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On λ-Commuting Operators

J.M. Khalagai and M. Kavila
School of Mathematics, University of Nairobi, P.O. Box 30197 - 00100, Nairobi
Email: khalagai@uonbi.ac.keormutiekavila@gmail.com

Abstract: *Two bounded linear operators A and B on a complex Hilbert space are said to λ-commute for $\lambda \in \mathbf{C}$ provided that: AB = λBA. In this paper we look for some properties satisfied by the operators A and B so that λ = 1. It is shown among other results that if one of the operators raised to some power is normal and 0 does not belong to the interior of the numerical range of the other operator then: λ = 1*

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1. Notation and Terminology

Given an operator A we shall denote the spectrum, the approximate point spectrum, the point spectrum and the closure of the numerical range by: $\sigma(A)$, $\sigma_{ap}(A)$, $\sigma_p(A)$ and $\overline{W(A)}$ respectively. For $A, B \in B(H)$ the commutator of A and B will be denoted by $[A, B]$. Thus $[A, B] = AB - BA$. The commutant of A will be denoted by $\{A\}'$.

Thus $\{A\}' = \{X \in B(H): [A, X] = 0\}$. Re$A$ and ImA will denote real and imaginary parts of A. The operator X is said to intertwine operators A and B if $AX = XB$.

The operator A will be said to be:

Dominant if to each $\lambda \in \mathbf{C}$ there corresponds a number $M_\lambda \geq 1$ such that $\|(A-\lambda)^* x\| \leq M_\lambda \|(A-\lambda)x\|$ for all $x \in H$.

Hyponormal if $A^*A \geq AA^*$.

Normal if $A^*A = AA^*$.

Self-adjoint if $A = A^*$.

We have the following inclusions of classes of operators:

{Self-adjoint} ⊆ {Normal} ⊆ {Hyponormal} ⊆ {Dominant}

2. Introduction

Let $B(H)$ denote the Banach algebra of bounded linear operators on a complex Hilbert space H. For any $T \in B(H)$ the numerical range of T denoted by $W(T)$ is the image of the unit sphere of H under the quadratic form $x \to \langle Tx, x \rangle$ associated with the operator.

More precisely, $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$. Thus the numerical range of an operator, like the spectrum, is a subset of the complex plane whose topological properties carry some information about the operator. We now note that λ-commuting operators have been considered by a number of authors. Among them are Brook *et al*, (2002) who proved the following results:

Theorem A: Let $A, B \in B(H)$ such that $AB \neq 0$ and $AB = \lambda BA$ for $\lambda \in \mathbf{C}$. Then:

(ii) Similarly given $AB = \lambda BA$ we have:

$A^n B = \lambda^n BA^n$ for $n \in J^+$

i.e., $B(\lambda^n A^n) = A^n B$

But $\lambda^n A^n$ and A^n are commuting normal operators. Since $0 \notin W(B)$, we have by theorem D again that:

$$A^n = \lambda^n A^n$$

Since $A^n \neq 0$ then $\lambda^n = 1$

i.e., $[A^n, B] = 0$

Corollary 1: Given $AB = \lambda BA$ we have that $[A, B] = 0$ under any one of the following conditions:

(i) A is normal and $0 \notin W(B)$

(ii) B is normal and $0 \notin W(A)$

Proof: We put $n = 1$ in the proof of the theorem above.

Remarks: (i) We note that for the operator equation $AB = \lambda BA$ the condition $[A, B] = 0$ trivially implies that $\lambda = 1$.

(ii) We also note that the condition that A or B is positive is more stringent than a mere requirement that $0 \notin W(A)$ or $0 \notin W(B)$. More precisely the following corollary is an improvement of theorem A above.

Corollary 2: Let A and B be self-adjoint operators such that $AB = \lambda BA$. Then $[A, B] = 0$ under any one of the following conditions:

(i) $\sigma(A) \cap \sigma(-A) = \varnothing$

(ii) $0 \notin W(A)$

(iii) $\sigma(\text{Re}A) \cap \sigma(-\text{Im}A) = \varnothing$

(iv) $\sigma(B) \cap \sigma(-B) = \varnothing$

(v) $0 \notin W(B)$

(vi) $\sigma(\text{Re}B) \cap \sigma(-\text{Im}B) = \varnothing$

Proof: Given $AB = \lambda BA$ we have:

$$\begin{aligned} A^2B &= A\lambda BA \\ &= \lambda ABA \\ &= \lambda.\lambda BAA \\ &= \lambda^2 BA^2 \end{aligned}$$

Now by **theorem A** above we have that $\lambda^2 = 1$. Thus $A^2B = BA^2$ or $[B, A^2] = 0$. We also have:

$$\begin{aligned} AB^2 &= \lambda BAB \\ &= \lambda B\lambda BA \\ &= \lambda^2 B^2 A \end{aligned}$$

By theorem A again $\lambda^2 = 1$. Thus $AB^2 = B^2A$ or $[A, B^2] = 0$.

Now in view of **theorem E** above each of the conditions (i) to (vi) implies $[A, B] = 0$ and consequently $\lambda = 1$.

Theorem 2: Let $A, B \in B(H)$ be such that $AB = \lambda BA$. Then we have:

(i) A is self-adjoint implies $B^*B \in \{A\}'$ and $BB^* \in \{A\}'$

(ii) B is self-adjoint implies $A^*A \in \{B\}'$ and $AA^* \in \{B\}'$

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