DIRECT SUM DECOMPOSITION AND CANONICAL FACTORIZATION OF OPERATORS IN HILBERT SPACES

By

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Declaration

This thesis is my original work and has not been presented for a degree award in any other University.

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This thesis has been submitted for examination with our approval as University supervisors.

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Abstract

The existence of direct sum decompositions and factorizations of bounded linear operators acting on a Hilbert space appears to be one of the most difficult questions in the theory of linear operators. The direct sum decomposition problem is closely related to the invariant subspace problem, which to date has very few affirmative answers regarding it. In this thesis we study the direct sum decomposition and factorization of some classes of operators in Hilbert spaces with a view to determining properties of the direct summands of these operators, their invariant and hyperinvariant subspace lattices and factors for such operators.

This thesis is organized as follows: Chapter 1 is an introduction and is devoted largely to notations and terminology and examples of various concepts that we shall use in the rest of this thesis.

Chapter 2 deals with the orthogonal direct sum decomposition of an arbitrary operator into a normal and a completely non-normal part. In this chapter we show that a general operator T decomposes in this manner. We give conditions under which an operator has nontrivial normal and direct summands. We study this decomposition for operators in the same equivalence classes (quasisimilar, similar, unitarily equivalent, almost-similar operators). We give conditions under which a non-normal operator is normal.

Chapter 3 is on the direct sum decomposition of a contraction operator into a unitary and a completely non-unitary (c.n.u.) part. We give conditions under which a non-unitary operator is unitary. We show that a general operator enjoys this decomposition upon re-normalization (by dividing the operator by its norm). In so doing we show that the problem of decomposing an operator into a normal and a c.n.n. part can be deduced from the decomposition of a contraction operator. We pay special attention to the c.n.u parts of an operator and the shift operators which play a very important role in this kind of decomposition. We use the canonical backward shift as a model to aid the decomposition of such operators. We also deduce the characteristic functions of some classes of operators and use them to determine the nature of the original contraction

operator.

In Chapter 4, we study the invariant and hyperinvariant subspaces for some classes of operators. We show that these subspaces reveal a lot of information about the direct sum decompositions of linear operators. We investigate the topological structure of $Lat(T)$ and $Hyperlat(T)$ for some operator classes containing T. We show that there is a one-to-one correspondence between the invariant lattice and the regular factorization of the characteristic function of a contraction operator T. We generalize this result to arbitrary operators.

Chapter 5 is on the factorization of some operators as a product of simpler operators (self-adjoint, unitary, normal, projections, idempotents, n-th roots of the identity, cyclic, scalar, etc.). We find necessary and sufficient conditions under which an operator can be expressed as a product of such simpler operators. We give necessary and sufficient conditions on the minimal number of such operator factors by improving on some known results.

By a canonical model of an operator we mean a natural representation of the operator in terms of simpler operators, and in a context in which more structure is present.

Most of the results in the direct sum decompositions of an operator T will revolve around its *nearness* to being a normal operator ($[T^*, T] = T^*T - TT^* \equiv 0$) and its *nearness* to being a unitary operator ($D_T^2 = I - T^*T \equiv 0$ or $D_{T^*}^2 = I - TT^* \equiv 0$).

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Nairobi, Kenya **Bernard M. Nzimbi**

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Dedication

This work is dedicated to my family.

List of abbreviations

$B(\mathcal{H})$: Banach algebra of bounded linear operators on $\mathcal{H}$

$\mathbb{D}$: Open unit disc in $\mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$\partial\mathbb{D}$: Unit circle in $\mathbb{C}$, $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$

$\langle\ ,\rangle$: Inner product on the Hilbert space $\mathcal{H}$

$\ell^2(\mathbb{C})$: The Hilbert space of sequences $(\xi_1, \xi_2, ...)$ of complex numbers for which $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$ with the inner product $\langle,\rangle$ defined on $\ell^2 \times \ell^2$ by $\langle x, y\rangle = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i}$ where $x = (\xi_i)$ and $y = (\eta_i)$ induced from the norm $\|x\| = \langle x, x\rangle^{1/2} = (\sum_{i=1}^{\infty} |\xi|^2)^{1/2}$.

$L^2([a,b])$: The Hilbert space of all complex valued Lebesgue measurable functions f defined in the interval $a \leq x \leq b$ with the property that $|f|^2$ is Lebesgue integrable and inner product $\langle f, g\rangle = \int_a^b f(x)\overline{g}(x)dx$.

$\hat{f}(k)$: the k^{th} Fourier coefficient of the function f defined on the unit circle $\partial\mathbb{D}$

$\mathbb{H}^2$, $\mathbb{H}^\infty$: Hardy classes of analytic functions, where

$\mathbb{H}^p = \{f \in L^p(\partial\mathbb{D}) : \hat{f}(k) = \int_{\partial\mathbb{D}} f(z)z^{-k}\frac{|dz|}{2\Pi} = 0, \quad for\ k < 0\}$

or

$\mathbb{H}^2 = \{f : \mathbb{D} \longrightarrow \mathbb{C} : f(z) = \sum_{n=0}^{\infty} a_n z^n, \ \sum_{n=0}^{\infty} |a_n|^2 < \infty, \ for\ all\ z \in \mathbb{D}\}$

$\mathbb{H}^\infty$: space of all functions that are analytic and bounded on the open unit disk $\partial\mathbb{D}$ with the norm of $f \in \mathbb{H}^\infty$ defined by $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$

Θ_T : Characteristic function of a contraction operator T

$r(T)$: The spectral radius of T, which is the radius of the smallest circle in the complex plane $\mathbb{C}$ and contains the spectrum of T.

$\mathcal{M}(\mathbb{D})$: the set of all injective, analytic mappings of the open unit disc $\mathbb{D}$ onto itself.

c.n.u. : Completely non-unitary

c.n.n. : Completely non-normal

$\{T\}'$: Commutant of T

$\{T\}''$: Double commutant of T

$\mathcal{M} \oplus \mathcal{N}$: Direct sum of $\mathcal{M}$ and $\mathcal{N}$

$\mathcal{M}^\perp$: Orthogonal complement of $\mathcal{M}$

$P_\mathcal{M}$: Orthogonal projection onto a closed subspace $\mathcal{M}$

$m, |.|$: the normalized (i.e $m(\partial\mathbb{D}) = 1$) Lebesgue measure on $\partial\mathbb{D}$

Chapter 1

Preliminaries

1.1 Introduction

In this thesis we study the direct sum decompositions and factorizations of some classes of operators on Hilbert spaces. The idea of decomposing an operator (or an operator-valued function) into parts, which are easier to investigate than the original operator, is fundamental to the theory of operators. The so-called direct sum decomposition is one of many known kinds of decompositions. The direct sum decomposition has been largely motivated by the work of Nagy and Foias [53] from which it results that any operator can be decomposed as a direct sum of normal and completely non-normal (c.n.n) parts and that a contraction operator can be decomposed as a direct sum of a unitary and completely non-unitary (c.n.u) parts (where any of these direct summands could be missing). Wold [77], while studying stationary stochastic processes, discovered the decomposition of an isometry into the unitary and the completely non-unitary parts, which has since been referred to as the von Neumann-Wold decomposition of an isometry.

Canonical decompositions are often the first step in constructing models of operators. By a description of the structure of an operator usually means one of the following: the determination of an equivalent operator on a prescribed class of concrete (often functional) models; a specific method of reconstructing it from a class of simpler operators (for example, in the form of a direct sum or factors); the discovery of a basis in which the operator has its simplest form; a complete description of the lattice of invariant and hyperinvariant subspaces; the identification of maximal chains of invariant subspaces (triangular representation) or maximal chains of reducing subspaces (diagonal or direct-sum representation); or the construction of a sufficiently wide functional calculus. The investigation of invariant subspaces is a natural first step in the attempt to understand the structure of operators. The powerful structure theorems that are known for finite-dimensional operators (the Jordan form) and normal operators (the spectral theorem) provide, in essence, decompositions into invariant subspaces of special kinds. No comparable theorem exists for general operators on an infinite dimensional Hilbert space. Although the general operator remains a mystery, one can say quite a bit about the invariant subspaces of a handful of specific operators.

The study of the structure and properties of an arbitrary operator on a Hilbert space is essentially equivalent to the study of its complementary parts, its invariant and hyperinvariant lattices, its characteristic function and its factors.

Several Mathematicians have proved some interesting results on operator decomposition and factorization. Williams [76] has demonstrated that every operator T is unitarily equivalent to the direct sum $T_1 \oplus T_2$ where T_1 is normal and T_2 is pure (completely non-normal) and that if $\mathcal{M}$ is a reducing subspace for T_2 and $T_2|_{\mathcal{M}}$ is normal, then $\mathcal{M} = \{0\}$. Stampfli and Wadhwa [66] while working on hyponormal operators proved that a hyponormal operator which is similar to a normal operator must actually be normal. Lee and Lee [50] studied a larger class -that of p-quasihyponormal operators and proved that if a p-quasihyponormal operator T has a finite defect index then it is normal. Nagy and Foias [53] have introduced a classification of contraction operators that depends on the asymptotic iterates of T and T^*. They proved that a contraction $T \in B(\mathcal{H})$ is a direct sum of a unitary and a completely non-unitary part and that for an isometry, this decomposition coincides with the von Neumann-Wold decomposition for isometries, where the completely non-normal part in this case is a unilateral shift (cf. Wold [77]). A similar result was proved by Fuhrmann [24] that any contraction $T \in B(\mathcal{H})$ has a unique decomposition with respect to the decomposition of $\mathcal{H}$ into a direct sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of reducing subspaces of T such that $T|_{\mathcal{H}_0}$ is unitary and $T|_{\mathcal{H}_1}$ is completely non-unitary. Wu [85] proved that if T is a contraction with finite defect indices then T is quasisimilar to an isometry if and only if the completely non-unitary part of T is quasisimilar to an isometry.

While working on the problem of writing an operator as a product of "nice" or simpler (normal, self-adjoint (hermitian), unitary, etc.) operators, Wu [84] proved that a unitary operator on an infinite dimensional space is a product of (sixteen) positive operators, an unexpected result in a finite dimensional Hilbert space, given the results of Ballantine [7] that if an operator is identified with a finite square matrix, then it is the product of positive operators precisely when its determinant is non-negative. Wu's [84] result was improved by Phillips [57] by showing that every unitary operator on an infinite-dimensional Hilbert space is a product of six positive operators.

Some operator theorists have studied the open question of the existence of nontrivial invariant subspaces. Kubrusly [45] has shown that if a contraction has no nontrivial invariant subspace, then it is either a C_{00}, a C_{01} or a C_{10} contraction. A similar result was proved for the class of hyponormal contractions by Kubrusly and Levan [44] that if a hyponormal contraction T has no nontrivial invariant subspace, then it is either a C_{00} or a C_{10} contraction. Duggal and Kubrusly [16] characterized the completely non-unitary part of a contraction using the Putnam-Fuglede(PF) Theorem. Hoover [39] proved that quasisimilarity preserves the existence of nontrivial hyperinvariant subspaces and Herrero [37] has shown that quasisimilarity does not preserve the full hyperlattice.

- Few results exist in the literature on the intrinsic properties of the pure direct summand of an operator in the direct sum decomposition of an operator. Some authors have given a classification of an operator depending on its direct summands. However, to-date open questions remain un-answered: Does every operator decompose into a direct sum? Which classes of operators decompose into non-trivial direct summands? This question is motivated by the most celebrated invariant subspace problem: Does every operator on a (separable) Hilbert space of dimension greater than one have a nontrivial invariant subspace? In this thesis we will give an attempt at the invariant subspace problem and come up with a partial solution to the problem for some classes of operators. We show that for any non-zero operator T, the invariant subspace problem is reduced to the class of contractions: Does every contraction have a nontrivial invariant subspace?

- There are many results in the literature about the direct sum decomposition of a contraction operator into a unitary and a completely non-unitary part but very few results linking this decomposition to an arbitrary operator are known. We look at how we can subject an arbitrary operator to this decomposition. We note that any operator T divided by its norm (normalized) is always a contraction operator.

- In this thesis, we show that there is a one-to-one correspondence between the invariant lattice and the regular factorization of the characteristic function associated with a contraction operator.
- Topological properties of invariant and hyperinvariant subspaces of operators have not been extensively studied. We shall describe these special subspaces for some classes of operators and use them to achieve a direct sum decomposition of an operator on a Hilbert space.
- We show that if an operator is invertible it factors into a product of simpler operators. We give necessary and sufficient conditions for an operator to enjoy such a product factorization. We determine a criterion for the optimal number of factors a given operator could have in its factorization.

We aim to generalize results in order to apply them to a wider class of operators. Finally, we establish a connection between direct sum decompositions, invariant, hyperinvariant and reducing subspaces and the factorizations of an operator. For instance, as an application we show that any direct sum decomposition of a contraction operator into a unitary part and a completely non-unitary part can be directly discerned from the direct sum decomposition of an operator into normal part and a completely non-normal direct part. We investigate an operator whose invariant subspace lattice satisfies a certain purely lattice-theoretic condition and whether or not it has a nontrivial hyperinvariant subspace and determine how this relates with the decomposition of such an operator.
We shall describe up to unitary equivalence all the c.n.u. contractions which possess a constant characteristic function. In case one of the defect indices is finite, we show that the characteristic function is constant if and only if the c.n.u. operator admits a direct sum decomposition such that each summand is one of the bilateral weighted shifts with weight sequence $\{..., 1, \lambda, 1, ...\}$, $0 < \lambda < 1$, or the unilateral shift or the adjoint of the unilateral shift. A consequence of this is that the characteristic function of an irreducible contraction is constant if and only if it is one of the shift operators described

above, which are examples of homogeneous operators. We use the (c.n.u.) contraction operators to put the notion of decomposition on a rigorous footing and obtain a decomposition of any operator, by first re-normalizing it to a contraction.

1.2 Notation and Terminology

In what follows, capital letters $\mathcal{H}$, $\mathcal{H}_1, \mathcal{H}_2$, $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$, etc denote Hilbert spaces or subspaces of Hilbert spaces, and T, T_1, T_2, A, B, etc denote bounded linear operators where by an operator we mean a bounded linear transformation from $\mathcal{H}$ into $\mathcal{H}$. By $B(\mathcal{H})$ we denote the Banach algebra of bounded linear operators on $\mathcal{H}$. $B(\mathcal{H}_1, \mathcal{H}_2)$ denotes the set of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. For an operator T, we denote by T^*, $\|T\|$, $Ran(T)$, $Ker(T)$ the adjoint, norm, range and kernel of T, respectively. We reserve the symbols $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{D}$, $\partial\mathbb{D}$ for the sets of integers, positive integers, real numbers, complex numbers, open unit disc in $\mathbb{C}$, and unit circle in $\mathbb{C}$, respectively. By $\sigma(T)$, $W(T)$, $w(T)$, $r(T)$ we denote the spectrum, numerical range, numerical radius and spectral radius of T, respectively where $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}$, (i.e. $Ker(\lambda I - T) \neq \{0\}$ or $Ran(\lambda I - T) \neq \mathcal{H}$, $W(T) = \{\langle Tx, x\rangle : \|x\| = 1, \quad x \in \mathcal{H}\}$, $r(T) = Sup\{|\lambda| : \lambda \in \sigma(T)\} = \max\{|\lambda| : \lambda \in \sigma(T)\} = \lim_n \|T^n\|^{\frac{1}{n}}$, $w(T) = Sup\{|\lambda| : \lambda \in W(T)\}$. We denote by $\sigma_P(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) \neq \{0\}\}$, which is the set of all eigenvalues of T and is called the *point spectrum* of T. The set of those λ for which $(\lambda I - T)$ has a densely defined but unbounded inverse is the *continuous spectrum*: $\sigma_C(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}\}$, $\overline{Ran(\lambda I - T)} = \mathcal{H}$ and $Ran(\lambda I - T) \neq \mathcal{H}$. If $(\lambda I - T)$ has an inverse that is not densely defined, then λ belongs to the *residual spectrum*: $\sigma_R(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}\}$ and $\overline{Ran(\lambda I - T)} \neq \mathcal{H}$. The parts $\sigma_P(T), \sigma_C(T)$ and $\sigma_R(T)$ are pairwise disjoint and $\sigma(T) = \sigma_P(T) \cup \sigma_C(T) \cup \sigma_R(T)$. We also define the approximate point spectrum of T : $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : (\lambda I - T) \text{ is not bounded below}\}$.

A subspace (closed linear manifold) $\mathcal{M} \subset \mathcal{H}$ is said to be *invariant* under an operator $T \in B(\mathcal{H})$ if $x \in \mathcal{M} \Rightarrow Tx \in \mathcal{M}$ or $T\mathcal{M} \subset \mathcal{M}$, and T is said to have a *nontrivial invariant subspace* (n.i.s) if there is a subspace $\{0\} \neq \mathcal{M} \neq \mathcal{H}$ invariant for T. A subspace $\mathcal{M} \subset \mathcal{H}$ is said to be a *reducing subspace* for T or reduces T if it is invariant under both T and T^*. An operator T on a Hilbert space $\mathcal{H}$ is *reductive* if every invariant subspace of T reduces T. We denote by $\overline{\mathcal{M}}$ the closure of a subspace $\mathcal{M}$ of $\mathcal{H}$.

A lattice $\mathfrak{L}$ is a partially ordered set such that every pair of elements of $\mathfrak{L}$ has a supremum and an infimum in $\mathfrak{L}$ (i.e. if there exists a unique $a \in \mathfrak{L}$ and a unique $b \in \mathfrak{L}$ such that $a = x \vee y$ and $b = x \wedge y$ for every pair $x \in \mathfrak{L}$ and $y \in \mathfrak{L}$). The lattice of all invariant subspaces of T will be denoted by $Lat(T)$. If Λ is any subset of $B(\mathcal{H})$, we denote by Λ' the commutant of Λ, i.e. $\Lambda' = \{T \in B(\mathcal{H}) : ST = TS \ for\ every\ S\ in\ \Lambda\}$. Specifically, $\{T\}' = \{S \in B(\mathcal{H}) : ST = TS\}$. The bicommutant or double commutant of $T \in B(\mathcal{H})$ is defined and denoted by $\{T\}'' = \Big\{A \in B(\mathcal{H}) : AS = SA,\ for\ all\ S \in \{T\}'\Big\} = \{p(T) : T \in B(\mathcal{H}),\ p\ a\ polynomial\}$. A subspace $\mathcal{M} \subset \mathcal{H}$ is said to be a *nontrivial hyperinvariant subspace* (n.h.s) for a fixed operator in $T \in B(\mathcal{H})$ if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$ and $S\mathcal{M} \subset \mathcal{M}$ for each S in $\{T\}'$. The lattice of all hyperinvariant subspaces of T will be denoted by $Hyperlat(T)$.

A subspace lattice $\mathfrak{L}$ is called *commutative* if for every pair of subspaces $\mathcal{M}, \mathcal{N} \in \mathfrak{L}$, the corresponding projections $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ commute. A lattice $\mathfrak{L}$ is said to be *totally ordered* if for every $\mathcal{M}$ and $\mathcal{N}$ in $\mathfrak{L}$, either $\mathcal{M} \subseteq \mathcal{N}$ or $\mathcal{N} \subseteq \mathcal{M}$. The *height* of a lattice $\mathfrak{L}$ of subspaces of $\mathcal{H}$ is defined to be the length of the longest path from $\{0\}$ to $\mathcal{H}$. In general, the lattice $\mathfrak{L} = (\wp(\Omega), \subseteq)$ has height $card(\Omega)$, for a finite set Ω where $\wp(\Omega)$ denotes the power set of Ω and $card(\Omega)$ denotes the cardinality of Ω.

For subspaces (closed linear manifolds) $\mathcal{M}, \mathcal{N}$ of a Hilbert space $\mathcal{H}$, $\mathcal{M}^{\perp}$ and $\mathcal{M} \oplus \mathcal{N}$ will denote the orthogonal complement of $\mathcal{M}$ and the orthogonal direct sum of $\mathcal{M}$ and $\mathcal{N}$, respectively.

An operator is said to be *reducible* if it has a nontrivial reducing subspace (equivalently, if it has a proper nonzero direct summand).

An operator T is said to be:

an **involution** if $T^2 = I$,

self-adjoint if $T = T^*$,

a **projection** if $T^2 = T$ and $T^* = T$,

unitary if $T^*T = TT^* = I$,

normal if $T^*T = TT^*$,

an **isometry** if $T^*T = I$,

co-isometry if $TT^* = I$,

a **partial isometry** if $T = TT^*T$,

quasinormal if $[T^*T, T] = 0$,

compact if for each bounded sequence $\{x_n\}$ in the domain $\mathcal{H}$, the sequence $\{Tx_n\}$ contains a subsequence converging to some limit in the range,

hyponormal if $T^*T \geq TT^*$,

seminormal if either T or T^* is hyponormal,

p-**hyponormal** if $(T^*T)^p \geq (TT^*)^p$, where $0 < p \leq 1$,

semi-hyponormal if $(T^*T)^{\frac{1}{2}} \geq (TT^*)^{\frac{1}{2}}$,

quasihyponormal if $T^{*2}T^2 - (T^*T)^2 \geq 0$, equivalently if $T^*(T^*T - TT^*)T \geq 0$,

M-**hyponormal** if $\|(zI - T)^*x\| \leq M\|(zI - T)x\|$, for all complex numbers z and for all $x \in \mathcal{M} \subset \mathcal{H}$ and M some positive number (i.e. $M > 0$),

paranormal if $\|Tx\|^2 \leq \|T^2x\|$, for all unit vectors $x \in \mathcal{H}$, equivalently if, $\|Tx\|^2 \leq \|Tx\|\|x\|$, for every $x \in \mathcal{H}$,

k-**paranormal** if $\|Tx\|^k \leq \|T^kx\|\|x\|^{k-1}$, for all $x \in \mathcal{H}$ and $k \geq 2$ some integer,

k-**quasihyponormal** if $T^{*k}\Big(T^*T - TT^*\Big)T^k \geq 0$, for $k \geq 1$ some integer, and every $x \in \mathcal{H}$,

p-**quasihyponormal** if $T^*\Big((T^*T)^p - (TT^*)^p\Big)T \geq 0$,

(p,k)-**quasihyponormal** if $T^{*k}\Big((T^*T)^p - (TT^*)^p\Big)T^k \geq 0$, where $0 < p \leq 1$ and k a positive integer,

dominant if for each $\lambda \in \mathbb{C}$ there corresponds a number $M_\lambda \geq 1$ such that $\|(T - \lambda I)^*x\| \leq M_\lambda\|(T - \lambda I)x\|$, for all $x \in \mathcal{H}$,

subnormal if it has a normal extension. That is, if there exists a normal operator B on a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ is a subspace of $\mathcal{K}$ and the subspace $\mathcal{H}$ is invariant under the operator B and the restriction of B to $\mathcal{H}$ coincides with T. That is, $T = B|_{\mathcal{H}}$,

a **contraction** if $\|T\| \leq 1$,

left shift operator if $Tx = y$, where $x = (x_1, x_2, ...)$ and $y = (x_2, x_3, ...) \in \ell^2$,

right shift operator if $Tx = y$, where $x = (x_1, x_2, ...)$ and $y = (0, x_1, x_2, ...) \in \ell^2$.

An operator $T \in B(\mathcal{H})$ is a **unilateral shift** if there exists a sequence $\{\mathcal{H}_0,\ \mathcal{H}_1,\ ...\}$ of pairwise orthogonal subspaces of $\mathcal{H}$ such that:

(a) $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus ...$

(b) T spans $\mathcal{H}_n$ isometrically onto $\mathcal{H}_{n+1}$.

An operator $T \in B(\mathcal{H})$ is:

a **scalar** if it is a scalar multiple of the identity operator (i.e. $T = \alpha I,\ \alpha \in \mathbb{C}$),

positive if $\langle Tx, x\rangle > 0$, for all $0 \neq x \in \mathcal{H}$.

Hilbert-Schmidt if $\|T\|_2 < \infty$ where $\|T\|_2 = \left\{\sum_{n=1}^{\infty} \|Te_n\|^2\right\}^{\frac{1}{2}}$ is the 2-norm and $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$,

an n**-th root of identity** if $T^n = I$, n a positive integer,

2-normal if $T^*T^2 = T^2T^*$.

An operator T is **quasinilpotent** if $\sigma(T) = \{0\}$. That is, if $r(T) = \lim_n \|T^n\|^{1/n} = 0$.

An operator T is **nilpotent** if $T^n = 0$ for some positive integer n.

Given a contraction $T \in B(\mathcal{H})$, both $(I - T^*T)$ and $(I - TT^*)$ are positive operators and hence have unique square roots. We define $D_T = (I - T^*T)^{\frac{1}{2}}$ and $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$ and call them the *defect operators* of T. The respective dimensions (ranks) d_T and d_{T^*} are called the *defect indices* of T.

An operator $T \in B(\mathcal{H})$ is *strongly stable* if the power sequence $\{T^n\}$ converges strongly (in the sense of the strong operator topology (SOT)) to the null operator (equivalently, $T^n \longrightarrow O$ strongly or $\|T^n x\| \longrightarrow 0$ for every $x \in \mathcal{H}$).

A contraction operator $T \in B(\mathcal{H})$ is of class:

$C_{1.}$ if $\lim \|T^n x\| \nrightarrow 0$, *strongly as* $n \to \infty$, *for every* $x \neq 0$,

$C_{.1}$ if $\lim \|T^{*n} x\| \nrightarrow 0$, *strongly as* $n \to \infty$, *for every* $x \neq 0$,

$C_{0.}$ if $\lim \|T^n x\| \to 0$, *strongly as* $n \to \infty$, *for every* $x \in \mathcal{H}$.

$C_{.0}$ if $\lim \|T^{*n} x\| \to 0$, *strongly as* $n \to \infty$, *for every* $x \in \mathcal{H}$,

C_{ij} if $T \in C_{i.} \cap C_{.j}$, $(0 \leq i, j \leq 1)$.

An operator $T \in B(\mathcal{H})$ is called a proper contraction if $\|Tx\| < \|x\|$, for every $0 \neq x \in \mathcal{H}$.

The maximum(largest) subspace in $\mathcal{H}$ which reduces an operator T to a unitary(respectively, normal) operator is called the unitary(normal) subspace in $\mathcal{H}$ of T.

An operator $T \in B(\mathcal{H})$ is said to be *pure* or *completely non-normal* (c.n.n.) if there exists no nontrivial reducing subspace $\mathcal{M} \subset \mathcal{H}$ such that $T|_{\mathcal{M}}$ (the restriction of T to $\mathcal{M}$) is normal, that is, if T has no direct normal summand, equivalently if the normal subspace is $\{0\}$. When the subspace $\mathcal{M}$ is invariant under the operator T, then T induces a linear operator $T_{\mathcal{M}} = T|_{\mathcal{M}}$ on the space $\mathcal{M}$. The linear operator $T_{\mathcal{M}}$ is defined by $T_{\mathcal{M}}(x) = T(x)$, for $x \in \mathcal{M}$. A *part* of an operator is a restriction of the operator to an invariant subspace.

A contraction $T \in B(\mathcal{H})$ is said to be *completely non-unitary* (c.n.u) if there exists no

nontrivial reducing subspace of $\mathcal{M} \subset \mathcal{H}$ of T on which T acts unitarily, or equivalently if its unitary part acts on the zero space $\{0\}$.
If $\mathcal{K}$ is a Hilbert space, $\mathcal{H} \subset \mathcal{K}$ is a subspace, $S \in B(\mathcal{K})$, and $T \in B(\mathcal{H})$, then S is a *dilation* of T (and T is a *power-compression* of S) provided that $T^n = P_{\mathcal{H}} S^n|_{\mathcal{H}}$, $n = 0, 1, 2, ...$ where $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.
Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. An operator $X \in B(\mathcal{H}, \mathcal{K})$ is invertible if it is injective (one-to-one) and surjective (onto); equivalently if $Ker(X) = \{0\}$ and $Ran(X) = \mathcal{K}$. We denote the class of invertible linear operators by $\mathcal{G}(\mathcal{H}, \mathcal{K})$. The commutator of two operators A and B, denoted by $[A, B]$ is defined by $[A, B] = AB - BA$. Two operators $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ are *similar* (denoted $T \approx S$) if there exists an operator $X \in \mathcal{G}(\mathcal{H}, \mathcal{K})$ such that $XT = SX$ (i.e. $T = X^{-1}SX$ or $S = XTX^{-1}$).
Linear operators $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ are *unitarily equivalent* (denoted $T \cong S$), if there exists a unitary operator $U \in \mathcal{G}(\mathcal{H}, \mathcal{K})$ such that $UT = SU$ (i.e. $T = U^*SU$ or equivalently $S = UTU^*$). Two operators are considered the "*same*" if they are unitarily equivalent since they have the same properties of invertibility, normality, spectral picture (norm, spectrum, spectral radius).
An operator $X \in B(\mathcal{H}, \mathcal{K})$ is *quasiinvertible* or a *quasi-affinity* if it is an injective operator with dense range (i.e. $Ker(X) = \{0\}$ and $\overline{Ran(X)} = \mathcal{K}$; equivalently, $Ker(X) = \{0\}$ and $Ker(X^*) = \{0\}$- thus $X \in B(\mathcal{H}, \mathcal{K})$ is quasiinvertible if and only if $X^* \in B(\mathcal{K}, \mathcal{H})$ is quasiinvertible).
An operator $T \in B(\mathcal{H})$ is a *quasiaffine transform* of $S \in B(\mathcal{K})$ if there exists a quasiinvertible $X \in B(\mathcal{H}, \mathcal{K})$ such that $XT = SX$. Two operators $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ are *quasisimiliar* (denoted $T \sim S$) if they are quasiaffine transforms of each other (i.e., if there exist quasiinvertible operators $X \in B(\mathcal{H}, \mathcal{K})$ and $Y \in B(\mathcal{K}, \mathcal{H})$ such that $XT = SX$ and $YS = TY$).
It is easily verified that quasisimilarity is an equivalence relation and also that T^* is quasisimilar to S^* whenever T is quasisimilar to S and that similar operators are, of course, quasisimilar but not conversely ([45]).
Quasisimilarity was introduced by Nagy and Foias [53] in their theory on infinite-dimensional analogue of the Jordan form for certain classes of contractions as a means of studying their invariant subspace structures. It replaces the familiar notion of similarity which is the appropriate equivalence relation to use with finite dimensional Hilbert

spaces. In finite dimensional spaces quasisimilarity is the same thing as similarity, but in infinite dimensional spaces it is a much weaker relation.
Two operators A and B are said to be *almost-similar* (denoted $A \overset{a.s}{\approx} B$) if there exists an invertible operator N such that the following two conditions are satisfied.

$$\begin{aligned} A^*A &= N^{-1}(B^*B)N \\ A^* + A &= N^{-1}(B^* + B)N \end{aligned}$$

An operator $X \in B(\mathcal{H}, \mathcal{K})$ *intertwines* $T \in B(\mathcal{H})$ to $S \in B(\mathcal{K})$ if $XT = SX$. In such a case we say that T is intertwined to S. Note that T is a quasiaffine transform of S if there exists a quasiinvertible operator intertwining T to S. T is said to be *densely intertwined* to S if there exists an operator with dense range intertwining T to S.
The *multiplicity* $\mu(T)$ of $T \in B(\mathcal{H})$ is the minimum cardinality of a set $\mathcal{K} \subset \mathcal{H}$ such that

$$\mathcal{H} = \bigvee_{n=0}^{\infty} T^n \mathcal{K}.$$

For the unilateral shift operator S_+, $\mu(S_+) = dim(Ker(S_+^*))$ and for the backward shift, $\mu(S_+^*) = dim(Ran(S_+^*)^{\perp}) = dim(Ker(S_+))$. We view the Hardy space $\mathbb{H}^2$ as a subspace of L^2 on the unit circle by replacing convergent power series by their boundary functions. The Lebesgue spaces L^p are defined with respect to normalized Lebesgue measure on $\partial\mathbb{D}$. It is known that the Hardy space $\mathbb{H}^\infty$ is the space of bounded analytic functions on $\mathbb{D}$ in the supremum norm, which we sometimes view as a subspace of L^∞.
A bounded linear operator T acting on the complex separable Hilbert space $\mathcal{H}$ is called *homogeneous* if $\sigma(T) \subset \overline{\mathbb{D}}$ and $\varphi(T)$ is unitarily equivalent to T for every $\varphi \in \mathcal{M}(\mathbb{D})$, the group of Möbius functions on the unit disc.

1.3 Inclusions of Operator Classes and Examples

It is well known that the following inclusions hold and are proper (see [1], [25]).

$$Normal \subseteq Quasinormal \subseteq Subnormal \subseteq Hyponormal \subseteq M - hyponormal$$

$Hyponormal \subseteq p\!-\!hyponormal(\frac{1}{2} < p < 1) \subseteq semi\!-\!hyponormal \subseteq p\!-\!hyponormal(0 < p < \frac{1}{2})$

$Hyponormal \subseteq M - hyponormal \subseteq Dominant$

$Unitary \subseteq Normal \subseteq Quasinormal \subseteq Binormal$

$Unitary \subseteq Isometry \subseteq Partial\ Isometry \subseteq Contraction$

$Unitary \subseteq Isometry \subseteq 2 - normal \subseteq Binormal$

$Projection \subseteq Self - adjoint \subseteq Normal \subseteq hyponormal$

$Hyponormal \subseteq \begin{cases} k-quasihyponormal \\ k-paranormal \end{cases}$

$quasihyponormal \subseteq paranormal$

$p - quasihyponormal \subseteq (p, k) - quasihyponormal$

$k - quasihyponormal \subseteq (p, k) - quasihyponormal$

$Normal \subseteq Quasinormal \subseteq Subnormal \subseteq Hyponormal \subseteq Paranormal$

$Hilbert - Schmidt \subseteq Compact.$

We note that a $(p, 1)$-quasihyponormal operator is p-quasihyponormal and that a $(1, k)$-quasihyponormal operator is k-quasihyponormal.
We give some examples to show that these inclusions are proper. The following example shows an isometry which is not unitary.

Example 1.1

Define $T : \ell^2 \longrightarrow \ell^2$ by $T(x) = T(x_1, x_2, x_3, ...) = (0, x_1, x_2, ...)$ for $x = (x_1, x_2, x_3, ..) \in \ell^2$. Then $T^*(y) = (y_2, y_3, ...)$ for $y = (y_1, y_2, ...) \in \ell^2$ and $(T^*T)(x) = T^*(0, x_1, x_2, x_3, ...) = (x_1, x_2, x_3, ...) = x$. This shows that $T^*T = I$. That is, T is an isometry. On the other hand, $(TT^*)(x) = T(x_2, x_3, x_4, ...) = (0, x_2, x_3, x_4, ...)$. Thus $TT^* \neq I$ and so $I = T^*T \neq TT^* \neq I$. This shows that T is an isometry which is not unitary. Indeed, it is clear in this case that T is a partial isometry which is not unitary: since $T(x) = T(x_1, x_2, ...) = (0, x_1, x_2, ...)$ and $TT^*T(x) = TT^*T(x_1, x_2, ...) = TT^*(0, x_1, x_2, ..) = T(x_1, x_2, ...) = (0, x_1, x_2, ..)$. Thus, T is a partial isometry which is not unitary. Also, T is a contraction which is not unitary, since $\|T\| \leq 1$.

Remark 1.1

We observe that in Example 1.1 above, T^* is a partial isometry which is not an isometry and that it is a contraction which is not an isometry. This is from the fact that $\|T^*\| = 1$ and hence a contraction but not an isometry since $TT^* \neq I$.

Example 1.2

There are operators which are self-adjoint but are not projections.

Consider $T = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. It easy to check that T is self-adjoint but not a projection.

Example 1.3

There are operators that are projections but are not unitary.

Consider $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then T is a projection which is not a unitary operator.

Example 1.4

Every hyponormal operator is paranormal. This assertion follows from:

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*Tx\| \leq \|T^2x\|,$$

for every unit vector $x \in \mathcal{H}$.

Example 1.5

Consider $S_+ : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $S_+(x,y) = (0,x)$. Then S_+ is a unilateral shift given by the matrix $S_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The multiplicity of S_+, $\mu(S_+) = dim(Ker(S_+)) = dim(span\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}) = 1$. Thus, in this example, S_+ is a unilateral shift of multiplicity one.

Example 1.6

Trivial examples of C_{00}, C_{01}, C_{10} and C_{11} are: the null operator, a backward shift, a unilateral shift and the identity operator.

Chapter 2

On normal and completely non-normal summands of an operator

In this chapter we study the decomposition of an (arbitrary) operator into a direct sum of its normal and completely non-normal(c.n.n.) parts . By a decomposition we mean separation of an operator into parts. This matches a requirement of isolating "simple" direct summands of an operator. We give a deeper characterization of the normal and completely non-normal summands for $T \in B(\mathcal{H})$. We use the properties of the direct summands to classify the original operator T.

Unlike other forms of decompositions of an operator, such as polar and cartesian decompositions which do not transfer invariant subspaces from the parts (factors or ordinary summands) to the original (decomposed) operator, direct sum decomposition does have this property. In fact an invariant subspace for a direct summand is invariant for the direct sum. By the Spectral Theorem [46, Theorem 6.43], for every compact normal operator $T \in B(\mathcal{H})$ there exists a countable resolution of the identity $\{P_k\}$ on $\mathcal{H}$ and a set of scalars $\{\lambda_k\}$ such that $T = \sum_k \lambda_k P_k$, where $\{\lambda_k\} = \sigma_P(T)$, the set of all (distinct) eigenvalues of T, and each P_k is the orthogonal projection onto the eigenspace $Ker(\lambda_k I - T)$. Since by [45, Theorem 0.14 and page 75], this ordinary sum of projections can be translated into a direct sum, we find that a normal operator is unitarily equivalent to a direct sum of simpler operators, which are indeed normal since projection operators are normal. This means that a normal operator is unitarily equivalent to a direct sum of scalar operators, which are easy to handle.

Every bounded linear operator T on a Hilbert space has an orthogonal decomposition $T = T_1 \oplus T_2$, implemented through a restriction of T to a reducing subspace, where T_1 is normal and T_2 is pure or completely non-norml (c.n.n.). which means that no part or restriction of T_2 to a reducing subspace is normal. It is well known that either of these two summands may be absent.

It is well known [45, page 22] that in a finite-dimensional setting quasinormality, subnormality and hyponormality all collapse to normality. This means that in such a setting such operators will have no pure direct summands.

The following results will be useful in the sequel. We start with the following known result due to [53].

Lemma 2.1 [53] *For any operator $T \in B(\mathcal{H})$, if $|\lambda| = \|T\|$ is an eigenvalue of T then $Ker(T - \lambda I)$ is reducing.*

Lemma 2.1 gives rise to the following consequences.

Corollary 2.2 *If T is pure(c.n.n) and if $\|T\| = r(T)$, then there are no eigenvalues λ for which $|\lambda| = \|T\|$.*

Remark 2.1

We note that $\sigma(T) = \sigma_P(T)$ for operators T acting on a finite dimensional space. However, $\sigma_P(T)$ may be empty in an infinite dimensional space.

Example 2.1

Consider the unilateral shift $T : \ell^2 \longrightarrow \ell^2$ on the Hilbert space ℓ^2 of all square summable infinite sequences of complex numbers, given by

$T(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$ for every $(x_1, x_2, x_3, ...) \in \ell^2$. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of T. Then there exists a non-zero eigenvector $(x_1, x_2, x_3, ...) \in \ell^2$ such that $(0, x_1, x_2, x_3, ...) = \lambda(x_1, x_2, x_3, ...) = (\lambda x_1, \lambda x_2, \lambda x_3, ...)$, so that $\lambda x_1 = 0$ and $\lambda x_i = x_{i-1}$ for every $i > 1$. If $\lambda = 0$, then the second condition implies that $x_1 = x_2 = x_3 = ... = 0$, a contradiction again. It follows that the operator T, which is a unilateral shift, has no eigenvalues and thus $\sigma_P(T) = \emptyset$.

Corollary 2.3 *If $T \in B(\mathcal{H})$ is dominant with $T = T_1 \oplus T_2$ where T_1 is normal and T_2 is pure, then T_2 is dominant.*

Remark 2.2

We note that Corollary 2.3 applies to all subclasses of dominant operators: hyponormal, M-hyponormal. We now investigate the decomposition of (p, k)-quasihyponormal operators, which are an extension of p-hyponormal operators, k-quasihyponormal operators and p-quasihyponormal operators.
Aluthge [1], Arora and Arora [5] and Kim[43] introduced p-hyponormal, p-quasihyponormal and (p, k)-quasihyponormal operators, respectively. These operators share many interesting properties with hyponormal operators. We give and prove conditions under which such an operator is normal.

Theorem 2.4 *If $T \in B(\mathcal{H})$ is a (p, k)-quasihyponormal operator and $S^* \in B(\mathcal{H})$ is a p-hyponormal operator, and if $TX = XS$ where $X : \mathcal{K} \longrightarrow \mathcal{H}$ is an injective bounded linear operator with dense range (a quasiaffinity), then T is a normal operator unitarily equivalent to S.*

Remark 2.3

Theorem 2.4 says that a (p, k)-quasihyponormal operator which is a quasiaffine transform of a co-p-hyponormal operator is always normal. We need the following result.

Proposition 2.5 [65] *If $T \in B(\mathcal{H})$ is a hyponormal operator and $S^{-1}TS = T^*$ for an operator S, where $0 \notin \overline{W(S)}$, then T is self-adjoint.*

Remark 2.4

From Proposition 2.5 we conclude that T is normal, since a self-adjoint operator is normal. From this result, we also deduce that if a hyponormal operator is similar to its adjoint, then it must be normal. We extend the result of Proposition 2.5 to the class of p-hyponormal operators as follows.

Theorem 2.6 *If T or T^* is p-hyponormal, and S is an operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint and hence normal.*

To prove this theorem we use the following lemma.

Lemma 2.7 [75] *If $T \in B(\mathcal{H})$ is any operator such that $S^{-1}TS = T^*$, where $0 \notin \overline{W(S)}$, then $\sigma(T) \subseteq \mathbb{R}$.*

Proof of Theorem 2.6. Suppose that T or T^* is hyponormal. Since $\sigma(S) \subseteq \overline{W(S)}$, S is invertible and hence $ST = T^*S$ becomes $S^{-1}T^*S = T = (T^*)^*$. Applying Lemma 2.5 to T^* we get $\sigma(T^*) \subset \mathbb{R}$. Then $\sigma(T) = \overline{\sigma(T^*)} = \sigma(T^*) \subset \mathbb{R}$. Thus $m_2(\sigma(T)) = m_2(\sigma(T^*)) = 0$, for the planar Lebesque measure m_2. Applying Putnam's inequality for p-hyponormal operators T or T^* (depending upon which is p-hyponormal), we get

$$\|(T^*T)^p - (TT^*)^p\| \leq \frac{p}{\Pi}\int\int_{\sigma(T)} r^{2p-1}drd\theta = 0$$

or

$$\|(TT^*)^p - (T^*T)^p\| \leq \frac{p}{\Pi}\int\int_{\sigma(T^*)} r^{2p-1}drd\theta = 0.$$

It follows that T or T^* is normal. Since $\sigma(T) = \sigma(T^*) \subset \mathbb{R}$, T must be self-adjoint. We extend the result of Theorem 2.6 to the class of p-quasihyponormal operators. We use the following lemma.

Lemma 2.8 [43] *If $T \in B(\mathcal{H})$ is a (p,k)-quasihyponormal operator, then T has the following matrix representation:*
$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ *with respect to the decomposition $\mathcal{H} = \overline{Ran(T^k)} \oplus Ker(T^{*k})$, where T_1 is p-hyponormal on $\overline{Ran(T^k)}$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1)\bigcup\{0\}$.*

The following result due to [43] is useful.

Theorem 2.9 [43] *If T is a (p,k)-quasihyponormal operator and S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is a direct sum of a self-adjoint (and hence normal) and a nilpotent operator.*

Corollary 2.10 *If T or T^* is a p-quasihyponormal operator and S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint (and hence normal).*

Proof. If T is p-quasihyponormal, then by Lemma 2.8, for $k = 1$, T has the matrix representation
$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ where T_1 is p-hyponormal on $\overline{Ran(T^k)}$ and $\sigma(T) = \sigma(T_1)\bigcup\{0\}$.

Since T_1 is self-adjoint and $T_2 = 0$ by Theorem 2.9, $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$ is also self-adjoint. On the other hand, if T^* is (p,k)-quasihyponormal, then using Theorem 2.9 we conclude that T is self-adjoint (and hence normal).

Lemma 2.11 *The restriction $T|_{\mathcal{M}}$ of a (p,k)-quasihyponormal operator T on $\mathcal{H}$ to an invariant subspace $\mathcal{M}$ is also a (p,k)-quasihyponormal operator.*

Remark 2.5

From Lemma 2.11, we conclude that the direct summands of a (p,k)-quasihyponormal operator T are again (p,k)-quasihyponormal.
We need the following results.

Theorem 2.12 *(Löwner-Heinz Theorem [40, Proposition A]) If A and B are operators such that $A \geq B \geq 0$ then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$.*

Theorem 2.13 *(Hansen's Inequality [40, Proposition B]) If $A \geq 0$ and $B \leq 1$, then $(B^*AB)^\delta \geq B^*A^\delta B$ for all $\delta \in (0,1]$.*

Lemma 2.14 *If $T \in B(\mathcal{H})$ is a (p,k)-quasihyponormal operator and $\mathcal{M}$ is an invariant subspace of T for which $T|_{\mathcal{M}}$ is an injective normal operator, then $\mathcal{M}$ reduces T.*

Proof. Suppose that P is an orthogonal projection of $\mathcal{H}$ onto $\overline{Ran(T^k)}$. Since T is (p,k)-quasihyponormal, we have $T^{*k}\Big((T^*T)^p - (TT^*)^p\Big)T^k \geq 0$. If we let $S = PT|_{\mathcal{M}}$, then clearly, $P\Big((T^*T)^p - (TT^*)^p\Big)P \geq 0$. Put $T_1 = T|_{\mathcal{M}}$ and $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Clearly, $S = T_1$ if $\mathcal{M} = \overline{Ran(T^k)}$. Since by assumption T_1 is an injective normal operator, we have $Q \leq P$ for the orthogonal projection Q of $\mathcal{H}$ onto $\mathcal{M}$ and $\overline{Ran(T_1^k)} = \mathcal{M}$, because T_1 has dense range. Therefore, $\mathcal{M} \subseteq \overline{Ran(T^k)}$ and hence $Q\Big((T^*T)^p - (TT^*)^p\Big)Q \geq 0$.
Since T is (p,k)-quasihyponormal, using the Löwner-Heinz inequality and Hansen's inequality, we have
$\begin{pmatrix} (T_1T_1^*)^p & 0 \\ 0 & 0 \end{pmatrix} = Q(TQT^*)^pQ \leq Q(TT^*)^pQ \leq (QT^*TQ)^p = \begin{pmatrix} (T_1^*T_1)^p & 0 \\ 0 & 0 \end{pmatrix}$. Since T_1 is normal, we have, by Löwner's inequality,

$(TT^*)^{\frac{p}{2}} = \begin{pmatrix} (T_1T_1^*)^{\frac{p}{2}} & A \\ A^* & B \end{pmatrix}$. So $\begin{pmatrix} (T_1T_1^*)^p & 0 \\ 0 & 0 \end{pmatrix} = Q(TT^*)^pQ = \begin{pmatrix} (T_1T_1^*) + AA^* & 0 \\ 0 & 0 \end{pmatrix}$ and hence $A = 0$ and $TT^* = \begin{pmatrix} T_1T_1^* & 0 \\ 0 & B^{\frac{2}{p}} \end{pmatrix}$. Since $TT^* = \begin{pmatrix} T_1T_1^* + T_2T_2^* & T_2T_3^* \\ T_3T_2^* & T_3T_3^* \end{pmatrix}$, it follows that $T_2 = 0$ and hence T is reduced by $\mathcal{M}$.

Remark 2.6

Lemma 2.14 says that $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}$, where $T_1 = T|_{\mathcal{M}}$. That is, T decomposes into a direct sum of nontrivial complementary parts.

We state the following result important result which has been proved in [46].

Theorem 2.15 [46] *Let $T \in B(\mathcal{H})$. The following assertions are pairwise equivalent.*

(a) $\mathcal{M}$ reduces T.

(b) $T = T|_{\mathcal{M}} \oplus T|_{\mathcal{M}^\perp} = \begin{pmatrix} T|_{\mathcal{M}} & 0 \\ 0 & T|_{\mathcal{M}^\perp} \end{pmatrix} : \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp \longrightarrow \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$.

(c) $PT = TP$, where $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp \longrightarrow \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ is the orthogonal projection onto $\mathcal{M}$.

From Theorem 2.15 we note that, if $\mathcal{M}$ reduces T, then the investigation of T is reduced to the investigation of the restrictions $T|_{\mathcal{M}}$ and $T|_{\mathcal{M}^\perp}$, which have a simpler structure than that of T.

The following result proved in [64] will come in handy in this chapter.

Theorem 2.16 [64, *Lemma* 1] *If T is a pure p-hyponormal operator, then $\sigma_P(T) = \emptyset$.*

Lemma 2.17 *If $T \in B(\mathcal{H})$ is paranormal, then the restriction $T|_{\mathcal{M}}$ to an invariant subspace $\mathcal{M}$ is also paranormal.*

Proof. Let $x \in \mathcal{M}$ be an arbitrary vector. Then we have,

$$\|T|_{\mathcal{M}}x\|^2 = \|Tx\|^2 \leq \|T^2x\|\|x\| = \|(T|_{\mathcal{M}})^2x\|\|x\|.$$

This implies that $T|_{\mathcal{M}}$ is paranormal.

Theorem 2.18 *If $\mathcal{H}$ is finite-dimensional and T is an M-hyponormal operator on $\mathcal{H}$, then T is normal.*

Remark 2.7

We note that Theorem 2.18 can be extended to the class of dominant operators and in general to any arbitrary operator acting on a finite-dimensional Hilbert space.
The next result due to [51] is useful.

Theorem 2.19 [51] *If $T \in B(\mathcal{H})$, then there exists a reducing subspace $\mathcal{M}$ of $\mathcal{H}$ (possibly trivial) such that $T|_{\mathcal{M}}$ is normal and $T|_{\mathcal{M}^{\perp}}$ is completely non-normal. Furthermore, the decomposition is unique, and*

$$\begin{aligned}\mathcal{M} &= \bigcap_{m,n=0}^{\infty} Ker(T^n T^{*m} - T^{*m} T^n) \\ &= \bigcap_{m=0}^{\infty}\bigcap_{n=0}^{\infty} Ker(T^n T^{*m} - T^{*m} T^n)\end{aligned}$$

Remark 2.8

Theorem 2.19 is a version of Theorem 2.15 and gives uniqueness of the decomposition. In the following result, we use the notation $[T^*, T] = T^*T - TT^*$. We use Theorem 2.19 to prove the following result.

Theorem 2.20 *Let T be an operator on $\mathcal{H}$. If $\mathcal{K} = Ran(T^*T - TT^*)$ is the smallest reducing subspace of T, then $T|_{\mathcal{K}}$ is the completely non-normal summand of T.*

Proof. Let $\mathcal{K}$ be as defined in the theorem. From Theorem 2.19, $T = T_1 \oplus T_2$ on $\mathcal{M} \oplus \mathcal{M}^{\perp}$, where T_1 is completely non-normal and T_2 is normal. Since $[T^*, T] = [T_1^*, T_1] \oplus [T_2^*, T_2]$, $[T_2^*, T_2] = 0$ and so it is clear that $\mathcal{K} \subseteq \mathcal{M}$, because $\mathcal{M}$ is a reducing subspace of T containing the range of $[T^*, T]$. If this containment is proper, then T_1 itself could further be reduced into $T_{11} \oplus T_{12}$ on $\mathcal{K} \oplus \mathcal{K}^{\perp}$, where $\mathcal{K}^{\perp}$ is the orthogonal complement of $\mathcal{K}$ in $\mathcal{M}$. But the definition of $\mathcal{K}$ implies that $[T_{12}^*, T_{12}] = 0$, which contradicts the fact that T_1 is completely non-normal. Therefore, $\mathcal{K} = \mathcal{M}$, which completes the proof.

Remark 2.9

Nzimbi, Pokhariyal and Khalagai [56, Theorem 2.9] have shown that Theorem 2.20 holds for the class of 2-normal operators. We give an example of a basic non-normal operator.

Example 2.2

A unilateral shift is a non-normal operator.

Proposition 2.21 *An isometry is pure if and only if it is a unilateral shift.*

Proof. Suppose that T is an isometry and that $T = T_1 \oplus T_2$, where T_1 is normal and T_2 is completely non-normal. Since T is pure, T_1 is missing and T_2 is pure. Hence $TT^* \neq T^*T = I = T_2^*T_2$. Thus T is an isometry which is not a co-isometry and hence must be a unilateral shift. Conversely, suppose T is a unilateral shift. Then $T^*Tx \neq TT^*x$ for every $0 \neq x \in \mathcal{M} \subset \mathcal{H}$. This proves that T is pure.

Remark 2.10

Proposition 2.21 is a special case of Theorem 2.16. Indeed, by [45, Remark 5.5] any pure operator is a unilateral shift or a direct sum of unilateral shifts. We investigate the direct sum decomposition for similar, quasisimilar and unitarily equivalent operators.

Lemma 2.22 *Let $T \in B(\mathcal{H}_1)$ be a p-quasihyponormal operator and $N \in B(\mathcal{H}_2)$ be a normal operator. If $X \in B(\mathcal{H}_2, \mathcal{H}_1)$ has dense range and satisfies $TX = XN$, then T is also a normal operator.*

Proof. By [69] and Lemma 2.8, $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} N_1 & 0 \\ 0 & 0 \end{pmatrix}$, with respect to the decompositions $\mathcal{H}_1 = \overline{Ran(T)} \oplus Ker(T^*)$ and $\mathcal{H}_2 = \overline{Ran(N)} \oplus Ker(N^*)$, respectively. Since $TX = XN$ and X has dense range, we have $\overline{X(\overline{Ran(N)})} = \overline{Ran(T)}$. If we denote the restriction of X to $\overline{Ran(N)}$ by X_1, then $X_1 : \overline{Ran(N)} \longrightarrow \overline{Ran(T)}$ has dense range and for every $x \in \overline{Ran(N)}$

$$X_1N_1x = XNx = TXx = T_1X_1x,$$

so that $X_1N_1 = T_1X_1$. Since T_1 is p-hyponormal by Lemma 2.8, there exists a hyponormal operator $\tilde{T}_1$ corresponding to T_1 and a quasiaffinity Y such that $\tilde{T}_1Y = YT_1$, where $\tilde{T}_1 = |\hat{T}_1|^{1/2}V|\hat{T}_1|^{1/2}$, with $T_1 = U|T_1|$ and $\hat{T}_1 = |T_1|^{1/2}U|T_1|^{1/2}$.
Thus, we have

$$\tilde{T}_1YX_1 = YT_1X_1 = YX_1N_1.$$

Since YX_1 has dense range, $\tilde{T}_1$ is normal, and so T_1 is normal. Thus the inequality

$$(T_1^*T_1)^p \geq (T_1T_1^* + T_2T_2^*)^p \geq (T_1T_1^*) = (T_1^*T_1)^p$$

implies that $T_2 = 0$. Hence T is normal.

We state the next result for quasisimilar p-quasihyponormal operators.

Theorem 2.23 *Let $T_i \in B(\mathcal{H}_i)$ $(i = 1,2)$ be injective p-quasihyponormal operators such that T_1 and T_2 are quasisimilar and let $T_i = N_i \oplus V_i$ on $\mathcal{H}_i = \mathcal{H}_{i1} \oplus \mathcal{H}_{i2}$, where N_i and V_i are the normal and pure parts of T_i, respectively. Then N_1 and N_2 are unitarily equivalent and there exist $X_* \in B(\mathcal{H}_{22}, \mathcal{H}_{12})$ and $Y_* \in B(\mathcal{H}_{12}, \mathcal{H}_{22})$ with dense ranges such that $V_1X_* = X_*V_2$ and $Y_*V_1 = V_2Y_*$.*

Proof. There exist quasiaffinities $X \in B(\mathcal{H}_2, \mathcal{H}_1)$ and $Y \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $T_1X = XT_2$ and $YT_1 = T_2Y$. Let $X := \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ and $Y := \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$. We show that $X_* = X_4$ and $Y_* = Y_4$. A simple matrix calculation shows that $V_1X_3 = X_3N_2$ and $V_2Y_3 = Y_3N_1$. We claim that $X_3 = Y_3 = 0$. Indeed, letting $\mathcal{M} = \overline{Ran(X_3)}$, the subspace $\mathcal{M}$ is invariant under V_1. So let $V_1^{'} = V_1|_{\mathcal{M}}$ and let $X_3^{'} : H_{21} \longrightarrow \mathcal{M}$ be defined by $X_3^{'}x = X_3x$ for each $x \in \mathcal{H}_{21}$. Since $V_1^{'}$ is injective p-quasihyponormal (since by Lemma 2.11 the restriction of a p-quasihyponormal operator to an invariant subspace is also p-quasihyponormal), $X_3^{'}$ has dense range, and $V_1^{'}X_3^{'} = X_3^{'}N_2$. Hence $V_1^{'}$ is normal by Lemma 2.22 and hence $\mathcal{M}$ reduces V_1. Since V_1 is pure, we have that $\mathcal{M} = \{0\}$, and hence $X_3 = 0$. Similarly, we have $Y_3 = 0$. Thus, it follows that X_1 and Y_1 are injective. Since $N_1X_1 = X_1N_2$ and $Y_1N_1 = N_2Y_1$, by [76, Lemma 1.1], we have that N_1 and N_2 are unitarily equivalent. Also, we can notice that X_4 and Y_4 have dense ranges and $V_1X_4 = X_4V_2$ and $Y_4V_1 = V_2Y_4$, which completes the proof.

Remark 2.11

For any operator $T \in B(\mathcal{H})$ the self-commutator of T, $[T^*, T] = T^*T - TT^*$ is always self-adjoint. Recall that an operator T is normal if it commutes with its adjoint. It is easy to check that the operator $[T^*, T]$ is normal. We use this notion to give a characterization of normal and quasinormal operators.

Theorem 2.24 *Let $T \in B(\mathcal{H})$ such that $T = T_1 \oplus T_2$ with T_1 normal and T_2 pure. T is normal if and only if $[T_2^*, T_2] = 0$.*

Proof. Suppose $T \in B(\mathcal{H})$ is normal. Then $T^*T - TT^* = [T_2^*, T_2] = 0$. Conversely,suppose $[T_2^*, T_2] = T_2^*T_2 - T_2T_2^* = 0$. Since T_2 is pure this holds only if $T_2 = 0$. A simple computation shows that $T^*T = TT^*$. This proves that T is normal.

Remark 2.12

We note that Theorem 2.24 can be proved easily by using the fact that T has no pure part. Recall that an operator $T \in B(\mathcal{H})$ is quasinormal if it commutes with T^*T. Equivalently, if $(T^*T - TT^*)T = 0$. We use this fact to prove the following result.

Theorem 2.25 $T \in B(\mathcal{H})$ *is quasinormal if and only if* $[T^*, T]T = 0$.

Proof. The proof follows easily by imitating Theorem 2.24 above.
We investigate the direct summands of a quasinormal operator.

Theorem 2.26 *Every direct summand of a quasinormal operator is again quasinormal.*

Proof. Let $\mathcal{M}$ be a reducing subspace for $T \in B(\mathcal{H})$. Suppose that $T = T_1 \oplus T_2$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ where $T_1 = T|_{\mathcal{M}}$ and $T_2 = T|_{\mathcal{M}^{\perp}}$ and T is quasinormal. Then $T^*TT = T_1^*T_1T_1 \oplus T_2^*T_2T_2 = T_1T_1^*T_1 \oplus T_2T_2^*T_2 = TT^*T$.
This shows that $T_1^*T_1T_1 = T_1T_1^*T_1$ and $T_2^*T_2T_2 = T_2T_2^*T_2$ (i.e $[T_1^*, T_1]T_1 = 0$ and $[T_2^*, T_2]T_2 = 0$) and hence by Theorem 2.25, T_1 and T_2 are both quasinormal.

Remark 2.13

Theorem 2.26 says that the restriction of a quasinormal operator to a reducing subspace is always quasinormal. We give conditions under which a hyponormal operator is quasisimilar to an isometry.

Corollary 2.27 *Let T be a hyponormal operator whose c.n.n part has finite multiplicity. Then T is quasisimilar to an isometry if and only if its normal part is unitary and its c.n.n. part is quasisimilar to a unilateral shift.*

Proof. Let T be hyponormal with the decomposition $T = T_1 \oplus T_2$ and suppose that T is quasisimilar to an isometry $V = U \oplus S$, where T_1 is normal, T_2 is c.n.n., U is unitary and S is a unilateral shift. By [33, Proposition 3.5], T_1 is unitarily equivalent to U and hence unitary. Since by assumption T is quasisimilar to V, and by Clary [14] quasisimilar hyponormal operators have the same spectra, and by [29], $\|T\| = r(T) = r(V) = 1$, where $r(T)$ and $r(V)$ are the spectral radii of T and S, respectively, then this proves that T_2 is quasisimilar to S.

Remark 2.14

We need the following result to give and prove conditions under which a p-quasihyponormal contraction is normal. This result will also be useful in Chapter Three.

Theorem 2.28 [44] *The c.n.u part of a paranormal contraction is of class $C_{.0}$.*

Remark 2.15

Since by [25] the class of p-quasihyponormal operators is contained in the class of paranormal operators , the c.n.u part of a p-quasihyponormal contraction is of class $C_{.0}$.We use this assertion to prove the following result.

Theorem 2.29 *Let $T \in B(\mathcal{H})$ be a p-quasihyponormal contraction. If $d_T < \infty$, then T is normal.*

Proof. Since T is a contraction, we have the decomposition $T = T_1 \oplus T_2$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $T_1 = T|_{\mathcal{H}_1}$ is unitary (hence injective with $Ker(T_1) = \{0\}$) and $T_2 = T|_{\mathcal{H}_2}$ is c.n.u. p-quasihyponormal. Since by Theorem 2.28 the c.n.u. part of p-quasihyponormal contraction is $C_{.0}$, T_2 is of class $C_{.0}$. On the other hand, since T_1 is unitary, the deficiency index $d_{T_1} = 0$ and hence $d_{T_2} = rank(D_{T_2}) = rank(D_T) < \infty$. Since $T_2 \in C_{.0}$, $T \in C_0$. Thus, there exists an inner function f such that $f(T) = 0$. By [66, Theorem 2], the planar Lebesgue measure of the spectrum of T is zero. This proves that T is normal.

Remark 2.16

Takahashi and Uchiyama [67] proved that if T is a hyponormal contraction with Hilbert-Schmidt defect operator, then T is c.n.n. and T is of class C_{10} are equivalent.
We give the following generalization to the class of p-quasihyponormal operators.

Theorem 2.30 *Let $T \in B(\mathcal{H})$ be a p-quasihyponormal contraction such that the defect operator D_T is of Hilbert-Schmidt class. Then T is completely non-normal if and only if T is of class C_{10}.*

Proof. Suppose that T is a c.n.n. p-quasihyponormal contraction. Then by Theorem 2.27, T is of class $C_{.0}$. Define $\mathcal{M} = \{x \in \mathcal{H} : T^n x \longrightarrow 0,\ n = 1, 2, ...\}$. Then $\mathcal{M}$ is a T-invariant subspace and the restriction operator $T|_{\mathcal{M}}$ is of class C_{00} and $D^2_{T_1} = I_{\mathcal{M}} - T_1^* T_1$

is of trace-class. That is, $\sum_n \langle |D_T| e_n, e_n \rangle < \infty$ or equivalently, $\sum_n \left\| |D_T|^{1/2} e_n \right\|^2 < \infty$ for an orthonormal basis $\{e_n\}$ of $\mathcal{M}$. By Nagy and Foias [53, Theorem 2], the planar Lebesgue measure of the spectrum of T is zero. But by Lee and Lee [50] the planar Lebesgue measure of the spectrum of any c.n.n. p-quasihyponormal operator is positive. This implies that $\mathcal{M} = \{0\}$, and hence T is of class C_{10}. Conversely, suppose T is of class C_{10}. For a normal operator N, $N \in C_{\cdot 0}$ and $N \in C_{0\cdot}$ are equivalent. So the condition $T \in C_{10}$ excludes the existence of any non-trivial normal direct summand. This means that T is completely non-normal.

Corollary 2.31 *Let A and B be hyponormal operators. Assume that the c.n.n. part of A has finite multiplicity. If A is quasisimilar to B then their normal parts are unitarily equivalent.*

Proof. The result follows easily from Corollary 2.27 and by the application of the fact that quasisimilar normal operators are unitarily equivalent (see Hastings [33], Williams [76]).

Note that from Corollary 2.31, we cannot conclude that quasisimilar hyponormal operators have quasisimilar pure parts. By [33], if A and B are quasisimilar hyponormal operators and A is pure, then B is also pure.

Definition 2.1

An operator $T \in B(\mathcal{H})$ is called *quasitriangular* if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank (orthogonal) projections such that $P_n \longrightarrow I$ (strongly, $n \longrightarrow \infty$) and $\|TP_n - P_nTP_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ (see [37],[54]). Recall that an operator T is *reductive* if every T-invariant subspace reduces T. We give some decomposition results for quasitriangular operators.

Theorem 2.32 *If A is completely nonnormal and reductive, and if T commutes with A, then T is quasitriangular .*

Proof. By Hoover [39], since A is reductive, and $AT = TA$, then T is reductive and hence every invariant subspace of T is reducing. Thus every invariant subspace of T is also an invariant subspace for T^*. If T^* has an eigenvalue λ, let $\mathcal{M}_\lambda = Ker(\lambda I - T^*)$. Clearly, $\mathcal{M}_\lambda$ is T^*-invariant and A^*-invariant. Thus $\mathcal{M}_\lambda$ is hyperinvariant for T^* and thus reduces T. Now suppose that T is nonquasitriangular and let $\mathcal{M}$ be the span of all eigenvectors of T^*. The subspace $\mathcal{M}$ reduces T and $T|_{\mathcal{M}}$ is diagonal, so it must be that $T|_{\mathcal{M}^\perp}$ is nonquasitriangular. But then by [3] $T^*|_{\mathcal{M}^\perp}$ would have an eigenvector, a contradiction of the choice of $\mathcal{M}$. Thus T is quasitriangular.

Remark 2.17

We note that Theorem 2.32 is not generally true for all reductive operators. For consider $T = I$. Thus T commutes with every linear operator A but $T = I$ is not quasitriangular. Thus the complete nonnormality of A cannot be dropped.
This fact leads to the following result.

Corollary 2.33 *Every reductive operator $T \in B(\mathcal{H})$ is quasitriangular.*

Proof. Suppose T is reductive. Then $T = T_1 \oplus T_2$, where T_1 is normal (hence) quasitriangular, and the completely non-normal T_2 commutes with itself and is therefore quasitriangular by Theorem 2.32.

Remark 2.18

We now look into the normal parts of a dominant operator. We first note that a hyponormal operator which is similar to a normal operator must be normal. It is known from the definition that every hyponormal operator is dominant. First, we need the following two corollaries due to Stampfli and Wadhwa [65].

Corollary 2.34 [66]. *Let $T \in B(\mathcal{H})$ be hyponormal. If T is similar to a normal operator, then T is normal.*

Remark 2.19

We note that Corollary 2.34 also follows easily from Proposition 2.5 and the Proof of Lemma 2.7. We give a simplified but detailed proof to the following corollary.

Corollary 2.35 [66]. *Let $T \in B(\mathcal{H})$. Let $TW = WN$ where N is normal and W is any non-zero operator in $B(\mathcal{H})$. Then T has a nontrivial invariant subspace.*

Remark 2.20

We note that Corollary 2.35 applies to quasiaffine transforms of all reducible operators with a finite-dimensional direct summand (remember: normal operators are reducible).

Corollary 2.36 [19]. *Let $A \in B(\mathcal{H}_1)$, $B \in B(\mathcal{H}_2)$ and $X \in B(\mathcal{H}_2, \mathcal{H}_1)$ be such that $AX = XB$. If either A is a pure dominant operator or B^* is a pure M-hyponormal operator, then $X = 0$.*

Remark 2.21

The following results show that the results on decompositions of (p, k)-quasihyponormal operators can be strengthened to subclasses.

Theorem 2.37 *An operator $T \in B(\mathcal{H})$ is k-quasihyponormal if and only if $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ with respect to the decomposition $\mathcal{H} = \overline{Ran(T^k)} \oplus Ker(T^{*k})$, where $T_1^*T_1 - T_1T_1^* \geq T_2T_2^*$ and $T_3^k = 0$.*

Proof. This result follows easily from Lemma 2.8.

Corollary 2.38 *If T is k-quasihyponormal and the spectrum of T has zero Lebesgue measure, then T is a direct sum of a normal operator and a nilpotent operator.*

Proof. The hypothesis implies that T is of the form in Theorem 2.37 with spectrum of T_1 of zero area measure. Therefore T_1 is normal and hence $T_2 = 0$. Hence, $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}$, where T_1 is normal and $T_3^k = 0$, i.e. T_3 is nilpotent. This proves the result.

Remark 2.22

From Corollary 2.38, it is clear that the direct summand T_3 is completely non-normal.

Corollary 2.39 *If T is k-quasihyponormal and the spectrum of T has zero Lebesgue area measure, and $Ker(T) \subset Ker(T^*)$ (equivalently, $Ker(T) \cap Ran(T) = \{0\}$), then T is normal.*

Proof. Suppose that T satisfies all the assumptions in Corollary 2.38. From Theorem 2.37 and Corollary 2.38, $T = T_1 \oplus T_3$ where T_1 is normal. If $T_3^k = 0$ and $T_3 \neq 0$, then $Ran(T) \cap Ker(T) = \{0\}$. Thus $T_3 = 0$ and thus $T = T_1 \oplus 0$, which is normal.

Theorem 2.40 *If A is a completely non-normal operator of norm one, such that A^*A-AA^* is a projection, then A is a unilateral shift. The conclusion is also true if the norm condition is not assumed.*

Proof. We show that

(1) A is quasinormal,

(2) $\mathcal{M} = Ker(I - A^*A)$ reduces A.

(3) $Ker(I - A^*A) \subseteq Ker(A^*A - AA^*)$.

Write $P = A^*A - AA^*$. Since for all $x \in \mathcal{M}$

$$\|x\|^2 \geq \|Ax\|^2 = \langle A^*Ax, x\rangle = \langle AA^*x, x\rangle + \langle Px, x\rangle = \|A^*x\|^2 + \|Px\|^2,$$

it follows that if $P = I$, then $A^*P = 0$. This implies that $(A^*P)^* = 0$, and hence $PA = 0$. This shows that $(A^*A)A = A(A^*A)$, which proves (1). If $x \in \mathcal{M}$, then $x - A^*Ax = 0$. Thus

$$Ax - (A^*A)Ax = Ax - A(A^*A)x = A(x - A^*Ax) = 0,$$

so that $\mathcal{M}$ is invariant under A^*. Similarly, replacing x by A^*x we get that $\mathcal{M}$ is invariant under A. This proves (2). Finally, since P is idempotent, it follows that

$$A^*A - AA^* = A^*AA^*A - AA^*A^*A - A^*AAA^* + AA^*AA^*.$$

Since A^*A commutes with both A and A^*, this can be written as

$$A^*A - AA^* = A^*A(A^*A - AA^*).$$

In other words, $P = A^*AP$, or, $(I - A^*A)P = 0$. It follows that $Ran(P) \subseteq \mathcal{M}$, or, $\mathcal{M} \perp Ker(P)$(since range and kernel of a projection operator are algebraic complements) and since A is completely non-normal, $Ker(P)$ includes no non-zero subspace that reduces A. Thus $\mathcal{M}^\perp = \{0\}$, which means that A is a unilateral shift.

Remark 2.23

From [56] an operator $T \in B(\mathcal{H})$ is 2-normal (denoted $T \in [2N]$) if $T^*T^2 = T^2T^*$. We investigate the direct sum decomposition of 2-normal operators.

Proposition 2.41 [56, *Proposition* 2.1] *Let $T \in B(\mathcal{H})$ have the direst sum decomposition $T = T_1 \oplus T_2$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. If $T \in [2N]$ then each direct summand T_i, $i = 1, 2$ is 2-normal.*

Proof. Suppose that $T^*T^2 = T^2T^*$. Then a simple operator multiplication shows that $T^*T^2 = T_1^*T_1^2 \oplus T_2^*T_2^2$ and $T^2T^* = T_1^2T_1^* \oplus T_2^2T_2^*$. Since $T \in [2N]$, we have $T_1^*T_1^2 \oplus T_2^*T_2^2 = T_1^2T_1^* \oplus T_2^2T_2^*$. Equating respective direct summands gives $T_1^*T_1^2 = T_1^2T_1^*$ and $T_2^*T_2^2 = T_2^2T_2^*$. Hence $T_i \in [2N]$, $i = 1, 2$.

Remark 2.24

Nzimbi, Pokhariyal and Khalagai [56] have claimed that the converse of Proposition 2.41 is also true. This leads to the following result.

Proposition 2.42 [56, *Proposition* 2.2] *Let T be a normal operator. Then T is 2-normal.*

Proof. Since T is normal, so is T^*. Thus $T^*T^2 = (T^*T)T = (TT^*)T = T(T^*T) = T(TT^*) = T^2T^*$. This completes the proof.

Remark 2.25

The converse of Proposition 2.42 does not hold in general. For instance, if $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then a simple matrix multiplication shows that T is 2-normal but not normal (indeed, in this case T is pure). This shows that a 2-normal operator T decomposes as $T = T_1 \oplus T_2$ where T_1 is normal and T_2 is completely non-normal and any of these direct summands could be missing. We give conditions under which a 2-normal operator turns out to be normal (That is, T has no non-trivial pure summand).
We give a condition for which a 2-normal or quasinormal operator is normal.

Proposition 2.43 [56, *Proposition* 2.6] *If $T \in B(\mathcal{H})$ is 2-normal and quasinormal and injective on $Ran([T^*, T])$, then T is normal.*

Proof. The assumption that T is 2-normal and quasinormal implies that $T^*T^2 = T^2T^*$ and $(T^*T - TT^*)T = 0$. A simple calculation shows that T is normal.

Remark 2.26

By Proposition 2.43, an operator which is both 2-normal and quasinormal has no nonzero c.n.n direct summand. We note that we can not merely drop any or both of the 2-normality or quasinormality conditions in Proposition 2.43. If T is the unilateral shift on ℓ^2 then T has an infinite matrix representation $T = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$. A simple computation shows that T is quasinormal but not 2-normal. Also, $T^*T - TT^* = diag(1, 0, 0, ...) \neq diag(0, 0, 0, ...)$. Thus T is not normal.

Chapter 3

On unitary and completely non-unitary summands of a contraction operator

In this chapter we study the decomposition of a contraction operator into a direct sum of unitary and completely non-unitary parts. We investigate properties of the c.n.u summands of contraction operators.

Nagy and Foias [53] have shown that every contraction operator T can be written as a direct sum of a unitary and a completely non-unitary (c.n.u.) part and that any of the direct summand could be missing. Recall that a contraction is completely nonunitary (c.n.u.) if it has no nonzero unitary direct summand; equivalently, if the restriction of it to any nonzero reducing subspace is not unitary. We start with some preliminary results. The following result which appears in [24] is useful.

Proposition 3.1 [24] *If $T \in B(\mathcal{H})$ is an isometry and $\mathcal{M}$ is an invariant subspace for T such that $T\mathcal{M} = \mathcal{M}$, then $\mathcal{M}$ reduces T and $T|_{\mathcal{M}}$ is unitary and*

$$\mathcal{M} \subseteq \bigcap_{n=0} T^n\mathcal{H}.$$

Remark 3.1

We give some results on contractions with defect indices. We begin with following well known result.

Lemma 3.2 [85] *Let T be a contraction with finite defect indices. Then the following statements are equivalent:*
(i) T is quasisimilar to a unilateral shift.
(ii) T is of class C_{10}.

Lemma 3.3 *Let T be a $C_{1.}$ contraction with finite defect indices. Let T have the triangulation $T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$ be of type $\begin{pmatrix} C_{.1} & * \\ 0 & C_{.0} \end{pmatrix}$. Then T_1 and T_2 are of class C_{11} and C_{10}, respectively.*

From Lemma 3.3 we state the following result about isometries.

Lemma 3.4 *Let $T \in B(\mathcal{H})$ be an isometry. If T has the decomposition $T = T_1 \oplus T_2$ with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, with $\mathcal{M}$ reducing, then $T_1 \in C_{11}$ and $T_2 \in C_{10}$.*

Remark 3.2

Note that any of the direct summands in Lemma 3.4 may be missing. This is the famous von Neumann-Wold decomposition of an isometry (see [45], [47]) which is a consequence of the Nagy-Foias-Langer decomposition of a general contraction operator as a direct sum of a unitary operator and a c.n.u. operator. If T_1 is missing in Lemma 3.4, then T is a pure isometry or a completely nonunitary isometry and hence a unilateral shift. From Lemma 3.4 we can conclude that every isometry on a Hilbert space is either a unitary operator, a unilateral shift or a direct sum of a unitary operator and a unilateral shift operator. Nagy and Foias [53] have proved that every contraction T of class C_{11} is quasisimilar to a unitary operator U and that since quasisimilar unitary operators are unitarily equivalent, the operator U is uniquely determined up to unitary equivalence.
Proof of Lemma 3.4. If T is an isometry then $A = \lim_{n\to\infty} T^{*n}T^n = I$ and $Ker(I - A) = \mathcal{H}$ and hence $\mathcal{M} = Ker(I - A) \cap Ker(I - A_*) = Ker(I - A_*)$ is a reducing subspace, where $A_* = \lim_{n\to\infty} T^n T^{*n}$. By the Nagy-Foias-Langer decomposition with $\mathcal{M} = Ker(I - A_*)$ we have that $T|_{\mathcal{M}}$ is unitary and $T|_{\mathcal{M}^{\perp}}$ is a completely non-unitary isometry on $\mathcal{M}^{\perp}$, which means a unilateral shift.
Note that the proof also follows immediately from Proposition 2.21 and Corollary 2.27.

Lemma 3.5 *Let T be a c.n.u contraction with finite defect indices and let $\mathcal{M}$ be an invariant subspace of T.*

(i) If T is quasisimilar to an isometry, so is $T|_{\mathcal{M}}$.

(ii) If T is quasisimilar to a unilateral shift, so is $T|_{\mathcal{M}}$.

Lemma 3.6 [80, *Corollary* 3.9]. *Let $T = T_1 \oplus T_2$ and $S = S_1 \oplus S_2$ be contractions, where T_1 and S_1 are of class C_{11} , T_2 and S_2 are of class $C_{.0}$ and T_2 has finite multiplicity. Then T is quasisimilar to S if and only if T_1 is quasisimilar to S_1 and T_2 is quasisimilar to S_2.*

Remark 3.3

We use Lemma 3.6 to prove the following result for hyponormal contractions. We use the fact that quasisimilar normal (unitary) operators are unitarily equivalent.

Corollary 3.7 *Let T and S be hyponormal contractions. Assume that the c.n.u part of T has finite multiplicity. Then T is quasisimilar to S if and only if their unitary parts are unitarily equivalent and c.n.u parts are quasisimilar to each other.*

Proof. The conclusion follows from [16, Lemma 1], Lemma 3.6 and the fact that completely non-normal (and hence completely non-unitary) hyponormal contractions are of class $C_{.0}$ by Theorem 2.28 since every hyponormal operator is paranormal.

Remark 3.4

Duggal and Kubrusly [16] have proved that the completely non-unitary direct summand of a contraction T is of class $C_{.0}$ if and only if T has the PF (short for Putnam-Fuglede) commutativity property: whenever $TX = XJ^*$ holds for some isometry $J \in B(\mathcal{K})$ and some $X \in B(\mathcal{H}, \mathcal{K})$, then $T^*X = XJ$. Contractions with the PF property include dominant and paranormal operators . This result has been extended by Duggal and Kubrusly [16] to the class of k-paranormal operators.

Corollary 3.8 *If a contraction $T \in B(\mathcal{H})$ is k-quasihyponormal or k-paranormal, then the completely non-unitary direct summand of T is of class $C_{.0}$.*

Remark 3.5

We now characterize contraction operators using shift operators on Hilbert spaces. This characterization makes the analysis of contraction operators easier to handle since we investigate their action on function Hilbert spaces rather than vector Hilbert spaces. We study the universal model of operators on finite and infinite dimensional Hilbert spaces. Shift operators have the following remarkable property: Up to unitary equivalence and multiplicative constants, the class of operators $T = S^*|_{\mathcal{M}}$, where S is a shift operator and $\mathcal{M}$ is an invariant subspace for S^*, includes every bounded linear operator on a Hilbert space. This result was proved by Rota [63].
Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\ ,\ \rangle_{\mathcal{H}}$ and norm $|\ .\ |_{\mathcal{H}}$. By $\mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$ we mean the Hardy space of all $\mathcal{H}$-valued holomorphic functions

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

on $\mathbb{D}$ for which the quantity

$$\frac{1}{2\Pi}\int_0^{2\Pi} |f(re^{i\theta})|^2_{\mathcal{H}} d\theta = \sum_{j=0}^{\infty} |a_j|^2_{\mathcal{H}} r^{2j} < \infty, \quad 0 \leq r < 1.$$

It is easy to see that $\mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$ is a Hilbert space with inner product given by

$$\langle f, g\rangle_2 = \lim \tfrac{1}{2\Pi}\textstyle\int_0^{2\Pi}\langle f(re^{i\theta}), g(re^{i\theta})\rangle_{\mathcal{H}} d\theta = \sum_0^{\infty}\langle a_j, b_j\rangle_{\mathcal{H}}$$

as $r \uparrow 1$ for any $f(z) = \sum_0^{\infty} a_j z^j$ and $g(z) = \sum_0^{\infty} b_j z^j$ in the space.
Thus $\mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$ is isomorphic with $\ell^2_{\mathcal{H}}$ via the correspondence between function and its Taylor coefficients. Where there is no confusion we write $\mathbb{H}^2(\mathbb{D})$ for $\mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$ and ℓ^2 for $\ell^2_{\mathcal{H}}$. As a result of this isomorphism we obtain the following results.

Theorem 3.9 *The operator of multiplication by z on $\mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$ defined by $S : f(z) \longrightarrow zf(z)$ for all $f(z)$ in $\mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$ is a shift operator of multiplicity $\dim(\mathcal{H})$ and the adjoint of S is $S^* : f(z) \longrightarrow \frac{[f(z)-f(0)]}{z}$.*

Proof. Note that $(Sf)(z) = zf(z), \quad f \in \mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$. Define the Fourier transform from $\mathcal{H}$ to $\mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$ by $(a_1, a_2, a_3 ...) \longrightarrow f$ where $f(z) = \sum_{j=0}^{\infty} a_j z^j$. Thus, $S(\sum_{j=0}^{\infty} a_j z^j) = \sum_{j=0}^{\infty} a_j z^{j+1} = \sum_{j=0}^{\infty} a_{j-1} z^j$. Clearly, the Fourier transform is a unitary equivalence between a shift operator and the operator S. Since every operator unitarily equivalent to a shift is a shift, S must be a shift operator. The rest of the assertion follows from the fact that the multiplicity of a shift on $\mathcal{H}$ is equal to $dim(\mathcal{H})$ and the adjoint of a shift operator is unitarily equivalent to S^*.

Corollary 3.10 *Every shift operator on a Hilbert space $\mathcal{H}$ is unitarily equivalent to multiplication by z on $\mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$ for some choice of $\mathcal{H}$.*

Remark 3.6

Let $\mathcal{H}$ be a separable Hilbert space. To every completely non-unitary (c.n.u) contraction T on $\mathcal{H}$, Nagy and Foias [53] associated a contraction-valued holomorphic function Θ_T on the open unit disk $\mathbb{D}$ such that $\Theta_T(0)$ is a pure contraction. We call this function the *characteristic function* of the operator T. Conversely, given any holomorphic function Θ on $\mathbb{D}$, there exists a completely non-unitary contraction T_Θ whose characteristic function coincides with Θ.

We will try to characterize completely non-unitary contractions in terms of their characteristic functions. We first investigate completely non-unitary contractions with constant characteristic functions.

Recall that $D_T = (I - T^*)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$ are the defect operators associated with a c.n.u contraction T and the range closures in the norm topology of $\mathcal{H}$, $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ of D_T and D_{T^*}, respectively, are called the defect spaces. We say that two operators $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in B(\mathcal{K}_1, \mathcal{K}_2)$ coincide if there exist unitary operators $U : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ and $V : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$ such that $VAU = B$. The operator-valued functions $\Theta_i(z) : \mathcal{H}_i \longrightarrow \mathcal{K}_i$, $i = 1, 2$ are said to coincide if there exist unitary operators $U : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ and $V : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$ such that $V\Theta_1(z)U = \Theta_2(z)$, for all z.

We construct an inner-outer factorization for operator-valued holomorphic functions that are of bounded type on a disk or half-plane. Let $\Omega = \mathbb{D}$ or $\mathbb{T}$, where $\mathbb{D} = \{z : |z| < 1\}$ and $\mathbb{T} = \{z : Im\ z > 0\}$, the unit disk and half-plane, respectively. If $A \in \mathbb{H}^\infty_{B(\mathcal{H})}(\Omega)$, then:

(i) A is an *inner function* if the operator $T(A) : f \longrightarrow Af$, $f \in \mathbb{H}^2_{\mathcal{H}}(\Omega)$, is a partial isometry on $\mathbb{H}^2_{\mathcal{H}}(\Omega)$.

(ii) A is an *outer function* if $\bigvee\{Af : f \in \mathbb{H}^2_{\mathcal{H}}(\Omega)\} = \mathbb{H}^2_{\mathcal{M}}(\Omega)$ for some subspace $\mathcal{M}$ of $\mathcal{H}$.

A function $\varphi \in \mathbb{H}^\infty$ is said to be inner provided $|\varphi| = 1$ almost everywhere on $\mathbb{R}$. For such φ, the set $\varphi\mathbb{H}^2 = \{\varphi f : f \in \mathbb{H}^2\}$ is a closed subspace of $\mathbb{H}^2$. Inner functions of the form

$$B(z) = \left(\frac{z-i}{z+i}\right)^m \prod_n \frac{|z_n^2+1|}{z_n^2+1}\frac{z-z_n}{z-\overline{z_n}}$$

for m and n nonnegative integers and $\{z_n\}$ a sequence in $\partial\mathbb{D}\backslash\{i\}$, with $\sum_n \frac{y_n}{1+|z_n|^2} < \infty$, $z_n = x_n+iy_n$ are *Blaschke products* with zeros $\{z_n\}$ and their multiplicity is defined to be the number of factors if this number is finite, and infinite if not. In general, we define the multiplicity of an inner function to be the dimension of the subspace $\mathbb{H}^2\ominus\varphi\mathbb{H}^2$. It can be shown that an inner function φ has finite multiplicity if and only if it is a finite Blaschke product, in which case the elements of $\mathbb{H}^2 \ominus \varphi\mathbb{H}^2$ are all rational functions. The function identically zero is both inner and outer. The canonical shift operators on $\mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$ and $\mathbb{H}^2_{\mathcal{H}}(\partial\mathbb{D})$ are defined by $S : f(z) = zf(z)$ on $\mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$ and $S : f(z) = \frac{z-i}{z+i}f(z)$ on $\mathbb{H}^2_{\mathcal{H}}(\mathbb{T})$.

These operators are unitarily equivalent by means of the isomorphism

$$f(z) \longrightarrow F(z) = \Pi^{-1/2}(z+i)^{-1}f(\frac{z-i}{z+i}),$$

from $\mathbb{H}^2_{\mathcal{H}}(\mathbb{D})$ onto $H^2_{\mathcal{H}}(\partial\mathbb{D})$.

We recall that a contraction T is said to be *proper* if $\|Tx\| < \|x\|$ for all nonzero vectors $x \in \mathcal{H}$. Recall that an operator $X \in B(\mathcal{H},\mathcal{K})$ is a quasi-invertible or a quasi-affinity if it has trivial kernel and dense range, (i.e, $Ker(X) = \{0\}$ and $\overline{Ran(X)} = \mathcal{K}$).

Using the preceding concepts we give the following results.

Lemma 3.11 *Let T be a contraction between two Hilbert spaces. Then the following are equivalent.*

(i) T is a proper contraction.

(ii) T^* is a proper contraction.

(iii) $(I-T^*T)^{1/2}$ is quasi-invertible.

(iv) $(I-TT^*)^{1/2}$ is quasi-invertible.

Proof. (i)$\Longleftrightarrow$(ii): First note that the contraction T is a proper contraction if and only if $Ker(I-T^*T)^{1/2} = \{0\}$. If T is a proper contraction, then polar decomposition shows that $(I-T^*T)^{1/2}$ and $(I-TT^*)^{1/2}$ are unitarily equivalent, which implies the stated equivalence.

(i)$\Longleftrightarrow$ (iii): If T is a proper contraction, then $Ker(I-T^*T)^{1/2} = \{0\}$.

This means that $(I-T^*T)^{1/2}$ is injective and hence has dense range. Thus $(I-T^*T)^{1/2}$ is quasi-invertible.

(i) $\Longrightarrow$(iv), (ii)$\Longleftrightarrow$ (iii) and (ii) $\Longrightarrow$ (iv): The equivalence of (i) and (ii) shows that $(I-TT^*)^{1/2}$ is quasi-invertible as well. Note that the equivalence of (ii) and (iii) implies the equivalence of (iii) and (iv). Finally, if $(I-T^*T)^{1/2}$ is quasi-invertible, then it is obvious that T is a proper contraction, which shows that (iv)$\Longrightarrow$ (i), (iv)$\Longrightarrow$ (ii).

Definition 3.1

$T \in B(\mathcal{H})$ is *strongly stable* if the power sequence $\{T^n\}$ converges strongly to the null operator (equivalently, $T^n \longrightarrow O$ strongly or $\|T^n x\| \longrightarrow 0$ for every $x \in \mathcal{H}$). Clearly every strongly stable contraction is clearly completely non-unitary and is thus a proper contraction and satisfies Lemma 3.11.

Example 3.1

The operator $B = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$ is strongly stable since $B^{*n}B^n = \begin{pmatrix} (1/2)^{2n} & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ as $n \longrightarrow \infty$. It is clear that B is completely non-unitary. A simple matrix computation shows that T satisfies Lemma 3.11.

Definition 3.2

A bounded linear operator T acting on the complex separable Hilbert space $\mathcal{H}$ is called *homogeneous* if $\sigma(T) \subset \overline{\mathbb{D}}$ and $\varphi(T)$ is unitarily equivalent to T for every $\varphi \in \mathcal{M}(\mathbb{D})$.

Remark 3.7

Homogeneous operators were investigated in several works (see [6], [13]). It was proved by Bagchi and Misra [6] that if T is a homogeneous contraction such that the restriction $T|_{\mathcal{D}_T}$ of T to the defect space $\mathcal{D}_T$ is of Hilbert-Schmidt class, then T has a constant characteristic function. Let $\mathcal{M}(\mathbb{D})$ denote the set of all injective, analytic mappings of the open unit disc $\mathbb{D}$ onto itself. That is, $\mathcal{M}(\mathbb{D}) = \{f : \mathbb{D} \longrightarrow \mathbb{D},\ f\ 1-to-1,\ \textit{analytic on}\ \mathbb{D}\}$. Any element of $\mathcal{M}(\mathbb{D})$ is of the form

$$\varphi_{k,a}(z) = \frac{k(z-a)}{(1-\overline{a}z)}, \quad z \in \mathbb{D},\ k \in \partial\mathbb{D},\ a \in \mathbb{D}.$$

These mappings are known as the Möbius transformations of the unit disk $\mathbb{D}$.
Nagy and Foias [53] have shown that for any $\varphi \in \mathcal{M}(\mathbb{D})$ and for any contraction $T \in B(\mathcal{H})$, the Möbius transform $\varphi(T)$ is unitary (c.n.u) if and only if T is unitary (c.n.u, respectively). Thus the contraction T is homogeneous if and only if the unitary and completely non-unitary parts are both homogeneous.
Let us assume that $T \in B(\mathcal{H})$ is a completely non-unitary contraction. We consider the characteristic function Θ_T of T defined by

$$\Theta_T(z) = (-T + zD_{T^*}(I - zT^*)^{-1}D_T)|_{\mathcal{D}_T} \in B(\mathcal{D}_T, \mathcal{D}_{T^*}), \quad z \in \mathbb{D},$$

where $D_T = (I - T^*T)^{1/2}$, $D_{T^*} = (I - TT^*)^{1/2}$ are the defect operators, and $\mathcal{D}_T = \overline{Ran(D_T)}$, $\mathcal{D}_{T^*} = \overline{Ran(D_{T^*})}$ are the defect spaces of T. The mapping

$$\Theta_T : \mathbb{D} \longrightarrow B(\mathcal{D}_T, \mathcal{D}_{T^*}),$$

is a contraction-valued, analytic function, and by [78], $\Theta_T(0) = -T|_{\mathcal{D}_T}$ is a pure (proper) contraction, that is, $\|Tx\| < \|x\|$ for every $0 \neq x \in \mathcal{D}_T$. We note that c.n.u contractions $T_1 \in B(\mathcal{H}_1)$ and $T_2 \in B(\mathcal{H}_2)$ are unitarily equivalent if and only if their characteristic functions Θ_{T_1} and Θ_{T_2} coincide. That is, there exist unitary transformations $U : \mathcal{D}_{T_1} \longrightarrow \mathcal{D}_{T_2}$ and $V : \mathcal{D}_{T_2^*} \longrightarrow \mathcal{D}_{T_1^*}$ such that $\Theta_{T_1}(z) = V\Theta_{T_2}(z)U$, $z \in \mathbb{D}$. We show that if the characteristic function of a c.n.u contraction T is constant, i.e, $\Theta_T(z) = \Theta_T(0)$, for every $z \in \mathbb{D}$, then T is a homogeneous contraction.
Nagy and Foias [53] have proved that a c.n.u. contraction $T \in B(\mathcal{H})$ of constant characteristic function is a weighted bilateral shift with special operator weights. We claim, however, that all unitary operators also have this property.

Theorem 3.12 *Let T be a homogeneous c.n.u. contraction. If $T|_{\mathcal{D}_T}$ is compact, then the characteristic function of T is constant.*

Proof. Suppose T is a homogeneous c.n.u. contraction with chracteristic function Θ_T. By [6, Theorem 2.9], $\Theta\circ\varphi^{-1}$ coincides with Θ for each $\Theta \in \mathcal{M}(\mathbb{D})$. By [6, Theorem 2.10], since $T|_{\mathcal{D}_T}$ is compact, T is unitarily equivalent to a c.n.u. contraction with constant characteristic function. This proves that Θ_T is constant.

Example 3.2

Some homogeneous contractions with non-constant characteristic functions are those of class C_{00}, i.e. those that are strongly stable. These are backward weighted shifts with weight sequence $\{w_n = (1+n)^{1/2}(c+n)^{-1/2}\}_{n=0}^{\infty}$, where $c > 1$.
We also note that if T is a c.n.u. contraction and Θ_T is constant, then T is the orthogonal direct sum of a unilateral shift , a backward shift and a C_{11}-contraction.

Remark 3.8

Isometries comprise a rather special class of C_1 - contractions. They are the contractions $T \in B(\mathcal{H})$ for which $\|Tx\| = \|x\|$, for every $x \in \mathcal{H}$. It is easy to show that every isometry has a nontrivial invariant subspace. Since a unitary operator is a normal isometry, it follows that the unitary contractions comprise a set of particularly well-known operators. By Kupin [48] every invariant subspace $\mathcal{M}$ of a model operator T_Θ defines a certain regular factorization of the characteristic function $\Theta = \Theta_2\Theta_1$ of a contraction T. These model operators have been characterized as multiplication or composition operators in $\mathbb{H}^2$. The following canonical decomposition will be useful.

Theorem 3.13 [53, *Theorem I.3.2*]. *To every contraction T on a Hilbert space $\mathcal{H}$ there corresponds a uniquely determined decomposition of $\mathcal{H}$ into an orthogonal sum of subspaces reducing T, say $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, such that $T_0 = T|_{\mathcal{H}_0}$ is unitary and $T_1 = T|\mathcal{H}_1$ is c.n.u. In particular, for an isometry, this canonical decomposition coincides with the von Neumann-Wold decomposition.*

Remark 3.9

For a contraction T with decomposition $T = T_0 \oplus T_1$ as in Theorem 3.13, we have $D_T = 0 \oplus D_{T_1}$, $D_{T^*} = 0 \oplus D_{T_1^*}$, $\mathcal{D}_T = \mathcal{D}_{T_1}$ and $\mathcal{D}_{T^*} = \mathcal{D}_{T_1^*}$. These results lead us to the following result.

Corollary 3.14 *Let $T \in B(\mathcal{H})$ have the decomposition in Theorem 3.13, then* $\Theta_T(\lambda) = \Theta_{T_1}(\lambda)$.

Proof. A simple computation shows that

$$\begin{aligned}\Theta_T(\lambda) &= \Theta_{T_0\oplus T_1}(\lambda)\\ &= [-(T_0\oplus T_1)+\lambda(D_{T_0^*\oplus T_1^*})(I-\lambda(T_0^*\oplus T_1^*))^{-1}D_{T_0\oplus T_1}]\\ &= \Theta_{T_0}(\lambda)+\Theta_{T_1}(\lambda)\\ &= \Theta_{T_1}(\lambda).\end{aligned}$$

Remark 3.10

The conclusion of Corollary 3.14 follows from the fact that the characteristic function of a unitary contraction operator is identically zero.

Recall that a function Θ_T is an *outer function* if $\overline{\Theta_T\mathbb{H}^2(\mathcal{D}_T)} = \mathbb{H}^2(D_{T^*})$. Recall that if $\mathcal{K}$ is a Hilbert space, $\mathcal{H}\subset\mathcal{K}$ is a subspace, $S\in B(\mathcal{K})$, and $T\in B(\mathcal{H})$, then S is a *dilation* of T (and T is a *power-compression* of S) provided that $T^n = P_{\mathcal{H}}S^n|_{\mathcal{H}}$, $n=0,1,2,\ldots$ and $P_{\mathcal{H}}$ is the projection on $\mathcal{H}$. We now characterize contractions T with unitary quasi-affine transforms in terms of their characteristic function Θ_T.

Proposition 3.15 *Let $\mathcal{H}$ be a Hilbert space and let T be a contraction on H such that $Ker(T)=\{0\}$. The following assertions are equivalent:*
(a) T has unitary quasi-affine transforms;
(b) T belongs to the class $C_{1.}$

Proposition 3.16 *Let $\mathcal{H}$ be a separable Hilbert space and let T be a contraction on $\mathcal{H}$ such that $Ker(T)=\{0\}$. The following assertions are equivalent:*
i) T has unitary quasi-affine transforms;
ii) the characteristic function Θ_T of T is outer and $Ker(\Theta_T)\cap\mathbb{H}^2(\mathcal{D}_T)=\{0\}$.

Proof. Assume without loss of generality that T is a c.n.u contraction. It is well known that the following are equivalent: (a) T is of class $C_{.1}$; (b) Θ_T is an outer function; (c) the operator $A = P_H|_{\mathcal{R}}:\mathcal{R}\longrightarrow\mathcal{H}$ (where $\mathcal{R}=\cap_{n=0}^{\infty}U_+^n\mathcal{H}$, $U_+ = U|\mathcal{K}_+$ and U is unitary dilation of T, U_+ is isometric dilation of T and $\mathcal{K}_+ = \bigvee_0^{\infty}U_+^n\mathcal{H}$) has dense range. Also the following assertions are equivalent: (1) A is injective; (2) $Ker(\Theta_T)\cap\mathbb{H}^2(\mathcal{D}_T)=\{0\}$.

Remark 3.11

From Theorem 3.13, we conclude that $T\in C_{.1}$ if and only if its characteristic function Θ_T is outer. In the decomposition in Theorem 3.13 it is not excluded that $\mathcal{H}_0$ or $\mathcal{H}_1$ is possibly the subspace $\{0\}$. Furthermore, $\mathcal{H}_0$ is given by

$$\begin{aligned}\mathcal{H}_0 &= \{x \in \mathcal{H} : \|T^n x\| = \|x\| = \|T^{*n} x\|, \quad n = 1, 2, ...\}\\ &= \bigcap_{k=1}^{\infty} \{x \in \mathcal{H} : T^{*k}T^k x = x = T^k T^{*k} x\}\end{aligned}$$

is called the *unitary subspace* of T and it is the maximal reducing subspace on which its restriction is unitary. We give another characterization of the unitary space: If T is a contraction, then $\|T^{n+1}x\| \leq \|T^n x\|$ for all $x \in \mathcal{H}$ and the sequences $\{T^{*n}T^n\}$ and $\{T^n T^{*n}\}$ are monotonically decreasing and hence converge strongly to positive contractions A^2 and A_*^2, respectively with $T^* A^2 T = A^2$ and $T^* A_*^2 T = A_*^2$. By using the unique positive square roots A and A_* of A^2 and A_*^2, respectively, we can represent $\mathcal{H}_0$ as follows.

$$\begin{aligned}\mathcal{H}_0 &= \{x \in \mathcal{H} : \|Ax\| = \|A_* x\| = \|x\|\}\\ &= \{x \in \mathcal{H} : A^2 x = A_*^2 x = x\}\\ &= Ker(I - A) \cap Ker(I - A_*).\end{aligned}$$

It is clear that $Ker(A) = \{x \in \mathcal{H} : Ax = 0\}$ and

$$\begin{aligned}Ker(I - A) &= \{x \in \mathcal{H} : Ax = x\}\\ &= \{x \in \mathcal{H} : \|T^n x| = \|x\|, \quad n = 1, 2, 3, ...\}\end{aligned}$$

are invariant under T and $T|_{Ker(I-A)}$ is an isometry and $Ker(A - A^2) = Ker(A) \oplus Ker(I - A)$. We give an example to illustrate this fact.

Example 3.3

Let T be the backward unilateral shift on $\mathcal{H} = \ell^2$. It is not difficult to show that T is a contraction on ℓ^2. A simple calculation gives that $A = 0$ and $I - A = I$, $Ker(A) = \mathcal{H}$ and $Ker(I - A) = \{0\}$. Hence, $Ker(A - A^2) = Ker(A) \oplus Ker(I - A) = \mathcal{H} \oplus \{0\}$ which we identify with $\mathcal{H}$.

Remark 3.12

If T is completely non-unitary (has no nonzero unitary direct summand), then the unitary subspace $\mathcal{H}_0 = Ker(I - A) \cap Ker(I - A_*) = \{0\}$.

Recall that a contraction $T \in C_{00}$ if $A = A_* = O$. We give some results on the nature of direct summands of some classes of completely non-unitary contractions.

Proposition 3.17 *If $T \in B(\mathcal{H})$ is a normal contraction, then the c.n.u part of T is of class C_{00}.*

Proof. Suppose T is normal and $T = T_1 \oplus T_2$ with T_1 unitary and T_2 completely non-unitary. It is easily verified by Mathematical Induction that $T^{*n}T^n = T^nT^{*n}$ for every $n \geq 1$. By Theorem 2.28, T_2 is of class $C_{\cdot 0}$. It suffices to show that T_2 is of class $C_{0\cdot}$. Since T_2 is of class $C_{\cdot 0}$, then $\|T_2^{*n}\| \longrightarrow 0$. Since T is normal, $A = \lim_n T_2^{*n}T_2^n = \lim_n T_2^nT_2^{*n} = A_* = O$. Thus $T_2 \in C_{00}$.

Remark 3.13

T and T^* have the Putnam-Fuglede property if and only if $A = A_*$. We note that Proposition 3.17 also follows from the fact that T has the Putnam-Fuglede property (see Nzimbi, Pokhariyal and Khalagai [55, Corollary 2.3]).

Proposition 3.18 *If $A \in B(\mathcal{H})$ is a normal contraction and $B \in B(\mathcal{H})$ be similar to A then the c.n.u part of B is of class C_{00}.*

Proof. The result uses Proposition 3.17.

Remark 3.14

We note that by Lemma 3.6 and Corollary 3.7, Proposition 3.18 also holds when we replace similarity with either quasisimilarity or unitary equivalence. The following result gives a further decomposition of normal operators and appears it in [17].

Lemma 3.19 [17, *Lemma* 1] *If A is a normal contraction such that A^n is normal for some integer $n \geq 2$, then there exist direct sum decompositions $\mathcal{H} = \mathcal{H}_n \oplus \mathcal{H}_p$ and $A_n = A|_{\mathcal{H}_n}$ is a normal $C_{11} \oplus C_{00}$ type contraction and $A_p = A|_{\mathcal{H}_p}$ is a pure C_{00} contraction.*

Remark 3.15

Note that if a contraction is pure then it must be c.n.u but the converse is not generally true. For consider the operator with the matrix $T = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$. Then $T \in C_{00}$ and T is normal. Thus not all C_{00} contractions are pure. Equivalently, there is no C_{00} contraction with a unitary part. A pure C_{00} contraction is c.n.u and has a triangulation as in Lemma 3.19.

Lemma 3.20 . *Let T be a C_{11} contraction on $\mathcal{H}$ and U be a unitary operator on $\mathcal{K}$. If there exists a one-to-one operator $X : \mathcal{H} \longrightarrow \mathcal{K}$ such that $XT = UX$, then T is quasisimilar to the unitary operator* $U|_{\overline{X\mathcal{H}} \subset \mathcal{K}}$.

Proof. Since T, being a C_{11} contraction, is quasisimilar to a unitary, the assertion follows.

Remark 3.16

We note that there are C_{11} contractions which are not normal (and hence, not unitary). For consider the operator with matrix $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. A simple computation shows that $T \in C_{11}$ but T is non-normal. The conclusion in Lemma 3.20 cannot be extended to cover similarity. For instance, take an arbitrary integer $n \geq 1$ and let $T_n = shift(\{\omega_k\}_{k=-\infty}^{\infty})$ be a bilateral weighted shift on ℓ^2 with weights $\omega_k = 1$ for all k except for $k = 0$ where $\omega_0 = (n+1)^{-1}$. Each T_n is a non-unitary C_{11}-contraction similar to a unitary operator, and $T = \bigoplus_{n=1}^{\infty} T_n$ is a C_{11}-contraction not similar to any unitary operator. Thus if $T \in C_{11}$, then T is quasisimilar to a unitary operator, in which case , there exists an increasing sequence $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ of T-invariant subspaces that span $\mathcal{H}$ (i.e., $\bigvee_{n\in\mathbb{N}} \mathcal{M}_n = \mathcal{H}$) such that each part $T|_{\mathcal{M}_n}$ is similar to a unitary operator. This remark leads to the following result.

Corollary 3.21 . *A nonunitary C_{11} contraction is similar to a unitary operator if it is invertible.*

Proof. We prove this assertion by the method of contradiction. Suppose a c.n.u. $T \in C_{11}$ is such that $T = X^{-1}UX$, where U is unitary and suppose that T is not invertible. Then this is an absurdity since the right hand side is invertible while the left hand side in not. This completes the proof.

Remark 3.17

We extend Corollary 3.7 on hyponormal contractions and unilateral shifts to wider classes of operators: that of p-quasihyponormal contractions and isometries.

Theorem 3.22 . *Let T be a p-quasihyponormal contraction whose c.n.n part has finite multiplicity. Then T is quasisimilar to an isometry if and only if its normal part is unitary and the c.n.n (hence c.n.u) part is quasisimilar to a unilateral shift.*

Proof. Assume that $T = T_n \oplus T_p$ is quasisimilar to an isometry $V = U \oplus S$, where T_n is normal, T_p is c.n.n., U is unitary and S is a unilateral shift. By the Putnam-Fuglede theorem [25], T_n unitarily equivalent to U whence unitary. By ([14], Theorem 2),([64], Corollary 12) and ([40, Theorem 5]), V and T have the same spectra (since they are quasisimilar p-hyponormal operators),

$$\|T\| = r(T) = r(V) = 1.$$

By Corollary 3.5 and Wu ([80], [85]), T_p is quasisimilar to S.

Remark 3.18

Takahashi and Uchiyama [67] have proved that if T is a hyponormal contraction with Hilbert-Schmidt defect operator, then the following assertions are equivalent.
(a) T is c.n.n.;
(b) T is of class C_{10}.
We give a generalization of this result to p-quasihyponormal contractions.

Theorem 3.23 *Let T be a p-quasihyponormal contraction such that the defect operator D_T is of Hilbert-Schmidt class. Then T is completely nonunitary if and only if T is of class C_{10}.*

Proof. The proof follows from the proof of Theorem 2.30 with c.n.n. replaced with c.n.u.
We now characterize some contraction operators in terms of their characteristic functions.

Corollary 3.24 *If $T \in B(\mathcal{H})$ is an isometry then the characteristic function Θ_T is identically zero almost everywhere.*

Proof. We first prove the result for a unitary T. For a unitary operator T, $D_T = D_{T^*} = 0$ and $\mathcal{D}_T = \overline{D_T\mathcal{H}} = \{0\}$. Therefore, $\Theta_T = -T|_{\mathcal{D}_T} = -T|_{\{0\}} = 0, \quad \forall\ \lambda \in \mathbb{D}$. Since $D_T = 0$ for a unilateral shift T, we also have that $\Theta_T = 0$. Since by the von

Neumann-Wold decomposition an isometry is a direct sum of a unitary and a unilateral shift, the result follows.

Remark 3.19

Note that in the proof of Corollary 3.24 Θ_T is constant since $\Theta_T(0) = \Theta_T(\lambda)$.

We prove the following important property of the characteristic function of an operator.

Theorem 3.25 *Let $T \in B(H)$. Then $\Theta_{kT}(\lambda) = k\Theta_T(\bar{k}\lambda), \quad \lambda \in \mathbb{D}$ holds for any $k \in \partial\mathbb{D}$.*

Proof. By a simple computation and using the definition we have

$$\Theta_{kT}(\lambda) = \Big[- kT + \lambda(I - k\bar{k}TT^*)^{1/2}(I - (\bar{k}\lambda)T^*)^{-1}(I - \bar{k}kT^*T)^{1/2}\Big].$$

But since $k \in \partial\mathbb{D}, \quad k = e^{in\theta}, \quad 0 \leq \theta \leq 2\Pi$ and $n = 0, \pm1, \pm2, \ldots$. Thus $\bar{k}k = 1$. Thus, the previous equality becomes

$$\Theta_{kT}(\lambda) = \Big[- kT + \lambda(I - TT^*)^{1/2}(I - (\bar{k}\lambda)T^*)^{-1}(I - T^*T)^{1/2}\Big].$$

Similarly,

$$k\Theta_T(\bar{k}\lambda) = \Big[- kT + (k\bar{k})\lambda(I - TT^*)^{1/2}(I - (\bar{k}\lambda)T^*)^{-1}(I - T^*T)^{1/2}\Big].$$

Once again, since $k \in \partial\mathbb{D}, \quad k\bar{k} = 1$. Thus the above equality simplifies to

$$k\Theta_T(\bar{k}\lambda) = \Big[- kT + \lambda(I - TT^*)^{1/2}(I - (\bar{k}\lambda)T^*)^{-1}(I - T^*T)^{1/2}\Big].$$

This completes the proof.

Definition 3.3

Two contraction-valued analytic functions $\theta_1 : \mathbb{D} \longrightarrow B(\mathcal{D}_{T_1}, \mathcal{D}_{T_1^*})$ and $\theta_2 : \mathbb{D} \longrightarrow B(\mathcal{D}_{T_2}, \mathcal{D}_{T_2^*})$ *coincide* if there exist unitary operators

$$U : D_{T_1^*} \longrightarrow D_{T_2^*},$$

$$V : D_{T_2} \longrightarrow D_{T_1}$$

such that $\theta_2(z) = U\theta_1(z)V$, for all $z \in \mathbb{D}$. This definition leads to the following result.

Corollary 3.26 *If c.n.u contractions T_1 and T_2 are unitarily equivalent, then their characteristic functions coincide.*

Proof. Suppose T_1 and T_2 are unitarily equivalent c.n.u contractions. Then there exists a unitary operator U such that $T_1 = U^*T_2U$. Using the definition of the characteristic function we have

$$\begin{aligned}\Theta_{T_1}(z) &= \Theta_{U^*T_2U}(z) \\ &= -(U^*T_2U) + z\Big[I - (U^*T_2U)(U^*T_2^*U)\Big](I - zU^*T_2^*U)^{-1}\Big(I - (U^*T_2^*U)(U^*T_2U)\Big) \\ &= -(U^*T_2U) + z\Big[I - U^*(T_2T_2^*)U\Big](I - zU^*T_2^*U)^{-1}\Big(I - U^*(T_2^*T_2)U\Big) \\ &= U^*\Theta_{T_2}(z)U, \qquad z \in \mathbb{D}.\end{aligned}$$

Without loss of generality, we let $V = U^*$.

Remark 3.20

Corollary 3.26 says that the characteristic function, modulo coincidence, is a complete unitary invariant for c.n.u. contractions. This indicates that it should be possible to recover a c.n.u. contraction, up to unitary equivalence, from its characteristic function. The following result characterizes c.n.u. contractions with constant characteristic functions.

Theorem 3.27 [41]. *If T is a c.n.u. contraction and the characteristic function Θ_T is constant, then T is the orthogonal sum of a unilateral backward shift and a C_{11}-contraction.*

Remark 3.21

Theorem 3.27 says that a c.n.u contraction with a constant characteristic function decomposes as a direct sum of a unilateral shift and an operator quasisimilar to a unitary operator. We note that since T is c.n.u., the C_{11} part cannot be unitary, otherwise this would contradict the complete non-unitarity of T. We now investigate conditions for a partial isometry implying quasinormality and paranormality.

Theorem 3.28 *Let $T \in B(\mathcal{H})$. Then T is a quasinormal partial isometry if and only if T is the direct sum of an isometry and zero.*

Proof. If T is a partial isometry and quasinormal, then $T = PT = TP$, where $P = T^*T$ is the projection on $\mathcal{M} = \overline{Ran(|T|)}$. Thus the space $\mathcal{M}$ reduces T and $T|_{\mathcal{M}}$ is an isometry. This means that $T = S \oplus 0$, where S is an isometry. Conversely, suppose $T = S \oplus 0$, where S is an isometry. Then

$$T^*TT = (S^*S \oplus 0)(S \oplus 0) = S \oplus 0 = T = (S \oplus 0)(S^*S \oplus 0) = TT^*T.$$

We now give a result on normal and subnormal partial isometries.

Theorem 3.29 *Let T be an operator on a Hilbert space $\mathcal{H}$. Then*
(i) T is normal partial isometry if and only if T is the direct sum of a unitary operator and zero.
(ii) T is subnormal partial isometry if and only if T is the direct sum of an isometry and zero.

Proof. (i). Since $T^*T = TT^*$ and $Ker(T)^\perp$ coincides with $Ran(T)$ and therefore $T|_{Ker(T)^\perp}$ is unitary, then $T = U \oplus 0$ on $Ker(T)^\perp \oplus Ker(T)$. The proof of the converse is obvious and we leave it.
(ii) If T is subnormal, then T is hyponormal. That is, $T^*T \geq TT^*$, so $Ker(T)^\perp \supset Ran(T)$. It follows that $Ker(T)^\perp$ is invariant under T, and hence it reduces T. Clearly $T|_{Ker(T)^\perp}$ is an isometry, so $T = S \oplus 0$ on $Ker(T)^\perp \oplus Ker(T)$, where S is an isometry. The converse follows from [25, §2.6.2].
We introduce the following useful definition.

Definition 3.4

Let $\mathcal{H}$ be a Hilbert space and $A \in B(\mathcal{H})$. An operator A is *left invertible* if it has a left inverse $X \in B(\mathcal{H})$ such that $XA = I$. An operator A is *right invertible* if it has a right inverse such that $AX = I$. An operator is said to be *semi-invertible* if it is left *or* right invertible.
We note that this definition makes sense only in an infinite dimensional Hilbert space since in finite dimensions every semi-invertible operator is invertible.

Theorem 3.30 *For any $T \in B(\mathcal{H})$ the following statements are equivalent:*
(a) T is left invertible,
(b) T is injective and $Ran(T)$ is closed,
(c) T^* is right invertible,
(d) T^* is surjective,
(e) $0 \notin \sigma_{ap}(T)$.

Remark 3.22

Recall that the spectrum of an operator is never an empty set. Note that if T is a finite-dimensional operator (i.e. $Ran(T)$ is finite-dimensional, if $\mathcal{H}$ is finite-dimensional), then $\sigma(T) = \sigma_P(T)$. A special example of a left invertible operator is an isometry since $T^*T = I$.

Definition 3.5

The *deficiency* of a left invertible operator T is $dim(Ran(T)^{\perp})$. The deficiency of a right invertible operator T is $dim(Ker(T))$. The deficiency of a semi-invertible operator is a nonnegative integer or $+\infty$. A semi-invertible operator is invertible if and only if its deficiency is zero. An isometry S is a unilateral shift if S^{*n} converges to 0 in the strong operator topology. The deficiency of a unilateral shift S is usually called the *multiplicity of* S. Equivalently, we take the multiplicity of a shift operator $S \in B(\mathcal{H})$ to be the minimum dimension of a cyclic subspace for S. Usually, multiplicity of S is $dim\Big(Ker(S^*)\Big)$ (cf. [61]). For any Hilbert space $\mathcal{H}$, the multiplicity of the shift operator

$$S(c_0, c_1, c_2, ...) = (0, c_0, c_1, c_2, ...)$$

on $\ell^2(\mathcal{H})$ is equal to the dimension of $\mathcal{H}$, since a simple calculation shows that $Ker(S^*) = \mathcal{H}$.

We now use the deficiency of a contraction operator to characterize its direct summands.

Proposition 3.31 *An isometry $T \in B(\mathcal{H})$ with deficiency 0 is a unitary operator.*

Proof. Suppose that T is an isometry with deficiency zero. Then $T^*T = I$ and $Ker(T) = \{0\}$. Since T has deficiency zero, $Ran(T) = \mathcal{H}$. Thus T is injective and surjective and hence invertible. This implies that $T^*T = I = TT^*$. Hence T is a unitary operator.

Remark 3.23

From Proposition 3.31, an isometry with deficiency zero has no nontrivial c.n.u. direct summand. This result gives a condition under which an isometry turns out to be unitary. Next we investigate isometries with nonzero deficiency indices and their decompositions.

Proposition 3.32 *(i) Every isometry with nonzero deficiency is a direct sum of a unilateral shift and a unitary operator.*
(ii) Two shifts are unitarily equivalent if and only if they have the same multiplicity.

Proof. Note that (i) gives the von Neumann-Wold decomposition and the proof follows from Proposition 2.21, Lemma 3.4 and [45, § 5.1].
(ii) Let $S_j \in B(\mathcal{H}_j)$ $j = 1,2$ be the shift operators. If S_1 and S_2 have the same multiplicity, then the subspaces $\mathcal{K}_j = Ker(S_j^*)$, $j = 1,2$, have the same dimension. Hence there is an isometry W_0 which maps $\mathcal{K}_1$ onto $\mathcal{K}_2$. For any $f \in \mathcal{H}_1$, define

$$Wf = \sum_0^\infty S_2^j W_0 k_j \quad if \quad f = \sum_0^\infty S_1^j k_j.$$

(This is because each $f \in \mathcal{H}$ has a unique representation $f = \sum_0^\infty S^j k_j$, where $k_j \in \mathcal{K}$, $j \geq 0$). Then W is an isomorphism (injective and surjective and indeed unitary by construction) from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $S_2 W = W S_1$. Thus S_1 and S_2 are unitarily equivalent. In the other direction, if S_1 and S_2 are unitarily equivalent, then $Ker(S_1^*)$ and $Ker(S_2^*)$ are isomorphic (and hence of the same dimension). Hence S_1 and S_2 have the same multiplicity.

Remark 3.24

Just like the unilateral shifts, all bilateral shifts of the same multiplicity are unitarily similar.
We now study the *universal model* of bounded linear operators.
Shift operators are very applicable in operator theory owing to the following remarkable property [61]: up to unitary equivalence and multiplicative constants, the class of operators $T = S^*|_{\mathcal{M}}$, where S is a shift operator and $\mathcal{M}$ is an invariant subspace for S^*, includes every bounded linear operator on a Hilbert space. Using [61] we give the following result.

Theorem 3.33 [61] *Let $T \in B(\mathcal{H})$ such that $\|T\| \leq 1$ and $\|T^n x\| \longrightarrow 0$ for each $x \in \mathcal{H}$. Let S be a shift operator on a Hilbert space $\mathcal{G}$ of multiplicity $\geq dim(\overline{(I - T^*T)\mathcal{H}})$. Then there exists an invariant subspace $\mathcal{M}$ of S^* such that T is unitarily equivalent to $S^*|_{\mathcal{M}}$.*

Remark 3.25

Theorem 3.33 says that any $T \in C_{0\cdot}$ is unitarily equivalent to a part of a backward unilateral shift. If $T \in B(\mathcal{H})$ does not satisfy the hypothesis of Theorem 3.33, then cT will satisfy the hypotheses for any $c \neq 0$ such that $\|cT\| < 1$. In this case, it is necessary to choose a shift operator S whose multiplicity is $dim(\mathcal{H})$.

Remark 3.26

We now show that the study of general contractions can be reduced to the study of the completely nonnormal (c.n.u.) contractions. First we need the following result.

Corollary 3.34 *For every contraction $T \in B(\mathcal{H})$ there exist reducing subspaces $\mathcal{H}_0$, $\mathcal{H}_1$ for T such that*
(i) $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$
(ii) $T|_{\mathcal{H}_1}$ is completely nonunitary, and
(iii) $T|_{\mathcal{H}_0}$ is a unitary operator.
The spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ are uniquely determined by conditions (i)-(iii).

Remark 3.27

Corollary 3.34 shows that the study of general contractions can be reduced in many cases to the study of the completely non-unitary ones. Clearly a unilateral shift is a completely nonunitary operator and we use it to study general contractions.
It is important to understand the structure and relative position of the invariant subspaces of an isometry. Recall that an isometry $V \in B(\mathcal{H})$ is a unilateral shift if there is a closed subspace $\mathcal{M} \subset \mathcal{H}$ (called a *wandering space*) such that the spaces $\{V^n\mathcal{M}\}_0^\infty$ are mutually orthogonal (that is, $\cap_{n=0}^\infty V^n\mathcal{H} = \{0\}$) and

$$H = \bigoplus_{n=0}^{\infty} V^n\mathcal{M}.$$

The dimension of the subspace $\mathcal{M}$ is called the *multiplicity* of V.
We show that the direct sum decomposition in Proposition 3.1 and Lemma 3.4 is unique.

Theorem 3.35 *Let V be an isometry on the Hilbert space $\mathcal{H}$. Then there exists a unique reducing subspace $\mathcal{H}_0$ for V such that*
(i) $V|_{\mathcal{H}_0}$ is a unilateral shift, and
(ii) $V|_{\mathcal{H}\ominus\mathcal{H}_0}$ is a unitary operator.

Proof. The sequence of subspaces $\{V^n\mathcal{H}\}_{n=0}^{\infty}$ is obviously decreasing so that we have

$$\mathcal{H} = \left(\oplus_{n=0}^{\infty} (V^n\mathcal{H} \ominus V^{n+1}\mathcal{H})\right) \oplus \left(\cap_{n=0}^{\infty} V^n\mathcal{H}\right);$$

We set $\mathcal{H}_0 = \oplus_{n=0}^{\infty}(V^n\mathcal{H} \ominus V^{n+1}\mathcal{H}) = \oplus_{n=0}^{\infty}V^n\mathcal{M}, \quad \mathcal{M} = \mathcal{H} \ominus V\mathcal{H}$. Thus $\mathcal{M}$ is a wandering subspace and $V|_{\mathcal{H}_0}$ is a shift. From the above equality, $\mathcal{H}_0$ is reducing and

$$\mathcal{H} \ominus \mathcal{H}_0 = \cap_{n=0}^{\infty}V^n\mathcal{H}.$$

Thus $V(\mathcal{H} \ominus \mathcal{H}_0) = \cap_{n=1}^{\infty}V^n\mathcal{H} = \cap_{n=0}^{\infty}V^n\mathcal{H} = \mathcal{H} \ominus \mathcal{H}_0$ and $\mathcal{H}_0$ has all the properties (i) and (ii). Now $V|_{\mathcal{H}_0}$ is completely nonunitary so that the uniqueness of $\mathcal{H}_0$ follows. This proves the result.

Corollary 3.36 *An isometry $V \in B(\mathcal{H})$ is a unilateral shift if and only if* $\lim_{n\to\infty} \|V^{*n}x\| = 0$ *for all $x \in \mathcal{H}$.*

Proof. Assume V is a shift so that

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} V^n\mathcal{M}, \quad \mathcal{M} \subset \mathcal{H}.$$

Then $V^{*n}x = 0$ for $x \in V^n\mathcal{M}$. Since the sequence $\{V^{*n}\}_{n=1}^{\infty}$ is bounded in norm and the spaces $\{V^n\mathcal{M}\}_{n=0}^{\infty}$ span $\mathcal{H}$, it follows that $\lim_{n\to\infty} \|V^{*n}x\| = 0$ for every $x \in \mathcal{H}$. Conversely, if V is not a shift and $\mathcal{H}_0$ is as in Theorem 3.35, then $\|V^{*n}x\| = \|x\|$ for every $x \in \mathcal{H} \ominus \mathcal{H}_0$. The corollary follows.

Remark 3.28

We give results characterizing c.n.u. contractions of class $C_{\cdot 0}$ in terms of their multiplicities. These results apply to hyponormal, quasihyponormal and paranormal contractions and to operators that have the Putnam-Fuglede (PF) property. Recall from Corollary 3.8 that the c.n.u. parts of these contractions are of class $C_{\cdot 0}$. We use $\mu(T)$ to denote the multiplicity of T. First we need the following result for a general operator T.

Lemma 3.37 *Let $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ and $X \in B(\mathcal{H}, \mathcal{K})$ be such that $SX = XT$ and $\overline{X\mathcal{H}} = \mathcal{K}$. Then $\mu(S) \leq \mu(T)$.*

Proof. If $\mathcal{M} \subset \mathcal{H}$, $card(\mathcal{M}) = \mu(T)$ and $\bigvee_{n=0}^{\infty} T^n\mathcal{M} = \mathcal{H}$, then $X\mathcal{M} \subset \mathcal{K}$, $card(X\mathcal{M}) \leq \mu(T)$, and $\bigvee_{n=0}^{\infty} S^n X\mathcal{M} = \bigvee_{n=0}^{\infty} XT^n\mathcal{M} = \overline{X\mathcal{H}} = \mathcal{K}$. Therefore $\mu(S) \leq card(X\mathcal{M}) = \mu(T)$, which was to be proved.

Remark 3.29

Lemma 3.37 says that if S and T are densely intertwined then $\mu(S) \leq \mu(T)$.

Corollary 3.38 *If $T \in B(\mathcal{H})$ is a contraction of class $C_{.0}$ then $\mu(T) \leq dim(\mathcal{D}_{T^*})$.*

Proof. It is clear that the minimal isometric dilation $U_+ \in B(\mathcal{K}_+)$ of T is a unilateral shift of multiplicity $dim(\mathcal{D}_{T^*})$. If P denotes the projection of $\mathcal{K}_+$ onto $\mathcal{H}$, we have $TP = PU_+$ so that $\mu(T) \leq \mu(U_+) = dim(\mathcal{D}_{T^*})$, by Lemma 3.37.

Lemma 3.39 *For every operator $T \in B(\mathcal{H})$ of class $C_{.0}$ there exists a unilateral shift $U \in B(\mathcal{K})$ and an operator $X \in B(\mathcal{K}, \mathcal{H})$ with dense range such that*
(i) $TX = XU$; and
(ii) The multiplicities of U and T are equal.

Proof. Let $U_+ \in B(\mathcal{H}_{1+})$ be the minimal isometric dilation of T, and choose a set $\mathcal{M} \subset \mathcal{H}$ such that $card(\mathcal{M}) = \mu(T)$ and $\bigvee_{n=0}^{\infty} T^n\mathcal{M} = \mathcal{H}$. Define $\mathcal{K} = \bigvee_{n=0}^{\infty} U_+^n\mathcal{M}$, $U = U_+|_{\mathcal{K}}$ and $X = P_{\mathcal{H}}|_{\mathcal{K}}$. The relation $TX = XU$ follows because $TP_{\mathcal{H}} = P_{\mathcal{H}}U_+$. Then $(\overline{X\mathcal{K}}) = \bigvee_{n=0}^{\infty} XU_+^n\mathcal{M} = \bigvee_{n=0}^{\infty} T^nX\mathcal{M} = \bigvee_{n=0}^{\infty} T^n\mathcal{M} = \mathcal{H}$ so that X has dense range. Thus U is a unilateral shift (as the restriction of a unilateral shift) and $\mu(U) \leq card(\mathcal{M}) = \mu(T)$. The opposite inequality $\mu(T) \leq \mu(U)$ follows from Lemma 3.37.

Remark 3.30

By Lemma 3.39, quasisimilar completely non-unitary contractions have the same multiplicity and hence are unitarily equivalent. Duggal and Kubrusly [16] have shown that c.n.u paranormal operators are of class $C_{.0}$. The preceding results can be used to characterize the c.n.u. parts of such contractions.
Let $T \in B(\mathcal{H})$ be a completely non-unitary contraction with minimal unitary dilation $U \in B(\mathcal{K})$. For every polynomial $p(\lambda) = \sum_{j=0}^{n} a_j\lambda^j$ we have

$$p(T) = P_{\mathcal{H}}p(U)|_{\mathcal{H}},$$

which shows that the functional calculus $p \longrightarrow p(T)$ might be extended to more general functions p. We generalize this result and define $f(T)$ by

$$f(T) = P_{\mathcal{H}}f(U)|_{\mathcal{H}}, \quad f \in L^{\infty}.$$

While the mapping $f \longrightarrow f(T)$ is linear, it is not generally multiplicative, and it is convenient to find a subalgebra in L^∞ on which the functional calculus is multiplicative. It turns out there is a unique maximal algebra that will do the job for all operators T; this algebra is $\mathbb{H}^\infty$, which is the set of all bounded analytic functions on $\mathbb{D}$. Every function $u \in \mathbb{H}^\infty$ can be extended a.e. on $\partial\mathbb{D}$ via taking radial limits:

$$u(e^{it}) = \lim_{r \to 1} u(re^{it}).$$

$\mathbb{H}^\infty$ is a subalgebra of L^∞. This follows easily since $\mathbb{H}^\infty$ is a subspace of L^∞ consisting of all bounded analytic functions u in $\mathbb{D}$ with norm $\|u\|_\infty = \sup_{|z|<1} |u(z)|$.

Lemma 3.40 *If $T \in B(\mathcal{H})$ is of class C_0 then T is of class $C_{.0}$.*

Proof. We need to prove that $\lim_{n\to\infty} \|T^{*n}x\| = 0$ for each $x \in \mathcal{H}$. Assume that $T \in C_0$. Then $u(T) = 0$ for some $u \in \mathbb{H}^\infty \backslash \{0\}$. Let $U_+ \in B(\mathcal{K}_+)$ be the minimal isometric dilation of T, where $\mathcal{K}_+ = \bigvee_0^\infty U^n\mathcal{H}$. Let $\mathcal{R} = \bigcap_{n=0}^\infty U_+^n\mathcal{H}$ be the residual part of $\mathcal{K}_+$, and $A = U_+|_\mathcal{R}$. We have

$$T(P_\mathcal{H}|_\mathcal{R}) = (P_\mathcal{H}|_\mathcal{R})A.$$

Consequently, we have

$$(P_\mathcal{H}|_\mathcal{R}) = u(T)(P_\mathcal{H}|_\mathcal{R}) = 0.$$

By the Fisher-Riesz theorem, the function $u(\xi)$ defined by $u(\xi) = \lim_{r\to1} u(r\xi)$ is different from zero for almost every $\xi \in \partial\mathbb{D}$ and by the spectral theorem $u(A) = \lim_{r\to1} u(rA) = \lim_{r\to1} \sum_0^\infty a_n r^n A^n$ has dense range. Thus $P_\mathcal{H}|_\mathcal{R} = 0$ and therefore $P_\mathcal{R}|_\mathcal{H} = 0$, and

$$0 = \|P_\mathcal{R}x\| = \|\lim_{n\to\infty} U_+^n T^{*n}x\| = \lim_{n\to\infty} \|T^{*n}x\|.$$

This proves the lemma.

The following result is a consequence of Lemma 3.40.

Corollary 3.41 *Every operator of class C_0 is also of class C_{00}.*

Proof. Let T be of class C_0. By Lemma 3.40, T is of class $C_{.0}$. The corollary follows from Lemma 3.40 applied to T^*.

Remark 3.31

We now study the relationship of the characteristic function of a C_0 contraction T and the characteristic functions of the direct summands or restrictions of T to invariant subspaces. Recall that a function $u \in \mathbb{H}^\infty$ is *inner* if $|u(e^{it})| = 1$ almost everywhere on $\partial\mathbb{D}$. Recall also that the inner function v such that $v\mathbb{H}^\infty = \{u \in \mathbb{H}^\infty : u(T) = 0\}$ is called the *minimal function* of T. We denote the minimal function of T by m_T. Let θ and θ' be two functions in $\mathbb{H}^\infty$. We say that θ *divides* θ', denoted $\theta|\theta'$ if θ' can be written as $\theta' = \theta.\phi$ for some $\phi \in \mathbb{H}^\infty$. Clearly, if θ and θ' are inner, then such ϕ must be an inner function.

Proposition 3.42 *Let $T \in B(\mathcal{H})$ be a c.n.u. contraction, $\mathcal{M}$ be an invariant subspace for T, and $\mathcal{N} = \mathcal{H} \ominus \mathcal{M}$. Let $T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$ be the matrix of T with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$. Then T is of class C_0 if and only if T_1 and T_2 are operators in class C_0. If T is of class C_0 then $m_{T_1}|m_T$ and $m_{T_2}|m_T$, and $m_T|m_{T_1}m_{T_2}$.*

Proof. We have $u(T) = \begin{pmatrix} u(T_1) & * \\ 0 & u(T_2) \end{pmatrix}$ for every $u \in \mathbb{H}^\infty$. If $u(T) = 0$, we conclude that $u(T_1) = 0$ and $u(T_2) = 0$ so that T_1 and T_2 are of class C_0 and m_{T_1} and m_{T_2} divide m_T.

Conversely, assume that T_1 and T_2 are of class C_0, $\theta_1 = m_{T_1}$ and $\theta_2 = m_{T_2}$. If $x_2 \in \mathcal{N}$ we have $0 = \theta_2(T_2)x_2 = P_{\mathcal{N}}\theta_2(T)x_2$ and therefore $\theta_2(T)x_2 \in \mathcal{M}$. Consequently, $(\theta_1\theta_2)(T)x_2 = \theta_1(T_1)\theta_2(T)x_2 = 0$. Since $(\theta_1\theta_2)(T)|_{\mathcal{M}} = \theta_2(T_1)\theta_1 = 0$ we conclude that $Ker(\theta_1\theta_2)(T) \supset \mathcal{M} \cup \mathcal{N}$ and this clearly implies that $(\theta_1\theta_2)(T) = 0$. We conclude that T is an operator of class C_0 and m_T divides $\theta_1\theta_2$. That is, $m_T|m_{T_1}m_{T_2}$.

Remark 3.32

We note that we do not in general have $m_T \equiv m_{T_1}m_{T_2}$. For consider $T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Clearly $T = T_1 \oplus T_2 = 0 \oplus 0$. A simple computation shows that $m_T = t$ while $m_{T_1} = t$ and $m_{T_2} = t$. Thus in this case $m_T \not\equiv m_{T_1}m_{T_2}$. We note that equality holds only when $\mathcal{M}$ is a hyperinvariant subspace for T.

Proposition 3.43 *Let $T \in B(\mathcal{H})$ be an operator of class C_0 and let θ be an inner divisor of m_T. If $T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$ is the matrix of T with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, with $\mathcal{M} = Ker(\theta(T))$, then $m_{T_1} \equiv \theta$ and $m_{T_2} \equiv m_T/\theta$.*

Proof. We have $\theta(T_1) = \theta(T)|_{Ker(\theta(T))} = \theta(T|_{\{x:\theta(T)x=0\}}) = \theta|_{\{0\}} = 0$ since θ is isometric on $\partial\mathbb{D}$. This shows that $m_{T_1}|\theta$. It is also clear that $\{0\} = m_T(T)\mathcal{H} = \theta(T)(m_T/\theta)(T)\mathcal{H}$ so that

$$(m_T/\theta)(T)\mathcal{N} \subset (m_T/\theta)(T)\mathcal{H} \subset Ker(\theta(T)) = \mathcal{N}$$

and consequently $(m_T/\theta)(T_2) = P_{\mathcal{N}}(m_T/\theta)(T)|_{\mathcal{N}} = 0$. We have $m_{T_1}|\theta$, $m_{T_2}|(m_T/\theta)$ and by Proposition 3.42, $\theta(m_T/\theta) = m_T|m_{T_1}m_{T_2}$. These relations imply $m_{T_1} \equiv \theta$ and $m_{T_2} \equiv m_T/\theta$.

The following is a consequence of Proposition 3.42.

Lemma 3.44 *Assume that $T \in B(\mathcal{H})$ is a contraction and $\mathcal{M}$ is an invariant subspace for T and $\mathcal{N} = \mathcal{H} \ominus \mathcal{M}$, and $T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$ is the triangulation of T with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$. If T is of class C_{00} then both T_1 and T_2 are of class C_{00}.*

Proof. Assume $T \in C_{00}$. Then we have $T_1^n = T^n|_{\mathcal{M}}$, $T_1^{*n} = P_{\mathcal{M}}T^{*n}|_{\mathcal{M}}$, $T_2^{*n} = T^{*n}|_{\mathcal{N}}$, and $T_2^n = P_{\mathcal{N}}T^n|_{\mathcal{N}}$ for $n \geq 1$. It clearly follows that T_1 and T_2 are of class C_{00}.

Alternative Proof. $T \in C_{00}$ implies that $T^n \longrightarrow 0$ and $T^{*n} \longrightarrow 0$ as $n \longrightarrow \infty$. Thus

$$\lim_{n\to\infty} (T^*T)^n = \lim_{n\to\infty} T_1^{*n}T^n \oplus T_2^{*n}T_2^n = 0 \oplus 0,$$

which implies that $T_1^n \longrightarrow 0$ and $T_2^n \longrightarrow 0$ as $n \longrightarrow \infty$. Thus both T_1 and T_2 are in class C_{00}.

The following results show that we can generalize the results on contractions to general operators. We use the fact that every operator suitably multiplied by a positive scalar becomes similar (in fact unitarily equivalent) to a part of a canonical backward unilateral shift[63].

Proposition 3.45 *Every part of a unilateral shift is again a unilateral shift.*

Remark 3.33

If $\mathcal{M}$ is an invariant subspace for S_+ (so that $S_+|_{\mathcal{M}}$ is a unilateral shift), then $\mathcal{M}^\perp$ is an invariant subspace for S_+^*. It is then natural to ask what kind of operator is $S_+^*|_{\mathcal{M}^\perp}$. More generally, which operators are parts of a backward unilateral shift? The answer to this question was given by Rota [63] and it was as surprising as it was remarkable: all operators, up to a scaling factor and up to similarity. That is, every operator suitably multiplied by a positive scalar becomes similar (in fact unitarily equivalent) to a part of a canonical backward unilateral shift. In other words, *canonical backward unilateral shifts are universal models.* This is Rota's Theorem. This gives us the motivation to give the following results which are consequences of Theorem 3.33.

Theorem 3.46 *Let T be an operator on a Hilbert space $\mathcal{H}$. If $r(T) < 1$, then T is similar to a part of the canonical backward unilateral shift on $\ell_+^2(\mathcal{H})$.*

Proof. Suppose $r(T) < 1$, or, equivalently, $\sum_{k=0}^{\infty} \|T^k\|^2 < \infty$, and set $W : \mathcal{H} \longrightarrow Ran(W) \subseteq \ell_+^2(\mathcal{H})$ as follows.

$$Wx = \bigoplus_{k=0}^{\infty} T^k x$$

so that

$$\|x\|^2 \leq \sum_{k=0}^{\infty} \|T^k x\| = \|Wx\|^2 \leq (\sum_{k=0}^{\infty} \|T^k\|^2)\|x\|^2$$

for all $x \in \mathcal{H}$. Therefore W is a bounded linear transformation which is also bounded below. Thus $Ran(W)$ is closed in $\ell_+^2(\mathcal{H})$ and hence $Ran(W)$ is a subspace of $\ell_+^2(\mathcal{H})$. Let S_+ be the canonical unilateral shift on $\ell_+^2(\mathcal{H})$ so that $S_+^* \overline{x} = \oplus_{k=0}^{\infty} x_{k+1}$ for all $\overline{x} = \oplus_{k=0}^{\infty} x_k \in \ell_+^2(\mathcal{H})$. Note that

$$WTx = \bigoplus_{k=0}^{\infty} T^{k+1} x = S_+^* Wx$$

for all $x \in \mathcal{H}$. Thus $Ran(W)$ is an invariant subspace for S_+^*, and hence $S_+^*|_{Ran(W)} : Ran(W) \longrightarrow Ran(W)$ is a part of S_+^* such that

$$T = W^{-1}(S_+^*|_{Ran(W)})W.$$

This proves the result.

Corollary 3.47 *An operator T is similar to a strict contraction if and only if $r(T) < 1$.*

Proof. The proof is the same as in Theorem 3.46.

Remark 3.34

Corollary 3.47 characterizes similarity to a strict contraction. By Halmos [32] similarity to a strict contraction is easier to handle than similarity to a general contraction. The following result is obtained by strengthening Theorem 3.33.

Theorem 3.48 *An operator T is unitarily equivalent to a part of a canonical backward unilateral shift if and only if* $\|T\| \leq 1$ *and* $T^n \longrightarrow 0$.

Proof. Use the Proof of Theorem 3.33.

Remark 3.35

We note that every operator suitably multiplied by a scalar becomes unitarily equivalent to a part of a canonical backward unilateral shift (equivalently, becomes a part of a backward unilateral shift). In other words, every operator $T \in B(\mathcal{H})$ is unitarily equivalent to a multiple of a part of the canonical backward unilateral shift S_+^* on $\ell_+^2(\mathcal{H})$, so that $S_+^* \in B(\ell_+^2(\mathcal{H}))$ is a universal model for $B(\mathcal{H})$. Since operators that are multiples of each other share the same invariant subspaces, the above result leads to a reformulation of the invariant subspace problem: Take any (nonzero) operator on a Hilbert space $\mathcal{H}$ (of dimension greater than one) and consider a multiple of it, say T, that is a strict contraction (e.g divide the original operator by the double of its own norm). Then by Theorem 3.49, T is unitarily equivalent to $S_+^*|_{\mathcal{N}}$, where S_+ is the canonical unilateral shift on $\ell_+^2(\mathcal{M})$, where $\mathcal{M}$ is any subspace of $\mathcal{H}$ such that $\overline{Ran(I - T^*T)} \subseteq \mathcal{M} \subseteq \mathcal{H}$ and $\mathcal{N} \subseteq \ell_+^2(\mathcal{M})$ is an invariant subspace for S_+^* (so that $dim(\mathcal{N}) = dim(\mathcal{H}) > 1$). Kubrusly [45] has given a generalization of these results.

Theorem 3.49 [45, *Theorem* 6.11] *A completely nonunitary contraction on a Hilbert space $\mathcal{H}$ is unitarily equivalent to a part of the direct sum of the canonical backward unilateral shift on $\ell_+^2(\mathcal{H})$ and the canonical backward bilateral shift on $\ell^2(\mathcal{H})$.*

Remark 3.36

We give results on unitary and c.n.u summands of almost-similar contraction operators. These results are due to [54].

Corollary 3.50 [54, *Corollary* 2.3] *Let $A \in B(\mathcal{H})$ and suppose that $A \overset{a.s}{\approx} S_+$, where S_+ denotes the unilateral shift of finite multiplicity. Then A is a completely non-unitary contraction such that $Re(A) \approx Q$ where Q is a quasidiagonal operator and $Re(A)$ denotes the real part of A.*

Proof. Since $A \overset{a.s}{\approx} S_+$, $A^*A = N^{-1}(S_+^*S_+)$ and $A^* + A = N^{-1}(S_+^* + S_+)N$, where N is an invertible operator. Since $S_+^*S_+ = I$, $A^*A = I$. That is, A is an isometry (indeed, a c.n.u isometry). It is clear by operator multiplication that $S_+^* + S_+$ a is quasi-diagonal operator Q. Hence $Re(A) \approx Q$.

Proposition 3.51 [54, *Proposition* 2.12] *Let $A \in B(\mathcal{H})$ such that A is almost similar to an isometry T. Then the unitary and completely non-unitary summands of A are isometric.*

Proof. Since T is an isometry, by the von Neumann-Wold decomposition ([45, § 5.2]), $T = S_+ \oplus U$, where U is unitary and S_+ is the unilateral shift. Since $A \overset{a.s}{\approx} T$, there exists an operator N such that

$$\begin{aligned} A^*A &= N^{-1}\Big[(S_+ \oplus U)^*(S_+ \oplus U)\Big]N \\ &= N^{-1}(S_+^*S_+ \oplus U^*U)N \\ &= N^{-1}(I \oplus I)N. \end{aligned}$$

Now, suppose $A = A_1 \oplus A_2$, then $A^*A = (A_1^*A_1 \oplus A_2^*A_2)$. This shows that $(A_1^*A_1 \oplus A_2^*A_2) \approx I \oplus I$. From this equation, it follows that $A_i^*A_i \approx I$, $i = 1, 2$. This means that there exists an operator N such that $A_i^*A_i = N^{-1}IN = I$. Thus $A_i^*A_i = I$. This proves that the direct summands of A are isometric.
The following is a consequence of Corollary 3.51.

Corollary 3.52 *If an operator $A \in B(\mathcal{H})$ is such that A^* is almost similar to a c.n.u. coisometry, then A has no unitary direct summand.*

Proof. Suppose $A = A_1 \oplus A_2$. By an application of the proof of Corollary 3.51, we get that the direct summands of A are unitary. But the c.n.u part of an operator cannot be unitary. This means that $A_1 = 0$ or A_1 acts on the null space $\{0\}$. Thus A has no unitary direct summand.

Corollary 3.53 [54, *Corollary* 2.13] *Let $A \in B(\mathcal{H})$ be a contraction. If A is unitarily equivalent to a unitary operator T, then A is unitary.*

Remark 3.37

Corollary 3.53 says that an operator which is unitarily equivalent to a unitary operator has no completely non-unitary direct summand.

Proposition 3.54 [54, *Proposition* 2.14] *If A, $B \in B(\mathcal{H})$ are contractions such that $A \overset{a.s}{\approx} B$ and B is c.n.u, then A is c.n.u.*

Proof. By the Nagy-Foias-Langer decomposition for contractions [45, § 5.1], $B = U \oplus C$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $U = B|_{\mathcal{H}_1}$ is the unitary part of B and $C = B|_{\mathcal{H}_2}$ is the completely non-unitary part of B. Since B is c.n.u, the unitary direct summand U is missing or $\mathcal{H}_1 = \{0\}$. Without loss of generality we suppose that $B = C$. Then $A^*A = N^{-1}(B^*B)N = N^{-1}(C^*C)N$. This shows that A^*A is similar to C^*C (i.e. $A^*A \approx C^*C$). Now suppose $A = A_1 \oplus A$. where A_1 is unitary and A_2 is c.n.u. Then $(A_1^*A_1 \oplus A_2^*A_2) \approx C^*C$. This holds if and only if the direct summand A_1 is missing. That is, $A = A_2$. Hence A is completely non-unitary.

Corollary 3.55 [54, *Theorem* 2.15] *If $A \in B(\mathcal{H})$ is normal, then $A \overset{a.s}{\approx} A^*$.*

Proof. The result follows easily from the fact that $AA^* = A^*A = N^{-1}(AA^*)N = N^{-1}(A^*A)N$ and $A + A^* = A^* + A = N^{-1}(A + A^*)N = N^{-1}(A^* + A)N$.

Remark 3.38

We note that the converse of Corollary 3.55 is not true in general, for consider $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By matrix computation gives $A^*A = N^{-1}(AA^*)N$ and $A^* + A = N^{-1}(A + A^*)N$. That is, $A \overset{a.s}{\approx} A^*$, although A is not normal.

Remark 3.39

We conjecture that Corollary 3.55 can be strengthened to the class of normal contractions as follows.

Conjecture 3.56 *If $A \in B(\mathcal{H})$ is a normal contraction and $B \in B(\mathcal{H})$ such that $A \overset{a.s}{\approx} B$, then the c.n.u part of B is of class C_{00}.*

Chapter 4

On invariant and hyperinvariant lattices of operators

In this chapter we study the invariant and hyperinvariant subspaces of some classes of operators in Hilbert spaces. We give a shot at the open invariant and hyperinvariant subspace problems : *Does every operator on an infinite dimensional Hilbert space have a nontrivial invariant (hyperinvariant) subspace* ? These two problems are both unresolved and are of importance for understanding the structure of Hilbert space operators. Invariant subspaces play a key role in studying the spectral properties and canonical forms of operators. The invariant and hyperinvariant lattices come in handy to determine whether it is possible to isolate the parts (direct summands) of a given linear operator. The basic motivations for the study of invariant subspaces come from the interest in the structure of operators . The well known Jordan-canonical-form theorem for operators on finite-dimensional spaces can be regarded as exhibiting operators (to within similarity) as direct sums of their restriction to certain invariant subspaces. The fact that every matrix on a finite-dimensional complex vector space is unitarily equivalent to an upper triangular matrix follows immediately from the existence of nontrivial invariant subspaces for operators on finite-dimensional spaces. We denote the lattice of invariant subspaces and hyperinvariant subspaces of T by $Lat(T)$ and $Hyperlat(T)$, respectively. If $\mathcal{H}$ is any Hilbert space and $T \in B(\mathcal{H})$, and $\mathcal{M} \in Lat(T)$, then the representation of

T with respect to the decomposition $\mathcal{M} \oplus \mathcal{M}^{\perp}$ of $\mathcal{H}$ is upper triangular:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

where $T_1 = T|_{\mathcal{M}}$ (the restriction of T to $\mathcal{M}$) and where T_2 and T_3 are operators mapping $\mathcal{M}^{\perp}$ into $\mathcal{M}$ and $\mathcal{M}^{\perp}$ respectively. Thus there are various relations between the structure of an operator T and $Lat(T)$. Also, knowledge of hyperinvariant subspaces of T can give information about the structure of the commutant of T. The commutant of an operator T is very useful since it contains all quasiaffine transforms of an operator and its very nature reveals information about operators quasisimilar, similar, or unitarily equivalent to T. Recall that a complete lattice is a partially ordered set(poset) in which all subsets have both supremum(join) and an infimum(meet). We show that if the algebra generated by T coincides with the commutant of T, then every invariant subspace is hyperinvariant. In particular, this is the case for injective unilateral shifts and in general any completely non-normal (or completely non-unitary) operator. Since the set of all the invariant subspaces of an operator T can be partially ordered by inclusion, then $Lat(T)$ is a complete lattice.

The following definitions will be useful in the rest of this chapter.

Let $T \in B(\mathcal{H})$. Then $Lat(T) = \{\mathcal{M} \subset \mathcal{H} : T\mathcal{M} \subset \mathcal{M}\}$, and $HyperLat(T) = \{\mathcal{M} \subset \mathcal{H} : S\mathcal{M} \subset \mathcal{M}.$ *whenever* $S \in B(\mathcal{H})$ *commutes with* $T\}$. If $\mathcal{M} \in Lat(T)$, then T has an upper triangular form relative to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$:

$$T = \begin{pmatrix} T|_{\mathcal{M}} & P_{\mathcal{M}}T|_{\mathcal{M}^{\perp}} \\ 0 & P_{\mathcal{M}^{\perp}}T|_{\mathcal{M}^{\perp}} \end{pmatrix},$$

where $P_{\mathcal{M}}$ denotes the orthogonal projection of $\mathcal{H}$ onto the subspace $\mathcal{M}$. Such a representation is called a triangulation of T.

We investigate the structure of invariant and hyperinvariant lattices of operators sharing a certain spectral property: quasisimilar, similar, unitarily equivalent, with a view to determining whether we can isolate the direct summands of one operator if the other operator enjoys this property. We will study a subset of $Lat(T)$, which we denote by $Red(T)$, of reducing subspaces for which T decomposes as a direct sum of two complementary parts. We note that $Red(T)$ is not in general a lattice. This is because both $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are in $Red(T)$ but the pair $\{\mathcal{M}, \mathcal{M}^{\perp}\}$ has no supremum and infimum in

$Red(T)$.

We show that if two operators are quasisimilar and if one of them has a nontrivial hyperinvariant subspace, then so has the other. That is quasisimilarity preserves nontrivial hyperinvariant subspaces. It is well known (see [8]) that similarity of operators preserves compactness, cyclicity, algebraicity, and the spectral picture (i.e. the spectrum, essential spectrum, and index function), and that similar operators have isomorphic lattices of invariant subspaces and hyperinvariant subspaces. That is, similarity of operators (which implies quasisimilarity of operators) not only preserves nontrivial hyperinvariant subspaces but also nontrivial invariant subspaces.

Definition 4.1

An operator $T \in B(\mathcal{H})$ is *cyclic* if there exists a vector $x \in \mathcal{H}$ for which the list $\{x, Tx, T^2x, ..., T^{n-1}x\}$ spans (and is therefore a basis) for a finite-dimensional Hilbert space $\mathcal{H}$.

Example 4.1

It is clear that if T has an eigenvalue, then the corresponding eigenspace is an invariant subspace. Since every operator $T : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ has eigenvalues it follows that T has nontrivial invariant subspaces whenever $n \geq 2$. However, on $\mathcal{H}$ there are many linear operators that do not have eigenvalues, e.g. the unilateral shift operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ defined by $T(a_1, a_2, a_3, ...) = (0, a_1, a_2, a_3, ...)$ as proved in Example 2.1. However, the unilateral shift T has nontrivial invariant subspaces. To prove this , we let $\mathcal{M}_n$ be the subspace of ℓ^2 of square summable sequences such that the first n components are zero. That is

$$\mathcal{M}_n = span\Big\{(0, 0, 0, ..., 0, a_{n+1}, a_{n+2}, ...)\Big\}.$$

Then it is clear that for each $x \in \mathcal{M}_n$, we have $Tx \in \mathcal{M}_{n+1} \subseteq \mathcal{M}_n$. Thus $\mathcal{M}_n$ is an invariant subspace for T. We show that the unilateral shift has plenty of invariant subspaces, by using the Fourier transform. We define the Hardy space $\mathbb{H}^2$ to be the space of complex-valued analytic functions on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \subseteq \mathbb{C}$. More precisely, we set

$$\mathbb{H}^2 = \{f : \mathbb{D} \longrightarrow \mathbb{C} : f(z) = \sum_{n=0}^{\infty} a_n z^n \ for \ z \in \mathbb{D}, \ \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

and determine the Fourier transform from $\mathcal{H}$ to $\mathbb{H}^2$ by

$$(a_1, a_2, a_3, ...) \longrightarrow f,$$

where

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Note that the Hardy space $\mathbb{H}^2$ contains all the polynomials. On the space $\mathbb{H}^2$ we can define the operator of multiplication by z as

$$(M_z f)(z) = zf(z), \quad f \in \mathbb{H}^2.$$

Then

$$M_z(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n z^{n+1} = \sum_{n=0}^{\infty} a_{n-1} z^n,$$

and it is easy to see that the Fourier transform provides a unitary equivalence between T and M_z. Now let $\{c_1, c_2, ..., c_n\}$ be a finite subset of $\mathbb{D}$. Then we can set

$$\mathcal{M} = \{f \in \mathbb{H}^2 : f(c_i) = 0, \ i = 1, 2, 3, ..., n\}.$$

It is easy also to see that $\mathcal{M}$ is a closed subspace of $\mathbb{H}^2$ and it is clear that $\mathcal{M}$ is invariant for M_z. It is also easy to see that $\mathcal{M}$ is a nontrivial subspace of $\mathbb{H}^2$. In fact, since $1 \in \mathcal{M}$ we have $\mathcal{M} \neq \mathbb{H}^2$, and we also note that

$$p(z) = \prod_{i=1}^{n} (c_i - z)$$

is a polynomial in $\mathcal{M}$, $p \neq 0$, hence $\mathcal{M} \neq \{0\}$. Thus we have a new invariant subspace for the unilateral shift T, and one can go on one step further: Let $\{c_1, c_2, ...\}$ be an infinite subset of $\mathbb{D}$, and set

$$\mathcal{M} = \{f \in \mathbb{H}^2 : f(c_i) = 0, \ i = 1, 2, 3, ...\}.$$

As before it is easily seen that $\mathcal{M} \subseteq \mathbb{H}^2$ is an invariant subspace of M_z, and that $\mathcal{M} \neq \mathbb{H}^2$. However, in this case, it is not clear that $\mathcal{M} \neq \{0\}$. It is easy to check that also, $\{0\}$ and $\mathbb{R}^n$ are always T-invariant and $span\left\{v_1, v_2, ..., v_m\right\}$ is T-invariant, where v_i are eigenvectors of T. Also, if T has the block upper triangulation $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$

with $T_{11} \in \mathbb{R}^{r \times r}$, then $\mathcal{M} = \left\{ \begin{pmatrix} z \\ 0 \end{pmatrix} : z \in \mathbb{R}^r \right\}$ is T-invariant.

Suppose $\mathcal{M}$ is a T-invariant subspace. Suppose that we pick a basis $\beta = \{v_1, v_2, ..., v_k\}$ of $\mathcal{M}$ and complete it to a basis of $\mathcal{H}$. Then with respect to this basis, the matrix representation of T takes the form $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$, where $T_{11} = T|_{\mathcal{M}}$, $T_{22} = T|_{\mathcal{M}^{\perp}}$ and $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. The operator $T_{12} = 0$ if and only if $\mathcal{M}$ reduces T, in which case the operator T is decomposed (reduced) into the (orthogonal) direct sum of the operators T_{11} and T_{22}: $T = T_{11} \oplus T_{22}$.

Thus, on a finite-dimensional complex Hilbert space every operator has an eigenvalue, and eigenspaces of non-scalar operators are nontrivial and hyperinvariant, so that every operator on a finite-dimensional complex Hilbert space of dimension greater than 1 has a nontrivial invariant subspace (hyperinvariant , actually, if it is non-scalar). The invariant subspace problem trivially has a negative answer in a real space. For instance, the operator $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbb{R}^2$ has no nontrivial invariant subspace (when acting on the Euclidean *real* space but, of course, it has a nontrivial invariant subspace when acting on the *complex* space $\mathbb{C}^2$). This is in general the case with rotations of a two-dimensional real vector space.

We need the following known results which are proved in [18].

Theorem 4.1 [18] *Let T be a k-th root of a p-hyponormal operator. If T is compact or T^n is normal for some integer $n > k$, then T is a (generalized) scalar operator.*

Corollary 4.2 [18, *Corollary* 2.2]. *Let $T \in B(\mathcal{H})$ be a k-th root of a p-hyponormal operator. If T is compact or T^n is normal for some integer $n > k$, then T has hyperinvariant subspaces.*

Definition 4.2

Let T, A, $B \in B(\mathcal{H})$. We say that T *intertwines* the pair (A, B) if $TA = BT$. If T intertwines both (A, B) and (B, A), we say that T doubly intertwines A and B.

Remark 4.1

We investigate the relationship of the invariant subspaces of such operators A and B, when T is an arbitrary operator and when T is a quasiaffinity. We try to answer the question whether quasisimilarity preserves nontrivial invariant subspaces.

Lemma 4.3 *If $T \in B(\mathcal{H})$ doubly intertwines A and B and $Lat(A) \cap Lat(B)$ is trivial then T is either 0 or a quasiaffinity. The same is true if T commutes with A and B and $Lat(A) \cap Lat(B)$ is trivial.*

Proof. T doubly commutes the pair (A, B) implies that $TA = BT$ and $TB = AT$. Since $TA = BT$ then $\overline{Ran(T)} \in Lat(B)$ and $Ker(T) \in Lat(A)$. Since $TB = AT$, we deduce that $\overline{Ran(T)} \in Lat(A) \cap Lat(B)$ and $Ker(T) \in Lat(A) \cap Lat(B)$. We run through the following two cases.

Case 1: If $\overline{Ran(T)} = \{0\}$ then $T = 0$. If $\overline{Ran(T)} = \mathcal{H}$, then $Ker(T) = \{0\}$ and hence T is one-to-one or injective and has dense range, hence a quasiaffinity.

Case 2: If T commutes with A and B, i.e. $TA = AT$ and $TB = BT$, then by the argument above, $\overline{Ran(T)} \in Lat(A) \cap Lat(B)$ and $Ker(T) \in Lat(A) \cap Lat(B)$. Thus by the same argument either $T = 0$ or T is a quasiaffinity.

Remark 4.2

The triviality of $Lat(A) \cap Lat(B)$ follows from the orthogonality of $Ker(T)$ and $\overline{Ran(T)}$. Strengthening Lemma 4.3 to similarity shows that $Lat(A)$ is isomorphic to $Lat(B)$.

Theorem 4.4 *Let $A, B \in B(\mathcal{H})$. If $A^2 = B^2$ and A has nontrivial invariant subspaces then B has nontrivial invariant subspaces.*

Proof. We prove this result by contradiction. Suppose $Lat(B)$ is trivial. That is, $Lat(B) = \{0\}$ or $\mathcal{H}$. Then $Lat(A) \cap Lat(B)$ is trivial also. Denote $T = A + B$. Then
$TA = (A + B)A = A^2 + BA, \quad BT = B(A + B) = BA + B^2;$
and
$AT = A(A + B) = A^2 + AB, \quad TB = (A + B)B = AB + B^2.$
This shows $TA = BT$ and $AT = TB$. Thus T doubly intertwines A and B.
If $T = 0$, then $A = -B$ or $A = B = 0$. Both cases imply that $Lat(B)$ is nontrivial, which is an absurdity, since we had assumed $Lat(B)$ to be trivial. We admit that $T \neq 0$, so by Lemma 4.3, T is a quasiaffinity doubly intertwining A and B. Using [58, Theorem 6.19], we deduce that B has nontrivial hyperinvariant subspaces, which is once more absurd. We conclude that $Lat(B)$ is nontrivial.

Theorem 4.5 *If A and B are nilpotent operators of nilpotency index 2 having no nontrivial common invariant subspaces, then A and B are quasisimilar.*

Proof. If $Lat(A) \cap Lat(B)$ is trivial, then $T = A + B$ is nonzero because if $T = 0$, then $Lat(A) = Lat(B)$ and nilpotent operators have nontrivial invariant subspaces. Consequently, A and B are quasisimilar since T is a quasiaffinity doubly intertwining A and B by Theorem 4.4.

The following theorem will come in handy in the sequel.

Theorem 4.6 (Spectral Theorem) [46, *Theorem* 0.14] *If $\mathcal{H}$ is a finite dimensional Hilbert space and $T \in B(\mathcal{H})$ is self-adjoint, then there exists an orthonormal basis $\varphi_1, \varphi_2, ..., \varphi_n$ for $\mathcal{H}$ and real numbers $\lambda_1, \lambda_2, ..., \lambda_n$ such that*

$$T\varphi_i = \lambda_i \varphi_i, \quad 1 \leq i \leq n.$$

The matrix $(t_{ij}) = (\langle T\varphi_j, \varphi_i \rangle)$ corresponding to T and $\varphi_1, \varphi_2, ..., \varphi_n$ is the diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

A natural question is whether this spectral theorem can be generalized to the case where T is self-adjoint and $\mathcal{H}$ is infinite dimensional. That is to say, is there an orthonormal basis $\varphi_1, \varphi_2, ...$ for $\mathcal{H}$ and numbers $\lambda_1, \lambda_2, ...$ such that

$$T\varphi_i = \lambda_i \varphi_i, \quad 1 \leq i\ ?$$

This means that the matrix corresponding to T and $\varphi_1, \varphi_2, ...$ is an infinite dimensional diagonal matrix. It is known that the spectral theorem admits an important generalization to compact self-adjoint operators. For an arbitrary operator T, the matrix is triangular. The following examples are useful in understanding the notion of an invariant subspace for T acting on infinite dimensional Hilbert spaces.

Example 4.2

- 1. Let $\{\varphi_1, \varphi_2, ...\}$ be an orthonormal basis for $\mathcal{H}$. Suppose the matrix corresponding to $T \in B(\mathcal{H})$ and $\{\varphi_n\}$ is upper triangular, that is,
  $t_{ij} = \langle T\varphi_j, \varphi_i \rangle = 0, \quad i > j.$

If the matrix (t_{ij}) is lower triangular, i.e., $t_{ij} = 0$ *if* $i < j$, then for each n, $span\{\varphi_{n+1}, \varphi_{n+2....}\}$ is an invariant subspace for T.

- 2. Define T on $L^2[a,b]$, the Hilbert space of square integrable functions on $[a,b]$, by $(Tf)(t) = \int_a^t k(t,s)f(s)ds, \qquad$ for $k \in (L^2[a,b] \times L^2[a,b])$.
  For each $t \in [a,b]$, the space
  $\mathcal{M}_t = \{f \in L^2[a,b] : f = 0 \text{ } a.e. \text{ } on \text{ } [a,t]\}$ is invariant under T.

Remark 4.3

We now investigate invariant subspaces of some classes of operators. We start with the following results for M-hyponormal operators.

Theorem 4.7 *If* $T \in B(\mathcal{H})$ *is an* M*-hyponormal operator on* $\mathcal{H}$*, then the set* $\mathcal{K} = \{x \in \mathcal{H} : \|(T^* - \bar{z}I)x\| = M\|(T - zI)x\|, \quad z \in \mathbb{C}\}$ *is a closed subspace of* $\mathcal{H}$.

Proof. For $x \in \mathcal{K}$, we have

$$\|(T^* - \bar{z}I)x\|^2 = M^2\|(T - zI)x\|^2,$$

which yields

$$\left\langle\Big(M^2(T^* - \bar{z}I)(T - zI) - (T - zI)(T^* - \bar{z}I)\Big)x, x\right\rangle \;=\; 0. \tag{4.1}$$

In view of the M-hyponormality of T, (4.1) holds if and only if

$$\Big(M^2(T^* - \bar{z}I)(T - zI) - (T - zI)(T^* - \bar{z}I)\Big)x \;=\; 0 \tag{4.2}$$

From (4.2) it follows that $\mathcal{K}$ is the kernel of the operator

$$M^2(T^* - \bar{z}I)(T - zI) - (T - zI)(T^* - \bar{z}I).$$

and since by Kubrusly [45, § 0.1], the kernel of any operator T is closed, $\mathcal{K}$ is a closed subspace as desired.

Remark 4.4

Note, similarly, that if T^* is M-hyponormal on $\mathcal{H}$, then the set $\mathcal{K} = \{x \in \mathcal{H} : \|(T - zI)x\| = M\|(T^* - \bar{z}I)x\|, \quad z \in \mathbb{C}\}$ is a closed subspace of $\mathcal{H}$. When $M = 1$ (i.e., T is a hyponormal operator), it is well known that the space $\mathcal{K}$ is an invariant subspace under

T, and the restriction $T|_{\mathcal{K}}$ is normal.

We investigate whether an M-hyponormal operator has a nontrivial invariant subspace. To do this we need to study the eigenspaces of M-hyponormal operators.

Theorem 4.8 *Suppose that the subspace $\mathcal{K}$ of $\mathcal{H}$ reduces an operator T on $\mathcal{H}$. Then T is M-hyponormal if and only if $T|_{\mathcal{K}}$ and $T|_{\mathcal{K}^{\perp}}$ are M-hyponormal.*

Proof. Let $T = T_1 \oplus T_2$, where $T_1 = T|_{\mathcal{K}}$ and $T_2 = T|_{\mathcal{K}^{\perp}}$. If T is M-hyponormal, then there exists a real number M such that

$$\|(T - zI)^*x\| \leq M\|(T - zI)x\|,$$

for all $x \in \mathcal{H}$ and for every complex number z. But on $\mathcal{K}$, $T = T_1$ and $T^* = T_1^*$. Thus for any vector $x \in \mathcal{K}$ we have

$$\|(T_1 - zI)^*x\| = \|(T - zI)^*x\| \leq M\|(T - zI)x\| = M\|(T_1 - zI)x\|.$$

This shows that T_1 is M-hyponormal.

Similarly, for $x \in \mathcal{K}^{\perp}$ we have,

$$\|(T_2 - zI)^*x\| = \|(T - zI)^*x\| \leq M\|(T - zI)x\| = M\|(T_2 - zI)x\|$$

showing that T_2 is M-hyponormal. Conversely assume that T_1 and T_2 are M-hyponormal operators. It is well known that for every $x \in \mathcal{H}$, $x = x_1 + x_2$, where $x_1 \in \mathcal{K}$ and $x_2 \in \mathcal{K}^{\perp}$. Hence for all complex z and for all vectors $x \in \mathcal{H}$ we have,

$$\begin{aligned}
\|(T - zI)^*x\|^2 &= \|(T - zI)^*x_1 + (T - zI)^*x_2\|^2 \\
&= \|(T_1 - zI)^*x_1 + (T_2 - zI)^*x_2\|^2 \\
&= \|(T_1 - zI)^*x_1\|^2 + \|(T_2 - zI)^*x_2\|^2 \\
&\leq M^2\|(T_1 - zI)x_1\|^2 + M^2\|(T_2 - zI)x_2\|^2 \\
&= M^2\|(T - zI)x_1\|^2 + M^2\|(T - zI)x_2\|^2 \\
&= M^2\|(T - zI)x\|^2
\end{aligned}$$

which proves the M-hyponormality of T.

Theorem 4.9 *Let T be an M-hyponormal operator. Then the span of all eigenvectors of T reduces T.*

Proof. We shall divide the proof into four steps.

Step 1- By the definition of M-hyponormality of T,

$$\{x \in \mathcal{H} : Tx = \lambda x\} \quad \subset \quad \{x \in \mathcal{H} : T^* x = \overline{\lambda} x\}$$

for all complex numbers λ.

Step 2- For each complex number λ , the subspace $\mathcal{K} = \{x \in \mathcal{H} : Tx = \lambda x\}$ reduces T since for any $x \in \mathcal{K}$, we have $T(Tx) = \lambda(Tx)$ which implies that Tx is in $\mathcal{K}$. Also $T(T^* x) = \overline{\lambda}(Tx) = \lambda(\overline{\lambda} x) = \lambda(T^* x)$ showing that $T^* x$ is in $\mathcal{K}$. This shows that $\mathcal{K}$ is invariant under both T and T^*. Hence $\mathcal{K}$ reduces T.

Step 3- If $\lambda_1 \neq \lambda_2$, then by [74, Proposition 2(i)], $\{x \in \mathcal{H} : Tx = \lambda_1 x\} \perp \{x \in \mathcal{H} : Tx = \lambda_2 x\}$.

*Step 4-*The span of all the eigenvectors of T reduces T and the restriction of T to that span is normal.

The proof follows from steps (1), (2), (3) and using the fact that the restriction of T to any of its eigenspaces is normal from step (2).

Remark 4.5

Note that Theorem 4.9 applies to all operator subclasses of the class of M-hyponormal operators. The subalgebra of all operators generated by an operator $T \in B(\mathcal{H})$, denoted by $W^*(T)$ will be called the (unital) weakly closed (von Neumann) algebra of T. We use this algebra to investigate the structures of the invariant and hyperinvariant lattices for various operators.

Theorem 4.10 *If an operator $A \in B(\mathcal{H})$ is in the weakly closed algebra generated by an operator $B \in B(\mathcal{H})$, then $Lat(B) \subseteq Lat(A)$.*

Proof. Since $A \in W^*(B)$, then $QP_{\mathcal{M}} = P_{\mathcal{M}}Q$ where $Q \in W^*(B)' = \{B\}' \cap \{B^*\}'$ is an orthogonal projection in $\{B\}'$, $\mathcal{M} \in Hyperlat(B)$, hence $P_{\mathcal{M}}AP_{\mathcal{M}} = P_{\mathcal{M}}A$, where $P_{\mathcal{M}} \in W^*(A)$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. This means that $\mathcal{M} \in Hyperlat(B) \subseteq Lat(B) \Longrightarrow \mathcal{M} \in Lat(A)$. This proves the result.

Remark 4.6

Theorem 4.10 can be strengthened to conclude that if $A \in W^*(B)$ then $Hyperlat(B) \subseteq Hyperlat(A)$. This follows from the fact that for any operator T, $Hyperlat(T) \subseteq Lat(T)$.

Theorem 4.11 *Similar operators have isomorphic lattices of invariant and hyperinvariant subspaces.*

Proof. Suppose $A, B \in B(\mathcal{H})$ such that $A = X^{-1}BX$. Then $XA = BX$ and $AX = XB$. The rest of the proof follows from Lemma 4.3 and Remark 4.2.

Remark 4.7

We want to show that quasisimilarity preserves hyperinvariant subspace lattices but does not in general preserve invariant subspace lattices. First we need the following results.

Proposition 4.12 *If* T_1 *and* $T_2 \in B(\mathcal{H})$ *are quasisimilar (with quasiaffinities* X *and* Y *in* $B(\mathcal{H})$*), then* $XY \in \{T_1\}'$ *and* $YX \in \{T_2\}'$.

Proof. Suppose $T_1 \sim T_2$ with quasiaffinities X and Y. Then $T_1X = XT_2$ and $T_2Y = YT_1$. Post-multiplying the first equation by Y and using the second equation we have $T_1XY = XT_2Y = XYT_1$, which proves that $XY \in \{T_1\}'$. Post-multiplying the second equation by X and using the first equation, we have $T_2YX = YT_1X = YXT_2$, which proves that $YX \in \{T_2\}'$.
We give some definitions which are useful in our next results.

Definition 4.3

A quasiaffinity X is said to have the *hereditary property with respect to an operator* $T \in B(\mathcal{H})$ if $X \in \{T\}'$ and $\overline{X(\mathcal{M})} = \mathcal{M}$ for every $\mathcal{M} \in Hyperlat(T)$.

Definition 4.4

If T_1 and T_2 are quasisimilar and there exists an implementing pair (X, Y) of quasiaffinities such that XY has the hereditary property with respect to T_1 and YX has the hereditary property with respect to T_2, then we say that T_1 is *hyper-quasisimilar* to T_2 and denote it by $T_1 \overset{h}{\sim} T_2$.

Remark 4.8

We note that hyperquasisimilarity is an equivalence relation which is strictly stronger than quasisimilarity. Clearly, from Definition 4.4. two operators T_1 and T_2 are hyper-quasisimilar if there exist quasiaffinites X and Y satisfying $XT_1 = T_2X$, $T_1Y = YT_2$, and the additional conditions that $\overline{YX\mathcal{M}_1} = \mathcal{M}_1$ and $\overline{XY\mathcal{M}_2} = \mathcal{M}_2$, for every

$\mathcal{M}_1 \in Hyperlat(T_1)$ and $\mathcal{M}_2 \in Hyperlat(T_2)$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are any two complete lattices, we write $\mathcal{L}_1 \equiv \mathcal{L}_2$ to signify that there exists an order preserving isomorphism of one onto the other. When we say that two operators have isomorphic lattices of invariant subspaces there are two things that can be meant. First, they are isomorphic as abstract lattices and second, they are isomorphic as lattices of subspaces of Hilbert space, that is, there is a bounded invertible operator from one Hilbert space onto the other that maps the first lattice onto the second.

Theorem 4.13 *If T_1 and T_2 are hyper-quasisimilar then $Hyperlat(T_1) \equiv Hyperlat(T_2)$.*

Proof. Since $T_1 \overset{h}{\sim} T_2$, we have quasiaffinities X and Y satisfying $\overline{YX\mathcal{M}_1} = \mathcal{M}_1$ and $\overline{XY\mathcal{M}_2} = \mathcal{M}_2$, for every $\mathcal{M}_1 \in Hyperlat(T_1)$ and $\mathcal{M}_2 \in Hyperlat(T_2)$. Using Proposition 4.12, $XY \in \{T_1\}'$ and $YX \in \{T_2\}'$, $\mathcal{M}_1 \in Hyperlat(T_2)$ for every $\mathcal{M}_1 \in Hyperlat(T_1)$ and $\mathcal{M}_2 \in Hyperlat(T_1)$ for every $\mathcal{M}_2 \in Hyperlat(T_2)$. This means that every hyperinvariant subspace of T_1 is a hyperinvariant subspace of T_2 and vice versa, which proves the result. □

Note that Theorem 4.13 also holds when $\equiv$ is replaced with $=$.

Theorem 4.14 *Suppose $X \in B(\mathcal{H})$ is a quasiaffinity and $0 \notin W(X)$, where $W(X)$ denotes the numerical range of X . Then X has a hereditary property with respect to every $T \in B(\mathcal{H})$ such that $X \in \{T\}'$.*

Proof. We prove the result by contradiction. Suppose that $X \in \{T\}'$ and $\mathcal{M} \in Hyperlat(T)$ such that $\overline{X\mathcal{M}} \neq \mathcal{M}$, there exists a vector $x \in \mathcal{M} \ominus \overline{X\mathcal{M}}$ and $\langle Xx, x\rangle = 0$. This is an absurdity. Hence $\overline{X\mathcal{M}} = \mathcal{M}$. Thus X has a hereditary property with respect to every $T \in B(\mathcal{H})$ such that $X \in \{T\}'$.

Corollary 4.15 *Suppose $X \in B(\mathcal{H})$ is a quasiaffinity and there exists $0 \leq \theta \leq \Pi$ such that $R = Re(e^{i\theta}X)$ is positive definite (i.e., $\langle Rx, x\rangle > 0$ for every $x \neq 0$ in $\mathcal{H}$). Then X has the hereditary property with respect to every T in $B(\mathcal{H})$ for which $X \in \{T\}'$.*

Proof. If $\langle Xx, x\rangle = 0$, then

$$\langle Rx, x\rangle = \langle \frac{1}{2}(e^{i\theta}X + e^{-i\theta}X^*)x, x\rangle = 0,$$

so $x = 0$. By Theorem 4.14, we conclude that X has the hereditary property with respect to every T in $B(\mathcal{H})$ for which $X \in \{T\}'$.

Remark 4.9

In the following result we denote by $\mathcal{H}^{(n)}$ the direct sum of n copies of $\mathcal{H}$, for any ordinal n satisfying $1 \leq n \leq \omega$. That is, $\mathcal{H}^{(n)} = \bigoplus_{0<k<n} \mathcal{H}_k$ with $\mathcal{H}_k = \mathcal{H}$ for every k.

Theorem 4.16 *Suppose $\{S_n\}_{n\in N}$ and $\{T_n\}_{n\in\mathbb{N}}$ are bounded sequences of operators in $B(\mathcal{H})$ with $\hat{S} = \bigoplus_{n\in\mathbb{N}} S_n$ and $\hat{T} = \bigoplus_{n\in\mathbb{N}} T_n$. Suppose, moreover, that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of invertible operators such that*

$$X_n^{-1} S_n X_n \;=\; T_n, \qquad n \in \mathbb{N}.$$

Then $\hat{S}$ and $\hat{T}$ are hyperquasisimilar and consequently

$$Hyperlat\hat{S} \;=\; Hyperlat\hat{T}.$$

Proof. As is well known,

$$\hat{X} = \bigoplus_{n\in\mathbb{N}} \frac{X_n}{\|X_n\|} \quad \text{and} \quad \hat{Y} = \bigoplus_{n\in\mathbb{N}} \frac{(X_n)^{-1}}{\|(X_n)^{-1}\|}$$

belong to $B(\mathcal{H}^{(\omega)})$ and satisfy $\hat{S}\hat{X} = \hat{X}\hat{T}$ and $\hat{Y}\hat{S} = \hat{T}\hat{Y}$. Moreover

$$\hat{X}\hat{Y} \;=\; \bigoplus_{n\in\mathbb{N}} \frac{1}{(\|X_n\|\;\|(X_n)^{-1}\|)} = \hat{Y}\hat{X}$$

is a positive operator and $\hat{X}\hat{Y}$ and $\hat{Y}\hat{X}$ have the appropriate hereditary properties by Theorem 4.14. The result follows from Theorem 4.13.

Definition 4.5

An operator $T \in B(\mathcal{H})$ such that there exists a nonzero polynomial p satisfying $p(T) = 0$ is called an *algebraic operator.*

Remark 4.10

Foias and Pearcy [22] have shown that the class of algebraic operators has a good supply of nontrivial hyperinvariant subspaces. Their result was motivated by the following well known theorem of Halmos [29].

Theorem 4.17 [29] *Suppose $T \in B(\mathcal{H})$ and p is monic polynomial of minimal degree such that $p(T) = 0$. If $p(z)$ has the factorization $p(z) = (z-\lambda_1)^{q_1}...(z-\lambda_k)^{q_k}$ where*

$\lambda_1, \lambda_2, ..., \lambda_k$ *are the distinct zeros of* p, *then* $\sigma(T) = \{\lambda_1, \lambda_2, ..., \lambda_k\}$ *and* T *is similar to an operator* T_1 *of the form*

$$T_1 = (\lambda_1 I + N_1) \oplus ... \oplus (\lambda_k I + N_k)$$

where $N_1, ..., N_k$ *are nilpotent operators.*

As an easy consequence of Theorem 4.17 we give the following result.

Corollary 4.18 *With the notation above,*
$Hyperlat(T) = Hyperlat(N_1) \oplus ... \oplus Hyperlat(N_k)$

Remark 4.11

The following concepts are useful.

Definition 4.6

A partially ordered set Ω is said to be a *directed set* if every pair $\{x, y\}$ of elements in Ω is bounded above. In this case, Ω is said to be *directed upward.* If every pair $\{x, y\}$ of elements in Ω is bounded below, then we say that Ω is *directed downward.*

Remark 4.12

We note also that since $Lat(T)$ and $Hyperlat(T)$ are lattices and since by Kubrusly [46, §1.6], lattices are directed both upward and downward, any subspace generated by two or more invariant or hyperinvariant subspaces is also an invariant or hyperinvariant subspace, respectively.
We need the following concepts.

Definition 4.7

A subalgebra $\mathcal{A} \subset B(\mathcal{H})$ is said to be *reflexive* if $\mathcal{A} = AlgLat(\mathcal{A})$, where $AlgLat(\mathcal{A}) = \{T \in B(\mathcal{H}) : Lat(\mathcal{A}) \subset Lat(T)\}$. An operator $T \in B(\mathcal{H})$ is said to be *reflexive* if the weakly closed (von Neumann) subalgebra generated by T in $B(\mathcal{H})$ is reflexive. We need the following terminology.
The *dual space* (or conjugate space) of a normed space $\mathcal{X}$, denoted by $\mathcal{X}^*$, is the normed space of all continuous linear functionals on $\mathcal{X}$ (i.e., $\mathcal{X}^* = B(\mathcal{X}, \mathbb{F})$, where $\mathbb{F}$ stands either for the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, depending on whether $\mathcal{X}$ is a real

or complex normed space, respectively). We note that $\mathcal{X}^*$ is a Banach space for every normed space $\mathcal{X}$, since $\mathcal{X}^* = B(\mathcal{X}, \mathbb{F})$ and $(\mathbb{F}, |\ |)$ is a Banach space. If $\mathcal{X} \neq \{0\}$, then $\mathcal{X}^* \neq \{0\}$ and hence $\mathcal{X}^{**} = (\mathcal{X}^*)^*$, the dual of $\mathcal{X}^*$ is again a non-zero Banach space, called the *second dual* of $\mathcal{X}$. It is easy to show that $\mathcal{X}$ can be identified with a linear manifold (a closed subset) of $\mathcal{X}^{**}$. That is, $\mathcal{X}$ is naturally embedded in its dual $\mathcal{X}^{**}$ (see [46], Theorem 4.66, p. 266).

Now, let $\Phi : \mathcal{X} \longrightarrow \mathcal{X}^{**}$ be the *natural embedding* of the normed space $\mathcal{X}$ into its second dual $\mathcal{X}^{**}$. If $\Phi(\mathcal{X}) = \mathcal{X}^{**}$ (i.e. Φ is surjective), then we say that $\mathcal{X}$ is *reflexive*. In Kubrusly [46, Example 3Q], it is shown that every finite-dimensional normed space is reflexive. It is clear that every reflexive normed space is a Banach space, but the converse is not true in general. There exist nonreflexive normed spaces. If $\mathcal{X}$ is separable and $\mathcal{X}^*$ is not separable, then $\mathcal{X} \ncong \mathcal{X}^{**}$ and $\mathcal{X}$ is not reflexive. This provides a necessary condition for reflexivity.

Remark 4.13

Consider the linear spaces ℓ_+^1 and ℓ_+^∞ equipped with their usual norms ($\|\ \|_\infty$ and $\|\ \|_1$, respectively). Since $\ell_+^{c_0}$ is a linear manifold of the linear space ℓ_+^∞, equip it with the sup-norm as well. Recall that $\ell_+^{c_0}$, the set of all scalar-valued sequences that converge to zero, and ℓ_+^1 are separable Banach spaces but the Banach space ℓ_+^∞ is not separable. It is not difficult to check that $(\ell_+^{c_0})^* \cong \ell_+^1$ and $(\ell_+^1)^* \cong \ell_+^\infty$ and so $(\ell_+^{c_0})^{**} \cong \ell_+^\infty$. Thus ℓ_+^1 is a separable Banach space with a nonseparable dual, since $(\ell_+^1)^*$ is not separable because $(\ell_+^1)^* \cong \ell_+^\infty$ and separability is a topology invariant. Hence, ℓ_+^1 is a nonreflexive Banach space. It is clear that $\ell_+^1 \subset \ell_+^2$ and that $\overline{(\ell_+^1)} = \ell_+^2$ in (ℓ_+^2, d_2), where $d_2(x, y) = (\sum_{k=1}^{\infty} |\xi_k - v_k|^2)^{1/2}$, for every $x = \{\xi_k\}_{k\in\mathbb{N}}$ and $y = \{v_k\}_{k\in\mathbb{N}}$ in ℓ_+^2. This shows that the set ℓ_+^1 is dense in the metric space (ℓ_+^2, d_2).

Positive results concerning the invariant subspace problem have been found for certain classes of operators on a Hilbert space (in particular for subnormal operators) and for a general Banach space. It is an open problem whether every non-normal operator $T \in B(\mathcal{H})$ has a nontrivial invariant subspace if $\mathcal{H}$ is an infinite-dimensional Hilbert space. It is easy to show that for non-separable Banach spaces the invariant subspace problem has an immediate affirmative answer. For finite dimensional complex Banach spaces the invariant subspace problem has an affirmative answer too. To see this, let

$T : X \longrightarrow X$ be a bounded operator on a finite dimensional complex Banach space ($dim(\mathcal{H}) > 1$). There is nothing to prove if T is a multiple of the identity operator, since then each subspace is invariant (and the only hyperinvariant subspaces are the trivial ones). So, we assume that T is not a multiple of the identity. Now if λ is an eigenvalue of T, then its eigenspace $\mathcal{M}_\lambda = \{x \in H : Tx = \lambda x\}$ is a non-trivial closed hyperinvariant subspace. Counter examples have been constructed to answer the invariant subspace problem, in some non-reflexive spaces and even in ℓ_1. For instance, Read [60] presented an example of a bounded operator on the real ℓ_1 space without non-trivial invariant subspaces. These examples have established that the invariant subspace problem in its general form has a negative answer.

As a consequence of the fundamental theorem of algebra, every linear operator on a complex finite-dimensional Banach space (in particular a Hilbert space) with dimension at least 2 has an eigenvector. Therefore every such linear operator has a non-trivial invariant subspace. The fact that the complex numbers are algebraically closed is required here. It is clear that the invariant subspaces of a linear operator is dependent upon the underlying scalar field of the Hilbert space $\mathcal{H}$.

The invariant and hyperinvariant subspaces of a linear operator T shed light on the structure of T. When $\mathcal{H}$ is finite dimensional Hilbert space over an algebraically closed field, a linear operator T acting on $\mathcal{H}$ is characterized (up to similarity) by the Jordan canonical form, which decomposes $\mathcal{H}$ into invariant subspaces of T. Many fundamental questions regarding T (including isolation of parts) can be translated to questions about invariant subspaces of T.

For subnormal operators, the existence of invariant subspaces was proved by Brown [11]. Chō and Huruya [12] have shown that a large class of hyponormal operators have invariant subspaces.

Definition 4.8

An algebra $\mathfrak{U}$ of operators on a Hilbert space is *reductive* if it is weakly closed, contains the identity operator, and that $Lat(\mathfrak{U}) = Lat(\mathfrak{U}^*)$.

It is clear that von Neumann algebras are reductive.

Corollary 4.19 *(Burnside)*[58]. *If $\mathcal{K}$ is a finite-dimensional Hilbert space and $\mathfrak{U}$ is a subalgebra of $B(\mathcal{K})$ with no nontrivial invariant subspace, then $\mathfrak{U} = B(\mathcal{K})$.*

Proof. Since $Lat(\mathfrak{U}) = \{\{0\}, \mathcal{K}\}$, it follows that $\mathfrak{U}$ is a reductive algebra. Now let $S \in B(\mathcal{H})$ be a unilateral shift of finite multiplicity, and define the reductive algebra $\mathfrak{W} = B(\mathcal{H}) \oplus \mathfrak{U}$. Since $\mathfrak{W}$ contains the operator $S \oplus 0$, $\mathfrak{W}$ is a von Neumann algebra, which implies that $\mathfrak{U}$ is a von Neumann algebra. It follows from this fact that $\mathfrak{U} = \mathfrak{U}''$, where $\mathfrak{U}''$ is the double commutant of $\mathfrak{U}$. The fact that $\mathfrak{U}$ has no nontrivial invariant subspace implies that $\mathfrak{A}'$, the commutant of $\mathfrak{U}$, consists of the scalar operators. Therefore, $\mathfrak{U} = \mathfrak{U}'' = (\mathfrak{U}')' = \{\lambda I : \lambda \in \mathbb{C}\}' = B(\mathcal{K})$. Thus $\mathfrak{U} = B(\mathcal{K})$, which was to be shown.

Remark 4.14

Wu [78] has studied the $Hyperlat(T)$ of C_{11} contractions with finite defect indices and characterized the elements of $Hyperlat(T)$ among invariant subspaces for T of their regular factorizations and has shown that the elements of $Hyperlat(T)$ are exactly the spectral subspaces of T defined by Nagy and Foias [53], which are T-invariant and matrix representation of T in the basis of $\mathcal{H}$ using the bases of these subspaces represent its Jordan canonical form. Nagy and Foias [53] have shown that if T_1 and T_2 are two such operators which are quasi-similar to each other, then $Hyperlat(T_1)$ is (lattice) isomorphic to $Hyperlat(T_2)$. Recently several authors studied $Hyperlat(T)$ for certain classes of contractions. Uchiyama ([70], [71]) has shown that $Hyperlat(T)$ is preserved, as a lattice, for quasi-similar $C_0(N)$ contractions and for completely injection-similar $C_{.0}$ contractions with finite defect indices. Wu [79], determined $Hyperlat(T)$ when T is a completely non-unitary (c.n.u) contraction with a scalar-valued characteristic function or a direct sum of such contractions. Wu [78] has shown that elements of $Hyperlat(T)$ are exactly the spectral subspaces $\mathcal{H}_{\mathcal{F}}$ defined by Nagy and Foias [53].

Using these results we can completely determine $Hyperlat(T)$ in terms of the well-known structure of the hyperinvariant subspace lattice of normal operators.

We know that for a C_{11} contraction T, $d_T = d_{T^*}$. Let Θ_T denote the characteristic function of an arbitrary contraction T. There is a one-to-one correspondence between the invariant subspaces of T and the regular factorizations of Θ_T. If $\mathcal{K} \subseteq \mathcal{H}$ is invariant for T with the corresponding regular

factorization $\Theta_T = \Theta_2\Theta_1$ and $T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$ is the triangulation on $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$, then the characteristic functions of T_1 and T_2 are purely contractive parts of Θ_1 and Θ_2, respectively.

We need the following notation. For any subset $\mathcal{E}$ of the unit circle, $\partial\mathbb{D}$, let $M_{\mathcal{E}}$ denote the operator of multiplication by e^{it} on the space $L^2(\mathcal{E})$ of square-integrable functions on E. It was proved in Wu [81] that any c.n.u. C_{11} contraction T with finite defect indices is quasi-similar to a uniquely determined operator, called the *Jordan model* of T, of the form $M_{\mathcal{E}_1}\oplus...\oplus M_{\mathcal{E}_k}$, where $\mathcal{E}_1,...,\mathcal{E}_k$ are Borel subsets of $\partial\mathbb{D}$ satisfying $\mathcal{E}_1 \supseteq \mathcal{E}_2 \supseteq ... \supseteq \mathcal{E}_k$. In this case $\mathcal{E}_1 = \{t : \Theta_T(t)$ *is not isometric*$\}$.
Here we use t to denote the argument of a function defined on the unit circle $\partial\mathbb{D}$. A statement involving t is said to be true if it holds for almost all t with respect to the Lebesgue measure. In particular, for $\mathcal{E},\mathcal{F} \in \partial\mathbb{D}$, $\mathcal{E} = \mathcal{F}$ means that $(\mathcal{E}\setminus\mathcal{F})\cup(\mathcal{F}\setminus\mathcal{E})$ has Lebesgue measure zero.
We start with the following lemma.

Lemma 4.20 [78, *Lemma* 1]. *Let T be a C_{11} contraction on a Hilbert space $\mathcal{H}$ and let U be a unitary operator on a Hilbert space $\mathcal{K}$. If there exists a one-to-one operator $X : \mathcal{H} \longrightarrow \mathcal{K}$ such that $XT = UX$, then T is quasi-similar to the unitary operator $U|_{\overline{XH}}$.*

Remark 4.15

Let T be a c.n.u C_{11} contraction on $\mathcal{H}$ with finite defect indices d_T and d_{T^*} and $U = M_{\mathcal{E}_1}\oplus...\oplus M_{\mathcal{E}_k}$ acting on $\mathcal{K} = L^2(\mathcal{E}_1)\oplus...\oplus L^2(\mathcal{E}_k)$ be its Jordan model. Let $X : \mathcal{H} \longrightarrow \mathcal{K}$ and $Y : \mathcal{K} \longrightarrow \mathcal{H}$ be quasi-affinities intertwining T and U. For any Borel subset $\mathcal{F} \subseteq \mathcal{E}_1$, let

$$\mathcal{K}_{\mathcal{F}} = L^2(\mathcal{E}_1\cap\mathcal{F})\oplus...\oplus L^2(\mathcal{E}_k\cap\mathcal{F})$$

be the spectral subspace of $\mathcal{K}$ associated with $\mathcal{F}$. For a contraction T we consider $\sigma(T) \subseteq \partial\mathbb{D}$ holds and there has been developed a spectral decomposition [53, p. 318]. Let $\mathcal{H}_{\mathcal{F}}$ denote the spectral subspace associated with $\mathcal{F} \subseteq \partial\mathbb{D}$. Indeed, $\mathcal{H}_{\mathcal{F}}$ is the (unique) maximal subspace of $\mathcal{H}$ satisfying (i) $T\mathcal{H}_{\mathcal{F}} \subseteq \mathcal{H}_{\mathcal{F}}$, (ii) $T_{\mathcal{F}} \equiv T|_{H\mathcal{F}} \in C_{11}$ and (iii) $\Theta_{T_{\mathcal{F}}}(t)$ is isometric for t in $\mathcal{F}^C$, the complement of $\mathcal{F}$, that is $t \in \partial\mathbb{D}\setminus\mathcal{F}$. Moreover $\mathcal{H}_{\mathcal{F}}$ is hyperinvariant for T. Such subspaces $\mathcal{H}_{\mathcal{F}}$ give all the elements in $Hyperlat(T)$. To show this, we need the following results.

Lemma 4.21 [78, *Lemma* 2] *For any Borel subset $\mathcal{F} \subseteq \mathcal{E}_1$, $\overline{X\mathcal{H}_{\mathcal{F}}} = \mathcal{K}_{\mathcal{F}}$.*

For any Borel subset $\mathcal{F} \subseteq \mathcal{E}_1$, let $q(\mathcal{K}_{\mathcal{F}}) = \bigvee_{ST=TS} SY\mathcal{K}_{\mathcal{F}}$. It is shown in [53 pp. 76-78] that $q(\mathcal{K}_{\mathcal{F}})$ is hyperinvariant for T and $\overline{Xq(\mathcal{K}_{\mathcal{F}})} = \mathcal{K}_{\mathcal{F}}$.

Lemma 4.22 [78, *Lemma* 3] *For any Borel subset $\mathcal{F} \subseteq \mathcal{E}_1$, let $q(\mathcal{K}_{\mathcal{F}})$ be defined as in Lemma 4.21. Then $q(\mathcal{K}_{\mathcal{F}}) = \mathcal{H}_{\mathcal{F}}$.*

Lemma 4.23 [78, *Lemma* 4] *Let $\mathcal{M} \subseteq \mathcal{H}$ be hyperinvariant for T with the corresponding factorization $\Theta_T = \Theta_2\Theta_1$ and let $\mathcal{F} = \{t : \Theta_1(t)$ is not isometric$\}$. Then $\mathcal{M} = \mathcal{H}_{\mathcal{F}}$.*

Remark 4.16

Lemma 4.23 says that the hyperinvariant subspaces of a C_{11} contraction are the spectral subspaces of T. Using the previous concepts we now state and prove the following result.

Theorem 4.24 *Let T be a c.n.u. C_{11} contraction on $\mathcal{H}$ with $d_T = d_{T^*} = n < \infty$. Let $\mathcal{K} \subseteq \mathcal{H}$ be an invariant subspace with the corresponding regular factorization $\Theta_T = \Theta_2\Theta_1$ and let $\mathcal{E} = \{t : \Theta_T$ is not isometric$\}$. Then the following are equivalent:*
(i) $\mathcal{K} \in Hyperlat(T)$
(ii) $\mathcal{K} = \mathcal{H}_{\mathcal{F}}$ for some Borel subset $\mathcal{F} \subseteq \mathcal{E}$
(iii) the intermediate space of $\Theta_T = \Theta_2\Theta_1$ is of dimension n and for almost all t, either $\Theta_2(t)$ or $\Theta_1(t)$ is isometric.

Proof. $(i) \Longrightarrow (ii)$. That $\mathcal{K} = \mathcal{H}_{\mathcal{F}}$, where $\mathcal{F} = \{t : \Theta_1(t)$ *is not isometric*$\}$, is proved in Lemma 4.23. It is a simple matter to check that $\mathcal{F} \subseteq \mathcal{E}$.
$(ii) \Longrightarrow (iii)$. Since $T|_{\mathcal{H}_{\mathcal{F}}} \in C_{11}$, the intermediate space of $\Theta_T = \Theta_2\Theta_1$ is of dimension n. For the proof of the remaining parts (see [53]).
$(iii) \Longrightarrow (i)$. Since the intermediate space $\Theta_T = \Theta_2\Theta_1$ is of dimension n and $det(\Theta_1) \not\equiv 0$ (otherwise $det(\Theta_T) \equiv 0$), we conclude that $T|_{\mathcal{K}}$ is of class C_{11} by [53, p. 318]. Therefore, Θ_1 is outer (from both sides). This, together with the other condition in (iii), implies that $\mathcal{K} = \mathcal{H}_{\mathcal{F}}$, where $\mathcal{F} = \{t : \Theta_1(t)$ *is not isometric*$\}$. Thus $\mathcal{K} \in Hyperlat(T)$.
These concepts lead to the following results.

Corollary 4.25 *Let T be as in Theorem 4.24 and let $U = M_{\mathcal{E}_1} \oplus ... \oplus M_{\mathcal{E}_k}$, acting on $\mathcal{K}$, be its Jordan model. Then $Hyperlat(T)$ is (lattice) isomorphic to $Hyperlat(U)$. Moreover, if $X : \mathcal{H} \longrightarrow \mathcal{K}$ and $Y : \mathcal{K} \longrightarrow \mathcal{H}$ are quasi-affinities intertwining T and U,* *then the mapping $\mathcal{M} \longrightarrow \overline{X\mathcal{M}}$ implements the lattice isomorphism from $Hyperlat(T)$ onto $Hyperlat(U)$, and its inverse is given by $\mathcal{N} \longrightarrow q(\mathcal{N}) = \bigvee_{ST=TS} SY\mathcal{N}$. In this case, $T|_{\mathcal{M}}$ and $U|_{\overline{X\mathcal{M}}}$ are quasi-similar to each other.*

Corollary 4.26 *Let T_1 and T_2 be c.n.u. C_{11} contractions with finite defect indices. If T_1 is quasi-similar to T_2, then $Hyperlat(T_1)$ is (lattice) isomorphic to $Hyperlat(T_2)$.*

Corollary 4.27 *Let T be a c.n.u. C_{11} contraction with finite defect indices. If $\mathcal{K}_1, \mathcal{K}_2 \in Hyperlat(T)$ and $T|_{\mathcal{K}_1}$ is quasi-similar to $T|_{\mathcal{K}_2}$, then $\mathcal{K}_1 = \mathcal{K}_2$.*

Proof. $T|_{\mathcal{K}_1}$ quasi-similar to $T|_{\mathcal{K}_2}$ implies that they have the same Jordan model, say $U = M_{\mathcal{E}_1} \oplus ... \oplus M_{\mathcal{E}_k}$. By Theorem 4.24, $\mathcal{K}_1 = \mathcal{H}_{\mathcal{E}_1} = \mathcal{K}_2$.

Remark 4.17

The following results give a characterization of invariant and hyperinvariant subspaces for some classes of operators. First we prove the following result which is an extension of Theorem 4.13.

Lemma 4.28 *Suppose $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are quasisimilar operators on $\mathcal{H}$. If B has a nontrivial hyperinvariant subspace then A has a nontrivial hyperinvariant subspace.*

Proof. Let $V : \mathcal{H} \longrightarrow \mathcal{K}$ and $W : \mathcal{K} \longrightarrow \mathcal{H}$ be quasi-affinities of A and B. That is, $BV = VA$ and $AW = WB$. Let $\mathcal{N}$ be a nontrivial invariant subspace for B. Define

$$\mathcal{M} = \bigvee\{XW\mathcal{N} : X \in \{A\}'\}.$$

Clearly $\mathcal{M}$ is B-hyperinvariant and $\mathcal{M} \neq \{0\}$ because $\mathcal{M} \supset W\mathcal{N}$. Moreover, $\mathcal{M} \neq \mathcal{H}$ because

$$\begin{aligned} V\mathcal{M} &= V\{\bigvee XW\mathcal{N} : X \in \{A\}'\} \\ &\subset \bigvee\{Y\mathcal{N} : Y \in \{B\}'\} \\ &\subset \mathcal{N} \neq \mathcal{K} = \overline{(V\mathcal{H})}. \end{aligned}$$

Thus $\mathcal{M}$ is nontrivial.

Remark 4.18

Recall from Definition 4.5 that an operator $A \in B(\mathcal{H})$ is *algebraic* if there exists a polynomial p other than 0 such that $p(A) = 0$. Every operator on a finite-dimensional space is algebraic. The algebraic operators on infinite-dimensional spaces can be characterized in terms of their invariant subspaces. An operator $T \in B(\mathcal{H})$ is algebraic if and only if the union of its finite-dimensional invariant subspaces is $\mathcal{H}$. We now investigate and analyze the invariant subspaces of an algebraic operator.

Theorem 4.29 *Let A_1 and A_2 be algebraic operators with minimal polynomials p_1 and p_2 on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then*
$Lat(A_1 \oplus A_2) = Lat(A_1) \oplus Lat(A_2)$ *if and only if* $g.c.d(p_1.p_2) = 1$.

Proof. In general case, for every operator $A_i \in B(H_i)$, $i = 1, 2$,
$Lat(A_1) \oplus Lat(A_2) \subset Lat(A_1 \oplus A_2)$ holds. For the inverse inclusion, let $g.c.d(p_1, p_2) = 1$. That is the two polynomials are relatively prime. We must show that $\mathcal{M} \in Lat(A_1 \oplus A_2)$ implies that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_i \in Lat(A_i)$, $i = 1, 2$. Given $\mathcal{M} \in Lat(A_1 \oplus A_2)$, let $\mathcal{M}_1 \oplus \{0\} = (1 \oplus 0)\mathcal{M}$ and $\{0\} \oplus \mathcal{M}_2 = (0 \oplus 1)\mathcal{M}$. Obviously, $\mathcal{M} \subset \mathcal{M}_1 \oplus \mathcal{M}_2$. To prove that $\mathcal{M}_1 \oplus \mathcal{M}_2 \subset \mathcal{M}$, let r_1 and r_2 be polynomials such that $r_1p_1 + r_2p_2 = 1$, and let $q_2 = r_2p_2$. We must have $q_2(A_1) = 1 - r_1(A_1)p(A_1) = 1$. So, $q_2(A_1 \oplus A_2) = q_2(A_1) \oplus q_2(A_2) = 1 \oplus 0$. Hence $\mathcal{M}_1 \oplus \{0\} = (1 \oplus 0)\mathcal{M} = q_2(A_1 \oplus A_2)\mathcal{M} \subset \mathcal{M}$.
Similarly, $\{0\} \oplus \mathcal{M}_2 \oplus \mathcal{M}$. Thus $\mathcal{M}_1 \oplus \mathcal{M}_2 \subset \mathcal{M}$, and it follows that $\mathcal{M}_1 \oplus \mathcal{M}_2 = \mathcal{M}$. The converse is easy to prove and we omit its proof.

Remark 4.19

Hoover [39] has indicated that the structure of an operator is not likely to be revealed by the presence of a single nontrivial hyperinvariant subspace for an operator T, but more likely by the presence of a collection of hyperinvariant subspaces $\{\mathcal{M}_\alpha\}_{\alpha \in I}$ for which the structure of $T|_{\mathcal{M}_\alpha}$ is well understood. Thus the ultimate value of the quasisimilarity relation may lie in the extent to which it preserves the lattices of the hyperinvariant subspaces of quasisimilar operators. Nagy and Foias [53, Prob 5.1, pg. 76] have shown that if A and B are quasisimilar and B is a unitary operator, then there exists an injective mapping of $HyperLat(B)$ into $HyperLat(A)$ which respects the lattice structures. They also proved that if S is unitary, then $HyperLat(T)$ contains a sublattice isomorphic to

$HyperLat(S)$. In Hoover [39], the result was extended to the case when B is a normal operator.

Using these facts as a motivation, we give a generalization of these results to completely non-normal operators. We note also that these results are valid for completely nonunitary contractions.

Theorem 4.30 *If* $T = T_1 \oplus T_2$ *where* T_1 *is normal and* T_2 *is c.n.n., then* $\{T\}' = \{T_1\}' \oplus \{T_2\}'$*, and all the invariant subspaces of* T_2 *are hyperinvariant, that is,* $Lat(T_2) = HyperLat(T_2)$.

Proof. Let $\mathcal{N}$ be the largest reducing subspace for T such that the restriction of T to $\mathcal{N}$ is normal. Then it is easy to see that there is a largest such subspace since $\mathcal{N}$ can be characterized as the span of the set $\{\mathcal{M} : \mathcal{M}\ \textit{is a reducing subspace for}\ T\ \textit{and}\ T|_{\mathcal{M}}\ \textit{is normal}\}$. Since T_2 is c.n.n, any operator A can be written as a matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ corresponding to the decomposition of the Hilbert space as $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^{\perp}$. If A commutes with T then $T_1 A_{12} = A_{12} T_2$ and $A_{21} T_1 = T_2 A_{21}$. This says that $T_2|_{Ker(A_{12})^{\perp}}$ is normal. Since T_2 is c.n.n., it has no normal direct summand, $Ker(A_{12})^{\perp} = \{0\}$. That is, $A_{12} = 0$. Similarly, $T_2|_{Ran(A_{21})}$ is normal and it follows that $A_{21} = 0$. This shows that $A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$. This shows that every invariant subspace of $A \in \{T\}'$ reduces A.
The fact that every invariant subspace of T_2 is hyperinvariant follows by the same proof, where the direct summands of T_2 play the role of T_1 and T_2.

The following is a consequence of Theorem 4.30.

Corollary 4.31 *If* $T \in B(\mathcal{H})$ *and* T *is completely non-normal (c.n.n.), then* $Lat(T) = HyperLat(T)$.

Corollary 4.32 *If* $T \in B(\mathcal{H})$ *is normal, then every hyperinvariant subspace of* T *is hyperinvariant for* T^**. That is,* $HyperLat(T) = HyperLat(T^*)$.

Proof. Since T is normal if and only if T^* is normal, the result follows from the fact that if $T^* \in \{T\}'$ then $T \in \{T^*\}'$.

Corollary 4.31 and Corollary 4.32 yield the following result.

Theorem 4.33 *If* $T = T_1 \oplus T_2$ *where* T_1 *is normal and* T_2 *is c.n.n.,* $\{T\}' = \{T_1\}' \oplus \{T_2\}'$ *and* $HyperLat(T) = HyperLat(T^*)$ *then* $Lat(T_2) = HyperLat(T_2) \cap HyperLat(T_2^*)$.

We note that for a normal operator T, $Lat(T) = Lat(T^*)$ holds if and only if T^* lies in the weakly closed algebra generated by T (i.e, the weak closure of the set of polynomials in T). Theorem 4.33 yields the following result.

Corollary 4.34 *Let T be a completely non-normal operator whose every invariant subspace is reducing. Then the normal direct summand T_1 is absent and $Lat(T) = HyperLat(T) \cap HyperLat(T^*)$.*

Remark 4.20

Note that if A commutes with T and $\mathcal{M}$ is a hyperinvariant subspace of A then $\mathcal{M}$ is invariant under T and hence, $\mathcal{M}$ reduces $\{T\}'$. In particular, $\mathcal{M}$ reduces A. In particular, every operator that commutes with a normal operator has a nontrivial hyperinvariant subspace that reduces it.

Corollary 4.35 *If T is completely non-normal and if every invariant subspace of T reduces T (that is, $Lat(T) = Red(T)$) and $AT = TA$, then every hyperinvariant subspace of A reduces A. That is $Hyperlat(A) = Red(A)$.*

Proof. By Corollary 4.34, $Lat(T) = HyperLat(T) \cap HyperLat(T^*)$ and since $AT = TA$, then $\mathcal{M}$ invariant for T is equivalent to $\mathcal{M}$ hyperinvariant for A. This means that $Lat(T) = Hyperlat(A)$. Since every invariant subspace of T reduces T, we have $Lat(T) = Red(T)$. By Remark 4.20, it follows that $Hyperlat(A) \subseteq Red(A)$. But the reverse inclusion $Hyperlat(A) \supseteq Red(A)$ is obvious. Thus $Hyperlat(A) = Red(A)$.

Remark 4.21

If $T \in B(\mathcal{H})$ is a contraction, then the operator $T^{*^n}T^n \longrightarrow A$ strongly to an operator A on $\mathcal{H}$, where $O \leq A \leq I$, $\|A\| = 1$ whenever $A \neq O$. We use this information to prove the following assertion by Nzimbi, Pokhariyal and Khalagai [55] about the characterization of invariant subspaces of some contraction operators.

Proposition 4.36 [55] *Let T be a contraction. If T and T^* have no nontrivial invariant subspace, then either $T \in C_{00}$ or $\|A\| < 1$ with $Ker(A - A^2) = \{0\}$.*

Proof. We prove the case for T. The case for T^* can be proved similarly by applying the adjoint operation. Now suppose that T is a contraction with no nontrivial invariant subspace. It suffices to show that $T \in C_{0\cdot}$. Using [55, Proposition 1.4], it is clear that $Ker(A-A^2)$ is invariant for T. By the hypothesis, this means that either $Ker(A-A^2) = \{0\}$ or $Ker(A-A^2) = \mathcal{H}$. The former case implies that A is a projection, and hence T can be decomposed as $T = G \oplus S_+ \oplus U$, where G is a strongly stable contraction, S_+ is a unilateral shift, and U is a unitary operator, where any of the direct summands may be missing. But by [45, § 0.5], S_+ and U have nontrivial invariant subspaces. Since T is assumed to have no nontrivial invariant subspace, these direct summands are missing in the decomposition of T. Thus $T = G$ and $T \in C_{0\cdot}$. To prove the latter case, we note that $\{x \in \mathcal{H} : \|Ax\| = \|x\|\} = Ker(I-A) \subsetneq Ker(A-A^2) = \{0\}$. Since T is strongly stable and A is also a contraction, $\|A\| \neq 1$. Hence $\|A\| < 1$. This completes the proof.

Remark 4.22

Note that direct sum decompositions of operators arises from the action of orthogonal projections of $\mathcal{H}$ onto invariant (in particular, reducing) subspaces. We now characterize invariant subspaces of an operator T in terms of orthogonal projections on such subspaces.

Theorem 4.37 *If $T \in B(\mathcal{H})$ and P is any projection onto $\mathcal{M} \subsetneq \mathcal{H}$ then $\mathcal{M} \in Lat(T)$ if and only if $TP = PTP$.*

Proof. If $\mathcal{M} \in Lat(T)$ and $x \in \mathcal{H}$, then TPx is contained in $T(\mathcal{M})$, and since $T(\mathcal{M}) \subset \mathcal{M}$ it follows that $P(TPx) = TPx$. Conversely, if $TP = PTP$ and $x \in \mathcal{M}$, then $Px = x$ and $Tx = PTPx$. Since $P(Tx) = Tx$, $Tx \in \mathcal{M}$, we have that $\mathcal{M} \in Lat(T)$.

Theorem 4.38 *If $T \in B(\mathcal{H})$ and P is the projection on $\mathcal{M} \subsetneq \mathcal{H}$ along $\mathcal{N} \subsetneq \mathcal{H}$ then $\mathcal{M}$ and $\mathcal{N}$ are both in $Lat(T)$ if and only if $TP = PT$.*

Proof. By Theorem 4.37, $\{\mathcal{M}, \mathcal{N}\} \subset Lat(T)$ if and only if $TP = PTP$ and $T(I-P) = (I-P)T(I-P)$, (since $I-P$ is a projection on $\mathcal{N}$). The second equation is equivalent to $T - TP = T - PT - TP + PTP$, or $0 = -PT + PTP$. The first equation gives $0 = -PT + TP$, which completes the proof.

Remark 4.23

Recall that $Red(T)$ is the collection of all subspaces of $\mathcal{H}$ which are invariant under both T and T^*. Equivalently, a subspace $\mathcal{M} \in Red(T)$ if $T\mathcal{M} \subset \mathcal{M}$ and $T\mathcal{M}^\perp \subset \mathcal{M}^\perp$.

It is easy to see that $\mathcal{M}$ reduces T if and only if $\mathcal{M} \in Lat(T) \cap Lat(T^*)$. These facts together with Theorem 4.38 lead to the following result.

Corollary 4.39 *Let $T \in B(\mathcal{H})$. A subspace $\mathcal{M} \in Red(T)$ if $PT = TP$, where P is the orthogonal projection onto $\mathcal{M}$.*

Remark 4.24

It is not difficult to see that a nontrivial subspace of $\mathcal{H}$ may be an invariant subspace for an operator $T \in B(\mathcal{H})$ but not reduce T. In fact, an operator may have many nontrivial invariant subspaces and no nontrivial reducing subspaces. For instance, the operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x, y) = (\frac{1}{2}x + \frac{1}{2}y, y)$ has $\mathcal{M} = span\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ as a nontrivial invariant subspace but $\mathcal{M}^{\perp} = span\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ is not invariant under T. So $\mathcal{M}$ does not reduce T.

Proposition 4.40 *For a self-adjoint $T \in B(\mathcal{H})$, invariance of $\mathcal{M}$ implies the invariance of $\mathcal{M}^{\perp}$.*

Remark 4.25

We note that Proposition 4.40 can be be extended to normal operators which contains the class of self-adjoint operators. Since similar operators have isomorphic invariant subspace lattices, the lattice of hyperinvariant subspaces of T is a similarity invariant. On the other hand, there may be T-invariant subspaces that are not T-hyperinvariant. For let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ acting on $\mathbb{R}^3$. Let $P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be the projection onto an A-invariant subspace $\mathcal{M}$, which is not B-invariant.
A simple computation shows that B commutes with A, and thus $Ran(P)$ is not hyperinvariant under A. By Kubrusly [47, Problem 1.3], since $AB = 0$, $Ker(A)$ and $\overline{Ran(B)}$ are nontrivial invariant invariant subspaces for both A and B. In this example, all invariant subspaces will be hyperinvariant since by construction $AB = 0$.
Proof of Proposition 4.40. Let $x \in \mathcal{M}^{\perp}$. Then $\langle x, y\rangle = 0$ for every $y \in M$. However, $Ty \in \mathcal{M}$ for $y \in \mathcal{M}$. Hence $\langle x, Ty\rangle = 0$. Since T is self-adjoint, $\langle Tx, y\rangle = 0$ for every

$y \in \mathcal{M}$. Consequently, $Tx \in \mathcal{M}^{\perp}$, as was required.
We give an application of the preceding facts.

Theorem 4.41 *Let $\mathcal{M}$ be an invariant subspace for $T \in B(\mathcal{H})$. If T is hyponormal, then $T|_{\mathcal{M}}$ is hyponormal.*

Proof. Let $\mathcal{M}$ be invariant under T. Then $(T|_{\mathcal{M}})^* = PT^*|_{\mathcal{M}}$, where P is the orthogonal projection onto $\mathcal{M}$. Thus

$$\|(T|_{\mathcal{M}})^*x\| = \|PT^*|_{\mathcal{M}}x\| \leq T^*|_{\mathcal{M}}x = \|T^*x\| \leq \|T|_{\mathcal{M}}x\|, \qquad x \in \mathcal{M}.$$

Remark 4.26

It is known (see Hoover [39]) that if A and B are quasisimilar operators and A has a nontrivial hyperinvariant subspace then so does B. Furthermore, if A is normal then quasisimilarity induces an injection from $Hyperlat(A)$ to $Hyperlat(B)$, so one could expect that quasisimilar operators always have isomorphic hyperlattices. An example will show that this is not necessarily true, even for simple operators.
To see this we investigate the hyperlattice of certain nilpotent operators.

Lemma 4.42 *Let T be a nilpotent operator of order three (i.e, $T^3 = 0$). Then $Ker(T^2)$ (respectively, $\overline{Ran(T^2)}$) is a maximal (respectively, minimal) hyperinvariant subspace of T.*

Proof. Let $T = \begin{pmatrix} 0 & T_{12} & T_{13} \\ 0 & 0 & T_{23} \\ 0 & 0 & 0 \end{pmatrix}$ be the matrix of T with respect to the orthogonal direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where $\mathcal{H}_1 = Ker(T)$, $\mathcal{H}_2 = Ker(T^2) \ominus Ker(T)$ and $\mathcal{H}_3 = \mathcal{H} \ominus Ker(T^2)$. Then T_{12} and T_{23} are injective operators and therefore their adjoints have dense ranges.
A straightforward computation shows that the commutant of T consists of all those operators $A \in B(\mathcal{H})$ of the form $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}$ such that $A_{11}T_{12} = T_{12}A_{22}$, $A_{22}T_{23} = T_{23}A_{33}$ and $A_{11}T_{13} + A_{12}T_{23} = T_{12}A_{23} + T_{13}A_{33}$, where A_{33} can be arbitrarily chosen. Let $\mathcal{M} \in Hyperlat(T)$ and assume that $\mathcal{M}$ is not contained in $Ker(T^2)$; then there exists a vector $v = (v_1, v_2, v_3)$ in $\mathcal{M}$ ($v_1 \in \mathcal{H}_1$, $v_2 \in \mathcal{H}_2$, $v_3 \in \mathcal{H}_3$) with $v_3 \neq 0$.

Let A be as above with $A_{jk} = 0$ for $(j,k) \neq (1,3)$; then the hyper-invariance of $\mathcal{M}$ implies that $Av = A(v_1, v_2, v_3) = (A_{13}v_3, 0, 0) \in \mathcal{M}$. Since A_{13} can be arbitrarily chosen, we conclude that $\mathcal{H}_1 \subset \mathcal{M}$. Hence $(0, v_2, v_3) \in \mathcal{M}$.

Since $Ran(T_{23}^*)$ is dense, there exits an $f_0 \in \mathcal{H}_2$ such that $\langle T_{23}^* f_0, v_3\rangle = 1$. Let f_2 be an element in $\mathcal{H}_2$ and define $B_{23} = f_2 \otimes T_{23}^* f_0$, $B_{12} = T_{12}f_2 \otimes f_0$ (where $x \otimes y$ denotes the operator defined by $x \otimes y(z) = \langle z, y\rangle x$) and $B_{jk} = 0$ for all $(j,k) \neq (1,2)$ or $(2,3)$. It is easily seen that $B = (B_{jk}) \in \{T\}'$, the commutant of T and therefore $B(0, v_2, v_3) = (B_{12}v_1, B_{23}v_3, 0) = (B_{12}v_1, f_2, 0) \in \mathcal{M}$. Thus $\mathcal{H}_1 \oplus \mathcal{H}_2 = Ker(T^2) \subset \mathcal{M}$. Hence $(0, 0, v_3) \in \mathcal{M}$.

Now use the fact that $\overline{Ran(T_{12}T_{23})}$ is dense in order to obtain an $e_0 \in \mathcal{H}_1$ such that $\langle (T_{12}T_{23})^* e_0, v_3\rangle = 1$. Let g_3 be an element in $\mathcal{H}_3$ and define $C_{33} = g_3 \otimes (T_{12}T_{23})^* e_0$, $C_{22} = T_{23}g_3 \otimes T_{12}^* e_0$, $C_{11} = T_{12}T_{23}g_3 \otimes e_0$, $C_{12} = T_{13}g_3 \otimes T_{12}^* e_0$, $C_{23} = T_{23}g_3 \otimes T_{13}^* e_0$ and $C_{jk} = 0$ for all $(j,k) \neq (1,1)$, $(1,2)$, $(2,2)$, $(2,3)$ or $(3,3)$. Then $C = (C_{jk}) \in \{T\}'$ and therefore $C(0,0,v_3) = (0, C_{23}v_3, C_{33}v_3) = (0, C_{23}v_3, g_3) \in \mathcal{M}$. Thus, we conclude that $\mathcal{M} = \mathcal{H}$. That is $\mathcal{M}$ is maximal.

The same arguments applied to T^* shows that $Ker(T^{*2})$ is a maximal hyperinvariant subspace of T^* and therefore $\left(Ker(T^{*2})\right)^{\perp} = \overline{Ran(T^2)}$ is a minimal hyperinvariant subspace of T.

Corollary 4.43 *If $\mathcal{M} \in Hyperlat(T)$, with $T^3 = 0$ and $\{0\} \neq \mathcal{M} \neq \mathcal{H}$, then $\overline{Ran(T^2)} \subset \mathcal{M} \subset Ker(T^2)$.*

Proof. The result follows from Lemma 4.42.

Definition 4.9

Let $T_k \in B(\mathbb{C}^k)$ be the nilpotent operator defined by $T_k e_1 = 0$, $T_k e_j = e_{j-1}$ for $j = 2, 3, ..., k$, with respect to the canonical orthonormal basis $\{e_j\}_{j=1}^{k}$ of $\mathbb{C}^k$ and let $T_k(\alpha_k)$ be the orthogonal direct sum of α_k copies of T_k acting in the usual fashion on the orthogonal direct sum of α_k copies of $\mathbb{C}^k$. An operator $J \in B(\mathcal{H})$ is a *Jordan operator* if it can be written as $J = \bigoplus_{k=1}^{n} T_k(\alpha_k)$ with respect to a suitable decomposition $\mathcal{H} = \bigoplus_{k=1}^{n}\left(\bigoplus_{j=1}^{\alpha_k} \mathbb{C}_j^k\right)$ of $\mathcal{H}$.

Clearly, every nilpotent operator $T \in B(\mathcal{H})$ is quasisimilar to a Jordan operator.

Proposition 4.44 *Let $T \in B(\mathcal{H})$ be such that $T^3 = 0$. Then $Hyperlat(T)$ is the chain of five elements*

$$\{0\} \subset \overline{Ran(T^2)} \subset Ker(T) \subset \overline{Ran(T)} = Ker(T^2) \subset \mathcal{H}.$$

Corollary 4.45 *There exist two quasisimilar nilpotent operators T and J of order three such that $Hyperlat(T)$ and $Hyperlat(J)$ do not contain the same (finite) number of elements. In particular, these lattices are not order-isomorphic.*

Remark 4.27

Fillmore, Herrero and Longstaff [21] have given an example of a nilpotent operator T such that $T^3 = 0$ and such that $Hyperlat(T)$ can only have four, six or eight elements and a Jordan operator J quasisimilar to T. However, by Proposition 4.44, the $Hyperlat(T)$ of a nilpotent operator T of nilpotence index 3 has five elements. Wu [83] has shown that if T is a C_0 contraction with finite defect indices, then $Hyperlat(T)$ is (lattice) generated by those subspaces which are either $Ker(\psi(T))$ or $\overline{Ran(\xi(T))}$, where ψ and ξ are scalar-valued inner functions. This result was extended to general operators by Fillmore, Herrero and Longstaff [21], who have shown that on a finite-dimensional space $\mathcal{H}$, $Hyperlat(T)$ is (lattice) generated by those subspaces which are either $Ker(p(T))$ or $Ran(q(T))$, where p and q are polynomials.
We give a simplified proof to the following result by Wu[81].

Theorem 4.46 [83] *Let $T \in B(\mathcal{H})$ be a contraction of class $C_{.0}$ with finite defect indices acting on a separable Hilbert space. Then $Hyperlat(T)$ is (lattice) generated by those subspaces which are either $Ker(\psi(T))$ or $Ran(\xi(T))$, where ψ and ξ are scalar-valued inner functions.*

Proof. The result follows easily since every unilateral shift on a Hilbert space $\mathcal{H}$ is unitarily equivalent to the operator of multiplication by z on the Hardy space $\mathbb{H}^2(\mathcal{H})$ and the fact that the unilateral shift is of class $C_{.0}$.
We give an extension of Theorem 4.46 to a general linear operator.

Corollary 4.47 *Let T be a linear transformation on a finite-dimensional space $\mathcal{H}$. Then $Hyperlat(T)$ is (lattice) generated by those subspaces which are either $Ker(p(T))$ or $Ran(q(T))$, where p and q are polynomials.*

Proof. For $0 < \alpha < 1/\|T\|$, $S \approx \alpha T$ is a strict contraction, hence a contraction of class $C_{\cdot 0}$. Theorem 4.46 implies that $Hyperlat(S) = Hyperlat(T)$ is (lattice) generated by those subspaces which are either $Ker(U)$ or $Ran(V)$, where U, V are operators in $\{S\}'' = \{T\}''$, the double commutants of S and T. Our assertion follows from the fact that $\{T\}''$ consists of all polynomials in T.

Definition 4.10

An operator is *reducible* if it has a nontrivial reducing subspace (equivalently, if it has a proper nonzero direct summand); otherwise it is said to be *irreducible* (e.g. a unilateral shift of multiplicity one is irreducible and so is the 2 by 2 operator matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$).
An operator T is said to be *reductive* if all its invariant subspaces are reducing. Note that an operator may be reducible but fail to be reductive. Thus the class of reductive operators is contained in the class of reducible operators. From Proposition 4.40, every self-adjoint (and by extension, normal) operator is reductive.
We state and prove the following result.

Theorem 4.48 *Every operator unitarily equivalent to a reducible operator is reducible.*

Proof. Let $\mathcal{H}$ and $\mathcal{K}$ be unitarily equivalent Hilbert spaces. Take $T,\ P \in B(\mathcal{H})$ and an arbitrary operator $U : \mathcal{K} \longrightarrow \mathcal{H}$. Put $S = U^*TU$ and $E = U^*PU$ in $B(\mathcal{K})$. The operator E is an orthogonal projection if and only if P is. Indeed $E^2 = U^*P^2U$ and $E^* = U^*P^*U$ so that $E = E^2$ if and only if $P = P^2$ and $E = E^*$ if and only $P = P^*$. Moreover, $E = U^*PU$ is nontrivial if and only if P is, and E commutes with S if and only if P commutes with T (since $ES - SE = U^*(PT - TP)U$). Thus S is reducible if and only if T is reducible.

Remark 4.28

We note that Theorem 4.48 does not hold under similarity. For consider the matrices

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

representing operators in $\mathbb{C}^3$. A simple matrix computation shows that $XA = BX$, X is invertible (thus A and B are similar) and B is a direct sum, $B = 1 \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ but

A is irreducible since the only one-dimensional invariant subspace $\mathcal{M} = span\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$ for A is not invariant for A^*.

We now extend and prove the celebrated Lomonosov theorem to complex Hilbert spaces.

Theorem 4.49 ***(Lomonosov Theorem)***([58, §8.3], [46, *Theorem* 0.12]). *If a non-scalar operator commutes with a nonzero compact operator, then it has a nontrivial hyperinvariant subspace.*

Proof. Let T be an operator on a complex Hilbert space. Suppose that there exists a nonzero compact operator A in $\{T\}'$, and suppose T has no nontrivial hyperinvariant subspace. The following assertions hold

(a) There exist an operator L in $\{T\}'$ such that $Ker(I - LA)$ is nonzero and T-invariant.

(b) T has an eigenvalue $\lambda \in \mathbb{C}$ such that $Ker(\lambda I - T) \neq \{0\}$.

But $Ker(\lambda I - T)$ is a hyperinvariant subspace for T. Therefore, if T has no nontrivial hyperinvariant subspace, then $Ker(\lambda I - T) = \mathcal{H}$. Equivalently, $T = \lambda I$; that is, T is scalar operator. This sums up the following: If an operator T has no nontrivial hyperinvariant subspace and commutes with a nonzero compact operator A, then T must be scalar. □

Note that any scalar operator T commutes with any operator in $B(\mathcal{H})$. Thus $\{T\}' = B(\mathcal{H})$. By Corollary 4.19, we have $Hyperlat(T) = Lat(\{T\}') = Lat(B(\mathcal{H})) = \{\{0\}, \mathcal{H}\}$.

We introduce the following notation: we let $A = \lim_{n \longrightarrow \infty} T^{*n}T^n$ and $A_* = \lim_{n \longrightarrow \infty} T^n T^{*n}$. We note that $A = 0$ if and only if $T^n x \longrightarrow 0 \ \ (n \longrightarrow \infty)$ and $A^* = 0$ if and only if $T^{*n}x \longrightarrow 0 \ \ (n \longrightarrow \infty)$.

Corollary 4.50 *If T is a contraction for which $A \neq 0$ and $A_* \neq 0$, then either T has a nontrivial hyperinvariant subspace or T is a scalar unitary.*

Proof. We consider two cases:

If $Ker(A) = Ker(A_*) = \{0\}$, then T is a C_{11}-contraction and hence it either has a nontrivial hyperinvariant subspace or it is a scalar unitary. This is because on a Hilbert space of dimension greater than one, a C_{11} has a nontrivial invariant subspace.

If $Ker(A) \neq \{0\}$, then by [45, Proposition 3.1(i)], $Ker(A)$ is a nontrivial hyperinvariant subspace for T (since $Ker(A) \neq \mathcal{H}$ because $A \neq 0$). Equivalently, if $Ker(A_*) \neq \{0\}$, then $Ker(A_*)$ is a nontrivial invariant subspace for T^*, so that $Ker(A_*)^{\perp}$ is a nontrivial hyperinvariant subspace for T. This completes the proof.

Remark 4.29

We now study the invariant subspaces of a shift operator. We note that the results can be extended to completely non-normal operators. This is true since every operator can be modeled using the backward shift by Remark 3.24 and Theorem 3.33.
The functions $e_n(z) = z^n$ for $n \in \mathbb{Z}$ form an orthonormal basis in $L^2(\partial\mathbb{D})$. The orthonormal expansions

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n)e_n, \quad \hat{f}(n) = \frac{1}{2\Pi}\int_0^{2\Pi} f(e^{it})e^{-int}dt, \quad n \in \mathbb{Z}$$

are just the classical Fourier series. Since $\widehat{(zf)}(n) = \hat{f}(n-1)$, for $n \in \mathbb{Z}$, the action of the operator $f \longrightarrow zf$ can be considered as a right translation or shift. Let $\mathcal{M} \subseteq L^2(\partial\mathbb{D})$, where $\mathcal{M}$ is a closed linear subspace. We want to find out how this $\mathcal{M}$ looks like. We distinguish two separate cases:

$$z\mathcal{M} = \mathcal{M}, \qquad or \qquad z\mathcal{M} \neq \mathcal{M}$$

We note that $z\mathcal{M} = \mathcal{M}$ if and only if $\bar{z}\mathcal{M} = \mathcal{M}$, since $z \in \partial\mathbb{D}$ and $z\bar{z} = |z|^2 = 1$. In this case when $z\mathcal{M} \subseteq \mathcal{M}$ and $\bar{z}\mathcal{M} \subseteq \mathcal{M}$ then $\mathcal{M}$ is a reducing subspace and in the case when $z\mathcal{M} \subset \mathcal{M}$, $z\mathcal{M} \neq \mathcal{M}$. $\mathcal{M}$ is simply invariant and not reducing.
First we consider the reducing subspaces of the shift.
We write $d\mu$ for normalized Lebesgue measure on the circle. That is $d\mu = \frac{1}{2\Pi}dt$.

Theorem 4.51 ***(Wiener)***[58]. *Let $\mathcal{M} \subseteq L^2(\partial\mathbb{D})$ satisfy $z\mathcal{M} = \mathcal{M}$. Then there is a unique measurable set $\sigma \subseteq \partial\mathbb{D}$ such that $\mathcal{M} = \chi_\sigma L^2(\partial\mathbb{D}) = \{f \in L^2(\partial\mathbb{D}) : f = 0$ a.e. outside $\sigma\}$, where χ_σ is the characteristic function (indicator function) of σ.*

Proof. Let $\chi = P_{\mathcal{M}}1$, $\chi \in \mathcal{M}$, where $P_{\mathcal{M}}$ is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto $\mathcal{M}$. We have the following:

$$1 - \chi = (I - P_{\mathcal{M}})1 \in \mathcal{M}^{\perp},$$

and so

$$z^n\chi \perp 1 - \chi, \quad \forall\, n \in \mathbb{Z},$$

that is,

$$\int_{\partial\mathbb{D}} z^n\chi(1-\overline{\chi})d\mu = 0, \quad \forall\, n\in\mathbb{Z}.$$

Since $\chi(1-\overline{\chi}) \in L^1(\partial\mathbb{D})$, the product $\chi(1-\overline{\chi})d\mu$ is a finite complex Borel measure on $\partial\mathbb{D}$ which annihilates the set $\mathcal{T}$ of trigonometric polynomials, the set of finite linear combinations of powers z^n with $n\in\mathbb{Z}$. But $\mathcal{T}$ is dense in the vector space $C(\partial\mathbb{D})$ of continuous functions on the unit circle, so, $\chi(1-\overline{\chi}) = 0$ a.e. Hence $\chi = |\chi|^2$ a.e. and this implies that χ takes only the values 0 and 1. Let $\sigma = \{t : \chi(t) = 1\}$. Then the set σ is well-defined up to a set of measure zero. Since $\chi \in \mathcal{M}$, we have $z^n\chi \in \mathcal{M}$ for all $n\in\mathbb{Z}$, and $\mathcal{T}\chi \subseteq \mathcal{M}$ and $\overline{\chi\mathcal{T}} \subseteq \mathcal{M}$. On the other hand, $\overline{\chi\mathcal{T}} = \chi L^2(\partial\mathbb{D})$, since $\overline{\mathcal{T}} = L^2(\partial\mathbb{D})$. Thus $\chi L^2(\partial\mathbb{D}) \subseteq \mathcal{M}$, and it only remains to show that these two spaces are equal.

Let $f\in\mathcal{M}$ with $f\perp \chi z^n$, $\forall\, n\in\mathbb{Z}$. Then $z^n f\in\mathcal{M}$ for all n, and $1-\chi \perp z^n f$, $\forall\, n\in\mathbb{Z}$, and these imply that

$$\int_{\partial\mathbb{D}} f\chi z^{-n}d\mu = 0 \qquad \textit{and} \qquad \int_{\partial\mathbb{D}} z^n f(1-\chi)d\mu = 0, \quad n\in\mathbb{Z}.$$

Hence $f\chi = f(1-\chi) = 0$ a.e., and $f = 0$ a.e. Hence $\chi L^2(\partial\mathbb{D}) = \mathcal{M}$.

We now study the structure of the simply invariant subspace of the shift operator.

Theorem 4.52 ***(Beurling, H. Helson)***[58]. *Let $\mathcal{M} \subseteq L^2(\partial\mathbb{D})$ with $z\mathcal{M} \subset \mathcal{M}$, $z\mathcal{M} \neq \mathcal{M}$. Then there exists a measurable function θ (unique up to a constant) such that $|\theta| = 1$ a.e. on $\partial\mathbb{D}$ and $\mathcal{M} = \theta\mathbb{H}^2$.*

Proof. First we note that $\theta\mathbb{H}^2$ is a closed subspace, since the transformation $f \longrightarrow \theta f$ is an isometry and even a unitary operator on $L^2(\partial\mathbb{D})$. Using the orthogonal projection method as in Theorem 4.51, we consider the orthogonal complement of $z\mathcal{M}$ in $\mathcal{M}$: $\mathcal{M}\ominus z\mathcal{M}$ is a nontrivial subspace of $\mathcal{M}$, so we take $\theta \in \mathcal{M}\ominus z\mathcal{M}$ with $\|\theta\| = 1$. Then $\theta\in\mathcal{M}$ and $\theta \perp z\mathcal{M}$, and so $z^n\theta \in z\mathcal{M}$, for $n\geq 1$, implying that $\theta \perp z^n\theta$. That is,

$$\int_{\partial\mathbb{D}} \overline{\theta}\theta z^n d\mu = 0, \quad n\geq 1 \quad i.e.$$

$$\int_{\partial\mathbb{D}} |\theta|^2 z^n d\mu = 0, \quad n\geq 1.$$

Taking complex conjugates we have:

$$\int_{\partial\mathbb{D}} |\theta|^2 \overline{z}^n d\mu = 0, \quad n\geq 1,$$

that is, $(\widehat{|\theta|^2})(n) = 0$ for $n \in \mathbb{Z}\backslash\{0\}$.

Thus $|\theta|^2 = const = c$ a.e. Since $1 = \|\theta\|_2^2 = \int_{\mathbb{D}} |\theta|^2 d\mu = c\mu(\partial\mathbb{D}) = c$, we have $|\theta| = 1$ a.e.

Thus $f \longrightarrow \theta f$ is an isometry in $L^2(\partial\mathbb{D})$. Thus we have $z^n\theta \in \mathcal{M}$, for $n \geq 0$. The linear span has the same property. We write $\mathcal{P}$ for the set of polynomials in z, so $\mathcal{P}\theta \subseteq \mathcal{M}$, and $\overline{\theta\mathcal{P}} = \theta\overline{\mathcal{P}} = \theta\mathbb{H}^2 \subseteq \mathcal{M}$. Thus we have a closed subspace of $\mathcal{M}$, $\theta\mathbb{H}^2 \subseteq \mathcal{M}$, and we want it to coincide with $\mathcal{M}$. To show this, consider $f \in \mathcal{M}$, $f \perp \theta\mathbb{H}^2$. We need to show that $f = 0$. Indeed, we have:

$$f \perp \theta\mathbb{H}^2 \quad \Longrightarrow \quad f \perp \theta z^n, \quad n \geq 0,$$

and

$$f \in \mathcal{M} \quad \Longrightarrow \quad z^n f \in z\mathcal{M}, \quad n \geq 1 \quad \Longrightarrow \quad z^n f \perp \theta, \quad n \geq 1.$$

It follows that

$$\int_{\partial\mathbb{D}} f\overline{\theta}\overline{z}^n d\mu = 0, \quad n \geq 0,$$

and

$$\int_{\partial\mathbb{D}} f\overline{\theta} z^n d\mu = 0, \quad n \geq 1.$$

Thus $(\widehat{f\overline{\theta}})(n) = 0, \quad \forall\, n \in \mathbb{Z}$ and $f\overline{\theta} \equiv 0$. But $|\theta| = 1$ a.e., and so $f = 0$ a.e. and $\mathcal{M} = \theta\mathbb{H}^2$.

To show uniqueness, let $\theta_1\mathbb{H}^2 = \theta_2\mathbb{H}^2$, where $|\theta_1| = |\theta_2| = 1$ a.e. on $\partial\mathbb{D}$. Then $(\theta_1\overline{\theta}_2)\mathbb{H}^2 = \mathbb{H}^2$, so $\theta_1\overline{\theta}_2 \in \mathbb{H}^2$, and, by symmetry, $\theta_2\overline{\theta}_1 \in \mathbb{H}^2$, or $\theta_1\overline{\theta}_2 \in \overline{\mathbb{H}^2}$. But $\mathbb{H}^2 \cap \overline{\mathbb{H}^2} = \{const\}$, since, for instance $f \in \mathbb{H}^2 \Longrightarrow \widehat{f}(n) = 0, n < 0$; and $\overline{f} \in \mathbb{H}^2 \Longrightarrow \widehat{\overline{f}}(n) = \overline{\widehat{f}(-n)} = 0, n < 0 \Longrightarrow f = const$.

Theorem 4.52 implies a particular result about closed shift-invariant subspaces of $\mathbb{H}^2$, generally referred to as Beurling's theorem.

Lemma 4.53 ***(Beurling)***[58] *Any closed shift-invariant subspace $\mathcal{M} \subseteq \mathbb{H}^2$ has the form $\mathcal{M} = \theta\mathbb{H}^2$, where θ is inner.*

Proof. Clearly, if $\theta\mathbb{H}^2 \subset \mathbb{H}^2$ and $|\theta| = 1$ a.e on $\partial\mathbb{D}$ then θ is in $\mathbb{H}^2$, and hence inner.

Remark 4.30

The Beurling Theorem characterizes invariant subspaces of the shift operator in terms of operator-valued inner functions on the unit disk. If $A \in B(\mathcal{H})$, then by Corollary 4.39

the reducing subspaces of A are the ranges of the orthogonal projections P such that $AP = PA$. Taking the adjoint of the last equality we obtain $PA^* = A^*P$. Thus, the problem of finding the reducing subspaces of A is contained in the problem of finding all operators that commute with A and A^*.
We solve this problem for the multiplication operator.

Theorem 4.54 *Let μ be a finite, positive, compactly supported Borel measure in the complex plane $\mathbb{C}$ and let A be the operator in $L^2(\mu)$ of multiplication by z,*

$$(Af)(z) = zf(z), \quad f \in L^2(\mu), \ z \in \mathbb{C}.$$

Then the operators that commute with A and A^* are precisely the operators on $L^2(\mu)$ of multiplication by the functions in $L^\infty(\mu)$.

Proof. One half of Theorem 4.54 is trivial: If A is as described and $\varphi \in L^\infty(\mu)$, then multiplication by φ obviously defines an operator on $L^2(\mu)$ that commutes with A and A^*. The other half of the theorem is easy to prove.

Remark 4.31

The unilateral shift S_+ is defined on ℓ^2 so that

$$(S_+f)(n) = \begin{cases} 0, & n = 0 \\ f(n-1), & n > 0 \end{cases}$$

The operator S_+ is an isometry and its adjoint, the backward shift, satisfies

$$(S_+^*f)(n) = f(n+1), \quad f \in \ell^2.$$

The sequence $\{S_+^{*n}\}$ converges strongly to 0. The minimal unitary (normal) extension U of S_+ is the bilateral shift defined on $L^2(\mathbb{D}) = \{f : \mathbb{Z} \longrightarrow \mathbb{D} \ \textit{and} \ \sum_{n=-\infty}^{\infty} \|f(n)\|^2 < \infty\}$ and U is defined by

$$(Uf)(n) = f(n-1), \quad \textit{for} \quad f \in L^2(\mathbb{D}).$$

It is easily verified that U is unitary and we identify ℓ^2 as a subspace of $L^2(\mathbb{D})$ in the obvious way , $S_+ = U|_{\ell^2}$. By the von Neumann-Wold decomposition, any isometry $T = W \oplus S_+$, where W is unitary and S_+ is a unilateral shift. The commutant of any isometry $T = W \oplus S_+$ consists of the restriction of operators with matrix $\begin{pmatrix} A_1 & 0 \\ A_3 & A_4 \end{pmatrix}$

where $A_1U = UA_1$, $A_1\ell^2 \subset \ell^2$, $A_3U = WA_3$ and $A_4W = WA_4$.
Shifts are of fundamental importance in Operator Theory. They can be considered as prototypes (models) of infinite-dimensional operators (i.e. operators with an infinite-dimensional range). Every operator is unitarily equivalent to a multiple of a part of the adjoint of a unilateral shift.

Remark 4.32

We now study some special T-invariant subspaces for arbitrary $T \in B(\mathcal{H})$. We define the kernels and ranges of the power T^n, $n = 0, 1, 2, ...$ of a linear operator T on a Hilbert space $\mathcal{H}$. We have the following two sequences of subspaces.

$$Ker(T^0) = \{0\} \subseteq Ker(T) \subseteq Ker(T^2) \subseteq ...$$

and

$$T^0(\mathcal{H}) = \mathcal{H} \supseteq T(\mathcal{H}) \supseteq T^2(\mathcal{H}) \supseteq ...$$

Generally, all these inclusions are strict. We note that for weighted unilateral shift operators one or both of these two sequences becomes constant.
We define the following special subspaces of $\mathcal{H}$.

$$T^\infty(\mathcal{H}) = \bigcap_{n=0}^{\infty} T^n(\mathcal{H})$$

$$Ker^\infty(T) = \bigcup_{n=0}^{\infty} Ker(T^n)$$

Proposition 4.55 *Both $T^\infty(\mathcal{H})$ and $Ker^\infty(T)$ are T-invariant.*

Proof. Let $x \in T^\infty(\mathcal{H})$. That is $x \in \bigcap_{n=0}^{\infty} T^n(\mathcal{H})$. Then

$$Tx \in T\Big(\bigcap_{n=0}^{\infty} T^n(\mathcal{H})\Big) = \bigcap_{n=0}^{\infty} T^{n+1}(\mathcal{H}) \subseteq \bigcap_{n=0}^{\infty} T^n(\mathcal{H})$$

Therefore $Tx \in \bigcap_{n=0}^{\infty} T^n(\mathcal{H})$. Thus $T^\infty(\mathcal{H})$ is invariant under T. The proof of showing that $Ker^\infty(T)$ is T-invariant is similar.

Theorem 4.56 *Let $T \in B(\mathcal{H})$ be a $C_{.0}$ contraction. Then $\mathcal{H} = T^\infty(\mathcal{H}) \oplus Ker^\infty(T)$.*
We note that this result can be extended to C_{00} contractions since $T \in C_{.0}$ implies $T^* \in C_{0.}$.

Remark 4.33

If $T \in B(\mathcal{H})$ is a nilpotent operator of nilindex n, then $Lat(T)$ $\left(Hyperlat(T)\right)$ is a chain and $Red(T) = \left\{\{0\}, \mathcal{H}\right\}$. The lattice of invariant subspaces of an operator T is a metric space. We give various topological conditions on a point in the lattice which ensure it is a hyperinvariant subspace for the operator $T \in B(\mathcal{H})$. We can extract some information about the structure or properties of an operator given its invariant subspace lattice.

We introduce the following terminology concerning subspace lattices.

Definition 4.11 *A lattice is* commutative *if for every pair of subspaces $\mathcal{M}$ and $\mathcal{N}$ in the lattice, the corresponding projections $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ commute.*

Theorem 4.57 *If $Lat(T)$ is commutative, then $Lat(T) = Hyperlat(T)$.*

Proof. This result follows from Theorem 4.37, Theorem 4.38 and Corollary 4.39 using Definition 4.11.

Example 4.3

For the unilateral shift A with multiplicity 1, $Lat(A) = Hyperlat(A)$. This follows clearly from Theorem 4.52 and Theorem 4.54.

Corollary 4.58 *If $dim(\mathcal{H}) < \infty$, then the following conditions are equivalent.*
(i) $Lat(T) = Hyperlat(T)$
(ii) $Lat(T)$ is finite

Proof. From Theorem 4.47, we have $Hyperlat(T) = \left\{Ker\left[(T-\lambda_1 I)...(T-\lambda_n I)\right] : \lambda_1, ..., \lambda_n \in \sigma(T)\right\} \cup \left\{Ran\left[(T-\lambda_1 I)...(T-\lambda_n I)\right] : \lambda_1, ..., \lambda_n \in \sigma(T)\right\} \cup \left\{\{0\}, \mathcal{H}\right\}$ and this is a finite set, since $\mathcal{H}$ is finite-dimensional.

Remark 4.34

The following results relate the lattices of invariant subspaces of similar operators. Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, and let $A : \mathcal{H} \longrightarrow \mathcal{H}$, $B : \mathcal{K} \longrightarrow \mathcal{K}$, and $X : \mathcal{H} \longrightarrow \mathcal{K}$ be operators such that X intertwines the operators A and B, i.e. $XA = BX$. Then the map $\Omega_X : Lat(A) \longrightarrow Lat(B)$ given by $\Omega_X(\mathcal{M}) = \overline{X(\mathcal{M})}$, $\mathcal{M} \in Lat(A)$ is well defined.

Theorem 4.59 *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, and let $A : \mathcal{H} \longrightarrow \mathcal{H}$, $B : \mathcal{K} \longrightarrow \mathcal{K}$, and $X : \mathcal{H} \longrightarrow \mathcal{K}$ be operators such that $XA = BX$. Then the map $\Omega_X : Lat(A) \longrightarrow Lat(B)$ given by $\Omega_X(\mathcal{M}) = \overline{X(\mathcal{M})}$, $\mathcal{M} \in Lat(A)$ has the following properties*

(1) Ω_X is a lattice isomorphism if and only if Ω_X is a bijection

(2) $\Omega_X(Lat(A)) = Lat(B)$ if and only if Ω_{X^*} is injective

(3) Ω_X is injective if $\mathcal{M}_1 = \mathcal{M}_2$ whenever $\mathcal{M}_1$, $\mathcal{M}_2 \in Lat(A)$, $\mathcal{M}_1 \subset \mathcal{M}_2$, and $\Omega_X(\mathcal{M}_1) = \Omega_X(\mathcal{M}_2)$.

Proof.

(1). Ω_X a lattice isomorphism if and only if it is an invertible lattice homomorphism that preserves spans(joins) and intersections(meets). This is equivalent to saying that Ω_X is a bijection of $Lat(A)$ onto $Lat(B)$.

(2). First note that $\Omega_{X^*} : Lat(B^*) \longrightarrow Lat(A^*)$ is well-defined by $\Omega_{X^*}(\mathcal{N}) = \overline{(X^*\mathcal{N})}$, $\mathcal{N} \in Lat(B^*)$. Since $XA = BX$, then $A^*X^* = X^*B^*$. Thus Ω_{X^*} is onto $Lat(A^*)$ if and only if $\Omega_{X^{**}} = \Omega_X$ is one-to-one on $Lat(A)$. It is clear that Ω_{X^*} is a lattice isomorphism since Ω_X is by (1). This proves that Ω_{X^*} is injective. Conversely, suppose Ω_{X^*} is injective. Then $Ker(\Omega_{X^*}) = \{0\}$ and $Ran(\Omega_{X^*}) = Lat(A^*)$, which shows that Ω_{X^*} is onto $Lat(A^*)$ and hence Ω_X is onto $Lat(B)$. Equivalently, $\Omega_X(Lat(A)) = Lat(B)$.

(3). This is trivial and follows from the definition.

Remark 4.35

In Theorem 4.59, if X is invertible, then A and B are similar. This is in accordance with the well known fact in [8] that similarity preserves the lattice of invariant subspaces. Recall from [8] that similarity preserves many other characteristics of the operators A and B, e.g, the multiplicity, spectra, etc. Quasisimilarity is weaker than similarity and it has been proved that quasisimilarity does not preserve invariant subspace lattices of quasisimilar operators A and B, in general. It preserves a sublattice of the invariant subspace lattice . But if A and B are quasisimilar weak contractions, then $Lat(A)$ and $Lat(B)$ are isomorphic. However, Hoover [39] has shown that quasisimilar operators have isomorphic hyperinvariant subspace lattices.

All normal operators have reducing subspaces ($\mathcal{M}$ and $\mathcal{M}^{\perp}$). This follows from the spectral theorem. Unless T is a multiple of the identity, these reducing subspaces are even hyperinvariant. Every subnormal operator, i.e., $T = N|_{\mathcal{M}}$, where $\mathcal{M}$ is an invariant subspace of the normal operator N, has invariant subspaces. These invariant subspaces need not be reducing, e.g., the unilateral shift (which is subnormal) has no reducing subspaces by Beurling's theorem although a unilateral shift has plenty of invariant subspaces.

Proposition 4.60 *For every $T \in B(\mathcal{H})$ and for every $\mathcal{M} \in Hyperlat(T)$, $P_{\mathcal{M}}$ belongs to $W^*(T)$, where $P_{\mathcal{M}}$ is the projection of T on $\mathcal{M}$ and $W^*(T)$ denotes the (weakly closed) (unital) von Neumann algebra generated by T.*

Proof. By the double commutant theorem, it suffices to show that if $Q = Q^2 = Q^* \in \{W^*(T)\}' = \{T\}' \cap \{T^*\}'$, then $P_{\mathcal{M}}Q = QP_{\mathcal{M}}$, or, equivalently, that $Q\mathcal{M} \subset \mathcal{M}$. Since $Q \in \{T\}'$ and $\mathcal{M} \in Hyperlat(T)$, this proves the result.

Proposition 4.61 *Let T be a normal operator in $B(\mathcal{H})$. Then*

$$Hyperlat(T) = \{\mathcal{M} \subset \mathcal{H} : P_{\mathcal{M}} \in W^*(T)\}.$$

Proof. By Proposition 4.60, if $\mathcal{M} \in Hyperlat(T)$, then $P_{\mathcal{M}} \in W^*(T)$. On the other hand, by Fuglede's theorem

$$\{T\}' = \{T\}' \cap \{T^*\}' = W^*(T).$$

Thus, if $P_{\mathcal{M}} \in W^*(T)'' = W^*(T)$ and $S \in \{T\}'$, then $P_{\mathcal{N}}S = SP_{\mathcal{N}}$ so $S\mathcal{N} \subset \mathcal{N}$ and $\mathcal{N} \in Hyperlat(T)$.

Remark 4.36

Recall from Corollary 4.47 that $Hyperlat(T)$ is generated as a lattice by the spaces $Ker(T^m)$ and $Ran(T^m)$, $m = 0, 1, 2, ..., n$. Lemma 4.42 can be generalized to show that $Ran(T^{n-1})$ (respectively, $Ker(T^{n-1})$) is the smallest (respectively, the largest) nontrivial hyperinvariant subspace for a nilpotent operator $T \in B(\mathcal{H})$, where n is the nilpotency of T. We give an illustration, where J_n denotes a Jordan operator.

Example 4.4

When nilpotency $n = 1$, we have $T = 0$ and the only hyperinvariant subspaces are the trivial ones: $T = J_1 : \{0\} \subset \mathcal{H}$.

When $n = 2$, there are two possible lattices:

$$
\begin{aligned}
T &= J_2 : \{0\} \subset Ker(T) = Ran(T) \subset \mathcal{H}, \\
T &= J_2 \oplus J_1 : \{0\} \subset Ker(T) \subset Ran(T) \subset \mathcal{H}
\end{aligned}
$$

while for $n = 3$, there are four possibilities:

$$
\begin{aligned}
T &= J_3 : \{0\} \subset Ran(T^2) = Ker(T) \subset Ran(T) = Ker(T^2) \subset \mathcal{H}. \\
T &= J_3 \oplus J_2 : \{0\} \subset Ran(T^2) \subset Ker(T) \subset Ran(T) \subset Ker(T^2) \subset \mathcal{H} \\
T &= J_3 \oplus J_1 : \{0\} \subset Ran(T^2) = Ran(T) \cap Ker(T) \subset Ran(T), \\
&\quad Ker(T) \subset Ran(T) \bigvee Ker(T) = Ker(T^2) \subset \mathcal{H} \\
T &= J_3 \oplus J_2 \oplus J_1 : \{0\} \subset Ran(T^2) \subset Ran(T) \cap Ker(T) \subset Ran(T), \\
&\quad Ker(T) \subset Ran(T) \bigvee Ker(T) \subset Ker(T^2) \subset \mathcal{H}.
\end{aligned}
$$

These results follow clearly from Remark 4.32 and Proposition 4.55. We give an example to show that quasisimilarity does not preserve the hyperlattice.

Example 4.5

Consider the operators

$$
\begin{aligned}
A &= J_1 \oplus J_2 \oplus J_2 \oplus \ldots \\
B &= J_2 \oplus J_2 \oplus J_2 \oplus \ldots
\end{aligned}
$$

where J_n denotes the Jordan operator associated with a nilpotent operator of nilpotency n. Clearly, A and B are quasisimilar but $Hyperlat(A)$ has *four* elements (i.e. is of height 4) and $Hyperlat(B)$ has *three* elements (i.e. is of height 3). In this case we have four lattices, where the first two are totally ordered and other two are not totally ordered. Recall that lattices are isomorphic if they have the same number of levels or heights. The following result strengthens Theorem 4.57. It says that the converse of Theorem 4.57 is also true.

Corollary 4.62 *Let $T \in B(\mathcal{H})$. $Hyperlat(T) = Lat(T)$ if $Lat(T)$ is any one of the following:*

(i) commutative

(ii) totally ordered.

Proof.

By assumption, $Hyperlat(T) = Lat(T)$ if $\{\mathcal{M} : T\mathcal{M} \subset \mathcal{M}\} = \{\mathcal{M} : S\mathcal{M} \subset \mathcal{M}, S \in \{T\}'\}$. Clearly $\mathcal{M}, \mathcal{N}$ are in $Lat(T)$ if and only if $P_{\mathcal{M}}TP_{\mathcal{M}} = TP_{\mathcal{M}}$ and $P_{\mathcal{N}}TP_{\mathcal{N}} = TP_{\mathcal{N}}$, where $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are the projections of $\mathcal{H}$ onto $\mathcal{M}$ and $\mathcal{N}$, respectively. Thus by the hypothesis, $P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{M}}P_{\mathcal{N}}$ and $\mathcal{M} \subseteq \mathcal{N}$ or $\mathcal{N} \subseteq \mathcal{M}$. This proves that $Lat(T)$ is commutative or totally ordered.

Remark 4.37

We note here that any element in $Hyperlat(T)$ is the kernel or range of some operators in $\{T\}''$. Note also that $T = \beta I$ (i.e. scalar operator) if and only if $Hyperlat(T) = \{\{0\}, \mathcal{H}\}$ if and only if $Red(T)$ is the collection (in this case, a lattice) of all subspaces of $\mathcal{H}$. Recall that $Red(T)$ is not necessarily a lattice.

Theorem 4.63 *If $Tx_k = x_{k+1}$ is the unilateral shift operator, the only closed linear subspaces which reduce T are $\{0\}$ and $\mathcal{H}$.*

Proof. Suppose $\mathcal{M} \neq \{0\}$ is a closed linear subspace which reduces T. For a non-zero vector $y = \sum_m^\infty \lambda_k x_k$, define the index of y to be the smallest subscript k such that $\lambda_k \neq 0$. Let m be the smallest index of any non-zero vector in $\mathcal{M}$, and choose any non-zero $y \in \mathcal{M}$ with index m. Clearly, $y = \sum_m^\infty \lambda_k x_k$. Necessarily, $m = 1$; otherwise, $\mathcal{M}$ would contain the non-zero vector $T^*y = \sum_m^\infty \lambda_k x_{k-1} = \sum_{m-1}^\infty \lambda_{k+1} x_k$, contrary to the minimality of m. We may suppose $\lambda_1 = 1$, thus $y = x_1 + \sum_2^\infty \lambda_k x_k$. One has $T^*y = T^*x_1 + \sum_2^\infty \lambda_k T^* x_k = 0 + \sum_2^\infty \lambda_k x_{k-1}$, hence $TT^*y = \sum_2^\infty \lambda_k Tx_{k-1} = \sum_2^\infty \lambda_k x_k = y - x_1$. Since $x_1 = y - TT^*y$, clearly $x_1 \in \mathcal{M}$. Clearly $\mathcal{M}$ also contains $Tx_1 = x_2$, $Tx_2 = x_3$, and so on. This shows that $\mathcal{M}^\perp = \{0\}$ and $\mathcal{M} = \mathcal{H}$. This completes the proof.

We note that by Theorem 4.63 that a unilateral shift or a direct sum of unilateral shifts has no nontrivial unitary summand in its decomposition.

Theorem 4.64 *If $\mathcal{M}$ reduces T, then $(T|_{\mathcal{M}})^* = T^*|_{\mathcal{M}}$.*

Proof. Let $R = T|_{\mathcal{M}}$ and $S = T^*|_{\mathcal{M}}$. For all $x, y \in \mathcal{M}$.

$$\langle R^*x, y\rangle = \langle x, Ry\rangle = \langle x, Ty\rangle = \langle T^*x, y\rangle = \langle Sx, y\rangle,$$

and since R^* and S are operators in $\mathcal{M}$, $R^* = S$.

Corollary 4.65 *If $\mathcal{M}$ reduces T, and T is normal (respectively, unitary), then $T|_{\mathcal{M}}$ is normal (respectively, unitary).*

Proof. By Theorem 4.64, since $\mathcal{M}$ reduces T, $(T|_{\mathcal{M}})^* = T^*|_{\mathcal{M}}$. The normality of T implies $T^*T = TT^*$. Hence $T_{\mathcal{M}}^*T_{\mathcal{M}} = (T|_{\mathcal{M}})^*(T|_{\mathcal{M}}) = (T_{\mathcal{M}})(T_{\mathcal{M}})^* = T_{\mathcal{M}}T_{\mathcal{M}}^*$. The case when T is unitary follows easily.

We investigate nontrivial invariant subspaces of operators. First, we need the following definition.

Definition 4.12

If T is an operator and λ is a scalar, then $Ker(\lambda I - T)$ is called the $\lambda-$th nontrivial subspace of T, and is denoted by $\mathcal{M}_T(\lambda) = \{x \in \mathcal{H} : Tx = \lambda x\}$. Clearly, $\mathcal{M}_T(\lambda)$ is a closed linear subspace of $\mathcal{H}$ and is different from $\{0\}$ if and only if λ is an eigenvalue of T.

Theorem 4.66 *If S and T are operators such that $ST = TS$, then the λ- subspaces of T are invariant under S.*

Proof. If $x \in \mathcal{M}_T(\lambda)$, then $T(Sx) = (TS)x = (ST)x = S(Tx) = S(\lambda x) = \lambda(Sx)$. This shows that $Sx \in \mathcal{M}_T(\lambda)$.

Theorem 4.67 *If T is a normal operator, then*
(i) the λ-spaces of T reduce T;
(ii) $\mathcal{M}_T(\lambda) = \mathcal{M}_{T^*}(\overline{\lambda})$;
(iii) $\mathcal{M}_T(\lambda) \perp \mathcal{M}_T(\alpha)$, whenever $\lambda \neq \alpha$.

Remark 4.38

Note that Theorem 4.67 is a combination of Theorem 4.66, Corollary 4.65 and Theorem 4.66 with S replaced with T^*. It is well known (see [58]) that every reductive operator is normal if and only if it has a nontrivial invariant subspace . If $\mathcal{M}$ reduces every operator in the commutant of T, we call $\mathcal{M}$ *hyperreducing* for T (equivalently, $\mathcal{M} \in Lat\{A\}' \cap Lat\{A^*\}'$).

Proposition 4.68 *If T is reductive, then every hyperinvariant subspace of T is hyperreducing.*

Proof. Suppose that $\mathcal{M}$ is hyperinvariant for T, and suppose that B commutes with T. Then $\mathcal{M}$ is invariant under B, and with respect to the decomposition $\mathcal{M} \oplus \mathcal{M}^{\perp}$ we can write T and B as operator matrices as follows:

$$T = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} E & F \\ 0 & G \end{pmatrix}.$$

Since $TB = BT$, it is true that $RF = FS$, and by the Putnam-Fuglede theorem $R^*F = FS^*$ as well. The last equation is the same as $F^*R = SF^*$ and this means that T commutes with the operator $D = \begin{pmatrix} 0 & 0 \\ F^* & 0 \end{pmatrix}$. But by hypothesis $\mathcal{M}$ is hyperinvariant for T, and hence is invariant under D. Thus $F^* = 0$, or, $F = 0$, and hence $\mathcal{M}$ reduces B.

Theorem 4.69 *If T is a reductive operator then T can be written as a direct sum $T_1 \oplus T_2$ where T_1 is normal, T_2 is reductive, $\{T\}' = \{T_1\}' \oplus \{T_2\}'$, and all the invariant subspaces of T_2 are hyperinvariant (in fact hyperreducing). Equivalently, $Lat(T_2) = Lat\{T_2\}' \cap Lat\{T_2^*\}'$.*

Proof. Mimic the proof of Theorem 4.30.

Remark 4.39

The second part of Theorem 4.69 says that if T is a completely non-normal reductive operator (that is, T has no normal direct summand) then $Lat(A) = Lat\{A\}'$. But by Remark 4.38, every operator has a nontrivial invariant subspace if and only if every reductive operator is normal. It may turn out there are no non-normal reductive operators and that we are dealing with an empty class. Remember that every scalar operator $T \in B(\mathcal{H})$ has a nontrivial invariant subspace if $dim(\mathcal{H}) > 1$.
Using Proposition 3.44, we deduce the following result.

Corollary 4.70 *Let $T \in B(\mathcal{H})$ be an operator in class C_0. If T is not a scalar then it has nontrivial hyperinvariant subspaces.*

Proof. By Proposition 3.44, an inner divisor θ of m_T is uniquely determined (up to a constant coefficient) by the hyperinvariant subspace $Ker(\theta(T))$. Thus, if $\theta \not\equiv 1$ and $\theta \not\equiv m_T$, $Ker(\theta(T))$ is a nontrivial hyperinvariant subspace for T. Assume that m_T has no nontrivial inner divisors. Then m_T must be a Blaschke factor: $m_T(\lambda) = \frac{a-\lambda}{(1-\bar{a}\lambda)}$, $\lambda \in \mathbb{D}$, which clearly means that $T = aI$.

Theorem 4.71 *Let $T \in B(\mathcal{H})$ and $\mathcal{M} \subseteq \mathcal{H}$. If $\mathcal{M}$ is $T-$invariant then $\overline{\mathcal{M}}$ is also $T-$invariant.*

Proof. Take an arbitrary $x \in \overline{\mathcal{M}}$ so that x is a limit point of $\mathcal{M}$ and hence there exists an $\mathcal{M}-$valued sequence, say $\{x_n\}$ that converges to x. If $\mathcal{M}$ is T-invariant, then $\{Tx_n\}$ is again an $\mathcal{M}$-valued sequence. Since T is continuous, $\{Tx_n\}$ converges to Tx. But $\overline{\mathcal{M}}$ is closed and each Tx_n lies in $\mathcal{M} \subseteq \overline{\mathcal{M}}$ so that Tx lies in $\overline{\mathcal{M}}$. This shows that $\overline{\mathcal{M}}$ is T-invariant whenever $\mathcal{M}$ is.

Corollary 4.72 *Let $T \in B(\mathcal{H})$. Then*
(i) $Ker(T)$ and $\overline{Ran(T)}$ are hyperinvariant subspaces for T.
(ii) If $dim(\mathcal{H}) > 1$ and T has no nontrivial invariant subspace, then $Ker(T) = \{0\}$ and $\overline{Ran(T)} = \mathcal{H}$.

Proof. (i) Suppose $L \in B(\mathcal{H})$ commutes with T. If $x \in Ker(T)$, then $TLx = LTx = 0$, and hence $Lx \in Ker(T)$. Thus $L(Ker(T)) \subseteq Ker(T)$ so that $Ker(T)$ is L-invariant. Since $LTx = TLx$ for every $x \in \mathcal{H}$, it follows that $L(Ran(T)) \subseteq Ran(T)$, and hence $L(\overline{Ran(T)}) \subseteq \overline{L(Ran(T))} \subseteq \overline{Ran(T)}$, so that $\overline{Ran(T)}$ is L-invariant.
(ii) Suppose $dim(\mathcal{H}) > 1$ and T has no nontrivial invariant subspace. Then $T \neq 0$ and has no nontrivial hyperinvariant subspace, so that $Ker(T)$ and $\overline{Ran(T)}$ are trivial subspaces by assertion (i). But since $T \neq 0$, it follows that $Ker(T) \neq \mathcal{H}$ and $Ran(T) \neq \{0\}$. Therefore, $Ker(T) = \{0\}$ and $\overline{Ran(T)} = \mathcal{H}$.

Remark 4.40

Corollary 4.72 says that if an operator has no nontrivial invariant subspace, then it is quasiinvertible.

Corollary 4.73 *Let S and T be nonzero operators on a Hilbert space $\mathcal{H}$. If $ST = 0$, then $Ker(S)$ and $\overline{Ran(T)}$ are nontrivial invariant subspaces for both S and T.*

Proof. If $ST = 0$, then $Ran(T) \subseteq Ker(S)$, and hence $T(Ker(S)) \subseteq T(\mathcal{H}) = Ran(T) \subseteq Ker(S)$. If $T \neq 0$, then $Ran(T) \neq \{0\}$ so that $Ker(S) \neq \{0\}$. If $S \neq 0$, then

$Ker(S) \neq \mathcal{H}$ and so $\overline{Ran(T)} \neq \mathcal{H}$ because $Ker(S)$ is closed. Therefore, $\{0\} \neq Ker(S) \neq \mathcal{H}$ and $\{0\} \neq \overline{Ran(T)} \neq \mathcal{H}$. Since S is continuous and since $S(Ran(T)) = \{0\}$,

$$S(\overline{Ran(T)}) \subseteq \overline{S(Ran(T))} \subseteq \overline{Ran(T)}.$$

Thus if $S \neq 0$, $T \neq 0$, and $ST = 0$, then $Ker(S)$ and $\overline{Ran(T)}$ are nontrivial invariant subspaces for S and T, respectively.

Chapter 5

On canonical factorization of an operator

In this chapter we investigate the factorization of an operator into two or more simpler factors and the properties shared by these factors. Although operator factorization is a live subject in Operator Theory, there is no general theory on how to carry on. A number of mathematicians have considered the problem of writing an operator as a product of "*nice*" operators, such as positive, hermitian (self-adjoint), unitary, cyclic, nilpotent, quasinilpotent, normal operators, projections, idempotents, cyclic, scalar, or n-th roots of identity. Operator factorization is a first hand tool in solving many problems in mathematical and theoretical physics and the diversity of the problems necessitates to keep improving it.
We start with the following result due to Ruan and Yan [64] which will be useful in the sequel.

Lemma 5.1 [64, *Lemma* 2]. *Let* $S, T \in B(\mathcal{H})$. *If* $A = TS$ *and* $B = ST$, *then* $dim\Big[Ker(A - \lambda I)\Big] = dim\Big[Ker(B - \lambda I)\Big]$, $\lambda \neq 0$; *moreover, if* $Ker(S) = Ker(T)$, *then* $\sigma_P(A) = \sigma_P(B)$.

Remark 5.1

Recall by Definition 4.12, Lemma 5.1 says that if $A = TS$ and $B = ST$ then $dim(\mathcal{M}_A(\lambda)) = dim(\mathcal{M}_B(\lambda))$.
Factorizations of matrices over a field are useful in quite a number of problems, both

analytical and numerical; for example in the numerical solution of linear equations and eigenvalue problems. Some well-known factorizations are QR, SVD, LU, Cholesky and Wiener-Hopf factorizations (see [68] for details). We need the following definitions and terminologies.

Definition 5.1

A vector $x \in \mathcal{H}$ such that $\bigvee\{T^n x\}_{n\geq 0} = \mathcal{H}$, then x is said to be a *cyclic vector* for T, where $\bigvee\{T^n x\}_{n\geq 0} = \mathcal{H} = \overline{span\{T^n x\}_{n\geq 0}}$, which is a subspace of $\mathcal{H}$. If $T \in B(\mathcal{H})$ has a cyclic vector, then it is a *cyclic operator*. Note that the (linear) span of the *orbit* of x under T (i.e. $\{T^n x\}$) is the set of the images of all nonzero polynomials of T at x, that is,

$$span\{T^n x\}_{n\geq 0} = \Big\{p(T)x \in \mathcal{H} : p \text{ is a nonzero polynomial}\Big\}.$$

A linear manifold $\mathcal{M}$ of $\mathcal{H}$ is *totally cyclic* for T if every nonzero vector in $\mathcal{M}$ is cyclic. Observe that T has no nontrivial invariant subspace if and only if every nonzero vector in $\mathcal{H}$ is a cyclic vector for T: For if $\mathcal{M} \subseteq \mathcal{H}$ is T-invariant, then $T^n(\mathcal{M}) \subseteq \mathcal{M}$; that is, if and only if $\bigvee\{T^n x\}_{n\geq 0} = \mathcal{H}$ for every $x \neq 0$ in $\mathcal{H}$; which means that $\mathcal{H}$ is itself totally cyclic for T. A diagonal operator D is said to have *multiplicity 1* if the diagonal sequence is made up of distinct elements.

Example 5.1

A unilateral shift of multiplicity 1 on $\mathbb{C}^2$ is $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. As has been shown in Example 1.5, $\mu(T) = 1$. This is an example of a nilpotent operator of nilpotence index 2.

Remark 5.2

We show that an operator T is the product of finitely many cyclic operators if and only if the $Ker(T^*)$ is finite-dimensional. That is, if the multiplicity $\mu(T^*)$ if finite. More precisely, if $dim(KerT^*) \leq k$ $(2 \leq k < \infty)$, then T is the product of at most $k+2$ cyclic operators. Wu [82] conjectured that in this case at most k cyclic operators would suffice. We verify this conjecture for some classes of operators.

A necessary condition for an operator T to be expressible as a product of k $(1 \leq k < \infty)$ cyclic operators is that $dim\Big(Ker(T^*)\Big) \leq k$. Indeed, for $k = 1$ this is trivial. Assuming its validity for k, we prove it for $k+1$. Let $T = T_1 ... T_{k+1}$ be a product of $k+1$ cyclic operators, and let $S = T_1 ... T_k$. Then $T^* = T^*_{k+1} S^*$ implies that $dim\Big(Ker(T^*)\Big) = dim\Big(Ker(S^*)\Big) + dim\Big(Ran(S^*) \cap Ker(T^*_{k+1})\Big) \leq dim\Big(Ker(S^*)\Big) + dim\Big(Ker(T^*_{k+1})\Big) \leq k+1$. In view of this, we conjecture that, when $k \geq 2$, this necessary condition is also sufficient. This leads to the following result.

Proposition 5.2 *An operator T is the product of k $(2 \leq k < \infty)$ cyclic operators if and only if $dim(Ker(T^*)) \leq k$.*

Lemma 5.3 *Every cyclic operator is the product of two other cyclic operators.*

Proof. Since T is cyclic, by ([29], Problem 167) it has a special matrix form

$T = \begin{pmatrix} a_1 & \dots & * \\ b_1 & a_2 & \\ \vdots & b_2 & \ddots \\ 0 & & \ddots \end{pmatrix}$, where all the b_n's are non-zero. We decompose

$T = \begin{pmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & \ddots \end{pmatrix} \begin{pmatrix} d_1 & & * \\ e_1 & d_2 & \\ & e_2 & \ddots \\ 0 & & \ddots \end{pmatrix}$, where the c_n's are distinct and the e_n's are all nonzero. Then T is the product of two cyclic operators.

Remark 5.3

We study some classes of operators where the factorizations have been investigated. Suppose $\mathcal{H}$ is a separable infinite-dimensional Hilbert space, and, for each positive integer n, let $\mathcal{P}_n$ denote the set of all operators on $\mathcal{H}$ that can be written as the product of n positive operators. It was shown in Wu [84] that the union of the $\mathcal{P}_n$ is the set of invertible operators and it equals $\mathcal{P}_{17}$.

Recall that an operator $T \in B(\mathcal{H}, \mathcal{K})$ is *invertible* if it has an inverse on $Ran(T) = \mathcal{K}$; and such an inverse must be bounded. For convenience we denote by $\mathcal{G}(\mathcal{H}, \mathcal{K})$ the class of all invertible operators in $B(\mathcal{H}, \mathcal{K})$. This class contains the class of unitary operators (i.e. an invertible operator for which $T^{-1} = T^*$). Note that $\mathcal{G}(\mathcal{H}) = \mathcal{G}(\mathcal{H}, \mathcal{H})$ is a group under multiplication and not an algebra and that $\mathcal{G}(\mathcal{H}) \subset B(\mathcal{H})$. $\mathcal{G}(\mathcal{H})$ contains the identity I. Hence it is a (unital) von Neumann algebra. Recall that an operator is said to be *positive* if $\langle Tx, x\rangle > 0$ for all nonzero $x \in \mathcal{H}$. We note that positive operators are not necessarily invertible (see [45], [46], [47]). If T is a positive operator, then $Ker(T) = \{0\}$ and $Ran(T)^{\perp} = Ker(T^*) = Ker(T) = \{0\}$, (for T is self-adjoint) so that $\overline{Ran(T)} = \{0\}^{\perp} = \mathcal{H}$, and T has an inverse on its dense range. However, $Ran(T)$ is not necessarily closed in $\mathcal{H}$. A positive scalar multiple of the identity operator is an example of an operator which is in both classes. In this case we say T is *strictly positive*. Kubrusly [46] has given another but equivalent characterization of a strictly positive operator T as one where $\alpha\|x\|^2 \leq \langle Tx, x\rangle$ for every $x \in \mathcal{H}$ and $\alpha > 0$. More recently, Phillips [57] improved on Wu's result and proved that $\mathcal{P}_7$ contains every invertible operator (i.e. $\mathcal{G}(\mathcal{H}) \subset \mathcal{P}_7$). That is, every invertible operator can be expressed as a product of seven positive operators. Finite-dimensional results had been obtained earlier by Ballantine [7] who proved that $\mathcal{P}_5$ is the set of all $n \times n$ matrices with positive determinant. Khalkhali *etal* [42], studied the norm closures of the $\mathcal{P}_n$, and membership in $\overline{\mathcal{P}_2}$ was characterized for certain classes of operators. It was shown that $\overline{\mathcal{P}_4}$ contains all biquasitriangular operators and that $\overline{\mathcal{P}_5}$ contains each $\mathcal{P}_n$ for $n \geq 5$; and hence $\overline{\mathcal{P}_5}$ contains every invertible operator. i.e. $\mathcal{G}(\mathcal{H}) \subset \overline{\mathcal{P}_5}$.

We need the following definitions.

An operator $T \in B(\mathcal{H})$ is called *quasidiagonal* (*quasitriangular*) if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank (orthogonal) projections such that $P_n \longrightarrow I$ (strongly as $n \longrightarrow \infty$) and $\|TP_n - P_nT\| \longrightarrow 0$, i.e. ($\|TP_n - P_nTP_n\| \longrightarrow 0$, respectively as $n \longrightarrow \infty$) (see [54]). The class of biquasitriangular operators is defined as $(BQT) = \{T \in B(\mathcal{H}) : T \textit{ and its adjoint } T^* \textit{ are quasitriangular}\}$. Quasitriangularity can be illustrated further as follows:

An operator matrix $Q = (q_{ij})$ is quastriangular(Hessenberg) matrix if $q_{ij} = 0$ whenever $i > j+1$. That is, if all entries of Q below the sub-diagonal are zero (see [54]). Compact operators, algebraic operators and quasinilpotent operators are biquasitriangular. Let S_+ denote a unilateral shift. Then the bilateral shift $S_+ \oplus S_+^*$ is a c.n.u biquasitriangular contraction (it is a partially isometric biquasitriangular contraction). Recall that bilateral shifts are unitary (i.e besides being isometries they are normal too).

Hadwin [27] has shown that $\overline{\mathcal{P}_4} \twoheadrightarrow \overline{\mathcal{P}_5}$ and completely characterized $\overline{\mathcal{P}_2}$ as the set of biquasitriangular operators T for which each component of $\sigma_e(T) \cup \sigma_0(T)$ intersects the set $\mathbb{R}^+$ of non-negative real numbers, where $\sigma_e(T)$ is the essential spectrum of T (a subset of the spectrum of T, i.e. those $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not Fredholm, i.e., its range is not closed and its kernel and range are not finite-dimensional) and $\sigma_0(T)$ is the set of normal eigenvalues of T (i.e, $\lambda \in \sigma_0(T)$ if λ is an isolated point of $\sigma(T)$ whose Riez-Dunford spectral projection is finite-dimensional). Hadwin [27] has also shown that $\overline{\mathcal{P}_3}$ contains every operator whose essential spectrum contains 0 and that the operators in $\overline{\mathcal{P}_3}$ all contain 0 in their essential numerical range. Hence, $\overline{\mathcal{P}_3} \neq \overline{\mathcal{P}_4}$. Wu [84] observed that $\mathcal{P}_2$ is closed under similarity since $\mathcal{P}_2$ is the set of operators similar to a positive invertible operator.

Theorem 5.4 *The following assertions are equivalent for an operator T:*
(i) $T \in \overline{\mathcal{P}_2}$
(ii) $T \in \overline{\{A : \sigma(A) \subset \mathbb{R}^+\}}$
(iii) T is biquasitriangular, and each component of $\sigma_e(T) \cup \sigma_0(T)$ intersects $\mathbb{R}^+$.

Proof. The equivalence of (ii) and (iii) is contained in [2 , Proposition 10.1] and the implication (i)$\Longrightarrow$ (ii) is obvious from the fact that $\mathcal{P}_2$ is the set of all operators similar to a positive invertible operator. This also follows from the fact that every positive operator is self-adjoint and hence has non-negative real spectra. Hence we need to show that if $\sigma(T) \subset \mathbb{R}^+$, then $T \in \overline{\mathcal{P}_2}$. If $\sigma(T) \subset \mathbb{R}^+$, then no point in the Fredholm resolvent can have non-zero index. Hence by [3], T is biquasitriangular. It follows from Voiculescu [73] that T is a norm limit of a sequence of algebraic operators. By the semicontinuity of the spectrum, for every positive number ε there is an algebraic operator A such that $\|T - A\| < \varepsilon$ and the imaginary parts of all the eigenvalues of A have absolute value less than ε. However, by [30], A has an upper triangular operator matrix whose diagonal entries are eigenvalues of A.
The equivalence (iii)$\Longrightarrow$ (i) follows immediately from Hadwin [27]. By perturbing the diagonal entries of A to make them positive and distinct, we obtain an operator B with $\|A - B\| < \varepsilon$ such that B has an upper triangular operator matrix with distinct positive diagonal entries. By Rosenblum's theorem [62], B is similar to a positive invertible operator and thus $B \in \mathcal{P}_2$. Since $\varepsilon > 0$ was arbitrary, $T \in \overline{\mathcal{P}_2}$.

Corollary 5.5 *If T is a bilateral operator-weight shift with weights that are all unitary or form a commuting family of diagonal operators $\{D_n\}$ with $D_n = D_1$ for all non-zero integers n, then $T \in \overline{\mathcal{P}_2}$.*

Proof. If the weights are all unitary, then T is a unitary operator whose spectrum is the unit circle. In the latter case, T is unitarily equivalent to a direct sum of scalar weighted bilateral shifts that are compact perturbations of scalar multiples of the unweighted shift. Hence by [3] T is biquasitriangular. Since $\sigma(T)$ and $\sigma_e(T)$ have circular symmetry, $\sigma_0(T) = \emptyset$ and each component of $\sigma_e(T)$ intersects $\mathbb{R}^+$. Hence, by Theorem 5.4, $T \in \overline{\mathcal{P}_2}$.

Theorem 5.6 [27] $\overline{\mathcal{P}_4} = \overline{\mathcal{P}_5}$

Remark 5.4

In Theorem 5.6, by showing that $\overline{\mathcal{P}_4} = \overline{\mathcal{P}_5}$ shows that $\mathcal{P}_4$ is not contained in the biquasitriangular operators. This result was improved by [27] by showing that $\overline{\mathcal{P}_3}$ is not contained in the biquasitriangular operators.

Theorem 5.7 [27] *The set $\overline{\mathcal{P}_3}$ contains every T whose essential spectrum contains 0.*

Remark 5.5

Theorem 5.7 shows that $\overline{\mathcal{P}_3}$ contains many operators; however, $\overline{\mathcal{P}_3}$ does not contain every invertible operator. Note that a number λ is in the essential numerical range of an operator T if there is an orthonormal sequence $\{e_n\}$ of vectors such that $\langle Te_n, e_n\rangle \longrightarrow \lambda$.

Proposition 5.8 *If $T \in \overline{\mathcal{P}_3}$ then the essential numerical range of T intersects $\mathbb{R}^+$.*

Proof. Suppose $T \in \overline{\mathcal{P}_3}$, and choose sequences $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ of positive invertible operators such that $A_nB_nC_n \longrightarrow T$. Since $A_nB_nC_n = (\frac{A_n}{\|A_n\|})(\|A_n\|B_n)C_n$, we can assume that $\|A_n\| = 1$ for each n. Since B_nC_n is similar to a positive operator, there is a positive λ_n in the left essential spectrum of B_nC_n. Hence we can construct a sequence $\{x_n\}$ of unit vectors such that $x_n \in \{x_1, x_2, \ldots, x_{n-1}\}$ for $n \geq 2$, and such that $\|B_nC_nx_n - \lambda_n\| \leq \frac{1}{n}$ for $n \geq 1$. Hence $dist[\langle Tx_n, x_n\rangle, \mathbb{R}^+] \leq |\langle Tx_n, x_n\rangle - \lambda_n\langle A_nx_n, x_n\rangle| \longrightarrow 0$. Thus any limit point of the sequence $\{\langle Tx_n, x_n\rangle\}$ is a point in the essential numerical range of T that lies in $\mathbb{R}^+$.

Remark 5.6

When we consider products of n invertible hermitian operators with at least one factor positive $\mathcal{H}_n$, we find that $T \in \mathcal{H}_2$ if and only if T is similar to an invertible hermitian operator.

(**Proof**. $AB = A^{\frac{1}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})A^{-\frac{1}{2}}$ and $S^{-1}DS = S^{-1}(S^*)^{-1}[S^*DS]$). This means that Theorem 5.4 remains valid with $\mathcal{P}_2$ replaced with $\mathcal{H}_2$ and $\mathbb{R}^+$ replaced with $\mathbb{R}$. From Theorem 5.6, it is clear that $\overline{\mathcal{H}_1} = \overline{\mathcal{P}_4}$. Proposition 5.8 remains valid when $\mathcal{P}_3$ is replaced with $\mathcal{H}_3$ and $\mathbb{R}^+$ replaced with $\mathbb{R}$.

Falling short of proving the Lemma 5.3, we are able to verify it for several classes of special operators. We start with the finite-dimensional case.

Proposition 5.9 *On a finite-dimensional Hilbert space $\mathcal{H}$, $T \in B(\mathcal{H})$ is the product of k $(2 \le k < \infty)$ cyclic operators if and only if $dim\Big(Ker(T^*)\Big) \le k$.*

Proof. We prove that $dim\Big(Ker(T^*)\Big) = k$ implies that T is the product of k cyclic operators. Since the property of cyclicity is preserved under similarity, we may assume that T is of the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \oplus \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \approx T_1 \oplus \dots \oplus T_k \oplus T_{k+1},$$

where T_j is of size n_j for $j = 1, \dots, k+1$ and the a_i's are all nonzero. Note that

$$T_1 \oplus \dots \oplus T_k = \begin{pmatrix} 0 & 2 & & & 0 \\ & 0 & 3 & & \\ & & & \ddots & \\ & & & \ddots & N \\ 0 & & & & 0 \end{pmatrix} \begin{pmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_N \end{pmatrix} \approx S_1 S_2, \text{ where } N = n_1 + n_2 +$$

$\dots + n_k$ and

$$b_j = \begin{cases} \frac{1}{j} & if \ \ 1 \le j \le N \text{ and } j \ne n_1 + 1, n_1 + n_2 + 1, \dots, n_1 + \dots + n_{k-1} + 1, \\ 0 & otherwise \end{cases}$$

$$\text{and that } T_{k+1} = \begin{pmatrix} b_{N+1} & & * \\ & \ddots & \\ 0 & & b_{N+n} \end{pmatrix} \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix} \equiv R_1 R_2,$$

where $b_{N+1}, \dots, b_{N+m}$ are all nonzero and distinct and the c_j's are nonzero and distinct and also distinct from the nonzero b_j's. Letting $A_j = S_j \oplus R_j$, $j = 1, 2$, we have

$T = A_1A_2$. Since $A_1 = \begin{pmatrix} L_1 & & * \\ & \ddots & \\ 0 & & L_n \end{pmatrix}$ where the L_i are all cyclic and have mutually disjoint spectra, A_1 must be cyclic. On the other hand, since A_2 is a diagonal operator with $k-1$ zero diagonals, we can express it as a product of $k-1$ diagonal operators each with distinct diagonals. Hence A_2 is the product of $k-1$ cyclic operators, and therefore T is the product of k cyclic operators.

Corollary 5.10 *On an n-dimensional Hilbert space, every operator is the product of n cyclic operators and n is the smallest such number.*

We now investigate operators on infinite-dimensional Hilbert spaces. We denote the multiplicity of an operator T by $\mu(T)$. Since $dim\Big(Ker(T^*)\Big) \leq \mu(T) = dim\Big(Ker(T)\Big)$ for any operator T, the next proposition is weaker than the Corollary 5.10. Recall that a ***multicyclic operator*** is one that has finite multiplicity. That is $\mu(T) < \infty$. The cyclic multiplicity is the number of cyclic subspaces for T that are needed to generate $\mathcal{H}$.

Proposition 5.11 *If T is a multicyclic operator with multiplicity m, then T is the product of m cyclic operators.*

Proof. We prove this result by Mathematical Induction on the multiplicity m. Obviously, this is true for $m = 1$. Assuming its validity for any operator T with multiplicity $\mu(T) = m$, we prove it for $m+1$. So let T be an operator with multiplicity $m+1$, then by Herrero and Wogen [36], we have the triangulation for $T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$, where T_1 is cyclic and $\mu(T_2) = m$. Hence $T_2 = S_1...S_{m-1}S_m'$ is a product of m cyclic operators by the induction process. On the other hand, using Lemma 5.3, we obtain $T_1 = R_1...R_{m+1}$, where the R_j's are all cyclic and each R_j, $j = 1, ..., m$, is diagonal and invertible with $\sigma(R_j)$ disjoint from $\sigma(S_j)$ when $j = 1, ..., m-1$. Moreover, let $S_m' = S_mS_{m+1}$, where both factors are cyclic, $\sigma(S_m)$ is disjoint from $\sigma(R_m)$ and S_{m+1} is diagonal and invertible with $\sigma(S_{m+1})$ disjoint from $\sigma(R_{m+1})$. Finally, we let $Q_j = \begin{pmatrix} R_j & 0 \\ 0 & S_j \end{pmatrix}$, $j = 1, ..., m$, and $Q_{m+1} = \begin{pmatrix} R_{m+1} & R_m^{-1}...R_1^{-1}X \\ 0 & S_{m+1} \end{pmatrix}$.
Then each Q_j is cyclic by Proposition 5.9 and $T = Q_1...Q_{m+1}$. This proves the result.

Remark 5.7

Recall that an operator is *triangular* if it can be represented in the matrix form $\begin{pmatrix} a_1 & & * \\ & a_2 & \\ 0 & & \ddots \end{pmatrix}$.

We give an improvement of Lemma 5.3.

Lemma 5.12 *If T is a cyclic operator with dense range, then $T = T_1T_2$, where T_1 is unitary cyclic and T_2 is triangular cyclic.*

Remark 5.8

This is essentially the infinite-dimensional QR decomposition: every operator with dense range is the product of a unitary operator and a triangular operator (see [68] for details).

Proposition 5.13 *If the spectrum of T does not surround 0, then T is the product of two cyclic operators.*

Proof. We assume that T is of the form $[T_{ij}]$, where T_{ii} is cyclic for all i and $T_{ij} = 0$ for $i > j$. It is easily seen from [58, Theorem 0.8] that $\sigma(T_{ii})$ is contained in the polynomially convex hull of $\sigma(T)$.
Our hypothesis implies that T_{ii} is invertible for all i. Thus by Lemma 5.12, we obtain $T_{ii} = U_iA_i$, where U_i is unitary cyclic and A_i is triangular cyclic. Let $\{r_i\}$ be a sequence of distinct real numbers between 1 and 2, and let $V_i = r_iU_i$ and $B_i = \frac{A_i}{r_i}$. Then $T_{ii} = V_iB_i$. Since the B_i's together with the A_i's are all invertible, we may make further adjustments so that the diagonals of all the B_i's are distinct. If

$$T_1 = \begin{pmatrix} V_1 & T_{12}B_2^{-1} & T_{13}B_3^{-1} & \\ & V_2 & T_{23}B_3^{-1} & \ddots \\ & & V_3 & \ddots \\ 0 & & & \ddots \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & B_3 & \\ 0 & & & \ddots \end{pmatrix},$$

then $T = T_1T_2$. The relation $B_i^{-1} = r_iA_i^{-1} = r_iT_{ii}^{-1}U_i$ implies that $\|B_i^{-1}\| \leq 2\|T^{-1}\|$ for all i, whence T_2 is invertible and therefore T_1 is indeed a bounded operator. Since the V_i's are cyclic and their spectra are mutually disjoint, Proposition 5.9 implies that T_1 is cyclic. On the other hand, if D_i is a diagonal operator whose diagonals are exactly those of B_i, then D_i is a quasiaffine transform of B_i. Hence $\sum_i \oplus D_i$ is a quasiaffine transform of T_2. Since $\sum_i \oplus D_i$ is itself a diagonal operator with distinct diagonals, it is cyclic, whence T_2 is cyclic. This completes the proof.

Theorem 5.14 *Let $T = \sum_n \oplus T_n$, where the T_n's are cyclic. If $k \geq 2$ and T_n has dense range for all $n \geq k$, then T is the product of k cyclic operators.*

Proof. By Lemma 5.3 we can express each T_n, $n = 1, 2, ..., k$, as a product $T_n = T_{n1}...T_{nk}$, where T_{nj}'s are all cyclic and each T_{nj}, $j \neq n$, is a diagonal operator with spectrum disjoint from the spectra of all the other T_{ij}'s. For the remaining T_n's, we use Lemma 5.12 to obtain $T_n = U_n A_n$, where U_n is unitary cyclic and A_n is triangular cyclic. We further express these T_n's as a product $T_n = T_{n1}...T_{nk}$ $(n > k)$, where each T_{n1} is a distinct multiple of U_n with spectrum disjoint from the spectra of $T_{11}, ..., T_{k1}$, and each T_{nj}, $j = 2, ..., k$, is either triangular cyclic or diagonal cyclic with the closure of its distinct diagonals disjoint from the spectra of $T_{1j}, ..., T_{kj}$. Let $S_j = \sum_n \oplus T_{nj}$, $j = 1, ..., k$. Obviously, $T = S_1...S_k$, with S_1 cyclic by the above construction. To prove the cyclicity of the remaining S_j's, let D_{nj} $(n > k)$ and $j = 2, ..., k$ be the diagonal operator with diagonals exactly those of T_{nj}. Since $T_{1j} \oplus ... \oplus T_{kj} \oplus \sum_{n=k+1}^{\infty} \oplus D_{nj}$, $j = 2, ..., k$, is cyclic and is a quasiaffine transform of S_j, using the above construction and Proposition 5.9, we conclude that S_j is cyclic as asserted.

We now use our results to give some factorizations for some special classes operators.

Corollary 5.15 *An isometry T is the product of k $(2 \leq k < \infty)$ cyclic operators if $dim\Big(Ker(T^*)\Big) \leq k$.*

Proof. This result follows from the application of Theorem 5.14 and the application of the spectral theorem and the von Neumann-Wold decomposition of an isometry that: Every isometry can be expressed as a direct sum of simple unilateral shifts (c.n.u) and cyclic unitary (unitary) operators. In a similar fashion, every co-isometry is the direct sum of some backward shift ((c.n.u) summand) and cyclic unitary (unitary summand) operators.

Corollary 5.16 *Every co-isometry is the product of two cyclic operators.*

Proof. This result follows by the application of the fact that the backward shift is cyclic and Theorem 5.14.

Remark 5.9

We note that Corollary 5.15 also follows from [44, Theorem 2] that an isometry T with $dim\Big(Ker(T^*)\Big) = k$ is the product of k simple unilateral shifts. Corollary 5.16 also

follows from [10, Theorem 3] that every co-isometry is the product of some backward shift and a simple unilateral shift. For a normal operator T with $dim\Big(Ker(T^*)\Big) = k$, we have, by the spectral theorem, the decomposition $T = \sum_{n=1}^{\infty} T_n$, where $T_1, ..., T_k$ are the zero operators on a one-dimensional space and every $T_n \ \ (n > k)$ is one-to-one with dense range.
This leads to the following result.

Corollary 5.17 *A normal operator T is the product of $k \ \ (2 \leq k < \infty)$ cyclic operators if and only if $dim\Big(Ker(T^*)\Big) \leq k$.*

The following result says that the product of finitely many cyclic operators condition can be characterized by the condition that the dimension of $Ker(T^*)$ is finite.

Theorem 5.18 *An operator T with $dim\Big(Ker(T^*)\Big) \leq k, \ \ 2 \leq k < \infty$ is the product of at most $k + 2$ cyclic operators.*

Proof. If $dim\Big(Ker(T)\Big) \leq dim\Big(Ker(T^*)\Big)$, then the polar decomposition of T yields $T = VP$, where V is an isometry with $dim\Big(Ker(V^*)\Big) = dim\Big(Ker(T^*)\Big) - dim\Big(Ker(T)\Big)$ and $P = (T^*T)^{1/2}$ satisfies $Ker(P) = Ker(T)$. By Corollaries 5.15 and 5.17, V and P are, respectively, the products of m and n cyclic operators, where $m = max\{dim\Big(Ker(T^*)\Big) - dim\Big(Ker(T)\Big), 2\}$ and $n = max\{dim\Big(Ker(T)\Big), 2\}$. It follows that T is the product of $k + 2$ cyclic operators.
On the other hand, if $dim\Big(Ker(T)\Big) > dim\Big(Ker(T^*)\Big)$, then consider the decomposition $T = PV$, where $P = (TT^*)^{1/2}$ and V is a co-isometry. Since $dim\Big(Ker(P)\Big) = dim\Big(Ker(T^*)\Big) \leq k$, Corollary 5.17 implies that P is a product of k cyclic operators. Also, V is the product of two cyclic operators by Corollary 5.16. This proves the result.

Remark 5.10

We note that Corollary 5.18 simply says that T is a product of k cyclic operators if $2 \leq dim\Big(Ker(T)\Big)$ and $dim\Big(Ker(T)\Big) + 2 \leq dim\Big(Ker(T^*)\Big) \leq k \quad (2 \leq k < \infty)$.
We also note that using Halmos [31], Corollary 5.15 can be sharpened to: Every isometry is either unitary or a shift or a product of two of these two kinds. Some results sharpening the preceding results have been given. For instance, Radjavi [59] has shown that a normal operator is the product of *four* self-adjoint operators, Halmos and Kakutani [28]

have proved that each unitary operator is the product of *four* symmetries. Recently, Phillips [57] has proved that every invertible operator is the product of *seven* positive operators. Very recently, Moslehian [52] worked on decomposition of an operator into a product of projections. It is well known (see [34]) that if T is a linear operator in a finite- dimensional Hilbert space having nonzero kernel, then T is the product of a finite number of projections.
We give a simple proof to the following important assertion.

Proposition 5.19 *An invertible operator T is a product of two self-adjoint operators if and only if T is similar to T^*.*

Proof. Suppose T is invertible with $T = AB$ with $A^* = A$ and $B^* = B$. Since T is invertible, then $I = TT^{-1} = (AB)(B^{-1}A^{-1})$. This shows that A and B are invertible also and hence BA is invertible.
$T^* = BA = BIA = BTT^{-1}A = BT(AB)^{-1}A = BTB^{-1}A^{-1}A = BTB^{-1}$. This shows that $T \approx T^*$. Conversely, suppose T is invertible and $T \approx T^*$. Since T is invertible, by the polar decomposition theorem, T has a unique polar decomposition $T = UP$, where U is unitary (and not necessarily self-adjoint) and $P = (T^*T)^{1/2}$ is a positive operator (self-adjoint). We use the similarity of T and T^* to show that U, must indeed, be self-adjoint. $T \approx T^*$ implies that $UP = X^{-1}(UP)^*X = X^{-1}PU^*X$. Without loss of generality, we can let $X = I$. In that case, $U = U^*$, which proves that U is self-adjoint. This completes the proof.

Remark 5.11

We denote by $\mathfrak{S}_0$ the set of all invertible products of self-adjoint operators A and B and by $\mathfrak{S}$ the set of invertible operators that are similar to their adjoints. It is clear that $\mathfrak{S}_0 \subseteq \mathfrak{S}$. Proposition 5.19 asserts that $\mathfrak{S} \subseteq \mathfrak{S}_0$ is also valid. By using the invariance of the classes $\mathfrak{S}_0$ and $\mathfrak{S}$ under similarity transformations $T = S^{-1}TS$. We notice that $\mathfrak{S}$ is strictly larger than the class of operators that are similar to self-adjoints. An example is the bilateral shift.

Theorem 5.20 *T is unitarily equivalent to its adjoint if and only if T is the product of a symmetry and a self-adjoint operator.*

Proof. If $T = JA$, where $J = J^* = J^{-1}$ is a symmetry and A is self-adjoint, then $JTJ = AJ = T^*$, so that T is unitarily equivalent to its adjoint. Conversely, suppose $TU = UT^*$, where U is unitary. Then T commutes with U^2. Let $\int e^{i\theta} dE_\theta$ be the spectral representation of U^2. If $V = \int e^{i\theta/2} dE_\theta$, then V is a unitary operator, $V^2 = U^2$, and V commutes with every operator that commutes with U^2. It follows that V commutes with U and T, therefore $J = V^{-1}U$ is a symmetry and $TJ = JT^*$. Hence $T = J(TJ)$ is the product of a symmetry and a self-adjoint operator.
Theorem 5.20 leads to the following assertion.

Corollary 5.21 *A unitary operator U is similar to its inverse if and only if U is the product of two symmetries.*

Remark 5.12

We now study operators that admit a factorization as a product of two self-adjoint operators.
We begin by considering the finite-dimensional case.

Theorem 5.22 *If $\mathcal{H}$ is a finite-dimensional Hilbert space, then the following are equivalent conditions for an operator T on $\mathcal{H}$.*
(i) T is a product of two self-adjoint operators.
(ii) T is a product of two self-adjoint operators, one which is invertible.
(iii) T is similar to T^*.

Remark 5.13

The implications $(iii) \Longleftrightarrow (ii) \Longrightarrow (i)$ are purely formal and hence they remain valid in the infinite-dimensional case. To show that (iii) does not imply (i), consider the operator $T = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$. It is clear that T is similar to T^* but $T = T_1T_2$, where $T_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 2 & 2 \\ -1 & 2 \end{pmatrix}$. Clearly, T_2 is not self-adjoint.
These results show that the invertibility condition of T in Proposition 5.19 cannot be dropped.

Theorem 5.23 *Two normal operators A and B that are similar are unitarily equivalent.*

Proof. Suppose A and B satisfy the relation $A = XBX^{-1}$ i.e $AX = XB$ for some invertible operator X. By the Putnam-Fuglede Theorem, if $AX = XB$ for some operator X, then $A^*X = XB^*$. Thus $A^* = X^{*-1}B^*X^*$ and $A^* = XB^*X^{-1}$, which means that $X^* = X^{-1}$ or X is a unitary operator. This proves the result.

Corollary 5.24 *Each normal operator in $\mathfrak{S}$ belongs to $\mathfrak{S}_0$.*

Proof. The proof follows from the fact that two normal operators that are similar are also unitarily equivalent, by Theorem 5.23.

Theorem 5.25 $T \in B(\mathcal{H})$ *is a product of two projections if and only if* $TT^*T = T^2$.

Proof. The necessity is trivial. To prove the sufficiency, let P_1 be the projection onto $\overline{Ran(T)}$, and let P_2 be the projection onto $Ker(T)^{\perp}$. Then $T = P_1P_2$.

Remark 5.14

We now study the factorization of invertible operators. Invertible operators include the unitary operators which are a first hand tool in solving many problems in mathematical and theoretical physics, with applications in quantum cryptography and quantum teleportation. We wish to factorize a unitary operator as $T = AB$, where A is unitary and is a product of (n-1) unitary operators and B is a little bit simpler operator than T. Recall that an operator T is similar to S if $T = X^{-1}SX$ for certain invertible X and T is congruent to S if $T = X^*SX$ for some operator X. Since unitary operators are invertible, we give the following general result.

Theorem 5.26 *(i) An invertible operator T is the product of a positive operator and a hermitian operator if and only if it is similar to a hermitian operator.*
(ii) An invertible operator is the product of a positive operator and an involution operator if and only if it is congruent to an involution operator.
(iii) Every invertible operator is a product of a positive operator and a unitary operator.

Proof. The proof of (i) is obvious.
(ii). If $T = X^*VX$ where V is an involution, then X^*X is positive, $X^{-1}VX$ is an involution and $T = (X^*X)(X^{-1}VX)$, as required. Conversely, if $T = PV$ where P is positive and V is an involution, then $P^{1/2}VP^{-1/2}$ is an involution and $T = P^{1/2}(P^{1/2}VP^{-1/2})P^{1/2}$, as required.

(iii) If $T = PU$ is the polar decomposition of T, then $P^{-1/2}TP^{1/2} = P^{1/2}UP^{1/2}$ and $P^{-1/2}TP^{-1/2} = P^{1/2}UP^{-1/2}$, as required.

Theorem 5.27 *An invertible operator T is*
(i) similar to its adjoint if and only if it is a product of two hermitian operators
(ii) similar to its inverse if and only if it is a product of two involution operators.

Lemma 5.28 *Let S and T be involution operators on $\mathcal{H}$. Then ST is an involution if and only if $TS = ST$.*

Proof. Since S and T are involutions, then $S^2 = I$ and $T^2 = I$ (equivalently, $S = S^{-1}$ and $T = T^{-1}$). Now ST an involution implies that $(ST)^2 = STST = I$. A simple computation shows that $ST = TS$. Conversely, suppose the involutions satisfy $TS = ST$. Then $STS = TSS = TS^2 = T$. Also, $ST^2TS = T$. Thus, $STTS = I$ or $STST = I$. This shows that $(ST)^2 = I$. Hence ST is an involution.

Remark 5.15

We introduce a special factorization of T which is applicable in solving linear systems. We conjecture that for commuting linear operators $P_0, P_1, ..., P_n$, any operator $T = P_0P_1...P_n$. This decomposition has an applicability in the solution of general inhomogeneous problems $Tx = y$. Using the above factorization this problem reduces to a system of simpler problems. These problems capture the structure of the solution and range spaces and, if the operators involved are differential, then this gives an effective way of lowering the differential order of the problem.

Theorem 5.29 *Let $T = P_0P_1...P_n$ where the P_i are commuting linear operators. Then*
(a) $Ker(T) = Ker(P_i)$
(b) $Ran(T) = Ran(P_i)$

Theorem 5.30 *The product of two projection operators P_M and P_N is also a projection operator if and only if P_M and P_N commute.*

Corollary 5.31 *Two subspaces $\mathcal{M}$ and $\mathcal{N}$ are orthogonal if and only if $P_{\mathcal{M}}P_{\mathcal{N}} = 0$.*

Theorem 5.32 *A finite sum of projection operators $P_{\mathcal{M}_1} + P_{\mathcal{M}_2} + ... + P_{\mathcal{M}_n} = Q$ is a projection if and only if $P_{\mathcal{M}_1}P_{\mathcal{M}_2}..P_{\mathcal{M}_n} = 0$. That is, if and only if $\mathcal{M}_i \perp \mathcal{M}_j$, $i \neq j$.*

Remark 5.16

We now study the factorization of operators into n-th roots of identity.

Corollary 5.33 *Let T be of the form $T_1 \oplus T_2 \oplus ... \oplus T_n$, where the product $T_1T_2...T_n$ is a group commutator. Then T is a product of three n-th roots of identity for any given $n > 2$.*

Proof. We write the product $T_1T_2...T_n$ as a commutator $UVU^{-1}V^{-1}$. Let w be the circulant matrix that sends the first basis vector to the last, the second to the first, the third to the second, etc. Let W be the tensor product of w with the identity operator I, i.e., $W = w \otimes I$. Define

$$K_1 = (U^{-1}T_1T_2 \oplus T_2^{-1}T_1^{-1}UVT_1 \oplus I \oplus ... \oplus I \oplus T_1^{-1}V^{-1})W$$

and $K_2 = (VT_1T_2 \oplus T_2^{-1}T_1^{-1}UT_1 \oplus T_1^{-1}V^{-1}U^{-1}T_1T_2T_3 \oplus T_4 \oplus ... \oplus T_{n-1} \oplus T_n)W$ Then $K_1^n = K_2^n = I$ and $T_1 \oplus T_3 \oplus T_4 \oplus ... \oplus T_n \oplus T_2 = K_1K_2W^{-2}$. Since T is block permutationally similar to $T_1 \oplus T_3 \oplus T_4 \oplus ... \oplus T_n \oplus T_2$, we are done.

Remark 5.17

It is clear from the proof that at least one of the n-th roots, namely W^{-2} is unitary and that for unitary T_i, $i = 1, 2, ..., n$, all n-th roots are unitary.

Proposition 5.34 *If an invertible normal operator N has a unitary direct summand acting on an infinite-dimensional subspace, then N is a product of three n-th roots of the identity.*

Proof. Write $N = N_1 \oplus U_2 \oplus U_3 \oplus ... \oplus U_{2n}$ with N_1 normal and U_i unitary, and let

$$T = (N_1 \oplus U_2)(U_3 \oplus U_4)...(U_{2n-1} \oplus U_{2n}) = (N_1U_3...U_{2n-2}) \oplus (U_2U_4...U_{2n}).$$

since the second direct summand is unitary and acts on an infinite-dimensional subspace, by [9, Theorem 2], it is a group commutator. Hence, T is a group commutator and the rest follows from Corollary 5.33.

Corollary 5.35 *Every unitary operator on an infinite-dimensional Hilbert space is a product of three unitary n-th roots of the identity, where $n > 2$.*

Proof. First, as a counterexample, let $U = \alpha I$ with $|\alpha| = 1$ and $\alpha^n \neq 1$. If $\alpha I = AB$ with $A^n = B^n = I$, then A and B commute and we have $\alpha^n I = A^n B^n = I$, which is impossible. For $n = 2$, it is not possible to have $\alpha I = ABC$ with $A^2 = B^2 = C^2 = I$ if $\alpha^2 \neq \pm 1$ since otherwise we would have $\alpha C = AB$ and $\alpha^{-1} C = BA \sim AB = \alpha C$. Hence, $\alpha^2 C$ would be similar to C which is impossible unless $\alpha^2 = 1$ or $\alpha^2 = -1$. Note however (see [40]) that every unitary operator is the product of four involutions.

Corollary 5.36 *A general invertible operator T on an infinite-dimensional Hilbert space is a product of five n-th roots of the identity for every $n > 2$ and three of the factors can be chosen to be unitary.*

Proof. By polar decomposition we have $T = PU$ where P is positive and U is unitary. By [38, Theorem 1], $P = K_1K_2V_1V_2$ with $K_1^n = I$, $K_2^n = I$, $V_1^n = I$, $V_2^n = I$ and V_1, V_2 unitary. Then $T = K_1K_2V_1V_2U = K_1K_2W$ and since $W = V_1V_2U$, being unitary, is also a product of three unitary n-th roots by Corollary 5.35.

Remark 5.18

The question is whether five factors are needed, or a general invertible operator T can be written as a product of fewer than five n-th roots for an $n > 2$.

Lemma 5.37 *If U be a bilateral shift on an infinite dimensional Hilbert space $\mathcal{H}$ with multiplicity equal to the dimension of $\mathcal{H}$ then*
(i) every invertible operator on $\mathcal{H}$ is a product of six operators similar to U.
(ii) every unitary operator on $\mathcal{H}$ is a product of two operators, unitarily equivalent to U.

Proof. Every unitary operator is a product of two bilateral shifts of multiplicity equal to dimension of the space ([28, Proof of Theorem 1]) and any two bilateral shifts of the same multiplicity are unitarily equivalent. This proves the second part.
Now choose an arbitrary integer $n \geq 3$. According to [38, Corollary 3] every invertible operator is a product of two n-th roots of identity and a unitary operator. The proof is concluded by noticing that every n-th root of the identity is similar to a unitary operator.

Remark 5.19

We use the terms multiplicity and deficiency interchangeably for isometries. An operator T is left invertible if there exists another operator S such that $ST = I$. Isometries are examples of left invertible operators.

Lemma 5.38 *Every left invertible operator $A \in B(\mathcal{H})$ with deficiency $4d$ can be expressed as a product of an operator similar to $\begin{pmatrix} B & \\ & C \end{pmatrix}$ and an operator similar to $\begin{pmatrix} D & \\ & E \end{pmatrix}$, where B, C, D and E are left invertible operators on H with deficiency d.*

Proof. Let $N \in B(\mathcal{H})$ be normal. Then by [28], $\mathcal{H}$ is the orthogonal sum of two subspaces of equal dimension, both of which reduce N. Now let T be an arbitrary isometry with nonzero deficiency. Then it is a direct sum of a shift and a unitary operator, by von Neumann-Wold decomposition. If the shift has multiplicity $d_1 + d_2$ it can obviously be expressed as a direct sum of shifts of multiplicities d_1 and d_2. Because A is left invertible its polar decomposition is $A = U|A|$, where U is an isometry with deficiency $4d$ and $|A|$ is normal (since it is positive and self-adjoint) and has two reducing subspaces. There exist an isometry V of deficiency $2d$ that leaves both these subspaces invariant and such that V restricted to each of these subspaces has deficiency d. Hence $V|A|$ has the same reducing subspaces as A and is similar to some $\begin{pmatrix} D & \\ & E \end{pmatrix}$ as required. There also exists another isometry W of deficiency $2d$ such that $U = WV$. But as stated above we can split W as a direct sum of two isometries, each with deficiency d and thus achieve the form $\begin{pmatrix} B & \\ & C \end{pmatrix}$.

We now study the factorization of unitary operators and isometries.

Proposition 5.39 *Let A be an isometry with deficiency d on a separable infinite dimensional Hilbert space $\mathcal{H}$. Suppose there exists a subspace $\mathcal{M}$ of $\mathcal{H}$ such that $dim(\mathcal{M}) = dim(\mathcal{H})$ and $A(\mathcal{M}) \perp \mathcal{M}$. Then every isometry with deficiency $6d$ is a product of 6 operators, unitarily equivalent to A.*

Proof. Let U be a bilateral shift with infinite multiplicity on $\mathcal{H}$. We know that an operator of the form $\begin{pmatrix} U & \\ & X \end{pmatrix}$ is a product of two operators unitarily equivalent to A where X is an isometry with deficiency $2d$. Let P and Q be arbitrary isometries on $\mathcal{H}$, with deficiencies $2d$ and $4d$, respectively. Since X^2 and Q have the same deficiencies, there exists a unitary operator V such that $VX^2 = Q$. We obtain $\begin{pmatrix} V & \\ & Y \end{pmatrix}$ as a product of two operators unitarily equivalent to A. There exists unitary W such that

$WY = P$, hence YW is unitarily equivalent to P.

Since a product of two operators unitarily equivalent to B equals W, a product of 6 operators unitarily equivalent to A is

$$\begin{pmatrix} Y & \\ & V \end{pmatrix} \begin{pmatrix} W & \\ & X^2 \end{pmatrix} \cong \begin{pmatrix} P & \\ & Q \end{pmatrix}.$$

The proof is completed by noticing that every isometry with deficiency $6d$ is unitarily equivalent to a direct sum of an isometry with deficiency $2d$ and one with deficiency $4d$ as in Lemma 5.39.

Theorem 5.40 *Let A be an isometry with deficiency $d > 0$ on a separable infinite dimensional Hilbert space $\mathcal{H}$. Then every isometry with deficiency index $6d$ is a product of 6 operators unitarily equivalent to A.*

Proof. Since $d \neq 0$, A is an orthogonal sum of a shift S and a unitary operator by Wold decomposition of an isometry. Let $S(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$. The space of all vectors of the form $(x_1, 0, x_3, 0, ...)$ clearly matches the requirements for $\mathcal{M}$ in Proposition 5.40, since $S(x_1, 0, x_3, 0, x_5, 0, ..) = (0, x_1, 0, x_3, 0, x_5, ...)$ is orthogonal to $span(x_1, 0, x_3, 0, ...) = \mathcal{M}$. The result follows from Proposition 5.39.

Chapter 6

Conclusion

6.1 Conclusion

The results in this thesis show that every linear operator acting on a Hilbert space has a direct sum decomposition into a normal part and a completely non-normal part and that either direct summand may be absent. Similarly, every contraction operator has a direct sum decomposition into a unitary part and a completely non-unitary part. The problem of decomposing some classes of operators as a direct sum has been solved in Chapter Two and Chapter Three. To aid in carrying out a direct sum decomposition of an operator, we studied its invariant and hyperinvariant lattices. This work was done in Chapter Three of this thesis. We have used simple operator theoretic tools to find out when some operators turn out to be normal or pure and when certain contractions are unitary or completely non-unitary. We have found a relationship between the direct sum decomposition of an operator and its invariant and hyperinvariant lattices. In Chapter Five we have studied the idea of factorizing a given linear operator into two or more factors. We have found conditions under which a certain operator factorizes into a certain number of factors.

6.2 Summary of Main Contributions

In this thesis we have made several key contributions about the spectral properties for some classes of operators. We have extended results on some classes of operators to higher classes of operators.

In Chapter Two we have developed a mechanism to determine conditions under which some higher classes of operators are normal. For example, in Theorem 2.6 we have shown that if a p-hyponormal operator is similar to its adjoint, then it has no completely non-normal direct summand. In Theorem 2.4 we have relaxed the condition of similarity and extended this result to a (p, k)-quasihyponormal operator which is a quasiaffine transform of a co-p-hyponormal operator. In particular, we have shown in Theorem 2.4 that a p-quasihyponormal operator which is a quasiaffine transform of a normal operator is normal. We have shown in Lemma 2.14 that if an operator is (p, k)-quasihyponormal such that the restriction of the operator to an invariant subspace is injective and normal, then the operator decomposes into a direct sum of nontrivial normal and completely non-normal (complementary) parts. In Lemma 2.22, we have shown that any p-quasihyponormal operator that densely intertwines a normal operator is also normal. We have proved in Proposition 2.43 that any linear operator T that is 2-normal and quasinormal and is injective on $Ran([T^*, T])$ has no c.n.n. part.

In Chapter Three we have solved a long standing open question: "When are the c.n.u parts of quasisimilar hyponormal contractions quasisimilar?", by investigating their c.n.u. parts. In this direction we have shown in Corollary 3.7 that this is the case when the c.n.u. part of one of the contractions has finite multiplicity. In Proposition 3.31, we have characterized isometries with deficiency index zero as unitary operators. Using Proposition 3.31, we have proved in Proposition 3.32 that every isometry with nonzero deficiency index is a direct sum of a unilateral shift and a unitary operator. This result agrees with the Wold Decomposition of an isometry. In Proposition 3.18, we have proved that the c.n.u. part of an operator which is similar to a normal contraction is of class C_{00}. Characterizing some contractions in terms of characteristic functions, we have shown in Corollary 3.23 that the characteristic function of any isometry is identically zero almost everywhere. We have proved in Corollary 3.52 that if two linear operators are almost similar and one operator is c.n.u, then the other is c.n.u.

Our work in Chapter Four was essentially to aid in carrying out the direct sum decompositions and is a first attempt in trying to give a partial solution to the long standing open problem: Does every operator have a nontrivial invariant subspace? In Lemma 4.28, we have shown that if two linear operators are quasisimilar and one operator has a nontrivial hyperinvariant subspace, then so is the other. We have characterized some classes of operators in terms of their invariant and hyperinvariant lattices. For instance, in Corollary 4.32, we have shown that if T is normal then every hyperinvariant subspace of T is also hyperinvariant for T^*. We have also shown in Corollary 4.35 that for any c.n.n. linear operator T, every invariant subspace is also hyperinvariant for T.

In Chapter five, we have proved several results on factorization of some operators. We have shown in Proposition 5.13 that any multicyclic operator with multiplicity m is a product of m cyclic operators. We have also proved in Corollary 5.17 that any operator T with $dim(Ker(T^*)) \leq k$, $(2 \leq k < \infty)$ is the product of at most $k+2$ cyclic operators. In Theorem 5.20, we have shown that any invertible operator is a product of two self-adjoint operators if and only if the operator is similar to its adjoint. We have proved interesting results about normal and unitary operators. For instance, in Corollary 5.34, we have shown that if an invertible normal operator has a unitary direct summand acting on an infinite dimensional subspace of a Hilbert space, then it is a product of three n-th roots of the identity. We have proved a consequence of this result in Corollary 5.35 for the case of a unitary operator.

6.3 Future Research

The results in this thesis clearly demonstrate that it is of considerable interest to carry out more analysis in order to determine structures and properties of operators. These results could be used to give more insight into the problem of determining the structure of operators in some classes of operators. For instance, given the subspace lattice and hyperinvariant lattice of an arbitrary linear operator, we may be able to discern the location of the spectrum of the operator. It is clear from this work that direct summands and factors of a linear operator reveal information about the operator. This thesis has produced many new results on direct sum decomposition and factorization of some classes of operators. The treatment of the topic is, however, far from complete. We give a list of some possibilities for future research.

1. It is well-known from results in this work that if the spectrum of a k-quasi-hyponormal operator has zero Lebesgue measure, then the operator can de decomposed as direct sum of a normal operator and a nilpotent operator. It is of considerable interest to find out more intrinsic properties of the nilpotent summand. For instance, what are the spectral properties of the operator if its nilpotent summand happens to be the zero operator or a non-zero nilpotent operator?

2. From the decomposition results in this thesis, it is clear that every c.n.n. contraction is completely non-unitary. It would be worthwhile if one were able to decompose any c.n.u contraction as a direct sum of a normal and a c.n.n. contraction.

3. Direct sum decomposition and factorization reveals spectral information about a linear operator. An interesting future research direction is to investigate how the spectrum, the numerical range and the norm of each direct summand and factor compares with that of the operator.

4. To date, special research emphasis has been on the direct sum decomposition of an operator into a normal and a c.n.n. part and a contraction into a unitary and a c.n.u. part. We anticipate other forms of direct sum decompositions where the direct summands have other properties.

5. Most of our results on factorization were on a single linear operator. An interesting future research direction would be to find out the relationship of such factorizations for operators which are unitarily equivalent, similar, quasisimilar, hyperquasisimilar, almost-similar, commute or are quasiaffine transforms of each other.

6. In the (numerical) solution of linear equations and eigenvalue problems.
The central theme in the decomposition of the abstract operator linear system $Ax = y$ into sets of linear subsystems (equations) which can be solved independently is to obtain a conceptual simplification of the system model. There are computational reasons for examining the decomposition process: Decomposition provides an alternative to inver-

sion as a technique for solving or analyzing the equations which describe a system. In particular, decomposition provides a practical technique for computing solutions to linear differential equations with arbitrary inputs. The ability to combine the solutions to small subproblems into a solution for the full system equation depends on the principle of linearity. It is known that we can decompose most linear systems into sets of simple scalar multiplications. It would be a research challenge to determine the optimal number of such subsystems.

7. Operator decomposition and factorization is applicable in the study of mathematical systems theory. It reduces the computational word length required in the operator computations. It is useful in easing the solution of linear operator equations. Results on the factorization of operators as products of self-adjoint operators in Hilbert space play a role in pure and applied mathematics. Problems which give rise to linear operator equations include linear regression, optimal resource allocation, optimal filtering, optimal control and solutions of integral and partial differential equations, which have lots of applications in control, signal and image processing. An interesting research direction would be to develop a real-time application of operator decompositions and factorizations in signal and image processing.

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