a natural choice.

1.3.3 The Standard Galerkin method for the heat equation

We will briefly look into the standard Galerkin finite element method for the approximate solution of initial-boundary value problem for the heat equation (note that the method is equally applicable to similar equations),

$$u_t - \Delta u = f$$

in Ω, for $t > O$

$u = 0$ on $d\Omega$, for $t > 0$ with $u(.,0) = v$

where Ω is a domain in the set of reals with smooth boundary $d\Omega$, and where $u = u(x,t)$, u_t denotes du/dt, and

$$\Delta = \sum_{j=1}^{d} \frac{d^2}{dx_j^2}$$

the Laplacian. Before we proceed with the discussion of this problem we need to revise on some basic material around the finite element method for its corresponding stationary problem, the Dirichlet problem for Poisson's equation,

$$-\Delta u = f$$

in Ω, with $u = 0$, on $d\Omega$. using variational formulation of this problem, we shall define an approximation of the solution u.as a function u_h which belongs to a finite dimensional linear space S_h of functions of x with certain properties. This function, is the simplest case a continuous, piecewise linear function on some partition of Ω, will be a solution of finance system of linear algebra equation. We show basic error estimates for this approximate solution in energy at least square norms. Looking back at the parabolic problem which we first write in a weak form, we continue to discretize this problem, first in the special variable x, which gate is result in an approximate solution $u_h(.,t)$ in the finite element space S_h, for $t \geq 0$, as a solution of an initial value problem for a finite-dimensional system of ordinary differential equations. We then define the fully discrete approximation by application of some finite difference time stepping method to this finite dimensional initial value problem. This yields an approximate solution $U = U_h$ which belongs to S_h at discrete-time levels. Errors estimates will be derived for both the special and fully discrete solutions.

1.4 Objectives

1. To study and understand previous works done on numerical methods for partial differential equations of conservation laws.
2. To study and understand the finite difference method (FDM) including its formulation and application.
3. To study and understand the shallow water equations and successfully apply FDM.

1.5 Outline

The outline of the thesis is as follows:

Chapter 2: Having research on the earlier numerical methods for PDEs, this chapter will focus on the strides made towards achieving the ultimate goal. The discontinuous galerkin method will be considered in particular where we will look at how the method has been applied in attempts to solve nonlinear equations like the Parabolic, Maxwell's and Visco-elastic flows in the recent past. We will briefly present other original numerical methods and describe their theoretical and computational developments in the framework of linear hyperbolic systems.

In addition we will review the application of the Finite Difference method in various equations including but not limited to the Viscid Shallow Water equations and the Isothermal Euler equations.

Chapter 3: In this chapter we will discuss the finite difference method in details, it's development will be considered. We will derive the and shallow water equations in non-conservative and conservative form and then using finite difference method we will solve it.

Chapter 4: For better understanding of how the finite difference method works we will run simulations of the results in chapter 3 using the MATLAB software before concluding our study in chapter 5.

2 Literature Review

Although the original DG method has been known since 1973, it was only recent that DG method have evolved in such a way that made them suitable for use in computational fluid dynamics and the aforementioned applications.

2.1 Early applications of the Discontinuous Galerkin method

Apart from the application of the Discontinuous galerkin method to the neutron transport equation and the ODEs, further applications of this method to analyze wave propagation within elastic medium was done between 1975 and 1977 by Oden and Wellford [8], it was also applied to optimal control in 1978 by Delfour and Trochu [11]. Other applications include

2.1.1 Time discretization of parabolic equations

In 1978, Jamet [12] applied the DG method to disctretize parabolic equations in time where he showed that the method was of order k. From there other authors have studied the method. In 1985, K. Eriksson, C. Johnson and V.Thomee [12] proved that the method was of order $2k+1$ at the nodes. Later Erikson and Johnson did further studies where they looked at the issue of error control in a series of articles starting from 1987 and completing it in 1995. In 1997, Babuska and Makridakis [13] looked at the effects of adaptive mechanisms on stability of the method.

2.1.2 Visco-elastic flows

For the first time in 1989 the DG method by Reed and Hill was applied on numerical computation of viscoelastic flows by Fortin [14]. Basically he wanted to apply the DG method to itutive laws which involved relating the extra-stress tensor in terms of velocity. Fortin together with other mathematicians revisited the development of the idea while Baaijiens, Bogaerds and Verbeeten [15] studied the failures and the successes of the methods in viscoelastic fluid analysis. A very recent application of the DG method to this problem was pursued by Sun, Smith, Armstrong and brown [16] in 1998. Mathematical analysis on these methods have been conducted in 1992 by Baranger and Sandri[17], in 1995 by Baranger and Wardi[18], in 1997 by Baranger and MachMoum [19] and in 1998 by Baranger, Bahhar and Sandri [20].

2.1.3 DG method for Maxwell's equations

The Maxwell's equations represent one of the most concise and elegant ways to state the fundamentals of magnetism and electricity. From the Maxwell's equations it has become possible to develop majority of the working relationships in the field of partial differential equations. Due to the concise nature, they cover a high level of mathematical sophistication.as a new development, the DG method has found use in this equations. The equations of magneto-hydrodynamics, which includes the Maxwell's equations, has been discretized by DG methods.

Apart from the application of the Discontinuous galerkin method to the neutron transport equation and the ODEs, further applications of this method to analyze wave propagation within elastic medium was done between 1975 and 1977 by Oden and Wellford [8], it was also applied to optimal control in 1978 by Delfour and Trochu [11]

2.2 Preliminaries

In this section we discuss some Numerical methods which have over the years been used in finding the solutions of Partial Differential equations.

2.2.1 Finite Volume Method

Recall that in the solution of linear problem [21] using the finite element method we normally solve simultaneous algebraic equations.

$$Ka = f$$

If the matrix of coefficients is non singular then we have a unique solution. For problems which are not linear we require to get a collection of algebraic equations; but those are the brightly questions will always be non-linear, and they are indicated as

$$\phi(a) = f - P(a) = 0$$

Here a is a collection of parameters used for discretization, f is a vector that does not depend on a while P depends on a. There can be multiple solutions for this equations . This means that if we get a solution that solution may or may not be the solution we are looking for. To achieve realistic answers we need to take physical look at the structure of the equations and run an algorithm that increases stepwise from the know solution. Such increments are fundamental if the constitutive law that relates changes in both strain and stress dependent on the route or if there are bifurcations in the path or different branches on some stages of the load .
The problem needs to be put miniature format as the solution of

$$\phi_{n+1} = \phi(a_{n+1}) = f_{n+1} - P(a_{n+1}) = 0$$

which starts off at a nearby solution

$$a = a_n, \phi_n = 0, f = f_n$$

and is often brought about by the variations in the function f_k to

$$f_{k+1} = f_k + \triangle f_k$$

The process of determining the change $\triangle a_n$ such that

$$a_{n+1} = a_n + \triangle a_n$$

is the main objective and generally the increments $\triangle f_n$ are kept reasonably small so that the dependence of the path is followed. Nevertheless, those incremental procedures becomes useful in an attempt to have minimal number of iterations and also when following the correct path physically. Apparently following the path $\triangle f_n$ will have both positive and negative as shown in Figure (**??**). Here if the function ϕ decreases it may cause a typical non-uniqueness and consequently increase as a increases. In a single increment of f we can obtain solutions in small non-linearity and independence on the path, i.e, with

$$f_k = 0,$$

$$\triangle f_k = f_{k+1} = f$$

2.2.2 Finite Element Method

For the numerical solutions for PDEs using the finite different methods, the differential system is replaced by the metric system A, which is usually square and has real elements. The linear equations system inrequiring the solution is

$$\sum_{j=1}^{n} a_{ij}x_j = b_i(i = 1,2,3,...,n)$$

which can be written as a matrix system

$$Ax = b$$

Where A is a matrix with rows and columns and the elements

$$a_{ij}$$

are all real numbers. The vectors b and x have any components. As usual the problem involves finding x where A and b are given. We get a unique solution which can be written in the form.

$$x = A^{-1}b$$

here A is non-singular, which is same as saying A has non-vanishing determinant. Since the matrix is just aa reprentation of a differential system, then A is sparse and contains a definite structure (given by the non-zero elements). The method that involves inventing A, in particular when the order the matrix is large (ie order n), is dependent largely on the structured A. Numerous techniques of inverting $A$are available. As n becomes larger and larger, the methods of iniversion becomes more and more efficient.

2.2.3 Parabolic Equations

Many problems that require numerical solution in physics and engineering involve special cases of linear parabolic partial differential equation given by

$$\alpha\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}a\frac{\partial u}{\partial x)} + b\frac{\partial u}{\partial x)} - cu \tag{13}$$

Which is only valid within some region R of the (x,t) space. Inside this region, the functions, α and a are strictly positive wheres c is non-negative,

The region is usually one of the three types given in the figures

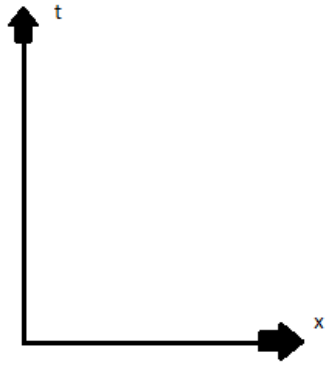

Figure 4. Semi-infinite plane

$$[-\infty < x < +\infty] \times [t \geq 0]$$

.

This region leads to an initial value problem called the **Cauchy problem** which consists of equation (13) together with initial condition given by

$$u = f(x),$$

$$(-\infty < x < +\infty)$$

$$\text{at } t = 0$$

2.2.4 Numerical Fluxes

For the problem to be defined completely, the remaining task is to choose the flux function (or the flow). According to Cockburn [22], we construct schemes which are perturbations of what is called the monotone schemes. By carrying out the perturbations on the monotone schemes, we can attain better accuracy which maintainthe properties of both instabilty and convergence. As a result, when discretization of space is constant, $K = 0$, the scheme of integration has to be monotonous

$$\frac{du_{0j}(t)}{dt} = \frac{\bar{f}_{j+1/2} - \bar{f}_{j-1/2}}{2}, \quad \forall j = 1, ..., N$$

$$u_{0j}(0) \frac{1}{2} \int_{-1}^{1} u_0(\psi) d\psi$$

This is the definition of a monotonous scheme if $\bar{f}(a,b)$ is a monotone flux, consistent and Lipschitz continuous. This means that

1. $\bar{f}(a,b)$ is locally Lipschitz, and with consistency $f(u)$ flux i.e $\bar{f}(u,u) = f(u)$
2. $\bar{f}(a,b)$ is a nondecreasing function of its frst argument
3. $\bar{f}(a,b)$ is a nonincreas- ing function of its second argument.

An example of a numerical flux is the Lax-Friedrichs [23] $\bar{f}^{LLF}$ which are confidered to be

$$\bar{f}^{LLF}(a,b) = \begin{cases} \frac{1}{2}[f(a) + f(b) - C(b-a)] \\ C = max|f\prime(s)|, \quad min(a,b) \leq s \leq max(a,b) \end{cases}$$

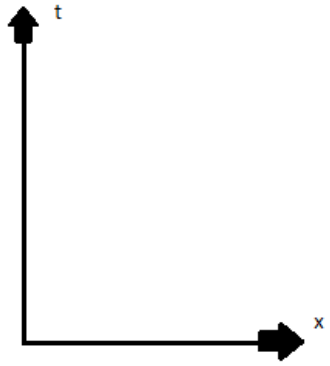

Figure 5. The quarter Plane

$$[x \geq 0] \times [t \geq 0]$$

.

The problem is an initial value problem which consist of a equation (13) together with the initial condition given by

$$u = f(x), (0 \leq x < +\infty)$$

at $t = 0$

Where the boundary condition is at

$$x = 0, t \geq 0$$

this consists of

$$a_0(0,t)u + a_1(O,t)\frac{\partial u}{\partial x} = a_2(O,t)$$

where

$$a_0(0,t) >> 0, a_1(O,t) << 0 and a_0 - a_1 > o$$

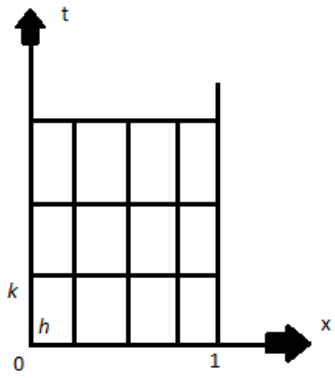

Figure 6. The open rectangle

$$[0 \leq x \leq 1] \times [t \geq 0]$$

.

Here the problem is made up of equation (13) together with the initial condition

$$u = f(x)(0 \leq x \leq 1$$

$$\text{at } t = 0$$

together with the boundary conditions

$$a_0(0,t)u + a_1(0,t)\frac{\partial u}{\partial x} = a_2(0,t)$$

$$\text{at } x = o, t \geq 0,$$

and

$$B_0(1,t)u + B_1(1,t)\frac{\partial u}{\partial x} = B_2(1,t)$$

$$\text{at } x = 1, t \geq O,$$

where the conditions for a, s are the same as those in (b) and where

$$B_0(1,t) \geq 0, B_1(1,t) \geq 0$$

$$\text{and } B_0 - B_1 > 0.$$

3 Numerical Solution of Shallow-water equation

3.1 Finite Difference Method

In this discussion [4] we will begin with numerical methods for partial differential equations. Consider the following initial-boundary-value problem

$$u_t = vu_{xx} \tag{14}$$

$$u(x,0) = F(x), x \in [0,1] \tag{15}$$

$$u(0,t) = a(t), v(1,t) = b(t), t \geq 0 \tag{16}$$

where

$$f(0) = a(0)$$

and

$$f(1) = b(0)$$

Of course this problem can be solved analytically. For illustration purposes we shall attempt to solve it numerically [24]. Our method involves reducing the problem above into a **discrete** problem which we will be able to solve. We start off by discretizing the spatial domain and we do that by placing a grid over it. By discretizing it means that we are going to approximate the solution using discrete quantity or quantities (in this case the solution(s)). To conveniently do so, introduce a fairly uniform grid, post spaced as

$$\triangle x = \frac{1}{M}$$

To refer to one of the points in a grid it is best we call the points

$$x_k, k = 0, ..., M$$

where

$$x_k = k\triangle x, k = 0, ..., M$$

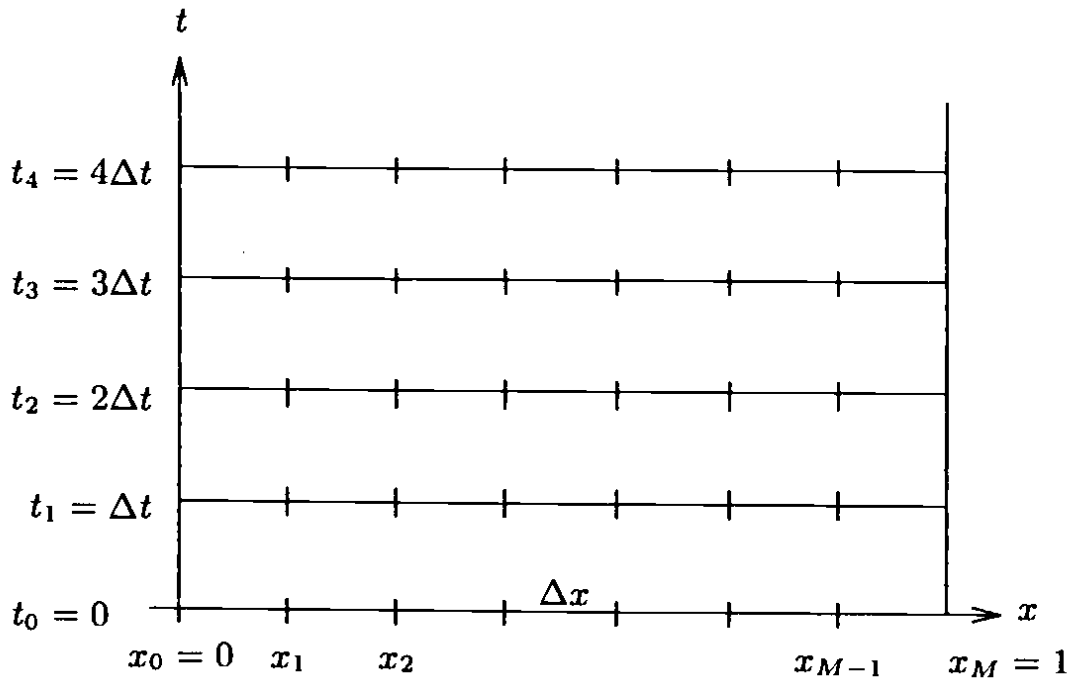

Figure 7. Grid on the time space domain

.

Similarly, we discretize time domain by placing a grid on the access with grid spacing $\triangle t$. The resulting grid is called the time-space domain as shown in Figure (7).

For our case the space-time domain is going to be approximated by a lattice of points in the Figure 7. We will try to approximate the solution to the problem at this points on the lattice. Here we let u_k^n be a function defined on the point $(k\triangle x, n\triangle t)$ or the lattice point (k,n). The function u_k^n will be considered to be the approximation of the solution to the problem (14) on the grid.
Noting that since

$$u_t(x,t) = \lim_{\triangle t \to 0} \frac{u(x,t+\triangle t) - u(x,t)}{\triangle t}$$

then an approximation of $u_t(k\triangle x, n\triangle t)$ can begin by

$$\frac{u_k^{n+1} - u_k^n}{\triangle t}.$$

$$\frac{u_k^{n+1} - u_k^n}{\triangle t} \tag{17}$$

In a similar way we can approximate u_{xx} at $(k\triangle x, n\triangle t)$ by

$$\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\triangle x^2} \tag{18}$$

To check that indeed this is a reasonable approximation consider that

$$\frac{(u_x)_{k+1/2}^n - (u_x)_{k-1/2}^n}{\triangle x}$$

is an approximation for $(u_{xx})_k^n$, and that $(u_x)_{k+1/2}^n$, and $(u_x)_{k-1/2}^n$, can be approximated using

$$\frac{u_{k+1}^n - u_k^n}{\triangle x}$$

and

$$\frac{u_k^n - u_{k-1}^n}{\triangle x}$$

respectively. Then

$$(u_{xx})_k^n \approx \frac{(u_x)_{k+1/2}^n - (u_x)_{k-1/2}^n}{\triangle x} \tag{19}$$

or

$$(u_{xx})_k^n \approx \frac{\frac{u_{k+1}^n - u_k^n}{\triangle t} - \frac{u_k^n - u_{k-1}^n}{\triangle t}}{\triangle t} \tag{20}$$

therefore the expression given by (17) and (18) approximates the pde (14) at the point $(k\triangle x, n\triangle t)$ using

$$\frac{u_k^{n+1} - u_k^n}{\triangle t} = u\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\triangle x^2} \tag{21}$$

or

$$u_k^{n+1} = u_k^n + u\frac{\triangle t}{\triangle x^2}(u_{k+1}^n - 2u_k^n + u_{k-1}^n) \tag{22}$$

Finally, we are able to see that the initial condition and boundary conditions of the problem are also reasonably approximated by

$$u_k^0 = f(k\triangle x), k = 0, ..., M \tag{23}$$

$$u_0^{n+1} = a((n=1)\triangle t), n = 0, ... \tag{24}$$

and

$$u_M^{n+1} = b((n=1)\triangle t), n = 0, ... \tag{25}$$

Our task will be to obtain the approximation to the problem (14) together with its initial and boundary conditions by solving problem (22)-(25). //

We begin by choosing

1. $\triangle x$ (or M) and
2. $\triangle t$

which are the ones to decide both the behavior and the accuracy of our solution.

If we temporary assume these details, we see that:

-Equation (23) gives u_k^0 for $k = 0, ..., M$;

-Equation (22) with $n = 0$ can be used to find u_k^1 for $k = 1, ..., M-1$;

and eventually,
-Equations (24) and (25) are used to determine u_0^1 and u_M^1.

Therefore, equations (22) (24) and (25) use the details at $n = 0$ to find u at the first Time Step.

Since $u_k^1, k = 0, ..., M$ is known , then equations (22) (24) and (25) can be applied to determine u for $n = 2$.

Continuing this process we can determine $u_k^n, k = 0, ..., M$ upto any desired time step n.

Notice that it was not possible to determine u_0^1 and u_M^1 using equation (22), because one of the subscripts $k+1$ and $k-1$ would have gone beyond the bounds ($\lneq 0$ or $\gneq 1$) for either one of the calculations. Thus it will always be necessary to have some boundary treatment equations (24) and (25). In this case the treatment was very easy and obvious which is not always the case.

This is now the scheme to approximate the solution to the initial-boundary value problem and we call it the **explicit scheme** since it is possible to solve for the variable at the $(n+1)$st time level explicitly.

3.2 The Shallow-Water Equations

The shallow-water equations (SWE) (also called the **Saint-Venant equations**), are a set of nonlinear hyperbolic equations, describing a thin layer of fluid having constant density within a hydro-static balance [25]. The layer is bounded from above by a free surface and below by the bottom topography.

These equations exhibit a variety of features, simply because they have infinite number of conservation laws. For instance to describe accurately the exact propagation of a tsunami until the wave reaches the shore we can use the shallow-water equations [25]. However while close enough or right at the shore, a model that is more complicated is required.

Another application of a shallow water equation model is to evaluate circulation patterns and minimum and maximum tides at the inside of the region in relation to a tidal force at the open boundaries of that region. These patterns of circulation are of basic interest in a model to analyze the transportation of different species in the environmental applications[26].

3.2.1 Derivation of shallow-water equations

The SWE's are derived from laws of physics on conservation-of-momentum and conservation-of-mass (the Navier-Stokes equations). These are harmonized into a set of non-linear differential equations [27].

To develop the shallow-water equations, we can begin with the Euler's equations with no surface tension,

free surface condition:

$$p = 0, \quad \frac{D_\eta}{D_t} = \frac{\partial \eta}{\partial_t} + \vec{v}. \bigtriangledown \eta = w, \quad on \quad z = \eta(x,y,t), \tag{26}$$

momentum equation:

$$\frac{D\vec{u}}{D_t} + \frac{1}{\rho} \bigtriangledown p + g\vec{z} = 0, \tag{27}$$

continuity equation:

$$\bigtriangledown .\vec{u} = 0, \tag{28}$$

bottom boundary condition:

$$\vec{u}. \bigtriangledown (z + h(x,y)) = 0, \quad on \quad z = -h(x,y) \tag{29}$$

Here, ρ is the density, η is the vertical displacement of the free surface, $\vec{u} = (u,v,w)$ is the velocity in three-dimensions, p the pressure, g the gravitational acceleration and $h(x,y)$ the bottom topography. As shown in figure (8)

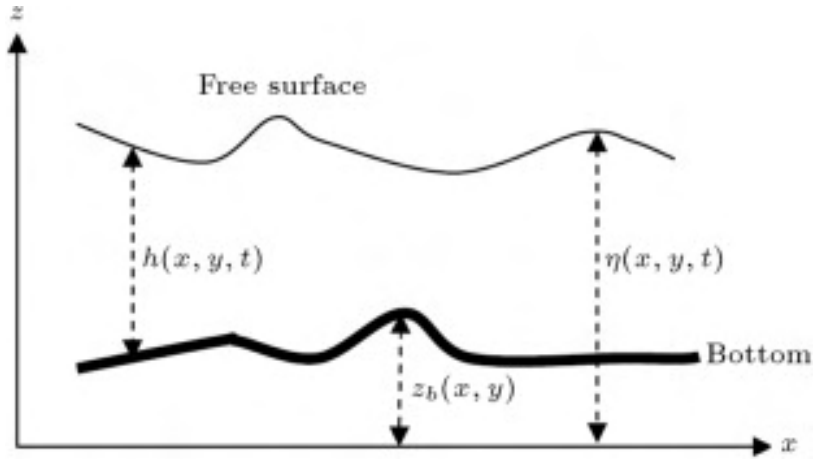

Figure 8. Methodical illustration of the Euler's system.

For the initial step of deriving the shallow-water equations, we consider the global conservation of mass. We first integrate the continuity equation (28) vertically,

$$0 = \int_{-h}^{\eta} [\bigtriangledown . \vec{u}] dz, \tag{30}$$

$$= \int_{-h}^{\eta} [\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}] dz, \tag{31}$$

$$= \frac{\partial}{\partial x} \int_{-h}^{\eta} u dz - u|_{z=\eta} \frac{\partial \eta}{\partial x} + u|_{z=-h} \frac{\partial (-h)}{\partial x}, + \frac{\partial}{\partial y} \int_{-h}^{\eta} v dz - v|_{z=\eta} \frac{\partial \eta}{\partial y} + v|_{z=-h} \frac{\partial (-h)}{\partial x}, + w|_{z=\eta} - w|_{z=-h}, \tag{32}$$

applying the bottom boundary condition (29) we get

$$0 = \frac{\partial}{\partial x} \int_{-h}^{\eta} u dz - u|_{z=\eta} \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial y} \int_{-h}^{\eta} v dz - v|_{z=\eta} \frac{\partial \eta}{\partial y} + w|_{z=\eta}. \tag{33}$$

Using the surface condition (26), equation (33) becomes

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \int_{-h}^{\eta} u dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} v dz = 0 \tag{34}$$

The next step involves making the long-wave approximation where we assuming that the length of the wave is way longer than the fluid depth. However, perturbations are

not assumed to have small amplitudes, so that we can keep nonlinear terms. The vertical acceleration term in (27) can be neglected all through the long-wave approximation, and derive the hydro-static pressure by integrating only the vertical component of the momentum equation (27),

$$\int_z^\eta \frac{\partial p}{\partial z} dz = -\int_z^\eta \rho g dz \tag{35}$$

$$p(x,y,\eta,t) - p(x,y,z,t) = -\rho g(\eta(x,y,t) - z) \tag{36}$$

$$p(x,y,z,t) = \rho g(\eta(x,y,t) - z). \tag{37}$$

here we used the surface condition $p(x,y,\eta,t) = 0$.

Assuming that there exist no vertical variations in (u,v), and using the expression of the hydro-static pressure (37), the momentum equations (horizontally) of the shallow-water system are found as follows,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial \eta}{\partial x} = 0 \tag{38}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial \eta}{\partial y} = 0 \tag{39}$$

The mass conservation given by (34) becomes

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(\eta + h)u] + \frac{\partial}{\partial y}[(\eta + h)v] = 0. \tag{40}$$

The equations (38), (39) and (40) are the Conservative **Shallow-water equations**.

For one-dimensional (say in the x-axis) flow, the Shallow-water equation in conservative form is

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = R(x,u) \tag{41}$$

$$u_t + F_x = R, \quad in \quad (0,L) \times (0,T), \tag{42}$$

with

$$u(x,t) = \begin{bmatrix} h \\ q \end{bmatrix},$$

$$F(w) = \begin{bmatrix} q \\ \frac{q^2}{h} + \frac{gh^2}{2} \end{bmatrix}$$

in which u is the velocity, h depth of flow, $q = uh$, discharge and g gravitational acceleration.

The right side of (42)is the sources and sinks of the momentum.

$$R(x,w) = \begin{bmatrix} 0 \\ gh(D - B(x)) \end{bmatrix} = \begin{bmatrix} 0 \\ ghH'(x) \end{bmatrix} \tag{43}$$

with $D = -z_x$,

where z_x = bed slope and it is the spatial partial derivative of the bottom elevation z. The loss of friction S_f is to be determined using an empirical formula [28].

Alternatively we can derive the one-dimensional SWE by contemplating a fluid (water) in a unit width channel. It is an assumption that the vertical velocity can be neglected while the velocity $u(x,t)$ horizontally is fixed through the entire cross sectional of the bed. This can be considered to be the case for significantly smaller waves with the length of the wave greater than it's depth. Further the fluid is assumed to have constant density ρ so that it is incompressible. The depth of fluid is given by $h(x,t)$ and velocity by $u(x,t)$ which are variables whose solution we seek. As shown in the figure below.

From Figure (3.2), at t all the mass in $x_1 \leq x \leq x_2$ can be written as

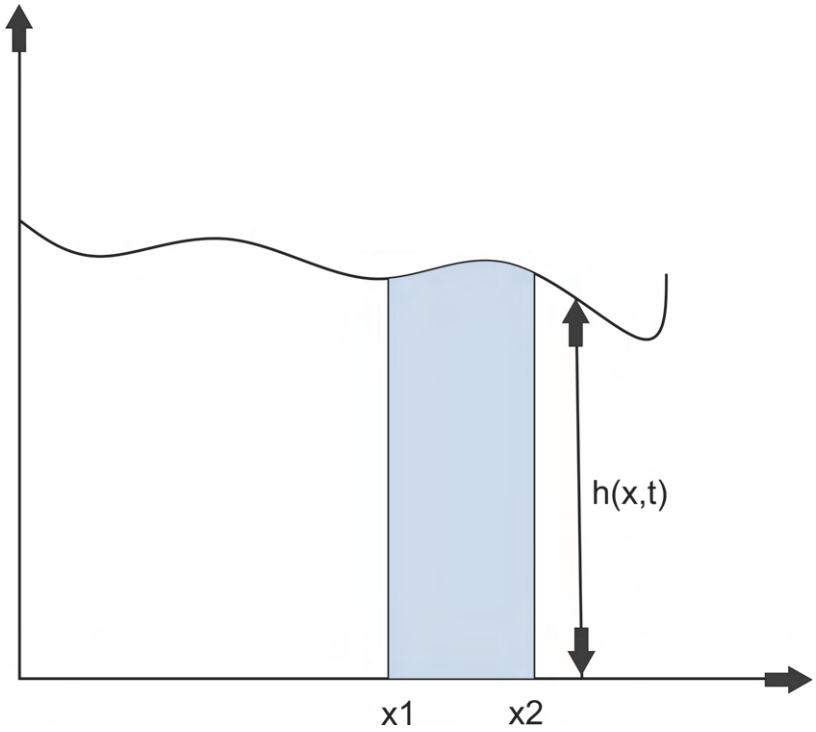

Figure 9. The graph of $h(x,t)$

$$\int_{x_1}^{x_2} \rho h(x,t)dx.$$

At each point, the Density of momentum is $\rho u(x,t)h(x,t)$

From the equation of conservation-of-mass the constant ρ is dropped which leads to the form

$$h_t + [uh]_x = 0 \tag{44}$$

Since also momentum is conserved we can write in the form of the gas dynamics equation

$$(\rho hu)_t + (\rho hu^2 + p)_x = 0.$$

We determine the pressure p by the hydrostatic law, which states that the pressure at distance $(h-y)$ below the surface is $\rho g(h-y)$, where g is the constant of gravitation. This pressure is caused by the weight of fluid above that point.

Taking the integral over the interval $0 \leq y \leq h(x,t)$ we get the total pressure that occurs at particular points in time and space.

In the momentum flux the correct pressure term is

$$p = \frac{1}{2}\rho g h^2$$

Cancelling ρ from this form we get

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x = 0 \tag{45}$$

Collecting equations (44) and (45) then writing as a system of differential equations we get

$$\begin{bmatrix} h \\ hu \end{bmatrix}_t + \begin{bmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = 0$$

.

3.3 Solution of the Shallow-Water Equation

Since mathematical model is a simplification of a physical problem, it is not known beforehand if its solution exists or it is unique or even if it can be obtained accurately. The existence and uniqueness of a solution is determined by the efficiency of the conditions, which depends on the system tigated and the solution. This section demonstrates numerical solutions to the Shallow-Water Equations (SWE) using the Finite Difference method for time and space discretization variables. Here we introduce a linearization error to enable us evaluate approximate and "(almost) accurate" numerical solutions [27]. Accurate numerical solutions can only be obtained by repositioning the mesh points efficiently to reduce the linearization error. Finding the solution for the SWE allows someone derive two necessary unknown components, **height**, $h(x,t)$ of the wave and its **velocity**, $u(x,t)$, which are individually functions of both space x and time t.

For a wave propagating through an incompressible media of density ρ, we can use given initial boundary conditions to calculate a collection of time-stepped numerical solutions, which provides us with a set of solutions to the unknown velocity and height of the wave.

Initial Data

We will consider one hump of water as in (44), using the **Initial condition**

$$h(x,0) = 1 + \frac{1}{5}\exp^{-2x^2} \tag{46}$$

the **Boundary condition** for height and velocity of $-8 \leq x \leq 8$,

gravitational force $g = 0.98$,

and

$$h(x_a,t) = h(x_b,t) = 1$$

$$u(x_a,t) = u(x_b,t) = 0$$

The integer $N \in \mathbb{Z}^+$ is the number of steps for both the space and spatial index i, for $1 \leq i \leq N+1$.

The step size is

$$\delta = \frac{x_b - x_a}{N}. \tag{47}$$

Thus

$$x_i = x_a + (i-1)\delta,$$

$$x_1 = x_a$$

and

$$x_{N+1} = x_b$$

, and there will exist N_1 nodes for N small intervals.

Equally for the time, $M \in \mathbb{Z}$ is the total number of steps. For time t_j, $j \in \mathbb{Z}^+ : 1 < j \leq M$.

3.3.1 FDMs FOR THE SOLUTION OF 1-D SSWE IN NONCONSERVATIVE FORM

ORDER- 1 EXPLICIT SCHEME

Consider the equation

$$w_{ij}^{n+1} = (I - \rho A_x \triangle_x) w_{ij}^n$$

If we combine the operators in the parenthesis into one operator, L, then the operands L equals the values of w at three nodes. Since forward differences are used explicitly with respect to t and x, the explicit scheme can be denoted by FFF. The truncation error has equal order as $\triangle t$ and $\triangle x$. If $\triangle x$ is replaced by centered difference $\delta_x/2$, the new 5-point scheme is unstable, bringing an additional order-2 viscosity term, $v\nabla^2 w$, is required on the right side. The accuracy in this case is of the same order as $(\triangle x)^2$ and stability condition is

$$(\rho_{A_x})^2 \triangle t \leq 4v$$

$$v\frac{\triangle t}{(\triangle x)^2} \leq \frac{1}{4},$$

where ρ_{A_x} is the spectral radius of matrix A_x
If we replace upwind difference in place of $\triangle x$, the stability condition becomes

$$\triangle t \leq \frac{(\triangle x)^2}{4v + (\rho_{A_x})\triangle x}$$

ORDER-2 EXPLICIT SCHEME

$$w_{ij}^{n+1} = (I - \rho A_x \triangle_x + \frac{\rho A_x}{2}(I + \rho A_x)\triangle_x^2 + \frac{\rho^2}{2}(A_x \triangle_x)) w_{ij}^n$$

This is a scheme of 9-points with an accuracy of order same to $(\triangle_x)^2$ [29].

3.3.2 Developing the difference equations

In order to develop the corresponding difference Shallow-Water equations (44) and (45) term-wise, we first assume that we know the values of $h(x,t)$ and $u(x,t)$ at time t_{j-1} and then find those values at time t_j. To accurately analyze 2–dimension shallow water equations however it is best to use the Generalized Finite Difference method (GFDM) [30].

Forward difference for time

$$\frac{d}{dt}h(x,t) \implies \frac{h(x_i,t_j) - h(x_i,t_{j-1})}{\psi}$$

$$\implies \frac{h_{i,j} - h_{i,j-1}}{\psi}$$

and by product rule

$$\frac{d}{dt}[h(x,t)u(x,t)] = h(x,t)\frac{d}{dt}u(x,t) + u(x,t)\frac{d}{dt}h(x,t)$$

$$\implies h(x_i,t_{j-1})\frac{d}{dt}u(x_i,t_{j-1}) + u(x_i,t_{j-1})\frac{d}{dt}h(x_i,t_{j-1})$$

$$\implies h_{i,j-1}\frac{u_{i,j} - u_{i,j-1}}{\psi} + u_{i,j-1}\frac{h_{i,j} - h_{i,j-1}}{\psi}$$

Central difference for space

$$\frac{d}{dx}[u(x,t)h(x,t)] = u(x,t)\frac{d}{dx}h(x,t) + h(x,t)\frac{d}{dx}u(x,t)$$

$$\implies u(x_i,t_{j-1})\frac{d}{dx}h(x_i,t_j) + h(x_i,t_{j-1})\frac{d}{dx}u(x_i,t_j)$$

$$\implies u_{i,j-1}\frac{h_{i+1,j} - h_{i-1,j}}{2\delta} + h_{i,j-1}\frac{u_{i+1,j} - u_{i-1,j}}{2\delta}$$

and by product rule

$$\frac{d}{dx}[h(x,t)u^2(x,t) + \frac{1}{2}gh^2(x,t)] = \frac{d}{dx}[h(x,t)u^2(x,t)] + \frac{1}{2}g\frac{d}{dx}h^2(x,t)$$

where using chain rule and product rule

$$\frac{d}{dx}[h(x,t)u^2(x,t)] = h(x,t)\frac{d}{dx}u^2(x,t) + u^2(x,t)\frac{d}{dx}h(x,t)$$

$$= 2h(x,t)u(x,t)\frac{d}{dx}u(x,t) + u^2(x,t)\frac{d}{dx}h(x,t)$$

$$\implies 2h_{i,j-1}u_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} + u^2_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta}$$

$$\implies h_{i,j-1}u_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{\delta} + u^2_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta}$$

and for the second term in the equation

$$\frac{d}{dx}\frac{1}{2}gh^2(x,t) = 2\frac{1}{2}gh(x,t)\frac{d}{dx}h(x,t)$$

$$= gh_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta}$$

Therefore the corresponding difference equations of (44)and (45) are as follows.

$$\begin{bmatrix} \frac{h_{i,j}-h_{i,j-1}}{\psi} \\ h_{i,j-1}\frac{u_{i,j}-u_{i,j-1}}{\psi} + u_{i,j-1}\frac{h_{i,j}-h_{i,j-1}}{\psi} \end{bmatrix}$$

$$+ \begin{bmatrix} u_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} \\ h_{i,j-1}u_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{\delta} + u^2_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + gh_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or in discrete form for **mass**

$$\frac{h_{i,j}-h_{i,j-1}}{\psi} + u_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} = 0 \tag{48}$$

and **momentum**

$$\begin{aligned} h_{i,j-1}\frac{u_{i,j}-u_{i,j-1}}{\psi} + u_{i,j-1}\frac{h_{i,j}-h_{i,j-1}}{\psi} + h_{i,j-1}u_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{\delta} \\ + u^2_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} + gh_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta} = 0 \end{aligned} \tag{49}$$

Multiplying (48) by $u_{i,j-1}$

$$u_{i,j-1}\frac{h_{i,j-h_{i,j-1}}}{\psi}+u_{i,j-1}^2\frac{h_{i+1,j}-h_{i-1,j}}{2\delta}+ u_{i,j-1}h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta} \tag{50}$$

Subtracting (50) from (49)

$$\begin{aligned} u_{i,j-1}\frac{h_{i,j}-h_{i,j-1}}{\psi}+u_{i,j-1}^2\frac{h_{i+1,j}-h_{i-1,j}}{2\delta}+h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta}=0 \\ h_{i,j-1}\frac{u_{i,j}-u_{i,j-1}}{\psi}+u_{i,j-1}h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta}+gh_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta}=0 \end{aligned} \tag{51}$$

Removing the $u_{i,j-1}^2$ element from first part of (51) by dividing all through by $u_{i,j-1}$

$$\begin{aligned} \frac{h_{i,j}-h_{i,j-1}}{\psi}+u_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta}+h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta}=0 \\ h_{i,j-1}\frac{u_{i,j}-u_{i,j-1}}{\psi}+u_{i,j-1}h_{i,j-1}\frac{u_{i+1,j}-u_{i-1,j}}{2\delta}+gh_{i,j-1}\frac{h_{i+1,j}-h_{i-1,j}}{2\delta}=0 \end{aligned} \tag{52}$$

Multiplying the equations (52) by $2\delta\psi$

$$\begin{aligned} 2\delta(h_{i,j}-h_{i,j-1})+\psi u_{i,j-1}(h_{i+1,j}-h_{i-1,j})+\psi h_{i,j-1}(u_{i+1,j}-u_{i-1,j})=0 \\ 2\delta h_{i,j-1}(u_{i,j}-)+\psi u_{i,j-1}h_{i,j-1}(u_{i+1,j}-u_{i-1,j})+\psi g h_{i,j-1}(h_{i+1,j}-h_{i-1,j})=0 \end{aligned} \tag{53}$$

Multiplying out the terms in (53)

$$2\delta h_{i,j}-2\delta h_{i,j-1}+\psi u_{i,j-1}h_{i+1,j}-\psi u_{i,j-1}h_{i-1,j}+\psi h_{i,j-1}u_{i+1,j}-\psi h_{i,j-1}u_{i-1,j}=0 \tag{54}$$

and in the second one

$$2\delta h_{i,j-1}u_{i,j} - 2\delta h_{i,j-1}u_{i,j-1} + \psi u_{i,j-1}h_{i,j-1}u_{i+1,j} - \psi u_{i,j}h_{i,j-1}u_{i-1,j} + \psi g h_{i,j-1}h_{i+1,j} - \psi g h_{i,j-1}h_{i-1,j} = 0 \tag{55}$$

Moving the known terms to right hand side for (54)

$$2\delta h_{i,j} - \psi u_{i,j-1}h_{i+1,j} + \psi u_{i,j-1}h_{i-1,j} - \psi h_{i,j-1}u_{i+1,j} + \psi h_{i,j-1}u_{i-1,j} = 2\delta h_{i,j} \tag{56}$$

and (55)

$$2\delta h_{i,j-1}u_{i,j} + \psi u_{i,j-1}h_{i,j-1}u_{i+1,j} - \psi u_{i,j-1}h_{i,j-1}u_{i-1,j} + \psi g h_{i,j-1}h_{i+1,j} - \psi g h_{i,j-1}h_{i-1,j} = 2\delta h_{i,j-1}u_{i,j-1} \tag{57}$$

3.3.3 RESULTS

Taking for example $N = 6$, we will find the initial solution for the discretized linear equations:

For $i = 1$

$$2\delta h_{1,j} - \psi u_{1,j-1}h_{2,j} + \psi u_{1,j-1}h_{0,j} - \psi h_{1,j-1}u_{2,j} + \psi h_{1,j-1}u_{0,j} = 2\delta h_{1,j-1} \tag{58}$$

and

$$2\delta h_{1,j-1}u_{1,j} + \psi u_{1,j-1}h_{1,j-1}u_{2,j} - \psi u_{1,j-1}h_{1,j-1}u_{0,j} + \psi g h_{1,j-1}h_{2,j} - \psi g h_{1,j-1}h_{0,j} = 2\delta h_{1,j-1}u_{1,j-1} \tag{59}$$

For $i = 2$

$$2\delta h_{2,j} - \psi u_{2,j-1}h_{3,j} + \psi u_{2,j-1}h_{1,j} - \psi h_{2,j-1}u_{3,j} + \psi h_{2,j-1}u_{1,j} = 2\delta h_{2,j-1} \tag{60}$$

and

$$2\delta h_{2,j-1}u_{2,j} + \psi u_{2,j-1}h_{2,j-1}u_{3,j} - \psi u_{2,j-1}h_{2,j-1}u_{1,j} + \psi g h_{2,j-1}h_{3,j} - \psi g h_{2,j-1}h_{1,j} = 2\delta h_{2,j-1}u_{2,j-1} \tag{61}$$

For $i = 3$

$$2\delta h_{3,j} - \psi u_{3,j-1}h_{4,j} + \psi u_{3,j-1}h_{2,j} - \psi h_{3,j-1}u_{4,j} + \psi h_{3,j-1}u_{2,j} = 2\delta h_{3,j-1} \tag{62}$$

and

$$2\delta h_{3,j-1}u_{3,j} + \psi u_{3,j-1}h_{3,j-1}u_{4,j} - \psi u_{3,j-1}h_{3,j-1}u_{2,j} + \psi g h_{3,j-1}h_{4,j} - \psi g h_{3,j-1}h_{2,j} = 2\delta h_{3,j-1}u_{3,j-1} \tag{63}$$

For $i = 4$

$$2\delta h_{4,j-1} - \psi u_{4,j-1}h_{5,j} + \psi u_{4,j-1}h_{3,j} - \psi h_{4,j-1}u_{5,j} + \psi h_{4,j-1}u_{3,j} = 2\delta h_{4,j} \tag{64}$$

and

$$2\delta h_{4,j-1}u_{4,j} + \psi u_{4,j-1}h_{4,j-1}u_{5,j} - \psi u_{4,j-1}h_{4,j-1}u_{3,j} + \psi g h_{4,j-1}h_{5,j} - \psi g h_{4,j-1}h_{3,j} = 2\delta h_{4,j-1}u_{4,j-1} \tag{65}$$

For $i = 5$

$$2\delta h_{5,j} - \psi u_{5,j-1}h_{6,j} + \psi u_{5,j-1}h_{4,j} - \psi h_{5,j-1}u_{6,j} + \psi h_{5,j-1}u_{4,j} = 2\delta h_{5,j} \tag{66}$$

and

$$\begin{aligned} 2\delta h_{5,j-1}u_{5,j} + \psi u_{5,j-1}h_{5,j-1}u_{6,j} - \psi u_{5,j-1}h_{5,j-1}u_{4,j} + \\ \psi g h_{5,j-1}h_{6,j} - \psi g h_{5,j-1}h_{4,j} = 2\delta h_{5,j-1}u_{5,j-1} \end{aligned} \tag{67}$$

Applying the boundary conditions at $I = (a,b) = (0,6)$, we get the initial velocity to be zero, i.e

$$u(x_a,t) = u(x_b,t) = 0,$$

and the initial height is

$$h(x_a,t) = h(x_b,t) = 1,$$

so that

$$u_{0,j} = u_{6,j} = 0,$$

$$h_{0,j} = h_{6,j} = 1.$$

The unknown terms therefore are $u_{1,j}, u_{2,j}, u_{3,j}, u_{4,j}, u_{5,j}, h_{1,j}, h_{2,j}, h_{3,j}, h_{4,j}, h_{5,j}$ and the ten equations can be simplified as:

For $i = 1$

$$\begin{aligned} 2\delta h_{1,j} - \psi u_{1,j-1}h_{2,j} - \psi h_{1,j-1}u_{2,j} = 2\delta h_{1,j-1} - \psi u_{1,j-1} \\ 2\delta h_{1,j-1}u_{1,j} + \psi u_{1,j-1}h_{1,j-1}u_{2,j} + \psi g h_{1,j-1}h_{2,j} = 2\delta h_{1,j-1}u_{1,j-1} + \psi g h_{1,j-1} \end{aligned} \tag{68}$$

For $i = 2$

$$\begin{aligned} 2\delta h_{2,j} - \psi u_{2,j-1} h_{3,j} + \\ \psi u_{2,j-1} h_{1,j} + \psi h_{2,j-1} u_{3,j} - \\ \psi h_{2,j-1} u_{1,j} = 2\delta h_{2,j-1} \end{aligned} \tag{69}$$

and

$$\begin{aligned} 2\delta h_{2,j-1} u_{2,j} + \psi u_{2,j-1} h_{2,j-1} u_{3,j} - \\ \psi u_{2,j-1} h_{2,j-1} u_{1,j} - \\ \psi g h_{2,j-1} h_{1,j} = 2\delta h_{2,j-1} u_{2,j-1} \end{aligned} \tag{70}$$

For $i = 3$

$$2\delta h_{3,j} - \psi u_{3,j-1} h_{4,j} + \psi u_{3,j-1} h_{2,j} - \psi h_{3,j-1} u_{4,j} + \psi h_{3,j-1} u_{2,j} = 2\delta h_{3,j-1} \tag{71}$$

and

$$\begin{aligned} 2\delta h_{3,j-1} u_{3,j} + \psi u_{3,j-1} h_{3,j-1} u_{4,j} - \psi u_{3,j-1} h_{3,j-1} u_{2,j} + \\ \psi g h_{3,j-1} h_{4,j} - \psi g h_{3,j-1} h_{2,j} = 2\delta h_{3,j-1} u_{3,j-1} \end{aligned} \tag{72}$$

For $i = 4$

$$2\delta h_{4,j} - \psi u_{4,j-1} h_{5,j} + \psi u_{4,j-1} h_{3,j} - \psi h_{4,j-1} u_{5,j} + \psi h_{4,j-1} u_{3,j} = 2\delta h_{4,j} \tag{73}$$

and

$$\begin{aligned} 2\delta h_{4,j-1} u_{4,j} + \psi u_{4,j-1} h_{4,j-1} u_{5,j} - \psi u_{4,j-1} h_{4,j-1} u_{3,j} + \\ \psi g h_{4,j-1} h_{5,j} - \psi g h_{4,j-1} h_{3,j} \\ = 2\delta h_{4,j-1} u_{4,j-1} \end{aligned} \tag{74}$$

For $i = 5$

$$2\delta h_{5,j} + \psi u_{5,j-1} h_{4,j} + \psi h_{5,j-1} u_{4,j} = 2\delta h_{5,j-1} + \psi u_{5,j-1} \tag{75}$$

and

$$\begin{aligned} 2\delta h_{5,j-1} u_{5,j} - \psi u_{5,j-1} h_{5,j-1} u_{4,j} - \psi g h_{5,j-1} h_{4,j} \\ = 2\delta h_{5,j-1} u_{5,j-1} - \psi g h_{5,j-1} \end{aligned} \tag{76}$$

Ordering and Sorting the above

$$\begin{bmatrix} 0 & \psi h_{1,j-1} & 0 & 0 & 0 & 2\delta & \psi u_{1,j-1} & 0 & 0 & 0 \\ 2\delta\psi h_{1,j-1} & \psi u_{1,j-1} h_{1,j-1} & 0 & 0 & 0 & 0 & \psi g h_{1,j-1} & 0 & 0 & 0 \\ -\psi h_{2,j-1} & 0 & \psi h_{2,j-1} & 0 & 0 & -\psi u_{2,j-1} & 2\delta & \psi u_{2,j-1} & 0 & 0 \\ -\psi u_{2,j-1} h_{2,j-1} & 2\delta h_{2,j-1} & \psi u_{2,j-1} h_{2,j-1} & 0 & 0 & -\psi g h_{2,j-1} & 0 & 0 & 0 & 0 \\ 0 & -\psi h_{3,j-1} & 0 & \psi h_{3,j-1} & 0 & 0 & -\psi u_{3,j-1} & 2\delta & \psi u_{3,j-1} & 0 \\ 0 & -\psi u_{3,j-1} - h_{3,j-1} & 2\delta h_{3,j-1} & \psi u_{3,j-1} h_{3,j-1} & 0 & 0 & -\psi g h_{3,j-1} & 0 & \psi g h_{3,j-1} & 0 \\ 0 & 0 & -\psi h_{4,j-1} & 0 & \psi h_{4,j-1} & 0 & 0 & -\psi u_{4,j-1} & 2\delta & \psi u_{4,j-1} \\ 0 & 0 & \psi u_{4,j-1} h_{4,j-1} & 2\delta h_{4,j-1} & \psi u_{4,j-1} h_{4,j-1} & 0 & 0 & -\psi g h_{4,j-1} & 0 & \psi g h_{4,j-1} \\ 0 & 0 & 0 & -\psi h_{5,j-1} & 0 & 0 & 0 & 0 & -\psi u_{5,j-1} & 2\delta \\ 0 & 0 & 0 & -\psi u_{5,j-1} h_{5,j-1} & 2\delta h_{5,j-1} & 0 & 0 & 0 & -\psi g h_{5,j-1} & 0 \end{bmatrix}$$

$$\begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ u_{4,j} \\ u_{5,j} \\ h_{1,j} \\ h_{2,j} \\ h_{3,j} \\ h_{4,j} \\ h_{5,j} \end{bmatrix}$$

$$= \begin{bmatrix} 2\delta h_{1,j-1} - \psi u_{1,j-1} \\ 2\delta h_{1,j-1}u_{1,j-1} + \psi g h_{1,j-1} \\ 2\delta h_{2,j-1} \\ 2\delta h_{2,j-1}u_{2,j-1} \\ 2\delta h_{3,j-1} \\ 2\delta h_{3,j-1}u_{3,j-1} \\ 2\delta h_{4,j} \\ 2\delta h_{4,j-1}u_{4,j-1} \\ 2\delta h_{5,j} + \psi u_{5,j-1} \\ 2\delta h_{5,j-1}u_{5,j-1} - \psi g h_{5,j-1} \end{bmatrix}$$

If we let

$$A = \begin{pmatrix} 0 & \psi h_{1,j-1} & 0 & 0 & 0 & 2\delta & \psi u_{1,j-1} & 0 & 0 & 0 \\ 2\delta\psi h_{1,j-1} & \psi u_{1,j-1}h_{1,j-1} & 0 & 0 & 0 & 0 & \psi g h_{1,j-1} & 0 & 0 & 0 \\ -\psi h_{2,j-1} & 0 & \psi h_{2,j-1} & 0 & 0 & -\psi u_{2,j-1} & 2\delta & \psi u_{2,j-1} & 0 & 0 \\ -\psi u_{2,j-1}h_{2,j-1} & 2\delta h_{2,j-1} & \psi u_{2,j-1}h_{2,j-1} & 0 & 0 & -\psi g h_{2,j-1} & 0 & 0 & 0 & 0 \\ 0 & -\psi h_{3,j-1} & 0 & \psi h_{3,j-1} & 0 & 0 & -\psi u_{3,j-1} & 2\delta & \psi u_{3,j-1} & 0 \\ 0 & -\psi u_{3,j-1} - h_{3,j-1} & 2\delta h_{3,j-1} & \psi u_{3,j-1}h_{3,j-1} & 0 & 0 & -\psi g h_{3,j-1} & 0 & \psi g h_{3,j-1} & 0 \\ 0 & 0 & -\psi h_{4,j-1} & 0 & \psi h_{4,j-1} & 0 & 0 & -\psi u_{4,j-1} & 2\delta & \psi u_{4,j-1} \\ 0 & 0 & \psi u_{4,j-1}h_{4,j-1} & 2\delta h_{4,j-1} & \psi u_{4,j-1}h_{4,j-1} & 0 & 0 & -\psi g h_{4,j-1} & 0 & \psi g h_{4,j-1} \\ & 0 & 0 & 0 & -\psi h_{5,j-1} & 0 & 0 & 0 & 0 & -\psi u_{5,j-1} \\ 0 & 0 & 0 & -\psi u_{5,j-1}h_{5,j-1} & 2\delta h_{5,j-1} & 0 & 0 & 0 & -\psi g h_{5,j-1} & 0 \end{pmatrix},$$

$$X = \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ u_{4,j} \\ u_{5,j} \\ h_{1,j} \\ h_{2,j} \\ h_{3,j} \\ h_{4,j} \\ h_{5,j} \end{pmatrix},$$

$$b = \begin{pmatrix} 2\delta h_{1,j-1} - \psi u_{1,j-1} \\ 2\delta h_{1,j-1}u_{1,j-1} + \psi g h_{1,j-1} \\ 2\delta h_{2,j-1} \\ 2\delta h_{2,j-1}u_{2,j-1} \\ 2\delta h_{3,j-1} \\ 2\delta h_{3,j-1}u_{3,j-1} \\ 2\delta h_{4,j} \\ 2\delta h_{4,j-1}u_{4,j-1} \\ 2\delta h_{5,j} + \psi u_{5,j-1} \\ 2\delta h_{5,j-1}u_{5,j-1} - \psi g h_{5,j-1} \end{pmatrix},$$

These equations now have the form

$$AX = b.$$

This linear form can be solved, starting with the known initial conditions.

For $j = 1$

$$A = \begin{pmatrix} 0 & \psi h_{1,0} & 0 & 2\delta & \psi u_{1,0} & 0 \\ 2\delta\psi h_{1,0} & \psi u_{1,0}h_{1,0} & 0 & 0 & \psi g h_{1,0} & 0 \\ -\psi h_{2,0} & 0 & \psi h_{2,0} & -\psi u_{2,0} & 2\delta & \psi u_{2,0} \\ -\psi u_{2,0}h_{2,0} & 2\delta h_{2,0} & \psi u_{2,0}h_{2,0} & -\psi g h_{2,0} & 0 & \psi g h_{2,0} \\ 0 & -\psi h_{3,0} & 0 & 0 & -\psi u_{3,0} & 2\delta \\ 0 & -\psi u_{3,0} - h_{3,0} & 2\delta h_{3,0} & 0 & -\psi g h_{3,0} & 0 \end{pmatrix},$$

$$X = \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ h_{1,1} \\ h_{2,1} \\ h_{3,1} \end{pmatrix}$$

and

$$b = \begin{pmatrix} 2\delta h_{1,0} - \psi u_{1,0} \\ 2\delta h_{1,0}u_{1,0} + \psi g h_{1,0} \\ 2\delta h_{2,0} \\ 2\delta h_{2,0}u_{2,0} \\ 2\delta h_{3,0} - \psi u_{3,0} \\ 2\delta h_{3,0}u_{3,0} - \psi g h_{3,0} \end{pmatrix},$$

3.4 Numerical Simulation Results

In this section we are discussing the outcomes of FDM for shallow water equations as displayed graphically using MATLAB. We consider the results within space interval $[-8,8]$ with $N+1$ been the total number of points in that interval including the two ends, that is the left end $x_1 = -8$, and right end $x_{N+1} = 8$.

The unknowns here are $u(x,t)$ the velocity, $h(x,t)$ which is the depth of water. Boundary conditions:

$$u(1;t) = u(N+1;t) = 0$$

and

$$h(1;t) = h(N+1;t) = 1$$

where t=time. The initial conditions are

$$h(x,0) = 1 + 1/5exp(-2x(k)^2)$$

and

$$u(x,0) = 0$$

The element δ will represent space step and is given by

$$\delta = \frac{(8-(-8))}{N}$$

The element ψ represent the time step size. The Dimension of our coefficient matrix A will be a 2N by 2N and b will be 2N by 1.

Results

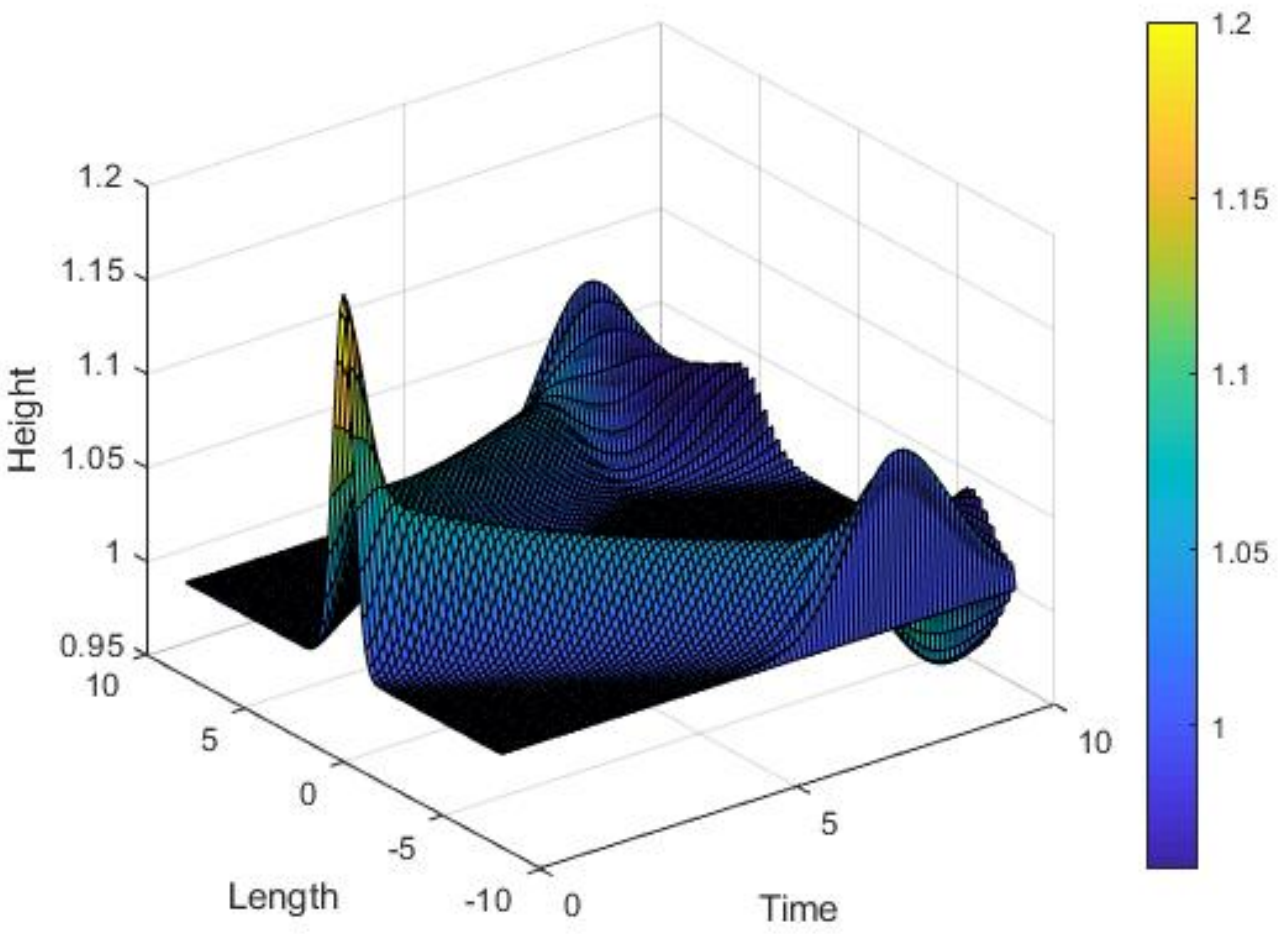

Figure 10. 3D graph of height h against time using finite difference.

Figure (10) describes water height contours at different time interval. It is a 3D representation of how h changes at different times steps realtime. It is seen than the solution starts off at the edge as a lump, it then develops a depression at the peak of the lump immediately. As time goes by the there is a significant split forming two lumps that move apart as the wave is headed to the edge of the bed. When wave has reached the boundary at time $t = 10$ the two wave bounces off the wall of the bed converging back towards the middle but this time with less momentum and less height.

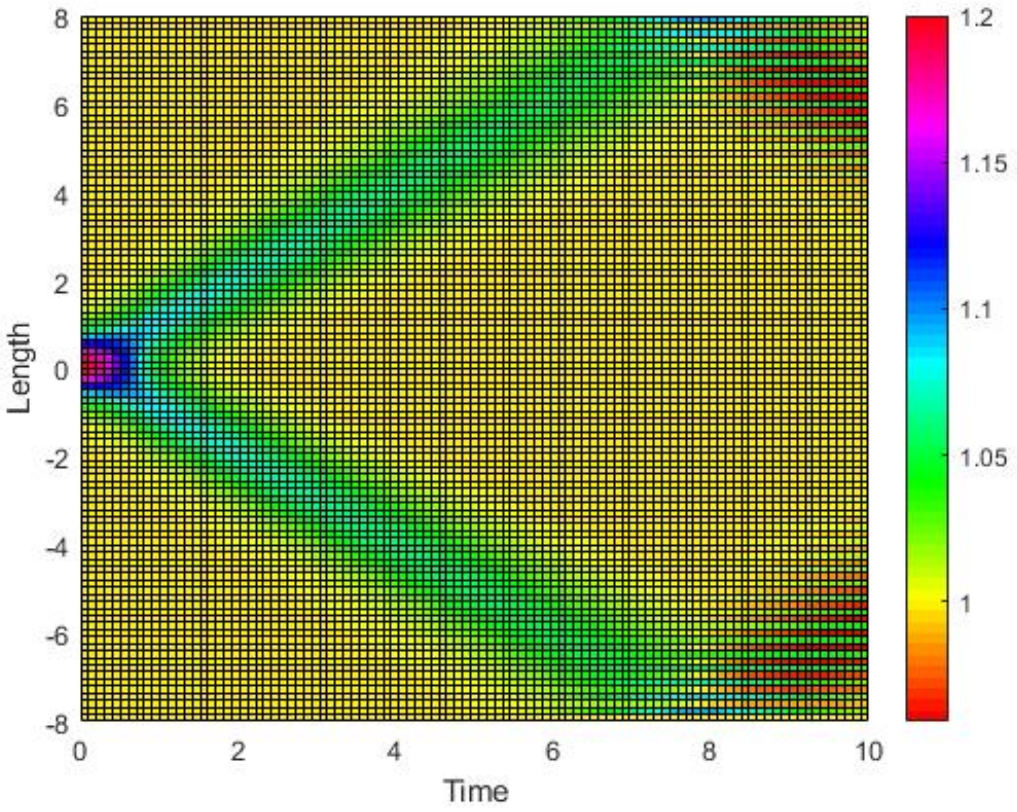

Figure 11. 2D graph of height h against time using finite difference.

Figure (11) is concurrent with figure 9. It shows the propagation of the wave in two dimension from an overhead view. It shows how the wave moves from the original point at time $t = 0$ splitting into two forming a ridge that is notably becoming wider as time increases. The waves are diverging up to the boundary where it experiences perturbations.

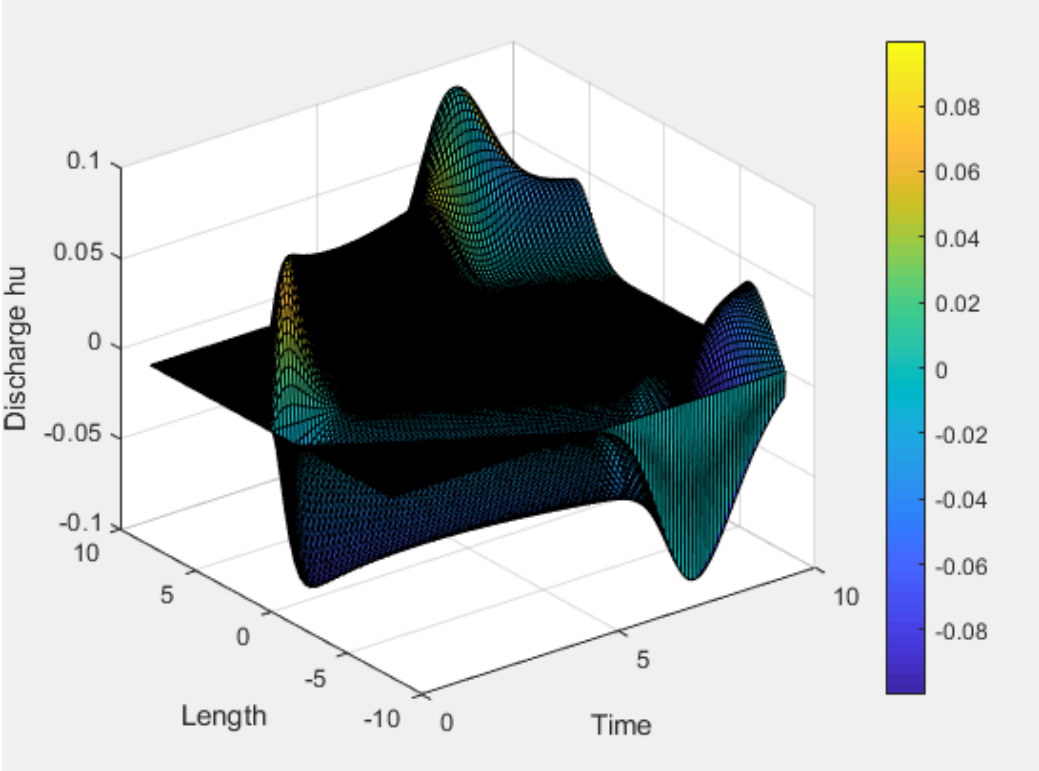

Figure 12. 3D graph of discharge hu against time using finite difference.

Figure (12) describes water discharge contours at different interval time. It is a 3D representation of how discharge hu changes at different times steps. It is seen that the solution starts off at the edge as a plane, it then develops a depression and a lump immediately. As time goes by the two tends to flatten while the wave is headed to the edge of the bed. Just when they are about to balance the wave has reached the boundary at time $t = 10$ where things escalate as the wave bounces back towards the original position.

Having given a generalized description of both the solutions of height and discharge. We are going to discuss particular observations at specific times

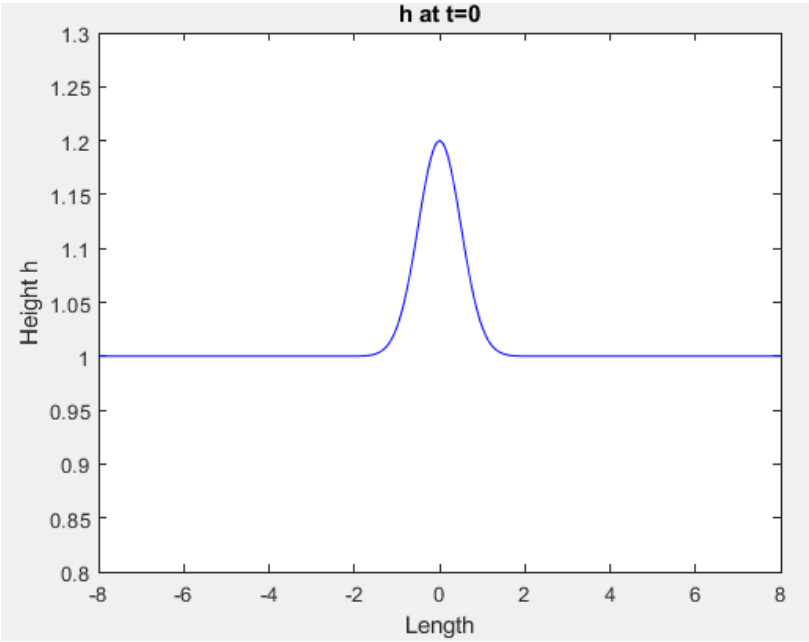

Figure 13. Graph of h at $t = 0$

$$h = 1.2.$$

Starting with the initial condition $h(x,0) = 1 + 1/5\exp(-2x^2), u(x,0) = 0$, Figure (15) shows the initial height at time $t = 0$. Here the wave starts off as a simple lump of height.

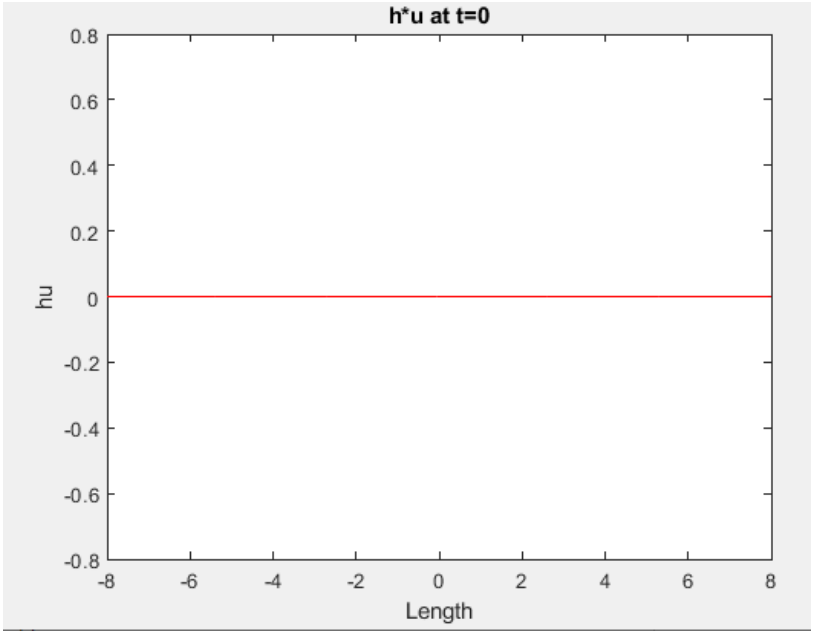

Figure 14. Graph of discharge hu at $t = 0$

Figure (16) shows the graph of discharge hu using finite difference at $t = 0$. The solution is just a straight line in 2D since we start with water in stationary positionat $u(x,0) = 0$ so that discharge $hu = 0$.

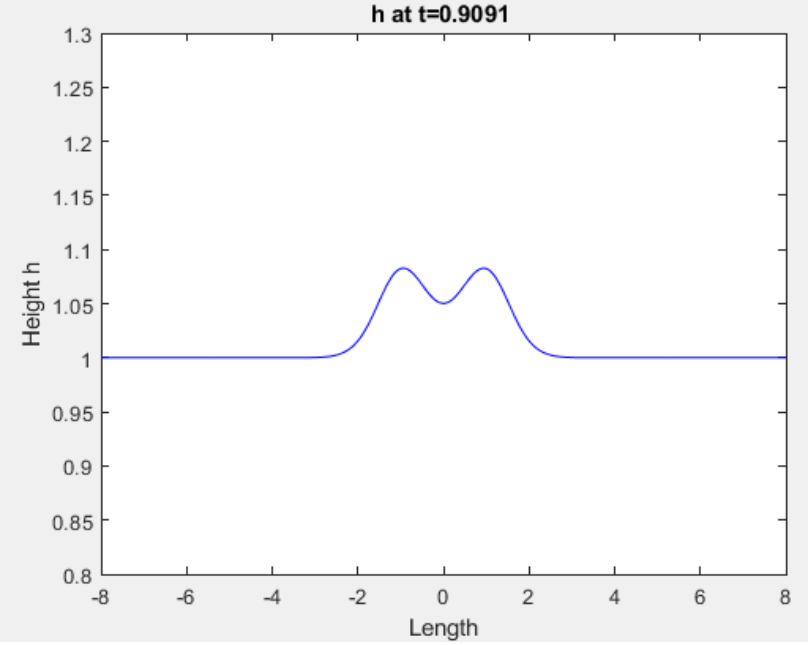

Figure 15. Graph of h at $t = 0.9091$

The lump distorts and begin to divide at the center. Consequently there is development of two wave like lumps.

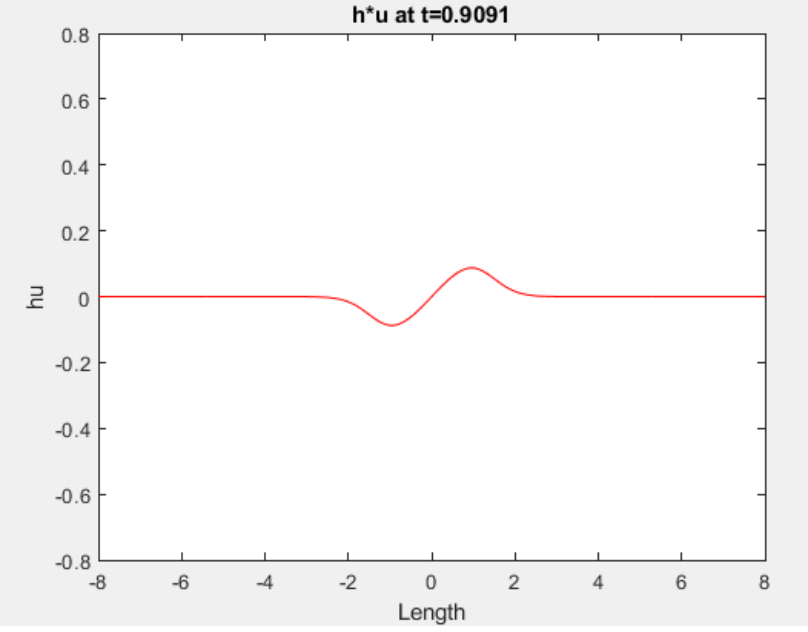

Figure 16. Graph of discharge hu at $t = 0.9091$

At time $t = 0.9091$, the solution for discharge hu the wave develops a depression and a lump that moves in different direction.

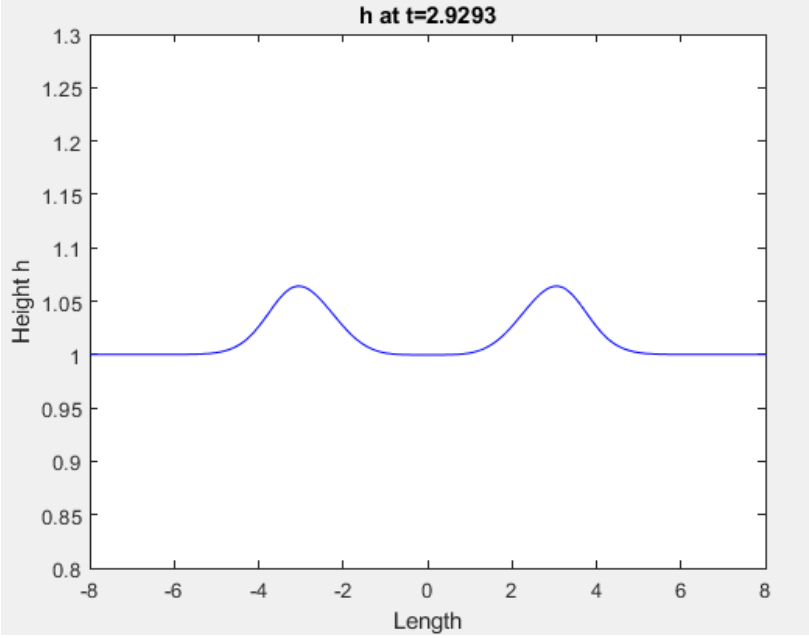

Figure 17. Graph of h at $t = 2.9293$

The lumps are moving further apart approaching the boundaries at time $t = 2.9293$.

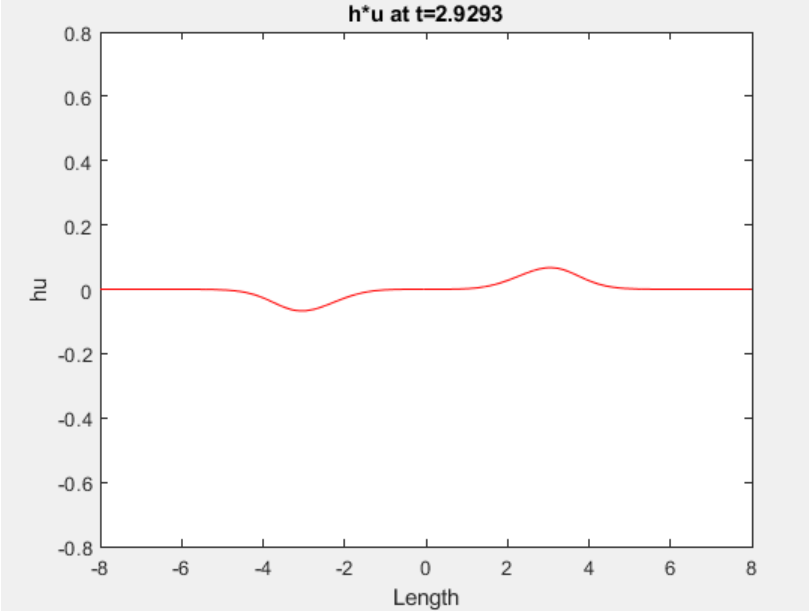

Figure 18. Graph of discharge hu at $t = 2.9293$

At time $t = 2.9293$, for discharge hu the depression and the lump tends to diminish as they move far apart towards the boundary.

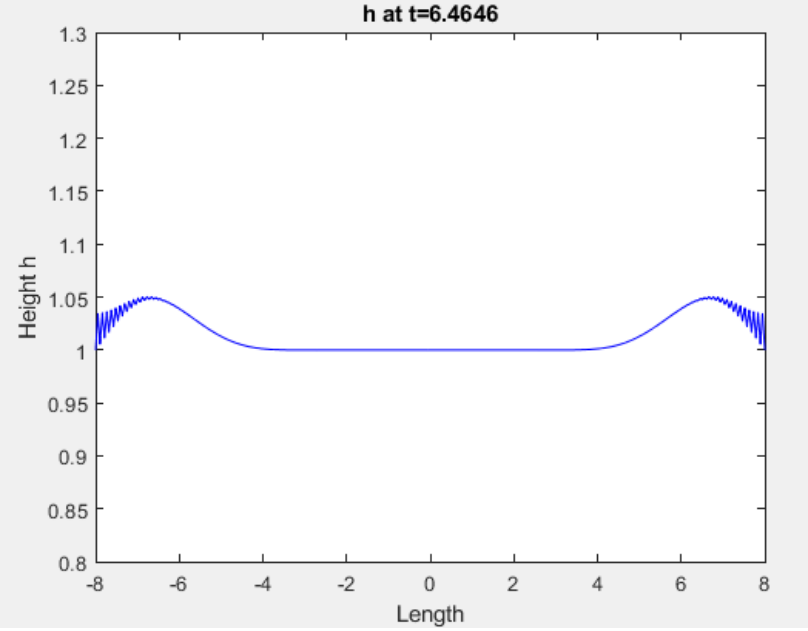

Figure 19. Graph of h at $t = 6.4646$

Upon reaching the boundaries the graph for h, shows that the water bounces of the wall and converges towards the center.

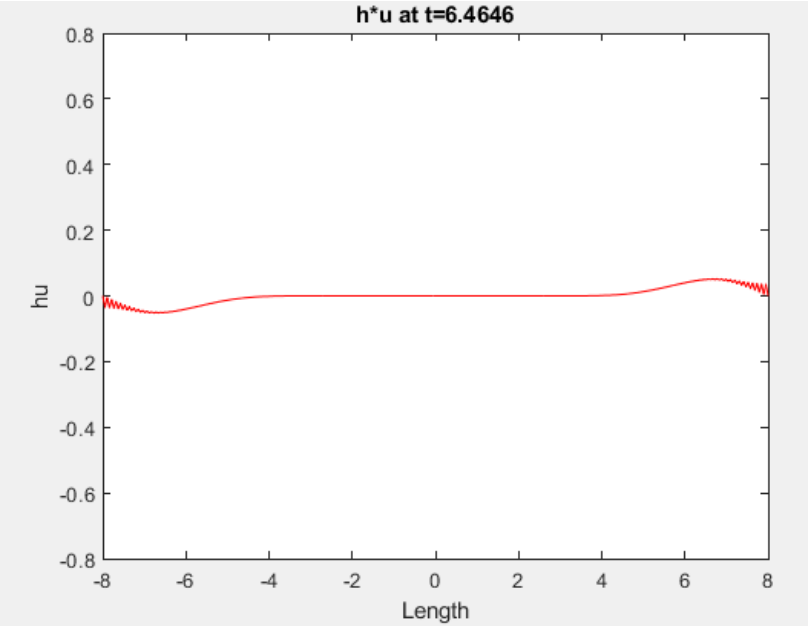

Figure 20. Graph of discharge hu at $t = 6.4646$

Equally for the graph of discharge hu the wave too experiences similar perturbations.

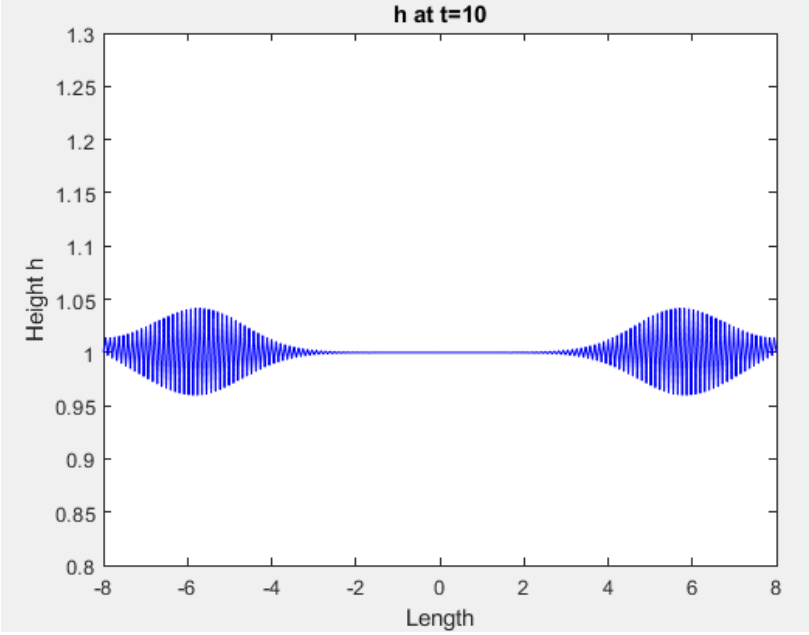

Figure 21. Graph of h at $t = 10$

This is a representation of the wave at the final time step for height h. Here the wave bounces off the walls and begin to move towards the center.

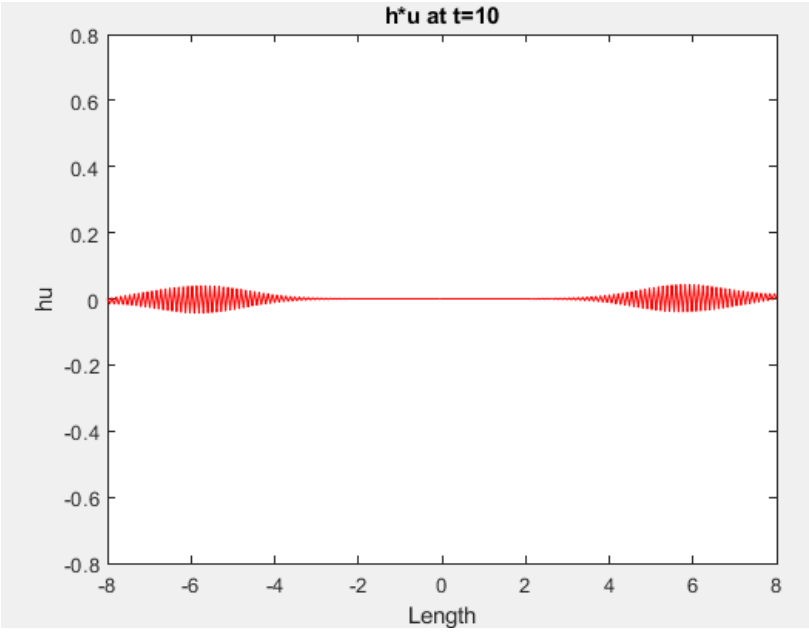

Figure 22. Graph of hu at $t = 10$

This is a representation of the wave at the final time step for discharge hu.

4 Conclusion

My current research has looked at the numerical methods to solve partial differential methods (pdes) of conservation laws. In particular we have expounded on the Finite difference (FD) method as our main methodology. We have applied this to solve the shallow water equations and results have been given adequately. We have established that it was possible to find the height and the velocity of the wave at different times using the FD method.

Having looked a particular cases of finite difference method and discontinuous galerkin methods as a numerical processes of solving partial differential equations we are at a better position to give a general overview of these methods as compared to analytical ones of seeking the solution. It has been established that getting an analytical solution for problem which are highly nonlinear is not possible one, and as such it is only practical to seek numerical schemes that can be able to solve the issue. For the shallow water equations for instance, finding the height and the velocity of the wave could have not been possible if we had depended on the analytical means. Different parameters were only utilized as we made assumptions that led to the formulation and application of the finite difference equation.

The finite difference method has proven itself of having the ability to give very good solutions which possesses an upper hand for improved accuracy by giving the optimal mesh adjustment.

4.1 Future research

Towards the end of the research, we have only considered Shallow Water Equatios in one dimension. Further work would build on applying uniform mesh refinement, local mesh refinement and applying finite difference method to higher orders (1.5D, 2D and 3D) while appplying mesh refinement to each. It will also be a great boost in development on ongoing research work surrounding application of the finite difference to solve the Dam break problem.

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Appendices

A MATLAB codes

clear; clc; close all; N =300; % number of steps in space
M = 100; % number of steps in time
$maxt = 25;$
t=linspace(0,maxt,M);
zeta=(maxt/M); % time step size
% draw profile g = 9.8; % gravitational constant,
delta = 16/N;
u(:,:) = zeros(N + 1,M); % create space for velocity
h(:,:) = zeros(N + 1,M); % creat space for height
x = -8 : delta : 8; % space step and range
$u(1,:) = 0; u(N+1,:) = 0;$ % velocity boundary conditions
u(:,1) = 0; % initial velocity
h(1,:) = 1; h(N + 1,:) = 1; % height boundary conditions
% initial displacement conditions for k = 2 : N
$h(k,1) = 1 + 1/5 * exp(-2 * x(k).^2);$ %Initial conditions end

% matrices for solving A = zeros(2 * (N - 1), 2 * (N - 1)); b = zeros(2 * (N - 1) 1);
for j = 2 : M A(1, N) = 2*delta ;
A(1,N + 1) = zeta*u(2,j-1);
A(1, 2) = zeta * h(2, j - 1);
%the first equation for i = 1 b(1, 1) = 2 * delta * h(2, j - 1) + zeta* u(2, j - 1) * h(1, j) + zeta * h(2, j - 1) * u(1, j)
A(2, 1) = 2 * delta * h(2, j - 1);
A(2, 2) = zeta * u(2, j -1) * h(2, j - 1);
%the second equation for i=1 A(2,N + 1) = zeta * g * h(2, j - 1);
b(2, 1) = 2 * delta * h(2, j - 1) * u(2, j -1) + zeta * u(2, j - 1) * h(2, j - 1) *u(1, j) + zeta * g * h(2, j - 1) * h(1, j);
A(2 * N - 3, 2 * N - 2) = 2 * delta;
A(2 * N - 3,N - 2 + N - 1) = -zeta * u(N,j - 1);
A(2 *N - 3,N - 1 - 1) = -zeta * h(N,j - 1);
%the first equation for i=3 b(2 *N -3, 1) = 2 * delta *h(N, j-1)-zeta *u(N, j-1) *h(N +1, j)-zeta*h(N,j-1) * u(N + 1, j);
A(2 * N - 2,N - 1) = 2 * delta * h(N, j - 1);
A(2 * N - 2,N - 1 - 1) = -zeta *u(N, j - 1) * h(N, j - 1);
A(2 * N - 2,N - 2 + N - 1) = -zeta * g * h(N, j - 1);

%the second equation for i=3 b(2 *N -2, 1) = 2 * delta * h(N, j -1) * u(N, j -1)-zeta * u(N, j -1) * h(N, j -1) *u(N + 1, j) - zeta * g * h(N, j- 1) * h(N + 1, j);
for i = 3 : N - 1

$A(2*i-3,N-2+i) = 2*delta;$
$A(2*i-3,N-2+i+1) = zeta*u(i,j-1);$
$A(2*i-3,N-2+i-1) = -zeta*u(i,j-1);$
$A(2*i-3,i+1-1) = zeta*h(i,j-1);$
$A(2*i-3,i-1-1) = -zeta*h(i,j-1);$
%the first equation for i = 2 $b(2*i-3,1) = 2*delta*h(i,j-1);$
$A(2*i-2,i-1) = 2*delta*h(i,j-1);$
$A(2*i-2,i+1-1) = zeta*u(i,j-1)*h(i,j-1);$
$A(2*i-2,i-1-1) = -zeta*u(i,j-1)*h(i,j-1);$
$A(2*i-2,N-2+i+1) = zeta*g*h(i,j-1);$
$A(2*i-2,N-2+i-1) = -zeta*g*h(i,j-1);$
%the second equation for i = 2 $b(2*i-2,1) = 2*delta*h(i,j-1)*u(i,j-1);$
end

% solving y=Ab;
y=Ab;
% applying the solution to the velocity and height spaces
$u(2:N,j) = y(1:N-1); h(2:N,j) = y(N:2*N-2);$
end

figure
plot(x, h(:, 1),' b-');
title('h at t=0');
axis([-8 8 0.8 1.8]);

figure
plot(x,h(:,25),'b-');
title('h at t=1.2060');
axis([-8 8 0.8 1.8]);

figure
plot(x,h(:,50),'b-');
title('h at t=2.4623');
axis([-8 8 0.8 1.8]);

```
figure
plot(x,h(:,75),'b-');
title('h at t=3.7186');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,100),'b-');
title('h at t=4.9749');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,1).*u(:,1),'r-');
title('h*u at t=0');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,25).*u(:,25),'r-');
title('h*u at t=1.2060');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,50).*u(:,50),'r-');
title('h*u at t=2.4623');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,75).*u(:,75),'r-');
title('h*u at t=3.7186');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,100).*u(:,100),'r-');
title('h*u at t=4.9749');
axis([-8 8 0.8 1.8]);
```

```
[T,X] = meshgrid(t,x);


figure
title('the meshgrid in 3 dimensions');
surf(T,X,h)
colorbar
xlabel('Time')
ylabel('Length')
zlabel('Height')


figure
surf(T,X,u)
colorbar
xlabel('Time')
ylabel('Length')
zlabel('Height')


figure
surf(T,X,h .*u)
colorbar
xlabel('Time')
ylabel('Length')
zlabel('Height')


figure
pcolor(T,X,h)
colormap(hsv(64))
colorbar
xlabel('Time')
ylabel('Length')
```