UNIVERSITY OF NAIROBI
DEPARTMENT OF MATHEMATICS

LAPLACE TRANSFORM SOLUTION OF HYDROMAGNETIC STEADY FLOW OF VISCOUS INCOMPRESSIBLE FLUID BETWEEN TWO PARALLEL INFINITE PLATES

A project paper presented in partial fulfillment of the requirements of the course
SMA 649: PROJECT IN APPLIED MATHEMATICS.

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July, 2010

DECLARATION

This is to confirm that the work presented in here is my original work and has not been presented to any other University.

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Declaration by Supervisor

This is to confirm that this project has been submitted for examination with my approval as supervisor.

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DEDICATION

I would like to dedicate this project with honor and love to my husband George Kimathi and my little girl Skylar Kimathi.

ACKNOWLEDGMENTS

I would like to acknowledge and extend my heartfelt gratitude to the following persons who have made this project possible.

First my supervisor Dr. C. B. Singh, for his unending assistance in helping me come up with the right equations and the project layout as a whole.

My classmate Idah Makena, for her great help especially in collecting books and the relevant materials.

Most especially to my beloved husband, George Kimathi not only for his encouragement and support, but also for his great help and guidance when I got stuck. Lastly to my daughter Skylar M. Kimathi for her understanding when I spent so much time in school away from her.

PROLOGUE

This project aims at getting the solution of hydromagnetic steady flow of viscous incompressible fluid between two parallel infinite plates using the Laplace transform method. The upper plate is moving with constant velocity and the lower plate is held stationary under the influence of inclined magnetic field.

We shall convert the boundary conditions of the fluid flow into initial conditions. The initial conditions are determined and the problem is solved by use of Laplace Transform.

In chapter one, we shall major in the definition of Hydromagnetic steady flow together with where it can be applied.

In chapter two we shall introduce the Laplace Integrals and Laplace transforms as well as some of their properties.

Finally in chapter three, we shall develop the equations governing the hydromagnetic flow and in this chapter we shall obtain these basic equations. We shall also get the solution of the hydromagnetic steady flow of viscous incompressible fluid between two parallel infinite plates using the Laplace transform and discus the results which are presented in form of a graph for selected Hartmann numbers.

TABLE OF CONTENTS

CHAPTER 1:

INTRODUCTION

1.1 HYDROMAGNETIC STEADY FLOW

Hydromagnetics involves the effect of externally impressed magnetic field on the onset of thermal instability in electrically conducting fluids. In broad terms, the subject of hydromagnetics is concerned with the ways in which magnetic fields can affect fluid behavior. These fluids include liquid metals and highly ionized gas-like substances called plasmas. When we consider a fluid which has the property of electrical conduction; and suppose also that magnetic fields are prevalent. The electrical conductivity of the fluid and the prevalence of magnetic fields contribute to effects of two kind: first, by motion of the electrically conducting fluid across the magnetic lines of force, electric currents are generated and the associated magnetic fields contribute to changes in the existing fields; and second, the fact that the fluid elements carrying currents transverse magnetic lines of force contributes to additional forces acting on the fluid elements. It is in this two fold interaction between the motions and the fields that is responsible for patterns of behavior which are often unexpected and striking.

The basic coordinate system describing magnetofluidmechanic phenomena is shown in figure 1.1 below.

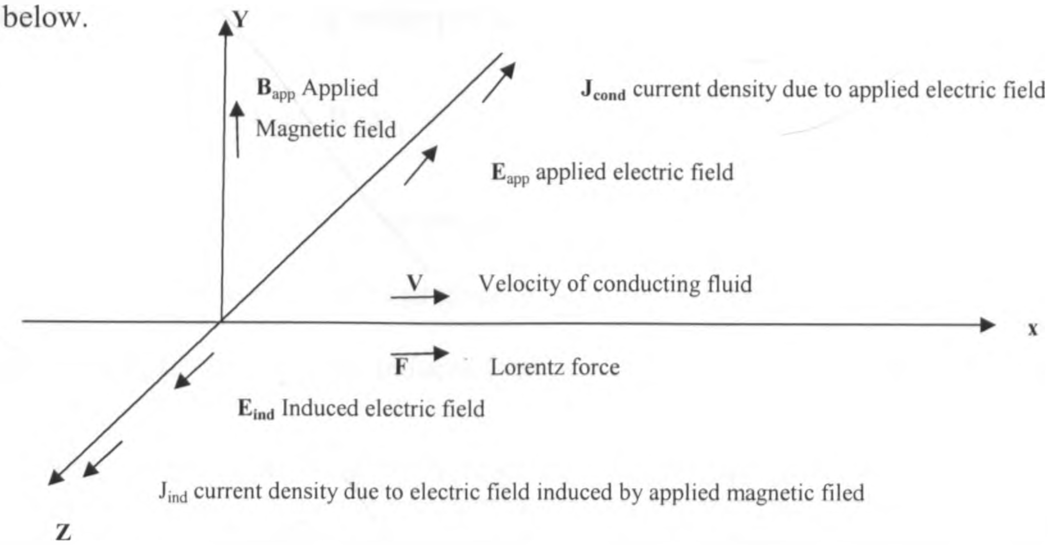


FIG 1.1 Vector diagram of Magnetofluidmechanics

We consider an electrically conducting fluid having a velocity vector **V**. At right angels to this we apply a magnetic field, the field strength of which is represented by the vector $\mathbf{B}_{app.}$ We assume that steady flow conditions have been attained. Because of the interaction of the two fields an electric field denoted by $\mathbf{E}_{ind}$ is induced at right angels to both **V** and $\mathbf{B}_{app.}$ This electric field is given by the following equation:

$$\mathbf{E}_{ind} = \mathbf{V} \times \mathbf{B}_{app} \qquad 1.1.1$$

If we assume that the conducting fluid is and remains isotropic in spite of the magnetic field, † we can denote its electrical conductivity by the scalar quantity σ. Then by Ohm's law the density of the current induced in the conducting fluid and denoted by $\mathbf{J}_{ind}$ is

$$\mathbf{J}_{ind} = \sigma \, \mathbf{E}_{ind} \qquad 1.1.2$$

Simultaneously occurring with the induced current is the induced ponder motive force $\mathbf{F}_{ind}$, which is given by the following vector product:

$$\mathbf{F}_{ind} = \mathbf{J}_{ind} \times \mathbf{B}_{app} \qquad 1.1.3$$

This force occurs because, as in an electric generator, the conducting fluid cuts the lines of the magnetic filed. Because the vector product of equation (1.1.3) yields a vector perpendicular to both $\mathbf{J}_{ind}$ and $\mathbf{B}_{app}$, the induced force is parallel to $\mathbf{V}$ but opposite in direction.

To make the configuration slightly more general, we now apply also an electric filed $\mathbf{E}_{app}$ at right angles to both B_{app} and $\mathbf{V}$, but opposite in direction to $\mathbf{J}_{ind}$. The current density due to this applied electric field we denote by $\mathbf{J}_{cond}$ and call it the direction current. The net current density $\mathbf{J}$ through the conducting field and the one that we would be measuring with a suitably placed ammeter is then

$$J = \sigma\ (\mathbf{E}_{app} + \mathbf{v} \times \mathbf{B}_{app}) = \sigma\,(\mathbf{E}_{app} + \mathbf{E}_{ind}) \qquad 1.1.4$$

The ponderomotive or Lorentz force associated with this current is t hen

$$F = J \times \mathbf{B}_{app} = \sigma\ (\mathbf{E}_{app} + \mathbf{v} \times \mathbf{B}_{app}) \times \mathbf{B}_{app} \qquad 1.1.5$$

In equation (1.1.5), $\mathbf{E}_{app} > \mathbf{v} \times \mathbf{B}_{app}$, we have an accelerator which may be used as a thrust- producing device. We note at once the potentialities of accelerating fluids by electromagnetic fields rather than by adding large quantities of thermal energy which would tend to result in the thermal deterioration of engine walls. At first this may delude the designer into believing that magnetodynamics circumvents the monumental problem of locating high-temperature-resistant solids. Unfortunately the matter is not so simple, because although the acceleration itself does not require high temperatures, producing a gas which is sufficiently well ionized to have a high conductivity susceptible to experiencing electromagnetic acceleration does necessitate very high temperatures. The engineer wishing to design magnetofluidmecahnic devices then has three alternatives. First, he might use conducting liquids. Unfortunately these are difficult to handle. Second, he might consider cooling his engine walls. However, high heat-transfer rates are not easily obtained. Third, he might displace the high –temperature gases from his engine walls by magnetic pressure leading to the so-called pinch effect. This scheme too is troublesome because of flow instabilities and practicalities associated with magnetic design, Ali [1963].

The study of hydromagnetics is closely related to the interaction of two branches, namely electromagnetic theory and fluid mechanics which in turn produces magnetohydrodynamics. The principles of hydrodmagnetics have many applications. A very good example is the construction of the magnetohydrodynamics pump. In this device, a conducting fluid in a pipe is forced to move by the Lorentz force created when mutually perpendicular magnetic fields and electric currents are applied perpendicular to the pipe. Such a device has been used to circulate liquid sodium carrying heat from the core of a fission reactor to the heat exchanger outside. Other related areas includes in the magnetohydrodynamics generator where the channel cross-section is normally circular with conducting walls.

The principles of hydromagnetic are also very important in designing controlled thermonuclear reactors (CRT). Much interest in hydrodynamics has arisen through strenuous efforts to release thermonuclear energy by controlled reactions between nuclei of elements (such as deuterium) in a hot ionized gas. Success in this field would be prelude to the construction of economically viable commercial power stations reacting on the same principle. A plasma of deuterium (i.e. a mass of highly ionized deuterium) must be generated; must be heated to a temperature so high that a significant fraction of nuclei have thermal energies sufficient to penetrate each others Coulomb barriers, fuse together and release energy; and must be maintained for time sufficient for a sizeable number of such reactions to take place. If the energy liberated by them mare than repays the energy required to produce and maintain the plasma, the difference can be made to perform useful work. The task of confining the hot plasma to a small reaction space (away from any materials walls which would contaminate it, and quench the reactions) has always been left to some kind of "magnetic bottle". This is a region of space walled in by strong magnetic fields which, as in the homopolar dynamo, will oppose by Lenz's law any motion of contained plasma across them. Clearly an understanding of the inter-relation between the motion of particles and the behavior of the electromagnetic fields is a key factor in the design of the bottle and therefore in the success of the project.

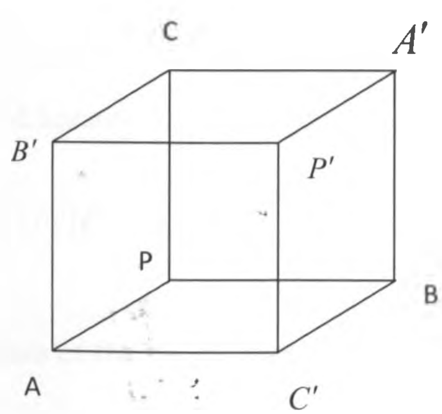
Another important area is the principles of hydromagnetic are greatly applied in the direct conversion of energy. Here electricity is usually produced from the chemical energy of flues such as coal or oil in the following way. The fuel is burned and the heat generated is used to create high pressure steam. Thus, in turn, drives a turbine linked to a dynamo.

The efficiency of such a heat engine cannot, of course exceed the theoretical maximum of the Carnot cycle i.e. $(T_1 - T)/T_1$ where T_1 and T_2 are the upper and lower absolute temperatures of the cycle; in practice, it is usually significantly less. The efficiency can be improved in theory by increasing T_1, but in practice the loss of strength in materials (eg the walls of the combustion chamber) at high temperatures is a limitation. Suppose, however, that the gases are burned at high temperatures in a flame far from the walls. The kinetic energy of the hot ionized gas in the flame can be converted directly into electrical energy by applying a perpendicular magnetic field: for a potential gradient is created in a direction perpendicular to field and motion, and currents can be drawn off by electrodes embedded in the gas.

Finally the principles of hydromagenetics are very useful in the construction of the flow meters. When conducting fluid passes down an insulating pipe across which a steady magnetic field is applied, a potential gradient (proportional to the flow speed) is created and can be measured by process embedded in the walls of the pipe. The flow rate can therefore be determined without, for example, contaminating the fluid in the pipe. It's this same technique that is used to measure the flow of blood.

1.2 EQUATION OF CONTINUITY

The equation of continuity for the incompressible flow in Cartesian coordinates is derived as follows by considering a parallelepiped fluid elemention as shown below;



Fluid flow with edges $PA = \delta x,\ PB = \delta y,\ PC = \delta z$ with one of the corners at P. let u, v, w be respectively x, y, z components of velocity along faces $CA'BP,\ CPAB'\ and\ PBC'A$ and the velocity components along faces $B'P'C'A,\ P'A'BC',\ B'CA'P'$ will then be given as $u + \frac{\partial u}{\partial x}\delta x,\ v + \frac{\partial v}{\partial y}\delta y,\ w + \frac{\partial w}{\partial z}\delta z.$ let $\rho(x, y, z, t)$ be the density.

Now mass of fluid flow inside parallelepiped per unit time through the face $PCA'B = \rho u \delta y \delta z$ and mass of fluid going out of the parallelepiped per unit through the face $B'P'C'A = \left[\rho u + \frac{\partial}{\partial x}(\rho u)\delta x\right]\delta y \delta z.$

Therefore increase in the mass of fluid per unit time because of flow across two faces is given by

$$\rho u \delta y \delta z - \left[\rho u + \frac{\partial}{\partial x}(\rho u)\delta u\right]\delta y \delta z = -\frac{\partial}{\partial x}(\rho u)\delta x \delta y \delta z \qquad 1.1.6$$

Also the mass of the fluid going inside parallelepiped per unit time through the face $B'CPA = \rho v \delta x \delta z.$ and mass of fluid going out of the parallelepiped per unit because of flow through the face $ABC'P' = \left[\rho v + \frac{\partial}{\partial y}(\rho v)\delta y\right]\delta x \delta z.$ Therefore increase in the mass of fluid inside parallelepiped because of flow across these two faces is given by:

$$\rho v \delta x \delta z - \left[\rho v + \frac{\partial}{\partial y}(\rho v)\delta y\right]\delta x \delta z = -\frac{\partial}{\partial y}(\rho v)\delta x \delta y \delta z$$

Similarly mass of fluid going inside parallelepiped per unit time through the face

$PBC'A = \rho w \delta x \delta y$ and mass of fluid going out of the parallelepiped pr unit time through the face

$B'CA'P' = \left[\rho w + \frac{\partial}{\partial z}(\delta z) \right] \delta x \delta y$. Therefore increase in the fluid per unit time because of flow across these two faces is given by:

$$\rho w \delta x \delta z - \left[\rho w + \frac{\partial}{\partial z}(\rho w)\delta z \right] \delta x \delta y = -\frac{\partial}{\partial z}(\rho w)\delta x \delta y \delta z$$

Also mass of fluid inside parallelepiped is given as $\rho \delta x \delta y \delta z$. Therefore increase in the mass of fluid inside parallelepiped per unit time

$$\frac{\partial \rho}{\partial t} \partial x \partial y \partial z$$

Now using principle of conservation of mass we have:

$$\frac{\partial \rho}{\partial t} \delta x \delta y \delta z \; \frac{\partial \rho}{\partial t} \delta x \delta y \delta z = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right] \delta x \delta y \delta z = 0$$

As the above expression is true of every elementary volume taken inside fluid flow, we have:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Which is equation of continuity in Cartesian coordinate system. The above equation may be given as:

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z} = 0$$

$$= \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left[\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0$$

$$= \left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]\rho + \rho\left[\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0$$

If we apply the definition of material derivative which is given by $\frac{D}{Dt}$ then the above equation becomes

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

This is yet another form of equation of continuity.

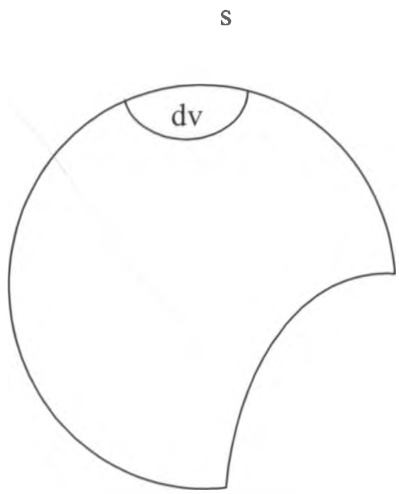
But if the fluid is incompressible i.e. $\rho = 0$ then, the above equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad 1.1.7$$

Where u, v, and w are components of velocity of the fluid in the x, y, and z directions.

1.3 EQUATION OF MOTION

The equation of motion that describes incompressible flow in each of the three directions is derived by considering the flow of non viscous fluid in a Cartesian region of space. We consider in that flow field a fluid volume v enclosed by surface s.



Let dv be an elementary volume enclosing fluid element p of density ρ. This mass ρdv remains constant throughout the motion.

Now if M is the momentum of the volume v and $\vec{q}$ is the fluid velocity of particle p, then

$$M = \iiint_v \bar{q}\rho dv \qquad 1.1.8$$

Where integral has been out for entire volume v.

Again let p be the pressure at a point on elementary surface ds then if $\bar{n}$ is the unit outward normal at the surface ds we have;

$$-\iint_s \rho\bar{n}ds = \iiint_v \nabla\bar{p}dv \qquad 1.1.9$$

Also if F is the external force per unit mass on volume v then the total external force is given by ;

$$\iiint_v F\rho dv$$

We also know that according to Newton's law, rate of change of linear momentum is equal to applied force. Therefore we have;

$$\frac{DM}{Dt} = -\iiint_v \nabla p dv + \iiint_v F\rho dv$$

$$= \frac{D}{Dt}\left[\iiint_v \vec{q}\rho dv\right] = -\iiint_v \nabla p dv + \iiint_v F\rho dv$$

By product rule we have;

$$\iiint_v \frac{D}{Dt}(\vec{q}\rho)dv = \iiint_v (F\rho - \nabla p)dv$$

The above relationship will be true for any considered volume v in the flow field. Therefore;

$$\frac{D\vec{q}}{Dt} = F - \frac{1}{\rho}\nabla p \qquad 1.2.0$$

We also know from the definition of material derivative that

$$\frac{D\vec{q}}{Dt} = \frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q}$$

Therefore equation 1.2.0 above may also be given as;

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} = F - \frac{1}{\rho}\nabla p \qquad 1.2.1$$

Again if

$q = u\hat{i} + v\hat{j} + w\hat{k}$ *and* $F = x\hat{i} + y\hat{j} + z\hat{k}$

Then from 1.2.0 above we have

$$\frac{D}{Dt}\left[u\hat{i} + v\hat{j} + w\hat{k}\right] = \left[x\hat{i} + y\hat{j} + z\hat{k}\right]\left[-\frac{1}{\rho}\left(\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z}\right)\right] \qquad 1.2.2$$

CHAPTER TWO

LAPLACE TRANSFORMS

2.1 INTRODUCTION

The Laplace transform operator L is very useful in the study of initial value problems involving linear differential equations with constants coefficients. The approach to the analysis and design of our system is based upon the solution of an ordinary linear differential equation with constants coefficients. To solve this system by Laplace transform, we first derive the ordinary linear differential equations which describe the motion of the system. We then obtain the initial conditions and get the Laplace transform of the differential equation. We then manipulate these algebraic equations and solve for the desired dependent variables. Finally we get the inverse Laplace transform of the functions and insert the boundary conditions to determine the initial conditions and the problem is subsequently solved. In this chapter, we discuss the definition of Laplace transform and its basic properties. Thomson $[1957]$

Let $f(t)$ be any function. The Laplace transform of $f(t)$ denoted by $L\{f(t)\}$ is defined by $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ 2.1.1

If $f_1(t)$ and $f_2(t)$ have Laplace transforms and if c_1 and c_2 are constants,

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}. \quad 2.1.2$$

2.2 SOME PROPERTIES OF LAPLACE TRANSFORMS

The properties are presented here without proof. Their proofs may be found in Thomson (1957) and in most books dealing with differential equations.

For brevity, we shall denote

$$s\int_0^\infty e^{-st} f(t)\,dt \;\; by \;\; c\{f(t)\}$$

- Property of linearity:

  Let $f(t)=\sum_{k=1}^{n} c_k f_k(t)$

  Where c_k are arbitrary constants. Then

$$L[f(t)] = L\left[\sum_{k=1}^{n} c_k f_k(t)\right] = \sum_{k=1}^{n} c_k L[f_k(t)]$$

$$=\sum_{k=1}^{n} c_k L\{f_k(t)\}$$

Thus

$$L\left[\frac{d}{d\lambda} f(t,\lambda)\right] = L\left[\frac{f(t,\lambda+d\lambda)-f(t,\lambda)}{d\lambda}\right]$$

$$=L\left\{\frac{f(s,\lambda+d\lambda)-f(s,\lambda)}{d\lambda}\right\}$$

$$=\frac{d}{d\lambda} L\{f(s,\lambda)\}$$

❖ Property of Similitude:-

For any constants α, we have

$$L\left\{f\left(\frac{t}{\alpha}\right)\right\}=\int_0^\infty f\left(\frac{t}{\alpha}\right)e^{-st}dt$$

$$=\alpha\int_0^\infty f(\tau)e^{-\alpha st}d\tau$$

$$=\alpha L\{f(\alpha s)\}$$

$$c\left\{f\left(\frac{t}{\alpha}\right)\right\}=s\int_0^\infty f\left(\frac{t}{\alpha}\right)e^{-st}dt$$

$$=\alpha s\int_0^\infty f(\tau)e^{-\alpha st}d\tau$$

❖ Laplace Transformation of derivatives:-

From intergration by parts, we can obtain

$$L\{f^n(t)\}=s^nL\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-sf^{n-2}(0)-sf^{(n-1)}(0).$$

$$c\{f^n(t)\}=s^nc\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-sf^{n-2}(0)-f^{n-1}(0).$$

Where n is a positive integer.

❖ Differentiation of Laplace Transforms

For a positive integer n,

$$\frac{d^nL\{f(t)\}}{ds^n}=(-1)^n\int_0^\infty t^nf(t)e^{-st}dt$$

$$=(-1)L\{t^nf(t)\}$$

$$\frac{d^n C\{f(t)\}}{ds^n} = (-1)^n C\left\{t^n f(t) - n\int_0^t t^{n-1} f(t)\,dt\right\}$$

❖ Laplace Transforms of integrals

$$L\left\{\int_0^t d\tau \int_0^\tau d\tau_1 \ldots \int_0^{\tau_n} f(\tau_{n-1})\,d\tau_{n-1}\right\} = \frac{L\{f(t)\}}{s^n}$$

Where n is a positive integer.

❖ Integration of Laplace Transforms

If

$$\int_p^\infty L\{f(\tau)\}d\tau$$

Is convergent, it is the Laplace transform of $\frac{f(t)}{t}$ that is we have

$$\int_p^\infty L\{f(t)\}\,d\tau = L\left\{\frac{f(t)}{t}\right\}.$$

❖ Given any positive τ, assuming that $f(t-\tau)=0$ for $t<\tau$, we obtain

$$L\{f(t-\tau)\} = \int_\tau^\infty f(t-\tau)e^{-st}\,dt$$

$$= \int_0^\infty f(u)\,e^{-s(u+t)}\,du$$

$$= e^{-s\tau}\int_0^\infty f(u)e^{-su}\,du$$

That is

$$L\{f(t-\pi)\} = e^{-st}L\{f(t)\}$$

2.3 SOME PROPERTIES OF LAPLACE INTEGRALS

1. If the integral defined by 2.1.1 is convergent at a point s_0, it is convergent at all points s for which $\operatorname{Re}(s-s_0)>0$. There are three possible cases for Laplace Integral.

    I. The integral is divergent everywhere

    II. The integral is convergent everywhere

    III. There exist a number σ_c such that the Integral is convergent for $\operatorname{Re} s>\sigma_c$ and divergent for $\operatorname{Re} s>\sigma_c$. The number σ_c is called the abscissa of convergence of integral 2.1.1.

2. If Integral 2.1.1 is absolutely convergent at the point $s_0=\sigma_0+i\tau_0$, it is absolutely and uniformly convergent in the half-plane $s \geq s_0$.

3. If 2.1.1 is convergent at the point $s_0=\sigma_0+i\tau_0$ and if $Q \geq 0$ and $k \geq 1$ are the given constants, the integral is uniformly convergent in the domain Δ given the inequalities

$$|s-s_0| \leq k(\sigma-\sigma_0)e^{Q(\sigma-\sigma_0)}, \quad \sigma \geq \sigma_0 \qquad 2.1.3$$

4. If $\sigma_c<\infty$ integral represents an analytic function if the variables s at all points of the half plane $\operatorname{Re} s>\sigma_c$ and

$$\frac{d^k L\{f(t)\}}{ds^k} \int_0^{\infty}(-t)^k f(t)e^{-st}dt. \qquad 2.1.4$$

An analytic function is a function of a complex variable which posses a derivative at every point of a region.

5. Let $L\{f_1(t), L\{f_2(t)\}\}$ be the Laplace transforms of functions $f_1(t)$ and $f_2(t)$. If both Laplace integrals are convergent at the point s_0 and

$$L\{f_1(s_0+nl)\} = L\{f_2(s_0+nl)\},$$

Where the constants $l > 0$ and $n = 0,1,2,........$, then $f_1(t) = f_2(t)$ almost everywhere.

6. If the integral 2.1.1 is that for some $\sigma_o > 0$ and $t \to \infty$, at the point

$s_0 = \sigma_0 + i\tau_0$, $\sigma > 0$, then

$$\lim_{t\to\infty} e^{-\sigma_0 t} \int_0^t f(u)du = 0$$

That is

$$\int_0^{t^1} f(u)du = 0 \quad as \quad t \to \infty$$

A necessary and sufficient condition for convergence of integral 2.1.1 is that for some $\sigma_0 > 0$ and $t \to \infty$,

$$f_1(t) = \int_0^t f(u)du = 0\left(e^{\sigma 0}\right)$$

That is

$$\lim_{t\to\infty} e^{-\sigma_0 t} \int_0^t f(u)du = 0$$

Theorem 1: if the integral (2.1.1) has an abscissa of convergence $\sigma_c < \infty$, we have the limit

$$\lim_{w \to \infty} \frac{1}{2\pi i} \int_{-i\omega}^{\gamma + i\omega} L\{f(t)\} \frac{e^{st}}{s} ds = \begin{cases} 0, & t < 0 \\ \int_0^t f(u)\, du, & t > 0 \end{cases} \qquad 2.1.5$$

Where $\gamma > \sigma_c, \gamma > 0.$ Hence for almost all t,

$$f(t) = \frac{d}{dt} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} L\{f(t)\} \frac{e^{st}}{s} ds,$$

Where the intergral is understood in the sense of the principle value. It follows from property 6 that

$$\frac{L\{f(t)\}}{s} = \int_0^{\infty} f_1(t) e^{-st} dt$$

Where

$$f_1(t) = \int_0^t f(u) du, \sigma > 0, \text{ and } s = \sigma + i\tau.$$

A constant Q exists such that $|f_1(t)| < Qe^{\sigma_0 t} < Qe^{\sigma 0} \; (\sigma > \sigma_c)$ for all t. Hence

$$\left| \frac{L\{f(t)\}}{s} \right| \le \frac{Q}{\sigma - \sigma_0} \qquad 2.1.6$$

Thus if

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt,\ \sigma > \sigma_c,\ \text{and}\ \ f_1(t) = \int_0^t f(u)du,$$

The Laplace transform of

$$f_1 t$$

will be

$$\frac{L\{f(t)\}}{s},$$

the Laplace transform being absolutely convergent for

$$\sigma > \sigma_c.$$

2.4 THE CONVOLUTION THEOREM

We often come across functions which are not the transforms of some known function, but then, they can possibly be expressed as a product of two functions, each of which is the transform of a known function. Let $a(t)$ and $b(t)$ be functions of real variable t. The convolution of these functions is the function $c(t)$ given by

$$c(t) = \int_0^t a(t-\tau)b(\tau)d\tau \qquad 2.1.7$$

Symbolically written as;

$$c(t)=a(t)*b(t) \tag{2.1.8}$$

The operation of obtaining the convolution is called the convolution.

Convolution are;

(1). Commutative,

$$a(t)*b(t)=b(t)*a(t). \tag{2.1.9}$$

(2). Associative,

$$(a*b)*c=a*(b*c). \tag{2.2.0}$$

(3). Distributive with respect t addition.

$$[a(t)+b(t)]*c(t)=a(t)*c(t)+b(t)*c(t). \qquad 2.2.$$

Theorem

If the integrals

$$L\{f_1(t)=\int_0^\infty f_1(t)e^{-st}dt$$

And

$$L\{f_2(t)=\int_0^\infty f_2(t)e^{-st}dt$$

Are absolutely convergent for $\operatorname{Re} s > \sigma_a$, then

$$L\{f(t)\} = L\{f_1(t)\} L\{f_2(t)\} \qquad 2.2.2$$

Is the Laplace transform of

$$f(t) = \int_0^t (t-\tau) f_2(\tau)\, d\tau, \qquad 2.2.3$$

And the integral

$$Lf(t) = \int_0^\infty f(t) e^{-st} dt \qquad 2.3.4$$

Is absolutely convergent for $\operatorname{Re} s > \sigma_a$.

If the convolution of functions $a(t)$ and $b(t)$, are continuous for $0 \le t < +\infty$, is identically zero, at least one of these functions is identically zero.

2.5 INVERSE LAPLACE TRANSFORMS

If the Laplace transform technique is to be useful in applications, we have to find the original function $f(t)$ when its Laplace transform is $F(s)$. Thus, if

$$L[f(t); s] = F(s)$$

Then

$$f(t) = L^{-1}[F(s); t]$$

Where L^{-1} is known as the inverse Laplace Transform operator.

In other words, the inverse Laplace Transform of a given function $F(s)$ is that function $F(t)$ whose Laplace Transform is $F(s)$.

There are three main methods of getting the Inverse Laplace transform:

- *Laplace Transform tables*

This is the most obvious method of finding the inverse Laplace Transform of a function s. A copy of the table of Laplace Transforms is presented at the appendix.

- *Inversion by Partial fractions*

The simplest method of inverting a Laplace-transformed equation is by use of the tables. If the function is too complex to find in the tables, we can use the method of partial fractions. Use of this method results in a sum of simple fractions, each of which can be inverse Laplace-transformed directly from memory or with the aid of the tables. Since the inverse Laplace transforms, we can treat each term separately.

- *The inversion integral.*

For functions which are not rational fractions, it is necessary to use the inversion integral. The inversion integral yields the time function directly. Its main use is for the inversion of more advanced functions e.g. $\left[\frac{1}{\sqrt{s}}, \frac{1}{\left(s^2+a^2\right)^{\frac{1}{2}}}\right]$ etc.

2.6 SOME PROPERTIES OF INVERSE LAPLACE TRANSFORMS

❖ *Linearity property.*

If c_1 and c_2 are any constants while $f_1(s)$ and $f_2(s)$ are the Laplace transforms of $F_1(t)$ and $F_2(t)$ respectively, then

$$L^{-1}\{c_1 f_1(s)+c_2 f_2(s)\}$$

$$=c_1 L^{-1}\{f_1(s)\}+c_2 L^{-1}\{f_2(s)\}$$

$$=c_1 F_1(t)+c_2 F_2(t)$$

❖ *First Shifting property*

If

$$L^{-1}\{f(s)\}=F(t),$$

Then

$$L^{-1}\{f(s-a)\}=e^{at}F(t)$$

❖ *Second Shifting property*

If

$$L^{-1}\{f(s)\}=F(t),$$

Then

$$**L^{-1}\{e^{-as}f(s)\}=\begin{cases}F(t-a), t>0\\ 0, t<0\end{cases}$$

❖ *Change of scale property*

Given that

$$L^{-1}\{f(s)\} = F(t),$$

Then

$$L^{-1}\{f(ks)\} = \frac{1}{k}F\left(\frac{t}{k}\right)$$

❖ *Inverse Laplace Transforms of derivatives*

If

$$L^{-1}\{f(s)\} = F(t),$$

Then

$$L^{-1}\{f^{(n)}(s)\} = L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}$$

$$=(-1)^n t^n F(t)$$

❖ *Inverse Laplace Transform of Integrals*

If

$$L^{-1}\{f(s)\} = F(t),$$

Then,

$$L\left\{\int_0^\infty f(u)\,du\right\} = \frac{F(t)}{t}$$

❖ *Multiplication by s^n*

Let

$$L^{-1}\{f(s)\} = F(t) \text{ and } F(0) = 0,$$

Then,

$$L^{-1}\{sf(s)\} = F'(t).$$

This is to say that multiplication by s has the effect of differentiating $F(t)$

If

$$F(0) \neq 0,$$

Then

$$L^{-1}\{sf(s) - F(0)\} = F'(t)$$

❖ *Division by s*

If

$$L^{-1}\{f(s)\} = F(t),$$

Then

$$L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)du$$

That is division by s (or multiplication by $\frac{1}{s}$) has the effect of integrating $F(t)$ from 0 to t.

❖ *The convolution property*

If

$$L^{-1}\{f_1(s)\} = F_1(t) \text{ and } L^{-1}\{f_2(s)\} = F_2(t),$$

Then,

$$L^{-1}\{f_1(s)f_2(s)\} = \int_0^t F_1(u)F_2(t-u)\,du = F_1(t) * F_2(t)$$

2.7 COMPUTATIONS OF SOME LAPLACE TRANSFORMS

With the above background we may now compute the Laplace Transforms of some functions that we are likely to come across as we solve our equation. They include:

➢ *Laplace of* e^{pt}

$$L(e^{pt}) = \int_0^\infty e^{-st}e^{pt}\,dt$$

This can be written as

$$= \int_0^\infty e^{-(s-p)t}\,dt$$

For $s \le k$, the exponent on e is positive or 0, and the integral diverges. For $s > k$, the integral converges, thus

$$\begin{aligned} L(e^{pt}) &= \int_0^\infty e^{-(s-p)t}\,dt \\ &= \left[\frac{-e^{-(s-p)t}}{s-p}\right]_0^\infty \\ &= 0 + \frac{1}{s-p} \end{aligned}$$

Thus

$$L\left(e^{pt}\right)=\frac{1}{s-p}, s>p$$

For $p=0$, we find that

$$L(1)=\frac{1}{s}, \text{ for } s>0.$$

Since the integral

$$\int_0^\infty e^{-(s-p)t}dt$$

Exists for $s>a$, the Laplace transforms exists for all functions $f(t)$ satisfying the inequality

$$\left|e^{-st}f(t)\right|<Ce^{(s-a)t}$$

Where C is a constant. This is to say that $f(t)$ does not grow more rapidly than Ce^{at}, or that $f(t)$ is of exponential order, and that

$$\lim_{t\to\infty} e^{-st}f(t)=0$$

- *Laplace transform of* $(\sin pt)$ *and that of* $(\cos pt)$

From elementary calculus,

$$\int e^{ax} \sin mx dx = \frac{e^{ax}(\sin mx - m\cos mx)}{a^2 + m^2} + c.$$

Therefore, the Laplace transform of $(\sin pt)$ is

$$L(\sin pt) = \int_0^{\infty} e^{-st} \sin pt dt$$

Which means that;

$$L(\sin pt) = \left[\frac{e^{-st}(-s\sin pt - p\cos pt)}{s^2 + p^2}\right]_0^{\infty}$$

For positive $s, e^{-st} \to o \ at \ t \to \infty$. Since $\sin(pt)$ and $\cos(pt)$ are bounded as $t \to \infty$, the above equation now becomes:

$$L(\sin pt) = 0 - \frac{1(0-p)}{s^2 + p^2}$$

$$= \frac{p}{s^2 + p^2}, for\ s > 0.$$

In a similar manner,

$$L(\cos pt) = \frac{s}{s^2 + p^2}, s > 0.$$

➢ *Laplace transform of* $\cosh t$ *and that of* $\sinh at$

$$L\{\cosh at\} = \int_0^\infty e^{-st} \cosh t dt$$

$$= \int_0^\infty e^{-st} \frac{e^{at} + e^{-at}}{2} dt$$

$$= \frac{s}{s^2 - a^2}$$

In a similar manner we can compute that;

$$L\{\sinh at\} = \frac{a}{s^2 - a^2}$$

CHAPTER 3

SOLUTION OF HYDROMAGNETIC STEADY FLOW OF VISCOUS INCOMPRESSIBLE FLUID BETWEEN TWO PARALLEL INFITE PLATES UNDER THE INFLUENCE OF INCLINED MAGNETIC FIELD

3.1 INTRODUCTION

The basic concept describing magnetohydrodynamic phenomena as follows. Consider an electrically conducting fluid having a velocity vector **V**. At an angel α with the direction of flow conditions have been attained. Because the interaction of two fields, an electric field denoted by **E** is induced at right angels to both **V** and $\mathbf{B}\sin\alpha$, this electric field is given by

$$\mathbf{E} = \mathbf{V} \times \mathbf{B}\sin\alpha \ .$$

If we assume that a conducting fluid is isotropic, we can denote its electrical conductivity by the scalar quantity σ. By Ohms Law, the density of the current induced in the conducting fluid, **J** is given by $\mathbf{J} = \mathbf{B}\ \sin\alpha$.

The Laminar flow of an eclectically conducting fluid through a channel under uniform inclined magnetic field is important because of the use of MHD generator, the MHD pump and the electromagnetic flow meter. The general model that is normally considered in these studies consists of an infinitely long channel of constant cross-section with a uniform static magnetic field applied transverse to the axis of the channel. The walls of channel are either insulator, conductors depending on the intended application. For example, in the MHD generator and pump, the channel cross-section is normally circular with c conducting walls.

There are several people who have looked at hydromagnetic steady flow equations: Sercif [1956] studied the steady motion of electrically conducting fluid in a pipe under transverse magnetic field. Drake [1965] considered the flow in a channel due to periodic pressure gradient and solved it by method of separation of variables. Singh and Ram [1977] considered the laminar flow of an electrically conducting fluid through a channel in the presence of a transverse magnetic field under the influence of a periodic pressure gradient and solved it by Laplace transform. Ram et al [1984] considered Hall effects on heat and mass transfer flow through porous medium. Simonura [1991] considered magnetohydrodynamic turbulent channel flow under uniform transverse magnetic field. Kazuyuki [1992] discussed inertia effects in two dimensional MHD channel flow. Singh [1998] considered unsteady MHD flow of liquid through a channel under variable pressure gradient and solved it by method of Laplace transform. Singh [2000] considered unsteady flow of liquid through a channel with pressure gradient changing exponentially under the influence of inclined magnetic field and solved it by method of Laplace transform. Kimathi [2001] solved magnetohydrodynanic flow problem by using Laplace Transform. Singh [2007] considered hydromagnetic steady flow of viscous incompressible fluid between two parallel infinite channel when upper plate is moving and lower plate is held stationary under the influence of inclined magnetic field by using method of solution of ordinary differential equations. Singh [2007]

In the present paper laminar hydromagntic steady flow of viscous incompressible fluid between two parallel infinite plates is considered when upper plate is moving and lower plate is held stationary under the influence of inclined magnetic field and the equation is solved by Laplace Transform. The changes in velocity profiles has been shown graphically for different Hartman numbers at different angels.

3.2 GOVERNING EQUATIONS

The equation of continuity for the incompressible flow is given from equation 1.1.7 as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

Where u, v, and w are components of velocity of the fluid in the x, y, and z directions. The equation of motion that describes incompressible flow in each direction is given by breaking up equation 1.2.1 and writing it in a form that describes flow in each of the direction as:-

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{F_x}{\rho} \tag{3.1.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{F_y}{\rho} \tag{3.1.2}$$

and

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+v\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{F_z}{\rho} \tag{3.1.3}$$

Where F_x, F_y, F_z are components of $\vec{J}\times\vec{B}\sin\alpha$ in the x, y, z directions, respectively.

For simplicity we shall consider a two-dimensional flow. In two- dimensions, equation 3.1.1 becomes;

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad 3.1.4$$

Since the plates are of infinite length, we assume that the flow is only along the x-axis and depends on y.

Thus,

$$\frac{\partial u}{\partial x}=0 \qquad 3.1.5$$

Since we have assumed a steady flow, the flow variables do not depend on time. Thus, equation 3.1.4 and 3.1.5 can be written as

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{F_x}{\rho} \qquad 3.1.6$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{F_y}{\rho} \qquad 3.1.7$$

Therefore, substitution of equation 3.1.6 into equation 3.1.7 leads to

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial y^2}\right)+\frac{F_x}{\rho} \qquad 3.1.8$$

Whereas combination of equation 3.1.5 and 3.1.6 and 3.1.8 yields;

$$0=\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{F_y}{\rho} \qquad 3.1.9$$

Since there is no component of body force in the y-direction and

$F_x=\vec{J}\times\vec{B}\sin\alpha,\ \ F_y+F_z=0$ as $v=w=0$ then equations of motion 3.1.8 and 3.1.9become

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial y^2}\right)+\frac{\vec{J}\times\vec{B}}{\rho}\sin\alpha \qquad 3.2.0$$

And

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad 3.2.1$$

Equation 3.2.0 implies that pressure does not depend on y.

We know from Ohm's law that the current $\vec{J}$ is proportional to the force per unit charge i.e.

$$\vec{J}=\sigma\vec{E}$$

And

$$\vec{E}=\vec{U}\times\vec{B}\sin\alpha$$

Where $\vec{U}$ is the fluid velocity along x-axis, which is the direction of fluid flow.

Thus,

$$\vec{J}\times\vec{B}\sin\alpha=\sigma\left[\left(\vec{U}\times\vec{B}\sin\alpha\right)\times\vec{B}\sin\alpha\right]$$

$$=\sigma\left[\left(\vec{U}\cdot\vec{B}\sin\alpha\right)\vec{B}\sin\alpha-\left(\vec{B}\sin\alpha\cdot\vec{B}\sin\alpha\right)\vec{U}\right]$$

Since $\vec{U}$ and $\vec{B}\sin\alpha$ are perpendicular vectors, we have

$$\vec{U}\cdot\vec{B}\sin\alpha=0$$

Giving

$$\vec{J}\times\vec{B}\sin\alpha=-\sigma B^2\vec{U}\sin^2\alpha$$

Hence $\dfrac{\vec{J}\times\vec{B}}{\rho}\sin\alpha=-\dfrac{\sigma B^2\vec{U}}{\rho}\sin^2\alpha$

Consequently the equation of motion 3.2.5 now reduces to

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\frac{\partial^2 u}{\partial y^2}-B^2u\sin^2\alpha \qquad 3.2.2$$

We set out to get the solution of the above equation 3.2.2 using the Laplace Transforms.

3.3 NON DIMENSIONLIZING

To simplify equation 3.2.2 further, we reduce the parameters in the equation by introducing the following non-dimensional quantities;

$$x' = \frac{x}{a}, \quad y' = \frac{y}{a}, \quad p' = \frac{pa^2}{\rho v^2}, \quad u' = u\frac{a}{v}$$

With these quantities we shall have;

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial u'}\frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = \frac{v}{a^2}\frac{\partial u'}{\partial y'} \qquad 3.2.8$$

Therefore,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial y'}\left[\frac{v}{a^2}\frac{\partial u'}{\partial y'}\right]\frac{\partial y'}{\partial y} \qquad 3.2.9$$

$$= \frac{\partial}{\partial y'}\left[\frac{v}{a^2}\frac{\partial u'}{\partial y'}\right]\frac{1}{a} = \frac{v}{a^3}\frac{\partial^2 u'}{\partial y'^2}$$

Similarly,

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial p'}\frac{\partial p'}{\partial x'}\frac{\partial x'}{\partial x} = \frac{\rho v^2}{a^3}\frac{\partial p'}{\partial x'}$$

And

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial p'}\frac{\partial p'}{\partial y'}\frac{\partial y'}{\partial y} = \frac{\rho v^2}{a^2}\cdot\frac{\partial p'}{\partial y'}\cdot\frac{1}{a} = \frac{\rho v^2}{a^3}\cdot\frac{\partial p'}{\partial y'} \qquad 3.3.0$$

Hence, substitution of equation 3.3.0 into 3.2.6 leads to

$$\frac{\partial p'}{\partial y'} = 0$$

Whereas combination of equation 3.2.6, 3.2.8and 3.2.9yeilds

$$0 = -\frac{1}{\rho}\frac{\rho v^2}{a^3}\frac{\partial p'}{\partial x'} + v\frac{v}{a^3}\frac{\partial u'}{\partial y'^2} - \frac{\sigma B^2}{\rho}\frac{v}{a}u'\sin^2\alpha \qquad 3.3.1$$

For convenience, we shall drop the primes in equation 3.3.1 and write equation simply as:

$$0 = \frac{v}{a^3}\left[-\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2 a^2}{\rho v}u\sin^2\alpha\right]$$

Or

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2 a^2}{\rho v}u\sin^2\alpha \qquad 3.3.2$$

We may write the above equation as:

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - M^2\sin^2\alpha u \qquad 3.3.3$$

Where

$$M\sin\alpha = Ba\sqrt{\frac{\sigma}{\mu}}\sin\alpha = M*$$

Where $\mu = \rho v$ and $M*$ is known as the Hartmann number. It is directly proportional to the where magnetic field B.

We can differentiate equation3.3.3 with respect to x to obtain

$$0 = \frac{\partial^2 p}{\partial x^2} \qquad 3.3.4$$

We can therefore see from equation 3.3.4 that

$$\frac{dp}{dx} = \text{constant} \qquad 3.3.5$$

Consequently, we can take the ordinary derivative of the equation of motion instead of partial derivative.

In the present case, the constant of equation 3.3.5 is zero because we are considering a situation where the pressure gradient is zero. Therefore, we have

$$\frac{d^2u}{dy^2} - M^2 \sin^2 \alpha u = 0 \qquad 3.3.6$$

3.4 SOLUTION OF THE EQUATION

We now set out to solve this equation 3.3.6 by Laplace method.

In so doing we shall set out to transform the equation into purely an algebraic equation in terms of the Laplace transform of the required solution. Once we have solved the algebraic equation for this Laplace transform, the general solution to the original ODE will be obtained by performing an inverse Laplace transform. Then using the given boundary conditions, we shall find the solution.

The Laplace transform of the first derivative of $f(t)$ is given by;

$$L\left\{\frac{du}{dt}\right\}=\int_0^\infty e^{-st}\frac{du}{dt}dt$$

Integrating by parts we have;

$$\lim_{p\to\infty}\int_0^p e^{-st}\frac{du}{dt}dt$$

$$=s\int_0^p e^{-st}u(x,t)dt-u(x,0)$$

$$=su(x,s)-u(x,0)$$

$$=su-u(x,0)$$

Where $u=u(x,s)=l\{u(x,t)\}$

To find

$$L\left\{\frac{d^2u}{dt^2}\right\},$$

We let $v=\frac{du}{dt}$

Such that;

$$L\left\{\frac{d^2u}{dt^2}\right\} \text{ becomes } L\left\{\frac{dv}{dt}\right\}$$

Thus;

$$L\left\{\frac{d^2u}{dt^2}\right\}=L\left\{\frac{dv}{dt}\right\}$$

$$=sL\{v\}-v(x,0)$$

$$=s\left[sL\{u\}-u(x,0)\right]-u'(x,0)$$

$$=s^2u-su(x,0)-u'(x,0)$$

While Laplace transform of u is $\overline{u}$

Therefore equation 3.3.6 now becomes;

$$s^2\overline{u}-su(0)-u'(0)-M^2\overline{u}\sin^2\alpha=0$$

but since we do not know the values of $u(0)$ *and* $u'(0)$ we let;

$$u(0) = c_1 \quad and \quad u'(0) = c_2$$

And equation (4) now becomes;

$$s^2\overline{u} - c_1 s - c_2 - M^2\overline{u}\sin^2\alpha = 0 \qquad 3.3.7$$

Simplifying yields;

$$s^2\overline{u} - M^2\overline{u}\sin^2\alpha = c_1 s + c_2 \qquad 3.3.8$$

or

$$\overline{u}(s^2 - M^2\sin\alpha) = c_1 s + c_2$$

Therefore;

$$\overline{u} = \frac{c_1 s + c_2}{s^2 - M^2\sin\alpha}$$

$$\overline{u} = \frac{c_1 s}{s^2 - M^2\sin\alpha} + \frac{c_2}{s^2 - M^2\sin\alpha} \qquad 3.3.9$$

We now get the inverse Laplace transform of the above equation 3.3.9

$$L^{-1}\{\bar{u}\} = u$$

$$L^{-1}\left\{\frac{c_1 s}{s^2 - M^2\sin^2\alpha}\right\} = c_1 L^{-1}\left\{\frac{s}{s^2 - M^2\sin^2\alpha}\right\} = c_1 \cosh M\sin\alpha(y)$$

$$L^{-1}\left\{\frac{c_2}{s^2 - M^2\sin^2\alpha}\right\} = \frac{c_2}{M\sin\alpha} L^{-1}\left\{\frac{M\sin\alpha}{s^2 - M^2\sin^2\alpha}\right\} = \frac{c_2}{M\sin\alpha}\sinh M\sin\alpha(y)$$

Therefore;

$$u = c_1\cosh M\sin\alpha(y) + \frac{c_2}{M\sin\alpha}\sinh M\sin\alpha(y) \qquad 3.4.0$$

To insert the boundary conditions we must note that the upper plate is moving with a constant velocity and the lower plate is held stationary under the influence of inclined magnetic field and therefore ; when

$$\begin{aligned} y &= -1; & u &= 0 \\ y &= 1 & u &= U \end{aligned} \qquad 3.4.1$$

The above equation now simplifies to;

$$0 = c_1\cosh M\sin\alpha - \frac{c_2}{M\sin\alpha}\sinh M\sin\alpha \qquad 3.4.2$$

And

$$U = c_1 \cosh M \sin\alpha + \frac{c_2}{M \sin\alpha} \sinh M \sin\alpha \qquad 3.4.3$$

Solving the two equations 3.4.2 and 3.4.3 simultaneously we shall have;

$$c_2 = \frac{U}{2} \frac{M \sin\alpha}{\sinh M \sin\alpha}$$

And

$$c_1 = \frac{U}{2 \cosh M \sin\alpha}$$

The above equation 3.4.0 now becomes;

$$u = \frac{U}{2 \cosh M \sin\alpha} \left(\cosh M \sin\alpha (y) \right) + \frac{U}{2 \sinh M \sin\alpha} \left(\sinh M \sin\alpha (y) \right) \qquad 3.4.4$$

And to simplify it further we shall use the following identities;

$$\sinh\theta \cosh\alpha + \sinh\alpha \cosh\theta = \sinh(\theta + \alpha)$$

And

$$2 \sinh\theta \cosh\theta = \sinh 2\theta$$

Therefore the equation becomes;

$$\frac{U\sinh\left(M\sin\alpha\right)\cosh\left(M\sin\alpha\left(y\right)\right)+\cosh\left(M\sin\alpha\right)\sinh\alpha\left(M\sin\left(y\right)\right)}{2\cosh\left(M\sin\alpha\right)\sinh\left(M\sin\alpha\right)} \qquad 3.4.5$$

Simplifying further we get;

$$u=\frac{U\sinh\left[\left(1+y\right)\left(M\sin\alpha\right)\right]}{\sinh\left(2M\sin\alpha\right)}$$

Therefore;

$$\frac{u}{U}=\frac{\sinh\left[\left(1+y\right)\left(M\sin\alpha\right)\right]}{\sinh\left(2M\sin\alpha\right)} \qquad 3.4.6$$

As expected.

3.5 THE RESULTING FIGURES

The resulting figures from equation 3.4.6 above are shown below:

FIGURE 1: M=1.0

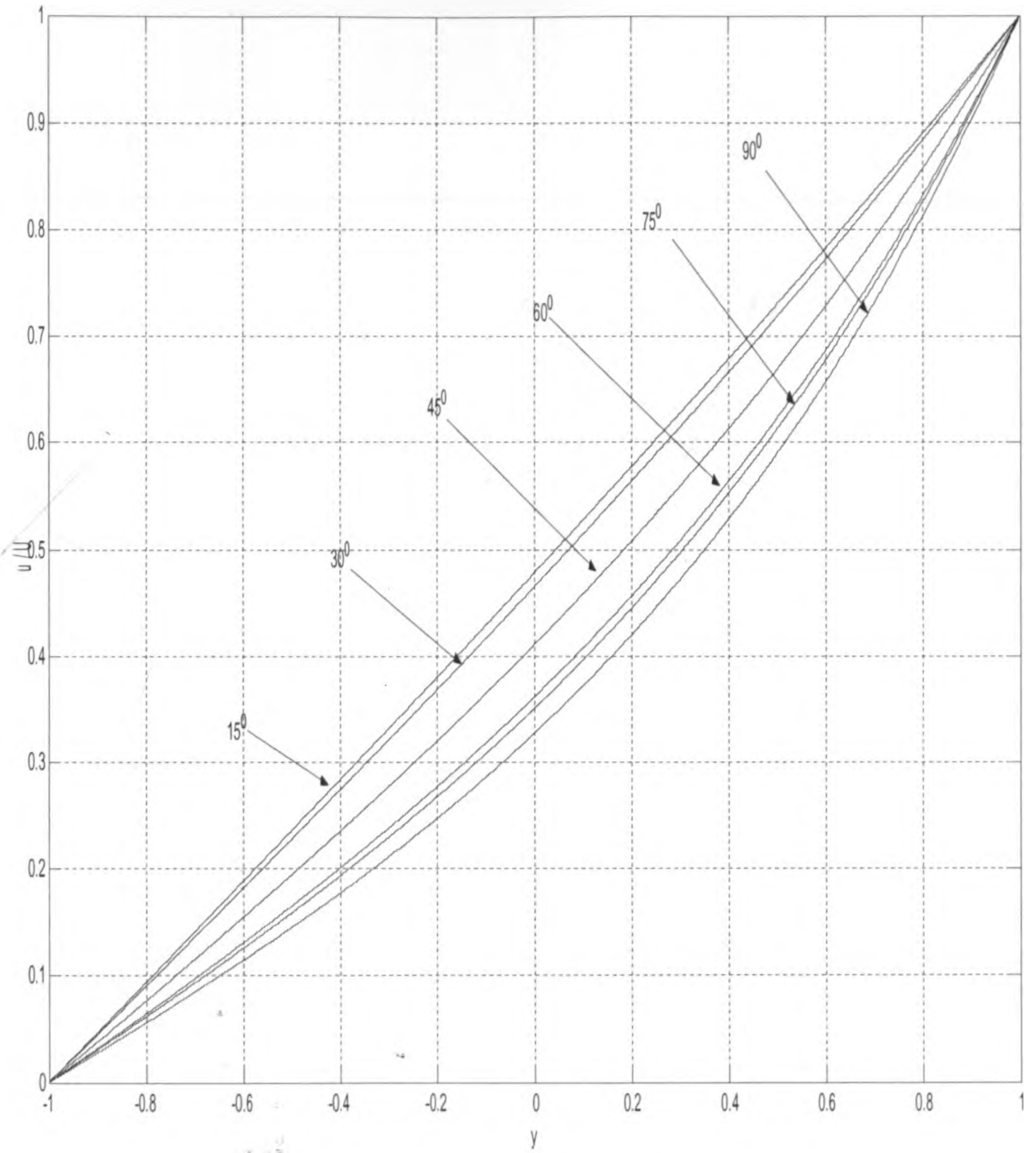
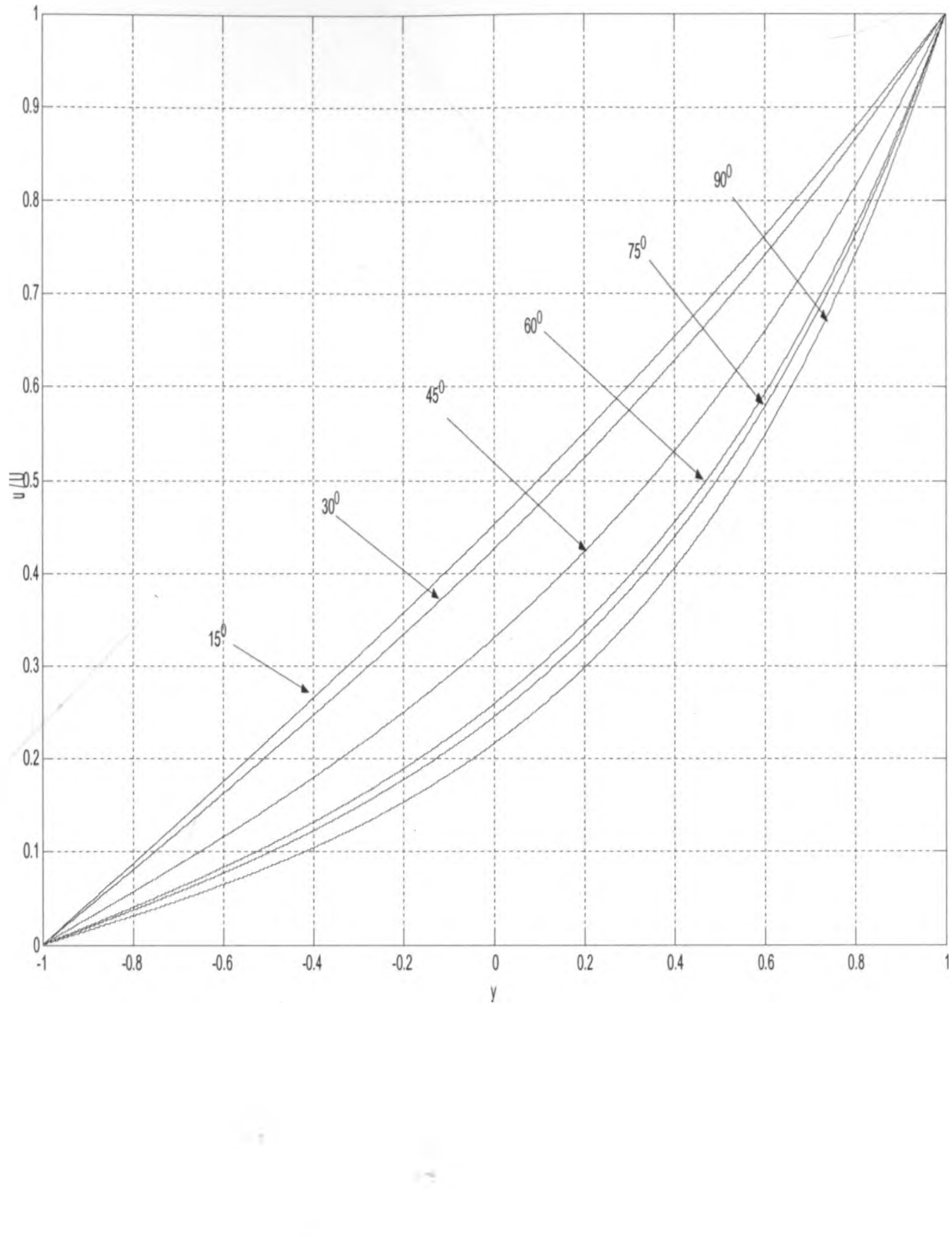


FIGURE 2: M=1.5



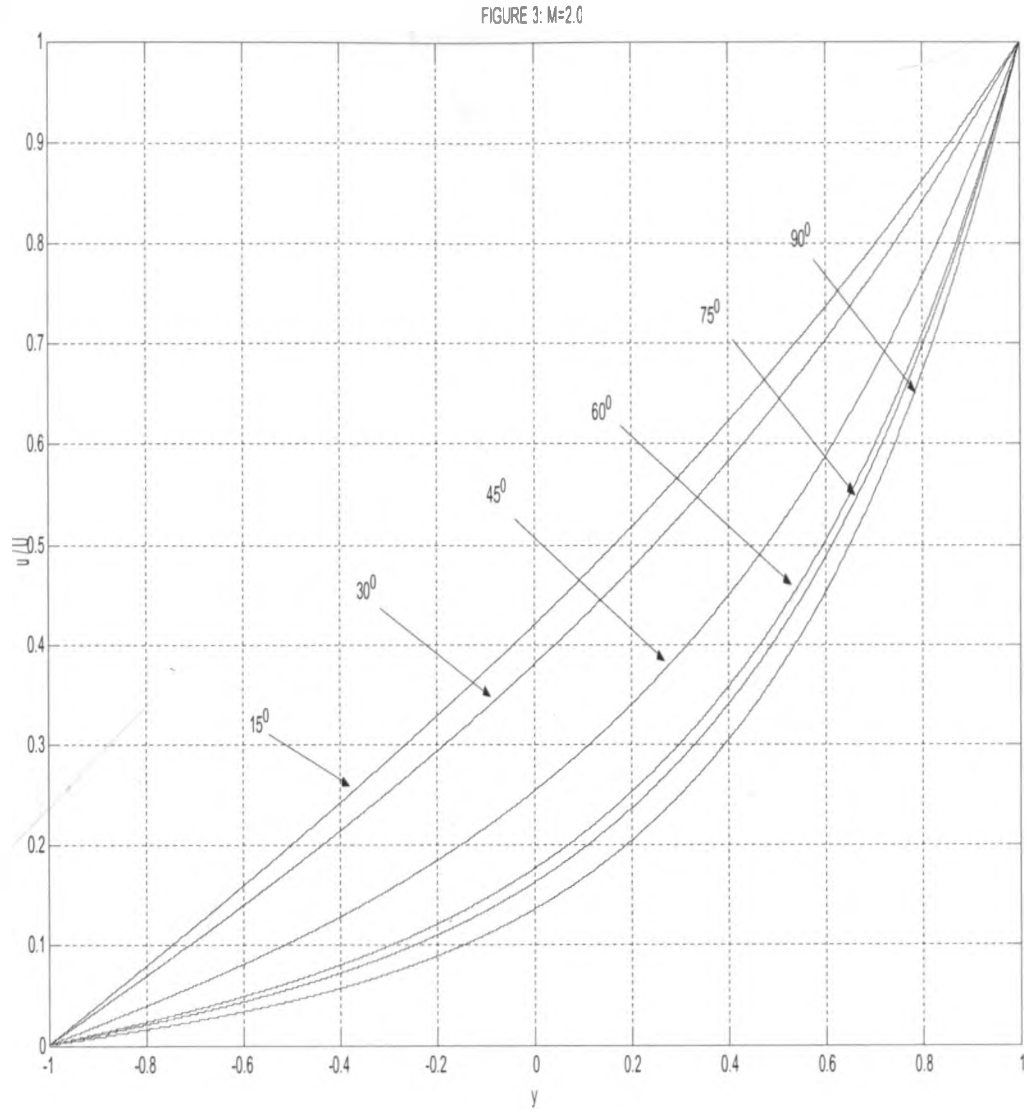


FIGURE 3: M=2.0
u / U
y
15⁰
30⁰
45⁰
60⁰
75⁰
90⁰
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

3.6 DISCUSSION OF RESULTS AND CONCLUSION

The problem has been solved by the method of Laplace Transforms. Analytic expression for velocity of fluid particle has been obtained. Figures 1, 2, and 3 are drawn for M=1, M=1.5, M=2 at the inclinations of $15^0, 30^0, 45^0, 60^0, 75$ *and* 90^0. These results are corresponding with Figures 1, 2, and 3 drawn for M=1, M=1.5, M=2 at the inclinations of $30^0, 45^0, 60^0,$ *and* 90^0 on the paper Singh, [2007].

It is evident from the figures that velocity profiles decrease as the strength of magnetic field is increased. It is also clear that with the increase of inclination of magnetic field, there is a decrease in the velocity profile. The velocity at 90^0 gives us the case of fluid flow under transverse magnetic field as a particular case of this problem. The results obtained here can be applied to the designs and operations of Magnetohyrodynamic generator, magnetohydrodynamic pump, electromagnetic flow meter and crude oil purification.

APPENDIX

KEY VECTOR IDENTITIES.

The following vector identities have been adapted from R. Dendy (1993),

Let $\vec{A}, \vec{B}, \vec{C}$ *and* $\vec{D}$ be vectors. The following identities hold.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$
$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$
$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right)$$
$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$$
$$\left(\vec{A} \times \vec{B}\right) \cdot \left(\vec{C} \times \vec{D}\right) = \left(\vec{A} \cdot \vec{C}\right)\left(\vec{B} \cdot \vec{D}\right) - \left(\vec{A} \cdot \vec{D}\right)\left(\vec{B} \cdot \vec{C}\right)$$

KEY RESULTS FROM VECTOR CALCULUS

Operator: $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

Gradient: $\nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

For a vector field: $\vec{A}\left(x, y, z\right)$,

Divergence: $\nabla \cdot \vec{A} = \frac{\partial \vec{A}_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial \vec{A}_z}{\partial z}$

Curl: $\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{A}_x & \vec{A}_y & \vec{A}_z \end{vmatrix}$

Where:

$$\vec{A} = \hat{I}\vec{A}_x + \hat{j}\vec{A}_y + \hat{k}\vec{A}_z$$
$$\nabla \cdot \left(\nabla \times \vec{A}\right) = 0$$
$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

For two vector fields: $\vec{A}(x, y, z)$ *and* $\vec{B}(x, y, z)$

$$\nabla \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\nabla \times \vec{A}\right) - \vec{A} \cdot \left(\nabla \times \vec{B}\right)$$
$$\nabla \times \left(\vec{A} \times \vec{B}\right) = \vec{A}\left(\nabla \cdot \vec{B}\right) - \vec{B}\left(\nabla \cdot \vec{A}\right) + \left(\vec{B} \cdot \nabla\right)\vec{A} - \left(\vec{A} \cdot \nabla\right)\vec{B}$$

Table of Laplace transforms

$f(t)$	$\bar{f}(s)$	s_0
c	$\frac{c}{s}$	0
ct^n	$\frac{c_n!}{s^{n+1}}$	0
$\sin bt$	$\frac{b}{\left(s^2+b^2\right)}$	0
$\cos bt$	$\frac{s}{\left(s^2+b^2\right)}$	0
e^{at}	$\frac{1}{(s-a)}$	a
$\sinh at$	$\frac{a}{\left(s^2-a^2\right)}$	$\lvert a\rvert$
$\cosh at$	$\frac{s}{\left(s^2-a^2\right)}$	$\lvert a\rvert$
$e^{at}\sin bt$	$\frac{b}{\left[(s-a)^2+b^2\right]}$	a
$e^{at}\cos bt$	$\frac{(s-a)}{\left[(s-a)^2+b^2\right]}$	a
$t^{1/2}$	$\frac{1}{2}\left(\frac{\pi}{s^3}\right)^{\frac{1}{2}}$	0
$t^{-\frac{1}{2}}$	$\left(\frac{\pi}{s}\right)^{\frac{1}{2}}$	0
$\delta(t-t_0)$	e^{-st_0}	0
$H(t-t_0)=\begin{cases}1 \text{ for } t\geq t_0\\ 0 \text{ for } t<t_0\end{cases}$	$\frac{e^{-st_0}}{s}$	0

The transformation are valid for $s > s_0$

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