THE GENERAL SOLUTION OF THE GEOPOTENTIAL

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ABSTRACT

Physical Geodesy aims at describing points on the surface of the earth and the requirement that such description be invariant makes it necessary for the physical vertical in one place be related to the corresponding vertical in another point on the surface and in space. In this way, the study of the geopotential, whose gradient defines the direction of the vertical, assumes great importance. All operationally useful parameters of the earth arise in a natural way.

In the geodetic sciences, one deals with the energy of a system containing no scientific observer and it is necessary that all observers confined to the surface of the earth and in space have the same experience of the system. It is clear that the parameters under study are measurements made on the surface of the earth and in space.

The methods employed for the study of the geopotential vary quite a lot but it is required that the conclusions made about the natural phenomenona be invariant. Currently, the majority of solutions give the geopotential in terms of series of Legendre polynomials and functions of the various kinds according to the degree of approximation. This is because, under certain conditions, these functions constitute a solution to Laplaces differential equation which is satisfied by the potential of the earth outside the gravitating masses. It is not usually clarified whether solutions given in terms these functions in special coordinate systems correspond to a potential as given by a body such as the physical earth. Instead several researchers point out that the exhibition of the potential of the earth in spherical harmonic series requires a coordinate surface to contain all the gravitating masses [Cook, 1967], a condition which is not satisfied by the physical surface of the earth. Apart from lack of proper treatment of the

problems of convergence, except recently by [Morrison, 1969, and Krarup, 1969], many artificial and probably unnecessary problems have been introduced by the desire to construct these co-ordinate surfaces, the most serious of these being the requirement to quantify the density of the matter involved in this regularisation.

In this solution, these problems will be offset by abandoning the dependence of the solution on the seperability conditions of Laplace's equation in terms of Legendre functions. Attention will be paid to the parameters measured and obtainable on the surface of the earth. In geodesy, one deals with points, lines, curves, angles, surfaces, etc., and these are primitive objects whose proper study and inter-relationship will yield information about the object (the earth) which contains them and the nature of the geopotential. In this way topological ideas enter the mathematical formulation. These are introduced and sketched out at the beginning of chapter one. Topology underlies much of the logic in analysing all geodetic systems. There is a common belief expressed by several geodesists [Hotine, 1969] that the mathematical foundations of geodesy are in "a state of crisis", to use the words of [Krarup, 1969]. The solution of the geopotential presented in this thesis is therefore based on rigorous mathematical exposition although detailed proofs are not presented, references being usually quoted (shown in square brackets by author's name and year).

Vector spaces have recently been used in their proper setting in mathematical geodesy as Hilbert spaces and Banach spaces spanned by Legendre and other function bases. A casual mention of these objects is made in (1.2). The notion of a manifold as a basic object of study in geodetic analysis is introduced in (1.3) with a description of local coordinates which are admitted by manifolds. An idea which is gathering strength in several geodetic analyses is the use of local coordinate neighbourhoods with a final transformation into a global co-ordinate system. This is obvious since all geodetic operations respond naturally to these geodetic local coordinate systems. Several approaches have been made to define suitable local systems but most of them depend on the deflexions of the vertical [Dufour, 1967]. As this solution allows only the natural parameters and therefore contains no spheroids or other approximations to the earth's figure, it is necessary to use parameters defined naturally on the surface of the earth. In this way the local system is chosen with respect to the geopotential. The geopotential is a function of position only in the definition but in subsequent operations only the magnitudes are required since one is dealing with equipotential surfaces. These level surfaces define loops on the surface of the earth which are integral curves of the geopotential. One of the most important result of this analysis is that an integral curve can be parametrised by specifying the parameterisation in one point only. This facilitates the abandonment of analytic solution and attention is paid to discrete solutions which are in keeping with the practical operations. It happens that these particular integral curves also have minimum energy (zero, since energy is positive in our case) and this confirms this result. The local system is therefore selected within the level surfaces with one vector normal to this surface (in the direction of the physical vertical). The transformation from local neighbourhoods to a global coordinate system is probably straight forward and can be found in much of the geodetic literature [Hotine, 1969].

An attempt is made to calculate vertical angles for the homogenous computation of geodetic triangulation. It is assumed that the area is gravimetrically well surveyed with enough orthometric height information from geodetic levelling. The assumptions made in this calculations have been shown by numerical tests to be innocuous for the distances of the order of 200 km. The measured distance (perhaps with some EDM equipment) is assumed geodesic for the prevailing atmospheric conditions of the geodetic operation. It is probably impossible to assess the nature of precision in a network without reference to some standard atmosphere (which yields relative precision). It is not possible to compare calculated vertical angles in this solution with classical analysis because the data required would not be compartible save in a very hypothetical setting. The construction of models is outside the scope of this work.

In chapter two some mathematical objects which are much used in geodesy are defined in a form tractable for use in discrete solutions. Tensor are therefore characterised in (2.1) without necessarily depending on definitions using indices and transformation laws.

To offset the necessity of regularising the topography, one studies the system of level surfaces as one progresses from sea level to cover the highest point on the topography. By functional identification, the geopotential is made to mirror the irregularities of the topography (as defined in (1.5)). Attention is therefore focused on the system of critical points of the geopotential as a function with domain on the surface of the earth. By a direct plausibility approach aimed at classifying these critical points, it turns out that the geopotential possesses topological invariants hitherto unknown in geodetic circles. These fix once for all the topological characteristics of the geopotential. It is possible to construct vector space with these invariants as dimensions. It is therefore clear that transformation within these spaces will have invariant ranks.

In (2.3) curvature properties of the earth are formulated in relation to loops formed by the level surfaces. It happens that the integral of absolute curvature over the surface of the earth (2-dimensional) is equal to the Euler-Poincare characteristic of the geopotential which is one of the invariants found in (2.2). Curvature being an intrinsic property of a surface, the geoid can be deformed continuously by an invariant transformation until the rugged surface is recovered. It appears that this invariant function is the absolute curvature of the surface of the earth.

The notion of the connexion in the fibre bundle is presented in (2.4) for the sake of geometrising the geopotential. This implies the construction of spaces which mirror the nature of the level surfaces of the geopotential and the vertical field. In this way it is possible to construct a space which can be decomposed into a linear and a nonlinear space. These spaces are possibly infinite dimensional and the linear space becomes a Banach space.

One of the growing techniques of study of the gravitational field of the earth entitles the study of the dynamics of artificial satellites. In classical solutions, it is the Keplarian orbit which is used as a norm and then subsequently perturbed in order to study the irregularities of the geopotential. In our approach (3.1), this point of view is unnecessary since the model problem in celestial mechanics can be realised as the study of flows generated by vector fields on a torus. The symplectic (or hamiltonian) vector fields preserve a torus and the orbit can be studied as the set of flows generated by one parameter subgroups of a group acting through a fixed point (the centre of gravity of the earth). In this same system, the total energy of a satellite under the geopotential is conserved. To any conserved quantity, there corresponds an invariant transformation. This transformation is calculated as an automorphism thus mapping one point into another automatically. The torus arises (as a multi-dimensional torus) out of the covering of the surface of the earth by a torus through spherical modifications in the critical submanifolds. This type of analysis using, Lie groups, suits quite well the discrete approach since the critical manifolds are countable. Any countable set can be paired with the integers and this therefore permits the use of discrete subgroups without recourse to magnitudes of the geopotential (the Morse numbers are integers).

It is envisaged that the calculation of the invariant transformation from geodetic data will facilitate a detailed study of external forces acting on a satellite which can be seperated from the geopotential (after considering other outside gravitational attractions) by comparison of the calculated and measured orbital parameters. Such a study would probably involve sophisticated statistical ideas (stochastic systems) and is therefore a study in itself beyond the scope of this work. The study of group representations, usually used to penetrate flows, is presented in (3.2).

A word about the nature of this solution is probably in order. The study is theoretical in its setting and aims at discovering the nature of parameters and suitable methods for use in the discrete solution of the geopotential. It is general not in the sense of a survey but in that it contains almost all the parameters which arise naturally within the energy of the delimited system of gravitating masses of the earth. Perhaps it could not be expected that final answers would be expected of a solution to a problem of this magnitude. Within this penetrating mathematical formulation which were in the past considered abstract, it will probably be possible to solve many practical problems which are usually made more difficult by philosophically inadmissible assumption about the geodetic system.

Finally a list of references is given at the end. The list cannot be considered to be exhaustive since most of the geodetic literature referred to in this study would seem irrelevant within the setting of this study. Many of the mathematical textbooks referred to were of background value and cannot all be listed at the end. Some papers, however, not specifically referred to in the text, are listed for their outstanding value in pinpointing the loopholes in the classical solutions.