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The COS Method for European Options in the Normal Inverse Gaussian Framework

By

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Abstract

The goal of this project is to explore the application of the Fourier-cosine expansion (COS) method within the framework of the Normal Inverse Gaussian (NIG) distribution for pricing European options, the COS-NIG model. The COS method, recognized as a highly efficient numerical tool, plays a pivotal role in the accurate pricing of European options. Our key insight lies in the close relationship between the characteristic function and the series coefficients derived from the Fourier-cosine expansion of the density function. Leveraging the known characteristic function of the NIG distribution, we develop a COS-NIG model for pricing of European options. The choice of the NIG distribution for modeling stock options is motivated by its ability to capture skewness and kurtosis, given the existence of higher moments, in contrast to the Gaussian distribution. Notably, the chosen distribution allows for a more accurate representation of the empirical density of log-returns. In our investigation, the COS-NIG model consistently surpasses the performance of the Black Scholes Model (BSM) especially for In the Money call options.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

29/11/2023

Signature Date

SHARON KOECH
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In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

10-December-2023

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Dedication

This thesis is dedicated to my lovely husband [Bethwel Kitum] and my son [Asher Kipkemoi],whose unwavering support and encouragement have been the driving force behind my academic journey. To my mum and my father-in-law your encouragement has been my inspiration, and your belief in my abilities has been my strength. Thank you for instilling in me the values of resilience, determination, and the importance of education. This achievement is as much yours as it is mine.

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This thesis is the culmination of collective contributions from these remarkable individuals, and for that, I am genuinely thankful.

Sharon Koech

Nairobi, 2023.

1 Introduction

1.1 Background

In the current complex economic world where prices fluctuate according to investor perception and there is numerous money exchange, trading of financial markets has become very uncertain. Derivative contracts have established themselves as the most popular method for investors to control risk while safeguarding their assets in the face of market volatility and uncertainty around future occurrences. These derivative contracts give investors the ability to diversify their holdings and use their money more effectively. Derivatives trading dates to early 1900's though these markets were then informal and unregulated. In the recent past, derivative markets have experienced tremendous growth across the globe. In fact, the government of Kenya through the Nairobi Stock Exchange approved the introduction of derivatives market in 2015 marking the beginning of a new market in the country. In June 2019, Nairobi Securities Exchange received the first license to trade derivatives from Kenya's capital markets regulator, CMA.

The value of financial derivatives depends on the value of another asset that they are based on. Some of the categories of financial derivatives includes futures, forwards, swaps, and options, which serve various purposes in speculating and managing risks within investments. When an individual purchases a financial asset with the expectation that its value will increase, they are said to have taken a long position. On the other hand, someone who sells a stock they do not yet own, hoping its value will decrease, is referred to as having taken a short position. Those who assume short positions typically borrow assets from significant financial institutions, selling them and later repurchasing the assets. An option is a type of financial contract that gives its owners the right, but not the obligation, to buy or sell another asset at a fixed price and within a certain period of time. The contract that allows the owner to buy the asset is called a call option, while the contract that allows the owner to sell the asset is called a put option. A stock option is a type of an option that derive its value from stocks.

Financial markets inherently carry risks, and inaccuracies in calculating option prices pose a considerable threat in option trading, potentially resulting in significant financial setbacks for investors. The primary objective of trading is to optimize anticipated returns while mitigating uncertainty. Investors instinctively gravitate towards companies poised to deliver substantial expected returns with commensurately low risks. The prosperity of an investor relies on adeptly navigating existing risks and making judicious investment choices.

Historically, Brownian motion, implying the normality of log returns, has been a fundamental concept in the field of option pricing. However, this model falls short in capturing extreme values in stock returns, as the distribution of stock returns exhibits higher kurtosis compared to a normal distribution. Moreover, the assumption of constant market volatility over time is often unrealistic. Recent studies have attempted to design models that better reflect market realities. GARCH models in (Aguilar, 2021) and (Madan & Seneta, 1990) for example, have been employed to account for conditional normality in stock returns, explaining phenomena like volatility clustering and ARCH effects. Nonetheless, these models struggle to capture the excess kurtosis and skewness present in real-world data. Exponential levy models have been proposed as the most suitable processes to incorporate this stylized facts of stock returns as well as maintaining their independence and stationarity. See for instance, (Barndorff-Nielsen, 1997), (Boyarchenko & Levendorskii, 2002), (Boyarchenko & Levendorski, 2002) and (Boyarchenko & Levendorski, 2022).

Levy processes do not have a convenient analytic tractability as the Geometric Brownian motion assumed in the Black Scholes model.The classic PDE techniques have been extended to the Partial Differential Integro Equations (PIDE) by (Matache, Von Petersdorff, & Schwab, 2004) and (Cont & Voltchkova, 2005). Montecarlo simulation techniques have proven viable also for the Variance Gamma(VG) and the NIG model as seen in (Avramidis & L'Ecuyer, 2006). Fourier based transform approach including the DFT, FFT and FrFFT have utilized the availability of a closed form characteristic function to obtain the option prices. In recent years, the Fourier-Based Cosine Method has emerged as a promising alternative for option pricing due to its computational efficiency and accuracy in handling complex market scenarios. Unlike other numerical methods, the Cosine Method operates in the frequency domain, providing analytic solutions to the option pricing equation. This approach allows for faster computation and improved performance compared to the former numerical techniques.

In this thesis, we aim to build upon (Fang & Oosterlee, 2009) and (Junike, 2023) work by exploring an alternative distribution for the log-returns of the underlying security, namely the Normal Inverse Gaussian (NIG) distribution. The motivation behind this research is rooted in the recognition of the deficiencies of option pricing models in dealing with complex market dynamics. Financial assets often exhibit jumps in prices and non-normal return distributions, particularly during periods of market turbulence and economic events. Therefore, it is essential to develop option pricing models that can effectively capture these features and provide more reliable valuation results.

The comparison of the Cosine Method's functionality with the classical Black-Scholes model will shed light on the trade-offs between accuracy and computational efficiency. This comparative analysis will provide valuable insights into the suitability of the Cosine Method for different market conditions and the potential advantages of incorporating more sophisticated return distributions. This research strives to improve the precision and applicability of option pricing in real-world financial markets by delving into the Normal Inverse Gaussian (NIG) distribution as a more fitting model for log-returns. The research endeavor is driven by the aspiration to advance option pricing methodologies, leading to more robust and realistic financial modeling approaches.

1.2 Statement of the problem

The Generalized Hyperbolic models, particularly the specialized subclass known as the NIG model of (Barndorff-Nielsen, 1997) is suggested as potential solutions to address the excess kurtosis and skewness in stock returns. The NIG model's ability to capture skewness and kurtosis makes it a promising candidate for describing the heavy-tailed return distributions observed in real-market scenarios.

The convenient analytic tractability observed in option pricing under the Geometric Brownian process assumption of the Black-Scholes-Merton model however, does not seamlessly extend to pricing models that incorporate stochastic volatility and Levy processes in modeling asset returns. These factors introduces complexities that challenge the straightforward analytic solutions characteristic of the Black-Scholes-Merton framework. This departure from the simplicity of geometric Brownian motion highlights the inherent challenges and increased computational demands associated with modeling more realistic market dynamics.Furthermore, the need for speed and accuracy in pricing European options, especially for a variety of strike prices at a single spot price, is essential for calibration at financial institutions. While existing integration methods are efficient for plain vanilla options, there is a constant pursuit in computational finance to improve the performance of the pricing methods. Quadrature rule-based techniques, however, are less efficient when solving Fourier transformed integrals, which often require a fine grid for accuracy. In this thesis, we will focus on Fourier-cosine expansions introduced by (Fang & Oosterlee, 2009) for numerical integration as an alternative approach to address these challenges and enhance the precision of option pricing methods.

1.3 Objectives

1.3.1 General objectives

The primary goal of my project is to create pricing formulas for European options using the Fourier-based cosine method, with a specific focus on incorporating the distribution of log-returns of the underlying security as a normal inverse Gaussian distribution.

1.3.2 Specific objectives

1. To model the log returns of stock prices using the Normal Inverse Gaussian Distribution
2. To develop a European Option pricing model based on the COS method when the log returns follow a NIG distribution
3. Compare the performance of the developed NIG Option pricing model with the Black Scholes model and the prevailing market option prices.

1.4 Significance of the study

The efficient pricing of financial options has been a central focus in the field of quantitative finance. Conventional option pricing models, like the Black-Scholes model, operate on the assumption of constant volatility and log-normal distributions for the returns of the underlying asset. However, these models often fall short in accurately capturing the dynamics of real-world financial markets, particularly when dealing with assets that exhibit jumps and non-normal return distributions.

The main objective of this thesis is to formulate pricing formulae for European options by employing the Fourier-Based Cosine Method, particularly when the distribution of log-returns of the underlying security adheres to a Normal Inverse Gaussian (NIG) model. The NIG model is recognized for its capacity to represent skewness and kurtosis, rendering it a more appropriate choice for characterizing financial return distributions with heavy tails

The Fourier-Based Cosine Method is known for its high computational efficiency. It typically requires a significantly smaller number of terms in the series expansion compared to other numerical methods, such as finite difference or Monte Carlo simulation, to achieve accurate results. This results in faster computation times, making it suitable for pricing options in real-time and large-scale applications. Moreover, the method is highly flexible and can be adapted to handle various types of options, including European, American, and exotic options, as long as their payoffs can be expressed analytically. It is particularly powerful when dealing with options on assets with non-normal return distributions, jumps, or stochastic volatility.

In recent literature, (Junike, 2023) employed the Cosine method to effectively price European options when the underlying return distribution followed a Variance Gamma (VG) model. Although this approach displayed potential, it was observed that the necessary bound for the number of terms, denoted as (N) in the Cosine method, was excessively large to be practically applicable. In this thesis, our objective is to expand upon Junike's work by substituting the VG model with the Normal Inverse Gaussian (NIG) model to address this limitation. The NIG distribution offers more flexibility and a broader range of possible return distributions, and it has the potential to overcome the limitations observed with the VG model. By using the NIG model, we seek to enhance the accuracy of option pricing and better reflect the complexities of real-world market dynamics.

The significance of this research lies in its potential contributions to the field of option pricing and quantitative finance. By incorporating the Fourier-Based Cosine Method with the NIG distribution, we expect to achieve more accurate pricing results for European options, particularly for assets with non-normal return distributions and jumps. This research can offer valuable insights for traders, investors, and risk managers, aiding them in making informed decisions and developing more effective hedging strategies in real-market scenarios.

Additionally, the comparison of the Cosine Method's functionality with the Black-Scholes model, known for its simplicity and widespread use, will shed light on the trade-offs between accuracy and computational efficiency. By showcasing the benefits of incorporating the NIG distribution in the option pricing framework, this research can potentially lead to advancements in the understanding and application of option pricing models with non-normal return distributions, paving the way for more realistic and robust financial modeling approaches. This thesis endeavors to contribute to the advancement of option pricing models in the literature by integrating the Fourier-Based Cosine Method with the NIG distribution. The research aims to provide a deeper understanding of option pricing with non-normal return distributions and jumps in asset prices, ultimately enhancing the accuracy and practicality of option valuation methods in real-world financial markets.

2 Literature Review

The widely recognized Black-Scholes Option Pricing models, introduced by Black and Scholes in 1973, are based on several crucial assumptions that have been demonstrated to be unrealistic in real-world financial markets. In response to this advancements in option pricing theory, a new category of models has emerged in the literature to better address the stylized features of actual markets. These novel models, belonging to the Lévy process family, are primarily distinguished by their probabilistic property of having infinitely divisible distributions(Kyprianou, 2014). Examples of these models which consider excess kurtosis and skewness are the Variance Gamma (VG), the Generalized Hyperbolic (GH) model and the Normal Inverse Gaussian (NIG). To estimate the degree of uncertainty in the return on the underlying asset, (Madan & Seneta, 1990) created the Variance Gamma (VG) model, a continuous-time stochastic process. This model served as a valuable and practical alternative to Brownian motion as the martingale component for modeling the dynamics of log prices. (Adeosun, Edeki, & Ugbebor, 2016) conducted an extensive analysis of the VG process, which is represented as a difference of two gamma processes. As a result of this analysis, (Adeosun et al., 2016) proposed a modified European call option VG model that incorporates the difference of two gamma processes, providing an improved framework for option pricing. (Harrison & Pliska, 1983) pointed out the significance of introducing a stochastic process for applications in European option pricing that do not require the separate computation of risk-neutral expectations. Instead, such processes take into consideration risk aversion through the identification of an exact change of measure. This approach provides a more comprehensive and integrated perspective on pricing options in the presence of risk aversion.(Eberlein & Keller, 1995) introduced the use of Generalized Hyperbolic(GH) distributions as a modeling approach to capture the peaked and skewed distribution characteristics often observed in equity returns. (Barndorff-Nielsen, 1997) used a normal variance-mean mixture where the mixing density is the inverse Gaussian distribution and proposed a normal-inverse Gaussian distribution (NIG) process.(YILMAZ & Hekimoglu, 2022) applied two pure jump models, the Variance Gamma (VG) and Normal-Inverse Gaussian (NIG) models, to fit the BIST30 index, which represents an emerging market. Their research confirmed that these models outperformed the classical Black-Scholes (BS) model in the context of option pricing, demonstrating their effectiveness in capturing the unique features of emerging market data.

The mathematical analysis of the jump models has been proved to be difficult. Monte Carlo simulations, Partial (integro) differential equations (PIDE) and the Fourier methods (Eberlein, 2013) are some of the commonly known methods for numerical evaluation.

(Ano & Ivanov, 2012) utilized the NIG process in modeling and optimizing investment strategies. Some specific option valuation methods also have been proposed which include the numerical quadrature method (Fusai & Meucci, 2008) for Asian options and Wilkinson Approximation for exotic options (Albrecher & Predota, 2004). (Ivanov, 2013) introduced analytical formulas for closed form pricing of European options for a family of NIG processes. These formulas were derived based on the values of the degenerate Appell hypergeometric function.Recent studies in the field of option pricing have continued to advance the methodology and expand the range of models that can be efficiently used to price options. For example, (Kirkby & Nguyen, 2020) developed an innovative and efficient transform approach for pricing Asian options for general asset dynamics, such as Regime Switching Levy processes and stochastic volatility algorithms with jumps, using duality and FFT-based density projection implementation. The use of Mellin Transform and residue calculus to obtain closed-form solutions for vanillas (European), digitals, power, and log options was introduced by (Aguilar & Kirkby, 2022). Under Lévy-driven models, Fourier transform-based strategies for option pricing are among the most popular and generally effective methods. Since the first use of Fast Fourier Transform (FFT method) by (Carr & Madan, 1999), several extensions have been made including (Lewis, 2001), (Jackson, Jaimungal, & Surkov, 2008) and (Lord, Fang, Bervoets, & Oosterlee, 2008).

In February 2008,(Fang & Oosterlee, 2009) proposed a fast and efficient way to compute the prices of European-style options. Their method hinged on a crucial insight, emphasizing the close correlation between the characteristic function and the series coefficients derived from the Fourier-Cosine expansion of the density function. This insight paved the way for a more efficient valuation of European options.Furthermore, (Fang & Oosterlee, 2009) extended their approach to pricing early-exercise and discretely monitored barrier options. They also devised an effective numerical method based on Fourier analysis for pricing Bermudan options and discretely monitored barrier options under the Heston Stochastic Volatility model. This extended technique employed the Fourier-Cosine series expansion in conjunction with high-order quadrature rules in the other dimension. This research, presented by (Fang & Oosterlee, 2011), significantly contributed to the field of quantitative finance by enhancing the precision and computational efficiency of option pricing models. (Alexander & Venkatramanan, 2012) extended the Cos method to higher dimensions with a multidimensional asset price process allowing the algorithm to be applied to rainbow options and basket options. In addition, (B. Zhang & Oosterlee, 2013) introduced an effective pricing algorithm for swing options using the Cos method. Their work specifically addressed the pricing of discretely monitored geometric and arithmetic Asian options that incorporated early exercise features. This algorithm was aptly named ASCOS. It employed Richardson extrapolation to determine the prices of continuously monitored options. In 2018,(Leitao, Oosterlee, Ortiz-Gracia, & Bohte, 2018) introduced the data-driven cosine (ddCOS) method, which stands as a model-independent approach, obviating the need for an analytically known characteristic function. In lieu of this requirement, the method relies on the estimation of coefficients that represent the distribution of the terminal stock price. These coefficients are determined through the analysis of historical data values.

3 Fundamentals of derivative pricing

In this chapter, we delve into the core principles of derivatives pricing. One of the main concepts in derivative pricing is the understanding that the behavior of underlying assets, such as stocks, commodities, interest rates, and exchange rates, is best described by stochastic processes. We explore the intricacies of the Geometric Brownian Motion first introduced by (Black & Scholes, 1973) in 1973 and the Normal Inverse Gaussian Model of (Barndorff-Nielsen, 1997) used in modeling the underlying assets and how these models facilitate the valuation of derivative contracts. Additionally, we describe the computational techniques employed to price derivatives under various assumptions about the underlying stochastic processes.

3.1 Geometric Brownian Motion

3.1.1 Stochastic Differential Equation

Instantaneous price movements are characterized as:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

S_t is the current stock price, μ is the drift of the expected return of the asset, σ is the volatility of the asset and dW is a wiener process (Brownian motion) increment.

The solution to this SDE is

$$S_T = S_0 \exp\left((\mu - \frac{\sigma^2}{2})(T-0) + \sigma(W_T - W_0)\right) \tag{2}$$

The mean and variance of S_T are:

$$E(S_T) = S_0.\exp(\mu T)$$
$$Var(S_T) = S_0^2.\left[\exp(2\mu T)\right]\left[\exp(\sigma^2 T) - 1\right]$$

and

$$\log_e S_t \sim N\left(\log_e S_0 + (\mu - \frac{\sigma^2}{2})t, \sigma W_t\right) \tag{3}$$

3.1.2 Black Scholes Model

The fundamental concept of the BSM model is the assumption that price movements follow a Geometric Brownian Motion. Other key assumptions in the model are:

1. **Assumption of Constant Volatility:** It is assumed that the volatility of returns for the underlying asset remains stable throughout the analysis.
2. **Constant and Known Risk-Free Interest Rate:** The presence of a constant and well-defined risk-free interest rate is considered as part of the modeling assumptions.
3. **No Dividend Payments by the Underlying Stock:** During the lifespan of the option under consideration, it is presumed that the underlying stock does not issue any dividend payments.
4. **Efficient Market and Absence of Transaction Costs:** The market is regarded as efficient, and it is assumed that no transaction costs, taxes, or short-selling restrictions exist.

Black Scholes Equation

Let S be an Ito process that has the SDE given in equation 1 and let f be a twice differentiable scalar function and a derivative dependent on S then f is an Ito process and

$$df = \left(\mu S\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial s^2}\right)dt + \sigma S\frac{\partial f}{\partial s}dW \tag{4}$$

where,

$$\begin{aligned}(dt)^2 &= 0\\ dt.dW &= 0\\ (dW)^2 &= 0\end{aligned}$$

Consider a risk free portfolio that replicates the payoffs of the option. The portfolio consists of the option itself and a certain amount of the underlying asset say a. Adjusting the proportion of the option and the underlying asset continuously can eliminate risk and achieve the same payoff as the option. This leads to the concept of risk neutral pricing. From equation 4,

$$d(f+aS) = \left(\mu S\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial s^2} + a\mu S\right)dt + \sigma S\left(\frac{\partial f}{\partial s} + a\right)dW \tag{5}$$

To hedge away all risk in our portfolio, we let $a = -\frac{\partial f}{\partial s}$

$$d(f+aS) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial s^2}\right)dt \tag{6}$$

The portfolio is now risk free and it must therefore grow at a risk free interest rate r.

$$\frac{d}{dt}(f+aS) = r(f+aS) = r\left(f - S\frac{\partial f}{\partial S}\right) \quad (7)$$

$$r\left(f - S\frac{\partial f}{\partial S}\right) = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial s^2} \quad (8)$$

If we rearrange equation 8, we get the Black Scholes Equation.

Black Scholes Equation. Given a stock price S that follow an Ito process, then the value of an option f of S is evaluated from the following equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial s^2} - rf = 0 \quad (9)$$

Inorder to obtain a unique solution to the Black Scholes Equation 9, we must specify the boundary conditions.

European Call and Put option prices

The price of a European call option at time t, $C(t)$, under the risk neutral probability measure can be easily obtained from equation 9. A step by step derivation of the price is given in (Nielsen, 1992). If we have K as the strike price of an option and S_t as the current price of the underlying stock, then:

$$C(t) = S_t N(d_1) - K e^{-r\tau} N(d_2) \quad (10)$$

Where:

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r - 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau} \quad (11)$$

$$d_2 = d_1 - \sigma\sqrt{\tau} \quad (12)$$

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{\frac{1}{2}s^2}\, ds \quad (13)$$

For a European put option valued as $P(t)$ with the same strike price K and current price of the underlying asset S_t as the call option $C(t)$, the following relation holds:

$$P(t) - C(t) = Ke^{-r\tau} - S_t \quad (14)$$

This relation must hold otherwise arbitrage opportunities will exist. From this equation 14, $P(t)$ is given by;

$$\begin{aligned} P(t) &= C(t) + Ke^{-r\tau} - S_t \\ &= K\,e^{-r\tau}N(-d_2) - S_tN(-d_1) \end{aligned} \tag{15}$$

The BSM has been widely used since its inception. It has however not been without critique. Studies have shown that it cannot effectively accommodate the inherent characteristics of financial assets, such as non-normality and jumps in asset prices.The Generalized hyperbolic models have been suggested as one of the approaches that could deal with the excess kurtosis and skewness of stock returns.In this study we propose the use of a specialized subclass of the Generalized Hyperbolic model known as the Normal Inverse Gaussian (NIG)model as an alternative.

3.2 Normal Inverse Gaussian (NIG) distribution

3.2.1 Inverse Gaussian Distribution

The inverse Gaussian distribution is a continuous probability distribution parameterized by two parameters mean μ and shape λ. Its pdf is given by:

$$f_{IG}(y) = \sqrt{\frac{\lambda}{2\pi y^3}} e^{-\frac{1}{2\mu^2 y}\lambda(y-\mu)^2} \tag{16}$$

The IG distribution is the distribution of the first passage time of a Brownian motion with drift $\gamma > 0$ for a barrier $\delta > 0$. If we let the barrier to be a linear function of the time t, we obtain the IG process which is defined as:

$$\begin{aligned} dZ_t &= \gamma dt + dW_t \\ g_t &= inf\{s > 0, Z_s = \delta t\} \end{aligned} \tag{17}$$

where $\{Z_s; s > 0\}$ is a Brownian motion with drift $\gamma > 0$. The distribution of the IG process at time t is given as

$$g_t \sim IG\left(\frac{\delta t}{\gamma}, \delta^2 t^2\right)$$

with mean(μ) and variance (κ). The variance of the IG distribution is given by $\frac{\mu^3}{\lambda}$. By changing variables, we get the variance of the IG process at time t as $\frac{\delta t}{\gamma^3}$. We can then have $\kappa = \frac{\delta}{\gamma^3}$. Consider a Brownian motion with drift

$$S_t = \theta t + \sigma W_t$$

If we substitute the time variable with an inverse Gaussian r.v

$$g_t \sim \Gamma(t, \kappa t)$$

we get the Normal Inverse Gaussian process,

$$S_t = \theta g_t + \sigma W_{g_t} \tag{18}$$

We choose

$$g_t \sim IG\left(\mu = t, \lambda = \frac{t^2}{\kappa}\right)$$

with $\delta = \gamma$ and $\kappa = \frac{1}{\gamma^2}$

3.2.2 NIG density

A random variable x is said to be NIG distributed if it has the pdf given as:

$$\begin{aligned} f_{\text{NIG}}(x) = & \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^2 K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \cdot \left((x-\mu)^2+\delta^2\right)^{\frac{\lambda}{2}-\frac{1}{4}} \\ & \times K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{(x-\mu)^2+\delta^2}\right)e^{(x-\mu)\beta} \end{aligned} \tag{19}$$

with parameters

$$\begin{aligned} \beta &= \frac{\theta}{\sigma^2} \\ \alpha &= \sqrt{\beta^2+\frac{1}{\kappa\sigma^2}} \\ \delta &= \frac{T\sigma}{\sqrt{\kappa}} \\ \mu &= cT \end{aligned} \tag{20}$$

A four-dimensional parameter vector determines the geometry of the NIG-density. Given the flexibility of this parameterization, it is possible to model a wide range of geometries and tail decay rate.

3.2.3 Properties of the NIG distribution

Moment Generating Function

Using (Barndorff-Nielsen, 1997) we derive the moment generating function of the NIG distribution and extend the result to determine its characteristic function.
The Moment Generating Function (MGF) of a random variable is defined as:

$$M_x(t) = E[e^{(tx)}] \tag{21}$$

where t is a real number and $E[\,]$ denotes the expectation (mean) operator.
From equation 21, the MGF of the NIG distribution can be derived as follows:

$$\begin{aligned} M_x(t) &= E[e^{(tx)}] \\ &= \int_{-\infty}^{\infty} e^{(tx)} f_{NIG}(x)\,dx \\ &= \int_{-\infty}^{\infty} e^{(tx)} \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^2 k_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \cdot \left((x-\mu)^2+\delta^2\right)^{\frac{\lambda}{2}-\frac{1}{4}} \cdot \\ &\quad K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{(x-\mu)^2+\delta^2}\right) e^{(x-\mu)\beta}\,dx \\ &= \frac{\exp\left\{\delta\sqrt{\alpha^2-\beta^2}-\beta\mu\right\}}{\exp\left\{\delta\sqrt{\alpha^2-(\beta+t)^2}-(\beta+t)\mu\right\}} \\ &= \exp\{\mu t\}\exp\left\{\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+t)^2}\right)\right\},\ |\beta+t|<\alpha \end{aligned} \tag{22}$$

Characteristic function

The characteristic function of a NIG random variable x is given by

$$\varphi(v) = M_x(iv)$$

where $i=\sqrt{-1}$. Thus we have that:

$$\begin{aligned} \varphi_{NIG}(v) &= M_{NIG}(iv) \\ &= \exp\{\delta\sqrt{\alpha^2-\beta^2}\}.\exp\{-\delta\sqrt{\alpha^2-(\beta+iv)^2}\}.\exp\{i\mu v\} \end{aligned} \tag{23}$$

and the characteristic exponent will be:

$$\phi(v) = \delta\left\{\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+iv)^2}\right\} \tag{24}$$

Cumulants

The cumulant generating function is given by:

$$\begin{aligned} K_x(v) &= \ln(\varphi_x(v)) \\ &= \delta\left[\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+iv)^2}\right]+i\mu v. \end{aligned}$$

The n^{th} cumulant is thus given by:

$$\begin{aligned}
K_x^{(n)} &= (-i)^n \frac{d^n \varphi(v=0)}{dv^n} \\
K_x^{(1)} &= \mu + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}} \\
K_x^{(2)} &= \frac{\alpha^2\delta}{(\alpha^2-\beta^2)^{\frac{3}{2}}} \\
K_x^{(3)} &= \frac{3\beta\delta\alpha^2}{(\alpha^2-\beta^2)^{\frac{5}{2}}} \\
K_x^{(4)} &= \frac{3\delta\alpha^2(\alpha^2+4\beta^2)}{(\alpha^2-\beta^2)^{\frac{7}{2}}}
\end{aligned} \tag{25}$$

Central moments

$$\begin{aligned}
E[X] = K_x^{(1)} &= \mu + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}} \\
Var[X] = K_x^{(2)} &= \frac{\alpha^2\delta}{(\alpha^2-\beta^2)^{\frac{3}{2}}} \\
Skew[X] = K_x^{(3)} &= \frac{3\beta\delta\alpha^2}{(\alpha^2-\beta^2)^{\frac{5}{2}}} \\
Kurt[X] = K_x^{(4)} &= \frac{3\delta\alpha^2(\alpha^2+4\beta^2)}{(\alpha^2-\beta^2)^{\frac{7}{2}}}
\end{aligned}$$

Convolution Property

If X and Y are two independent random variables with $x \sim NIG(\alpha,\beta,\delta_x,\mu_x)$ and $Y \sim NIG(\alpha,\beta,\delta_y,\mu_y)$, then we have;

$$\begin{aligned}
M_{X+Y}(t) &= M_X(t).M_Y(t) \\
&= \exp\{\mu_x t\} \exp\left\{\delta_x\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+t)^2}\right)\right\} \\
&. \exp\{\mu_y t\} \exp\left\{\delta_y\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+t)^2}\right)\right\} \\
&= \exp\{(\mu_x+\mu_y)t\} \exp\left\{(\delta_x+\delta_y)\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+t)^2}\right)\right\},\ |\beta+t|<\alpha \\
&\sim NIG(\alpha,\beta,\mu_x+\mu_y,\delta_x+\delta_y)
\end{aligned}$$

Semi-heavy tails

The Bessel function exhibits the following behavior asymptotically;

$$K_1(x) \sim \frac{1}{\sqrt{2}} e^{-x} \sqrt{\frac{\pi}{x}} \quad \text{as} \quad |x| > \infty.$$

It then follows that

$$f_{NIG}(x) \sim \frac{1}{\sqrt{x}^3} e^{-\alpha|x|} e^{\beta x} \quad \text{as} \quad x > \pm\infty \quad \text{and} \quad \alpha - |\beta| > 1$$

3.2.4 Estimating NIG parameters

The four parameters of the NIG distribution are;
Location Parameter (μ)**:** This parameter represents the location or central value of the distribution. It corresponds to the mean of the distribution.
Scale Parameter (α)**:**This parameter controls the scale or spread of the distribution. It is similar to the standard deviation in a normal distribution but can also be negative, which allows for asymmetry.
Shape Parameter (β)**:**This parameter controls the shape of the distribution, particularly its tail behavior. It can take positive or negative values. Positive values indicate a right-skewed distribution with heavier tails, while negative values indicate a left-skewed distribution with lighter tails.
Dispersion Parameter (δ)**:** This parameter controls the dispersion of the distribution. It is related to the precision of the distribution and influences the kurtosis. It can take positive values.
These parameters are estimated using the maximum likelihood method.
If we have $x_1, x_2, \cdots, x_n$ I.I.D random variables.Then,

$$\log L(\Omega) = \sum_{j=1}^{n} \log(f(x_j; \Omega))$$

Ω represents the set of model parameters.
The log-likelihood function of $x \sim NIG(\alpha, \beta, \delta, \mu)$ is:

$$\log L(\Omega) = n \log \alpha + n \log \delta + n \left[\delta \sqrt{\alpha^2 - \beta^2} - \beta \mu \right] + \beta \sum_{j=1}^{n} x_j - \frac{1}{2} \sum_{j=1}^{n} \log \left[\delta^2 + (x_j - \mu)^2 \right] + \sum_{j=1}^{n} \log k_1 \left[\alpha \sqrt{\delta^2 + (x_j - \mu)^2} \right] \tag{26}$$

The parameter estimation can be easily achieved using the *SciPy* package in python, particularly the *scipy.optimize* module. See Appendix A.1.2

4 The COS Method

4.1 Fourier-Cosine Expansion

In risk neutral pricing, the valuation of a derivative involves determining its cost by considering the anticipated value of its future payments when discounted under the risk-neutral measure. Specifically, for a derivative with a payoff function represented by $h(S_T)$, the calculation is done as follows:

$$\begin{aligned} V(t) &= e^{-r\tau} E^{\mathbb{Q}}\left[h(y_T|F_t)\right] \\ &= e^{-r\tau} \int_{\mathbb{R}} h(y)\, p(y,T|F_t) dy \end{aligned} \tag{27}$$

In this case:
$E^{\mathbb{Q}}[]$ represents the expectation under the risk-neutral measure, r is the risk-free interest rate, τ the time of expiration for the derivative $T-t$, $h(y_T)$ the payoff of the derivative at time T, F_t the information available at time t and $p(y,T|F_t)$ is the risk-neutral pdf of the asset price S(T) at time T, given the information available at time t.

In equation 27, evaluating the integral is straightforward when you have a known pdf in closed form. However, this isn't always the case in exponential Levy models. In such situations, you can instead work with the characteristic function, which is the continuous Fourier transform of the density function. To recover the density, you then need to apply the inverse continuous Fourier transform. The advantage here is that the characteristic function is often readily available in closed form, making it easier to work with. Once you have the continuous Fourier integral, you can discretize it and efficiently compute it using the Fast Fourier Transform (FFT) algorithm. In (Carr & Madan, 1999) and (Reiner, 2000), Fourier transforms and damping techniques are used to handle the non-integrability nature of call options with respect to the logarithm of the strike price. While numerical integration methods combined with the FFT algorithm provide efficient ways to compute Fourier integrals, they come with trade-offs in terms of error convergence, grid size requirements, and limitations on grid coarseness (Fang, 2010).In this thesis, we explore an alternative approach for solving equation 28 proposed by (Fang, 2010). Using the findings by (Boyd, 2001) that Fourier-cosine series expansions usually give an optimal approximation of functions with a finite support, (Fang, 2010) restructured equation 28 as a function of cosines.

The Fourier relationship between the characteristic function $\varphi_x(v)$ and the pdf $f_x(y)$ is as follows:

$$\begin{aligned} \varphi_x(v) &= \int\limits_{\mathbb{R}} e^{iyv} f_x(y)\, dy \\ f_x(y) &= \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{-iyv} \varphi_x(v)\, dv \end{aligned} \tag{28}$$

For the Fourier Transform to exist, the following conditions must be met:

$$\begin{aligned} &1. \quad \text{Integrability:} \quad \int_{-\infty}^{\infty} |f(t)|\, dt < \infty \\ &2. \quad \text{Finite Energy:} \quad \int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty \\ &3. \quad \text{Piecewise Continuity:} \quad \text{The function should have bounded variation.} \end{aligned} \tag{29}$$

The Fourier expansion of a function $g(\theta)$ on the interval $[-1,1]$ is a representation of $g(\theta)$ as an infinite sum of trigonometric functions, specifically sines and cosines. Mathematically, it can be expressed as:

$$g(\theta) = \sum_{n=0}^{\infty} a_n \cos(n\pi\theta) + \sum_{n=1}^{\infty} b_n \sin(n\pi\theta) \tag{30}$$

$$a_n = \int\limits_{-1}^{1} g(\theta) \cos(n\pi\theta)\, d\theta$$

and

$$b_n = \int\limits_{-1}^{1} g(\theta) \sin(n\pi\theta)\, d\theta$$

Given a function $g : [0,\pi] \to \mathbb{R}$ defined on the interval $[0,\pi]$, we can extend it to interval $[-\pi,\pi]$ while preserving its symmetry around $\theta = 0$ as follows:

$$g(\theta) = \begin{cases} g(\theta), & \text{if } \theta \geq 0 \\ g(-\infty), & \text{if } \theta < 0 \end{cases}$$

Here, $g(\theta)$ takes the value of $g(\theta)$ for θ in the original interval $[0,\pi]$ and mirrors it for θ in the interval $[-\pi,0]$ to ensure that the function is even around $\theta = 0$. Setting $b_n = 0$ in 30, we have:

$$g(\theta) = {\sum_{n=0}^{\infty}}' \bar{a}_n \cos(n\theta)$$

with

$$\bar{a}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \bar{g}(\theta)\cos(n\theta)\,d\theta = \frac{2}{\pi}\int_{0}^{\pi} \bar{g}(\theta)\cos(n\theta)\,d\theta$$

where the $\sum'$ symbol means that the first term is multiplied by half.
By performing a change of variables, on the function $g(\theta)$ in any finite interval, say $[\alpha,\beta] \in \mathbb{R}$ we obtain the Fourier cosine expansion. Let

$$y = \frac{\beta-\alpha}{\pi}\theta + \alpha$$
$$\theta = \frac{y-\alpha}{\beta-\alpha}\pi$$

it then reads

$$g(y) = \sum_{n=0}^{\infty} \bar{a}_n.\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right) \tag{31}$$

with

$$\bar{a}_n = \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} g(y).\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy \tag{32}$$

Definition 4.1.1. *A truncated probability density function (pdf) is a probability distribution characterized by a finite support, where the density function is approximated by a finite number of terms in a cosine expansion (Fang & Oosterlee, 2009)*

To satisfy the conditions in equation 29, the integrands in equation 28 must approach zero as the integration limits extend to $\pm\infty$. This property allows us to truncate the integration range without sacrificing significant accuracy. Let's assume that we select the interval $[\alpha,\beta] \in \mathbb{R}$ in such a way that the truncated integral as described in 4.1.1 provides a very close approximation to the infinite counterpart.

$$\hat{\varphi}_x(v) = \int_{\alpha}^{\beta} e^{ivy} f_x(y)\,dy \approx \int_{\mathbb{R}} e^{ivy} f_x(y)\,dy = \varphi_x(v) \tag{33}$$

By Euler Formula,

$$e^{iv} = \cos(v) + i\sin(v)$$

Which simplifies to,

$$\mathbb{R}(e^{iv}) = \cos(v)$$

with $\mathbb{R}\{*\}$ the real part.
For a random variable x and $\alpha \in \mathbb{R}$

$$\varphi_x(v)\, e^{i\alpha} = E\left[e^{ivx+i\alpha}\right] = \int_{-\infty}^{\infty} e^{i(vy+\alpha)} f_X(y)\, dy$$

By taking the real parts,

$$\mathbb{R}\left\{\varphi_x(v)\, e^{i\alpha}\right\} = \mathbb{R}\left\{\int_{-\infty}^{\infty} e^{i(vy+\alpha)} f_x(y)\, dy\right\} \tag{34}$$

$$= \int_{-\infty}^{\infty} \cos\left(vy+\alpha\right) f_x(y)\, dy \tag{35}$$

substitute

$$v = \frac{n\pi}{\beta - \alpha}$$

and multiply 33 by

$$\exp\left(-i\frac{n\alpha\pi}{\beta-\alpha}\right)$$

that is,

$$\hat{\varphi}_x\left(\frac{n\pi}{\beta-\alpha}\right) \exp\left(-i\frac{n\alpha\pi}{\beta-\alpha}\right) = \int_{\alpha}^{\beta} \exp\left(iy\frac{n\pi}{\beta-\alpha} - i\frac{n\alpha\pi}{\beta-\alpha}\right) f_x(y)\, dy \tag{36}$$

Taking the real parts we have ,

$$\mathbb{R}\left\{\hat{\varphi}_x\left(\frac{n\pi}{\beta-\alpha}\right) \exp\left(-i\frac{n\alpha\pi}{\beta-\alpha}\right)\right\} = \int_{\alpha}^{\beta} f_x(y) \cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right) dy \tag{37}$$

It follows that $\bar{a}_n \approx \bar{f}_n$ from equation 32 with,

$$\bar{f}_n = \frac{2}{\beta-\alpha}\mathbb{R}\left\{\varphi_x\left(\frac{n\pi}{\beta-\alpha}\right) \exp\left(-i\frac{n\alpha\pi}{\beta-\alpha}\right)\right\} \tag{38}$$

Replace $\bar{a}_n$ by $\bar{f}_n$ in equation 31.

$$f_x(y) \approx {\sum_{n=0}^{\infty}}' \bar{f}_n \cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right) \tag{39}$$

and truncate the series summation so that

$$\hat{f}_x(y) \approx {\sum_{n=0}^{N-1}}' \bar{f}_n \cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right) \tag{40}$$

Therefore the COS method can be summarized as

$$f_x(y) = {\sum_{n=0}^{\infty}}' f_n \cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right) \tag{41}$$

with $x \in [\beta, \alpha] \subset \mathbb{R}$ and the coefficient f_n is defined as

$$f_n = \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} f(x)\cos(n\pi\frac{x-\alpha}{\beta-\alpha})\,dx \tag{42}$$

f_n has direct relation to the characteristic function $\varphi(v) = \int_{\mathbb{R}} f(x)e^{ivx}\,dx$

since $\left(\int_{\mathbb{R};[\alpha,\beta]} f(x)\,dx \approx 0\right)$

$$a_n \approx f_n = \frac{2}{\beta-\alpha}\int_{\mathbb{R}} f(x)\cos(n\pi\frac{x-\alpha}{\beta-\alpha})\,dx = \frac{2}{\beta-\alpha}\mathbb{R}\left[\varphi\left(\frac{n\pi}{\beta-\alpha}\right)\exp\left(-i\frac{n\alpha\pi}{\beta-\alpha}\right)\right] \tag{43}$$

4.2 Density Recovery with the COS method

The COS method is a valuable tool employed to approximate the pdf of random variables. This approach harnesses the power of characteristic functions, establishing a robust framework that finds extensive utility in the field of finance such as in calibration, the computation of forward starting options and static hedging. In Figures 4 and 5, we visually explore the random variables of normal and NIG distributions approximated using the COS method at different values of N. These figures vividly illustrate the prowess of the COS method in effectively modeling and comparing these distinct distributions. From the plotted figures we can see that the densities are very well approximated for large values of N.

4.3 Risk Neutral pricing

4.3.1 COS method Option under the risk Neutral measure

In equation 27, we defined the value of a derivative under the risk neutral measure as:

$$\begin{aligned} V(t) &= e^{-r\tau}E^{\mathbb{Q}}\left[h(y_T|F_t)\right] \\ &= e^{-r\tau}\int_{\mathbb{R}} h(y)\,p(y,T|F_t)dy \end{aligned}$$

Since the density $p(y,T|F_t)$ rapidly decays to zero as $y \to \pm\infty$ in 27, we truncate the infinite integration range to $[\alpha,\beta] \in \mathbb{R}$, without losing significant accuracy. That is:

$$\hat{V}(t) = e^{-r\tau} \int_{\alpha}^{\beta} h(y)\, p(y,T|F_t) dy \tag{44}$$

The density $p(y,T|F_t)$ is usually not known but its characteristic function can be found. We therefore replace the density by its cosine expansion as shown in equation 41.

$$\begin{aligned} \hat{V}(t) &= e^{-r\tau} \int_{\alpha}^{\beta} h(y) {\sum_{n=o}^{\infty}}' a_n \cos(n\pi \frac{y-\alpha}{\beta-\alpha})\, dy \\ &= e^{-r\Delta\tau} \sum_{n=o}^{\infty} a_n \int_{\alpha}^{\beta} h(y) \cos(n\pi \frac{y-\alpha}{\beta-\alpha})\, dy \end{aligned} \tag{45}$$

We can define

$$I_n = \frac{2}{\beta-\alpha} \int_{\alpha}^{\beta} h(y) \cos(n\pi \frac{y-\alpha}{\beta-\alpha})\, dy \tag{46}$$

We shall then have

$$\hat{V}(t) = \frac{\beta-\alpha}{2} e^{-r\tau} {\sum_{n=0}^{\infty}}' a_n I_n \tag{47}$$

Since the coefficients in this case decay rapidly, the sum can be further truncated to obtain:

$$\hat{V}(t) = \frac{\beta-\alpha}{2} e^{-r\tau} {\sum_{n=0}^{N-1}}' a_n I_n \tag{48}$$

Now, $a_n \approx f_n$ implying that:

$$f_n = \frac{2}{\beta-\alpha} \mathbb{R} \left\{ \varphi\left(\frac{n\pi}{\beta-\alpha}\right) e^{-in\pi\left(\frac{\alpha}{\beta-\alpha}\right)} \right\}$$

$$V(t) = e^{-r\tau} {\sum_{n=0}^{N-1}}' \mathbb{R} \left\{ \varphi\left(\frac{n\pi}{\beta-\alpha};x\right) e^{-in\pi\frac{\alpha}{\beta-\alpha}} \right\} I_n \tag{49}$$

Equation 49 is the summarized COS formula for pricing options. f_n has the information on the underlying Asset and can be applied to several models where the characteristic function is known in closed form. The key factor for achieving convergence is the convergence of the density function's cosine series, as the cosine series of the payoff is introduced due to the interchange of summation and integration.I_n has the information on the payoff function and this is pretty simple for plain vanilla options which we focus on in this study.

4.3.2 European Call Option

The payoff of a European call option is given by

$$h(y(T)) = \max(S_T - K, 0)$$

where S_T is the spot price of the underlying asset at the expiration (maturity) time T and K the Strike Price
If we have $x = \ln(\frac{S_0}{K})$ and $y = \ln(\frac{S_T}{K})$, then the the payoff function can be expressed as

$$h(y) = K(e^y - 1)^+$$

With the assumption that the characteristic function of the log asset price is known, the payoff function I_n for a European call option in equation 49 is given by:

$$\begin{aligned} I_n &= \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} K(e^y-1)^+ \cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy \\ &= \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} K\,e^y \cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy - \int_{\alpha}^{\beta} K\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy \end{aligned} \tag{50}$$

for some intervals $(\upsilon, \nu) \in (\alpha, \beta)$ the integrals can be solved explicitly.
Lets begin with the first integral

$$\int_{\upsilon}^{\nu} e^y \cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy.$$

To solve this integral, we use integration by parts. The formula for integration by parts is given by:

$$\int u\,dv = uv - \int v\,du$$

Lets choose

$$u = e^y \text{ and } dv = \cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy$$

From this we get

$$du = e^y dy \text{ and } v = \frac{\beta-\alpha}{n\pi}\sin\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)$$

then

$$\int_{\upsilon}^{\nu} e^y \cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy = e^y.\frac{\beta-\alpha}{n\pi}\sin\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right) - \int \frac{\beta-\alpha}{n\pi}\sin\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right).e^y dy$$

We'll apply integration by parts again to solve the remaining integral, choosing:

$$u = e^y \text{ and } dv = \frac{\beta-\alpha}{n\pi}.\sin\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy$$

We find du and v as:

$$du = e^y dy \text{ and } v = -\frac{\beta-\alpha}{n\pi}.\frac{\beta-\alpha}{n\pi}\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)$$

$$\int \frac{\beta-\alpha}{n\pi}`\sin\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right).e^y dy = e^y.-\left(\frac{\beta-\alpha}{n\pi}\right)^2\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right) + \int\left(\frac{\beta-\alpha}{n\pi}\right)^2\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)e^y dy$$

Now, we substitute this result back into the original integration by parts formula:

$$\int_{\upsilon}^{\nu} e^y\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy = e^y.\frac{\beta-\alpha}{n\pi}\sin\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right) + e^y.\left(\frac{\beta-\alpha}{n\pi}\right)^2\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right) - \int\left(\frac{\beta-\alpha}{n\pi}\right)^2\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)e^y dy$$

Notice that the term on the right side is the same as the original integral, so we can collect like terms and solve for the original integral:

$$\left(1+\left(\frac{\beta-\alpha}{n\pi}\right)^2\right)\int_{\upsilon}^{\nu} e^y\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy = e^y.\frac{\beta-\alpha}{n\pi}\sin\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right) + e^y.\left(\frac{\beta-\alpha}{n\pi}\right)^2\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)$$

This equates to:

$$\begin{aligned}\int_{\upsilon}^{\nu} e^y\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy &= \frac{1}{1+\left(\frac{n\pi}{\beta-\alpha}\right)^2}\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)e^y + \frac{n\pi}{\beta-\alpha}\sin\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)e^y \\ &= \frac{1}{1+\left(\frac{n\pi}{\beta-\alpha}\right)^2}\left[\cos\left(n\pi\frac{\nu-\alpha}{\beta-\alpha}\right)e^{nu} - \cos\left(n\pi\frac{\upsilon-\alpha}{\beta-\alpha}\right)e^{\upsilon}\right] \\ &+ \frac{n\pi}{\beta-\alpha}\left[\sin\left(n\pi\frac{\nu-\alpha}{\beta-\alpha}\right)e^{\nu} - \sin\left(n\pi\frac{\upsilon-\alpha}{\beta-\alpha}\right)e^{\upsilon}\right] \\ &= \xi_n(\upsilon,\nu)\end{aligned} \tag{51}$$

and

$$\int_{\upsilon}^{\nu} \cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy = \frac{\beta-\alpha}{n\pi}\sin\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)$$
$$= \begin{cases} \frac{\beta-\alpha}{n\pi}\left[\sin\left(n\pi\frac{\nu-\alpha}{\beta-\alpha}\right) - \sin\left(n\pi\frac{\upsilon-\alpha}{\beta-\alpha}\right)\right] & \texttt{for } n \neq 0 \\ (\nu-\upsilon) \ \texttt{for } n=0 \end{cases} \tag{52}$$
$$= \zeta_n(\upsilon,\nu)$$

Equation 50 is summarized as:

$$I_n = \frac{2}{\beta-\alpha}K\left(\xi_n(0,\beta) - \zeta_n(0,\beta)\right) \tag{53}$$

Combining equation 49 and equation 53 the price of a European call option is given as:

$$C(t) = e^{-r\tau}{\sum_{n=0}^{N-1}}' Re\left\{\varphi\left(\frac{n\pi}{\beta-\alpha};x\right)e^{-in\pi\frac{\alpha}{\beta-\alpha}}\right\}K\left(\xi_n(0,\beta) - \zeta_n(0,\beta)\right) \tag{54}$$

4.3.3 European Put option

The payoff of a European call option is given by

$$h(y(T)) = \max(K - S_T, 0)$$

where S_T is the spot price of the underlying asset at the expiration (maturity) time T and K the Strike Price
If we have $x = \ln\left(\frac{S_0}{K}\right)$ and $y = \ln\left(\frac{S_T}{K}\right)$, then the payoff function can be expressed as

$$h(y) = -K(e^y - 1)^+$$

. I_n for a European Put option is then given by:

$$I_n = \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} K(1-e^y)^+\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy$$
$$= \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} K\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy - \int_{\alpha}^{\beta} K\,e^y\cos\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right)dy \tag{55}$$

The integrals shall be solved similarly as that for a call option in equation 51 and equation 52. and

$$I_n = \frac{2}{\beta - \alpha} K \left(\zeta_n(\alpha, 0) - \xi_n(\alpha, 0) \right) \tag{56}$$

The price of a European Put option will then be:

$$P(t) = e^{-r\tau} \sum_{n=0}^{N-1}{}' Re \left\{ \varphi \left(\frac{n\pi}{\beta - \alpha}; x \right) e^{-in\pi \frac{\alpha}{\beta - \alpha}} \right\} K \left(\zeta_n(\alpha, 0) - \xi_n(\alpha, 0) \right) \tag{57}$$

4.4 The COS method Option Prices with Levy process

Equation 49 can be applied generally to several underlying processes including the exponential levy processes.

Definition 4.4.1. *A process $X(t)$ in a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with $X(0) = 0$ is called a levy process if:*

i) It has stationery increments

ii) It has independent Increments

iii) $\lim_{s \to t} P(|X(t) - X(s)| > \varepsilon) = 0$ *for $s \leq t$ and $\varepsilon > 0$.*

The Lévy-Khintchine theorem by (Sato, 2001) is a powerful result that provides a characterization of Lévy processes in terms of their underlying properties.

Theorem 4.4.2 (Lévy-Khintchine theorem)**.** *For a levy process $X(t)$ with characteristic function $\varphi(v) = E[e^{(ivX(t))}]$, $\varphi(v)$ has the following form:*

$$\varphi(v) = \exp \left\{ iv\mu - \frac{1}{2}\sigma^2 v^2 + \int [e^{(ivx)} - 1 - ivx1_{[|x|<1]} \nu(dx) \right\} \tag{58}$$

where: μ is the drift or location parameter, σ^2 is the variance of the process and $\nu(dx)$ is a Lévy measure, which is a measure that describes the jump characteristics of the Lévy process.

For proof see (Ng, 2008).

In derivatives pricing, especially when dealing with options and other financial derivatives for exponential levy models, the asset price S_t can be expressed generally as:

$$S_t = S_0 \exp(X(t)) \tag{59}$$

X_t is the stochastic process of the underlying asset return.

Theorem 4.4.3 (Martingale Theorem)**.** *Under the risk neutral pricing, we assume that if $S(t)$ is the price of an asset at time t, and r is the risk-free interest rate, then the process $M(t) = e^{(-rt)}S(t)$ is a $\mathbb{Q}$ martingale.*

Proof. We use the definition of martingale given in Definition 4.4.4. The expected value of $M(t)$ under the risk neutral measure $\mathbb{Q}$ is given by:

$$E^{\mathbb{Q}}[M(t)] = E^{\mathbb{Q}}[e^{(-rt)}S(t)]$$

$\forall\, s \leq t$,

$$E^{\mathbb{Q}}[M(t)|F(s)] = E^{\mathbb{Q}}[e^{(-rt)}S(t)|F(s)]$$

The key property of conditional expectation is that $E^{\mathbb{Q}}[X|F(s)]$ is the best predictor of X based on the information available at time s. Therefore, we can write:

$$E^{\mathbb{Q}}[M(t)|F(s)] = e^{(-rs)}E^{\mathbb{Q}}[S(t)|F(s)]$$

Since,$M(t) = e^{(-rt)}S(t)$, then

$$E^{\mathbb{Q}}[S(t)|F(s)] = S(s)$$

We shall then have:

$$E^{\mathbb{Q}}[M(t)|F(s)] = e^{(-rs)}S(s)$$

Similarly,

$$M(s) = e^{(-rs)}S(s)$$

□

Definition 4.4.4. *A martingale is a stochastic process where, on average, the expected future value is equal to the current value that is, A stochastic process $M(t), t \geq 0$ is a $\mathbb{Q}$ martingale if $\forall\, t \geq 0$:*

- $E^{\mathbb{Q}}[M(t)] < \infty$
- $\forall\, s \leq t$, $E^{\mathbb{Q}}[M(t)|F(s)] = M(s)$ *where $F(s)$ is the information available at time s.*

To ensure that the condition of a martingale is met under the risk neutral measure, then the characteristic function $\varphi(v)$ evaluated at $-i$ must be given by:

$$\varphi(-i) = E(e^{X(t)}) = e^{rt}$$

This condition is satisfied if we choose:

$$v = r - \frac{1}{2}\sigma^2 - \int [e^x - 1 - x1_{[|x|<1]}\nu(dx) \tag{60}$$

Asset returns are often modeled using the log transformation. Let $\Phi(v)$ be the characteristic function of the log of the underlying Asset process S_t.

$$\begin{aligned}\Phi(v) &= E\left[e^{iv\ln S_t}\right]\\ &= E\left[e^{iv(\ln S_0+rt+wt+X_t)}\right]\\ &= e^{iv(\ln S_0+rt+wt)}.E\left[e^{ivX_t}\right]\\ &= e^{iv(\ln S_0+rt+wt)}.\varphi(v)\end{aligned}$$

If we evaluate $\varphi(v)$ at $-i$, we shall have

$$E[S_t] = e^{(\ln S_0+rt+wt)}.\varphi(-i) \tag{61}$$

The expected value of S_t under the risk neutral measure is

$$E[S_t] = e^{(\ln S_0+rt)}. \tag{62}$$

Comparing equation 61 and 62, it can be seen that:

$$\begin{aligned}&e^{wt}.\varphi(-i) = 1\\ \text{and}\quad &w = -\frac{1}{t}\ln(\varphi(-i))\end{aligned}$$

This summarizes the characteristic function of the log of the asset price as:

$$\begin{aligned}\Phi(v) &= E\left[e^{iv\ln S_t}\right]\\ &= e^{iv(\ln S_0+rt+wt)}.\varphi(v)\\ &= e^{iv(\ln S_0+rt)}.\frac{\varphi(v)}{\varphi(-i)}\end{aligned} \tag{63}$$

We review the performance of two processes which fall under the class of exponential levy models, that is, the GBM and the NIG process. The characteristic functions of these processes can be presented in the form:

$$\varphi(v;x) = \Phi(v)\,exp(ivx) \tag{64}$$

with

$$\Phi(v) = \varphi(v;0)$$

Inserting this representation to equation 49, we shall have

$$V(t) = e^{-r\tau}\sum_{n=0}^{N-1}{}' \mathbb{R}\left\{\Phi\left(\frac{n\pi}{\beta-\alpha}\right)e^{-in\pi\frac{x-\alpha}{\beta-\alpha}}\right\}I_n \tag{65}$$

4.4.1 GBM Process

The instantaneous price movements under the GBM process are given in equation 1. Let $\Phi(v)$ be the characteristic function of the log of the underlying Asset process S_t and

$$y = \ln\left(\frac{S_t}{K}\right)$$

. The log of a GBM follows a normal distribution, and its characteristic function can be derived as follows:

$$\begin{aligned}
\Phi_{GBM}(v) &= E\left[\exp\left(ivy\right)\right] \\
&= E\left[\exp\left\{iv\left(\ln S_0 + (r - \frac{\sigma^2}{2})t + \sigma W_T\right)\right\}\right] \\
&= \exp\left\{iv\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T\right)\right\}.\exp\left(-\frac{v^2\sigma^2}{2}T\right) \\
&= \exp\left(iv\ln S_0\right).\exp\left\{ivT\left(r - \frac{\sigma^2}{2}\right) - \frac{v^2\sigma^2}{2}T\right\}
\end{aligned} \tag{66}$$

Since the characteristic function of the GBM process takes the form presented in equation 64, then equation 65 can be combined with equations 54 and 57 to obtain the call and put call option prices respectively under this process as follows:

$$C(t) = e^{-r\tau}{\sum_{n=0}^{N-1}}' \mathbb{R}\left\{\Phi_{GBM}\left(\frac{n\pi}{\beta-\alpha}\right)e^{-in\pi\frac{x-\alpha}{\beta-\alpha}}\right\}K\left(\xi_n(0,\beta) - \zeta_n(0,\beta)\right) \tag{67}$$

$$P(t) = e^{-r\tau}{\sum_{n=0}^{N-1}}' \mathbb{R}\left\{\Phi_{GBM}\left(\frac{n\pi}{\beta-\alpha}\right)e^{-in\pi\frac{x-\alpha}{\beta-\alpha}}\right\}K\left(\zeta_n(\alpha,0) - \xi_n(\alpha,0)\right) \tag{68}$$

4.4.2 NIG Process

The characteristic function of a NIG pdf is given in equation 23 and its nth cumulants in equation 25. Choosing the correct martingale measure as presented in equation 60 is key for the utilization of this process. Several useful measures have been suggested including(Gerber & Shiu, 1996) and (Elliott, Chan, & Siu, 2005). In (Hirsa, 2012), an underlying process S_t can be presented as:

$$S_t = S_0 \exp\left(rt + wt + x_t\right)$$

Assuming that the characteristic function of the process is known that is $\varphi(v) = E(e^{ivx_t})$, then w is chosen such that the expected value of the asset price process is a martingale as defined in theorem 4.4.3.

Let

$$y = \ln\left(\frac{S_t}{K}\right)$$

where S_t is the price of an underlying asset at time t, and K is the strike price, the NIG model has a closed form characteristic function given by

$$\Phi(v) = \exp\left[ivy + iv(r+w)\tau - \phi(v)\right] \tag{69}$$

$\phi(v)$ is the characteristic exponent given in equation 24 and ω is risk neutral compensator that ensures that the discounted stock price follows a martingale.

$$w = \delta\left(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}\right) \tag{70}$$

To solve for w , we set $\varphi(v=-1) = \exp\left(x_t + r\tau\right)$

The call and put option prices under this process from equation 54 and equation 57 can be combined with equation 65 to obtain;

$$C(t) = e^{-r\tau}\sum_{n=0}^{N-1}{}' \mathbb{R}\left\{\Phi_{NIG}\left(\frac{n\pi}{\beta-\alpha}\right)e^{-in\pi\frac{x-\alpha}{\beta-\alpha}}\right\}K\left(\xi_n(0,\beta) - \zeta_n(0,\beta)\right) \tag{71}$$

$$P(t) = e^{-r\tau}\sum_{n=0}^{N-1}{}' \mathbb{R}\left\{\Phi_{NIG}\left(\frac{n\pi}{\beta-\alpha}\right)e^{-in\pi\frac{x-\alpha}{\beta-\alpha}}\right\}K\left(\zeta_n(\alpha,0) - \xi_n(\alpha,0)\right) \tag{72}$$

4.5 Truncation range

(Fang & Oosterlee, 2009) Proposed a formula for determining the intervals of integration $[\alpha, \beta]$ in the COS method. That is:

$$\alpha = (x+K_1) - L\sqrt{K_2 + \sqrt{K_4}} \quad \text{and} \quad \beta = (x+K_1) + L\sqrt{K_2 + \sqrt{K_4}} \tag{73}$$

x is $\ln\left(\frac{S_T}{K}\right)$
K_n is the n^{th} cumulant of the of x
L is deliberately selected from the range $[6, 12]$, with the choice contingent upon the acceptable variation or error specific to each case. For exponential Levy processes, (Oosterlee

& Grzelak, 2019) recommends a value of $L = 8$, determined through meticulous numerical experiments assessing performance when N is chosen large enough upto $N = 2^{14}$. Formula 73 is however not very convenient when pricing of options with multiple strikes due to the complexity introduced in the dependence of x on K. (Oosterlee & Grzelak, 2019) determined an alternative approach that does not depend on x but works equally likely.

$$\alpha = -L\sqrt{\tau} \text{ and } \beta = L\sqrt{\tau} \tag{74}$$

We want to price options for multiple range of strikes at once in our research and therefore we use formula 74 to get the intervals of integration $[\alpha, \beta]$.

5 Data Analysis

5.1 Data source and summary

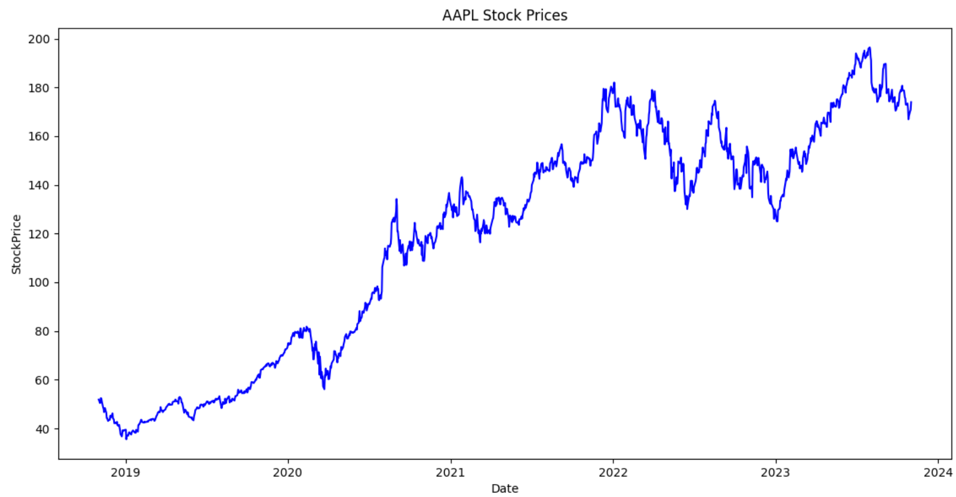
The data set utilized for this analysis comprises daily closing prices of the two stocks Amazon.com and Apple Inc classified as among the most active stocks on Nov 2, 2023. The daily closing stock prices were obtained for the period 02/11/2018 to 02/11/2023. These data were sourced from Yahoo Finance, a reputable financial data platform, providing reliable and up-to-date information on stock market performance. To ensure data integrity and suitability for analysis, certain preprocessing steps were applied. This included handling missing values and calculating daily log-returns, which are commonly used in financial analyses for their ability to transform non-stationary price data into a more stable format. The corresponding option data(call and put option prices) were obtained from SPX Options section on MarketWatch. Data cleaning was carried out; outliers and option pairs with zero bid and ask prices were excluded from the study. It was assumed that the "true" market prices of calls and puts were obtained by averaging their respective bid and ask prices. The options expiring in 30 days were classified as short dated options whereas those expiring in 90 days were classified as long dated options according to (Singh, 2013).

5.2 Descriptive statistics

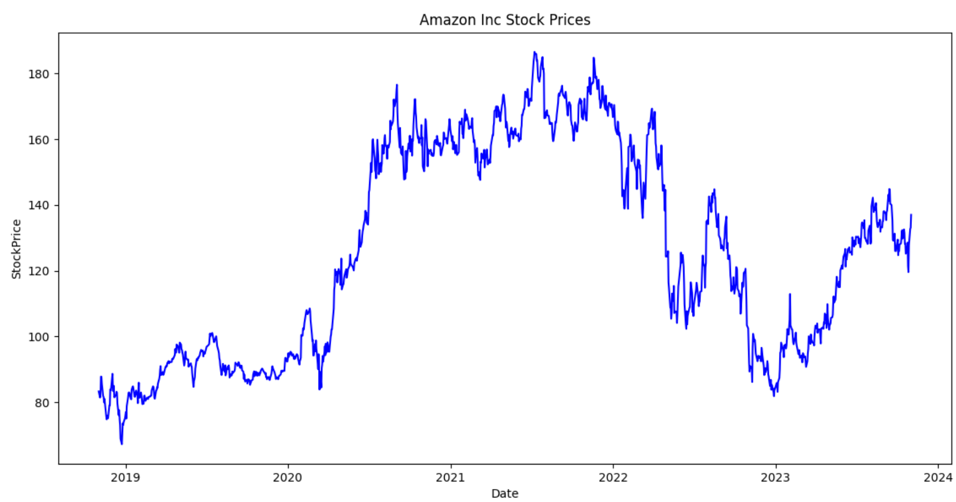
The basic summary statistics for the Amazon Inc and AAPL daily closing stock prices are shown in Table 1 and plots of the data in 1. Both stocks have positive means and have asymmetric distributions as shown by the skewness. The AAPL stocks are left-skewed while the Amazon Inc stocks are right-skewed.

	No.	Mean	standard Deviation	Skewness	Kurtosis
AAPL	1257	118.4456106	46.81040687	-0.3490124	-1.2734562
AMZN	1257	125.5498735	32.08939419	0.1349467	-1.4597373

Table 1. Summary statistics for AAPL and Amazon Inc daily stock prices



(a) AAPL stock prices



(b) Amazon stock prices

Figure 1. Plot of the stock prices for the period from Nov 2018 to Nov 2023

Similarly, the basic summary statistics for the log-returns are presented in Table 2. The means of the log returns is positive while the skewness is negative for both stocks. This implies that the daily log returns for both AAPL and AMAZON stocks are is asymmetric and skewed to the left. The log returns for both stocks also have positive kurtosis which is greater than 3 indicating heavy tails and more peaked distributions (leptokurtic). This means that the data have more extreme values than a normal distribution.

	No.	Mean	Standard Dev.	Skewness	Kurtosis
AAPL	1256	0.0009635	0.002376514	-0.2353210	4.8637848
AMZN	1256	0.001090046	0.02289795	-0.0628474	3.9082357

Table 2. Summary statistics for AAPL and Amazon Inc daily log returns

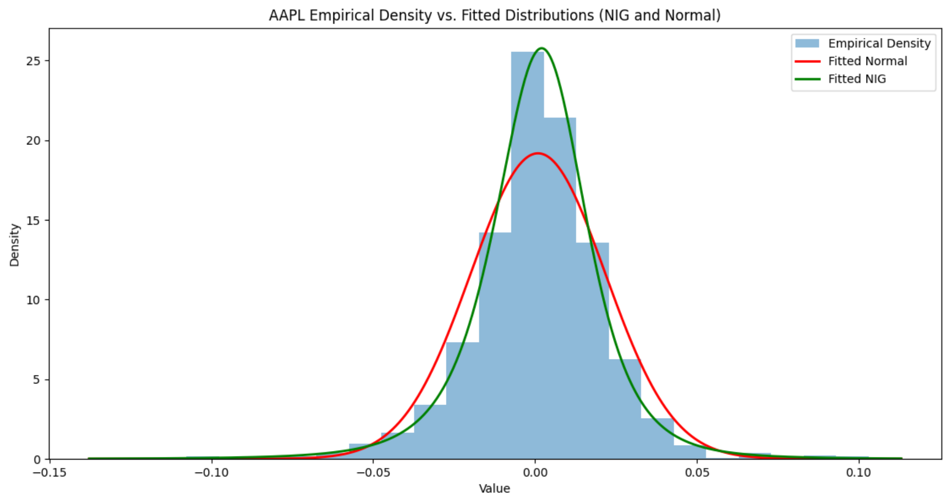
5.3 Goodness of fit test

A Shapiro-Wilk test performed on the log-returns gives a p-value which is significantly less than the common significance level of 0.05 for both data. This confirms our assumption that the data significantly deviates from the normal distribution. see Table 3.

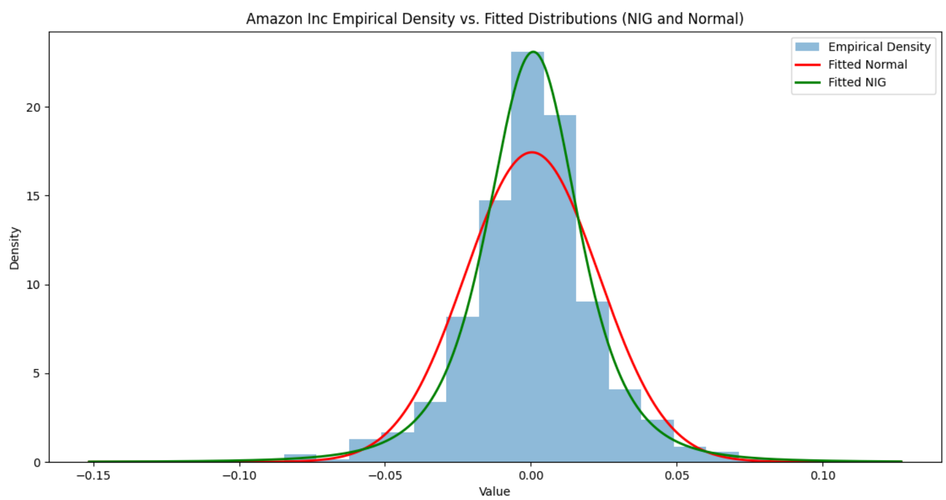
	K statistic	P-value(KS)	P-value (Shapiro-Wilk test)
AAPL	0.0133922	0.9757089	5.42E-21
AMZN	0.0124373	0.9888023	2.98E-18

Table 3. Goodness of fit test for the log returns

The Kolmogorov-Smirnov (KS) test for the fit of the data to the Normal Inverse Gaussian distribution was performed. The p-value for both stocks is high (closer to 1) and it shows that the log-returns data is not significantly different from the NIG distribution.
The stock data is further described by plotting qq plots on the normal distribution and the empirical density versus the normal and NIG distributions. The plots show that the empirical densities for both stocks are different from the normal distribution and seem to follow the NIG distribution. The results in figure 3 are confirmed by the Shapiro wilk test and the KS tests shown in Table 3.

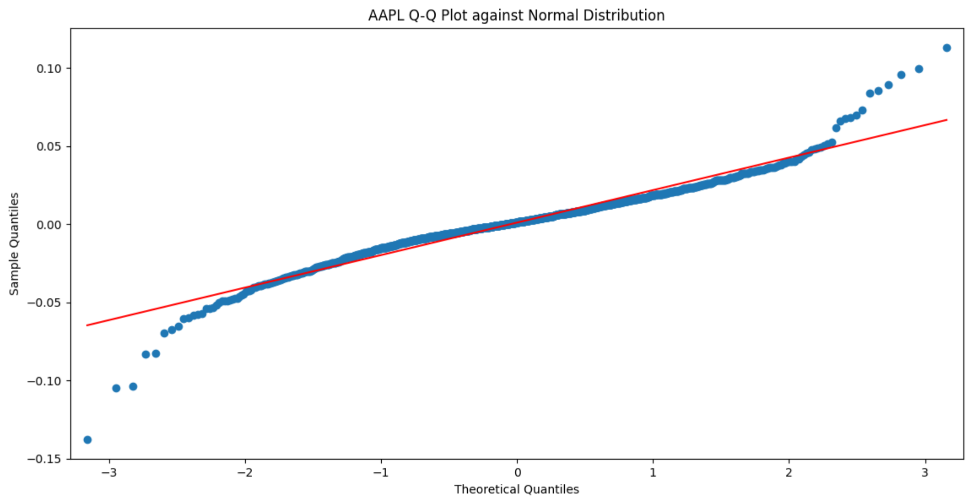


(a) AAPL empirical versus fitted densities

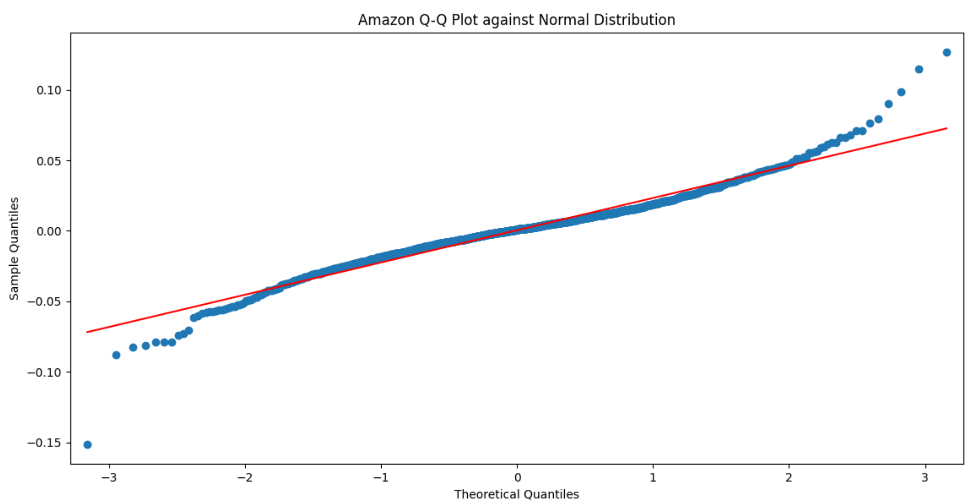


(b) Amazon empirical versus fitted densities

Figure 2. Empirical versus NIG and Normal densities for the log returns



(a) AAPL Q-Q plot



(b) Amazon Q-Q plot

Figure 3. Q-Q plots against the normal distribution

Assuming that the log returns are I.I.D, we approximate the parameters using the MLE estimator as described in equation 26. The parameterizations used in the *Scipy.optimize* package in *python* are as described in equation 20. The parameters are presented in Table 4.

	μ	δ	α	β
AAPL	0.0009635	0.018082	42.89208	-3.329876
Amazon	0.001090046	0.02005154	38.26604	-1.321365

Table 4. Estimated NIG parameters

5.4 Density recovery

In section 4.2, we noted that the Cos method is very useful in density recovery. In this section we show how the density recovered from the COS method differs from the random densities using our data.

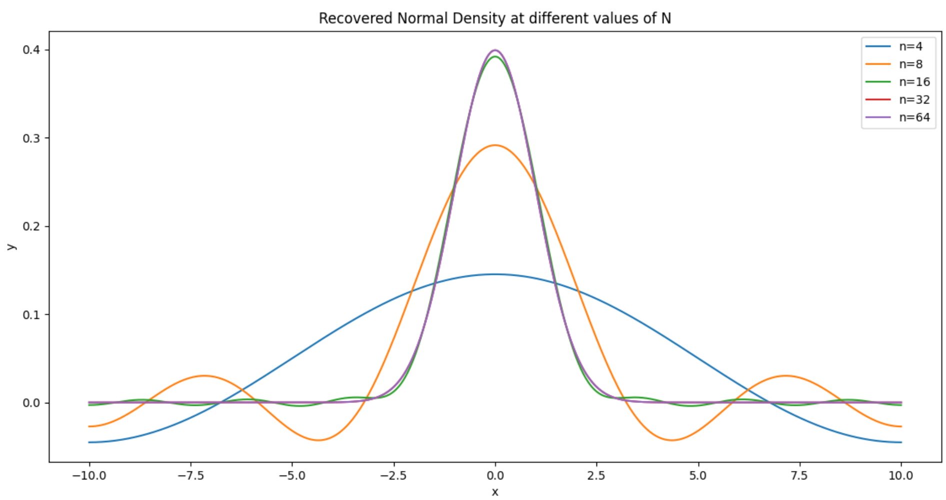


Figure 4. Normal density recovered from the COS method

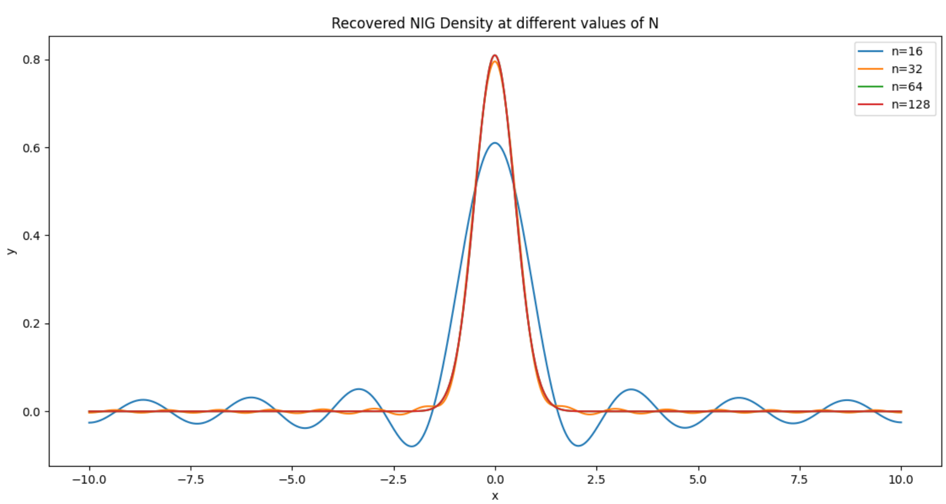


Figure 5. NIG density recovered from the COS method

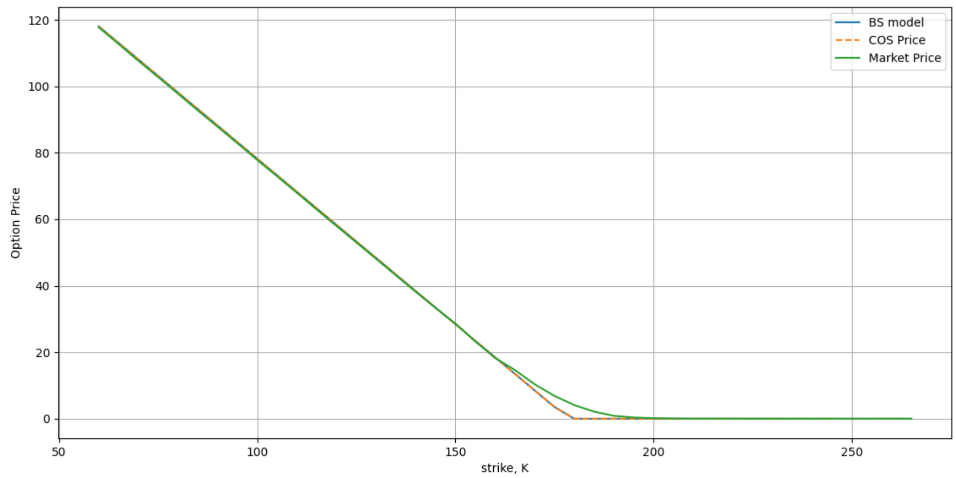
The plotted figures 4 and 5, shows the prowess of the COS method in efficiently approximating densities at different values of N. It can be noted that the plots are much closer to their distributions at larger values of N. It is then important to note that the value of N should be chosen such that it is large enough to approximate the densities nicely.

5.5 Option prices

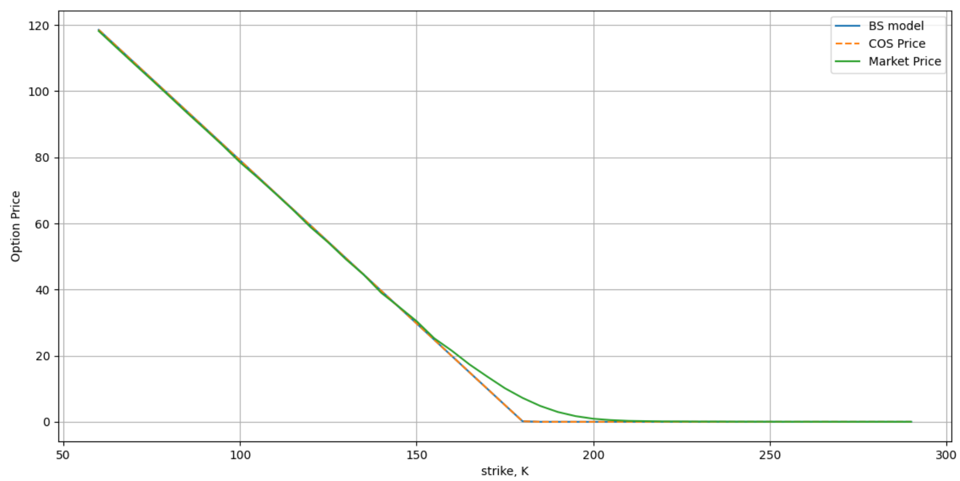
Black-Scholes model parameters were obtained as follows: the current stock price (as at 2nd November, 2023) was \$138.07 for the Amazon data and \$177.97 for AAPL; r was assumed to be 5.38% p.a.(This is the current value of a 3 month U.S treasury bill), σ was obtained from the historical stock data; the strike prices,K, and the time to maturity were obtained from the market option data.

5.5.1 GBM model

Call and put option prices were calculated from equation 67 and 68 for the GBM model with the COS method. They were compared with the BSM - 73 model prices in equations 10 and 15 and the prevailing market prices. A plot of the graphs are shown in figures 6, 7 8 and 9.

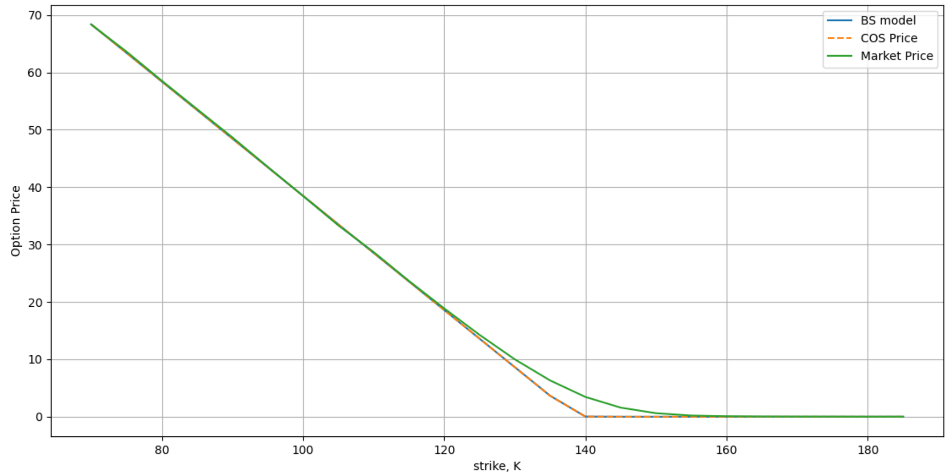


(a) call prices for options with 30 days expiry

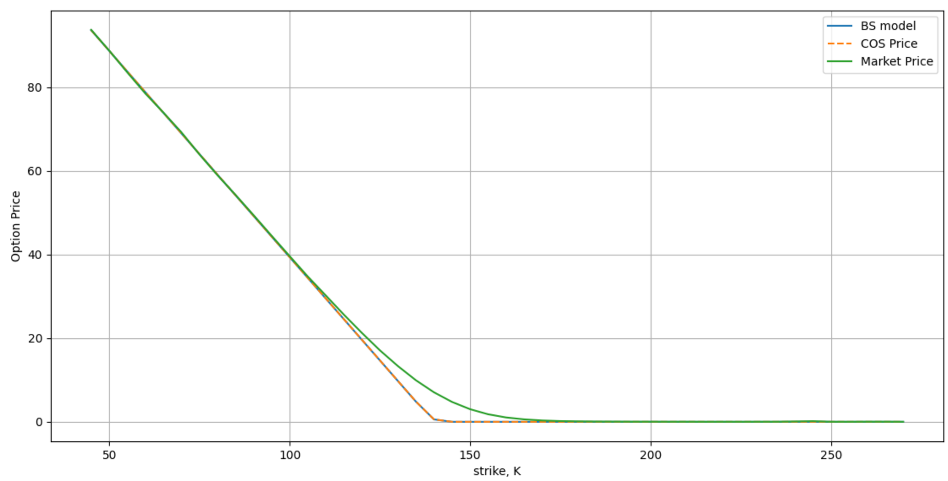


(b) call prices for options with 90 days expiry

Figure 6. AAPL call Price Model Comparison: COS-GBM, BSM, MKT

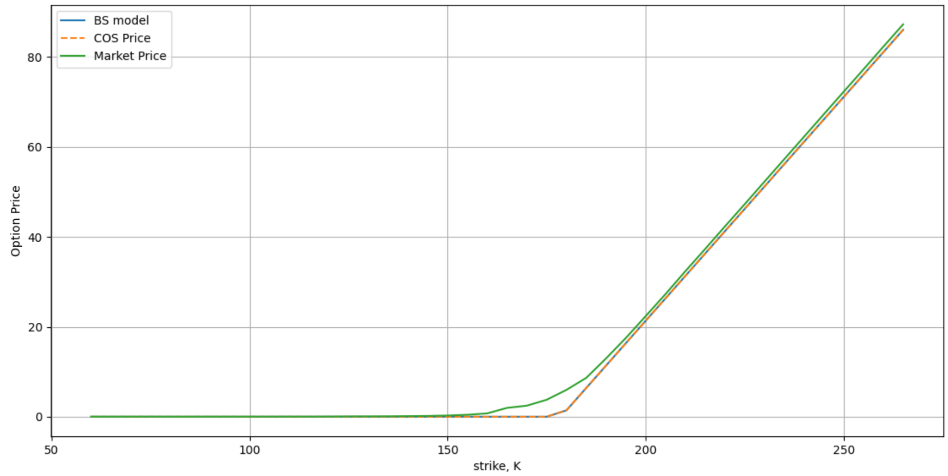


(a) call prices for options with 30 days expiry

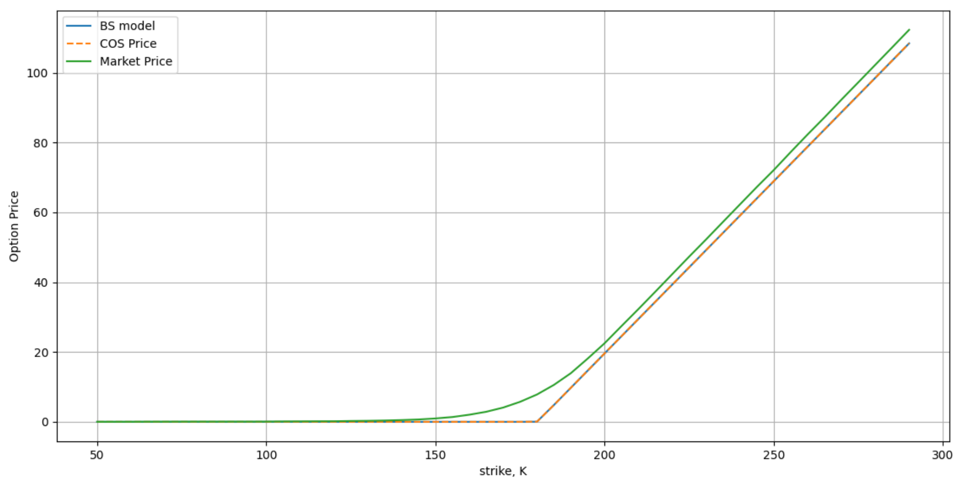


(b) call prices for options with 90 days expiry

Figure 7. Amazon Inc call Price Model Comparison: COS-GBM, BSM, MKT

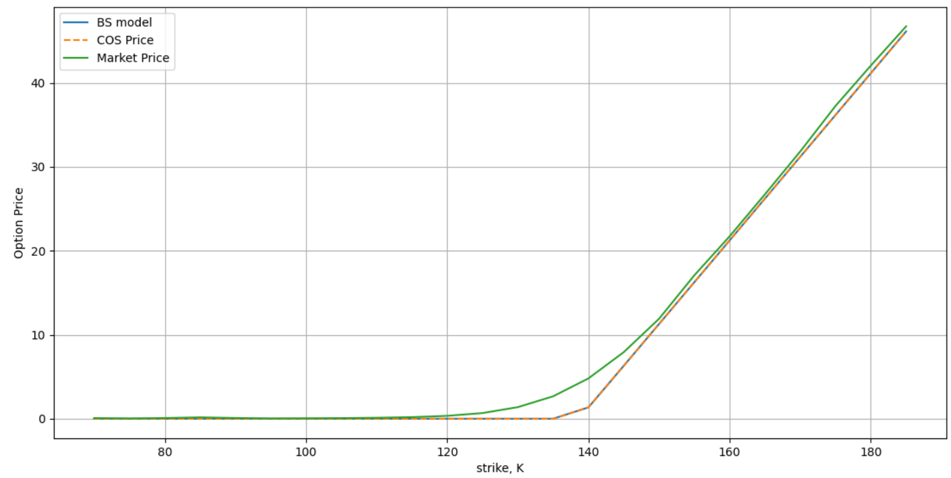


(a) Put prices for options with 30 days expiry

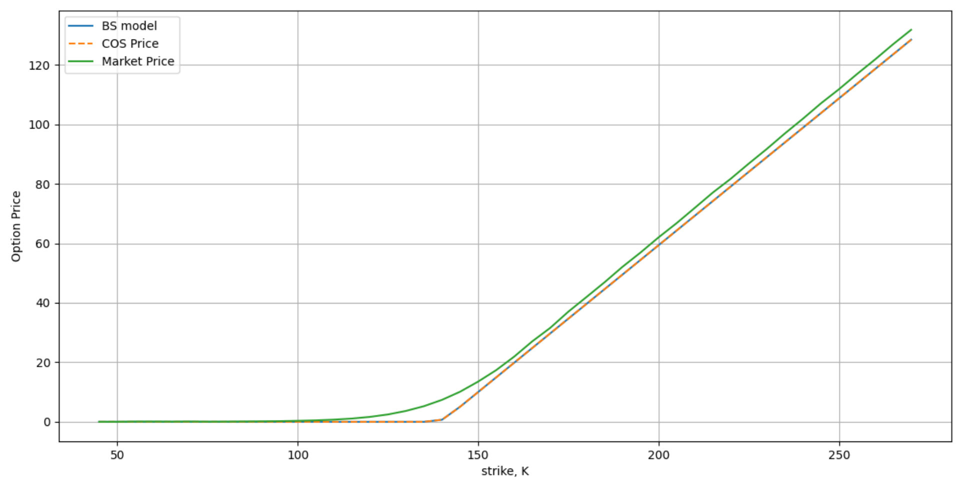


(b) Put prices for options with 90 days expiry

Figure 8. AAPL Put Price Model Comparison: COS-GBM, BSM, MKT



(a) Put prices for options with 30 days expiry

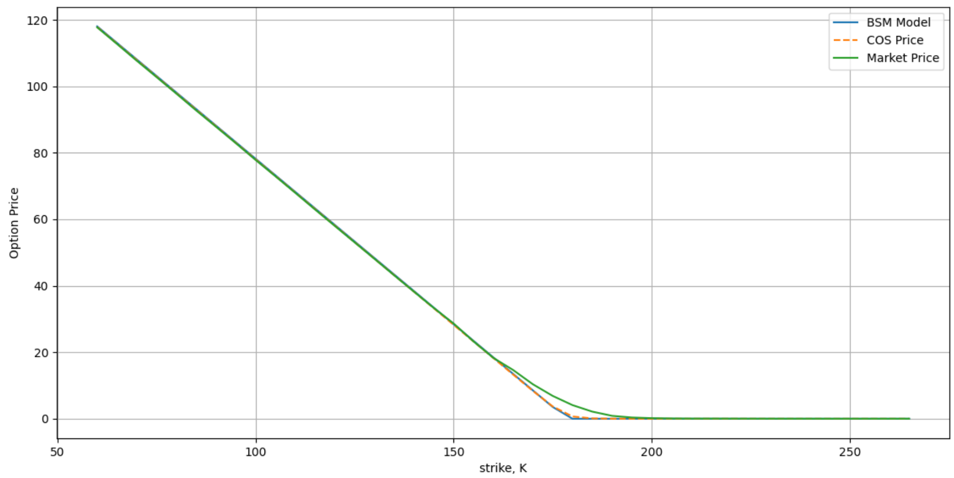


(b) Put prices for options with 90 days expiry

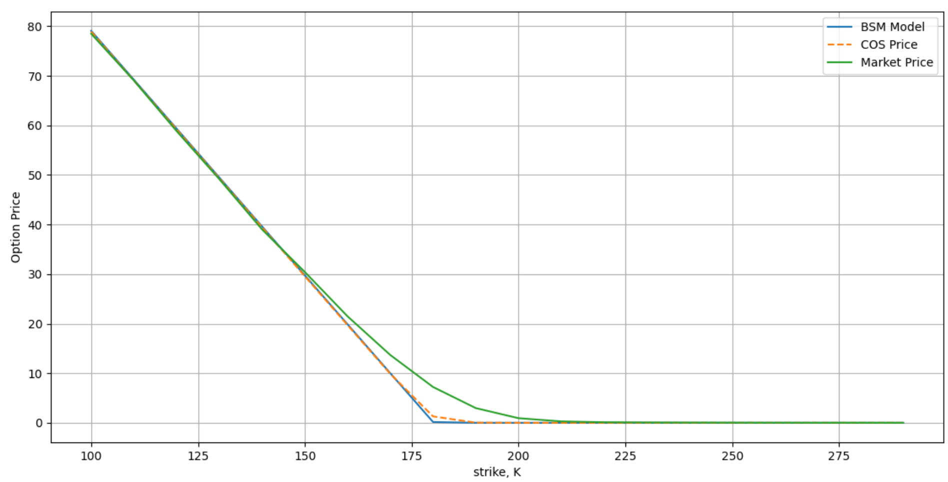
Figure 9. Amazon Inc Put Price Model Comparison: COS-GBM, BSM, MKT

5.5.2 NIG model

Similarly, Call and put option prices for the NIG model with the COS method were calculated from equation 71 and 72 and compared with the BSM - 73 model prices in equations 10 and 15 and the prevailing market prices. Results are presented in figure

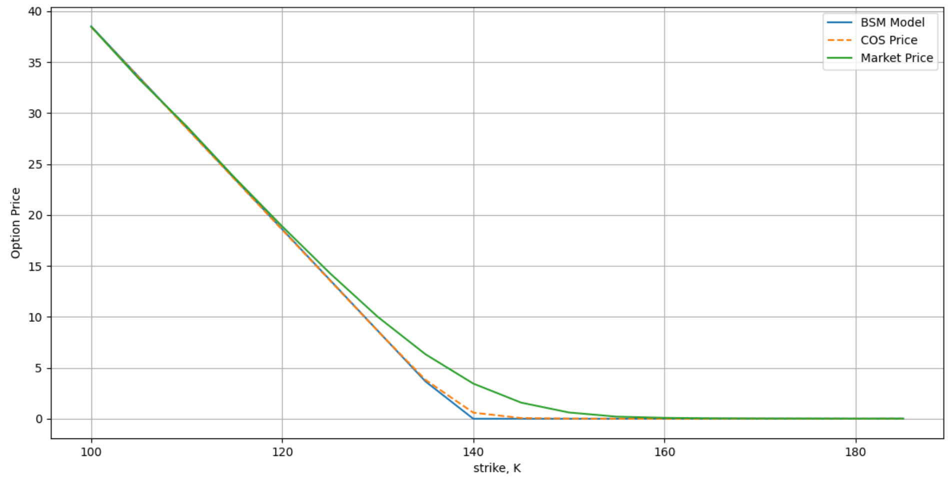


(a) call prices for options with 30 days expiry

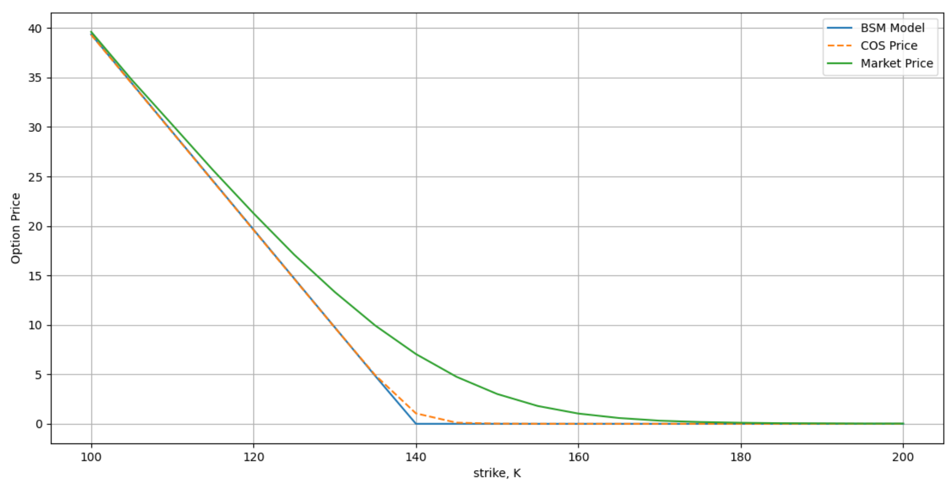


(b) call prices for options with 90 days expiry

Figure 10. AAPL call Price Model Comparison: COS-NIG, BSM, MKT

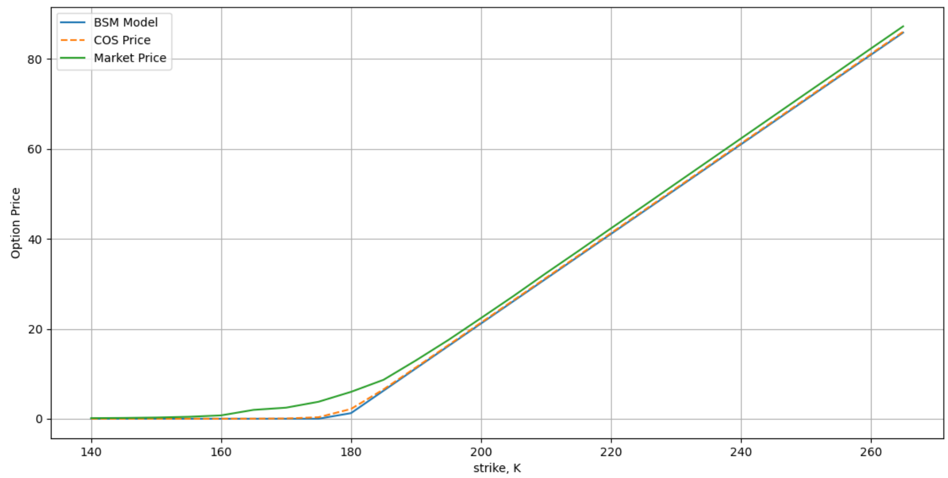


(a) call prices for options with 30 days expiry

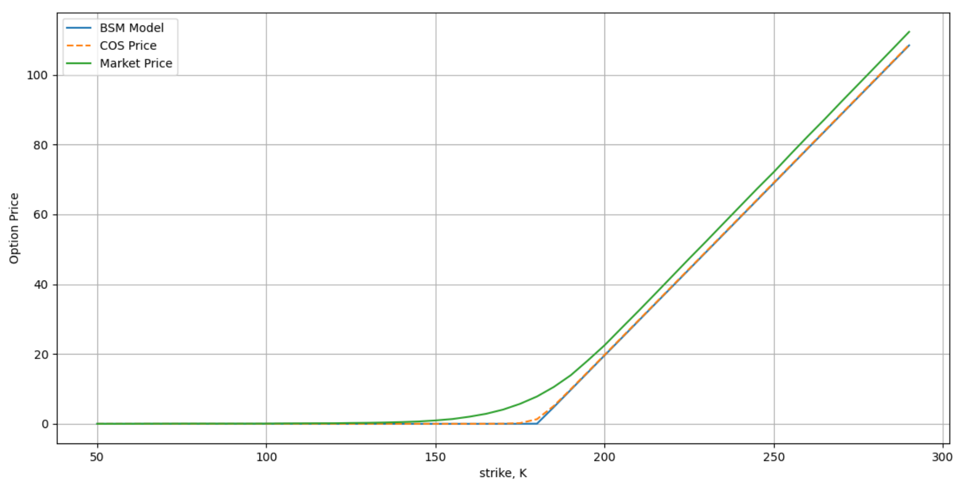


(b) call prices for options with 90 days expiry

Figure 11. Amazon Inc call Price Model Comparison: COS-NIG, BSM, MKT

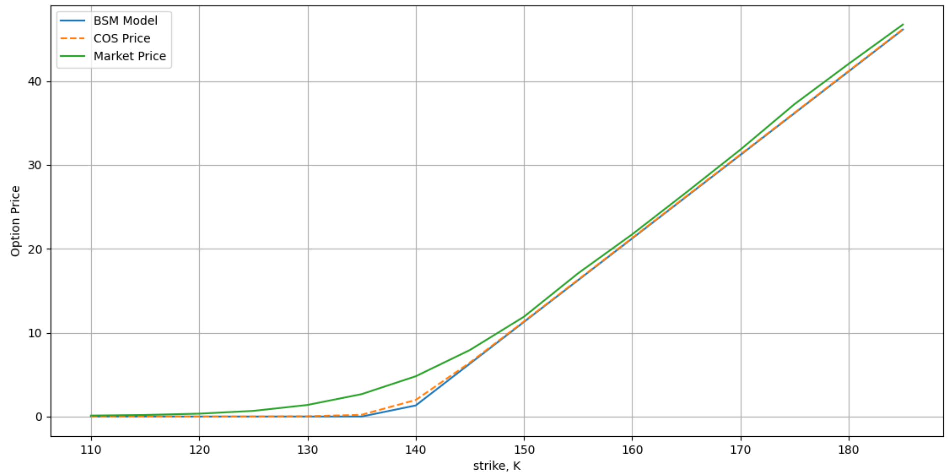


(a) Put prices for options with 30 days expiry

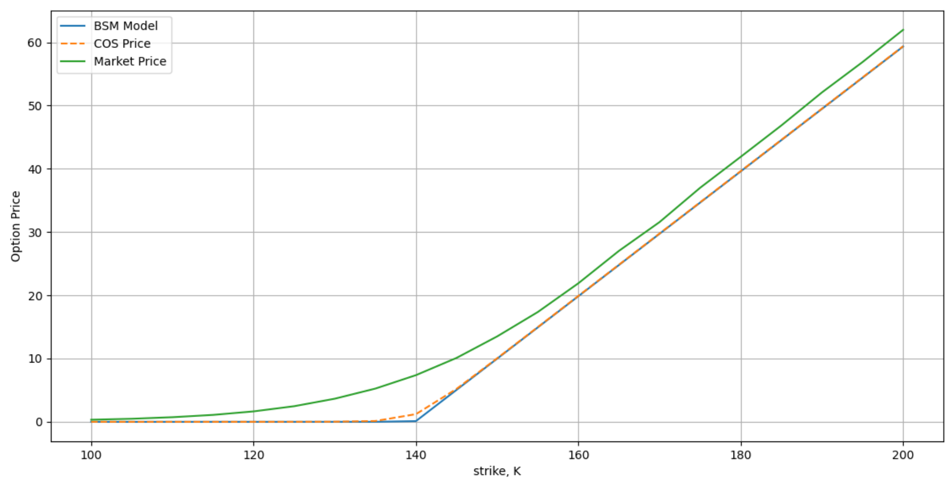


(b) Put prices for options with 90 days expiry

Figure 12. AAPL Put Price Model Comparison: COS-NIG, BSM, MKT



(a) Put prices for options with 30 days expiry



(b) Put prices for options with 90 days expiry

Figure 13. Amazon Inc Put Price Model Comparison: COS-NIG, BSM, MKT

5.6 Results and Discussion

To further analyze the performance of our model, the results were compared with the prevailing market prices(AAPL) using the RMSE(Root Mean Square Error) and ARPE(Absolute Relative Percentage Error) metrics. The formulas in (H. Zhang et al., 2016) were used as follows:

$$RMSE = \sqrt{\sum_{j=1}^{n} \frac{(C_j^{mkt} - C_j^{mdl})^2}{n}} \tag{75}$$

$$ARPE = \frac{1}{n}\sum_{j=1}^{n}\frac{|C_j^{mkt} - C_j^{mdl}|}{C_j^{mkt}} * 100 \tag{76}$$

The RMSE and ARPE values for the call and put prices are presented in the table 5 and 7.

Table 5. RMSE and ARPE values for AAPL call option prices

	30 day expiry		90 day expiry	
	RMSE	ARPE	RMSE	ARPE
BSM	0.9759665	44.8621748	1.7054365	51.2452230
COS GBM	0.9759658	44.8621775	1.7054158	51.2451604
COS NIG	0.8876795	44.1444565	1.5746820	50.71490517

Table 6. RMSE and ARPE values for Amazon call option prices

	30 day expiry		90 day expiry	
	RMSE	ARPE	RMSE	ARPE
BSM	1.0073523	44.3416605	1.6677955	60.9309833
COS GBM	1.0066088	44.3354124	1.6656613	60.9207284
COS NIG	0.9140173	43.2807972	1.6204024	60.7258384

Table 7. RMSE and ARPE values for AAPL put option prices

	30 day expiry		90 day expiry	
	RMSE	ARPE	RMSE	ARPE
BSM	1.3076478	60.7950897	2.9047063	57.7306935
COS GBM	1.3076666	60.7951603	2.9047770	57.7308533
COS NIG	1.1344210	59.8618154	2.8338572	57.2922026

Table 8. RMSE and ARPE values for Amazon put option prices

	30 day expiry		90 day expiry	
	RMSE	ARPE	RMSE	ARPE
BSM	1.0907780	63.1197414	2.6161520	47.3433311
COS GBM	1.0919698	63.1277813	2.6218729	47.356694
COS NIG	0.9680107	61.9969443	2.5488699	47.0365657

The superior performance of the NIG COS method can be seen from the analysis of the pricing errors.It has the best overall performance for both the put and call options. For both the long term and short term maturity date options, the NIG model still performs better than the rest of the models. The BSM model and the COS GBM model have very close RMSE values indicating a similar performance. This is based on the fact that their underlying asset price process follow a GBM process.
The options data was then classified based on the moneyness of the underlying asset in relation to the options' strike prices.The moneyness is calculated as the ratio of the current asset price S_0 to the option's strike price K, that is

$$M = S_0/K$$

The following three categories were obtained, In the Money (ITM),At the Money (ATM) and Out of the Money (OTM);

- If $M < 1$ then a put option is ITM and a call option is OTM.
- If $M = 1$ an option is ATM
- If $M > 1$ then a put option is ATM and a call option is ITM

In tables 9 and 10 we see that for options with short maturity dates, using the COS method with the NIG model gives better perfomance compared to the BSM model for ITM options in both calls and put options. However, the BSM model is the best choice when pricing OTM call options. For long dated options, the NIG model is the best choice for OTM call options.

Table 9. RMSE for ITM, ATM and OTM call options

		BSM	COS GBM	COS NIG
30 day expiry	ITM	0.3818145	0.3818363	0.3221362
	ATM	2.7060797	2.7060640	2.4751220
	OTM	0.1107249	0.1107249	0.1097956
90 day expiry	ITM	0.7178890	0.7178932	0.7177176
	ATM	4.9219719	4.9219020	4.4944945
	OTM	0.4600584	0.4600584	0.4586550

Table 10. RMSE for ITM, ATM and OTM Put options

		BSM	COS GBM	COS NIG
30 day expiry	ITM	1.1105865	1.1106277	0.9157909
	ATM	3.1160274	3.1160496	2.7110326
	OTM	0.4655605	0.4655605	0.4632732
90 day expiry	ITM	3.3129074	3.3130306	3.3321299
	ATM	5.6711569	5.6712096	5.2761748
	OTM	0.8590891	0.8590891	0.8579360

6 Conclusion and Recommendations

6.1 Conclusion

In conclusion, this thesis has explored the effectiveness of the COS-NIG option pricing model for European options, leveraging the unique characteristics of the NIG distribution and Fourier cosine expansions to derive analytic solutions to the risk-neutral integral equation. The goodness-of-fit tests revealed that the log returns of stock prices exhibit a superior fit to the NIG distribution in comparison to the Gaussian distribution. This foundational insight underscores the relevance of the NIG distribution in capturing the underlying dynamics of financial stock markets.

Furthermore,a model comparison of the COS-NIG model with the BSM model shows that the COS-NIG model generally performs better than the BSM model. This finding emphasizes the importance of considering alternative models, such as the COS-NIG, which not only align with empirical data but also provide more accurate pricing predictions for European options. It is however worth noting that for In the Money (ITM) put options, the BSM model performs better than the COS-NIG model.

In essence, the integration of the COS-NIG model into option pricing frameworks offers a more robust and reliable tool for financial practitioners and researchers. The analytical solutions derived from this model, coupled with its demonstrated superiority over the BSM model in empirical tests, contribute significantly to the advancement of option pricing theory. As financial markets continue to evolve, the insights from this research provide a valuable foundation for refining and enhancing option pricing models, ultimately aiding in better risk management and decision-making processes in financial markets.

6.2 Future Research

This research has mainly focused on the European Option pricing. Its performance can be extended in pricing exotic options such as Asian, barrier, and compound options. Assessing its adaptability to non-standard features can broaden its applicability in diverse financial scenarios.

Comparing the speed and accuracy of the COS-NIG model with other computational methods, such as the fast Fourier transform, and other existing techniques, should be a good venture.

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Appendix A: Python Codes

A.1 Data description

A.1.1 Descriptive statistics

Listing A.1. Stock Prices

```
import matplotlib.pyplot as plt
import matplotlib.dates as mdates
import numpy as np
from scipy.stats import norm, norminvgauss, kstest, nct
from scipy.stats import shapiro
import pandas as pd
from scipy.stats import skew, kurtosis
from statsmodels.graphics.gofplots import qqplot

data = pd.read_csv("C:\\Users\\user\\Documents\\AMZN.csv")
x = data['Date']
x = pd.to_datetime(x, format='%d-%b-%y')
y = data['Price']
plt.plot(x, y, linestyle='-', color='b', label='StockPrices')
plt.gca() .xaxis.set_major_locator(mdates.YearLocator())
plt.gca().xaxis.set_major_formatter(mdates . DateFormatter('%Y'))
plt.xlabel('Date')
plt.ylabel('StockPrice')
plt.title('Amazon␣Inc␣Stock␣Prices')
plt.show()

stock_data = np.array(y)
# Calculate mean and variance using NumPy
num_data_points = len(stock_data)
mean = np.mean(stock_data)
variance = np.var(stock_data)

# Calculate skewness and kurtosis using SciPy
skewness = skew(stock_data)
kurt = kurtosis(stock_data)

print(f"Number␣of␣data␣points:␣{num_data_points}")
print(f"Mean:␣{mean}")
print(f"Variance:␣{variance}")
print(f"Skewness:␣{skewness}")
print(f"Kurtosis:␣{kurt}")

# Calculate log-returns
log_returns = np.log(stock_data[1:] / stock_data[:-1])
# Calculate mean and variance using NumPy
num_data = len(log_returns)
mean_returns = np.mean(log_returns)
```

```
variance_returns = np.var(log_returns)

# Calculate skewness and kurtosis using SciPy
skewness_returns = skew(log_returns)
kurt_returns = kurtosis(log_returns)

print(f"Number of data points: {num_data}")
print(f"Mean_L: {mean_returns}")
print(f"Variance_L: {variance_returns}")
print(f"Skewness_L: {skewness_returns}")
print(f"Kurtosis_L: {kurt_returns}")

# Perform the Shapiro-Wilk test
statistic, p_value = shapiro(log_returns)

print("Shapiro-Wilk Test:")
print("Statistic:", statistic)
print("p-value:", p_value)

if p_value > 0.05:
    print("The data appears to be normally distributed.")
else:
    print("The data does not appear to be normally distributed.")
#KS test
# Estimate NIG distribution parameters from log-returns
params = norminvgauss.fit(log_returns)
alpha, beta, mu, delta = params
print(params)
#params = [38.26604, ]
# Generate random samples from the estimated NIG distribution
n = len(log_returns)
nig_samples = norminvgauss.rvs(*params, size=n)
# Perform the KS test
ks_statistic, p_value = kstest(log_returns, 'norminvgauss', params)

print("KS Statistic:", ks_statistic)
print("P-Value:", p_value)
#Histogram
hist, bin_edges = np.histogram(log_returns, density=True)

mu, std = norm.fit(log_returns)
params = nct.fit(log_returns)
x = np.linspace(min(log_returns), max(log_returns), 1000)
pdf_normal = norm.pdf(x, mu, std)
pdf_nig = nct.pdf(x, *params)
#Q-Qplots
qqplot(log_returns, line = "s")
plt.title('Amazon Q-Q Plot against Normal Distribution')
plt.show()

plt.figure(figsize=(10, 6))
plt.hist(log_returns, bins=25, density=True, alpha=0.5, label='Empirical Density')
plt.plot(x, pdf_normal, 'r-', lw=2, label='Fitted Normal')
plt.plot(x, pdf_nig, 'g-', lw=2, label='Fitted NIG')
plt.legend()
plt.title('Amazon Inc Empirical Density vs. Fitted Distributions (NIG and Normal)')
```

```
plt.xlabel('Value')
plt.ylabel('Density')
plt.show()
```

A.1.2 NIG Parameters

Listing A.2. Estimated NIG Parameters

```
import numpy as np
from scipy.optimize import minimize
from scipy.stats import norminvgauss

# Generate some example data
data = np.random.normal(0, 1, 1000)

# Define the log-likelihood function for the NIG distribution
def neg_log_likelihood(params, data):
    alpha, beta, mu, delta = params
    return -np.sum(norminvgauss.logpdf(data, alpha, beta, mu, delta))

# Initial parameter guesses
initial_guess = [1, 1, 0, 1]

# Perform Maximum Likelihood Estimation
result = minimize(neg_log_likelihood, initial_guess, args=(data,),
method='Nelder-Mead')
estimated_params = result.x

alpha, beta, mu, delta = estimated_params

print("Estimated␣Parameters:")
print("Alpha:", alpha)
print("Beta:", beta)
print("Mu:", mu)
print("Delta:", delta)
```

A.1.3 Density Recovery

Normal Density recovery

Listing A.3. Normal Density

```
# %%
"""
Normal density recovery using the COS method
"""
import numpy as np
import matplotlib.pyplot as plt
import scipy.stats as st

def COSDensity(cf, x, N, a, b):
```

```
    i = complex(0.0, 1.0)  # assigning i=sqrt(-1)
    k = np.linspace(0, N - 1, N)
    u = np.zeros([1, N])
    u = k * np.pi / (b - a)

    # F_k coefficients
    F_k = 2.0 / (b - a) * np.real(cf(u) * np.exp(-i * u * a));
    F_k[0] = F_k[0] * 0.5;  # adjustment for the first term

    # Final calculation
    f_X = np.matmul(F_k, np.cos(np.outer(u, x - a)))

    # we output only the first row
    return f_X


def mainCalculation():
    i = complex(0.0, 1.0)  # assigning i=sqrt(-1)

    # setting for the COS method
    a = -10.0
    b = 10.0

    # define the range for the expansion points
    N = [2 ** x for x in range(2, 7, 1)]

    # setting for normal distribution
    mu = 0.0
    sigma = 1.0

    # Define characteristic function for the normal distribution
    cF = lambda u: np.exp(i * mu * u - 0.5 * np.power(sigma, 2.0) * np.power(u, 2.0));

    # define domain for density
    x = np.linspace(-10.0, 10, 1000)
    f_XExact = st.norm.pdf(x, mu, sigma)

    fig, ax = plt.subplots(figsize=(8, 6))
    for n in N:
        f_X = COSDensity(cF, x, n, a, b)
        error = np.max(np.abs(f_X - f_XExact))
        print("For {0} expanansion terms the error is {1}".format(n, error))

        # Add the current curve to the same subplot
        ax.plot(x, f_X, label=f'n={n}')

    # Set labels, title, and legend
    ax.set_xlabel('x')
    ax.set_ylabel('y')
    ax.set_title('Recovered Normal Density at different values of N')
    ax.legend()

    # Show the combined plot with all curves
    plt.show()
```

```
mainCalculation()
```

NIG Density Recovery

Listing A.4. NIG Density

```
# %%
"""
NIG density recovery using the COS method
"""
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norminvgauss

def COSDensity(cF, x, N, a, b):
    i = complex(0.0, 1.0)  # assigning i=sqrt(-1)
    k = np.linspace(0, N - 1, N)
    u = np.zeros([1, N])
    u = k * np.pi / (b - a)

    # F_k coefficients
    F_k = 2.0 / (b - a) * np.real(cF(u) * np.exp(-i * u * a));
    F_k[0] = F_k[0] * 0.5;  # adjustment for the first term

    # Final calculation
    f_X = np.matmul(F_k, np.cos(np.outer(u, x - a)))

    # we output only the first row
    return f_X


def mainCalculation():
    i = complex(0.0, 1.0)  # assigning i=sqrt(-1)

    # setting for the COS method
    a = -10.0
    b = 10.0

    # define the range for the expansion points
    N = [16, 32, 64, 128]

    # setting for NIG distribution
    alpha = 16.29
    beta = 0.76
    mu = -0.19
    delta = 3.99

    # Define characteristic function for the NIG distribution
    cF = lambda u: np.exp(i * mu * u - delta * np.sqrt(alpha**2
    - (beta + i * u)**2) + delta * np.sqrt(alpha**2 - beta**2));

    # define domain for density
    x = np.linspace(-10, 10, 1000)
```

```
    f_XExact = norminvgauss.pdf(x, alpha, beta, mu, delta)

    fig, ax = plt.subplots(figsize=(8, 6))
    for n in N:
        f_X = COSDensity(cF, x, n, a, b)
        error = np.max(np.abs(f_X - f_XExact))
        print("For {0} expansion terms the error is {1}".format(n, error))

        # Add the current curve to the same subplot
        ax.plot(x, f_X, label=f'n={n}')

    # Set labels, title, and legend
    ax.set_xlabel('x')
    ax.set_ylabel('y')
    ax.set_title('Recovered NIG Density at different values of N')
    ax.legend()

    # Show the combined plot with all curves
    plt.show()

mainCalculation()
```

A.2 Option pricing

A.2.1 GBM model

Listing A.5. Option pricing with GBM model

```
# %%
"""
COS GBM model
"""
import numpy as np
import matplotlib.pyplot as plt
import scipy.stats as st
import time
import pandas as pd

def cosnigmethod(chf, CP, S0, delta, aalpha, sigma, beeta, r, tau, K, N, L):
    # chf   - characteristic function
    # CP    - C for call and P for put
    # S0    - Initial stock price
    # r     - interest rate
    # tau   - time to maturity
    # K     - Strikes
    # N     - Number of expansion terms
    # L     - size of truncation domain

    # reshape K to a column vector
    K = np.array(K).reshape([len(K), 1])

    # assigning i=sqrt(-1)
    i = complex(0.0, 1.0)
```

```
    x0 = np.log(S0 / K)

    # truncation domain
    aalpha = 0 - L * np.sqrt(tau)
    beeta = 0 + L * np.sqrt(tau)

    # sumation from n = 0 to n=N-1
    n = np.linspace(0, N-1, N).reshape([N, 1])
    v = n * np.pi / (beeta - aalpha)

    # Determine coefficients for Put Prices
    I_n = callputcoefficients(CP, aalpha, beeta, n)

    mat = np.exp(i * np.outer((x0 - aalpha), v))

    temp = chf(v) * I_n
    temp[0] = 0.5 * temp[0]

    value = np.exp(-r * tau) * K * np.real(mat.dot(temp))

    return value


"""
Determine coefficients for Put Prices
"""
def callputcoefficients(CP, aalpha, beeta, n):
    if str(CP).lower() == "c" or str(CP).lower() == "1":
        upsilon = 0
        nu = beeta
        coef = xi_zeta(aalpha, beeta, upsilon, nu, n)
        xi_n = coef["xi"]
        zeta_n = coef["zeta"]
        if aalpha < beeta < 0.0:
            I_n = np.zeros([len(n), 1])
        else:
            I_n = 2.0 / (beeta - aalpha) * (xi_n - zeta_n)

    elif str(CP).lower() == "p" or str(CP).lower() == "-1":
        upsilon = aalpha
        nu = 0
        coef = xi_zeta(aalpha, beeta, upsilon, nu, n)
        xi_n = coef["xi"]
        zeta_n = coef["zeta"]
        I_n = 2.0 / (beeta - aalpha) * (- xi_n + zeta_n)

    return I_n

def xi_zeta(aalpha, beeta, upsilon, nu, n):
    zeta = (np.sin(n * np.pi * (nu - aalpha) / (beeta - aalpha))
            - np.sin(n * np.pi * (upsilon - aalpha)/(beeta - aalpha)))
    zeta[1:] = zeta[1:] * (beeta - aalpha) / (n[1:] * np.pi)
    zeta[0] = nu - upsilon

    xi = 1.0 / (1.0 + np.power((n * np.pi / (beeta - aalpha)), 2.0))
```

```
    expr1 = (np.cos(n * np.pi * (nu - aalpha)/(beeta - aalpha))
    * np.exp(nu) - np.cos(n * np.pi * (upsilon - aalpha)
     / (beeta - aalpha)) * np.exp(upsilon))
    expr2 = (n * np.pi / (beeta - aalpha)
     * np.sin(n * np.pi * (nu - aalpha) / (beeta - aalpha))
             - n * np.pi / (beeta - aalpha)
             * np.sin(n * np.pi * (upsilon - aalpha) / (beeta - aalpha))
             * np.exp(upsilon))
    xi = xi * (expr1 + expr2)

    value = {"xi": xi, "zeta": zeta}

    return value


def bsmmethod(CP, S_0, K, sigma, tau, r):
    # Black-Scholes Call option price
    cp = str(CP).lower()
    K = np.array(K).reshape([len(K), 1])
    d1 = (np.log(S_0 / K)
    + (r + 0.5 * np.power(sigma, 2.0)) * tau) / float(sigma * np.sqrt(tau))
    d2 = d1 - sigma * np.sqrt(tau)
    if cp == "c" or cp == "1":
        value = st.norm.cdf(d1) * S_0 - st.norm.cdf(d2) * K * np.exp(-r * tau)
    elif cp == "p" or cp == "-1":
        value = st.norm.cdf(-d2) * K * np.exp(-r * tau) - st.norm.cdf(-d1)*S_0
    return value

def maincalculation():
    i = complex(0.0, 1.0)

    CP = "p"
    S0 = 177.79
    r = 0.053
    tau = 90/365
    sigma = 0.002376514
    K = list(range(50, 291, 5))
    N = 2**14
    L = 8

    chf = lambda v: np.exp((r - 0.5 * np.power(sigma,2.0)) * i * v * tau - 0.5
                           * np.power(sigma, 2.0) * np.power(v, 2.0) * tau)

    # Timing results
    NoOfIterations = 100
    time_start = time.time()
    for k in range(0, NoOfIterations, 1):
        val_COS = cosnigmethod(chf, CP, S0, delta, alpha, sigma, beta, r, tau, K, N, L)
    time_stop = time.time()
    print("It␣took␣{0}␣seconds␣to␣price."
    .format((time_stop-time_start)/float(NoOfIterations)))
    # evaluate analytical Black Scholes equation
    val_BSM = bsmmethod(CP, S0, K, sigma, tau, r)
    # import market option prices for comparison
    market_option_prices =
    pd.read_excel("C:\\Users\\user\\Desktop\\Msc␣Thesis\\My␣Thesis\\Put_Prices_90.xlsx")
```

```
    mkt_price = market_option_prices['Market␣Price']

    plt.plot(K, val_COS, '--', label='COS')
    plt.plot(K, val_BSM, label='BSM')
    plt.plot(K, mkt_price, label='Market␣Price')

    plt.xlabel("strike,␣K")
    plt.ylabel("Option␣Price")
    plt.legend(["COS␣Price", "BS␣model", "Market␣Price"])
    plt.grid()
    plt.show()

    #  Data
    data = {'Strike␣Price': K,
            'COS␣Value': [val_COS[i][0] for i in range(len(K))],
            'Exact␣Value': [val_BSM[i][0] for i in range(len(K))]}

    result_df = pd.DataFrame(data)

    # Save the results to an Excel file
    result_df.to_excel('Put_Price_GBMAAPL_90.xlsx', index=False)

maincalculation()
```

A.2.2 NIG model

Listing A.6. Option pricing with NIG model

```
# %%
"""
COS NIG model
"""
import numpy as np
import matplotlib.pyplot as plt
import scipy.stats as st
import time
import pandas as pd

def cosnigmethod(chf, CP, S0, delta, aalpha, sigma, beeta, r, tau, K, N, L):
    # chf   - characteristic function
    # CP    - C for call and P for put
    # S0    - Initial stock price
    # r     - interest rate
    # tau   - time to maturity
    # K     - Strikes
    # N     - Number of expansion terms
    # L     - size of truncation domain

    # reshape K to a column vector
    K = np.array(K).reshape([len(K), 1])

    # assigning i=sqrt(-1)
    i = complex(0.0, 1.0)

    x0 = np.log(S0 / K)
```

```
    # truncation domain
    aalpha = 0 - L * np.sqrt(tau)
    beeta = 0 + L * np.sqrt(tau)

    # sumation from n = 0 to n=N-1
    n = np.linspace(0, N-1, N).reshape([N, 1])
    v = n * np.pi / (beeta - aalpha)

    # Determine coefficients for Put Prices
    I_n = callputcoefficients(CP, aalpha, beeta, n)

    mat = np.exp(i * np.outer((x0 - aalpha), v))

    temp = chf(v) * I_n
    temp[0] = 0.5 * temp[0]

    value = np.exp(-r * tau) * K * np.real(mat.dot(temp))

    return value


"""
Determine coefficients for Put Prices
"""
def callputcoefficients(CP, aalpha, beeta, n):
    if str(CP).lower() == "c" or str(CP).lower() == "1":
        upsilon = 0
        nu = beeta
        coef = xi_zeta(aalpha, beeta, upsilon, nu, n)
        xi_n = coef["xi"]
        zeta_n = coef["zeta"]
        if aalpha < beeta < 0.0:
            I_n = np.zeros([len(n), 1])
        else:
            I_n = 2.0 / (beeta - aalpha) * (xi_n - zeta_n)

    elif str(CP).lower() == "p" or str(CP).lower() == "-1":
        upsilon = aalpha
        nu = 0
        coef = xi_zeta(aalpha, beeta, upsilon, nu, n)
        xi_n = coef["xi"]
        zeta_n = coef["zeta"]
        I_n = 2.0 / (beeta - aalpha) * (- xi_n + zeta_n)

    return I_n

def xi_zeta(aalpha, beeta, upsilon, nu, n):
    zeta = (np.sin(n * np.pi * (nu - aalpha) / (beeta - aalpha))
          - np.sin(n * np.pi * (upsilon - aalpha)/(beeta - aalpha)))
    zeta[1:] = zeta[1:] * (beeta - aalpha) / (n[1:] * np.pi)
    zeta[0] = nu - upsilon

    xi = 1.0 / (1.0 + np.power((n * np.pi / (beeta - aalpha)), 2.0))
    expr1 = (np.cos(n * np.pi * (nu - aalpha)/(beeta - aalpha)) * np.exp(nu)
           - np.cos(n * np.pi * (upsilon - aalpha) / (beeta - aalpha))
```

```
            * np.exp(upsilon))
    expr2 = (n * np.pi / (beeta - aalpha) * np.sin(n * np.pi
    * (nu - aalpha) / (beeta - aalpha))
     - n * np.pi / (beeta - aalpha) * np.sin(n * np.pi
    * (upsilon - aalpha) / (beeta - aalpha))
            * np.exp(upsilon))
    xi = xi * (expr1 + expr2)

    value = {"xi": xi, "zeta": zeta}

    return value


def bsmmethod(CP, S_0, K, sigma, tau, r):
    # Black-Scholes Call option price
    cp = str(CP).lower()
    K = np.array(K).reshape([len(K), 1])
    d1 = (np.log(S_0 / K) + (r + 0.5 * np.power(sigma, 2.0)) * tau)
    / float(sigma * np.sqrt(tau))
    d2 = d1 - sigma * np.sqrt(tau)
    if cp == "c" or cp == "1":
        value = st.norm.cdf(d1) * S_0 - st.norm.cdf(d2) * K * np.exp(-r * tau)
    elif cp == "p" or cp == "-1":
        value = st.norm.cdf(-d2) * K * np.exp(-r * tau) - st.norm.cdf(-d1)*S_0
    return value

def maincalculation():
    i = complex(0.0, 1.0)

    CP = "p"
    alpha = 42.89208
    beta = -3.329876
    delta = 0.018082
    S0 = 177.79
    r = 0.053
    tau = 90/365
    sigma = 0.002376514
    K = list(range(50, 291, 5))
    N = 2**14
    L = 8
    w = delta * (np.sqrt(alpha**2
    - (beta + 1)**2) - np.sqrt(alpha**2 - beta**2))

    chf = lambda v: (np.exp(r * i * v * tau + i * v * tau * w)
    * np.exp(delta * (np.sqrt(alpha**2 - beta**2)
    - np.sqrt(alpha**2 - (beta + i * v)**2))))


    # Timing results
    NoOfIterations = 100
    time_start = time.time()
    for k in range(0, NoOfIterations, 1):
        val_COS = cosnigmethod(chf, CP, S0, delta,
         aalpha, sigma, beeta, r, tau, K, N, L)
    time_stop = time.time()
    print("It␣took␣{0}␣seconds␣to␣price."
```

```
    .format((time_stop-time_start)/float(NoOfIterations)))
    # evaluate analytical Black Scholes equation
    val_BSM = bsmmethod(CP, S0, K, sigma, tau, r)
    # import market option prices for comparison
    market_option_prices =
     pd.read_excel("C:\\Users\\user\\Desktop
␣␣␣␣␣\\Msc␣Thesis\\My␣Thesis\\Put_Prices_90.xlsx")
    mkt_price = market_option_prices['Market␣Price']

    plt.plot(K, val_COS,'--', label='COS')
    plt.plot(K, val_BSM, label='BSM')
    plt.plot(K, mkt_price, label='Market␣Price')

    plt.xlabel("strike,␣K")
    plt.ylabel("Option␣Price")
    plt.legend(["COS␣Price", "BS␣model", "Market␣Price"])
    plt.grid()
    plt.show()

    # data
    data = {'Strike␣Price': K,
            'COS␣Value': [val_COS[i][0] for i in range(len(K))],
            'Exact␣Value': [val_BSM[i][0] for i in range(len(K))]}

    result_df = pd.DataFrame(data)

    # Save the results to an Excel file
    result_df.to_excel('Put_Price_NIGAAPL_90.xlsx', index=False)

maincalculation()
```