International Electronic Journal of Pure and Applied Mathematics
Volume 5 No. 3 2012, 99-104

ON COMMUTANTS AND OPERATOR EQUATIONS

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Abstract: Let B(H) denote the algebra of bounded linear operators on a Hilbert Space H into itself. Given $A, B \in B(H)$ define $C(A,B)$ and $R(A,B) : B(H) \longrightarrow B(H)$ by $C(A,B)X = AX - XB$ and $R(A,B)X = AXB - X$. Our task in this note is to show that if A is one-one and B has dense range then $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A,B)X = 0$ for some $X \in B(H)$. Similarly, if $R(A^2, B^2)X = 0$ and $R(A^3, B^3)X = 0$ then $R(A,B)X = 0$ for some $X \in B(H)$.

AMS Subject Classification: 47B47, 47A30, 47B20
Key Words: commutant, quasiaffinity and normal operator

1. Introduction

Let $B(H)$ denote the algebra of operators, i.e. bounded linear transformations on the complex Hilbert space H into itself.

Given $A, B \in B(H)$, let $C(A,B) : B(H) \longrightarrow B(H)$ be defined by $C(A,B)X = AX - XB$ and $R(A,B)X = AXB - X$. Moajil [5] proved that if N is a normal operator such that $N^2X = XN^2$ and $N^3X = XN^3$ for some $X \in B(H)$, then $NX = XN$. Thus for a normal operator N, if $N^2 \in \{X\}'$ and $N^3 \in \{X\}'$, then $N \in \{X\}'$ for some $X \in B(H)$.

Kittaneh [4] generalized this result to cover subnormal operators by taking A and B^* to be subnormal operators, i.e. if $A^2X = XB^2$ and $A^3X = XB^3$ for some $X \in B(H)$, then $AX = XB$. Thus if $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ then $C(A,B) = 0$ for some $X \in B(H)$.

Bachir [1] generalized these results to cover the classes of dominant and p-hyponormal operators as follows:

Received: July 31, 2012

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Theorem A. *Let A be a dominant operator and B^* be a p-hyponormal operator or log-hyponormal. If $A^2X = XB^2$ and $A^3X = XB^3$ then $AX = XB$, for some $X \in B(H)$. Thus we have that if A is dominant and B^* is either p-hyponormal or log-hyponormal then $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A, B)X = 0$*

In this note we consider any operator $A, B \in B(H)$ without necessarily specifying the classes in which they belong and look for other conditions under which we can get similar results on the operator equation $C(A, B)X = 0$. We will also investigate similar results on the operator equation $R(A, B)X = 0$. Khalagai & Nyamai, [3] also had the following theorem and corollaries on the operator equation $R(A, B)X = 0$.

Theorem B. *Let A, B and $X \in B(H)$ be such that $R(A, B)X = 0$. Then B is one to one whenever X is one to one.*

Corollary A. *Let A, B and $X \in B(H)$ be such that $R(A, B)X = 0$ where X is quasiaffinity. Then both B and A^* are one to one.*

Corollary B. *Let A, B and $X \in B(H)$ be such that $R(A, B)X = 0$ implies $R(A^*, B^*)X = 0$ where X is a quasiaffinity. Then both A and B are also quasiaffinities.*

Goya & Saito [2] had the following result:

Theorem C. *Let $A, B, X \in B(H)$ where A is a paranormal contraction, B a coisometry and X has a dense range. Assume $C(A, B)X = 0$. Then A is a unitary operator. In particular, if X is injective and has a dense range, then B is also a unitary operator.*

2. Notation and Terminology

Given an operator $A \in B(H)$ we shall denote the spectrum of A by $\sigma(A)$. Thus $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \, is\, not\, invertible\}$. The numerical range of A is denoted by $W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$. The commutator of any two operators A and B is defined by $[A, B] = AB - BA$. The commutant of A is given by $\{A\}' = \{X \in B(H) : [A, X] = 0\}$. An operator A is said to be:

- Dominant if to each $\lambda \in \mathbb{C}$ there corresponds a number $M_\lambda \geq 1$ such that for all $x \in H, \|(A - \lambda I)^*x\| \leq M_\lambda \|(A - \lambda I)x\|$.
- M-hyponormal if there is a constant M such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$ such that $\|(A - \lambda I)^*x\| \leq M\|(A - \lambda I)x\|$
- Hyponormal if from above $M = 1$
- P-hyponormal if $(A^*A)^p \geq (AA^*)^p$ for $0 < p < 1$

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- Log-hyponormal if A is an invertible operator such that $log(A^*A) \geq log(AA^*)$
- Paranormal if $\|A^2x\| \leq \|Ax\|^2$ for any unit vector $x \in H$
- Normal if $A^*A = AA^*$
- Subnormal if A has a normal extension
- Partial isometry if $A = AA^*A$
- Isometry if $A^*A = I$
- Co-isometry if $AA^* = I$
- Unitary if $A^*A = AA^* = I$
- Compact if for each bounded sequence $\{x_n\}$ in the domain H, the sequence $\{Ax_n\}$ contains a sub sequence converging to some limit in the range.
- Contraction if $\|A\| \leq 1$.

3. Results

Theorem 1. *Let $A, B \in B(H)$ be any pair of operators such that A is one-one and B has a dense range. Then we have that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A, B)X = 0$ for some $X \in B(H)$.*

Proof. Let $T = AX$ and $S = XB$. Then from $A^2X = XB^2$ and $A^3X = XB^3$, we have $AT = SB$ and $A^2T = SB^2$ and moreover:

$$A(AT) = ASB = (SB)B,$$

$$ASB - (SB)B = 0,$$

$$(AS - SB)B = 0.$$

Since B has dense range we have that $B \neq 0$ and hence $AS - SB = 0$. Therefore

$$AS = SB,$$

$$AT = SB = AS,$$

$$AT - AS = 0,$$

i.e. $T - S = 0$ since A is one-one, $T = S$. Thus $AX = XB$.

Hence $C(A, B)X = 0$. □

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Corollary 1. *If A and B are quasi-affinities such that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ then $C(A, B)X = 0$ for some $X \in B(H)$.*

Proof. If A and B are quasi-affinities then each one of them is both one-one and has dense range. Hence the proof of Theorem 1 can easily be traced to give the required result. □

Corollary 2. *If A is a quasi-affinity such that $C(A^2, A^{*2})X = 0$ and $C(A^3, A^{*3})X = 0$ then $C(A, A^*)X = 0$ for some $X \in B(H)$.*

Proof. If A is quasi-affinity then A^* is also quasi-affinity. Hence by Corollary 1 the result follows. □

Corollary 3. *Let $\wp$ be the class of operators defined as follows:*

$$\wp = \{A \in B(H) : 0 \notin W(A)\}.$$

If $A, B \in \wp$ such that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ then $C(A, B)X = 0$ for some $X \in B(H)$.

Proof. We only have to note that for any operator A with $0 \notin W(A)$, A is both one-one and has a dense range. □

Corollary 4. *If A is a quasi-affinity such that $A^2 \in \{X\}'$ and $A^3 \in \{X\}'$ then $A \in \{X\}'$ for some $X \in B(H)$.*

Proof. We only have to note that in Theorem 1 we let $A = B$. □

Theorem 2. *Let $A, B \in B(H)$ be a pair of operators such that A is one-one and B has dense range. Then $R(A^2, B^2)X = 0$ and $R(A^3, B^3)X = 0$ imply $R(A, B)X = 0$ for some $X \in B(H)$.*

Proof. Given $A^2XB^2 = X$ and $A^3XB^3 = X$ we have $A^2XB^2 = A^3XB^3$,

$$A^3XB^3 - A^2XB^2 = 0,$$

$$A(A^2XB^2 - AXB)B = 0.$$

Since A is one-one and B has dense range we have:

$$A^2XB^2 - AXB = 0$$

i.e. $A(AXB - X)B = 0$. Since A is one-one and B has dense range we have that $AXB - X = 0$.

Hence $R(A, B)X = 0$. □

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Corollary 5. *If $A, B \in B(H)$ are quasi-affinity such that $R(A^2, B^2)X = 0$ and $R(A^3, B^3)X = 0$, then $R(A, B)X = 0$.*

Proof. We note that the quasi-affinity is both one to one and has dense range. Hence the result is immediate by Theorem 2 above. □

Corollary 6. *A is quasi-affinity such that:*

$R(A^2, A^{*2})X = 0$ and $R(A^3, A^{*3})X = 0$ then $R(A, A^*)X = 0$ for some $X \in B(H)$.

Proof. It is immediate from Theorem 2 above and the fact that if A is a quasi-affinity then A^* is also quasi-affinity. □

Corollary 7. *If $R(A, B)X = 0$ implies $R(A^*, B^*)X = 0$ for some X which is quasi-affinity then $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A, B)X = 0$.*

Proof. $R(A, B)X = 0$ implying $R(A^*, B^*)X = 0$ where X is a quasi-affinity implies A and B are quasi-affinities from Corollary B. From Theorem 1, the result follows since quasi-affinities are both one to one and have a dense range. □

Corollary 8. *Let $A, B, X \in B(H)$ where A is a paranormal contraction, B a coisometry and X is a quasi-affinity. If $C(A, B^*)X = 0$, then $R(A, B)X = 0 = R(A^*, B^*)X$.*

Proof. First note that B is unitary from Theorem C. Therefore

$$C(A, B^*)X = 0 \Rightarrow AX = XB^*,$$

$$\Rightarrow AXB = XB^*B, \Rightarrow AXB = X,$$

$$\Rightarrow AXB - X = 0,$$

$$R(A, B)X = 0.$$

We also have that A is unitary by theorem C. Thus:

$$C(A, B^*)X = 0 \Rightarrow AX = XB^*,$$

$$\Rightarrow A^*AX = A^*XB^*,$$

$$\Rightarrow X - A^*XB^* = 0,$$

$$\Rightarrow A^*XB^* - X = 0,$$

$$R(A^*, B^*)X = 0.$$

□

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