SOME EXAMPLES AND PROBLEMS IN MARKOV CHAINS

By

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DECLARATION BY CANDIDATE

This project is my original work and it has not been presented for a degree in any other University.

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DECLARATION BY SUPERVISOR

This project has been submitted for examination with my approval as University Supervisor.

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DEDICATION

To the future generation of our beloved country, Kenya, in whose hands the better future of our country lies.

ABSTRACT

This project is an equivalent of one course in masters programme in Mathematics department.

In chapters I and II some theory on markov chains is considered. Specifically, chapter I is about definition and properties of markov chains.

In chapter II states of markov chains are classified into different categories. Here irreducible, ergodic and absorbing markov chains are studied.

In chapter III examples of markov chains are studied. In this chapter the following chains are considered:

(i) Random walk with absorbing barriers

(ii) Random walk with one reflecting barrier

(iii) Random walk with two reflecting barriers.

(iv) Cyclical random walk

(v) Two state random walk.

(vi) The Ehrenfest chain.

In the study of these examples of markov chains the theory of chapters I and II is applied.

In the fourth chapter some problems relevant to the topics covered in the other chapters are solved.

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CHAPTER ONE

1.1. Definition of Markov Chain

A markov chain, like a branching process is an area of stochastic processes where conditional probability is substantially used.

Let E_j and E_k be two events. Using conditional probability, the joint probability is given by

$$P(E_j, E_k) = P(E_k|E_j)\ P(E_j)$$

$$= P(E_j).\ P(E_k|E_j$$

$$= a_j P_{jk} \qquad (1.1)$$

where

$$a_j = P(E_j),$$

and

$$P_{jk} = P(E_k|E_j)$$

For three events E_j, E_k and E_r,

$$P(E_j, E_k, E_r) = P(E_k, E_r|E_j).P(E_j)$$

$$= P(E_r|E_k)\ P(E_k|E_j).P(E_j)$$

$$= P(E_j)\ P(E_k|E_j)\ P(E_r|E_k)$$

$$= a_j\ P_{jk}\ P_{kr} \qquad (1.2)$$

Where

a_j and P_{ij} are defined as above

and $P_{kr} = P(E_r|E_k)$

Extending the idea to four events, we have

$$P(E_j, E_k, E_r, E_s) = P(E_k, E_r, E_s|E_j)\ P(E_j)$$

$$= P(E_s|E_r)\ P(E_r|E_k\ P(E_k|E_j)\ P(E_j)$$

$$= P(E_j)\ P(E_k|E_j)\ P(E_r|E_k)\ P(E_s|E_r)$$

$$= a_j\ p_{jk}\ P_{kr}\ P_{rs} \qquad (1.3)$$

etc.

Using this notation of conditional probability we can now define a markov chain as follows.

Def:

A sequence of trials with possible outcomes $E_{jo}, E_{j1}, E_{j2}, \ldots, E_{jn-1}\ E_{jn}$ is called a markov chain if

$$P(E_{jo}, E_{j1}, \ldots, E_{jn-1}, E_{jn}) = a_{jo}\ P_{joj_1} \cdot P_{j1j2} \cdots\cdots P_{jn-1}\ P_{jn} \qquad (1.4)$$

Where

$$a_{jo} = P(E_{jo}),$$

is the probability at the initial or zeroth trial, and P_{jk} is the fixed conditional probability of E_k given that E_j has occurred at the preceding trial.

<u>Terminology:</u> The event E_j shall be called state E_j or simply state j. The conditional probability P_{jk} which is the probability of E_k given that E_j has occurred will be called the transition probability the transition probability from state E_j to E_k. Thus

$$P_{jk} = P(E_k \mid E_j)$$

$$= P(E_j \rightarrow E_k)$$

The transitional probabilities P_{jk} can be arranged in a matrix form as follows:

$$\underline{P} = \begin{array}{c} \\ E_o \\ E_1 \\ E_2 \\ \vdots \end{array} \begin{array}{cccc} E_o & E_1 & E_2 \cdots\cdots \\ \left[\begin{array}{ccc} p_{oo} & p_{o1} & p_{o2} \cdots\cdots\cdots \\ p_{1o} & p_{11} & p_{12} \cdots\cdots\cdots \\ p_{2o} & p_{12} & p_{22} \cdots\cdots \cdots \\ \vdots & \vdots & \vdots \end{array}\right. \end{array} \qquad (1.5)$$

Or

$$\underset{\sim}{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 \ldots\ldots \end{array} \\ \begin{bmatrix} p_{11} & p_{12} & p_{13} \ldots\ldots \\ p_{21} & p_{22} & p_{23} \ldots\ldots \\ p_{31} & p_{32} & p_{33} \ldots\ldots \\ \vdots & \vdots & \vdots \end{bmatrix} \end{array}$$

We should note that

(1) A ~~finite~~ transition probability matrix can be finite or infinite

(2) $p_{jk} \geq o$

and

(3). $\sum_k p_{jk} = 1$ (1.6)

is called a stochastic matrix. In case each column also adds up to unity then we have a double stochastic matrix.

So any stochastic matrix $\{P_{jk}\}$ with initial distribution $\{a_n\}$ completely defines a markov chain.

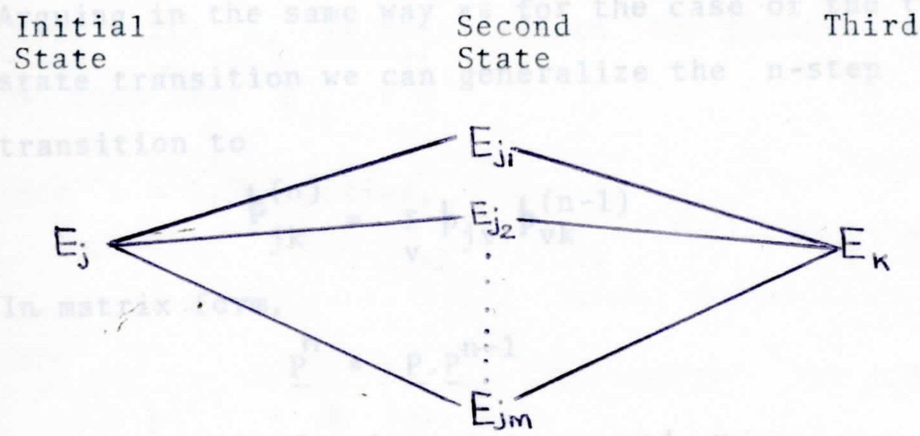
1.2. HIGHER ORDER TRANSTION PROBABILITIES

The probability of a process passing from E_j to E_k in exactly n steps is called an n-step transition probability and is denoted by $p_{jk}^{(n)}$.

In matrix form we have

$$\underline{P}^n = p_{jk}^{(n)} \qquad (1.7)$$

Let us consider a two-step transition probability $p_{jk}^{(2)}$. This can occur via different paths as shown in the diagram below:



The diagram shows that there are m mutually exclusive paths, namely

$$E_j \to E_{j1} \to E_k \, , \; E_j \to E_{j2} \longrightarrow E_k , \ldots \; E_j \to E_{jm} \to E_k$$

The corresponding conditional probabilities are :

$P(E_j \to E_v \to E_k) = P_{jv} \, P_{vk}$

for

$V = j_1, j_2, j_3, \ldots\ldots j_m .$

So the probability of moving from E_j to E_k in two steps is

$$p_{jk}^{(2)} = \sum_{v=1}^{m} p_{jv}\, p_{vk} \qquad (1.7a)$$

But by definition of a matrix multiplication the sum given above is the (j-k)th element of $\underline{P}^2$.

Thus

$$\underline{P}^2 = p_{jk}^{(2)} = \underline{P}.\underline{P} \qquad (1.7b)$$

Arguing in the same way as for the case of the two state transition we can generalize the n-step transition to

$$p_{jk}^{(n)} = \sum_{v} p_{jv}\, p_{vk}^{(n-1)} \qquad (1.8a)$$

In matrix form,

$$\underline{P}^n = \underline{P}.\underline{P}^{n-1} \qquad (1.8$$

More generally for integers m and n,

$$p_{jk}^{(m+n)} = \sum_{v} p_{jv}^{(m)} . p_{vk}^{(n)} \qquad (1.9$$

In matrix form

$$\underline{P}^{m+n} = \underline{P}^m . \underline{P}^n \qquad (1.$$

These equations are called chapman-kolmogorov equations. In order to have the chapman-kolmogorov true for all n positive we shall define $p_{jk}^{(n)}$ by

$$p_{jj}^{(o)} = 1 \qquad (1.5)$$

and (1.10)

$$p^{(o)}_{jk} = 0 \quad , \quad j \neq k$$

1.3 RECURRENT EVENTS

1.3.1 Definition of reachability

A transition from one state to another is not always possible depending upon the type of states. The state E_k is said to be reachable or accessible from state E_j if there exists some positive integer n such that $p^{(n)}_{jk} > 0$

For $n = 0$, we define,

$$p^{(o)}_{jj} = 1$$

and

$$p^{(o)}_{jk} = 0 \quad \text{for} \quad k \neq j.$$

If E_k is reachable from E_j and E_j is reachable from E_i, then there exist some positive integers m and n such that $p^{(m)}_{ij} > 0$ and $p^{(n)}_{jk} > 0$.

Using the chapman-kolmogorov equation:

$$p^{(m+n)}_{ik} = \sum_{\ell} p^{(m)}_{i\ell} \cdot p^{(n)}_{\ell k} \geq p^{(m)}_{ij} \cdot p^{(n)}_{jk} > 0$$

Thus E_k is reachable from E_i .

1.3.2. THE RELATIONSHIP BETWEEN $p_{jj}^{(n)}$ AND $f_{jj}^{(n)}$

We have defined $p_{jk}^{(n)}$ to be the probability that starting from E_j we enter E_k in n steps regardless of the number of entrances into E_k prior to n. Let us now define $f_{jk}^{(n)}$ to be the probability of entering E_k from E_j in n steps for the first time. We wish to find the relationship between the two types of probabilities, in particular between $p_{jj}^{(n)}$ and $f_{jj}^{(n)}$ as follows:

Let the first return to state E_j occur at the r^{th} step. The probability of this first return to E_j in r-steps is $f_{jj}^{(r)}$. In the remaining (n-r) steps, the state E_j will be reached once again with probability $p_{jj}^{(n-r)}$. Thus

$$p_{jj}^{(n)} = \sum_{r=o}^{n} f_{jj}^{(r)} p_{jj}^{(n-r)} \qquad n \geq 1 \tag{1.11}$$

In terms of generating functions, let

$$F(s) = \sum_{v=o}^{\infty} f_{jj}^{(v)} s^v$$

$$\text{and,} \quad P(s) = \sum_{\mu=o}^{\infty} p_{jj}^{(\mu)} s^\mu \tag{1.12}$$

Multiplying the relation (1.12) by s^n and summing up the result over n, we have :

$$\sum_{n=1}^{\infty} p_{jj}^{(n)} S^n = \sum_{n=1}^{\infty} \left(\sum_{r=o}^{n} f_{jj}^{(r)} p_{jj}^{(n-r)} \right) S^n \quad (1.13)$$

Since

$$p_{jj}^{(o)} = 1,$$

as given in (1.10) which is the probability of remaining in state E_j in no steps at all, the L.Hs. of (1.13) becomes

$$\sum_{n=1}^{\infty} p_{jj}^{(n)} S^n = P(s) - p_{jj}^{(o)}$$

$$= P(s) - 1 \quad (1.14)$$

For the RHs of (1.13) we recall the notion of convolution. That is, if

$$A(s) = \sum_{k=o}^{\infty} a_k S^k$$

and (1.15)

$$B(s) = \sum_{k=o}^{\infty} b_k S^k$$

then,

$$A(s).B(s) = \sum_{n=o}^{\infty} \left(\sum_{k=o} a_r b_{n-r} \right) S^n \quad (1.16)$$

Therefore

$$\sum_{n=o}^{\infty} \left(\sum_{r=o}^{n} f_{jj}^{(r)} p_{jj}^{(n-r)} \right) S^n = F(s) P(s) \quad (1.17)$$

Alternatively

$$\sum_{n=o}^{\infty} \sum_{r=o}^{n} f_{jj}^{(r)} p_{jj}^{(n-r)} S^n$$

$$= \sum_{n=o}^{\infty} \sum_{r=o}^{\infty} f_{jj}^{(r)} \; p_{jj}^{(n-r)} \; s^{n}$$

$$= \sum_{n=o}^{\infty} \sum_{r=o}^{\infty} f_{jj}^{(r)} \; p_{jj}^{(n-r)} \; s^{r} \; s^{n-r}$$

$$= \sum_{n=o}^{\infty} \left(\sum_{n=o}^{\infty} p_{jj}^{(n-r)} \; s^{(n-r)} \right) f^{(r)} \; s^{r}$$

$$= \sum_{r=o}^{\infty} \left(\sum_{n=r}^{\infty} p_{jj}^{(n-r)} \; s^{n-r} \right) f_{jj}^{(r)} \; s^{r}$$

$$= \sum_{r=o}^{\infty} P(s) \; f_{jj}^{(r)} \; s^{r}$$

$$= P(s) \sum_{r=o}^{\infty} f_{jj}^{(r)} \; s^{r}$$

$$= P(s) \, . \, F(s) \qquad (1.18)$$

So applying (1.14) and (1.18), (1.13) becomes

$$P(s) - 1 = F(s) P(s)$$

which implies

$$p(s) = \frac{1}{1 - F(s)} \qquad (1.19)$$

1.3.3. PERSISTENT AND TRANSIENT STATES

Let

$$f_{jk} = \Sigma f_{jk}^{(n)} \qquad (1.20)$$

which is the probability of ever passing through E_k from E_j.

If

$$f_{jk} = 1 ,$$

then $\{f_{jk}^{(n)}\}$ is a proper distribution called "the first passage distribution". In particular if

$$f_{jj} = 1 ,$$

then $f_{jj}^{(n)}$ is called the distribution of recurrence times. So the expectation E(n) is given by

$$\mu_j = \Sigma_n n f_{jj}^{(n)} \qquad (1.21)$$

called the mean recurrence time.

A state E_j is said to be persistent or recurrent if

$$f_{jj} = 1$$

That is, there's eventual return. Further if

$$\mu_j = \infty ,$$

then E_j is null. And if

$$\mu_j < \infty$$

then E_j is non-null. A state E_j is said to be transient or non-recurrent if

$$f_{jj} < 1.$$

We can define persistent and transient states in terms of $p_{jj}^{(n)}$ using the relationship (1.19).

That is

$$P(s) = \frac{1}{1-F(s)}$$

Where

$$P(s) = \sum_{n=o}^{\infty} p_{jj}^{(n)} s^n$$

and

$$F(s) = \sum_{n=o}^{\infty} f_{jj}^{(n)} s^n$$

Putting $s = 1$, we have

$$P(1) = \frac{1}{1-F(1)}$$

Where

$$P(1) = \sum_{n=o}^{\infty} p_{jj}^{(n)}$$

and (1.22)

$$F(1) = \sum_{n=o}^{\infty} f_{jj}^{(n)} \quad f_{jj}$$

Thus if E_j is persistent then,

$$P(1) = \frac{1}{1-1} = \infty$$

Since $f_{jj} = 1$

And if E_j is transient then

$$f_{jj} < 1$$

In this case,

$$p(1) = \frac{1}{1-f_{jj}^{(n)}} < \infty$$

So a state E_j is persistent if

$$\sum_{n=o}^{\infty} p_{jj}^{(n)} = \infty$$

and E_j is transient if

$$\sum_{n=o}^{\infty} p_{jj}^{(n)} < \infty$$

Let us now look at the asymptotic behaviour if $p_{jj}^{(n)}$.

From

$$P(s) = \frac{1}{1-F(s)}$$

We have,

$$(1 - s)\ P(s) = \frac{1 - s}{1-F(s)} \qquad (1.23)$$

Expanding (1-s) P(s) we get

$$(1-s)\ P(s) = (1-s) \sum_{n=o}^{\infty} p_{jj}^{(n)} s^n$$

$$= \sum_{n=o}^{\infty} p_{jj}^{(n)} s^n - \sum_{n=o}^{\infty} p_{jj}^{(n)} s^{n+1}$$

$$= p_{jj}^{(o)} + p_{jj}^{(1)} s + p_{jj}^{(2)} s^2 + .. + p_{jj}^{(n-1)} s^{n-1} + p_{jj}^{(n)} s^n$$

$$- p_{jj}^{(o)} - p_{jj}^{(1)} s^2 - p_{jj}^{(2)} s^3 - ..... - p_{jj}^{(n-1)} s^n - ....$$

putting

$$s = 1$$

and noting that

$$p^{(o)} = 1 ,$$

we have

$$\lim_{s=1} (1-s) P(s) = \lim_{n=\infty} p_{jj}^{(n)}$$

and

$$\lim_{s=1} \frac{1 - s}{1 - p(s)} = \frac{1 - 1}{1 - 0} = \frac{0}{0}$$

which is undefined.

Using L' Hospital's rule, we have

$$\lim_{s=1} \frac{1 - s}{1 - F(s)} = \frac{-1}{-F'(s)} = \frac{-1}{-\mu_{jj}} = \frac{+1}{\mu_j}$$

Therefore

$$\lim_{s=1} (1-s) P(s) = \lim_{s=1} \frac{(1-s)}{1-F(s)}$$

which implies that

$$\lim_{n=\infty} p_{jj}^{(n)} = \frac{1}{\mu_j}$$

For null state,

$$\mu_j = \infty$$

In this case

$$\lim_{n=\infty} p_{jj}^{(n)} = o$$

In general

$$p_{ij}^{(n)} = \sum_{v} f_{jj}^{(v)} \; p_{jj}^{(n-v)}$$

Therefore

$$\lim_{n=\infty} p_{ij}^{(n)} = \sum_{v} f_{ij} \left(\lim_{n=\infty} p_{jj}^{(n-r)} \right)$$

$$= \sum_{v} f_{ij} \left(\lim_{n=\infty} p_{jj}^{(n-v)} \right)$$

$$= \frac{f_{ij}}{\mu_j}$$

We can now summarize some facts about the persistent and transient states in the following theorems.

<u>Theorem 1.1.</u>

$$\sum_{n=o}^{\infty} p_{jj}^{(n)} = \infty$$

But

$$\lim_{n=\infty} p_{jj}^{(n)} = o$$

for null state

<u>Theorem 1.2.</u>

A state E_j is absorbing iff

$$f_{jj}^{(1)} = 1 \; ; \; f_{jj} = 1$$

and

$$\mu_j = 1$$

<u>Theorem 1.3.</u>

E_j is transient iff

$$\sum_{n=o}^{\infty} p_{jj}^{(n)} < \infty$$

1.3.4. PERIODICITY OF STATES

A state E_k is said to be of period t, if t is the greatest common divisor of n for which $p_{kk}^{(n)} > o$.

Alternatively E_k is said to be of period t if

$$p_{kk}^{(n)} = o$$

unless $n = vt$

i.e. n is a multiple of t.

If $t = 1$, then E_k is said to be aperiodic, otherwise it is periodic.

1.3.5. ERGODIC STATE

This is a state with the following characteristics:

(i) the state must be aperiodic

(ii) the state must be persistent

(iii) the state must be non-null.

In other words the state must have a finite mean recurrance time. Such a state is called Ergodic.

CHAPTER TWO

CLASSIFICATION OF CHAINS

2.1. CLOSED SETS

A set c of states is closed if no state outside c can be reached from any state in c. Alternatively a set c of states is closed if each state in c communicates only with others in c.

So if,

$$E_j \in c \text{ and } E_k \notin c,$$

then,

$$p_{jk} = 0$$

and in general

$$p_{jk}^{(n)} = 0 \text{ for } n \geq 1.$$

If E_j, $E_k \in c$,

then

$$\sum_k p_{jk}^{(n)} = 1$$

and generally

$$\sum_k p_{jk}^{(n)} = 1, n \geq 1.$$

So the importance of closed sets is that a markov chain can be split into sub-markov chains which can be studied independently of other states.

A closed set of communicating states is a class. So if c is a class, then every pair E_j and E_k in c, there exists a positive integer n for which $p_{jk}^{(n)} > 0$.

Remark

We should note from the above definitions that:

(i) the totality of all states that can be reached from a given state E_j form a closed set.

(ii) A closed set may contain states which may not communicate.

An absorbing state

A single state E_k forming a closed set is called an absorbing state. It is a state once reached cannot be left. Further, an absorbing state is considered a class.

<u>**Examples of closed sets.**</u>

Let,

$P =$	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9
E_1	0	0	0	*	0	0	0	0	*
E_2	0	*	*	0	*	0	0	0	*
E_3	0	0	0	0	0	0	0	*	0
E_4	*	0	0	0	0	0	0	0	0
E_5	0	0	0	0	*	0	0	0	0
E_6	0	*	0	0	0	0	0	0	0
E_7	0	*	0	0	0	*	*	0	0
E_8	0	0	*	0	0	0	0	0	0
E_9	0	0	0	*	0	0	0	0	0

Where entries with * signs show that $p_{jk} > 0$.

(i)

$E_1 \rightarrow E_4 \rightarrow E_1$

$\downarrow$

$E_9 \rightarrow E_4 \rightarrow E_1$

E

Therefore $\{E_1, E_4, E_9\}$ is a closed set.

(ii) $E_5 \rightarrow E_5$.

This implies that $\{E_5\}$ is a closed set. Since this set contains a single set, it is called an absorbing state.

(iii) $E_3 \rightarrow E_8 \rightarrow E_3$

Thus $\{E_3, E_8\}$ does form a closed set.

(iv) $\{E_2, E_8, E_7\}$ does not form a closed set. They are not independent of the other six states, although the six are independent of them.

For example,

$$E_2 \rightarrow E_5 \text{ but } E_5 \rightarrow E_5$$

Having found some closed subsets, we can re-arrange the given markov chain by considering the closed sets as follows:

Let,

$E_5 = E'_1$, $E_3 = E'_2$, $E_8 = E'_3$,

$E_1 = E'_4$, $E_9 = E'_5$, $E_4 = E'_6$

$E_2 = E'_7$, $E_7 = E_8$, $E_6 = E'_9$

The matrix is now as shown below:

	E'_1	E'_2	E'_3	E'_u	E'_5	E'_6	E'_7	E'_8	E'_9
E'_1	*	o	o	o	o	o	o	o	o
E'_2	o	o	*	o	o	o	o	o	o
E'_3	o	*	o	o	o	o	o	o	o
E'_4	o	o	o	o	*	*	o	o	o
E'_5	o	o	o	o	*	*	o	o	o
E'_6	o	o	o	*	o	o	o	o	o
E'_7	*	*	o	o	*	o	*	o	o
E'_8	o	o	o	o	o	o	*	*	*
E'_9	o	o	o	o	o	o	*	o	o

2.2. Irreducible Markov chain

A markov chain is irreducible if there exists no closed subsets other than the set of all states.

Two states are of the same type if

(i) Both have the same period

(ii) Both are transient or else both are persistent.

(iii) Both are persistent, then further both have finite mean recurrence times or both have infinite mean recurrence times.

Theorem 2.1.

All states of an irreducible chain are of the same type.

Proof:

For any irreducible chain, all the states

Let

$$p_{ii}^{(N+n+m)} = K < \infty$$

Then

$$K \geq ab \; p_{jj}^{(N)}$$

from the first part of relationship (2.1).

This implies that

$$p_{jj}^{(N)} \leq \frac{k}{ab} < \infty$$

Therefore if state i is trancient then state j is also transient.

(b) Suppose i is persistent.

This means that,

$$p_{ii}^{(N)} = \infty$$

Then the second part of relationship (2.1) becomes:

$$p_{jj}^{(n+N+m)} \geq ab \times \infty = \infty.$$

This implies that state j is also persistent.

(c) Suppose state i is persistent-null. This means that,

$$\lim_{n=\infty} p_{ii}^{(N)} = 0;$$

Then substituting this relation in (2.1) we get

$$\lim_{n=\infty} p_{jj}^{(N)} = 0 .$$

So j is also null.

(d) Suppose that i is persistent non-null. Let its period be t. Then $p_{ii}^{(N)} > 0$ whenever N is a multiple of t. Now

$$p_{ii}^{(n+m)} \geq p_{ij}^{(n)} \; p_{ji}^{(m)} = ab > 0 ,$$

So that (n+m) is a multiple of t. Substituting in relationship (2.1) above we get,

$$p_{jj}^{(N+n+m)} \geq ab \; p_{ii}^{(N)} > 0.$$

Thus $(n+m+N)$ is a multiple of t and so t is the period of the state j. So i and j have the same period.

Theorem 2.2.

For a persistent state E_j, there exists a unique irreducible set c containing E_j, such that for every pair E_i, E_k in c,

$$f_{ik} = f_{ki} = 1.$$

Proof:

Let a_k be the probability that starting from state E_j, the state E_k is reached without previously returning to E_j. The probability of never returning to E_j from E_k is $1 - f_{jk}$.

The probability of reaching state E_k from E_j and never returning to state E_j is $a_k\,(1-f_{jk})$. Since state E_j is persistent the probability of no return to E_j is zero.

Therefore,

$$1 - f_{jk} = 0$$

This implies that

$$f_{jk} = 1,$$

for every E_k that can be reached from E_j. The three states are accessible from one another since the set c is irreducible. In an irreducible chain all states are of the same type. It follows from the foregoing statement that states E_i and E_k are persistent because E_j is.

Let the probability of reaching the state E_k from state E_i for the first time be b_k. The probability of never returning to E_i once E_k is reached $1 - f_{ik}$. The probability of the system reaching state E_k from state E_i without having returned to state E_i and never returning to E_i is $b_k(1 - f_{ik})$. Since E_i is persistent the probability of no return to E_i is zero.

Therefore,

$$1 - f_{ik} = 0$$

This implies that

$$f_{ik} = 1$$

for every E_k that can be reached from E_i.

The same result can be obtained if the process now starts in state E_k and ends up in E_i. That is

$$f_{ki} = 1.$$

<u>Corollary</u>

In a finite chain there exists no null states and it is impossible that all states are transient.

<u>Proof:</u>

Let $\underline{P}$ be a stochastic transition matrix chain with a finite number of states. In a finite chain, a state E_j is transient iff there exists another state E_k such that E_k is reachable from E_j after any number of steps but E_j cannot be reached from E_k. That is

$$p_{jk}^{(n)} \quad o,$$

but

$$p_{kj}^{(n)} = o$$

When n is the number of transitions, $p_{jk}^{(n)}$ for all j and k are the entrants of the matrix $\underline{P}^n$, which is a stochastic matrix.

Since the state E_j cannot be reached from the other state E_k then E_k is transient. This implies that some states in a finite Markov Chain are not transient. The other states that are not transient form an irreducible set of states which are of the same type. Within the irreducible set all states are reachable from one another. That is,

$$p_{ij}^{(n)} \neq 0$$

and $$p_{ii}^{(n)} > 0$$

Hence there's at least one non-null state. The presence of a non-null state in the irreducible set implies that all the states in the irreducible set have a finite mean of recurrence time.

Theorem 2.3.

The states of a markov chain can be divided in a unique manner into non-overlapping sets T, C_1, C_2,.......

Such that:

(i) T consists of all transient states

(ii) If E_j in C_v then

$$f_{jk} = 1,$$

for all E_k in C_v, while

$$f_{jk} = 0,$$

for all E_k outside C_v.

2.3 IRREDUCIBLE AND ERGODIC MARKOV CHAIN

Definition of invariant or stationary distribution

A probability distribution $\{v_i\}$ is called stationary or invariant for a given chain if,

$$v_k = \sum_j v_j \, p_{jk} \quad ,$$

such that,

$$v_j > o$$

and $$\sum v_j = 1 \qquad (2.2)$$

This implies that

$$v_k = \sum_j \left(\sum_i v_i \, p_{ij}\right) p_{jk}$$

$$= \sum_i v_i \left(\sum p_{ij} \, p_{jk}\right)$$

$$= \sum_i v_i \, p_{ik}^{(2)}$$

In generaly

$$v_k = \sum_i v_i \, p_{ik}^{(n)} \qquad (2.3)$$

In the matrix notation relationship (2.3) can put as follows:

$$\begin{bmatrix} v_1 & v_2 & v_3 \cdots \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 & \cdots\cdots \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \cdots \\ p_{21} & p_{22} & p_{23} \cdots \\ p_{31} & p_{32} & p_{33} \cdots \end{bmatrix}$$

We now study the behaviour of a markov process when it is repeated several times. In otherwords under what conditions if any does

$$p_{jk}^{(n)} \to v_k \quad n \to \infty$$

be independent of j ?

If such a limit exists, the system settles down and becomes stable.

Theorem:

For an irreducible and ergodic chain, the limits

$$v_k = \lim_{n=\infty} p_{jk}^{(n)}$$

exist and are independent of j.
The limits v_k are such that

$$v_k > 0$$

and (2.4)

$$\sum_k v_k = 1$$

That is, the limits v_k define a distribution.
Furthermore the limiting distribution $\{v_k\}$ is identical with the stationary distribution for the given chain so that

$$v_k = \sum_j v_j \, p_{jk}$$

and

$$\sum_k v_k = 1$$

Conversely suppose that the chain is irreducible and aperiodic and there exists

$$v_k \geq o$$

Such that

$$\sum_k v_k = 1$$

and

$$v_k = \sum_j v_j \, p_{jk}$$

Then the chain is ergodic and

$$v_k = \frac{1}{\mu_k} = \lim_{n=\infty} p_{jk}^{(n)}$$

Proof:

Since the states are ergodic then

$$\lim_{n=\infty} p_{jk}^{(n)} = \frac{f_{jk}}{\mu_k}$$

so the limits exist. But,

$$f_{jk} = 1$$

Since E_k can be reached from a persistent state E_j (from theorem 2.2).

$$\lim_{n\to\infty} p_{jk}^{(n)} = \frac{1}{\mu_k} > o$$

which is independent of j.

Next consider

$$p_{jk}^{(n+m)} = \sum_i p_{ji}^{(n)} \; p_{ik}^{(m)}$$

This implies that

$$\lim_{n=\infty} p_{jk}^{(n+m)} = \lim_{n=\infty} \sum_i p_{ji}^{(n)} \; p_{ik}^{(m)}$$

$$\geq \sum (\lim_{n=\infty} p_{ji}^{(n)}) p_{ik}^{(m)} ,$$

by Fatu's lemma. That is,

$$v_k \geq \sum_i \left[\lim_{n\to\infty} p_{ji}^{(n)}\right] p_{ik}^{(m)} = \sum_i v_i \, p_{ik}$$

Therefore,

$$v_k \geq \sum_i v_i p_{ik}^{(m)} \text{ for all m.}$$

Suppose,

$$v_k > \sum_i v_i p_{ik}^{(m)}$$

then

$$\sum_k v_k > \sum_k \sum_i p_{ik}^{(m)} v_i$$

$$= \sum_i v_i (\sum_k p_{ik}^{(m)})$$

$$= \sum_i v_i ,$$

Since $\sum p_{ik}^{(m)} = 1$.

Therefore,

$$\sum_k v_k > \sum_i v_i .$$

This is impossible. This implies that

$$\sum_k v_k = \sum_i v_i$$

This is,

$$v_k = \sum_i v_i \, p_{ik}$$

From

$$v_k = \sum_i p_{ik}^{(m)}$$

$$\lim_{n=\infty} v_k = \lim_{m=\infty} v_i \, p_{ik}^{(m)}$$

$$= \sum_i v_i (\lim \, p_{ik}^{(m)})$$

$$v_k = \sum_i v_i \, v_k$$

$$= (\sum_i v_i) v_k$$

This implies that

$$\Sigma\, v_k = 1,$$

by cancellation of v_k. So $\{v_k\}$ is a probability distribution since,

$$v_k = \lim_{n=\infty} p_{jk}^{(n)} > o\ .$$

2.4. THE ABSORBING MARKOV CHAIN

Definition 1:

A markov chain is absorbing if

(i) it has at least one absorbing state

(ii) from every state, it is possible to go to an absorbing state (not necessarily in one step).

Definition 2:

An absorbing markov chain is one whose state space consists only of transient states and absorbing states.

2.4.1. Canonical Form

In developing the theory of absorbing markov chain it is always convenient to use transition matrices in what is known as canonical form.

We renumber the states such that the absorbing states come first i.e. the rows and columns are rearranged so that all the absorbing states are dealt with first. Their transition probabilities appear together at the left hand corner of the matrix with the element 1 in the leading diagonal. Then the rows and columns for the transient states are placed in any order in the remaining positions of the matrix.

Example 2.4.1.1

Let

$$P = \begin{array}{c} \\ E_o \\ E_1 \end{array} \begin{array}{c} \begin{array}{cc} E_o & E_1 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \end{array}$$

In canonical form the matrix becomes:

$$\underline{P} = \begin{array}{c} \\ E_1 \\ E_o \end{array} \begin{array}{c} \begin{array}{cc} E_1 & E_o \end{array} \\ \left[\begin{array}{cc} 1 & o \\ \frac{1}{2} & \frac{1}{2} \end{array}\right] \end{array}$$

Example 2.412.

Let,

$$\underline{P} = \begin{array}{c} \\ E_o \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{ccc} E_o & E_1 & E_2 \end{array} \\ \left[\begin{array}{ccc} \frac{1}{2} & o & \frac{1}{2} \\ o & 1 & o \\ o & \frac{1}{4} & \frac{3}{4} \end{array}\right] \end{array}$$

In canonical form the matrix becomes:

$$\begin{array}{c} \\ E_1 \\ E_o \\ E_2 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_o & E_2 \end{array} \\ \left[\begin{array}{ccc} 1 & o & o \\ \frac{1}{2} & o & \frac{1}{2} \\ \frac{1}{4} & o & \frac{3}{4} \end{array}\right] \end{array}$$

Example 2.4.3.

Let,

$$\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

In canonical form the matrix becomes:

$$\begin{array}{c} \\ E_2 \\ E_4 \\ E_1 \\ E_3 \end{array} \begin{array}{c} \begin{array}{cccc} E_2 & E_4 & E_1 & E_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \end{array}$$

Usually the states in the canonical form are renumbered e.g. in example 2.4.3 so that the first state becomes E_1 and the second one becomes E_2 etc.

In general if there are s absorbing states and t transient states, then the canonical form of the transition matrix will be

$$\underline{P} = \begin{array}{c} s\{ \\ \\ \\ \end{array} \left[\begin{array}{c|c} I & O \\ \hline R & Q \end{array} \right] \begin{array}{c} \\ \\ \} t \end{array} \qquad (2.5)$$

$$\underbrace{}_{t}$$

Where

I is an (sxs) unit matrix

O is an (sxt) zero matrix

R is a (txs) matrix concerned with transformations from transient states to absorbing states

Q is a(txt) matrix concerned with transitions from transient to transient states.

From example 2.4.13.

$$\underline{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$\underline{Q} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

2.4.2. HIGHER TRANSITION MATRICES

Let,

$$\underline{P} = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$$

then

$$\underline{P}^2 = \underline{P}.\underline{P} = \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \begin{bmatrix} I & O \\ R & Q \end{bmatrix}$$

$$= \begin{bmatrix} I & O \\ R(I+Q) & Q^2 \end{bmatrix}$$

$$\underline{P}^3 = \underline{P}^2.\underline{P} = \begin{bmatrix} I & O \\ R(I+Q) & Q^2 \end{bmatrix} \begin{bmatrix} I & O \\ R & Q \end{bmatrix}$$

$$= \begin{bmatrix} I & O \\ R(I+Q+Q^2) & Q^3 \end{bmatrix}$$

$$\underline{P}^4 = \underline{P}^3.\underline{P} = \begin{bmatrix} I & O \\ R(I+Q+Q^2) & Q^3 \end{bmatrix} \begin{bmatrix} I & O \\ R & Q \end{bmatrix}$$

$$= \begin{bmatrix} I & O \\ R(I+Q+Q^2+Q^3) & Q^4 \end{bmatrix}$$

In general

$$\underline{P}^n = \begin{bmatrix} I & O \\ M & Q^n \end{bmatrix}$$

Where,

$$M = \left[I + Q + Q^2 + \ldots\ldots + Q^{n-1} \right] \underline{R} \qquad (2.6)$$

2.4.3. ASYMPTOTIC BEHAVIOUR OF $\underline{P}^n$

Theorem 2.5.

$$Q^n \rightarrow \underline{O} \; .$$

Proof:

By definition of absorbing markov chain, every state will have to go to an absorbing state.

Theorem 2.6.

If Q^n tends to zero matrix as n tends to infinity, then

$$(I - Q)$$

has got an inverse and,

$$(I - Q)^{-1} = I + Q + Q^2 + \ldots\ldots \sum_{k=o}^{\infty} Q^k \; .$$

Proof:

Consider the following identity:

$$(I - Q)(I + Q + Q^2 + \ldots + Q^{n-1}) = (I - Q^4) \; .$$

The result can be verified by multiplication.

By hypothesis,

$$(I - Q^n) \to I$$

Therefore

$$| I - Q^n | = 1.$$

This implies that,

$$| I - Q^n | \neq 0,$$

for sufficiently large n.

We know that,

$$| I - Q | | (I + Q + Q^2 + \ldots + Q^{n-1}) |$$

$$= | (I - Q) (I + Q + \ldots + Q^n) |$$

Since right handside equals to one, then,

$$| (I - Q) | | (I + Q + Q^2 + \ldots + Q^{n-1}) | = 1$$

This implies that determinants on the left hand side are non-zero.

Thus

$$| (I - Q) | \neq 0.$$

So

$$(I - Q)$$

has an inverse.

Multiplying both sides of the identiy, by the inverse of (I-Q):

$$(I-Q)^{-1}\ (I-Q)\ (I+Q+Q^2+.....+Q^{n-1}) = (I-Q)^{-1}\ (I-Q^n).$$

This implies that

$$(I+Q+Q^2+....+Q^{n-1}) = (I-Q)^{-1}(I-Q^n)$$

But R.Hs of this identiy approaches $(I-Q)^{-1}$ for large n.

Therefore,

$$(I + Q + Q^2+....+ Q^{n-1}) = (I - Q)^{-1}$$

Thus

$$\lim_{n=\infty} \underline{P}^n = \begin{bmatrix} I & O \\ (I-Q)^{-1}R & O \end{bmatrix}$$

Defination of fundamental matrix

For an absorbing markov chain we define the fundamental matrix to be:

$$N = (I - Q)^{-1} \qquad (2.7)$$

2.4.4. Some interpretations and applications of the fundamental matrix

Let n_{ij} be the total number of times a process starting from state E_i is in state E_j.

Let x_{ij} be the event that the process is in state E_j from state E_i.

Therefore,

$$x_{ij}^{(k)} = 1,$$

if the process is in state E_j after k steps from E_i.

$$x_{ij}^{(k)} = 0 \quad \text{otherwise.}$$

Therefore,

$$x_{ij} = \sum_{k=o}^{\infty} x_{ij}^{(k)}$$

$$E(n_{ij}) = \sum_{k=o}^{\infty} E(x_{ij}^{(k)})$$

$$= \sum_{n=o}^{\infty} \left[O.P(x_{ij}^{(k)} = O) + 1.P(x_{ij}^{(k)}) = 1) \right]$$

$$= \sum_{k=o}^{\infty} p(x_{ij}^{(k)}) = 1)$$

$$= \sum_{k=o}^{\infty} p_{ij}^{(k)}$$

In matrix form,

$$((E(n_{ij}))) = \sum_{k=o}^{\infty} ((p_{ij}^{(k)}))$$

$$= \sum_{k=o}^{\infty} \underline{Q}^{k}$$

$$= (I-Q)^{-1}$$

$$= \underline{N} \qquad (2.8)$$

Thus $E(n_{ij})$ is the average number of times a process takes in state E_j before absorption given that it starts from state E_i. This means that the entries of the fundamental matrix give the expected number of times the process will be in each transient state.

<u>Theorem 2.7.</u>

$$\mathrm{Var}\left[n_{ij}\right] = (2\, N_{dg} - I)N - N_{sq}.$$

Where,

N is the fundamental matrix,

N_{sq} is the fundamental matrix with all the entries squared.

N_{dg} is the fundamental matrix with all entries being zero except the major diagonal entries.

I is an identity matrix

n_{ij} is the total number of times that a process is in a state E_j from state E_i.

Probability that a markov process starting from a non-absorbing will terminate in an absorbing state

Let E_{ij} be an event if E_i moves to E_j in one step. And A_{ik} be an event if E_i moves to E_k in one step. The states E_i and E_j are transient and E_k is absorbing.

The event A_{ik} can occur in two exclusive ways:

(i) direct transition from E_i to E_k

(ii) Transition from E_i to E_k via E_j.

Here,

$$A_{ik} = E_{ik} \cup \sum_j E_{ij} \quad A_{jk}$$

Therefore,

$$p(A_{ik}) = p(E_{ik}) + \sum_j p(E_{ij} \cap A_{jk})$$

$$= p(E_{ik}) + \sum_j p(E_{ij}).p(A_{jk})$$

Let,

$$p(A_{ik}) = b_{ik},$$

then

$$b_{ik} = p_{ik} + \sum_j p_{ij}\, p_{jk}$$

In matrix form

$$((b_{ik})) = ((p_{ik})) + ((\sum_j p_{ij}\, p_{jk}))$$

Let,

$$\underline{B} = ((b_{ik}))$$

and note that,

$$\underline{P} = \begin{bmatrix} I & O \\ R & Q \end{bmatrix}$$

So that we have,

$$\underline{B} = \underline{R} + \underline{Q}\ \underline{B}$$

From where

$$\underline{B} = (\underline{I}-\underline{Q})^{-1}\underline{R}$$

$$= \underline{N}\ \underline{R} \tag{2.9}$$

The elements of $\underline{B}$ i.e. b_{ik} are probabilities of moving from a transient state E_i to an absorbing state E_k.

<u>The expected number of steps that the process is in a non-absorbing state before absorption</u>

Let us denote by $\sum_j E(n_{ij})$

the expected number of steps the process is in a transient state before absorption. To obtain $\Sigma E(n_{ij})$ we have to post - multiply the fundamental matrix by a column matrix of ones. Let us denote this column matrix by,

$$\underline{C} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$$

Then,

$$(I-\underline{Q})^{-1}\underline{C} = \begin{bmatrix} n_{11} & n_{12} & n_{13}\cdots \\ n_{21} & n_{22} & n_{23}\cdots \\ n_{31} & n_{32} & n_{33}\cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} n_{11} & n_{12} & n_{13}\cdots \\ n_{21} & n_{22} & n_{23}\cdots \\ n_{31} & n_{32} & n_{33}\cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Clearly this is the sum of the row elements of the fundamental matrix.

Let t be the number of times an absorbing chain moves among the transient States (including the original position) before passing to one of the absorbing states.

Then,

$$E(t = \tau) = (I-\underline{Q})^{-1}\,\underline{C}$$

$$= \underline{N}\cdot\underline{C}$$

and (2.10)

$$Var(t) = (2\underline{N} - I)\underline{\tau} - \tau_{sq}$$

CHAPTER III

EXAMPLES OF MARKOV CHAINS

In this chapter we intend to give some illustrative examples of Markov Chains. For some of these examples we shall try to classify the states of the Markov Chains and study the asymptotic behaviour of their transitional probabilities by applying the theories developed in the previous chapters.

3.1. Random walk with absorbing barriers

Let us consider a boy moving on a straight line x - axis, say. The boy moves on the points 1, 2,...,m on the x - axis. If he arrives at points 1 and m he remains there permanently i.e. he is absorbed - thus the term absorbing barriers. Let us denote the points on the x - axis by $E_1, E_2, E_3, \ldots\ldots, E_m$ and refer to these points as states of the system. States E_1 and E_m are called absorbing states.

Consider a situation when the boy is in state E_i different from the absorbing states. The boy can move to state E_{i+1} with a probability p or he can move to states E_{i-1} with probability q.

The transition probabilities of this system are:

$$p_{11} = p_{mm} = 1$$

$$p_{ij} = \begin{cases} p, & \text{for } j = i+1,\ i= \ ,2,3,.. \\ q, & \text{for } j = i-1,\ i=2,3,.... \\ o & \text{otherwise} \end{cases}$$

In matrix form,

$$\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_m \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & \cdots & E_m \end{array} \\ \left[\begin{array}{cccccc} 1 & o & o & o & \cdots & o \\ q & o & p & o & \cdots & o \\ o & q & o & p & o \cdots & o \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ o & o & o & o & o \cdots & 1 \end{array}\right] \end{array}$$

In studying the properties of this chain let us take a four - state chain. That is

$$\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \left[\begin{array}{cccc} 1 & o & o & o \\ q & o & p & o \\ o & q & o & p \\ o & o & o & 1 \end{array}\right] \end{array}$$

Classification of states

States E_1 and E_4 are absorbing. Since this is an absorbing Markov Chain then the rest of the states are transient.

Application of the theory absorbing Markov Chain

In canonical form,

$$\underset{\sim}{P} = \begin{array}{c} E_1 \\ E_4 \\ E_2 \\ E_3 \end{array} \left[\begin{array}{cccc} 1 & o & o & o \\ o & 1 & o & o \\ \hline q & o & o & p \\ o & p & q & o \end{array} \right]$$

$$\begin{array}{cccc} E_1 & E_4 & E_2 & E_3 \end{array}$$

Where

$$\begin{bmatrix} 1 & o \\ o & 1 \end{bmatrix} = \underset{\sim}{I}$$

$$\begin{bmatrix} o & o \\ o & o \end{bmatrix} = \underset{\sim}{O}$$

$$\begin{bmatrix} q & o \\ o & p \end{bmatrix} = \underline{R}$$

$$\begin{bmatrix} o & p \\ q & o \end{bmatrix} = \underline{Q}$$

The importance of the canonical form of an absorbing markov chain is the applicability of its sub-chain, the fundamental matrix. The fundamental matrix is

$$\underline{N} = (I - \underline{Q})^{-1}$$

$$= \frac{1}{1-pq} \begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix}$$

<u>What is the expected number of times that the boy will be in each non-absorbing state before reaching an absorbing state</u> ?

Here states E_2 and E_3 are non-absorbing. This question can be answered with the help of $\underline{N}$.

$$\underline{N} = \frac{1}{1-pq} \begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix}$$

$$\qquad\qquad E_2 \qquad E_3$$

(i) Assume the boy starts his movement from state E_2, then

$$E(n_{22}) = \frac{1}{1-pq} .$$

This is the average number of times that the boy will be in state E_2 before reaching absorption

(ii) Assuming the boy starts in E_2 how many times will he be in E_3 before reaching absorption ? That is,

$$E(n_{23}) = \frac{p}{1-pq}$$

(iii) If the boy starts from state E_3 how many times will he be in E_2 before reaching absorption ?

$$E(n_{32}) = \frac{q}{1-pq}$$

(iv) If he starts in E_3 how many times will he be there on average before reaching absorption ?

$$E(n_{33}) = \frac{1}{1-pq}$$

<u>What is $Var(n_{ij})$?</u>

We know that

$$Var\ (n_{ij}) = N \left[2\ N_{dg} - I\right] - N_{sq} .$$

From the matrix in question,

$$\underline{N} = \frac{1}{1-pq} \begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix}$$

Therefore,

$$\underline{N}_{dg} = \frac{1}{1-pq} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$N_{sq} = \left(\frac{1}{1-pq}\right)^2 \begin{bmatrix} 1 & p^2 \\ q^2 & 1 \end{bmatrix}$$

Substituting these, we get,

$$Var(n_{ij}) = \frac{1}{1-pq} \begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{1-pq} - 1 & 0 \\ 0 & \frac{2}{1-pq} - 1 \end{bmatrix}$$

$$- \left(\frac{1}{1-pq}\right)^2 \begin{bmatrix} 1 & p^2 \\ q^2 & 1 \end{bmatrix} =$$

$$= \left(\frac{1}{1-pq}\right)^2 \begin{bmatrix} 1+pq & p(1+pq) \\ q(1+pq) & 1+pq \end{bmatrix}$$

$$- \frac{1}{(1-pq)^2} \begin{bmatrix} 1 & p^2 \\ q^2 & 1 \end{bmatrix} =$$

$$= \frac{1}{1-pq)^2} \begin{bmatrix} pq & p+p^2(q-1) \\ q+q^2(p-1) & pq \end{bmatrix}$$

Specifically,

(i) $Var(n_{22}) = \frac{pq}{(1-pq)^2}$

(ii) $Var(n_{23}) = \frac{p+p^2(q-1)}{(1-pq)^2} = \frac{p-p^3}{(1-pq)^2}$

(iii) $Var(n_{32}) = \frac{q+q^2(p-1)}{(1-pq)^2} = \frac{q-q^3}{(1-pq)^2}$

(iv) $Var(n_{33}) = \frac{pq}{(1-pq)^2}$

<u>What is the expected number of times that the boy is in a non-absorbing state before reaching an absorbing state ?</u>

Let t be number of times that the boy is in a non-absorbing state (E_2 or E_3).

$$E(t) = = \underset{\sim}{N} \cdot \underset{\sim}{C}$$

Where,

$$C = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \cdot \end{bmatrix}, \quad \underline{N} = \frac{1}{1-pq} \begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix}$$

Therefore,

$$E(t) = \underset{\sim}{\tau} = \frac{1}{1-pq} \begin{bmatrix} 1+p \\ 1+q \end{bmatrix}$$

(i) If the boy starts on state E_2, on average the boy will either be in E_2 or E_3 $\left(\frac{1+p}{1-pq}\right)$ times before reaching an absorbing state.

(ii) Starting from state E_3 the boy will either be in E_3 or E_2 $\left(\frac{1+q}{1-pq}\right)$ times before reaching an absorbing state.

<u>What is Var(t) ?</u>

We know that,

Var(t) = (2N - I)

$$= \left(\frac{1}{1 - pq}\right)^2 \begin{bmatrix} 3pq + p - p^3 \\ 3pq + q - q^3 \end{bmatrix}$$

The variance of the number of times that the boy is in a non-absorbing state if he starts in state E_2 is

$$\frac{3pq + p - p^3}{(1 - pq)^2}$$

Similarly if he starts from state E_3 the variance of the number of states that he will be in an absorbing state is

$$\frac{3pq + q - q^3}{(1 - pq)^2}$$

<u>What is the probability that the boy will end up in an absorbing state if he starts from a transient state ?</u>

This question is answered by using the

matrix,

$$\underline{B} = \underline{N} \cdot \underline{R},$$

because it was stated in chapter two that $\underline{R}$ is a matrix concerned with transition from transient states to absorbing states.

$$\underline{B} = \frac{1}{1-pq} \begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix} \cdot \begin{bmatrix} q & o \\ o & p \end{bmatrix}$$

$$= \frac{1}{1-pq} \begin{bmatrix} \overset{E_2}{q} & \overset{E_2}{p^2} \\ q^2 & p \end{bmatrix} \begin{matrix} E_2 \\ E_3 \end{matrix}$$

(i) $p(E_2 \rightarrow E_1) = \left(\frac{q}{1 - pq}\right)$

(ii) $p(E_2 \rightarrow E_4) = \left(\frac{p^2}{1 - pq}\right)$

(iii) $p(E_3 \rightarrow E_1) = \left(\frac{q^2}{1 - pq}\right)$

(iv) $p(E_3 \rightarrow E_4) = \left(\frac{p}{1 - pq}\right)$

3.2. Random Walk with two reflecting barriers

Consider a boy moving on a straight line. The boy may be at one of the points $1,2,3,\ldots,m-1,m$ on the straight line. At the ends of the line segment the boy bounces back once he reaches there.

When the boy is at point i, he moves to point $i+1$ with probability p. He remains at point 1 once he reaches there with probability q. His probability of moving to point $i-1$ from point i is q. Thus

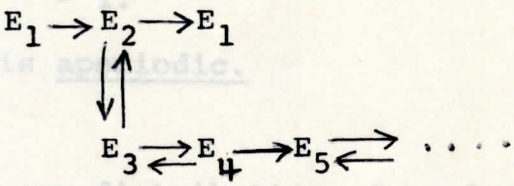
$$p_{ij} = \begin{cases} p, & j=i+1 \quad \text{for } i=1,2,\ldots \\ q, & \text{for } i=2,3,\ldots j=i-1 \\ o & \text{otherwise.} \end{cases}$$

The transition matrix is

$$\begin{array}{c|cccccccc|} & E_1 & E_2 & E_3 & E_4 & \cdots & & & E_m \\ \hline E_1 & q & p & o & o & \cdots & & & o \\ E_2 & q & o & p & o & \cdots & & & o \\ E_3 & o & q & o & p & o & \cdots & & o \\ E_4 & o & o & q & o & p & \cdots & & o \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ E_{m-1} & o & o & o & o & \cdot & q & o & p \\ E_m & o & o & o & o & \cdots & & q & p \end{array}$$

The points $1,2,\ldots.m$ are represented by $E_1, E_2, \ldots\ldots\ldots, E_m$.

Classification of states



It is clear that the states of this chain are reachable from one another. Therefore the chain is irreducible.

Periodicity

Consider state E_1.

We find all n such that

$$f_{11}^{(n)} > o$$

$$f_{11}^{(1)} = p(E_1 \to E_1) = q > o$$

$$f_{11}^{(2)} = p(E_1 \to E_2 \to E_1) = pq > 0$$

$$f_{11}^{(3)} = p(E_1 \to E_2 \to E_2 \to E_1) = o$$

$$f_{11}^{(4)} = p(E_1 \to E_2 \to E_3 \to E_2 \to E_1) = (pq)^2$$

etc.

Therefore,

$$t = \text{H.C.F of } \; 1,\; 2,\; 4,\; 6,\ldots.$$
$$= 1.$$

The chain is <u>aperiodic.</u>

<u>The stationary distribution of random walk with two reflecting barriers</u>

We find a matrix $\underline{V}$ such that

$\underline{V} = \underline{V}.\underline{P}$.

$$P = \begin{array}{c|cccccccc|} & E_o & E_1 & E_2 & E_3 & \cdot\,\cdot & E_{m-2} & E_{m-1} & E_m \\ \hline E_o & q & p & o & o & & o & o & o \\ E_1 & q & o & p & o & & o & o & o \\ E_2 & o & q & o & p & & o & o & o \\ E_3 & o & o & q & o & p.. & o & o & o \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ E_{m-2} & o & o & o & o & o..q & o & p & o \\ E_{m-1} & o & o & o & o & o..o & q & o & p \\ E_m & o & o & o & o & o..o & o & q & p \end{array}$$

$\underline{V} = \underline{V}.\underline{P}$

becomes

$$(V_o \; V_1 \ldots V_3) = \begin{bmatrix} V_o V_1 \ldots V_m \end{bmatrix} \begin{bmatrix} q & p & o & o \ldots\ldots o & o & o \\ q & o & p & o \ldots\ldots o & o & o \\ o & q & o & p \ldots\ldots o & o & o \\ o & o & q & o \;\; p \;\; o & o & o \\ \vdots & \vdots & \vdots & \vdots \;\; \vdots \;\; \vdots & \vdots & \vdots \\ o & o & o & o \ldots q \;\; o & p & o \\ o & o & o & o \ldots o \;\; q & o & p \\ o & o & o & o \ldots o \;\; o & q & p \end{bmatrix}$$

which implies

$$V_o = q\,V_o + qV_1$$

Therefore

$$V_1 = \left(\frac{p}{q}\right) V_o$$

$$V_1 = pV_o + qV_2$$

Therefore,

$$V_2 = \left(\frac{p}{q}\right)^2 V_o$$

$$V_2 = p\,V_1 + q\,V_3 \, .$$

Therefore,

$$q\,V_3 = V_2 - p\,V_1 .$$

$$V_3 = \frac{1}{q}\left[\left(\frac{p}{q}\right)^2 - \frac{p^2}{q}\right] V_o = \frac{V_o}{q}\,\frac{p^2}{q^2}\left[1-q\right] = \left(\frac{p}{q}\right)^3 V_o$$

$$V_3 = pV_2 + qV_4$$

which implies

$$qV_4 = V_3 - pV_2$$

$$= \left[(\tfrac{p}{q})^3 - (\tfrac{p}{q})^2 p\right] V_o$$

$$V_4 = \left(\frac{p^3}{q^3}\right)\frac{1}{q}\left[1 - q\right] V_o$$

$$= (\frac{p}{q})^4 V_o$$

$$\vdots \qquad \vdots$$

$$V_{m-1} = (\tfrac{p}{q})^{m-1} V_o$$

$$V_m = (\tfrac{p}{q})^3 V_o$$

But,

$$V_o + V_1 + V_2 + \ldots . V_3 = 1.$$

This implies that,

Therefore,

$$1 = V_o \left[1 + \frac{p}{q} + (\frac{p}{p})^2 + (\frac{p}{q})^3 + \ldots + (\frac{p}{q})^m \right]$$

$$= V_o \left[\frac{1 - (p/q)^{m+1}}{1 - \frac{p}{q}} \right]$$

which implies,

$$1 - \frac{p}{q} = V_o \left[1 - (\frac{p}{q})^{m+1} \right]$$

Therefore,

$$V_o = \frac{1 - p/q}{1 - (\frac{p}{q})^{m+1}}$$

Since,

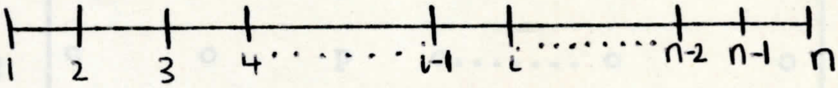
$$V_i = (\frac{p}{q})^i V_o$$

then,

$$V_i = (\frac{p}{q})^i \frac{(1 - \frac{p}{q})}{1 - (\frac{p}{q})^{m+1}}$$

3.3. Random Walk with one reflecting barrier

Consider a boy moving on a straight line e.g. the x - axis between points 1 and n.



Let the points 1,2,...n denote the states of the boy. In this type of walk one of the end points must be an absorbing state. Let E_1 be the absorbing state and the other one be a reflecting barrier. When the boy comes to state E_n he walks back to E_{n-1} with probability q or he can stay at E_n with probability p.

The boy moves from one state to another without skipping any of the states. The transition probabilities of the movement are:

$$p_{11} = 1$$

$$p_{nn} = p$$

$$p_{ij} = \begin{cases} p \quad , & i = 1,2,\ldots, \; j = i+1 \\ q \quad , & i = 2,3,\ldots, \; j = i-1 \\ o & \text{otherwise} \end{cases}$$

The transition probability matrix is

$$
\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ \vdots \\ E_{n-1} \\ E_n \end{array}
\begin{array}{c}
\begin{array}{cccccc} E_1 & E_2 & E_3 \ldots\ldots & \ldots\ldots & E_{n-1} & E_n \end{array} \\
\left[\begin{array}{cccccc}
1 & o & o \ldots\ldots & \ldots\ldots & o & o \\
q & o & p & o \ldots\ldots & o & o \\
o & q & o & p \ldots\ldots & o & o \\
o & o & q & o \quad p \ldots & o & o \\
\vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots \\
o & o & o & o \ldots q & o & p \\
o & o & o & o \ldots o & q & p
\end{array}\right]
\end{array}
$$

Classification of states

It is possible to enter into any state of this chain starting from any state except state E_1. Therefore this chain is absorbing.

This implies that all states are transient except state state E_1 which is absorbing.

Since this chain is absorbing it can be written in canonical form.

$$
\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{n-1} \\ E_n \end{array}
\begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 \ldots\ldots & E_{n-1} & & E_n \end{array} \\
\left[\begin{array}{cccccc}
1 & o & o \ldots\ldots\ldots & o & & o \\
q & o & p\ o \ldots\ldots & o & & o \\
o & q & o\ p\ o \ldots & o & & o \\
\vdots & & & & & \\
o & o & o \ldots\ldots q & o & & p \\
o & o & o \ldots\ldots & q & & p
\end{array}\right] \end{array}
$$

From the canonical form of $\underline{P}$ we get

$$\underline{I} = (1)$$

$$\underline{O} = \left[o \quad o \ldots\ldots\ o \quad o \right]$$

$$\underline{R} = \begin{bmatrix} q \\ o \\ o \\ \vdots \\ o \\ o \end{bmatrix}$$

and

$$
\underline{Q} = \begin{bmatrix}
o & p & o \ldots\ldots & o & o \\
q & o & p\ o \ldots\ldots & o & o \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
o & o & o \ldots\ldots q & o & p \\
o & o & o \ldots\ldots o & q & p
\end{bmatrix}
$$

From these subchains it is possible to get $\underline{N}$ from

$$\underline{N} = (I - \underline{Q})^{-1}$$

<u>The stationary distribution of random Walk with one reflecting barrier.</u>

Here we find $\underline{V}$, such that,

$$\underline{V} = \underline{V}.\underline{P}$$

$$\begin{bmatrix} V_1 & V_2 & V_3 \cdots V_{n-1} & V_n \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \cdots V_{n-1} & V_n \end{bmatrix} \begin{bmatrix} 1 & o & o \cdots & o & o \\ q & o & p\; o \cdots & o & o \\ o & q & o\; p \cdots & o & o \\ \vdots & \vdots & & & \\ o & o & o\; o \cdots & q & p \end{bmatrix}$$

Expanding the right hand side we get,

$$V_1 = V_1 + q V_2 \Longrightarrow V_2 = o$$

$$V_2 = q V_3 \Longrightarrow V_3 = o$$

$$V_3 = p V_2 + q V_4 \Longrightarrow V_4 = o$$

$$V_4 = p V_3 + q V_5 \Longrightarrow V_5 = o$$

$$\vdots$$

$$V_{n-1} = p V_{n-2} + q V_n \Longrightarrow V_n = o.$$

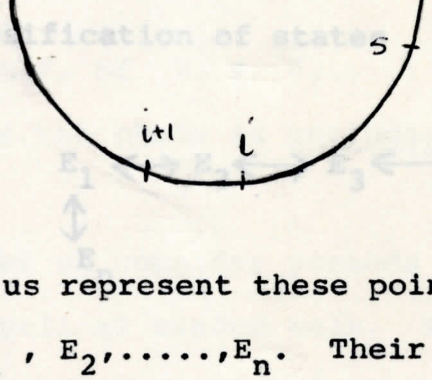
This means that stationary distribution for a random walk with one absorbing state does not exist.

3.4. Cyclical Random Walk.

Let us now consider a situation where a boy moves on a circle.

We suppose that the boy moves one step clockwise with probability p and counter - clockwise with probability q.

See diagram below.



Let us represent these points on the circle with E_1 , $E_2, \ldots, E_n$. Their transition probabilities are:

$$p(E_i \rightarrow E_{i+1}) = p$$

$$p(E_i \rightarrow E_{i-1}) = q$$

$$p(E_n \rightarrow E_1) = p$$

$$p(E_1 \rightarrow E_n) = q.$$

The corresponding transitional probability matrix is

$$
\begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ \vdots \\ E_{n-1} \\ E_n \end{array}
\begin{array}{c}
\begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 \cdots\cdots & E_{n-1} & E_n \end{array} \\
\begin{bmatrix}
o & p & o & o \cdots\cdots & o & q \\
q & o & p & o \cdots\cdots & o & o \\
o & q & o & p \quad o \cdots & o & o \\
o & o & q & o \quad p \cdots & o & o \\
\vdots & & & & & \\
o & o & o & o \cdots\cdots q & o & p \\
p & o & o & o \cdots\cdots o & q & o
\end{bmatrix}
\end{array}
$$

Properties of a cyclical random walk

(i) Classification of states

$$
\begin{array}{l}
E_1 \leftrightarrow E_2 \leftrightarrow E_3 \longleftrightarrow E_4 \cdots\cdots \leftrightarrow E_n \\
\updownarrow \\
E_n
\end{array}
$$

All the states of this chain are reachable. Therefore the chain is irreducible. This means that all the states have the same properties.

Let us consider state E_1 for the properties of the chain.

(ii) Periodicity

find,

$$f_{11}^{(n)} > o, \text{ for } n \geq 1.$$

$f_{11}^{(1)} = p(E_1 \rightarrow E_1) = o$

$f_{11}^{(2)} = p(E_1 \rightarrow E_2 \rightarrow E_1) = pq$

$f_{11}^{(3)} = o$

$f_{11}^{(4)} = p(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) = (pq)^2$

$f_{11}^{(5)} = p(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_4 \rightarrow E_1) = o$

$f_{11}^{(6)} = p(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_4 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) = (pq)^3$

etc.

t = H.C.F. of 2, 4, 6,..... = 2

Therefore the chain is <u>periodic</u>.

Let us consider periods of specific examples of the cyclical random walk. We consider a case where the total number of steps of the walk, n is even and a case where n is odd.

(i) Let $n = 3$.

Then,

$$\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \left[\begin{array}{ccc} o & p & q \\ q & o & p \\ p & q & o \end{array}\right] \end{array}$$

$$f_{11}^{(1)} = p(E_1 \longrightarrow E_1) = o$$

$$f_{11}^{(2)} = p(E_1 \rightarrow E_2 \longrightarrow E_1) \quad o$$

$$f_{11}^{(3)} = p(E_1 \rightarrow E_2 \longrightarrow E_3 \longrightarrow E_1) \quad o$$

$$f_{11}^{(4)} = p(E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1) \quad o$$

$$f_{11}^{(5)} = p(E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_1) \quad o$$

etc.

$$p_{kk}^{(n)} \neq 1 \quad \text{for} \quad n \neq 1$$

Therefore,

$$t = \text{H.C.F of } \{2, 3, 4, 6, \ldots\ldots\} = 1$$

Therefore a three - state cyclical random walk is <u>aperiodic.</u>

(ii) Further let us consider a five - state cyclical random walk.

	E_1	E_2	E_3	E_4	E_5
E_1	o	p	o	o	q
E_2	q	o	p	o	o
E_3	o	q	o	p	o
E_4	o	o	q	o	o
E_5	p	o	o	q	o

$$f_{11}^{(1)} = p(E_1 \rightarrow E_1) = o$$

$$f_{11}^{(2)} = p(E_1 \rightarrow E_5 \rightarrow E_1) \quad o$$

$$f_{11}^{(4)} = p(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) \quad o$$

$$f_{11}^{(5)} = p(E_1 \rightarrow E_5 \rightarrow E_4 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1)$$

$$f_{11}^{(6)} = p(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_4 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) \quad o$$

$$f_{11}^{(7)} = p(E_1 \rightarrow E_5 \rightarrow E_4 \rightarrow E_3 \rightarrow E_2 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1)$$

etc.

Then,

$$t = \text{H.C.F. of } \{2, 4, 5, 6, 7, \ldots\} = 1$$

Therefore this chain is <u>aperiodic</u>.

(iii) Seven - state cyclical random walk.

	E_1	E_2	E_3	E_4	E_5	E_6	E_7
E_1	o	p	o	o	o	o	q
E_2	q	o	p	o	o	o	o
E_3	o	q	o	p	o	o	o
E_4	o	o	q	o	p	o	o
E_5	o	o	o	q	o	p	o
E_6	o	o	o	o	q	o	p
E_7	p	o	o	o	o	q	o

$f_{11}^{(1)} = p(E_1 \rightarrow E_1) = o$

$f_{11}^{(2)} = p(E_1 \rightarrow E_7 \rightarrow E_1) > o$

$f_{11}^{(4)} = p(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) > o$

$f_{11}^{(6)} = p(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_4 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) > o$

$f_{11}^{(7)} = p(E_1 \rightarrow E_7 \rightarrow E_8 \rightarrow E_9 \rightarrow E_4 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) > o$

etc.

t = H C F of $\{2, 4, 6, 7, \ldots\ldots\} = 1.$

Therefore the chain is <u>aperiodic</u>.

(iv) Four - state cyclical Random Walk

$$\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{bmatrix} o & p & o & q \\ q & o & p & o \\ o & q & o & p \\ p & o & q & o \end{bmatrix} \end{array}$$

$f_{33}^{(2)} = p(E_3 \rightarrow E_2 \rightarrow E_3) > o$

$f_{33}^{(4)} = p(E_3 \rightarrow E_4 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3) > o$

$f_{33}^{(6)} = p(E_3 \rightarrow E_4 \rightarrow E_1 \rightarrow E_4 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3) > o$

etc.

$t = \text{H.C.F. of } \{2, 4, 6, \ldots\ldots\} = 2.$

Therefore the chain is periodic.

(v) Six - state cyclical random walk

$$\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{bmatrix} o & p & o & o & o & q \\ q & o & p & o & o & o \\ o & q & o & p & o & o \\ o & o & q & o & p & o \\ o & o & o & q & o & p \\ p & o & o & o & q & o \end{bmatrix} \end{array}$$

$$f_{11}^{(2)} = p(E_1 \to E_6 \to E_1) > o$$

$$f_{11}^{(3)} = o$$

$$f_{11}^{(4)} = p(E_1 \to E_2 \to E_3 \to E_2 \to E_1) > o$$

$$f_{11}^{(6)} = p(E_1 \to E_6 \to E_5 \to E_4 \to E_3 \to E_2 \to E_1) > o$$

etc.

$$t = \text{H.C.F. of } \{2, 4, 6, \ldots..\}$$

$$= 2$$

Therefore the chain is periodic.

Conclusion.

The specific examples (i) to (v) considered above contradict the property that a general cyclical random walk is periodic. We can now see that cyclical random walks with odd number of states are aperiodic. Cyclical random walks with an even number of states are periodic.

This is a contradiction.

Note:

From the examples of cyclical walks considered above one can conclude that cyclical random walks with an odd number of states e.g. 3 - state, 5 - state, 7 - state etc. are all aperiodic.

4 - state, 6 - state etc. cyclical random walks are periodic.

3.5. Two state Markov Chains

Consider a sequence of Bernoulli trials which can be represented as a two state chain. The two states are E_1 and E_2 representing head and tail of a coin respectively.

First let us assume that a coin is tossed and a head is obtained. We suppose that the probability of the next toss resulting into a head is p and that of resulting into a tail is q.

Secondly let us assume that the coin is tossed and the result is a tail. We suppose that the probability of the next trial resulting into a tail is q and that of resulting into a head is p. Thus

$$p_{11} = p$$

$$p_{12} = q$$

$$p_{21} = p$$

$$p_{22} = 2$$

The corresponding matrix is

$$P = \begin{array}{c} \\ E_1 \\ E_2 \end{array}\begin{array}{c} E_1 \quad\quad E_2 \\ \begin{bmatrix} p & q \\ p & q \end{bmatrix} \end{array}$$

<u>Classification of states</u>

$$E_1 \rightarrow E_1 \leftrightarrow E_2 \rightarrow E_2.$$

Since all states are reachable from one another then the chain is said to be <u>irreducible</u>.

<u>Periodicity</u>

$$f_{11}^{(1)} = p(E_1 \rightarrow E_1) = p$$

$$f_{11}^{(2)} = p(E_1 \rightarrow E_2 \rightarrow E_1) = pq$$

$$f_{11}^{(3)} = p(E_1 \rightarrow E_2 \rightarrow E_2 \rightarrow E_1) = p\,q^2$$

$$f_{11}^{(4)} = p(E_1 \rightarrow E_2 \rightarrow E_2 \rightarrow E_2 \rightarrow E_1) = p \cdot q^3$$

etc.

t = H.C.F of $\{1, 2, 3, 4, \ldots\ldots\}$

= 1

Therefore the chain is <u>aperiodic</u>.

Persistent/transient

$$f_{11} = f_{11}^{(1)} + f_{11}^{(2)} + f_{11}^{(3)} + \ldots\ldots\ldots$$

From the foregoing section we get,

$$f_{11} = p + pq + pq^2 + pq^3 + \ldots\ldots\ldots\ldots\ldots$$

$$= p(1 + q + q^2 + q^3 + \ldots\ldots\ldots\ldots\ldots\ldots)$$

$$= \frac{p}{1 - q} = \frac{p}{p}$$

$$= 1 .$$

Therefore the chain is persistent.

Mean of recurrence time

$$E(n) = \sum_{n-1}^{\infty} nf_{jj}^{(n)}$$

If

$$E(n) < \infty$$

then the chain is said to be non-null otherwise it is null.

$$E(n) = p + 2pq + 3pq^2 + 4pq^3 + 5pq^4 + \ldots\ldots$$

$$= p(1 + 2q + 3q^2 + 4q^3 + 5q^4 + \ldots\ldots\ldots\ldots)$$

Let

$$\frac{dy}{dq} = 1 + 2q + 3q^2 + 4q^3 + 5q^4 + \ldots\ldots$$

Integrating,

$$y = q + q^2 + 3q^2 + q^4 + q^5 + \ldots\ldots\ldots$$

$$= \left(\frac{q}{1 - q}\right)$$

$$\frac{dy}{dq} = \left(\frac{1}{1 - q}\right)^2 = \frac{1}{p^2}$$

Substituting this differential in $E(n)$ we get,

$$E(n) = \frac{p}{p^2} = \frac{1}{p}$$

This implies that the chain is <u>non-null</u>. An aperiodic - persistent - non-null chain is said to be <u>ergodic</u>.

<u>Invariant distribution of two state Markov Chain</u>

We find V_k, such that

$$V_k = \Sigma\, v_j\, p_{jk}$$

where

$$v_j \geq o$$

and

$$\sum v_j = 1$$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} p & q \\ p & q \end{bmatrix}$$

Therefore

$$v_1 = (v_1 + v_2)p$$

$$v_2 = (v_1 + v_2)q$$

and

$$v_1 + v_2 = 1.$$

From these equations we get,

$$v_1 = p$$

$$v_2 = q$$

Therefore,

$$\underline{V} = \begin{bmatrix} p & q \\ p & q \end{bmatrix}$$

Limiting distribution

The limiting distribution of this chain exists because the chain is ergodic and has an invariant distribution.

$$\underline{P} = \begin{bmatrix} p & q \\ p & q \end{bmatrix}$$

$$\underline{P}^2 = \begin{bmatrix} p(p+q) & q(p+q) \\ p(p+q) & q(p+q) \end{bmatrix} = \begin{bmatrix} p & q \\ p & q \end{bmatrix}$$

$$\underline{P}^3 = \underline{P}^2.\underline{P} = \begin{bmatrix} p & q \\ p & q \end{bmatrix} \begin{bmatrix} p & q \\ p & q \end{bmatrix}$$

$$= \begin{bmatrix} p & q \\ p & q \end{bmatrix}$$

$$\underline{P}^4 = \begin{bmatrix} p & q \\ p & q \end{bmatrix}$$

Therefore

$$\underline{P}^n = \begin{bmatrix} p & q \\ p & q \end{bmatrix}$$

Let us consider a two state chain whose transitional probability matrix is:

$$\underline{P} = \begin{bmatrix} p & q \\ q & p \end{bmatrix}$$

All states of this chain are all aperiodic and persistent. Let us find its invariant distribution.

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} p & q \\ q & p \end{bmatrix}$$

$$V_1 = pV_1 + qV_2.$$

$$V_2 = qV_1 + pV_2$$

From these we obtain

$$V_1 = V_2$$

Therefore,

$$\underline{V} = \begin{bmatrix} V_1 & V_1 \\ V_1 & V_1 \end{bmatrix}$$

<u>Special cases of the two state Markov Chain</u>

(1) $$\underline{P} = \begin{matrix} & E_1 & E_2 \\ E_1 & p & 1-p \\ E_2 & 1-q & q \end{matrix} = \begin{bmatrix} p & q \\ p & q \end{bmatrix}, \quad p \neq o, q \neq o$$

$$E_1 \rightarrow E_1 \rightarrow E_2 \rightarrow E_2$$

Since the two states are reachable from one another, the chain is <u>irreducible</u>. Since the chain is irreducible, the states have the same properties. Let us take one of the states and study its properties. Let us consider state E_1.

<u>Periodicity</u>

$$P_{11}^{(1)} = p(E_1 \rightarrow E_1) = p$$

$$P_{11}^{(2)} = p(E_1 \rightarrow E_2 \rightarrow E_1) = pq$$

$$P_{11}^{(3)} = p(E_1 \rightarrow E_2 \rightarrow E_2 \rightarrow E_1) = pq^2$$

$$P_{11}^{(4)} = p(E_1 \rightarrow E_2 \rightarrow E_2 \rightarrow E_2 \rightarrow E_1) = pq^3$$

$$\vdots$$

$$P_{11}^{(n)} = pq^{n-1}$$

The period of the chain is the H.C.F. of all n for $P_{11}^{(n)} > 0$

Therefore,

$$t = \text{H.C.F. of } 1, 2, 3, 4, \ldots\ldots, n$$

$$= 1$$

The chain is aperiodic.

Persistent/transient

We need to find,

$$f_{11} = \sum_{n=1}^{\infty} p_{11}^{(n)}$$

If,

$$f_{11} = 1,$$

then E_1 is persistent otherwise it is transient.

$$f_{11} = p + pq + pq^2 + pq^3 + \ldots\ldots\ldots$$

$$= \frac{p}{1-q}$$

$$= \frac{p}{p} = 1$$

the chain is persistent.

Mean recurrence time

$$E(n) = \sum_{n=1}^{\infty} n \, p_{11}^{(n)}$$

If,

$$E(n) < \infty$$

the chain is said to be non-null otherwise it is null.

$$E(n) = p + 2pq + 3pq^2 + 4pq^3 + \dots\dots\dots$$

$$= p(1 + 2p + 3q^2 + 4q^3 + \dots\dots\dots)$$

Let us denote the bracket by $\frac{dy}{dq}$

Therefore,

$$\frac{dy}{dq} = 1 + 2q + 3q^2 + 4q^3 + \dots\dots\dots$$

Integrating we get,

$$y = q + q^2 + q^3 + q^4 + \dots\dots\dots$$

$$= \frac{q}{1 - q}$$

Differentiating with respect to q, we get,

$$\frac{dy}{dq} = \frac{(1-q) + q}{(1-q)^2} = \frac{1}{(1-q)^2}$$

Going back to our orginal equation, we get,

$$E(n) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\frac{1}{p} < \infty$$

therefore, the mean recurrence time is finite.

The chain is non-null.

We have found that the chain is irreducible, aperiodic and persistent-null. Such a chain is said to be <u>ergodic</u>

<u>Invariant distribution of a two - state Markov Chain</u>

Let,

$$V_k = \sum_{n-1}^{\infty} p_{jk}$$

where

$$V_j \geq 0$$

and

$$\sum_j V_j = 1$$

$$\underline{P} = \begin{bmatrix} p & q \\ p & q \end{bmatrix}$$

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} p & q \\ p & q \end{bmatrix}$$

$$= \begin{bmatrix} V_1p + V_2p & V_1q + V_2q \end{bmatrix}$$

$$= \begin{bmatrix} (V_1 + V_2)p & (V_1 + V_2)q \end{bmatrix}$$

This implies,

$$V_1 = p(V_1 + V_2) \text{ ——————(i)}$$

$$V_2 = q(V_1 + V_2 \text{ ——————(ii)}$$

and

$$V_1 + V_2 = 1 \text{ ——————————(iii)}$$

Substituting (iii) in (2) and (ii), we get,

$$V_1 = p$$

$$V_2 = q$$

Therefore the stable distribution of P is,

$$\begin{bmatrix} p & 1 - p \\ 1-q & q \end{bmatrix}$$

(2). $$\underline{P} = \begin{array}{c c} & \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

$$E_1 \rightarrow E_1$$

$$E_2 \rightarrow E_2$$

Each of the two states is an absorbing state.

THE INVARIANT DISTRIBUTION OF $\underline{P}$

$$\underline{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We find

$$V_k = \sum_j p_{jk}$$

Where,

$$V_j \geq 0$$

and

$$\sum_j V_j = 1$$

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

The invariant distribution does not exist because this chain is a trivial absorbing chain.

$$
(3) \qquad \underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{bmatrix} o & 1 \\ 1 & o \end{bmatrix} \end{array}
$$

$$E_1 \to E_2 \to E_1$$

The states of this chain are reachable from one another therefore the chain is irreducible. It is enough to study the properties of only one of the states.

<u>Periodicity of state E_1</u>

$$p_{11}^{(1)} = p(E_1 \to E_1) = o$$

$$p_{11}^{(2)} = p(E_1 \to E_2 \to E_1) = 1$$

$$p_{11}^{(3)} = p(E_1 \to E_2 \to E_2 \to E_1) = o$$

$$p_{11}^{(4)} = p(E_1 \to E_2 \to E_2 \to E_1) = o$$

$$\vdots$$

$$p_{11}^{(n)} = o.$$

Therefore

$$t = \text{H.C.F. of } 1$$

$$= 1$$

This chain is therefore aperiodic

Persistent/transient

Let us find

$$f_{11} = \sum_{n=1}^{\infty} p_{11}^{(n)}$$

$$= o + 1 + o + \ldots\ldots$$

$$= 1.$$

Therefore the chain is persistent.

Mean of recurrence time

$$E(n) = \sum_{n=1}^{\infty} n\, p_{11}^{(n)}$$

$$= o + 2 \times 1 + o + o + \ldots\ldots\ldots$$

$$= 2.$$

Since the mean is finite then, the chain is non-null. A persistent-non-null, aperiodic chain is called ergodic.

3.6. THE EHRENFEST MODEL

The Ehrenfest model is basically an urn model which can be described as follows:

Consider a container with K balls some of which are black and others white. A ball is picked from the container at random. Each time a ball is picked up it has to be replaced by another ball of the opposite colour so that the number of balls in the container remains to be K.

The state of system is determined by the number of black balls in the container. If there are j black balls in the container, then the system is in state E_j. Thus

$$P_{jj} = p(E_j \rightarrow E_j) = o$$

$$P_{j,j-1} = p(E_j \rightarrow E_{j-1}) = \frac{j}{k}$$

$$P_{j,j+1} = p(E_j \rightarrow E_{j+1}) = \frac{k-j}{k}$$

In particular,

$$P_{o1} = p(E_o \rightarrow E_1) = 1$$

and

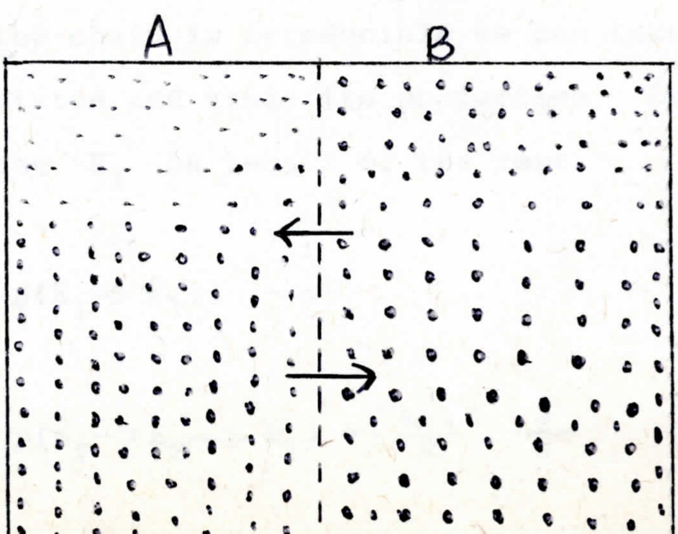
$$P_{k,k-1} = p(E_K \rightarrow E_{k-1}) = 1$$

In the matrix form,

$$
\underline{P} = \begin{array}{c} \\ E_o \\ E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{k-1} \\ E_k \end{array}
\begin{array}{c}
\begin{array}{cccccccc} E_o & E_1 & E_2 & E_3 & \cdots & E_{k-1} & & E_k \end{array} \\
\left[\begin{array}{cccccccc}
o & 1 & o & o & \cdots & o & & o \\
\frac{1}{k} & o & \frac{k-1}{k} & o & \cdots & o & & o \\
o & \frac{2}{k} & o & \frac{k-2}{k} & o & \cdots & o & o \\
o & o & \frac{3}{k} & o & \frac{k-3}{k} & o & & o \\
\vdots & & & & & & & \\
o & o & o & o & \cdots & \frac{k-1}{k} & o & \frac{1}{k} \\
o & o & o & o & \cdots & o & 1 & o
\end{array}\right]
\end{array}
$$

The physical situation of the Ehrenfest model is as follows:

There are two containers A and B which are separated by a permeable membrane. In these



two containers there are distributed k molcules, which move freely across the membrane. It is assumed that the number of molcules in the containers remains constant.

The state of the system is determined by the number of molecules in A. When there are i molecules in container A then the state of the system is E_i.

Properties of the Ehrenfest Model

(i) Classification of states.

$$E_o \leftrightarrow E_1 \leftrightarrow E_2 \leftrightarrow E_3 \leftrightarrow \cdots \leftrightarrow E_{k-1} \leftrightarrow E_k \, .$$

This shows that the chain is irreducible because all the states are reachable from one another.

(ii) Periodicity

Since the chain is irreducible we can take one of the states and study its properties. Let us study state E_1 on behalf of the rest.

$$p_{11}^{(1)} = p(E_1 \rightarrow E_1) = o$$

$$p_{11}^{(2)} = p(E_1 \rightarrow E_2 \rightarrow E_1) = \frac{k-1}{k} \cdot \frac{2}{k}$$

$$p_{11}^{(3)} = p(E_1 \rightarrow E_2 \rightarrow E_2 \rightarrow E_1) = o$$

$$p(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_1) = o$$

$$p_{11}^{(4)} = p(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) = \frac{k-1}{k} . \frac{k-2}{k} . \frac{3}{k} . \frac{2}{k}$$

$$p_{11}^{(5)} = p(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_4 \rightarrow E_3 \rightarrow E_1) = o$$

$$p_{11}^{(6)} = p(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_4 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1)$$

$$= \frac{k-1}{k} \cdot \frac{k-2}{k} \cdot \frac{k-3}{k} \cdot \frac{4}{k} \cdot \frac{3}{k} \cdot \frac{2}{k}$$

etc.

We want to get n such that,

$$p_{11}^{(n)} > 0$$

$$t = \text{H.C.F. of } \{2, 4, 6, \ldots\ldots\}$$

$$= 2$$

Therefore the chain is periodic.

STATIONARY DISTRIBUTION OF THE EHRENFEST CHAIN

We are looking for V_j such that

$$V_j = \Sigma\, v_{ji}\, p_i$$

and

$$\sum_j v_j = 1$$

In the matrix form, we are looking for $\underline{V}$ such that

$$\underline{V} = \underline{V}\cdot\underline{P} \qquad \text{(IB)}$$

i.e.

$$\left[V_o,\ V_1,\ V_2 \cdots\cdots V_k\right] = \left[V_o,\ V_1,\ V_2, \cdots\cdots\cdots V_k\right] \underline{P}$$

where $\underline{P}$ is given in (1) above.

Multiplying out these two matrices we get,

$$
\begin{aligned}
V_o &= \frac{1}{k} V_1 \\
V_1 &= \frac{k}{k} V_o + \frac{2}{k} V_2 \\
V_2 &= \frac{k-1}{k} V_1 + \frac{3}{k} V_3 \\
V_3 &= \frac{k-2}{k} V_2 + \frac{4}{k} V_4 \qquad\qquad (2) \\
&\vdots \\
V_j &= \frac{k-(j-1)}{k} V_{j-1} + \frac{j+1}{k} V_{j+1} \\
V_{k-1} &= \frac{k-(k-2)}{k} V_{k-2} + \frac{k}{k} V_k \\
V_k &= \frac{1}{k} V_{k-1}
\end{aligned}
$$

This implies

$$k\,V_o = V_1$$

$$k\,V_1 = k\,V_o + 2kV_2$$

$$kV_2 = (k-1)V_1 + 3V_3 \qquad (3)$$

$$\vdots$$

$$kV_j = k - (k-2 \quad V_{j-1} + (j+1)\,V_{j+1}$$

$$\vdots$$

$$kV_{k-1} = k-(k-2)\;V_{k-2} + kV_k$$

$$kV_{k-1} = V_{k-1}$$

Next get the probability generating function of the sequence V_j . Thus

$$G(s) = \sum_{j=o}^{k} v_j\, s^j \qquad (4)$$

Therefore,

$$G'(s) = \sum_{j=o}^{k} j\, v_j\, s^{j-1} \qquad (5)$$

In the set (3) of equations, multiply the first by s^o, the second by s^1 and the third by s^2 etc.

Thus we get

$$k\ V_o s^o = V_1 s^o$$

$$k\ V_1 s^1 = (k\ V_o + 2\ V_2) s^1$$

$$k\ V_2 s^2 = \left(k\ V_1 - V_1 + 3\ V_3\right) s^2$$

$$k\ V_3 s^3 = \left(k\ V_2 - 2\ V_2 + 4\ V_4\right) s^3 \qquad (6)$$

$$\vdots$$

$$k\ V_j s^j = \left[k\ V_{j-1} - (j-1)\ V_{j-1} + (j+1) V_{j+1}\right] s^{j+1}$$

$$\vdots$$

$$k\ V_{k-1} s^{k-1} = \left[k\ V_{k-2} - (k-2)\ V_{k-2}\ k\ V_k\right] s^{k-1}$$

This can be re-written as follows

$$k\ V_o s^o = \left(o - o - 1.V_1\right) s^o$$

$$k\ V_1 s = \left(k\ V_o - o.V_o + 2\ V_2\right) s^1$$

$$k\ V\ s^2 = \left(k\ V_1 - 1.V_1 + 3\ V_3\right) s^2$$

$$k\,V_3\,s^3 = k\,V_2 - 2\,V_2 + 4\,V_4\,s^3$$

$$k\,V_j\,s^j = k\,V_{j-1} - (j-1)V_{j-1} + (j+1)\,V_{j+1}\,s^j$$

$$\vdots$$

$$k\,V_{k-1}s^{k-1} = (k\,V_{k-2} - (k-2)V_{k-2} + k\,V_k\,s^{k-1}$$

$$k\,V_k\,s^k = k\,V_{k-1} - (k-1)V_{k-1} + (k+1)V_{k+1}S^k$$

$$kV_{k+1}S^{k+1} = k\,V_k - k\,V_{k+0}\,s^{k+1}$$

In the summation form we have

$$k\sum_{j=o}^{\infty} V_j\,s^j = k\,s\sum_{j=o}^{k} V_j\,s^j - s^2\sum_{j=o}^{k} j\,V_j s^{j-1} + \sum_{j=o}^{k} jV_j s$$

That is,

$$k\,G(s) = k\,s\,G(s) - s^2\,G'(s) + G'(s)$$

$$\therefore\ (1-s^2)\,G'(s) = k(1-s)\,G(s)$$

This implies that,

$$G'(s) = \frac{k}{1+s}\,G(s)$$

This also implies that

$$\frac{G'(s)}{G(s)} \; ds + \quad k \; \frac{ds}{1+s}$$

Therefore,

$$\text{In} \quad G(s) \; = \; k \; \text{In}(1+s) + c$$

But,

$$G(1) = 1$$

which implies that

$$\text{In } 1 \; = \; k \; \text{In}2 \; + \; c$$

That is

$$0 \; = \; k\text{In}2 \; + \; c$$

which implies,

$$c \; = \; - \; k\text{In}2$$

Therefore,

$$\text{In } G(s) \; = \; k \; \text{In}(1+s) \; - \; k \; \text{In}2$$
$$= \; k \; \text{In} \; (\frac{1+s}{2})$$

Therefore,

$$G(s) = \left(\tfrac{1}{2} + \tfrac{1}{2}s \right)^k$$

$$= \sum_{j=o}^{k} \binom{k}{j} (\tfrac{1}{2}s)^j \; (\tfrac{1}{2})^{k-j}$$

$$=$$

$$= \sum_{j=o}^{k} \binom{k}{j} (\tfrac{1}{2})^j \; s^j$$

Therefore,

$$V_j = \text{Coefficient of } s^j$$

$$= \binom{k}{j} (\tfrac{1}{2})^j \quad , \quad j=o,1,\ldots.,k.$$

CHAPTER IV

Problems and Solutions

In this chapter we shall try to solve some problems using the theory of chapters I and II.

Problem 1:

In a sequence of Bernoulli trials we say that at time n the state E_1 is observed if the trials number n-1 and n resulted in SS. Similarly E_2, E_3, E_4 stand for SF, SF, FF. Find the matrix $\underline{P}$ and all its powers. Generalize the scheme.

Solution .

$$\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{bmatrix} p & q & o & o \\ o & o & p & q \\ p & q & o & o \\ o & o & p & q \end{bmatrix} \end{array}$$

$$\underline{P}^2 = \underline{P}.\underline{P} = \begin{bmatrix} p & q & o & o \\ o & o & p & q \\ p & q & o & o \\ o & o & p & q \end{bmatrix} . \begin{bmatrix} p & q & o & o \\ o & o & p & q \\ p & q & o & o \\ o & o & p & q \end{bmatrix}$$

$$= \begin{bmatrix} p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \end{bmatrix}$$

$$\underline{P}^3 = \underline{P}^2 \cdot \underline{P} = \begin{bmatrix} p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \end{bmatrix} \begin{bmatrix} p & q & o & o \\ o & o & p & q \\ p & q & o & o \\ o & o & p & q \end{bmatrix}$$

$$= \begin{bmatrix} p^2(p+q) & pq(p+q) & qp(p+q) & q^2(p+q) \\ p^2(p+q) & pq(p+q) & qp(p+q) & q^2(p+q) \\ p^2(p+q) & pq(p+q & qp(p+q) & q^2(p+q) \\ p^2(p+q) & pq(p+q) & qp(p+q) & q^2(p+q) \end{bmatrix}$$

Since

$$p + q = 1,$$

then,

$$\underline{P}^3 = \begin{bmatrix} p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \end{bmatrix}$$

$$\underline{P}^4 = \underline{P}^3 \cdot \underline{P} = \underline{P}^3 = \underline{P}^2$$

$$= \begin{bmatrix} p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^3 \end{bmatrix}$$

In general,

$$\underline{P}^n = \begin{bmatrix} p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \\ p^2 & pq & qp & q^2 \end{bmatrix}$$

Problem 2:

Classify the states for the four chains whose matrices $\underline{P}$ have the rows given below. Find in each case $\underline{P}^2$ and the asymptotic behaviour of $p_{jk}^{(n)}$.

(a) $(0,\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, 0)$

(b) $(0,0,0,1)$ $(0, 0, 0, 1)$ $(\frac{1}{2}, \frac{1}{2}, 0,0)$, $(0, 0, 1, 0)$

(c) $(\frac{1}{2},0,0,0,)$, $(\frac{1}{4}, \frac{1}{2}. \frac{1}{4}, 0,0,)$, $(\frac{1}{2}, 0, \frac{1}{2}, 0, 0,)$,

$(0,0,0,\frac{1}{2},\frac{1}{2})$, $(0,0,0,\frac{1}{2},\frac{1}{2})$.

(d) $(0,\frac{1}{2},\frac{1}{2},0,0,0)$, $(0,0,0,\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(0,0,0, \frac{1}{3},\frac{1}{3};\frac{1}{3})$

$(1,0,0,0,0,0)$, $(1,0,0,0,0,0,)$, $(1,0,0,0,0,0)$.

Solution:

(a)

$$\underline{P} = \begin{array}{c|ccc} & E_0 & E_1 & E_2 \\ \hline E_0 & 0 & \frac{1}{2} & \frac{1}{2} \\ E_1 & \frac{1}{2} & 0 & \frac{1}{2} \\ E_2 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

Classification of states

(i) $E_0 \leftrightarrow E_1 \leftrightarrow E_2 \leftrightarrow E_0$.

All the states are reachable from one another, so the chain is irreducible. Since the chain is irreducible, all states have the same properties. We shall therefore consider only one state to be representative of the others.

(ii) Periodicity

$$p_{11}^{(2)} = p(E_1 \to E_o \to E_1) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$

$$= p(E_1 \to E_2 \to E_1) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$

$$P_{11}^{(3)} = p(E_1 \to E_o \to E_2 \quad E_1) = (\tfrac{1}{2})^3$$

$$P_{11}^{(4)} = p(E_1 \to E_o \to E_2 \to E_o \to E_1 = (\tfrac{1}{2})^4$$

$$t = \text{H.C.F. of } \{2, 3, 4, \ldots\}$$

$$= 1$$

The states are all aperiodic.

(iii) Persistent/transient

$$P_{11}^{(2)} = p(E_1 \to E_o \to E_1) + p(E_1 \to E_2 \to E_1) = (\tfrac{1}{2})^2 + (\tfrac{1}{2})^4$$

$$= (\tfrac{1}{2})$$

$$P_{11}^{(3)} = p(E_1 \to E_o \to E_2 \to E_1) + p(E_1 \to E_2 \to E_o \to E_1)$$

$$= (\tfrac{1}{2})^3 + (\tfrac{1}{2})^3$$

$$= (\tfrac{1}{2})^2$$

$$p_{11}^{(4)} = p(E_1 \to E_o \to E_2 \to E_o \to E_1) + p(E_1 \to E_2 \to E_o \to E_2 \to E_1)$$

$$= (\tfrac{1}{2})^4 + (\tfrac{1}{2})^4$$

$$= (\tfrac{1}{2})^3$$

etc.

Therefore

$$\Sigma p_{11}^{(n)} = \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \ldots\ldots$$

$$= \frac{\tfrac{1}{2}}{1 - \tfrac{1}{2}} = 1$$

Therefore the states are all <u>persistent</u>

(iv) <u>Mean of recurrence times</u>

$$E(n) = \sum_{n=1}^{\infty} n\, p_{11}^{(n)}$$

$$= 2 \times \tfrac{1}{2} \times 3x \tfrac{1}{4} + 4 \times \frac{1}{8} + 5 \times \frac{1}{16} + \frac{6}{32} +$$

$$= \sum_{n=2}^{\infty} n(\tfrac{1}{2})^{n-1}$$

If this sum converges then the mean is finite otherwise it is infinite.

Let us use the ratio test to check whether the sum is convergent or not.

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \rho$$

$$\text{or} \quad \left| \frac{u_{n+1}}{u_n} \right| \to +\infty$$

If $\rho < 1$ then the series converges

If $\rho > 1$ then the series diverges

If $\rho = 1$ the test gives no information

$$U_n = n\,(\tfrac{1}{2})^{n-1}$$

$$U_{n+1} = (n+1)\,(\tfrac{1}{2})^n$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\,(\frac{1}{2})^n}{n\,(\frac{1}{2})^{n-1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n}\,(\tfrac{1}{2}) \right|$$

$$= \lim_{n\to\infty} \tfrac{1}{2} \left| 1 + \frac{1}{n} \right|$$

$$= \tfrac{1}{2}$$

Therefore the sum converges. This implies that the mean of recurrent times is <u>finite</u> or <u>non-null</u>.

A chain with all these properties is called <u>ergodic</u>.

$$\underline{P}^2 = \underline{P} \cdot \underline{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

<u>Asymptotic behaviour of P</u>

$$\underline{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\underline{P}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$\underline{P}^3$

$$\underline{P}^3 = \underline{P}^2 \cdot \underline{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & 3/8 & 3/8 \\ 3/8 & \frac{1}{4} & 3/8 \\ 3/8 & 3/8 & \frac{1}{4} \end{bmatrix}$$

$$\underline{P}^4 = \begin{bmatrix} \frac{1}{4} & 3/8 & 3/8 \\ 3/8 & \frac{1}{4} & 3/8 \\ 3/8 & 3/8 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{6}{16} & \frac{5}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{6}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{5}{16} & \frac{6}{16} \end{bmatrix}$$

$$\underline{P}^5 = \frac{1}{32} \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{32} \begin{bmatrix} 10 & 11 & 11 \\ 11 & 10 & 11 \\ 11 & 11 & 10 \end{bmatrix}$$

$$\underline{P}^6 = \frac{1}{64} \begin{bmatrix} 10 & 11 & 11 \\ 11 & 10 & 11 \\ 11 & 11 & 10 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{64} \begin{bmatrix} 22 & 21 & 21 \\ 21 & 22 & 21 \\ 21 & 21 & 22 \end{bmatrix}$$

$$\underline{P}^7 = \frac{1}{128} \begin{bmatrix} 22 & 21 & 21 \\ 21 & 22 & 21 \\ 21 & 21 & 22 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{128}\begin{bmatrix} 42 & 43 & 43 \\ 43 & 42 & 43 \\ 43 & 43 & 43 \end{bmatrix}$$

Generally,

$$\underline{P}^n \simeq \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

2(b)

$$\underline{P} = \begin{array}{c} \\ E_o \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{cccc} E_o & E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} o & o & o & 1 \\ o & o & o & 1 \\ \frac{1}{2} & \frac{1}{2} & o & o \\ o & o & 1 & o \end{bmatrix} \end{array}$$

<u>Classification of states</u>

<u>Irreducible/reducible</u>

$$\begin{array}{l} E_o \rightarrow E_3 \rightarrow E_2 \rightarrow E_1 \rightarrow E_3 \\ \qquad\qquad\quad \downarrow \\ \qquad\qquad\quad E_o \end{array}$$

All the states of this chain are reachable from one another. The chain is therefore <u>irreducible</u>. This implies that all the states have the same properties. It is therefore enough to study the properties of only one state.

(i) <u>Periodicity</u> Consider state E_1

$$p_{11}^{(2)} = 0$$

$$p_{11}^{(3)} = p(E_1 \to E_3 \to E_2 \to E_1) = \tfrac{1}{2}$$

$$p_{11}^{(4)} = 0$$

$$p_{11}^{(5)} = 0$$

$$p_{11}^{(6)} = p(E_1 \to E_3 \to E_2 \to E_o \to E_3 \to E_2 \to E_1)$$

$$= (\tfrac{1}{2})^2$$

$$p_{11}^{(7)} = 0$$

$$p_{11}^{(8)} = 0$$

$$p_{11}^{(9)} = p(E_1 \to E_3 \to E_2 \to E_o \to E_3 \to E_2 \to E_o \to E_3 \to E_2 \to E_1$$

$$= (\tfrac{1}{2})^3$$

etc.

$$t = \text{H.C.F. of } \{3, 6, 9, \ldots\}$$

$$= 3.$$

This means that the chain is periodic.

(ii) Persistent/transient

$$p_{11}^{(3)} = p(E_1 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) = \tfrac{1}{2}$$

$$p_{11}^{(6)} = p(E_1 \rightarrow E_3 \rightarrow E_2 \rightarrow E_o \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) = (\tfrac{1}{2})^2$$

$$p_{11}^{(9)} = p(E_1 \rightarrow E_3 \rightarrow E_2 \rightarrow E_o \rightarrow E_3 \rightarrow E_2 \rightarrow E_o \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) = (\tfrac{1}{2})^3$$

$$p_{11}^{(12)} = (\tfrac{1}{2})^4$$

$$\Sigma\, p_{11}^{(n)} = \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{2})^4 + \ldots\ldots\ldots\ldots$$

$$= \frac{\tfrac{1}{2}}{1 - \tfrac{1}{2}}$$

$$= 1$$

Therefore the chain is persistent.

(iii) Mean of recurrence times

$$E(n) = \sum_{n=1}^{\infty} n\, p_{11}^{(n)}$$

$$= 3(\tfrac{1}{2}) + 6(\tfrac{1}{2})^2 + 9(\tfrac{1}{2})^3 + 12(\tfrac{1}{2})^4 + \ldots\ldots$$

$$= 3 \sum_{n=1}^{\infty} n(\tfrac{1}{2})^n$$

Let us use the ratio test to check for the convergence of this series

$$U_n = 3n\ (\tfrac{1}{2})^n$$

$$U_{n+1} = 3(n+1)\ (\tfrac{1}{2})^{n+1}$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{U_n} \right| = \lim_{n\to\infty} \left| \frac{3(n+1)\ (\tfrac{1}{2})^{n+1}}{3n\ (\tfrac{1}{2})^n} \right| = \lim_{n\to\infty} \tfrac{1}{2} \left| 1 + \frac{1}{n} \right|$$

$$= \tfrac{1}{2}.$$

Therefore the series converges. This implies that the mean is finite i.e. non-null.

$$\underline{P}^2 = \underline{P}.\underline{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 \end{bmatrix}$$

Asymptotic behaviour of $p_{jk}^{(n)}$

$$\underline{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\underline{P}^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\underline{P}^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{P}^4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\underline{P}^5 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$\underline{P}^6 = \underline{P}^5.\underline{P} = \underline{P}^2.\underline{P} = \underline{P}^3$$

$$\underline{P}^7 = \underline{P}^6.\underline{P} = \underline{P}^3.\underline{P} = \underline{P}^4 = \underline{P}$$

$$\underline{P}^8 = \underline{P}^7.\underline{P} = \underline{P}.\underline{P} = \underline{P}^2$$

$$\underline{P}^9 = \underline{P}^8.\underline{P} = \underline{P}^2.\underline{P} = \underline{P}^3$$

In general,

$$\underline{P}^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ For } n \text{ a multiple of 3.}$$

$$\underline{P}^n = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}, \text{ For } n = 2,5,8,11,14,\ldots$$

$$\underline{P}^n = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ For } n = 1,4,7,10,13\ldots.$$

2C.

$$\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{array} \begin{array}{c} \begin{array}{ccccc} E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

The states of this chain are not reachable from one another. Therefore the chain is <u>reducible</u>.

$\{E_1, E_3\}$ and $\{E_4, E_5\}$ are closed sets.

State E_2 is not a closed set.

Re-arranging these states we get,

$$E_1' = E_1$$

$$E_2' = E_3$$

$$E_3' = E_4$$

$$E_4' = E_5$$

$$E_5' = E_2$$

Now,

$$P = \begin{array}{c} \\ E_1' \\ E_2' \\ E_3' \\ E_4' \\ E_5' \end{array} \begin{array}{c} \begin{array}{ccccc} E_1' & E_2' & E_3' & E_5' & E_5' \end{array} \\ \left[\begin{array}{ccccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{array}\right] \end{array}$$

Incidentally this chain has two identical closed sets. Let us now Consider the properties of one of these sets.

$$P = \begin{array}{c} \\ E_1' \\ E_2' \end{array} \begin{array}{c} \begin{array}{cc} E_1' & E_2' \end{array} \\ \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array}\right] \end{array}$$

<u>Classification of states</u>

(i) Periodicity.

$$p_{11}^{(1)} = p(E_1' \to E_1') = \tfrac{1}{2}$$

$$p_{11}^{(2)} = p(E_1' \to E_2' \to E_1') = (\tfrac{1}{2})^2$$

$$p_{11}^{(3)} = p(E_1' \to E_2' \to E_2' \to E_1') = (\tfrac{1}{2})^3$$

etc.

$$t = \text{H.C.F. of } \{1, 2, 3, \ldots\ldots\}$$

$$= 1.$$

Therefore the states of the two closed sets are <u>aperiodic</u>.

(ii) Persistent/transient.

$$\sum_{n=1}^{\infty} p_{11}^{(n)} = (\tfrac{1}{2}) + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \ldots\ldots\ldots\ldots$$

$$= \frac{\tfrac{1}{2}}{1-\tfrac{1}{2}} =$$

$$= 1.$$

This means that all states of the two closed sets are persistent.

<u>Mean of recurrence times</u>

$$E(n) = \Sigma\, n\, p_{11}^{(n)}$$

$$= (\tfrac{1}{2}) + 2(\tfrac{1}{2})^2 + 3(\tfrac{1}{2})^3 + \ldots\ldots\ldots$$

$$= \Sigma\, n(\tfrac{1}{2})^n.$$

Using the ratio test, it can be seen that the series coverges. This means that the states are all non-null. The two closed sets are ergodic chains.

Asymptotic behaviour of the closed sets

$$\underline{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\underline{P}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\underline{P}^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

It can be seen that all powers of $\underline{P}$ are all equal to $\underline{P}$.

Therefore,

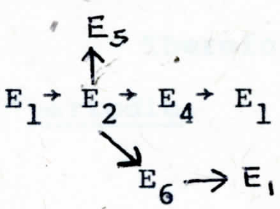
$$\underline{P}^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

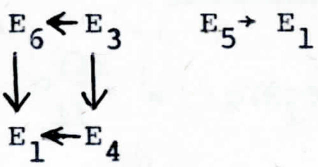
2d.

P =

	E_1	E_2	E_3	E_4	E_5	E_6
E_1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
E_2	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
E_3	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
E_4	1	0	0	0	0	0
E_5	1	0	0	0	0	0
E_6	1	0	0	0	0	0

Irreducible/Reducible





All the states of this chain are reachable from one another. Therefore the chain is irreducible.

(i) Periodicity

$$p_{11}^{(1)} = p(E_1 \rightarrow E_1) = 0$$

$$p_{11}^{(2)} = 0$$

$$p_{11}^{(3)} = p(E_1 \rightarrow E_2 \rightarrow E_4 \rightarrow E_1) = \frac{1}{6}$$

$$p_{11}^{(u)} = 0$$

$$p_{11}^{(5)} = p(E_1 \rightarrow E_2 \rightarrow E_u \rightarrow E_4 \rightarrow E_1) = 0$$

etc.

$t =$ H.C.F. of 3

$= 3$

Therefore all the states of the chain are periodic.

(ii) Persistent/transient

$$p_{11}^{(3)} = p(E_1 \rightarrow E_2 \rightarrow E_4 \rightarrow E_1) + p(E_1 \rightarrow E_2 \rightarrow E_5 \rightarrow E_1) +$$

$$p(E_1 \rightarrow E_2 \rightarrow E_6 \rightarrow E_1) + p(E_1 \rightarrow E_3 \rightarrow E_4 \rightarrow E_1) +$$

$$+ p(E_1 \rightarrow E_3 \rightarrow E_5 \rightarrow E_1) + p(E_1 \rightarrow E_3 \rightarrow E_6 \rightarrow E_1)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

$$= 1.$$

This means that the chain is persistent.

(iii) Mean of recurrence times

$$E(n) = \sum_{n=1}^{\infty} n\, p_{11}^{(n)}$$

$$= 3 \times 1$$

$$= 3.$$

Since the mean is finite it means that the states of this chain are non-null.

The Asymptotic behaviour of $p_{jk}^{(n)}$

$$P^2 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{P}^3 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\underline{P}^4 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \underline{P}$$

$\underline{P}^5 = \underline{P}^2$

$\underline{P}^6 = \underline{P}^3$

$\underline{P}^7 = \underline{P}^1$

Generally,

$$\underline{P}^n = \underline{P} \text{ , } n = 4, 7, 10, 13,$$

$$\underline{P}^n = \underline{P}^2 \text{ , } n = 5, 8, 11, 14,......$$

$$\underline{P}^n = \underline{P}^3 \text{ , } n=6, 9, 12, 15,........$$

Problem 3

A chain with states 1, 2,....n has a matrix whose first and last rows are

(q, p, o....,o)

and

(o, o,.....o, q, p).

In all other rows

$$p_{k,k+1} = p$$

and

$$p_{k-1} = q$$

Find the stationary distribution can the chain be periodic , Solution:

$$
\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ E_{n-1} \\ E_4 \end{array}
\begin{array}{cccccccc}
E_1 & E_2 & E_3 & E_4 & \cdots & \cdots & \cdots & E_n \\
q & p & o & o & \cdots & \cdots & \cdots & o \\
q & o & p & o & & & & o \\
o & q & o & p & & & & o \\
o & o & q & o & & & & o \\
\cdot & & & & & & & \\
\cdot & \cdot & o & q & & & & \cdot \\
\cdot & \cdot & \cdot & o & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & o & q & o & p \\
o & o & o & o & & o & q & p
\end{array}
$$

Stationary distribution

Here we find a vector $\underline{V}$ such that

1. $\underline{V} = \underline{V}.\underline{P}$

2. $\sum_j V_j = 1.$

$$V_1\ V_2\ V_3 \ldots\ldots = V_1\ V_2\ V_3 \begin{bmatrix} q & p & o & o \ldots\ldots & & o \\ q & o & p & o & & o \\ o & q & o & p & & o \\ o & o & o & o & & o \\ . & . & . & q & & . \\ . & . & . & . & & p \\ o & o & o & o & q & p \end{bmatrix}$$

Working out the multiplication

$$V_1 = q\ V_1 + q\ V_2 \qquad V_2 = \left(\frac{p}{q}\right) V_1$$

$$V_2 = p\ V_1 + q\ V_3 \qquad q_3 = \left(\frac{p}{q}\right)^2 V_1$$

$$V_3 = p\ V_1 + q\ V_4 \qquad V_4 = \frac{1}{q}\left[\left(\frac{p}{q}\right)^2 - \frac{p^2}{q}\right] V_1$$

$$= \frac{1}{q}\left[\frac{p^2}{q^2} - \frac{p^2}{q}\right] V_1$$

$$= \frac{\left(p^2 - p^2 q\right)}{q^3} V_1$$

$$= \frac{p^2}{q^3} (1-q)\ V_1$$

$$= \frac{p^3}{q^3} V_1$$

Since

$$\sum_{j=1}^{\infty} V_j = 1$$

then

$$V_1 + \frac{p}{q} V_1 + \left(\frac{p}{q}\right)^2 V_1 + \left(\frac{p}{q}\right)^3 V_1 + \ldots\ldots = 1.$$

Dividing all through by V_1, we get

$$\frac{1}{V_2} = 1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \left(\frac{p}{q}\right)^3 + \ldots\ldots$$

$$= \frac{1}{1-p/q} = \frac{q}{q-p}$$

Therefore

$$V_1 = \frac{q-p}{q}$$

$$V_2 = \left(\frac{p}{q}\right)\left(1 - \frac{p}{q}\right)$$

$$V_3 = \left(\frac{p}{q}\right)^2 \left(1 - \frac{p}{q}\right)$$

$$V_4 = \left(\frac{p}{q}\right)^3 \left(1 - \frac{p}{q}\right)$$

$$V_n = \left(\frac{p}{q}\right)^{n-1} \left(1 - \frac{p}{q}\right)$$

Therefore,

$$\underline{V} = \begin{bmatrix} 1-p/q & \frac{p}{q}(1-p/q) & (p/q)^2(1-p/q) \ldots (p/q)^{n-1}(1-p/ \\ 1-p/q & \frac{p}{q}(1-p/q) & (p/q)^k(1-p/q) \ldots (p/q)^{n-1}(1-p/ \\ 1-p/q & \frac{p}{q}(1-p/q) & (p/q)^2(1-p/q) \ldots (p/q)^{n-1}(1-p/ \\ . & . & . \quad . \quad\quad . \quad . \\ . & . & . \quad . \quad\quad . \quad . \\ . & . & . \quad . \quad\quad . \quad . \\ 1-p/q & p(q(1-p/q) & (p/q)^2(1-p/q) \ldots (p/q)^{n-1}(1-p/ \end{bmatrix}$$

Periodicity

The matrix is irreducible. We can therefore find the period of ones of the states.

$$p_{11}^{(1)} = p(E_1 \rightarrow E_1) = q$$

$$p_{11}^{(2)} = p(E_1 \rightarrow E_2 \rightarrow E_1) = pq$$

$$p_{11}^{(4)} = p(E_1 \quad E_2 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1) = p^2q^2$$

etc.

$$t = \text{H.C.F. of } \{1,2,4,\ldots\} = 1.$$

Therefore the chain is aperiodic .

This chain can be periodic if,

$$p_{11} = o$$

and

$$p_{12} = 1.$$

Problem 4:

Given the transitional probability matrix

$$\underline{P} = \begin{bmatrix} \alpha_{oo} & \alpha_{o1} \\ \alpha_{10} & \alpha_{11} \end{bmatrix}$$

Find $\underline{P}^n$ using the relationship,

$$P(s) = \frac{1}{1 - F(s)}$$

where

$$P(s) = \sum_{n=o}^{\infty} p^n_{jj} s^n$$

and

$$F(s) = \sum_{n=o}^{\infty} f^n_{jj} s^n.$$

Solution:

P^n is the coefficient of s^n in $P(s)$.

Now,

$$f^{(o)}_{oo} = 0$$

$$f_{oo}^{(2)} = p(E_o \to E_1 \to E_o) = \alpha_{01} \cdot \alpha_{10}$$

$$f_{oo}^{(1)} = p(E_o \to E_o) = \alpha_{oo}$$

$$f_{oo}^{(3)} = p(E_o \to E_1 \to E_1 \to E_o) = \alpha_{01} \cdot \alpha_{11} \cdot \alpha_{10}$$

$$f_{oo}^{(4)} = p(E_o \to E_1 \to E_1 \to E_1 \to E_o) = \alpha_{01} \cdot \alpha_{11}^2 \cdot \alpha_{10}$$

$$f_{oo}^{(5)} = p(E_o \to E_1 \to E_1 \to E_1 \to E_o) = \alpha_{01} \cdot \alpha_{11}^3 \cdot \alpha_{10}$$

$$\vdots$$

$$f_{oo}^{(n)} = p(E_o \to E_1 \to \ldots\ldots\ldots E_1 \to E_o) = \alpha_{01} \cdot \alpha_{11}^{n-2} \cdot \alpha_{10}, \; n \geq 2.$$

Therefore,

$$F_{oo}(s) = \sum_{n=o}^{\infty} f_{oo}^{(n)} s^n$$

$$= f_{oo}^{(o)} + f_{oo}^{(1)} s + f_{oo}^{(2)} s^2 + \ldots\ldots\ldots\ldots$$

$$= f_{oo}^{(o)} + f_{oo} s + \sum_{n=2}^{\infty} f_{oo}^{(n)} s^n$$

$$= \alpha_{oo} s + \sum_{n=2}^{\infty} \alpha_{01} \alpha_{11} \alpha_{10}^{n-2} s^n$$

$$= \alpha_{00}s + \alpha_{01} \alpha_{10}s^2 \sum_{n=2}^{\infty} \alpha_{11}^{n-2} s^{n-2}$$

$$= \alpha_{00}s + \alpha_{01} \alpha_{10} s^2 \sum_{n=2}^{\infty} \alpha_{11}^{n-2} s^{n-2}$$

$$= \alpha_{00}s + \alpha_{01} \alpha_{10} \sum_{n=2}^{\infty} (\alpha_{11}s)^{n-2}$$

$$= \alpha_{00}s + \alpha_{01} \alpha_{10} s^2 \left(\frac{1}{1-\alpha_{11} s}\right)$$

$$= \frac{\alpha_{00} s (1 - \alpha_{11}s) + \alpha_{01} \cdot \alpha_{10} s^2}{1 - \alpha_{11}s}$$

$$= \frac{\alpha_{00}s - \alpha_{00} \cdot \alpha_{11}s^2 + \alpha_{01} \cdot \alpha_{10}s^2}{1 - \alpha_{11} s^2}$$

In the stochastic matrix we have

$\alpha_{00} + \alpha_{01} = 1,$

which implies,

$\alpha_{00} = 1 - \alpha_{01}$

and

$\alpha_{10} + \alpha_{11} = 1,$

which also implies,

$$\alpha_{11} = 1 - \alpha_{10}$$

Therefore,

$$F_{00}(s) = \frac{(1-\alpha_{01})s - (1-\alpha_{01})(1-\alpha_{10})s^2 - \alpha_{01}\,\alpha_{10}\,s^2}{1-(1-\alpha_{10})s}$$

$$= \frac{(1-\alpha_{01})s - (1-\alpha_{01}-\alpha_{10})s^2}{1-(1-\alpha_{10})s}$$

Therefore,

$$1 - F_{00}(s) = 1 - \frac{(1-\alpha_{01})s - (1-\alpha_{01}-\alpha_{10})s^2}{1-(1-\alpha_{10})s}$$

$$= \frac{1-(1-\alpha_{10})s - (1-\alpha_{01})s + (1-\alpha_{01}-\alpha_{10})s}{1-(1-\alpha_{10})s}$$

$$= \frac{1-(2-\alpha_{01}-\alpha_{10})s + (1-\alpha_{01}-\alpha_{10})s^2}{1-(1-\alpha_{10})s}$$

Substituting this expression in $P_{oo}(s)$, we get

$$P_{oo}(s) = \frac{1 - (1 - \alpha_{10})s}{1 - (2 - \alpha_{01} - \alpha_{10})s + (1 - \alpha_{01} - \alpha_{10})s^2}$$

$$= \frac{1 - (1 - \alpha_{10})s}{1 - s - (1 - \alpha_{01} - \alpha_{10})s + (1 - \alpha_{01} - \alpha_{10})}$$

$$= \frac{1 - (1 - \alpha_{10})s}{1 - s - (1 - \alpha_{01} - \alpha_{10})s\left[1 - s\right]}$$

$$= \frac{1 - (1 - \alpha_{10})s}{(1-s)\left[1 - (1 - \alpha_{01} - \alpha_{10})s\right]}$$

Writing this expression in partial fractions,

$$P_{oo}(s) = \frac{A}{1-s} + \frac{B}{1 - (1 - \alpha_{01} - \alpha_{10})s}$$

Therefore,

$$A\left[1 - (1 - \alpha_{01} - \alpha_{10})s\right] + B - Bs = 1-(1- \alpha_{10})s$$

Let ,

$$s = o.$$

then we get,

$$A + B = 1$$

Let,

$$s = 1,$$

then we get,

$$A\left[1 - (1 - \alpha_{01} - \alpha_{10})\right] = 1 - (1 - \alpha_{10}$$

$$A\ (\alpha_{01} + \alpha_{10}) = \alpha_{10}$$

Therefore,

$$A = \frac{\alpha_{10}}{\alpha_{10} + \alpha_{01}}$$

Since,

$$A + B = 1$$

then

$$B = 1 - \frac{\alpha_{10}}{\alpha_{10} + \alpha_{01}}$$

$$= \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}$$

Substituting the values of A and B in $P_{oo}(s)$ we get,

$$P_{oo}(s) = \frac{1}{\alpha_{01}+\alpha_{10}}\left[\frac{\alpha_{10}}{1 - s} + \frac{\alpha_{01}}{1 - (1 - \alpha_{01} - \alpha_{10})s}\right]$$

We know that $P_{oo}^{(n)}$ is the coefficient of s^n in the expression $P_{oo}(s)$.

Therefore,

$$P_{oo}^{(n)} = \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} + \frac{\alpha_{01}}{\alpha_{10}+\alpha_{10}}\ (1 - \alpha_{10} - \alpha_{10})^n$$

$$= \frac{1}{\alpha_{01}+\alpha_{10}}\left[\alpha_{10} + \alpha_{01}(1 - \alpha_{01}(1 - \alpha_{01} - \alpha_{10})^n\right]$$

We know that,

$$p_{oo}^{(n)} + p_{o1}^{(n)} = 1$$

Therefore,

$$p_{o1}^{(n)} = 1 - p_{oo}^{(n)}$$

$$= 1 - \frac{1}{\alpha_{01}+\alpha_{10}}\left[\alpha_{10} + \alpha_{01}(1 - \alpha_{01} - \alpha_{10})^n\right]$$

$$= \frac{\alpha_{01} + \alpha_{10} - \alpha_{10} - \alpha_{01}(1 - \alpha_{01} - \alpha_{10})^n}{\alpha_{01} + \alpha_{10}}$$

$$= \frac{\alpha_{01} - \alpha_{01}(1 - \alpha_{01} - \alpha_{10})^n}{\alpha_{01} + \alpha_{10}}$$

$$= \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}\left[1 - (1 - \alpha_{01} - \alpha_{10})^n\right]$$

Next we find $P_{11}^{(n)}$.

We can show from the transition probability matrix above that,,

$$f_{11}^{(n)} = \alpha_{10}\,\alpha_{11}^{n-2}\alpha_{01} \quad , \quad n \geq 2.$$

If we compare $f_{11}^{(n)}$ and $f_{oo}^{(n)}$ we can see that

α_{01} and α_{10} and α_{oo} and α_{11} are replaceble .

Therefore

$$p_{11}^{(n)} = \frac{1}{\alpha_{10+}\ \alpha_{10}}\left[\alpha_{01} + \alpha_{10}\,(1 - \alpha_{01} - \alpha_{10})^n\right]$$

and

$$p_{10}^{(n)} = \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}}\left[1 - (1 - \alpha_{10} - \alpha_{01})^n\right]$$

Now,

$$\underline{p}^n = \begin{bmatrix} p_{11}^{(n)} & p_{01}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} \end{bmatrix}$$

$$\frac{1}{\alpha_{01} + \alpha_{10}}\begin{bmatrix} \alpha_{10} + \alpha_{01}\,(1 - \alpha_{01} - \alpha_{10})^n & \alpha_{01}(1 - (1- \alpha_{10} - \alpha_{10})^n \\ \alpha_{10}(1 - (1- \alpha_{01} - \alpha_{10})^n & \alpha_{01} + \alpha_{10}(1\ \alpha_{01} - \alpha_{10})^n \end{bmatrix}$$

<u>Problem 5:</u>

A student takes a 3-year diploma course. Each year he sits an examination to decide whether he has passed the year's course or not. If he passes he moves up and leaves the college at the end of the third stage. If he fails he repeats the year's course. The probabilities of his passing the various examinations are (i) 0.8 in the first one (ii) 0.7 in the second one and (iii) 0.5 in the third one.

Let E_1, E_2, E_3, E_4 represent the states 1st year, 2nd year, 3rd year and left college respectively.

(a) Write down the transition matrix for the year - year movement of the student.

(b) Determine the probability that the student is in state E_2 after his second examination.

(c) Determine the mean and variance of the number of years that a student with these transition probabilities spends at the college.

(d) Find the matrix

$$\underline{B} = \underline{N}.\underline{R}$$

and interpret the result.

Solution:

(a) The transition matrix is

$$
\underline{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array}
\begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\
\begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}
$$

(b) From the transition matrix the probability that the student is in state E_2 after his second examination is

$$p(E_2 \rightarrow E_2) = 0.3$$

(c) To be able to determine the variance and mean number of years that this students spends in college we need to put the transitional matrix in the canonical form:

$$
\underline{P} = \begin{array}{c} \\ E_n \\ E_1 \\ E_2 \\ E_3 \end{array}
\begin{array}{c} \begin{array}{cccc} E_4 & E_1 & E_2 & E_3 \end{array} \\
\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0.5 & 0 & 0 & 0.5 \end{array}\right] \end{array}
$$

The matrix

$$\underline{Q} = \begin{bmatrix} 0.2 & 0.8 & 0 \\ 0 & 0.3 & 0.7 \\ 0 & 0 & 0.5 \end{bmatrix}$$

denotes transitions from one transient state to another. Now,

$$(\underline{I} - Q) = \begin{bmatrix} 0.8 & -0.8 & 0 \\ 0 & 0.7 & -0.7 \\ 0 & 0 & 0.5 \end{bmatrix}$$

The fundamental matrix,

$$\underline{N} = (I - \underline{Q})^{-1}$$

$$= ((E(n_{ij}))),$$

gives the average number of times the student is in state E_j starting state E_i.

$$\underline{N} = \frac{\text{Adj } (I-Q)}{\det (I-Q)}$$

An adjoint of a square matrix is the transposed matrix of its co factors.

Cofactors of (I - Q)

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 0.7 & -0.7 \\ 0 & 0.5 \end{vmatrix}$$

$$= 0.35$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & -0.7 \\ 0 & 0.5 \end{vmatrix}$$

$$= -1 \times 0 = 0.$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 0.7 \\ 0 & 0 \end{vmatrix}$$

$$= 0.$$

$$A_{21} = (-1)^{1+2} \begin{vmatrix} -0.8 & 0 \\ 0 & 0.5 \end{vmatrix}$$

$$= 0.40$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 0.8 & 0 \\ 0 & 0.5 \end{vmatrix}$$

$$= 0.40$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 0.8 & -0.8 \\ 0 & 0 \end{vmatrix} = 0.$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -0.8 & 0 \\ 0.7 & -0.7 \end{vmatrix}$$

$$= 0.56$$

$$A_{32} = (-1)^5 \begin{vmatrix} 0.8 & 0 \\ 0 & -0.7 \end{vmatrix}$$

$$= 0.56$$

$$A_{33} = (-1)^6 \begin{vmatrix} 0.8 & -0.8 \\ 0 & 0.7 \end{vmatrix}$$

$$= 0.56$$

$$\mathrm{Adj}(I-Q) = \begin{bmatrix} 0.35 & 0.40 & 0.56 \\ 0 & 0.40 & 0.56 \\ 0 & 0 & 0.56 \end{bmatrix}$$

$$\mathrm{Det}(I-Q) = 0.8(0.35) + 0.8(0) + 0.$$

$$= 0.280$$

Now,

$$(I-Q)^{-1} = \frac{1}{0.28} \begin{bmatrix} 0.35 & 0.40 & 0.56 \\ 0 & 0.40 & 0.56 \\ 0 & 0 & 0.56 \end{bmatrix}$$

The mean number of years spend by the student,

$$E(n_{ij}) = \frac{1}{0.28} \begin{bmatrix} 0.35 & 0.40 & 0.56 \\ 0 & 0.40 & 0.56 \\ 0 & 0 & 0.56 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{35}{28} + \frac{40}{28} + \frac{56}{28} \\ \frac{40}{28} + \frac{56}{28} \\ \frac{56}{28} \end{bmatrix} = \begin{bmatrix} \frac{131}{28} \\ \frac{96}{28} \\ \frac{56}{28} \end{bmatrix}$$

This means:

(i) the student spends an average of $\frac{131}{28}$ years in the college once he is in 1st year.

(ii) Once the student is 2nd year he spends $\frac{96}{28}$ in college before going out.

(iii) Once the student is in 3rd year he spends 2 years in college before leaving.

$$\text{Var}(t) = (2\,\underset{\sim}{N} - I)\underline{\tau} - \tau_{sq}$$

where,

$$\underline{N} = \frac{1}{28} \begin{bmatrix} 35 & 40 & 56 \\ 0 & 40 & 50 \\ 0 & 0 & 56 \end{bmatrix}$$

$$\tau = \begin{bmatrix} \frac{139}{28} \\ \frac{96}{28} \\ 2 \end{bmatrix}$$

and,

$$\tau_{sq} = \frac{1}{28^2} \begin{bmatrix} 139^2 \\ 96^2 \\ 56^2 \end{bmatrix}$$

$$\text{Variance} = \frac{1}{28} \left(\begin{bmatrix} 70 & 80 & 112 \\ 0 & 80 & 112 \\ 0 & 0 & 112 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \frac{139}{28} \\ 96/28 \\ 2 \end{bmatrix} - \frac{1}{28^2} \begin{bmatrix} 139^2 \\ 96^2 \\ 56^2 \end{bmatrix}$$

$$= \frac{1}{28^2} \begin{bmatrix} 42 & 80 & 112 \\ 0 & 52 & 112 \\ 0 & 0 & 84 \end{bmatrix} \begin{bmatrix} 139 \\ 96 \\ 56 \end{bmatrix} - \frac{1}{28^2} \begin{bmatrix} 139^2 \\ 96^2 \\ 56^2 \end{bmatrix}$$

$$= \frac{1}{28^2} \begin{bmatrix} 42 \times 139 + 80 \times 96 + 112 \times 56 - 139 \times 139 \\ 52 \times 96 + 112 \times 56 - 96 \times 96 \\ 84 \times 56 \quad - 56 \times 56 \end{bmatrix}$$

$$\frac{1}{28^2} \begin{bmatrix} 459 \\ 2048 \\ 56 \times 28 \end{bmatrix}$$

(d) $\underline{B} = \underline{N}.R$

$$= \frac{1}{28} \begin{bmatrix} 35 & 40 & 56 \\ 0 & 40 & 56 \\ 0 & 0 & 56 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

This means that the student will eventually leave college .

<u>Problem 6:</u>

E_1, E_2, E_3, E_4 are four points in a circle and a boy steps from one to the other according to the following transition matrix.

$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ p & 0 & q & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(a) Describe his movements assuming he starts in E_2.

(b) Determine $\underline{N}$, $\underline{N}_2$, $\underline{\tau}$, τ_2 and $\underline{B}$

(c) If the starting point is E_2, what are:

(i) the mean and variance of the number of times the boy is at point E_3.

(ii) the mean and variance of the number of steps the boy takes before arriving in E_1 or E_4.

(iii) the probability that the boy is absorbed in E_4 ? (iv) Prob. of boy being absorbed in E_1 ?

Solution:

(a) From state E_2 the boy can go to E_1 where he stops or he can go to state E_3. From state E_3 he can go to state E_2 where he repeats the movement as described above or he can go to E_4 where he stops.

(b) We express the matrix in canonical form as follows:

$$P = \begin{array}{c} \\ E_1 \\ E_4 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_4 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 0 & q \\ 0 & q & p & 0 \end{bmatrix} \end{array}$$

From the matrix we get

$$Q = \begin{bmatrix} 0 & q \\ p & 0 \end{bmatrix}$$

$\underline{N} = (I-Q)^{-1}$

$$= \left(\frac{1}{1-pq}\right) \begin{array}{c} \\ E_2 \\ E_3 \end{array}\!\!\begin{array}{c} E_2 \quad\ E_3 \\ \begin{bmatrix} 1 & q \\ p & 1 \end{bmatrix} \end{array}$$

$$N_2 = (2\,\underline{N}_{dg} - I)\,\underline{N} - \underline{N}_{sq}$$

$$= \left(\frac{1}{1-pq}\right)\begin{bmatrix} 1 & q \\ p & 1 \end{bmatrix}\left(\frac{2}{1-pq}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) - \frac{1}{(1-pq)^2}\begin{bmatrix} 1 & q^2 \\ p^2 & 1 \end{bmatrix}$$

$$= \left(\frac{1}{1-pq}\right)\begin{bmatrix} 1 & q \\ p & 1 \end{bmatrix}\begin{bmatrix} \frac{1+pq}{1-pq} & 0 \\ 0 & \frac{1+pq}{1-pq} \end{bmatrix} - \frac{1}{(1-pq)^2}\begin{bmatrix} 1 & q^2 \\ p^2 & 1 \end{bmatrix}$$

$$= \left(\frac{1}{1-pq}\right)^2\begin{bmatrix} 1+pq & q(1+pq) \\ p(1+pq) & 1+pq \end{bmatrix} - \frac{1}{(1-pq)^2}\begin{bmatrix} 1 & q^2 \\ p^2 & 1 \end{bmatrix}$$

$$= \left(\frac{1}{1-pq}\right)^2 \begin{array}{c} E_2 \qquad\qquad E_3 \\ \begin{bmatrix} pq & q(1-q^2) \\ p(1-p^2) & pq \end{bmatrix} \end{array}\begin{array}{c} \\ E_2 \\ E_3 \end{array}$$

$\underline{\tau} = \underline{N}.\underline{C}$

$$= \frac{1}{1-pq}\begin{bmatrix} 1 & q \\ p & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$= \left(\frac{1}{1-pq}\right) \begin{bmatrix} 1+q \\ 1+p \end{bmatrix}$$

$$\tau_2 = (2\underline{N} - \underline{I})\underset{\sim}{\tau} - \tau_{sq}$$

$$\frac{2}{1-pq} \begin{bmatrix} 1 & q \\ p & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{1-pq} \begin{bmatrix} 1+q \\ 1+p \end{bmatrix} - \frac{1}{(1-pq)^2} \begin{bmatrix} (1+q)^2 \\ (1+p^2 \end{bmatrix}$$

$$= \left(\frac{1}{1-pq}\right)^2 \begin{bmatrix} 1+pq & 2q \\ 2p & 1+pq \end{bmatrix} \begin{bmatrix} 1+q \\ 1+p \end{bmatrix} - \left(\frac{1}{1-pq}\right)^2 \begin{bmatrix} 1+q)2 \\ 1+p)^2 \end{bmatrix}$$

$$= \left(\frac{1}{1-pq}\right)^2 \begin{bmatrix} q + 3pq - q^3 \\ p + 3pq - p^3 \end{bmatrix}$$

$$\underline{B} = \underline{N}.\underline{R}$$

$$= \left(\frac{1}{1-pq}\right) \begin{bmatrix} 1 & q \\ p & 1 \end{bmatrix} \cdot \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

$$= \left(\frac{1}{1-pq}\right) \begin{bmatrix} p & q^2 \\ p^2 & q \end{bmatrix}$$

(c) If the boy starts at state E_2,

(i) mean number of times the boy will be in E_3 is

$$E(n_{23}) = \frac{q}{1 - pq}$$

This is found the entries of $\underline{N}$.

The variance of the number of times that the boy is at E_3 starting from E_2 is given by the entries of $\underline{N}_2$.

$$Var(n_{22}) = \frac{q(1-q^2)}{1-pq)^2}$$

(ii) the mean number of steps that the boy takes before arriving in E_1 or E_4 is given by entries of $\underline{\tau}$.

Therefore the boy takes

$$E(n_{21}) = (\frac{1+q}{1-pq}) \text{ steps before entering } E_1.$$

The boy takes

$$E(n_{24}) = (\frac{1 + p}{1 - p})$$

before entering E_4

The variances are given by entries of τ_2.

The variance of the number of steps before entering E_1 is

$$\text{Var}(n_{21}) = \frac{q + 3pq - q^3}{(1 - pq)^2}$$

and

$$\text{Var}(n_{24}) = \frac{p + 3pq - p^3}{(1 - pq)^2}$$

(iii) The probability that the boy is absorbed in E_4 starting from E_2 is

$$b_{24} = \left(\frac{q^2}{1 - pq}\right)$$

(iv) Probability that the boy is absorbed in state E_1 given that he started at E_2 is

$$b_{21} = \left(\frac{p}{1 - pq}\right)$$

BIBLIOGRAPHY

1. Bailey, N.T.J. "The elements of stochastic Process", New York, John Wiley, 1964.

2. Childress, Robert L, "Fundamental Mathematics" Prentice-Hall, Inc. Englewood. Cliffs, New Jersey.

3. Feller, W. "An introduction to probality Theory and its applications" Vol. 1, John Wiley, New York, 1968.

4. Kemeny, John G., "Finite Markov Chains" D. Van Nostrand Company Inc., Princeton, New Jersey, 1963.

5. Medhi, J. "Stochastic Processes" New Dheli, Wiley, 1982.