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On The Cartesian and Direct Sum Decomposition of Operators In Hilbert Spaces

Research Report in Mathematics, Number 01, 2023

August 22, 2023

Department of Mathematics

Submitted to the Department of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

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On The Cartesian and Direct Sum Decomposition of Operators In Hilbert Spaces

Research Report in Mathematics, Number 01, 2023

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Master Thesis

Submitted to the Department of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

Submitted to: The Department of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I, the undersigned declare that this project is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

22/08/2023

Signature Date

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In my capacity as a supervisor of the candidate's project, I certify that this project has my approval for submission.

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Dedication

This project is dedicated to my loving parents, Dobenson Onyango and Jane Atieno, as well as my supportive siblings, Dalmas Odera, Rumbe John, Mercy Onyango and loved ones, who have always been a continual source of inspiration and motivation for me. Their unfailing love and encouragement have been invaluable in my academic career, and I will be eternally grateful for their assistance.

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Abstract

In this project, we study the Cartesian and the direct sum decomposition of operators in Hilbert Spaces with a view of determining the properties of their components.
We first show that an arbitrary operator T decomposes as $T = A + iB$ where A and B are self-adjoint operators. We give some of the properties of this decomposition in some classes of operators and finally study this decomposition in some equivalence classes (similar, almost-similar and unitarily equivalent)
We then show that an arbitrary operator $T \in B(H)$ decomposes into normal and completely non-normal parts and that a contraction operator $T \in B(H)$ decomposes into unitary and completely non-unitary parts. Further we study this decomposition of operators in some equivalence classes.

Contents

1 Preliminaries

In this chapter we present the necessary background, notations and terminologies needed in this project.

1.1 Introduction

This project studies the Cartesian and direct sum decomposition of some classes of operators in Hilbert Spaces. Decomposition of operators is an important tool for operator theory in functional analysis. Cartesian decomposition is a specific type of operator decomposition that is commonly used in mathematics. It involves breaking down an arbitrary operator $T \in B(H)$ into the form $T = A + iB$, where A and B are self-adjoint operators.

We discuss the properties of A and B and connect to some equivalence relation to see how A and B are related. Unitarily equivalent, similar, almost similar, metrically equivalent operators and their Cartesian decomposition are studied.

We will study the direct sum decomposition of an operator and investigate the connection between the direct summands and the arbitrary operator with this decomposition. We will look at normal and completely non-normal summands of an operator and the unitary and completely non-unitary summands of a contraction operator.

1.2 Notations

- H, K denote a Hilbert space.
- $B(H)$ denotes the Banach algebra of bounded linear operators from H into H.
- T denotes a bounded linear operator.
- T^* denotes the adjoint of T.
- $Ker(T)$ denotes the kernel of T.
- $Ran(T)$ denotes the range of T.
- M denotes a subspace of H.
- $\overline{M}$ denotes the closure of the subspace M of H.
- $M^{\perp}$ denotes the orthogonal complement of a closed subspace M of H.
- $||T||$ denotes the operator norm of T.

- $B(H,K)=\{T:H\to K\}$ where T is a linear and bounded operator.
- $\langle a,\, b\rangle$ denotes the inner product of a and b on a Hilbert space H.
- 0 denotes the zero operator on H.
- I denotes the identity operator on H.
- $\rho(T)$ denotes the resolvent set of T.
- $\sigma(T)$ denotes the spectrum of T.
- $\sigma_p(T)$ denotes the point spectrum of T.
- $\sigma_c(T)$ denotes the continuous spectrum of T.
- $\sigma_r(T)$ denotes the residual spectrum of T.
- $l^2(\mathbb{N})$ denotes the space of square-summable functions over $\mathbb{N}$.
- $W(T)$ denotes the numerical range of T.
- $w(T)$ denotes numerical radius of T.
- $r(T)$ denotes the spectral radius of T.
- $tr(A)$ denotes the trace of an $n\times n$ matrix A.
- $\lambda(A)$ denotes the eigenvalues of a matrix A.
- $ran(A)$ denotes the rank of a matrix A.
- $det(A)$ denotes the determinant of a matrix A.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{N}$ denotes the set of natural numbers.
- $\mathbb{C}$ denotes the set of complex numbers.
- $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}, \mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$.
- $\partial\mathbb{D}$ denotes the set of unit circle in $\mathbb{C}, \partial\mathbb{D}=\{z\in\mathbb{C}:|z|=1\}$.
- $T=V|T|$ denotes polar decomposition of an operator T.
- $T=A+iB$ denotes the Cartesian decomposition of an operator T.

1.3 Terminologies and Definitions

Definition 1.3.1. *An operator $T \in B(H,K)$ is a bounded linear transformation from H into K.*

Definition 1.3.2. *Let H and K be Hilbert spaces over the complex plane $\mathbb{C}$. A function $T : H \to K$ is called a linear operator if for all $x, y \in H$ and for all $\lambda \in \mathbb{C}$;*

- $T(x+y) = T(x) + T(y)$.
- $T(\lambda x) = \lambda T(x)$

Definition 1.3.3. *A subspace $M \subseteq H$ is said to be invariant under T if $TM \subseteq M$ or for any $x \in M, Tx \in M$.*

Definition 1.3.4. *A subspace $M \subseteq H$ is said to reduce T if M is invariant under both T and T^*.*

Definition 1.3.5. *An operator is reducible if it has a nontrivial reducing subspace.*

Definition 1.3.6. *An operator T is said to have nontrivial invariant subspace if $\{0\} \neq M \neq H$ invariant for T.*

Definition 1.3.7. *An operator $T \in B(H)$ is reductive if each invariant subspace of T reduces T.*

An **operator** $T \in B(H)$ is said to be:
an **isometry** If $T^*T = I$.

a **co-isometry** If $TT^* = I$.

partial isometry If $T = TT^*T$.

unitary If $TT^* = T^*T = I$.

normal If $TT^* = T^*T$.

quasinormal If $T(T^*T) = (T^*T)T$.

binormal If T^*T and TT^* commutes. That is $[T^*T, TT^*]$.

hyponormal If $T^*T \geq TT^*$.

symmetry If $T = T^* = T^{-1}$.

skew-adjoint If $T^* = -T$.

co-hyponormal if its adjoint is hyponormal.

p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, where $0 < p \leq 1$.

quasihyponormal If $T^*(T^*T - TT^*)T \geq 0$. or $T^{*2}T^2 \geq (T^*T)^2$.

paranormal if $||Tx|| \leq ||T||||x|| \; for \; all \; x \in H$.

p-quasihyponormal If $T^*((T^*T)^p - (TT^*)^p)T \geq 0$.

k-quasihyponormal If $T^{*k}(T^*T - TT^*)T^k \geq 0$, for some integer $k \geq 0$ and $x \in H$.

p,k-quasihyponormal If $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$, where $0 < p \leq 1$ and k is a positive number.

dominant if for any $\lambda \in \mathbb{C}$ corresponds a number $M_\lambda \geq 1$ such that $||(T - \lambda I)^* x|| \leq M_\lambda ||(T - \lambda I)x|| \; for \; all \; x \in H$.

left shift operator if $Tx = y$, where $x = (x_1, x_2, \cdots) \in l^2$ and $y = (x_2, x_3, \cdots) \in l^2$,

right shift operator if $Tx = y$, where $x = (x_1, x_2, \cdots)$ and $y = (0, x_1, x_2, x_3, \cdots) \in l^2$.

is a **unilateral shift** if there exists a sequence $\{H_0, H_1, H_2, \cdots\}$ of pairwise orthogonal subspaces of H such that :

- $H = H_0 \oplus H_1 \oplus \cdots$.
- T spans H_n isometrically onto H_{n+1}.

Hilbert Schmidt if $||T||_2 \leq \infty$ where $||T||_2 = \{\sum_{n=1}^{\infty} ||Te_n||^2\}^{1/2}$ is the $2-norm$ and $\{e_n\}$ is an orthonormal basis for H.

a **scalar** if it is a scalar multiple of the identity operator, that is $T = \mu I, \mu \in \mathbb{C}$.

subnormal operator if it has a normal extension. That is if there exists a normal operator N on a Hilbert space K with $H \subset K$ and the subspace H is invariant under the operator N and the restriction of N to H coincides with T.

an **involution** If $T^2 = I$.

contraction if $||Tx|| \leq ||x||$ for every $x \in H$.

2-normal If $T^*T^2 = T^2T^*$.

seminormal if it is either hyponormal or co-hyponormal.

Definition 1.3.8. *A lattice, L, is a partially ordered set where each pair of elements $a, b \in L$ has a least upper bound and a greatest lower bound. The lattice of all invariant subspaces of T will be denoted by $Lat(T)$ while for all reducing subspaces will be denoted as $Red(T)$.*

Definition 1.3.9. *The commutant of an operator $T \in B(H)$ is the set of all operators that commute with T, denoted by $\{T\}^{'} = \{S \in B(H) : ST = TS\}$.*

Definition 1.3.10. *A bounded linear operator $X : H \to K$ is called a quasi-affinity or quasi-invertible if it is injective and has a dense range, $Ker(X) = \{0\}$ and $\overline{RanX} = K$.*

Definition 1.3.11. *An operator X is said to intertwine two operators S and T if $XS = TX$.*

Definition 1.3.12. *If $T_1 \in B(H)$ and $T_2 \in B(K)$ and there exist quasi-affinities $X : H \to K$ and $Y : K \to H$ satisfying $XT_1 = T_2X$ and $T_1Y = YT_2$ then T_1 and T_2 are said to be quasisimilar.*

Definition 1.3.13. *Two operators $T \in B(H)$ and $S \in B(K)$ are unitarily equivalent to each other (denoted by $\overset{u}{\sim}$) if there exists a unitary operator $U \in B(H,K)$ such that $UT = SU$. (Equivalently, $T = U^*SU$).*

Definition 1.3.14. *Two operators $T \in B(H)$ and $S \in B(K)$ are said to be similar denoted by $S \sim T$ if there exists an invertible operator $N \in B(H,K)$ such that $NS = TN$*
That is $S = N^{-1}SN$.

Definition 1.3.15. *Two operators $T \in B(H)$ and $S \in B(K)$ are said to be almost similar (denoted as $\overset{a.s.}{\sim}$) if there exists an invertible operator N such that $S^*S = N^{-1}(T^*T)N$ and $S^* + S = N^{-1}(T^* + T)N$.*

Definition 1.3.16. *Two operators $T \in B(H)$ and $S \in B(K)$ are said to be metrically equivalent (denoted as $\overset{m}{\sim}$) If $T^*T = S^*S$ or equivalently $\langle Tx, Tx\rangle = \langle Sx, Sx\rangle$ for every $x \in H$.*

Definition 1.3.17. *An operator $T \in B(H)$ is said to be normaloid if $w(T) = ||T||$.*

Remark 1.3.18. *An operator T is a projection if it is self-adjoint and idempotent (i.e. $T^* = T$ and $T^2 = T$).*

Definition 1.3.19. *Let X be a non-empty set. The convex hull of X, denoted by $conv(X)$, is the smallest convex set containing X. It is precisely, the intersection of all convex sets containing X.*

Remark 1.3.20. *If X_1 and X_2 are convex sets, then $X_1 + X_2$ is a convex set.*

Example 1.3.21. *Let $X_1 = \{v \in \mathbb{C} : |v| = 1\}$ and $X_2 = \{0,1\}$. Then $conv(X_1) = \{v \in \mathbb{C} : |v| \leq 1\}$ and $conv(X_2) = [0,1]$.*

Definition 1.3.22. *A direct summand is a restriction of an operator to a reducing subspace of it.*

Remark 1.3.23. *If an operator M is an invariant subspace under $T \in B(H)$, then relative to the decomposition $H = M \oplus M^{\perp}, T$ can be written as*
$T = \begin{pmatrix} T|_M & X \\ 0 & Y \end{pmatrix}$, *where operators $X : M^{\perp} \to M, Y : M^{\perp} \to M^{\perp}$ and $T|_M : M \to M$.*
On the other hand, if $T \in B(H)$ can be written as

$T = \begin{pmatrix} Z & X \\ 0 & Y \end{pmatrix}$ *with respect to the decomposition* $H = M \oplus M^{\perp}$, *then* $Z = Y|_M$ *is a part of* T.
The operator $X = 0$ *if and only if* M *reduces* T. *In this case,* T *is reduced into the orthogonal direct sum of the operators* $Z = T|_M$ *and* $Y = T|_{M^{\perp}}$ *such that* $T = Z \oplus Y$.

If $\{T_j \in B(H_j)\}$ *is a bounded set of operators, then the direct sum of* $\{T_j\}$ *is the operator* $T \in B(H)$ *such that* $T|_{H_j} = T_j$ *for each* j. *We denote this by*

$$T = \bigoplus_j T_j$$

Definition 1.3.24. *The set* $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}$ *or equivalently* $(Ker(\lambda I - T) \neq \{0\})or\ Ran(\lambda I - T) \neq H))$ *is called the spectrum of* T.

Definition 1.3.25. *The set* $\sigma_p(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) \neq \{0\}\}$ *is called the point spectrum of* T.

Definition 1.3.26. *The set* $\sigma_c(T) = \{\lambda \in \mathbb{C} : \lambda I - T\ is\ injective, Ran(\lambda I - T) \neq H\ and\ \overline{Ran(\lambda I - T)} = H\}$ *is called the continuous spectrum of* T.

Definition 1.3.27. *The set* $\sigma_r(T) = \{\lambda \in \mathbb{C} : \lambda I - T\ is\ injective, Ran(\lambda I - T) \neq H\ and\ \overline{Ran(\lambda I - T)}\} \neq H$ *is called the residual spectrum of* T.

Remark 1.3.28. *Since the spectrum* $\sigma(T)$ *of a bounded linear operator* $T \in B(H)$ *is the set of all scalars* $\lambda \in \mathbb{C}$ *for which the operator* $\lambda I - T$ *fails to be invertible element of the Banach algebra, we can therefore split the spectrum of an operator* T *into three disjoint parts.*

That is, the set $\{\sigma_p(T), \sigma_c(T), \sigma_r(T)\}$ *forms a partition of* $\sigma(T)$. *Meaning they are pairwise disjoint and thus* $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$.

Remark 1.3.29. *We note that in finite dimensions, the* $\sigma_c(T)$ *and the* $\sigma_r(T)$ *are empty subsets of* $\mathbb{C}$ *and thus for any operator* T *acting on finite dimensional space* $\sigma(T) = \sigma_p(T)$.

Remark 1.3.30. *This is not the case in infinite dimensional Hilbert space since in an infinite dimensional space* $\sigma_p(T)$ *may be empty.*

Let us look at the following example that shows that in an infinite dimensional Hilbert space the $\sigma_p(T) = \emptyset$.

Example 1.3.31. *Consider the unilateral shift* $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ *on the Hilbert space* $l^2(\mathbb{N})$ *of all infinite sequences of complex numbers given by* $T(y_1, y_2, y_3, \cdots) = (0, y_1, y_2, y_3, \cdots)$ *for every* $(y_1, y_2, y_3, \cdots) \in l^2$. *Suppose now that* $\lambda \in \mathbb{C}$ *is an eigenvalue of* T. *Then there exists a non-zero eigenvector* $(v_1, v_2, v_3, \cdots) \in l^2(\mathbb{N})$ *such that* $(0, y_1, y_2, y_3, \cdots) = \lambda(v_1, v_2, v_3, \cdots)$ *so that* $\lambda y_i = v_{i-1}$ *for every* $i > 1$. *If* $\lambda = 0$, *then the second condition implies that* $v_1 = v_2 = v_3 = \cdots = 0$, *a contradiction again. It then follows that the operator* T, *which is a unilateral shift, has no eigenvalues and thus* $\sigma_p(T) = \emptyset$.

Definition 1.3.32. $\sigma_{ap} = \{\lambda \in \mathbb{C} : \lambda I - T$ *is not bounded below*$\}$ *is called the approximate point spectrum of an operator* T.

Definition 1.3.33 (Resolvent Set)**.** *Let* T *be a linear operator on a Hilbert Space* H. *The resolvent set of* T, *denoted* $\rho(T)$, *is the set of all complex numbers* λ *for which the operator* $T - \lambda I$ *is invertible, where* I *is the identity operator on* H.
In other words, $\rho(T)$ *is the set of all complex numbers outside the spectrum of* T *(The complement of the spectrum of* T*). Therefore,* $\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\}$.

1.4 Some Properties of Bounded Linear Operators

Definition 1.4.1. *Let* X *be a set and let* $x, y, z \in X$. *A relation* R *on* X *is called an equivalence relation if it satisfies the following properties:*

1. *reflexive:* $xRx \ \ \forall x \in X$.
2. *Symmetric:* $xRy \implies yRx \ \ \forall x, y \in X$.
3. *Transitive:* xRy *and* $yRz \implies xRz \ \ \forall x, y, z \in X$.

Remark 1.4.2. *Note that unitary equivalence, similarity, almost-similarity and metric equivalence of operators are equivalence relations on* $B(H)$.

Definition 1.4.3. *If* $T \in B(H)$, *then its adjoint* T^* *is an operator in* $B(H)$ *such that* $\langle x,\ Ty\rangle = \langle T^*x,\ y\rangle, \forall x, y \in H$.

Definition 1.4.3 implies that $(A^*)^* = A$ and $(AB)^* = B^*A^*$.

Definition 1.4.4. *Suppose* $\{T_n \in B(H) : n \geq 1\}$ *is a sequence of operators on a Hilbert Space H, then if one of the following conditions holds true, then the sequence* $\{T_n \in B(H) : n \geq 1\}$ *is weakly convergent and we denote it by* $T_n \xrightarrow{w} T$.

a *There exist* $T \in B(H) : \langle T_n x, y\rangle \to \langle Tx, y\rangle$ *as* $n \to \infty\ \forall x, y \in H$.

b *There exist* $T \in B(H) : \langle T_n x, x\rangle \to \langle Tx, x\rangle$ *as* $n \to \infty\ \forall x \in H$.

c *The scalar sequence* $\{\langle T_n x, x\rangle \in \mathbb{C} : n \geq 1\}$ *converges to* $H\ \forall x \in H$.

d *The scalar sequence* $\{\langle T_n x, y\rangle \in \mathbb{C} : n \geq 1\}$ *converges to* $H\ \forall x, y \in H$.

Definition 1.4.5. *The sequence* $\{T_n\}_{n\geq 1}$ *is strongly convergent denoted by* $T_n \xrightarrow{s} T$ *if one of the following equivalent conditions holds true.*

a *There exist* $T \in B(H) : ||(T_n - T)x|| \to 0$ *as* $n \to \infty \ \ \forall x \in H$.

b *The sequence* $\{T_n x \in H, n \geq 1\}$ *converges in* $H \ \ \forall x \in H$.

Definition 1.4.6. *A sequence* $\{T_n\}_{n\geq 1}$ *is uniformly convergent denoted by* $T_n \xrightarrow{u} T$ *if it converges in* $B(H)$.

Definition 1.4.7. *An operator* $T \in B(H)$ *is weakly stable if the power sequence* $\{T^n\}_{n\geq 1}$ *converges to the null operator. That is* $T^n \xrightarrow{w} 0$.

Definition 1.4.8. *An operator* $T \in B(H)$ *is said to be positive if* $\langle Tx,\ x\rangle \geq 0, \forall x \in H$ *and also* T *is self adjoint.*

Remark 1.4.9. *Many operators between Hilbert spaces in classical analysis and operator theory are positive linear operators.*

Example 1.4.10. *Let* $H = \mathbb{R}^2$ *and* $T : H \to H$ *defined by;*

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

In this case, T has a matrix representation of $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the standard basis of $\mathbb{R}^2$ and it is easy to check that T is positive.

Definition 1.4.11. *An operator* $S \in B(H)$ *is called a square root of an operator* $T \in B(H)$ *if* $S^2 = T$.

Example 1.4.12. *Let* $T \in B(\mathbb{R}^3)$. *Then consider the matrix* $M(T) = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 25 \end{pmatrix}$. *Then the operator* $S \in B(\mathbb{R}^3)$ *with matrix* $M(S) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ *is a square root of* T.

Theorem 1.4.13. *Let* $T \in B(H)$. *Then the following are equivalent:*

1. T *is positive.*
2. T *is self-adjoint and all eigenvalues of* T *are nonnegative.*
3. T *has a positive square root.*

4. *T has a self-adjoint square root.*

5. *There exist an operator S such that $T = S^*S$.*

Proof. We need to show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$.

To begin, we first prove that $1 \Rightarrow 2$. Assume that 1 holds. That is T is positive. Without Loss of Generality, T is self-adjoint. Now suppose λ is an eigenvalue of T. Let x be an eigenvector of T corresponding to the eigenvalue λ. Then,

$$\begin{aligned} 0 \leq & \langle Tx, x \rangle \\ = & \langle \lambda x, x \rangle \\ = & \lambda \langle x, x \rangle. \end{aligned}$$

Thus λ is a nonnegative number.
Hence condition 2 holds.

$(2) \Rightarrow (3)$: Assume 2 holds so that T is self-adjoint and all eigenvalues of T are non-negative. By the spectral theorem, there exist an orthonormal basis $e_1, e_2, \cdots, e_n \; of \; H$ consisting of eigenvectors of T. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the corresponding eigenvalues. Each $\lambda_i \geq 0$. Let S be such that $Se_i = \sqrt{\lambda_i} e_i \; for \; i = 1, 2, 3, \cdots, n$. Then S is a positive operator. Furthermore, $S^2 e_i = \lambda_i e_i = Te_i \; for \; each \; i$, which implies that $S^2 = T$.
Thus S is a positive square root of T. This proves 3.

$(3) \Rightarrow (4)$: This follows from the definition (every positive operator is self-adjoint)

$(4) \Rightarrow (5)$: Suppose that 5 holds true, that is there exists a self-adjoint operator S on H such that $T = S^2$. Then $T = S^*S$ because $R^* = R$. This proves 5.

$(5) \Rightarrow (1)$: Lastly, assume 5 holds. Let $S \in B(H)$ be such that $T = S^*S$. Now

$$\begin{aligned} T^* = & (S^*S)^* \\ = & S^*(S^*)^* \\ = & S^*S \\ = & T \end{aligned}$$

Hence T is self-adjoint.

Note that

$$\begin{aligned}\langle Tx, x\rangle &= \langle S^*Sx, x\rangle \\ &= \langle Sx, Sx\rangle \\ &\geq 0.\end{aligned}$$

For every x.
Thus T is positive.

□

Definition 1.4.14. *Let $T : X \to Y$. Then $||T|| = sup\{||Tx||_Y : x \in X, ||x|| = 1\}$. $||T||$ is the smallest constant such that $||Tx|| \leq ||T||||x||, \forall x \in X,$ where $||T||$ is the norm of T.*

$||.||$ is a norm on $B(X,Y)$, the space of all bounded linear operators $T : X \to Y$.

For example, suppose $S, N \in B(H)$, then $||(S+N)x||$ is a norm on X as shown below.

$$\begin{aligned}||(S+N)x|| &= ||Sx + Nx|| \\ &\leq ||Sx|| + ||Nx|| \\ &\leq ||S||||x|| + ||N||||x|| \\ &= (||S|| + ||N||)||x||\end{aligned}$$

$\implies ||S+N|| \leq ||S|| + ||N||$.

Similarly, let $S \in B(X,Y)$ and $N \in B(Y,Z)$, then $SN \in B(X,Z)$ is a norm.

Proof.

$$\begin{aligned}||(SN)x|| &= ||S(Nx)||_Z \\ &\leq ||S||_{B(Y,Z)}||Nx||_Y \\ &\leq ||S||_{B(Y,Z)}||N||_{B(X,Y)}||x||_X\end{aligned}$$

$$\implies ||SN|| \leq ||S||_{B(Y,Z)}||N||_{B(X,Y)}. \qquad \square$$

Definition 1.4.15. *A normed space Y is called Banach if every Cauchy sequence $(x_n) \subset Y$ converges to an element $y \in Y$.*

Remark 1.4.16. *A Banach space is a complete normed space.*

2 Litrature Review

In functional analysis, the idea of Cartesian decomposition of operators is crucial, especially when studying Banach and Hilbert spaces. Researchers have found that in order to study the structures of any operator in Hilbert spaces, it is necessary to break the operator down into its simplest forms. Some examples of these simple form decompositions with respect to separable Hilbert spaces include Direct Sum decomposition, Polar decomposition, and Cartesian decomposition.

In a recent article, Bhatia and Kittaneh [3] who studied the Cartesian decomposition and Schatten norms investigated the properties of its components.

In 2015, Kittaneh et al [9] contributed to the study of the Cartesian decomposition. They studied the Cartesian decomposition and numerical radius inequality where they showed that If $T = A + iB$ is the Cartesian decomposition of $T \in B(H)$, then for $\alpha, \beta \in \mathbb{R}, sup_{\alpha^2+\beta^2=1}||\alpha A + \beta B|| = w(T)$. They then used this result to find the upper and lower bound of the numerical radius $w(Re(A)X - XRe(B)),\ where\ A, B, X \in B(H)\ and\ 0 \leq mI \leq X$.

We will further investigate the direct sum decomposition of some operators. Direct sum decomposition is one of many decompositions that has been significantly influenced by the work of Nagy and Foias [13]. Their work showed that every operator may be broken down into a direct sum of normal and completely non-normal (c.n.n) parts.

Williams [19] showed an important results on direct sum decomposition of operators. He proved that every operator T is unitarily equivalent to the direct sum $T = T_1 \oplus T_2$ with respect to $H = M \oplus M^{\perp}$ where T_1 is normal and T_2 is completely non-normal.

Nzimbi [15] who studied the direct sum decomposition and factorization of some classes of operators in Hilbert spaces, determined the properties of the direct summands of these operators, their invariant and hyperinvariant subspace lattices.

Nagy and Foias [13] have also shown that every contraction operator T can he written as a direct sum of a unitary and a completely non-unitary $(c.n.u.)$ part and that any of the direct summands could be missing.

Wold [20] discovered the decomposition of an isometry into a unitary and a completely non-unitary parts while researching stationary stochastic processes, which has since been referred to as the Von Neumann-Wold decomposition of an isometry.

In this project, our goal is to check the properties of A and B and answer the following questions:

If $T \in B(H)$ belongs to any of the classes of operators and T decomposes as $T = A + iB$, where A and B are self-adjoint operators:

- How are $T, A \textit{ and } B$ related?
- How are $\sigma(T), \sigma(A), \sigma(B)$ related?
- How are $W(T), W(A), W(B)$ related?
- How are $w(T), w(A), w(B)$ related?

We will also extend the work of Kittaneh et al of 2015 and look at different cases of α and β such that $\alpha^2 + \beta^2 = 1$.
Finally, we analyse the spectral picture of these decompositions.

3 On Cartesian Decomposition of Operators

In this chapter, we study the Cartesian decomposition of an operator $T \in B(H)$. Our interest is to investigate the properties of its components.

Suppose $T \in B(H)$, and $T = A + iB$. Then T is called the Cartesian decomposition. A is called the real part of T denoted by $Re(T)$ while B is the imaginary part of T and denoted by $Im(T)$. Thus T can as well be represented as $T = Re(T) + iIm(T)$.

Remark 3.0.1. *Note that if $T = A + iB$, then $T^* = A - iB$. Therefore $Re(T) = \frac{T+T^*}{2}$ and $Im(T) = \frac{T-T^*}{2i}$.*

Remark 3.0.2. *If $A = 0$, then $T = iB$ and we say T is purely imaginary and if $B = 0$, then $T = A$, and we say T is real.*

3.1 Cartesian Decomposition of Some Classes of Operators

Our aim is to check the properties of A, B, and those of T if $T = A + iB$.

Theorem 3.1.1. *If $T = A + iB$ and T is normal, then $[A, B] = 0$.*

Proof. Let $T = A + iB$. Then $T^* = A^* - iB^* = A - iB$ since A and B are self-adjoint. Therefore,

$$\begin{aligned} T^*T &= (A - iB)(A + iB) \\ &= A^2 + iAB - iBA + B^2 \end{aligned}$$

$$\begin{aligned} TT^* &= (A + iB)(A - iB) \\ &= A^2 - iAB + iBA + B^2 \end{aligned}$$

Since T is normal, we have $T^*T = TT^*$.
$\implies A^2 + iAB - iBA + B^2 = A^2 - iAB + iBA + B^2$
$\implies i[A, B] = -i[A, B]$
$\implies 2i[A, B] = 0$
$\implies [A, B] = 0$.
Therefore $AB = BA$. $\square$

Theorem 3.1.2. *Let $T = A + iB$ and T be binormal. Then $[A,B] = 0$, if A and B are projection operators.*

Proof. An operator is binormal if T^*T commutes with TT^*.

$$\begin{aligned} T^*T =& (A - iB)(A + iB) \\ =& A^2 + iAB - iBA + B^2 \end{aligned}$$

$$\begin{aligned} TT^* =& (A + iB)(A - iB) \\ =& A^2 - iAB + iBA + B^2 \end{aligned}$$

A simple computation shows that

$$A^2B - B^2A = 0. \tag{1}$$

Since A and B are projection, Equation 1 becomes $AB - BA = 0$. Therefore we conclude that $[A,B] = 0$ if A and B are projections.

□

Theorem 3.1.3. *If $T = A + iB$ and T is hyponormal, then $[A,B] \geq 0$.*

Proof. Let $T = A + iB$ and $T^* = A - iB$ since A and B are self-adjoint.
Therefore,

$$\begin{aligned} T^*T =& (A - iB)(A + iB) \\ =& A^2 + iAB - iBA + B^2 \end{aligned}$$

$$\begin{aligned} TT^* =& (A + iB)(A - iB) \\ =& A^2 - iAB + iBA + B^2 \end{aligned}$$

Since T is hyponormal, then $T^*T \geq TT^*$.
$\implies A^2 + iAB - iBA + B^2 \geq A^2 - iAB + iBA + B^2$

$$\begin{aligned} i[A,B] &\geq -i[A,B] \\ 2i[A,B] &\geq 0 \\ [A,B] &\geq 0. \end{aligned}$$

This tells us that $AB - BA \geq 0$. □

Remark 3.1.4. *[1] If $[A,B] \geq 0$, where A and B are self adjoint operators, we can say that A and B have a certain degree of "compatibility."*

If $[A,B] \geq 0$, then A and B have a common set of eigenvectors, and their eigenvalues are ordered in such a way that the eigenvalues of A are greater than or equal to the eigenvalues of B.

To see this, let $\vec{u}$ be a common eigenvector of A and B with eigenvalue a and b respectively. Then :

$$\begin{aligned} [A,B]\vec{u} &= AB\vec{u} - BA\vec{u} \\ &= ab\vec{u} - ba\vec{u} \\ &= [a,b]\vec{u} \end{aligned}$$

Since $[A,B] \geq 0$, we have $[a,b] \geq 0$
$\implies ab - ba \geq 0$.

Therefore, the eigenvalues of A are greater than or equal to the corresponding eigenvalues of B for any common eigenvalue $\vec{u}$ of A and B.

Theorem 3.1.5. *If $T = A + iB$ and T be cohyponormal, then $[A,B] \leq 0$.*

`Proof.`

$$\begin{aligned} TT^* &= (A+iB)(A-iB) \\ &= A^2 - iAB + iBA + B^2 \end{aligned}$$

$$\begin{aligned} T^*T &= (A-iB)(A+iB) \\ &= A^2 + iAB - iBA + B^2 \end{aligned}$$

Since T is cohyponormal, then $TT^* \geq T^*T$.
$\implies A^2 - iAB + iBA + B^2 \geq A^2 + iAB - iBA + B^2$.

$$
\begin{aligned}
-i[A,B] &\geq i[A,B] \\
-2i[A,B] &\geq 0 \\
[A,B] &\leq 0.
\end{aligned}
$$

This shows that $AB - BA \leq 0$. □

Theorem 3.1.6. *If $T = A + iB$ and T is quasinormal, then $[A,B] = 0$.*

`Proof.` Let $T = A + iB$. Then $T^* = A - iB$.
Therefore,

$$
\begin{aligned}
T^*T &= (A - iB)(A + iB) \\
&= A^2 + iAB - iBA + B^2
\end{aligned}
$$

$$
\begin{aligned}
TT^* &= (A + iB)(A - iB) \\
&= A^2 - iAB + iBA + B^2
\end{aligned}
$$

Since T is quasinormal, then $T(T^*T) = (T^*T)T$.
$\implies TT^*T - T^*TT = 0$

$$
\begin{aligned}
(TT^* - T^*T)T &= 0 \\
(A^2 - iAB + iBA + B^2 - A^2 - iAB + iBA - B^2)T &= 0 \\
(-2iAB + 2iBA)T &= 0 \\
-2i[A,B]T &= 0 \\
[A,B]T &= 0
\end{aligned}
$$

Let $T \neq 0$, then $[A,B] = 0$ and hence A, B commutes. □

Corollary 3.1.7. *Suppose T is quasinormal and $T = A + iB$. Then $[A,B]$ is skew-adjoint.*

Theorem 3.1.8. *Suppose $T = A + iB$. If T is skew-adjoint, then $A = 0$, hence T is purely imaginary.*

Proof.

$$\begin{aligned} A - iB &= -(A + iB) \\ A - iB &= -A - iB \\ 2A &= 0 \\ A &= 0 \end{aligned}$$

This shows that $A = 0$ and therefore $T = iB$. □

Corollary 3.1.9. *If T is a projection and $T = A + iB$, then $B = 0$.*

Corollary 3.1.10. *If $T = A + iB$, then A and B have a real spectra. That is $\sigma(A), \sigma(B) \subseteq \mathbb{R}$.*

To understand Corollary 3.1.10, let us give this example:

Example 3.1.11. *Let* $T = \begin{pmatrix} 1 & 2+i \\ 2+i & 4 \end{pmatrix}$. *This implies that T decomposes as:*

$T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Thus,

$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The eigenvalues of A can be found by solving the characteristic equation:

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$$

Thus, the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 5$, which are both real. Therefore, $\sigma(A) = \{0, 5\} \subseteq \mathbb{R}$.

The eigenvalues of B can be found in a similar way. The characteristic equation is:

$$\begin{vmatrix} \lambda & -i \\ -i & \lambda \end{vmatrix} = \lambda^2 - 1 \implies \lambda^2 = 1$$

Thus, the eigenvalues of B are $\lambda_1 = 1$ and $\lambda_2 = -1$, which are both real. Therefore, $\sigma(B) = \{1, -1\} \subseteq \mathbb{R}$.

We then look at their relationship with the operator $T = \begin{pmatrix} 1 & 2+i \\ 2+i & 4 \end{pmatrix}$.

A simple computation shows that the eigenvalues of T are $\lambda_1 = \frac{5+\sqrt{21+16i}}{2}$, and $\lambda_2 = -\frac{-5+\sqrt{21+16i}}{2}$, and hence $\sigma(T) = \{\frac{5+\sqrt{21+16i}}{2}, \frac{-5+\sqrt{21+16i}}{2}\}$.

Remark 3.1.12. *Cartesian decomposition does not transfer the spectral properties from the component parts to the arbitrary operator T.*

We now introduce the notion of numerical range of operators.

3.2 Numerical Range

The numerical range is an important aspect in operator theory since it can be used to study the behavior of linear operators. For example, the numerical range can be used to determine if an operator is self-adjoint, normal, or unitary. In particular:

T is self-adjoint if and only if $W(T)$ is real.
In this section we will describe the numerical range of different matrices and link it to our decomposition later in this chapter.

Let us begin by defining the numerical range of an operator.

Definition 3.2.1 (numerical range)**.** *The numerical range of a bounded linear operator T on a Hilbert Space H is defined as:*
$W(T) = \{\langle Tx, x\rangle : x \in H, ||x|| = 1\}$

Remark 3.2.2. *The numerical range of $T \in B(H)$ is a subset of the complex plane and provides information about the behavior of T.*

We want to investigate how the numerical range behaves under Cartesian and direct sum decomposition.

3.2.1 Examples of Numerical Ranges

Theorem 3.2.3. *[16] One of the following forms applies to the numerical range of a 2×2 matrix.*

- *If the operator is a scalar multiple of the identity operator, then the numerical range is a single point.*

- *A section of a line connecting the eigenvalues if the operator has two unique eigenvalues and is normal.*

- *A unique elliptical disc with foci at the eigenvalues, if the operator is non-normal but has the eigenvalue.*

Example 3.2.4. *Let* $T = \begin{pmatrix} \omega_1 & \alpha \\ 0 & \omega_2 \end{pmatrix}$.

Then the numerical range of T *is:*

1. *An ellipse with foci* ω_1 *and* ω_2 *having a minor axis of length* $|\alpha|$, *if* $\omega_1 \neq \omega_2$.
2. *A closed disc centred at* $\omega_i, i = 1,2$ *if* $\omega_1 = \omega_2$.
3. *A line segment joining* ω_1 *and* ω_2 *if* $\omega_1, \omega_2 \in \mathbb{R}$, *are distinct and* $\alpha = 0$.

Example 3.2.5. *[5] Let* $T = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$,

where $a,b \in \mathbb{C}$. *Then* $W(T)$ *is an ellipse with Foci at* $F = \pm\sqrt{ab}$.

We see that

$L = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\alpha-\beta}{2}} \end{pmatrix}$, *then* L *is unitary and so* $LTL^{-1} = e^{i\frac{\alpha+\beta}{2}} \begin{pmatrix} 0 & |a| \\ |b| & 0 \end{pmatrix}$, *and see that* $W(T)$ *is an ellipse with foci at* $\pm\sqrt{|a||b|}e^{i\frac{\alpha+\beta}{2}} = \pm\sqrt{ab}$, *which are the eigenvalues of* T.

Theorem 3.2.6 (The elliptic theorem)**.** *If* T *is a linear operator in* $\mathbb{C}^2$, *then* $W(T)$ *is an (possibly degenerate) elliptic disc.*

Example 3.2.7. *Let* $M_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

Then $W(M_1)$ *is the ellipse with foci* $F_1 = -1$ *and* $F_2 = 1$ *and minor axis 1 and major axis 2.23 and* $w(M_1) = 1.118$ *as seen in the figure below.*

Example 3.2.8. *Let* $M_2 = \begin{pmatrix} -2 & 4 \\ 0 & 2 \end{pmatrix}$

Then M_2 *is the ellipse with foci* $F_1 = -2$ *and* $F_2 = 2$ *and minor axis 4 and major axis 5.64 and* $w(M_2) = 2.828$ *as seen in the figure below.*

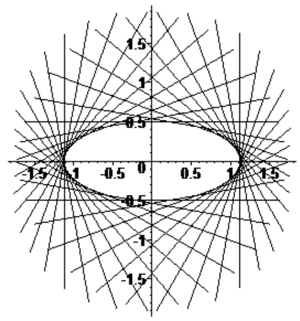

Figure 1. Numerical range of M_1.

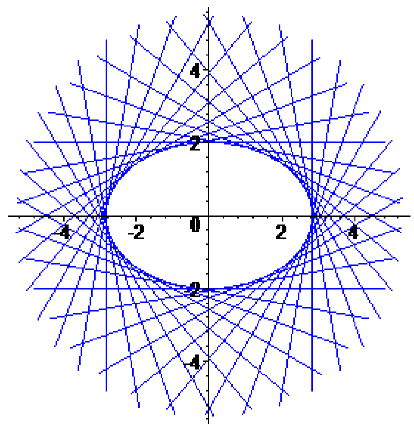

Figure 2. Numerical range of M_2.

Example 3.2.9. *Let $M_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $W(M_3)$ is a circular disc centred at the origin with radius $1/2$. We can see clearly from the figure that $w(M_3) = 1/2$.*

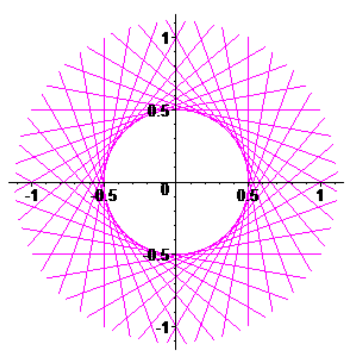

Figure 3. Numerical range of M_3.

3.2.2 Numerical Range of 3×3 matrices

In this subsection, we will look at a number of tests that can be used to determine a $W(T)$ of 3×3 matrices. By now, it should be evident that an operator's numerical range is a complex subset of the complex plane $\mathbb{C}$, which has all of the operator's eigenvalues and as a result, its convex hull, indicated by $conv(\sigma(T))$. Also, we should be able to remember that for a normal operator $T; W(T) = conv(\sigma(T))$.

With a self-adjoint operator, the ellipse transforms into a line segment that connects the two eigenvalues λ_1 and λ_2 when $L = 0$.

Example 3.2.10. *Consider the following normal matrix* $M_4 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

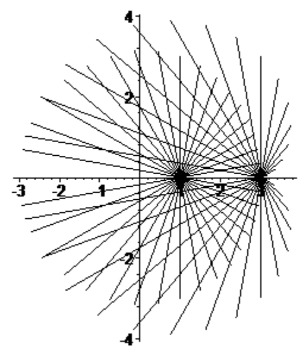

Figure 4. Numerical range of M_4.

$W(M_4)$ is a line segment with minor axis 0. We can see clearly from the figure that $w(M_4) = 3$.

Theorem 3.2.11. *[16] Let* $T = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

1. *If $|\alpha| = 1$ then T is unitary and hence normal and $W(T)$ is the equilateral triangle (interior and boundary) whose vertices are the three cube roots of α.*
2. *If $|\alpha| > 1$ then $W(T)$ is a distorted equilateral triangle (interior and boundary) whose vertices are the three cube roots of α.*
3. *If $|\alpha| \to \infty$ then $W(T)$ is the circular disc centered at the origin with radius 1.*

We demonstrate these results using the following examples.

Example 3.2.12. *For $\lambda = 1$, That is* $M_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *we have the following figure*

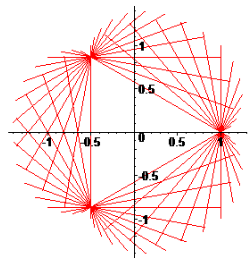

Figure 5. Numerical range of M_5.

We can see that the $w(M_5) = 1$ and $W(M_5)$ is region enclosed by the equilateral triangle whose vertices are the cube roots of α.

Similarly, if we let $\alpha = -1$, we get an equilateral triangle with the numerical radius equal to 1 as demonstrated below:

Example 3.2.13. $M_6 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

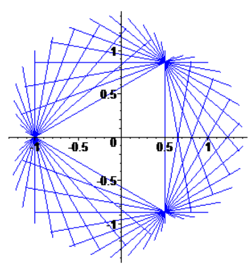

Figure 6. Numerical range of M_6.

Example 3.2.14. *For $\alpha = 3$, that is* $M_7 = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *we have the following figure*

We can see that it is a distorted equilateral triangle with $w(M_7) = 1.78077640673685344$.

Example 3.2.15. *When $\alpha \to \omega$, where $\omega \to \infty$, then we get a circular disc centered at the origin with radius $\omega/2$ as demonstrated in the following figure where we take $\omega = 2000$.*

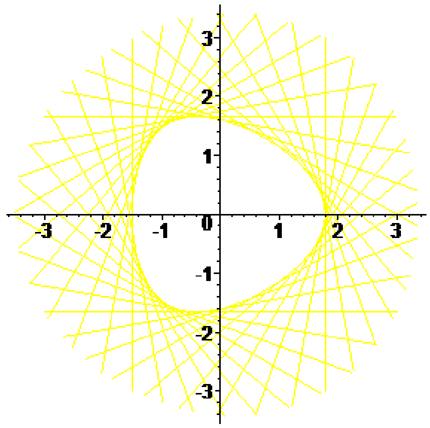

Figure 7. Numerical range of M_7.

$$M_8 = \begin{pmatrix} 0 & 0 & 2000 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

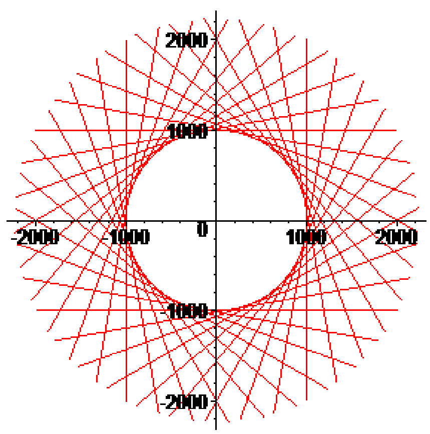

Figure 8. Numerical range of M_8.

Keeler et al (1997) [7] also classified the numerical ranges of a 3×3 matrices. They argued that the numerical range of 3×3 matrices take either of the following forms:

- An ovular shape
- A shape with a flat portion on the body
- The convex hull of its eigenvalues
- The convex hull of an ellipse and a point

3.2.3 Numerical Range of Operators in Higher Dimensions

We point out in this section that the higher the dimension, the stranger the numerical range.

To give full understanding of this statement, we are going to give out an example for clarity.

Example 3.2.16. *Consider* $M_9 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

A simple computation shows M_9 is a normal operator and therefore $W(M_9)$ is the convex hull of its eigenvalues as shown in the figure below.

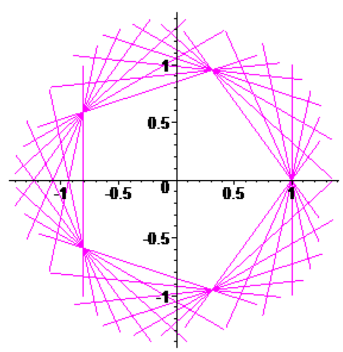

Figure 9. Numerical range of M_9.

From the figure, we can see that the corners are the eigenvalues of this operator M which are 1, $-\frac{1}{4}+\frac{1}{4}\sqrt{5}+\frac{1}{4i}\sqrt{2}\sqrt{5+\sqrt{5}}$, $-\frac{1}{4}-\frac{1}{4}\sqrt{5}+\frac{1}{4i}\sqrt{2}\sqrt{5-\sqrt{5}}$, $-\frac{1}{4}-\frac{1}{4}\sqrt{5}-\frac{1}{4i}\sqrt{2}\sqrt{5-\sqrt{5}}$, *and* $\frac{1}{4}+\frac{1}{4}\sqrt{5}-\frac{1}{4i}\sqrt{2}\sqrt{5+\sqrt{5}}$.

Using Maple, we find that $w(M_9) = 0.9999998\ ||M|| = 1$.
From here we conclude therefore that $w(M_9) = ||M_9||$.
Thus the operator M_9 is referred to as normaloid since all those operators exhibiting this behavior are referred to as normaloid.

After this discussion on numerical range of an operator, we now look at equivalence relations which is useful in the study of Cartesian decomposition.

3.3 Equivalence Relations

In this section, we introduce equivalence relations and connect to the Cartesian decomposition of operators.

Proposition 3.3.1. *If $T,S \in B(H)$ are normal operators in a Hilbert space H, then S is unitarily equivalent to T if and only if S is similar to T.*

The following corollary is an immediate result to Proposition 3.3.1.

Corollary 3.3.2. *Two similar normal operators S and T are unitarily equivalent.*

Proof. The proof of Corollary 3.3.2 follows immediately from Proposition 3.3.1. □

Remark 3.3.3. *Unitarily equivalent operators share many properties including same spectrum, same spectral radius, same numerical range and same numerical radius.*

To make sense of this remark, let us give the following examples:

Example 3.3.4. $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ *are unitarily equivalent operators.*

$$\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0$$

$\Longrightarrow \lambda_1 = 3$ *and* $\lambda_2 = -1$

A simple calculation shows that $\sigma(A) = \sigma(B) = \{-1,3\}$, *The numerical range* $W(A) = W(B) = [-1,3]$, *and finally, the numerical radius* $r(A) = r(B) = 3$.

The following diagrams shows the numerical ranges of A and B, respectively.

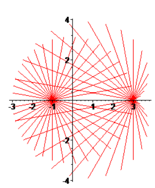

Figure 10. Numerical range of A

Clearly, their numerical range is the same. This clarifies Remark 3.3.3.

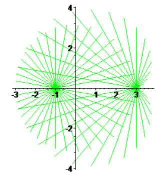

Figure 11. Numerical range of B

Theorem 3.3.5. *Let S and T be almost-similar self-adjoint operators, then S and T are similar.*

Proof. Since T and S are almost similar operators, then there exists an invertible operator $N \in B(H)$ such that:

$$T^*T = N^{-1}(S^*S)N. \tag{2}$$

$$T^* + T = N^{-1}(S^* + S)N. \tag{3}$$

□

Because T and S are self-adjoint operators $(i.e. T^* = T, S^* = S)$. Having this, equation 2 become

$$T^2 = N^{-1}S^2N \tag{4}$$

and equation 3 becomes

$$2T = N^{-1}(2S)N. \tag{5}$$

Dividing equation 5 both sides by 2, we obtain $T = N^{-1}SN$.
This shows that T is similar to S.

Remark 3.3.6. *The notion of almost similarity and similarity of operators coincide for self-adjoint operators.*

Proposition 3.3.7. *If T is an isometry and $P \in B(H)$ is metrically equivalent to T, then P is an isometry.*

Proof.

$$\begin{aligned} P^*P =& T^*T \\ =& I \end{aligned}$$

This implies that P is an isometry. □

We connect the Cartesian decomposition with these equivalence relations.

Theorem 3.3.8. *Suppose $S,T \in B(H)$ and $S = S_1 + iS_2$ and $T = T_1 + iT_2$ with $T_i, S_i, i = 1,2$ self-adjoint. If T is unitarily equivalent to S, then T_i is unitarily equivalent to $S_i, i = 1,2$.*

Proof. By Definition 1.3.13, two operators $T \in B(H)$ and $S \in B(K)$ are unitarily equivalent to each other and we denote by $T \overset{u}{\sim} S$ if there exists a unitary operator $U \in B(H,K)$ such that $T = U^*SU$.

$$\begin{aligned} T =& U^*SU \\ =& U^*(S_1 + iS_2)U \\ =& U^*S_1U + iU^*S_2U \\ T_1 + iT_2 =& U^*S_1U + iU^*S_2U \end{aligned}$$

$\implies T_1 = U^*S_1U$ and $iT_2 = iU^*S_2U$
$\implies T_1 = U^*S_1U$ and $T_2 = U^*S_2U$
$\implies T_1 \overset{u}{\sim} S_1$ and $T_2 \overset{u}{\sim} S_2$.
This shows that T_i is unitarily equivalent to $S_i, i = 1,2$.

Remark 3.3.9. *The converse of Theorem 3.3.8 is not true in general. That is, having $S_i \overset{u}{\sim} T_i, i = 1,2$, where $T = T_1 + iT_2$ and $S = S_1 + iS_2$ does not necessarily mean that $S \overset{u}{\sim} T$.*

The following example follows immediately from Remark 3.3.9.

Example 3.3.10. *Let $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, S_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.*

This imply that $T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix}$.
It is easy to check that a unitary operator U such that $T = U^*SU$ does not exist. Therefore the converse of Theorem 3.3.8 is not true in general.

□

Theorem 3.3.11. *Suppose $S,T \in B(H)$ and $S = S_1 + iS_2$ and $T = T_1 + iT_2$ with $T_i, S_i, i = 1,2$ self-adjoint. If T is similar to S, then T_i is similar to $S_i, i = 1,2$.*

Proof. By Definition 1.3.14, two operators $T \in B(H)$ and $S \in B(K)$ are said to be similar denoted by $S \sim T$ if there exists an invertible operator $N \in B(H,K)$ such that $S = N^{-1}SN$. Now $S = N^{-1}SN$.

$$\begin{aligned} S =& N^{-1}SN \\ S =& N^{-1}(T_1 + iT_2)N \\ =& N^{-1}T_1N + iN^{-1}T_2N \\ S_1 + iS_2 =& N^{-1}T_1N + iN^{-1}T_2N. \end{aligned}$$

$\implies S_1 = N^{-1}T_1N$ and $iS_2 = iN^{-1}T_2N$
$\implies S_1 = N^{-1}T_1N$ and $S_2 = N^{-1}T_2N$
$\implies S_1 \sim T_1$ and $S_2 \sim T_2$. □

Remark 3.3.12. *The converse of Theorem 3.3.11 is not true in general. That is, having $S_i \overset{s}{\sim} T_i, i = 1,2,$ where $T = T_1 + iT_2$ and $S = S_1 + iS_2$ does not necessarily mean that $S \overset{s}{\sim} T$.*

Let us consider the following example to illustrate Remark 3.3.12

Example 3.3.13. *Let* $T_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
This imply that $T = \begin{pmatrix} 2 & -i \\ -i & 1 \end{pmatrix}$ *and* $S = \begin{pmatrix} 2 & i \\ i & 1 \end{pmatrix}$.
It is easy to check that an invertible operator N such that $T = N^{-1}SN$ does not exist. Therefore the converse of Theorem 3.3.11 is not true in general.

Theorem 3.3.14. *[14] If T is a normal operator and $S \in B(H)$ is unitarily equivalent to T, then S is normal.*

Proof. Suppose $S = U^*TU$, where U is unitary and T is normal.
Then

$$\begin{aligned} S^*S =& (U^*T^*U)(U^*TU) \\ =& U^*T^*TU \\ =& U^*TT^*U \ (Putnam - Fuglede\ Theorem) \\ =& SU^*T^*U \\ =& SU^*US^* \\ =& SS^*. \end{aligned}$$

□

Theorem 3.3.15. *If $T,S \in B(H,K)$ are metrically equivalent and $T = T_1 + iT_2$ and $S = S_1 + iS_2$ with $T_i, S_i, i = 1,2$ self-adjoint operators, then T_i is metrically equivalent to S_i.*

Proof. This follows immediately from the definition of metrically equivalent operators. We know that two operators $T \in B(H)$ and $S \in B(K)$ are metrically equivalent If $T^*T = S^*S$. Therefore,

$$\begin{aligned}(T_1+iT_2)^*(T_1+iT_2) =&(S_1+iS_2)^*(S_1+iS_2).\\ (T_1-iT_2)(T_1+iT_2) =&(S_1-iS_2)(S_1+iS_2).\\ T_1^2+T_2^2 =&S_1^2+S_2^2.\\ \textit{that is } |T| = |S|.\end{aligned}$$

Since T,S are self-adjoint operators we know that $T^*T = T^2$ and $S^*S = S^2$.

We then conclude that $T_1^*T_1 = S_1^*S_1$ and $T_2^*T_2 = S_2^*S_2$

$\Longrightarrow$ T_1 is metrically equivalent to S_1 and T_2 is metrically equivalent to S_2. □

3.4 Spectral Picture of Operators with Cartesian Decomposition

3.4.1 Spectral Picture of Almost Similar Operators

To determine the spectral picture of almost similar operators we require the following result by Halmos which is crucial in proving the results that follow.

Lemma 3.4.1. *[6] Suppose that A and B are similar operators on a Hilbert space H, then A and B have the same:*

i *Spectrum*

ii *Point spectrum*

iii *Approximate point spectrum*

Theorem 3.4.2. *If T and S are almost similar projections, then $\sigma(T) = \sigma(S)$.*

Proof. To prove this theorem we begin by showing that T and S are similar.
Suppose that N is an invertible operator such that:

$$T^*T = N^{-1}(S^*S)N \tag{6}$$

and

$$T^* + T = N^{-1}(S^* + S)N \tag{7}$$

According to Theorem 3.3.5 equations 6 and 7 collapses to the equality $T = N^{-1}SN$. This shows that T is similar to S and by Lemma 3.4.1 we conclude that $\sigma(T) = \sigma(S)$. □

Remark 3.4.3. *Since projection operators are self-adjoint then Theorem 3.4.2 simplifies to the following result.*

Corollary 3.4.4. *If T and S are almost similar and self-adjoint operators, then $\sigma(T) = \sigma(S)$.*

Remark 3.4.5. *We note that almost similarity does not preserve the spectrum of operators because almost similarity does not in general imply similarity.*

Definition 3.4.6. *Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of a matrix $T \in \mathbb{C}^{n\times n}$. Then its spectral radius denoted by $r(T)$ is defined as $r(T) = max\{|\lambda_1|, \cdots |\lambda_n|\}$.*

Corollary 3.4.7. *If T and S are self-adjoint operators and almost similar, then $r(T) = r(S)$.*

`Proof.` Since T and S are almost similar self-adjoint operators, then by Corollary 3.4.4 $\sigma(T) = \sigma(S)$ and by Definition 3.4.6 implies that $r(T) = r(S)$. □

We now want to investigate whether two almost similar self-adjoint operators have equal norms. To know about this, let us begin with the following propositions:

Proposition 3.4.8. *[6] Let $T \in B(H)$ be a self-adjoint operator. Then $w(T) = r(T)$.*

Proposition 3.4.9. *[18] Let $T \in B(H)$ be a self-adjoint operator. Then $w(T) = ||T||$.*

Proposition 3.4.10. *Let $S, T \in B(H)$ be almost similar self-adjoint operator. Then $||T|| = ||S||$.*

`Proof.` Because T and S are almost similar self-adjoint operators, then by Corollary 3.4.4 $\sigma(T) = \sigma(S)$ and by Definition 3.4.6 implies that $r(T) = r(S)$. Using Proposition 3.4.8, then $w(T) = w(S)$ and by Proposition 3.4.9

$$\begin{aligned} ||T|| &= w(T) \\ &= w(S) \\ &= ||S||. \end{aligned}$$

That is $||T|| = ||S||$. □

Remark 3.4.11. *Almost similar self-adjoint operators have equal norms.*

Theorem 3.4.12. *Let $S,T \in B(H)$ be self adjoint. Then $w(T) = w(S)$.*

Proof. Since the operators are self-adjoint, hence normaloid. Then we have:

$$\begin{aligned} w(T) =& ||T|| \\ =& ||S|| \\ =& w(S) \end{aligned}$$

We then conclude that two self-adjoint almost similar normaloid operators have equal numerical radius. □

Let us look at the following example.

Example 3.4.13. *Let $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*
Both operators are self-adjoint and a simple computation shows that T and S are almost-similar with $N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. A simple computation shows that $\sigma(T) = \sigma(S) = \{-1, 1\}$. $\Longrightarrow$ $r(T) = r(S) = 1$.

But from Proposition 3.4.8:

$$\begin{aligned} w(T) =& r(T) \\ =& r(S) \\ =& w(S) \\ =& 1 \end{aligned}$$

and from Proposition 3.4.9

$$\begin{aligned} ||T|| =& w(T) \\ =& w(S) \\ =& ||S|| \\ =& 1 \end{aligned}$$

To make sense of Example 3.4.13 we give their respective numerical range.

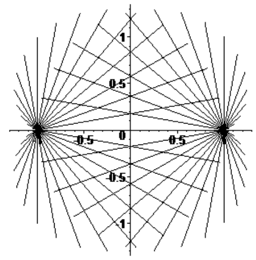

Figure 12. Numerical range of T.

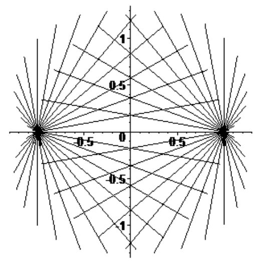

Figure 13. Numerical range of S.

Lemma 3.4.14. *If T is normal operator, then $\sigma_r(T) = \emptyset$*

Proof. By contradiction, suppose $\sigma_r(T) \neq \emptyset$ and let $\lambda \in \sigma_r(T)$. By definition, $\lambda \in \sigma_r(T)$ if $(\lambda I - T)^{-1}$ exists. This actually means that there exists a non-zero vector x such that:

$$(\overline{\lambda} I - T^*)x = 0 \; for \; all \; x \neq 0. \tag{8}$$

Since T is normal, then so is $\lambda I - T$.

$$||(\lambda I - T)x|| = ||(\overline{\lambda} I - T^*)x|| \; for \; all \; x \neq 0. \tag{9}$$

From Equations 8 and 9 $||(\lambda I - T)x|| = 0 \; for \; all \; x \neq 0$ or $(\lambda I - T)x = 0 \; for \; all \; x \neq 0$. This is a contradiction since $\sigma_r(T) \cap \sigma_p(T) = \emptyset$. Therefore $\sigma_r = \emptyset$. □

Corollary 3.4.15. *If T is normal operator, then $\sigma_{ap}(T) = \sigma(T)$.*

Proof. By the definition of approximate point spectrum, we have that $\sigma_{ap}(T) \supseteq \sigma_p(T) \cup \sigma_c(T)$ and since $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$. Then the result follows easily. □

Theorem 3.4.16. *[9] let $T = A + iB$ be the Cartesian decomposition of $T \in B(H)$. Then for all $\alpha, \beta \in \mathbb{R}$,*

$$sup_{\alpha^2+\beta^2=1} ||\alpha A + \beta B|| = w(T) \tag{10}$$

More specifically,

$$\frac{1}{2}||T+T^*|| \leq w(T) \text{ and } \frac{1}{2}||T-T^*|| \leq w(T) \tag{11}$$

Remark 3.4.17. *By using 11, we get some known inequalities,*

i $||T|| = ||A+iB|| \leq ||A|| + ||B|| \leq 2w(T)$. *Hence* $\frac{1}{2}||T|| \leq w(T)$.

ii *If* $T = T^*$, *then* $T = A$. *Hence* $||T|| = ||A|| \leq w(T) \leq ||T||$ *and* $w(T) = ||T||$ *and* $w(T) = w(A)$.

iii *Through a simple computation, we have* $\frac{T^*T+TT^*}{2} = A^2 + B^2$. *Hence* $\frac{1}{4}||T^*T+TT^*|| = \frac{1}{2}||A||^2 + ||B||^2|| \leq \frac{1}{2}(||A||^2 + ||B||^2||) \leq w^2(T)$ *(see also [10]).*

iv *Let* $\alpha, \beta \in \mathbb{R}$ *satisfy* $\alpha^2 + \beta^2 = 1$. *Then for any unit vector* $x \in H$, *we have*

$$||(\alpha A + \beta B)x|| = ||\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha x \\ \beta x \end{pmatrix}|| \leq ||\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}|| = ||\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}||^{1/2}$$
$$= ||A^2 + B^2||^{1/2} = \frac{1}{\sqrt{2}}||T^*T + TT^*||^{1/2}.$$

Thus we have, $w^2(T) = sup_{\alpha^2+\beta^2=1}||\alpha A + \beta B||^2 \leq \frac{1}{2}|T^*T + TT^*||$ *(see also [10]).*

We now want to extend this Theorem 3.4.16 by investigating some cases for α, β such that $\alpha^2 + \beta^2 = 1$.

case 1 $\alpha = 0, \beta = 1$.

Proposition 3.4.18. *Let* $T = A + iB$ *be the Cartesian decomposition of* $T \in B(H)$. *Then for all* $\alpha, \beta \in \mathbb{R}$, *with* $\alpha = 0, \beta = 1$,

$$sup_{\alpha^2+\beta^2=1}||B|| = w(B) \tag{12}$$

`Proof.` We first note that the numerical radius $w(T) = sup_{\theta \in \mathbb{R}}\{||\mathbb{R}e(e^{i\theta}T)\}||$. But $\mathbb{R}e(e^{i\theta}T) = \frac{e^{i\theta}T + e^{-i\theta}T^*}{2}$.

$$\begin{aligned}\mathbb{R}e(e^{i\theta}T) = \frac{e^{i\theta}T + e^{-i\theta}T^*}{2} &= 1/2\{(cos\theta + isin\theta)T + (cos\theta - isin\theta)T^*\} \\ &= (cos\theta)\frac{T+T^*}{2} - (sin\theta)\frac{T-T^*}{2i}.\end{aligned}$$

$$= (cos\theta)A + (sin\theta)B. \tag{13}$$

Therefore by putting $\alpha = cos\theta = 0$ and $\beta = -sin\theta = 1$, we get 12

□

case 2 $\alpha = 1, \beta = 0.$

Proposition 3.4.19. *let $T = A + iB$ be the Cartesian decomposition of $T \in B(H)$. Then for all $\alpha, \beta \in \mathbb{R}$, with $\alpha = 1, \beta = 0$,*

$$sup_{\alpha^2+\beta^2=1}||A|| = w(A). \tag{14}$$

`Proof.` The proof of this proposition is found by mimicking the proof of Proposition 3.4.18. □

case 3 $\alpha^2 = 1, \beta = 0.$

Proposition 3.4.20. *let $T = A + iB$ be the Cartesian decomposition of $T \in B(H)$. Then for all $\alpha, \beta \in \mathbb{R}$, with $\alpha^2 = 1, \beta = 0$,*

$$sup_{\alpha^2+\beta^2=1}||A|| = w(A) \tag{15}$$

case 4 $\alpha = 0, \beta^2 = 1.$

Proposition 3.4.21. *let $T = A + iB$ be the Cartesian decomposition of $T \in B(H)$. Then for all $\alpha, \beta \in \mathbb{R}$, with $\alpha = 0, \beta^2 = 1$,*

$$sup_{\alpha^2+\beta^2=1}||B|| = w(B) \tag{16}$$

We now want to find the relationship between the spectrum, numerical range, and numerical radius of $T, A,$ *and* B with $T = A + iB$.

Example 3.4.22. *Let us consider an arbitrary operator* $T = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. *This operator decomposes as* $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We begin by investigating the spectrum of $T, A,$ and B. A simple computation shows that their spectrum $\sigma(T) = \{1+i, 1-i\}, \sigma(A) = \{1\}, \sigma(B) = \{1, -1\}$.

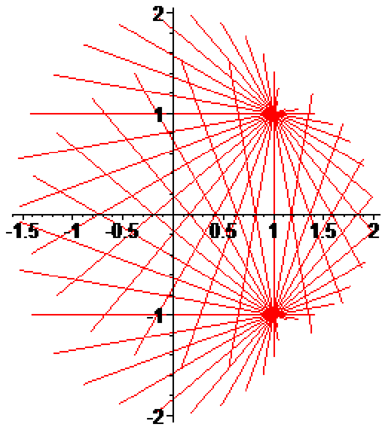

Figure 14. Numerical range of T.

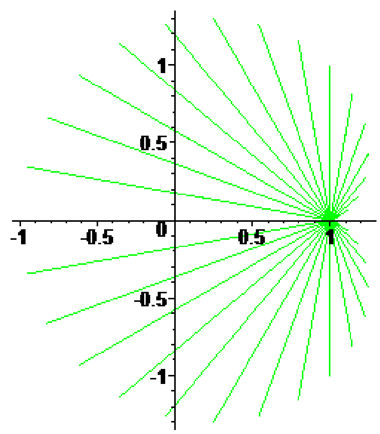

Figure 15. Numerical range of A.

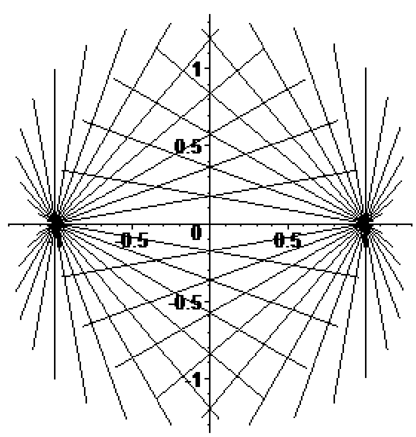

Figure 16. Numerical range of B.

We now look at their numerical ranges and numerical radii respectively.

From these figures we can see that the numerical radius of T,A *and* B *are as follows:* $w(T) = 1.40883205280000001, w(A) = 1,$ *and* $w(B) = 1.$

4 On Direct Sum Decomposition of Operators

Introduction

In this chapter, we study direct sum decomposition of some classes of operators in Hilbert spaces. The direct sum decomposition was greatly influenced by the work of Nagy and Foias [13] whose major result was that an operator can be decomposed into a direct sum of normal and completely non normal (c.n.n) parts.
A part of an operator is a restriction of it to an invariant subspace, and a direct summand is a restriction of it to a reducing subspace.
We study direct sum decomposition of an operator into normal and completely non-normal parts and direct sum decomposition of a contraction operators into unitary and completely non-unitary (c.n.u) parts. An operator $T = T_1 \oplus T_2$ is classified by using the properties of the direct summands T_1 *and* T_2. Direct sum decomposition is an interesting form of decomposition as it transfers invariant subspaces from the direct summands to the original operator. This property lacks in the Cartesian and the Polar decomposition. Every bounded linear operator T on a Hilbert Space H has an orthogonal decomposition $T = T_1 \oplus T_2$, which is implemented through a restriction of T to a reducing subspace M of H, with T_1 normal and T_2 completely non-normal $(c.n.n)$ or sometimes referred to as pure. This means that no restriction or part of T_2 to a reducing subspace is normal.

4.1 On Normal and Completely Non-normal Summands of an Operator.

In this section we want to investigate the direct sum decomposition of operators into normal and completely non-normal parts.

4.1.1 Direct Summands of Normal and Quarsinormal Operators.

We begin with the following result.

Theorem 4.1.1. *Let $T \in B(H)$ such that $T = T_1 \oplus T_2$ with respect to the decomposition $H = H_1 \oplus H_2$ and T_1 normal and T_2 c.n.n. Then T is normal if and only if $H_2 = \{0\}$ or $T_2|_{M^{\perp}} = 0$.*

Proof. Suppose $T \in B(H)$ is normal. Then $T^*T = TT^* = [T_2^*, T_2] = 0$. This implies that $T_2|_{H_2} = 0$. Conversely, suppose $[T_2^*, T_2] = T_2^*T_2 = T_2T_2^* = 0$. Since T_2 is pure this holds only if $T_2 = 0$. A simple calculation shows that $T^*T = TT^*$. Hence T is normal. □

Quasinormal operators were first studied by Brown [4] and from 1.3, it is clear that quasinormal $\supset$ normal.

Remark 4.1.2. *An operator T can be quasinormal and not normal.*

Let us give an example to illustrate Remark 4.1.2

Example 4.1.3. *Let $H = l^2(\mathbb{N})$ and let T be the unilateral shift given by the following matrix.*

$$\begin{pmatrix} 0 & 0 \cdots\cdots\cdots \\ 1 & 0 \cdots\cdots\cdots \\ 0 & 1 \ddots \ddots \\ \vdots & \vdots \ddots \ddots \\ 0 & 0 \cdots \ddots \ddots \end{pmatrix}$$

Then $T^*T = I$. This implies that $T(T^*T) = (T^*T)T$. Thus T is quasinormal.
A simple computation shows that $T^*T - TT^* = diag(1,0,0,\cdots)$. This shows that T is not normal.

Theorem 4.1.4. *Every direct summand of a quasinormal operator is quasinormal.*

Proof. Let N be a reducing subspace for $T \in B(H)$. Suppose now that $T = T_1 \oplus T_2$ on $H = N \oplus N^{\perp}$ where $T_1 = T|_N$ and $T_2 = T|_{N^{\perp}}$ and T is quasinormal, then $T^*TT = T_1^*T_1T_1 \oplus T_2^*T_2T_2 = T_1T_1^*T_1 \oplus T_2T_2^*T_2 = TT^*T$.

This shows that $T_1^*T_1T_1 = T_1T_1^*T_1$ and $T_2^*T_2T_2 = T_2T_2^*T$ ($That\ is\ [T_1^*, T_1]T_1 = 0\ and\ [T_2^*, T_2]T_2 = 0$) and hence T_1 and T_2 are both quasinormal. $\square$

Definition 4.1.5. *[2] An operator $T \in B(H)$ is said to be quasi-paranormal if for each vector $x \in H, ||T^*Tx||^2 \leq ||T2x||||x||$.*

Theorem 4.1.6. *[12] let $T \in B(H)$ be a quasi-paranormal operator, then T can be expressed uniquely as a direct sum $T = T_1 \oplus T_2$ defined on $H = H_1 \oplus H_2$ such that the following properties are satisfied;*

- *T_1 is normal.*
- *T_2 is a quasi-paranormal.*

Proof. Refer to Theorem 4.1 in [12]. □

Remark 4.1.7. *Theorem 4.1.6 shows that the operator $T \in B(H)$ decomposes into normal and completely non-normal part.*

4.1.2 Direct Summands of 2-normal Operators

In this subsection, we investigate the direct sum decomposition of 2-normal operators. As we stated earlier, a $2-normal$ operator T is an operator such that $T^*T^2 = T^2T^*$.

We begin this subsection with the following result.

Proposition 4.1.8. *Suppose $T \in B(H)$ decomposes as $T = T_1 \oplus T_2$, with respect to a decomposition $H = H_1 \oplus H_2$. If $T \in B(H)$ is a 2-normal operator, then each direct summand is 2-normal.*

Proof. Since T is 2-normal, then we have $T^*T^2 = T^2T^*$. Now $T^*T^2 = T_1^*T_1^2 \oplus T_2^*T_2^2$ and $T^2T^* = T_1^2T_1^* \oplus T_2^2T_2^*$. Since T is a 2-normal operator, we have that $T_1^*T_1^2 \oplus T_2^*T_2^2 = T_1^2T_1^* \oplus T_2^2T_2^*$.

This shows that $T_1^*T_1^2 = T_1^2T_1^*$ and $T_2^*T_2^2 = T_2^2T_2^*$.
Hence $T_i, i = 1,2$ is 2-normal as required. □

Proposition 4.1.9. *Suppose T is a normal operator, then T is 2-normal.*

Proof. Since T is normal, then $T^*T = TT^*$. This tells us that If T is normal, then T^* is also normal. Multiplying to the right by T, we have that $(T^*T)T = (TT^*)T = T(T^*T) = T(TT^*) = T^2T^*$. Hence T is 2-normal. □

Remark 4.1.10. *The converse of Proposition 4.1.9 is not true in general.*

To make sense of Remark 4.1.10, let us consider the following example.

Example 4.1.11. *Let $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. A simple calculation shows that $T^*T^2 = T^2T^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. This shows that T is 2-normal. Another simple computation shows that $T^*T \neq TT^*$. This shows that T is not normal. This example shows that If T is 2-normal then it is not generally true that T is normal.*

We proceed to give a condition for which a 2-normal operator $T \in B(H)$ or quasinormal operator $T \in B(H)$ is normal.

Proposition 4.1.12. *If $T \in B(H)$ is 2-normal and quasinormal and injective on $Ran([T^*, T])$, then T is normal.*

4.1.3 Equivalence Relations in Direct Sum Decomposition

In this subsection we want to look at direct sum decomposition of equivalence relations and properties the direct summands transfers to the arbitrary operator.
We start with the following results.

Theorem 4.1.13. *Let $S,T \in B(H)$. Suppose $T = T_1 \oplus T_2$ and $S = S_1 \oplus S_2$, where T_1, S_1 are normal and T_2, S_2 are c.n.n. and suppose that T and S are unitarily equivalent, then $S_i, i = 1,2$ are unitarily equivalent to $T_i, i = 1,2$. (Direct summands are unitary equivalent).*

`Proof.` Since T and S are unitarily equivalent, then there exists a unitary operator U such that $US = TU$ (equivalently $S = U^*TU$). Therefore:

$$S_1 \oplus S_2 = U^*(T_1 \oplus T_2)U$$
$$= U^*T_1U \oplus U^*T_2U$$
$$S_1 \oplus S_2 = U^*T_1U \oplus U^*T_2U$$

$\implies S_1 = U^*T_1U$ and $S_2 = U^*T_2U$ This shows that S_1 is unitarily equivalent to T_1 and S_2 is unitarily equivalent to T_2 □

Corollary 4.1.14. *[17] If an operator $T \in B(H)$ is similar (unitarily equivalent) to a direct summand of an operator $L \in B(K)$, then H is a direct summand of an operator similar (unitarily equivalent) to L*

Corollary 4.1.15. *[17] If an operator $T \in B(H)$ is unitarily equivalent to a direct sum $L \in B(K)$, then it is a direct sum itself with direct summand unitarily equivalent to each direct summand of L.*

Corollary 4.1.16. *[17] Every operator unitarily equivalent to a reducible operator is reducible.*

Remark 4.1.17. *Corollary 4.1.16 does not hold under similarity.*

Let us make sense of Corollary 4.1.16 by giving an example.

Example 4.1.18. *Let us consider the 3×3 matrices representing the operators $M, N,$ and O on $\mathbb{C}^3$.*

$$M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, O = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A simple calculation shows that $OM = NO, O$ *is invertible (Thus* M *and* N *are similar) and* N *is a direct sum, that is* $N = 1 \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *but* M *is irreducible since the only 1-dimensional invariant subspace* $W = span \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ *for* W *is not invariant for* M^*.

Theorem 4.1.19. *Let* $S, T \in B(H)$. *Suppose* $T = T_1 \oplus T_2$ *and* $S = S_1 \oplus S_2$ *where* T_1, S_1 *are normal and* T_2, S_2 *are c.n.n. and suppose that* T *and* S *are similar, then* S_i, *is similar to* $T_i, i = 1, 2$.

Proof. Since T and S are similar, then there exists an invertible operator N such that $NS = TN$ (equivalently $S = N^{-1}TN$). Therefore:

$$\begin{aligned} S_1 \oplus S_2 =& N^{-1}(T_1 \oplus T_2)N \\ =& N^{-1}T_1N \oplus N^{-1}T_2N \\ S_1 \oplus S_2 =& N^{-1}T_1N \oplus N^{-1}T_2N \end{aligned}$$

$\implies S_1 = N^{-1}T_1N$ and $S_2 = N^{-1}T_2N$
This shows that S_1 is similar to T_1 and S_2 is similar to T_2 □

Theorem 4.1.20. *Let* $S, T \in B(H)$. *Suppose* $T = T_1 \oplus T_2$ *and* $S = S_1 \oplus S_2$ *with* T_1, S_1 *are normal and* T_2, S_2 *c.n.n. and suppose that* T *and* S *are almost similar, then* $S_i, T_i, i = 1, 2$ *are almost similar.*

Proof. Since T and S are almost similar, then there exists an invertible operator N such that $S^*S = N^{-1}(T^*T)N, S^* + S = N^{-1}(T^* + T)N$. Therefore:

$(S_1 \oplus S_2)^*(S_1 \oplus S_2) = N^{-1}((T_1 \oplus T_2)^*(T_1 \oplus T_2))N = N^{-1}T_1^*T_1N \oplus N^{-1}T_2^*T_2N$
$S_1^*S_1 \oplus S_2^*S_2 = N^{-1}T_1^*T_1N \oplus N^{-1}T_2^*T_2N$

This implies that $S_1^*S_1 = N^{-1}T_1^*T_1N$ and $S_2^*S_2 = N^{-1}T_2^*T_2N$ This shows that S_1 is almost similar to T_1 and S_2 is almost similar to T_2.

Similarly,

$S^* + S = N^{-1}(T^* + T)N$
$(S_1 \oplus S_2)^* + (S_1 \oplus S_2) = N^{-1}((T_1 \oplus T_2)^* + (T_1 \oplus T_2))N$
$S_1^* + S_1 \oplus S_2^* + S_2 = N^{-1}(T_1^* + T_1)N \oplus N^{-1}(T_2^* + T_2)N$
This shows that $S_1^* + S_1 = N^{-1}(T_1^* + T_1)N$ and $S_2^* + S_2 = N^{-1}(T_2^* + T_2)N$

$\square$

4.2 On Unitary and Completely Non Unitary Summands of a Contraction Operator

In this section, we study the decomposition of a contraction into a direct sum of unitary and completely non-unitary parts. A contraction operator $T \in B(H)$ decomposes as $T = T_1 \oplus T_2$ where T_1 is unitary and T_2 is completely non-unitary part. If $T_1 = 0$, then T is a completely non-unitary contraction. We will further investigate the properties of completely non-unitary $(c.n.u)$ summands of this operator.
An operator $T \in B(H)$ is strongly stable if the sequence $\{T^n\}$ converges to the null operator.
We will take B to be the strong limit of $\{T^{*n}T^n\}_{n \geq 0}$.

4.2.1 Nagy-Foias Classes of Contraction

1 A contraction T is said to belong to class $C_{0.}$ if T is a strongly stable contraction.

2 A contraction T is said to belong to class $C_{.0}$ if T^* is a strongly stable contraction.

3 A contraction T is said to belong to class $C_{1.}$ if $T^n x \not\to 0$ for every nonzero vector $x \in H$.

4 A contraction T is said to belong to class $C_{.1}$ if $T^{*n} x \not\to 0$ for every nonzero vector $x \in H$.

Remark 4.2.1. *A contraction T is of class C_{00} if T and T^* are both strongly stable contraction. That is $B = B_* = 0$.*

From these classes of contractions, we have a generalization as follows.

- $T \in C_{00}$ if and only if $B = B_* = 0$.
- $T \in C_{01}$ if and only if $B = 0$ and $N(B_*) = \{0\}$.
- $T \in C_{10}$ if and only if $N(B) = \{0\}$ and $B_* = 0$.
- $T \in C_{11}$ if and only if $N(B) = \{0\}$ and $N(B_*) = \{0\}$.

Example 4.2.2. **1** *Let* $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *This operator is a unitary operator and therefore it decomposes as* $T = T_1 \oplus T_2$, *where* T_1 *is unitary and* T_2 *is completely non unitary. That is* $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $T_2 = 0$. *This means that completely non-unitary part is missing. This operator belongs to class* C_{11}.

2 *Let* $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *This operator decomposes as* $T = 1 \oplus \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, *where* $1 \oplus \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ *is unitary and the c.n.u part is missing.*

Remark 4.2.3. **1** *Every unitary operator is a contraction.* $(i.e. ||Tx|| = ||T^*x|| = ||x||, \; for\, all\; x \in H)$.

2 *Every isometry is a contraction operator since* $||Tx|| = ||x|| \forall x \in H$.

3 *Every co-isometry is a contraction operator since* $||T^*x|| = ||x|| \forall x \in H$.

4.2.2 Some Classes of Completely Non-Unitary Contraction

Note that a contraction T is of class C_{00} if $B = B_* = 0$. In this subsection, we want to have a look at some results on the nature of direct summands of some classes of c.n.u. operators.

We begin with the following result.

Theorem 4.2.4. *Paranormal contractions are a direct sum of unitary and* $C.0$ *completely non-unitary contraction.*

The following result follows immediately from Theorem 4.2.4

Theorem 4.2.5. *Let* $T = T_1 \oplus T_2$ *be a paranormal contraction where* T_1 *is unitary and* T_2 *is c.n.u. Then the completely non unitary part is of class* $C.0$.

Proposition 4.2.6. *Let* $T \in B(H)$ *be a normal contraction and that* $T = T_1 \oplus T_2$. *Then* T_2 *is of class* C_{00}.

`Proof.` Since T is normal, then $T^*T = TT^*$. Also since T decomposes as $T = T_1 \oplus T_2$, with T_1 unitary and T_2 completely non-unitary. From Theorem 4.2.5 we have that

T_2 is of class $C_{.0}$. We now need to check that T_2 is of class $C_{0.}$. Since T is normal, $B = Lim_{n\to\infty}T^{*n}T^n = Lim_{n\to\infty}T^nT^{*n} = B_* = 0$. Thus clearly T_2 is of class C_{00}. □

Let us consider the following example where T belongs to C_{00}.

Example 4.2.7. *let* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
This operator is completely non-unitary and therefore belong to class C_{00}.

Using this contraction operator, we answer the following question:
Can we find a nonzero reducing subspace M *such that* $T|_M$ *is unitary?*

Answer: NO.
The operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *is completely non-unitary.*

$$T^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

This shows that $T^n \to 0, T \in C_{0.}$.
Similarly, $T^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. *This shows that* $T^{*2} = 0$.
This implies that $T^{*n} \to 0, T \in C_{.0}$.
Therefore in this example we see that $T \in C_{00}$ *and thus the unitary part of the direct summand is missing.*

Remark 4.2.8. *We note that if a contraction operator is pure, then it is completely non-unitary but the converse is not true in general.*

Example 4.2.9. $T = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$. *This operator* T *belongs to the class* C_{00} *and* T *is normal.*
This means that not all C_{00} *contractions are pure. Similarly, there is no* C_{00} *contraction with a unitary part as in Example 4.2.7*

Theorem 4.2.10. *Let* $T \in B(H)$ *be a normal contraction and* $S \in B(H)$ *be similar to* T, *then* S *has a completely non-unitary part of class* C_{00}.

`Proof.` The proof to this theorem follows from Proposition 4.2.6 □

Corollary 4.2.11. *[15] A nonunitary C_{11} contraction is similar to a unitary operator if it is invertible.*

Proposition 4.2.12. *(Wold decomposition)[11] Every isometry is a direct sum of a unitary operator and a unilateral shift.*

The following result follows from Proposition 4.2.12

Lemma 4.2.13. *If $T \in B(H)$ is an isometry, then T decomposes as $T = T_1 \oplus T_2$, where T_1 is unitary and T_2 is completely non-unitary (unilateral shift) with respect to the decomposition $H = M \oplus M^{\perp}$.*

Proof. Since T is an isometry, we have that $B = Lim_{n\to\infty} T^{*n}T^{n} = I$ and $N(I-B) = H$ and M is a reducing subspace such that $M = Ker(I-B) \cap Ker(I-B_*) = Ker(I-B_*)$. This shows that $T|_M$ is unitary and $T|_{M^{\perp}}$ is a unilateral shift or c.n.u. □

Proposition 4.2.14. *An isometry is completely non-normal if and only if it is a unilateral shift.*

Proof. This follows immediately from the inclusion unitary $\subset$ normal. □

Theorem 4.2.15. *Suppose $M \in B(H)$ is metrically equivalent to an isometry $T \in B(H)$, then the unitary and completely non-unitary summands of M are isometric.*

Proof. First note that T is isometry. Then $T = T_1 \oplus T_2$ where T_1 is unitary and T_2 is a unilateral shift by von Neumann-Wold decomposition. Since M is metrically equivalent to T, then we write:

$$
\begin{aligned}
M^*M =& T^*T \\
=& (T_1 \oplus T_2)^*(T_1 \oplus T_2) \\
=& T_1^*T_1 \oplus T_2^*T_2 \\
=& I \oplus I
\end{aligned}
$$

Now suppose $M = M_1 \oplus M_2$, then

$$
\begin{aligned}
M^*M =& (M_1 \oplus M_2)^*(M_1 \oplus M_2) \\
=& M_1^*M_1 \oplus M_2^*M_2.
\end{aligned}
$$

Therefore, $M_1^*M_1 = I$, $M_2^*M_2 = I$. This proves that the direct summands of M are isometrics. $\square$

4.3 Spectral Picture in a Direct Sum Decomposition

In this section, we study the spectral picture and to be more specific, with use of examples we want to investigate the relationship between the arbitrary operator T and the direct summands in relation to the spectrum, numerical range, numerical radius and spectral radius.

We first look at the following example.

Example 4.3.1. *Let* $T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. *It is clear that* T *decomposes as* $T = T_1 \oplus T_2$.

This is equivalent to $T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \oplus 2$. *We start by examining the spectrum of* T, T_1 *and* T_2.

As we can see, $\sigma(T) = \{-1, 1, 2\}$, $\sigma(T_1) = \{-1, 1\}$ *and* $\sigma(T_2) = \{2\}$.

We can see that $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$.

We now investigate their numerical ranges.

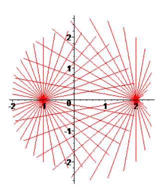

Figure 17. Numerical range of T.

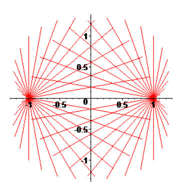

Figure 18. Numerical range of T_1.

Remark 4.3.2. *The numerical range of* T *is union of the numerical range of* T_1 *and* T_2.

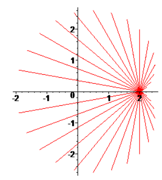

Figure 19. Numerical range of T_2.

Example 4.3.3. *Let us now consider the following square matrix* $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

T *decomposes as* $T = T_1 \oplus T_2$.

This is equivalent to $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Using the same concept, we begin by investigating their spectra.

As we can see that $\sigma(T) = \{1,1,0,0,\}$, $\sigma(T_1 = \{1,0\})$ *and* $\sigma(T_2) = \{0,1\}$.

We can see that $\sigma(T) \subseteq \sigma(T_1) \cup \sigma(T_2)$.

From these two examples we have the following result.

Theorem 4.3.4. *Suppose* T *decomposes as* $T = T_1 \oplus T_2$, *then the spectrum of* T *is a subset of* $\sigma(T_1) \cup \sigma(T_2)$. *(Equivalently,* $\sigma(T) \subseteq \sigma(T_1) \cup \sigma(T_2)$*).*

We now investigate the conditions under which the spectrum of T will be equal to the union of the spectra of T_1 *and* T_2.

Theorem 4.3.5. *The spectrum of the direct sum of an arbitrary operator* T *is equal to the spectra of its complementary parts if and only if the complementary parts commute.*

To make sense of this theorem, let us look at the following example:

Example 4.3.6. *Let* $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$. *This matrix decomposes as* $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \oplus \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

This implies that $[T_1, T_2] = 0$.

We then calculate their respective spectra.

We find that the spectrum of $T = \{1,2,3,4\}$ *and the spectra of* T_1 *and* T_2 *are* $\{1,2\}$, $\{3,4\}$ *respectively.*

We can clearly see that $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$.

Remark 4.3.7. *Note that the only condition when the spectrum of the arbitrary operator T is equal to the union of the spectra of direct summands parts is when the direct summands commute and* $\sigma(t) = \sigma(T_1) \cup \sigma(T_2)$.

Example 4.3.8. *let* $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. *This matrix decomposes as* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *Here* T_1 *is normal and* T_2 *is completely non-normal.*

We determine their spectrum, numerical radius and numerical range. To begin lets calculate their numerical ranges. Using Maple, we get the figures below.

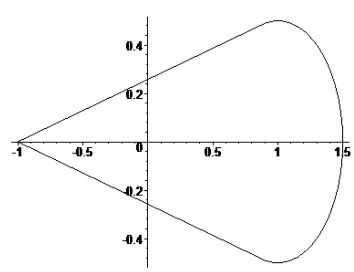

Figure 20. Numerical range of T.

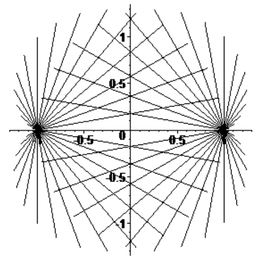

Figure 21. Numerical range of T_1.

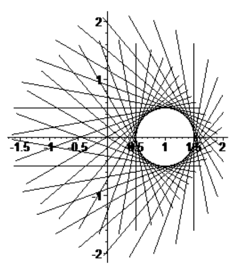

Figure 22. Numerical range of T_2.

From these figures we note that the numerical radius of $T = 1.5, T_1 = 1$ *and* $T_2 = 1.5$. Their numerical range and radius are not related in any way.

Remark 4.3.9. *We note that* $W(T) = conv\{W(T_1) \cup W(T_2)\}$.

5 Conclusion and Recommendations

5.1 Conclusion

In this project, we have shown that every operator has a Cartesian decomposition and a direct sum decomposition into normal and completely non-normal (c.n.n) parts (and that either direct summand may be missing). Likewise, every contraction operator has a direct sum decomposition into a unitary part and a completely non-unitary (c.n.u) part.

In chapter 3, we have shown some properties of an operator $T \in B(H)$ with a Cartesian decomposition $T = A + iB$, and its component parts A and B. For instance, in Theorem 3.1.1, we have shown that if T is normal and $T = A + iB$, then A commutes with B. We have also introduced the equivalence relations and connected it to the Cartesian decomposition. For example, in Theorem 3.3.8, we have proved that for any two arbitrary operators S and T, where $S = A + iB$ and $T = A + iB$ are unitarily equivalent. This result demonstrates that there exists a unitary operator U such that $T = U^*SU$, highlighting the significant relationship between these operators and their unitary equivalence under appropriate conditions.
In chapter 4, we have shown in Theorem 4.3.5 that if $T = T_1 \oplus T_2$, then $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$ if $[T_1, T_2] = 0$. This result provides a valuable insight into the spectral behavior of the direct sum of operators, illustrating that the spectrum of an operator T is the union of the spectra of its components T_1 and T_2 under the condition of commuting. These results contribute to a greater comprehension of operator theory and its uses in a wider mathematical setting.

In conclusion, our research has addressed a question proposed by one of the researchers in Theorem 4.1.19, which focused on investigating the similarities between two operators, S and T, where $S = S_1 \oplus S_2$ and $T = T_1 \oplus T_2$. Specifically, we aimed to determine the implications for the direct summands of both T and S in such cases.

5.2 Recommendation

The decomposition of operators has proven to be a valuable tool in the field of mathematical systems theory, as it allows for a more manageable analysis of complex systems. By decomposing T and S into their respective direct summands, $T_1, T_2, S_1, and S_2$, we were able to focus on studying the individual components of these operators.
We proved that if $T \overset{s}{\sim} S$ and $T = T_1 \oplus T_2$ and $S = S_1 \oplus S_2$, then $T_i \overset{s}{\sim} S_i, i = 1, 2$. But the converse has not been proved.

Question 1: If $S = S_1 \oplus S_2$ and $T = T_1 \oplus T_2$ and $S_i \overset{s}{\sim} T_i, i = 1, 2$, is it true that $S \overset{s}{\sim} T$?

Question 2: If $T = A + iB$, what is the relationship between $\sigma(T), \sigma(A)$ and $\sigma(B)$?

Question 3: If A and B are self adjoint operators and $T = A + iB$, can the information on $\sigma(A)$ and $\sigma(B)$ be enough to describe $\sigma(T)$?

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