FRAILTY MODELS

APPLICATIONS IN PENSION SCHEMES

BY

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DECLARATION

This project is my original work and has never been presented in any learning institution for any academic award.

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DEDICATION

I would like to dedicate this project to my family, who have given me every reason and opportunity to be happy and to all actuaries who have worked to fairly price insurance products in the light of uncertainty.

ACKNOWLEDGEMENTS

First and foremost, I would like to acknowledge the pioneering work of Beard (1959) who is considered a forerunner in mortality modeling allowing for frailty as this formed a basis for this study.

Thank you to my project supervisor, Prof Jam, for providing me with the proper guidance throughout, and for allowing me to work on my own schedule. This project couldn’t be finished without his assistance. I attended all of his classes and they have been a great learning experience.

ABSTRACT

Heterogeneity in a population of assured lives in respect of mortality can be explained by differences among the individuals; some of these are observable, while others, for instance an individual's attitude towards health and/or all genetic factors having influence on survival are difficult to monitor and measure. This undermines usage of observable risk factors as the only rating factors for life insurance. Insurance companies have not taken proper care of unobservable risk factors possibly due to difficulties inherent in their modeling. This heterogeneity exposes insurers to adverse selection if only the healthiest lives purchase annuities, so standard annuities are priced with a mortality table that assumes above-average longevity. This makes standard annuities expensive for many individuals. To avoid biases in valuation a better understanding of heterogeneity in required.

Frailty models are extensions of the Cox proportional hazards model which is popular in survival studies. In many applications, the study population needs to be considered as a heterogeneous sample. Sometimes, due to lack of knowledge or for economical reasons, some covariates related to the event of interest are not measured. The frailty approach is a statistical modeling method which aims to account for the heterogeneity caused by unmeasured covariates. It does so by adding random effects which act multiplicatively on the hazard.

This study carries out an extensive review of frailty models and is aimed at extending this work by considering other distributions that can be used in modeling. In particular, the non-central gamma distribution is proposed for frailty modeling.

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CHAPTER 1

GENERAL INTRODUCTION

1.1 Background Information

The concept of frailty modeling is based on mixture distributions and survival analysis.

1.1.1 Finite mixtures

Mixture distributions consist of mixing a distribution with another. This can be achieved by taking k different distributions with probability densities $f_1(x), f_2(x), \ldots, f_k(x)$ with mixing weights: $w_1, w_2, \ldots, w_k$

Where $w_j > 0 \; and \; \sum_{j=1}^{k} w_j = 1$

The new density function or mass function is:

$$f(x) = \sum_{j=1}^{k} w_j f_j(x)$$

This is a finite mixture (Johnson et.al. 2005 p.344)

Instead of k different distributions we can have k different components of a distribution.

1.1.2 Varying parameter and unknown covariates

A mixture distribution also arises when the density/mass function of a random variable depends on a parameter.

Consider a random variable x depending on its parameter θ then the conditional probability density function can be written as:

$$f(x) = \int f(x|\theta)g(\theta)\, d\theta$$

Johnson (2005, p.345) states that this mixture distribution includes a situation where the source of a random variable is unknown. Thus instead of considering a parameter θ , we consider an unknown covariate.

1.1.3 Survival analysis

Survival analysis is a branch of statistics which deals with time to the occurrence of a given event of interest. For insurers this event could be time to death, ill health or retirement. It's different from other fields of statistics in that we are observing something that develops dynamically over time and takes censoring into consideration which is partial information about the variable of interest.

Three important functions of time to the event are:

- The survival function, $S(t)$, describes the probability that an individual survives longer than time t.
- The probability density function, $f(t)$
- The hazard function, $h(t)$, describes the instantaneous death rate

Estimating the survival function using non-parametric methods such as the Kaplan-Meier technique leads to obtaining the median time to the event under investigation.

Determining factors affecting the hazard, the Cox PH model has been widely used.

1.2 Problem Statement

The Cox PH model is given by:

$$h_i(t) = h_o(t)\, e^{\beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k}$$

Where:

$h_i(t)$ = is the hazard function at time t

$h_o(t)$ = is the baseline hazard function at time t

$\beta's$ = are the regression coefficients.

X's = are vector of covariates.

One reason why this model is so popular is because of the ease with which technical difficulties such as censoring and truncation are handled. This is due to the appealing interpretation of the hazard function as a risk that changes over time.

The concept allows for the entering of covariates in order to describe their influence and to model different levels of risk for different sub-groups.

However, we may not know all relevant risk factors. It may also be costly to measure them.

Ordinary life tables assume that populations are homogeneous implying that all individuals have the same risk. Yet in reality we have a mixture of individuals with different risks. Frailty models tackle such issues.

In an insurance setting ignoring heterogeneity could lead to model error i.e. using the wrong life tables to price insurance products which could be costly to both the insurer and the insured.

1.3 Research Objectives

Main Objective:

The main objective of the study is to review univariate frailty models.

Specific Objectives:

- Extend the work by considering other distributions that can be used in modeling.
- To construct the hazard function when the insured population is considered to be heterogeneous.
- Propose the non-central gamma distribution to represent heterogeneity in an insurance setting.

1.4 Significance of the study

Risks within an insurance contract are heterogeneous. Therefore, to come up with premiums that are consistent with the insured risk all relevant factors affecting mortality need to be considered. Buhlmann (1970) developed models to accounting for features of the insured risk in non-life insurance which is unknown but relevant to explain overall claim frequencies. In another context, Olivieri (2006) has applied frailty models to pensions and life annuities to account for unobserved heterogeneity.

1.5 Outline

This thesis is organized as follows: Chapter 1 gives a general introduction to survival analysis and mixture distributions, chapter 2 is an introduction to different aspects of frailty modeling and chapter 3 reviews distributions that have been on frailty models. In chapter 4 Gompertz model parameters are estimated using R program and in chapter 5 the proposed model properties are discussed. Chapter 6 describes an application of frailty to life insurance and chapter 7 is on conclusions and recommendation for further research. Finally, the reference materials used in the study are listed.

1.6 Review of the Literature

Estimating the survival function

Existing literature on estimating the survival function includes the parametric, semi-parametric and non-parametric methods. Frailty models are extension of Cox-proportional hazard model where the relative risk is replaced with a random variable called the ‘frailty term’.

1.6.1 Parametric Methods

For parametric inference, it is necessary to make assumptions about the distribution of failure times. Parametric approaches such as Weibull, lognormal, exponential, etc can be used to estimate the survival function for homogeneous populations. Basically, any distribution of non-negative random variables can be used.

1.6.2 Non-Parametric Methods

Non-parametric approaches such as Kaplan-Meier (1958) and Aalen-Nelson (1978) can be used to estimate the survival function when assumption of the failure time distribution is to be avoided. An advantage of non-parametric models is their good fit and their ability to deal with any distribution without any additional assumptions.

Important consideration when estimating the hazard function is to investigate the relation between the survival time and some risk factors (covariates). These risk factors might be fixed variables, or they may change over time. Examples include; age, gender, socio-economic status, education, blood pressure, body mass index, smoking habits, nutrition, physical activity level, heart rate and so forth. Their influence on the survival is of great interest for insurers and can be estimated by statistical models.

The effect of X_i can be either parameterized as proportional hazards (PH) or accelerated failure time (AFT).

PH assumes $h(t_i) = h_o(t_i)exp\{x_i'\beta\}$ for some baseline hazard $h_o(t_i)$

AFT assumes $S(t_i) = S_o(exp\{-x_i'\beta\}t_i)$ for some baseline survival function $S_o(t)$

Parametric survival models assume some function for $h_o(t)$ and hence for $S_o(t)$

1.6.3 Semi-Parametric Methods

Cox proportional hazard model (1972)

A Cox model is a technique for exploring the relationship between the survival time of an individual and several explanatory variables. The hazard function for each individual is proportional to the baseline hazard $h_o(t)$ and thus the hazard is fully determined by the covariate vector. The hazard function for individual i at time (age) t is written as:

$$h_i(t) = h_o(t)\, e^{\beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k}$$

$h_o(t)$ is the baseline hazard function and corresponds to the probability of dying (or reaching an event) when all the explanatory variables are zero. In this model it is left unspecified. $exp(\beta' X_i)$ is the relative risk of individual i, where $x_i = (x_1, \dots, x_k)$ are vector of covariables and $\beta' = (\beta_1, \dots, \beta_k)$ are vector of regression coefficients that give the proportional change that can be expected in the hazard, related to changes in the explanatory variables.

Cox (1972) proposed the partial likelihood method to estimate the β parameter of this model. The partial likelihood is a product over the uncensored failure times written as:

$$L(\beta) = \prod_{Yi\ uncensored} \frac{exp\ (\beta' X_i)}{\sum_{Yj \geq Yi} exp\ (\beta' X_j)}$$

Each factor can be interpreted as the conditional probability that individual i dies at time ti, given the risk set R_i. The first and second derivatives of the log likelihood of the model can be derived. Parameter estimates can then be obtained by maximizing $L(\beta)$

The log partial likelihood is given by

$$l(\beta) = \log\big(L(\beta)\big) = \sum_{Yi\, uncensored} \{\beta' X_i - log \sum_{Yj \geq Yi} exp\,(\beta' X_j)\}$$

1.6.4 Frailty Models

Vaupel et al. (1979) introduced the term frailty and used it in individual survival models. Clayton (1978) promoted the model by its application to multivariate situations. Ordinary life table methods implicitly assume that the population under study is homogenous. This means that all individuals in that study are subject under the same risk (e.g., risk of death, risk of accident). Basic observation of medical statistics shows that individuals differ greatly. Thus, the study population cannot be assumed to be homogeneous but must be considered as a heterogeneous sample.

A random effect model takes into account the effects of unobserved or unobservable heterogeneity, i.e. an individual's attitude towards health or some genotypic personal characteristics. Thus, the role of "frailty" is to include all unobservable factors acting on the individual mortality. The random effect denoted by Z is the term that describes the individual heterogeneity.

Differential Mortality Models

The Multiplicative approach.

Vaupel et al. (1979) described the model as

$h(x|z) = zh_o(x)$

$h_o(x)$ is the baseline mortality considered to be a known function of x that is to be specified. The frailty, Z is meant to quantify uncertainty associated with the hazard rate which acts in a multiplicative manner

The Aalen Additive Model.

Aalen (1980;1989) described a nonparametric additive hazard model given by

$h(x|Z) = h_o(x) + \beta Z \qquad \beta > 0$

This model is useful in dealing with right censored survival data, especially in the presence of time-varying covariates.

Age Shifting Model.

$h'(x) = h(x + z)$

The age shift model was proposed by Humphreys (1874). Who argued that the mortality experience of a group of impaired lives accepted for life insurance should have an increased premium rating determined by assuming that the insured's age is higher than the real current age, hence adopting the "age shift".

Constant mortality model.

$h'(x) = h(x) + b\ , b > 0$

Mortality increase is constant and independent of the initial age; such model is consistent with extra-mortality due to accidents (related either to occupation or to extreme sports).

TYPES OF FRAILTY MODELS

1.6.5 Frailty models without observed covariates.

This model is used when only survival data is available for the analysis, or when additional information is of no interest.

$h(t, Z) = Zh_o(t)$

The non-negative random variable Z is called frailty and $h_o(t)$ is the baseline hazard. This model is non-identifiable from survival data, since different combinations of $h_o(t)$ and frailty distributions may produce the same marginal hazard rate $h(t)$. The model becomes identifiable when the parametric structure of $h_o(t)$ is fixed and Z is assumed to belong to some parametric distribution family.

1.6.6 Frailty models with observed covariates.

If covariates are known, they can be included in the analysis; however it is nearly impossible to include all factors. Therefore, the individual hazard function with frailty factor Z and covariates X is given by

$h(t, x|Z) = Z\, h_o(t)\ e^{\beta' X}$

The conditional survival function is;

$s(t, x|Z) = e^{-ZHo(t)e^{\beta' X}}$

β is a vector of regression coefficients characterizing the measure of influence of X on the hazard rate. X and Z are assumed to be independent. However, incorporation of unobservable factors (such as frailty) into Cox PH models poses theoretical difficulties in the estimation and inference procedures (Therneau and Grambsch, 2000).

1.6.7 Univariate Frailty Models

This model describes the influence of unobserved covariates in a proportional hazards model for independent lifetimes. The variability can be split into a part that depends on observable risk factors, and is therefore theoretically predictable, and a part that is theoretically unpredictable, even when all relevant information is known. This model has been used by Hougaard (1991) to show that these two sources of variability can explain some unexpected results or gives an alternative explanation of some results.

1.6.8 Bivariate Frailty Models

Bivariate frailty models are used to analyze the effects of dependence between life spans of two related individuals with random effect. This model estimates the impact of dependence on the regression coefficients of the Cox-proportion hazard model

(Clayton, 1978; Hougaard, 1995).

The bivariate survival function is given by:

$$S(t_1, t_2|Z) = S_1(t_1)^Z S_2(t_2)^Z$$

1.6.9 Multivariate Frailty Models

The aim for multivariate analysis is to account for the dependence in clustered event times. A natural way to model dependence of clustered event times is through the introduction of a cluster-specific random effect - the frailty. This random effect explains the dependence in the sense that had we known the frailty, the events would be independent. This approach can be used for survival times of related individuals like family members, parent-child, twins) or recurrent observations on the same person.

Clayton (1978) used this approach to model statistical dependence for clustered events.

The survival function is given by;

$$S(t_1, t_2, \ldots, t_n) = \int_0^{\infty} S(t_1 \mid z, X_1) S(t_2 \mid z, X_2), \ldots, S(t_n \mid z, X_n) g(z) dz$$

$$S(t_1, t_2, \ldots, t_n) = L_Z(H_{oi}(t_i))$$

1.6.10 Shared Frailty Models

This model was introduced by Clayton (1978) it is relevant to event times of related individuals or observations that are clustered into groups such as cities that are assumed to share the same frailty Z. The survival times are assumed to be conditional independent with respect to the shared (common) frailty.

The hazard function for a shared frailty model is given by:

$h_{ij}(t|Z_i) = Z_i \, h_o(t) \; e^{\beta' X_{ij}}$ Z_i is the random effect associated with the ith group

1.6.11 Correlated Frailty Models

In correlated frailty models, the frailty of each individual in a pair is defined by a measure of relative risk where two associated random variables are used to characterize the frailty effect for each pair. For example, one random variable is assigned for the husband and one for the wife so that they would no longer be constrained to have a common frailty. These two variables are associated and have a joint distribution. For two individuals in a pair, frailties are not necessarily the same, as they are in the shared frailty model. The hazard of individual *j* (*j* = 1*;* 2) in pair *i* (*i*= 1*; : : : ; n*) has the form

$$h(t) = Z_{ij} h_{oj}(t) e^{\beta' X_{ij}}$$

$h_{oj}(t)$are baseline hazard functions, and $Z_{ij}$are unobserved (random) effect or frailty.

Yashin et al. (1993, 1995) introduced the correlated gamma frailty model and applied to related lifetimes.

1.6.12 Nested Frailty Models

Nested frailty models account for hierarchical clustering of the data by including two nested random effects that act multiplicatively on the hazard function. Such models are appropriate when observations are clustered at several hierarchical levels such as in geographical areas (Rondeau et al. 2006).

The hazard function is given by:

$$h_{ijk}(t|Z_i, U_{ij}) = Z_i \, U_{ij} h_o(t) \;\; e^{\beta' X_{ijk}}$$

The cluster random effect Z_i and the sub-cluster random effect U_{ij} are both independently and identically distributed.

1.6.13 Joint Frailty Models

Recurrent events across time for subjects in a study may be terminated by loss to follow-up, end-of-study, or a major failure event such as death. Here, the major failure event could be correlated with the recurrent events. Joint frailty models provides a way to study the joint evolution over time of two survival processes by considering the terminal event as informative censoring (Rondeau et al. 2007).

1.6.14 Discrete Frailty Models

The distribution of the frailty factor is normally assumed to be continuous. In some cases, it may be appropriate to express heterogeneity as a discrete mixture. Having zero frailty can be interpreted as being immune, and population heterogeneity may be analyzed using discrete frailty models. Continuous frailty distributions do not allow having zero risks. Nickell(1979) used the binary discrete model to account for heterogeneity in unemployment spell data. In this study however continuous frailty models will be considered.

Frailty Distributions

First, neither theory nor data typically provides much guidance for choosing a specific distribution from which to draw the frailty, thus any distribution with positive support and finite mean is suitable to represent the frailty distribution. However, for tractability reasons the choice of distribution is limited to those that provide a closed form expression for the frailty survivor function, density and hazard functions.

The choice of parametric distributions for Z is often a matter of computational convenience and it should be strictly positive support, since negative frailty leads to negative mortality rates. Some of the distributions considered in this study are:

- Gamma distribution Vaupel et.al (1979)
- Inverse-Gaussian distribution Manton et al (1986)
- Positive Stable distribution; Hougaard (1986)
- Lognormal distribution (McGilchrist and Aisbett, 1991).
- Compound Poisson distribution (Aalen 1988, 1992)
- Inverse Gamma distribution
- Reciprocal Inverse-Gaussian distribution
- Non Central Gamma distribution

Baseline Hazard Distributions

Two different approaches are possible. In the parametric case the baseline hazard is chosen in the class of parametric lifetime distributions. The model also works without any specification of the baseline hazard function. However, there has been no study or survival experiment, which restricts estimates for the parametric form of the baseline hazard. Baselines with monotone increasing hazards are often used because one is often interested in the life of a device in the period of its life when an aging process is in force. Models with monotone decreasing hazard functions are used less often but can have application in the study of early lifetimes of devices. Constant hazard functions can be used as baseline distributions to which other distributions are compared.

The baseline hazards considered in the study includes;

- The Gompertz model (1825) (Vaupel et.al. 1979)
- The Weibull distribution (Manton and Stellard, 1988),
- Exponential distribution
- Log-logistic distribution
- Lognormal distribution
- Exponential Power distribution
- Pareto distribution

1.7 Applications in Life Insurance

Frailty models are used in life insurance to represent heterogeneity in a population due to unobservable risk factors. Heterogeneity due to observable risk factors is addressed at policy issue during the underwriting process to ensure that each contract is assigned premium consistent with the insured risk. Neglecting such factors may lead to biased valuation of insurance products.

Actuaries have developed models for valuing life insurance that only consider observable risk factors. However, in general insurance models accounting for unobservable risk has been developed to explain overall claim frequency i.e. the Poisson-Gamma model.

1.7.1 Life Insurance

Life insurance contracts with benefits contingent on the lifetime of an individual and whose benefit is stated in advance is considered. Heterogeneity can be classified as emerging from observable risk factors (at issue) i.e. age, sex, health status, profession, smoke habits, sport activities, and so on. Or unobservable risk factors like an individual's attitude towards risk.

For immediate annuities, the relation of premium and annuity depends on the health of the insured at the time the contract is taken out. However, in deferred annuities (pension schemes), the insurer has to perform some kind of underwriting at the end of the deferment period.

The underwriting process:

The purpose of underwriting is to assign each insured a frailty factor $\tilde{Z}$ as an estimate of Z to determine the pricing mortality rates. These underwriting factors are observable characteristics, such as smoking status, that explain mortality heterogeneity. Underwriting is done to ensure that premiums and benefits are fairly priced.

The tests carried out during the underwriting process are based on;

- Biological and physiological factors, such as age, gender, genotype;
- Features of the living environment; in particular: climate and pollution, nutrition standards population density, hygienic and sanitary conditions;

- Occupation, in particular in relation to professional disabilities or exposure to injury, and educational attainment;
- Individual lifestyle, in particular with regard to nutrition, alcohol and drug consumption, smoke, physical activities and pastimes;
- Current health conditions, personal and/or family medical history, civil status, and so on.

This assessment can be performed through proper questions in the application form and as to health conditions through a medical examination.

Modeling unobservable factors:

In addition to observable factors, heterogeneity may be caused by unobservable individual-specific factors, referred to as frailty. Frailty models may provide an appropriate description of the age-specific mortality shape, as well as the estimate of parameters of the relevant models according to mortality observed within the portfolio. However, there is no data available that can be linked to the choice of the distribution of Z since it is unobserved

1.7.2 Pension Schemes

Let x be the age at entry (time 0), N_t the number of annuitants at time $t, t \geq 0$; at the valuation time $T, T \geq 0$, the number N_T is known, while N_t is random for $t > T$. The amount of benefit at time t is denoted by δ_t. Assuming a deterministic financial setting, the short interest rate at time t is assumed to be deterministic, thus $\delta_t = \delta$ for any t.

The value at time T of one monetary unit at time $t, t > T$, is $e^{-\delta(t-T)}$

In case the population is considered homogeneous the future lifetimes $\{Ti_{(x+t)}\ ;\ i\ =\ 1, 2, \ldots, n\}$ are independent and identically distributed.

If the population is heterogeneous, then the future lifetimes are correlated through Z_{x+t} and can be assumed conditionally independent and identically distributed. This means dependence between survival times is only due to unobservable covariates or frailty.

CRITICAL LITERATURE REVIEW:

2.1 Probability Tools

Some of the common probability tools used in survival analysis that will be used in the study are described below. Let T_x be the future lifetime variable i.e. the remaining duration of life of a person aged x, which is a positive real valued variable, having a continuous distribution with finite expectation. Several functions characterize the distribution of T_x:

- $f_x(t), t \geq 0$ is the probability density of T_x;
- $S_x(t) = P(T_x > t) = \int_t^\infty f_x(x)dx = 1 - F_x(t)$ is the survival function, which is sometimes denoted with ${}_tP_x$
- $F_x(t) = P(T_x \leq t)$ Expresses the probability of dying within t years for a person age x and is denoted by ${}_tq_x$
- $h(t) = \frac{f(t)}{S(t)} = lim_{\delta t \to 0} \frac{P(t \leq T < t + \delta t | T \geq t)}{\delta t} = \frac{-\partial S(t)/\partial t}{S(t)}$ is the hazard function, which represents the probability that an individual alive at t experiences the event in the next period σt. (also called the instantaneous death rate)
- $H(t) = \int_0^t h(x)dx$ is the cumulative hazard function

2.1.1 Relationships between $f(t)$, $h(t)$ and $s(t)$

Let $h(t)$ denote the hazard function, defined by

$$h(t) = \lim_{(dt \to 0)} \frac{pr(t < T \leq t + dt \mid T > t)}{dt}$$

T is nonnegative and represents the future lifetime of an individual

$$h(t) = \lim_{(dt \to 0)} \frac{pr(t < T \leq t + dt \mid T > t)}{prob(T > t) * dt}$$

$$h(t) = \lim_{(dt -> 0)} \frac{pr(t < T \leq t + dt)/dt}{prob(T > t)}$$

$$h(t) = \frac{f(t)}{1 - F(t)}$$

$$h(t) = \frac{f(t)}{S(t)}$$

By definition;

$$S(t) = 1 - F(t)$$

$$f(t) = F'(t) = -S'(t)$$

$$Substitute\ in\ (1)\ h(t) = \frac{-S'(t)}{S(t)}$$

$$h(t) = -\frac{d}{dt}\ ln(S(t))$$

$$-\int h(t)\ dt = ln\ s(t)$$

$$S(t) = exp(-\int h(t)\ dt)$$

$$S(t) = \exp(-H(t))$$

2.1.2 Laplace Transform

The Laplace transform is crucial in this study since it makes computations of the survival and hazard functions from the density function easy.

The Laplace transform of a random variable Z with density function $f(z)$ is given by;

$$L_Z(s) = E[e^{-sZ}]$$

$$L_Z(s) = \int e^{-sZ} f(z) dz$$

2.2 FRAILTY MODELS

Frailty models are extensions of the Cox-proportional hazards model. In many cases it is impossible to measure all relevant covariates related to the subject of interest, sometimes because of economical reasons or sometimes the importance of some covariates is still unknown.

The frailty approach aims to account for heterogeneity, caused by unmeasured covariates in the Cox-proportional model which is described by a mixture variable Z called frailty.

The Cox-proportional model is given by;

$h(t, x) = h_o(t)\, exp(\beta' X)$.

The hazard is modified to the frailty model by substituting the relative risk $exp(\beta' X_i)$ by a random variable Z which represents the unobserved covariates X_i i.e.

$h(t, z) = h_o(t) * Z$

The frailty Z is then assumed to follow some distribution with positive support and has a multiplicative effect on the baseline hazard function which is common to all individuals.

2.3 THE MULTIPLICATIVE MODEL

This model describes the population as a mixture and assumes that each individual correspond a frailty quantity Z, describing the individual's relative risk. The non-negative quantity z encompasses all other factors affecting mortality other than age.

The hazard at age x conditional on Z is assumed to be $Z\, h_o(x)$

I.e. $h(x|z) = z * h_o(x)$ where $h_o(x)$ is the ‘standard hazard function’ corresponding to a ‘standard individual’, conventionally those with frailty $z = 1$

Individuals with $Z > 1$ experience a force of mortality that is proportionally higher than $h(x)$ at all ages. Individuals with $Z < 1$ experience proportionally lower mortality rates. $Z = 1$Correspond to the standard hazard function.

The composition of a cohort with respect to the frailty Z changes as a cohort grows older because the more frail (susceptible) individuals tend to die earlier than the least frail individuals.

Due to the stochastic nature of Z, the random effect or frailty model is stochastic.

The survival function of an individual with frailty Z is given by

$$S(t|Z) = exp(-\int h(t|Z)\, dt)$$

$$= exp(-\int Zh_o(t)\, dt)$$

$$S(t|Z) = exp\{-ZH_o(t)\}$$

Since, the individual model $S(t|Z)$ is not observable as each individual Z is unobserved; it is ‘integrated out’ by specifying a distribution and obtaining the unconditional survival function.

The survival function of the total population is the mean of individual survival functions with respect to the frailty distribution. It can be viewed as the survival function of a randomly drawn individual, and corresponds to what can actually be observed.

Integrating over the range of frailty variable Z having density $f(z)$, we get marginal survival function representing the population as,

$$S(t) = \int S(t|Z)\, f(Z)\, dz$$

$$S(t) = E[S(t|Z)]$$

$$S(t) = E[exp\{-ZH_o(t)\}]$$

$S(t) = L_Z(H_o(t))$

$f(z)$ is the density of Z and $L_Z(s)$ is the Laplace transform of Z.

To obtain the marginal density function $f(t)$

Consider the relationship;

$$h(t|Z) = \frac{f(t|Z)}{s(t|Z)} = Zh_o(t)$$

$$f(t|Z) = Zh_o(t)S(t|Z)$$

Since, $f(t|Z) = \frac{f(t,Z)}{f(z)}$

$$f(t,Z) = Zh_o(t)S(t|Z)f(z)$$

Also, $f(t) = \int f(t,Z)f(z)\,dz$

$$f(t) = h_o(t)\int ZS(t|Z)f(z)\,dz \quad \dots\dots\dots\dots..(2)$$

$$= h_o(t)E[ZS(t|Z)]$$

$$f(t) = -h_o(t)L'(H_o(t))$$

2.3.1 Model Assumptions.

- The frailty Z has a multiplicative effect on the mortality rate of the individuals :

  $h(t;Z) = Zh_o(t)$

- The frailty Z_x is stationary. i.e. the frailty of an individual keeps constant throughout the whole lifetime span (but the probability distribution does depend on the age, and this justifies the suffix *x*)
- Z is distributed independent of age(x) or time(t)
- Z has a strictly positive support since negative hazards are impossible.

2.3.2 Individual Vs Population hazards

In frailty modeling the individual hazard rate increases over time while the population hazard obtained by averaging over all the survivors decreases. This is because the population becomes populated by more and more robust individuals as the frail members fail. In a homogeneous assumption, the population hazard is the same thing since all individuals are assumed to be identical. Whereas in a heterogeneous setting, it turns out that the population hazard can fall while the individual hazards all rise.

2.4 DISCRETE FRAILTY MODELS

There are some situations in which a discrete distribution may be appropriate. For example, heterogeneity in lifetime arises because of exposure to damage on a random number of occasions. Having zero frailty can be interpreted as being immune, and population heterogeneity may be analyzed using discrete frailty models. There are two kinds of discrete frailty models in the literature. One kind of discrete frailty model is constructed by separating the frailty into ones with fixed and random numbers of mass points. The second kind of discrete frailty model is based on a fixed number of components and with masses at integers.

CONSTRUCTION

Survival functions for the frailty model with discrete distributions:

The unconditional survival function for the discrete frailty distribution is given by;

$$S(t) = \sum_{z=0}^{\infty} S(t|Z)P(z)$$

$$S(t) = E[S(t|Z)]$$

$$S(t) = E[e^{(-ZH(t))}]$$

Z is a discrete random variable with the probability function $P\ (Z = z) = P\ (z)$.

Standard discrete distributions such as Geometric, Poisson and Negative Binomial distributions have been considered by Carolini et.al (2010)

2.4.1 GEOMETRIC DISTRIBUTION

If Z is a geometric-distributed random variable with the probability function

$$P(z) = p(1 - p)^z, z = 0,1,2...$$

The unconditional survival function is given by

$$S(t) = E[e^{(-ZH(t))}]$$

$$S(t) = \sum_{Z=0}^{\infty} e^{-ZH(t)} * p(1 - p)^z$$

$$S(t) = p\sum_{z=0}^{\infty} ((1-p)e^{-H(t)})^z$$

$let\ S_o(t) = e^{-H(t)}$ and $1 - p = q$ where $S_o(t)$ is the baseline survival function.

$$S(t) = p\sum_{z=0}^{\infty} (qS_o(t))^z$$

$$S(t) = p\{1 + qS_o(t) + (qS_o(t)) + (qS_o(t))^2 + (qS_o(t))^3 + \cdots\}$$

$$S(t) = \frac{p}{1 - qS_o(t)}$$

2.4.2 POISSON DISTRIBUTION

If Z is a Poisson distributed random variable with parameter μ > 0 and probability density function;

$$P(z) = \frac{\mu^z e^{-\mu}}{Z!} \quad z = 0,1,2,..$$

The unconditional survival function is given by;

$$S(t) = E[e^{(-ZH(t))}]$$

$$S(t) = \sum_{Z=0}^{\infty} e^{-ZH(t)} \frac{\mu^{z} e^{-\mu}}{Z!}$$

$$S(t) = e^{-\mu} \sum_{Z=0}^{\infty} \frac{\left(e^{-H(t)}\mu\right)^{z}}{z!}$$

$$S(t) = e^{-\mu} * e^{\left(e^{-H(t)}\mu\right)}$$

$$S(t) = e^{-\mu} * e^{\left(S_o(t)\mu\right)}$$

$$S(t) = e^{\mu(S_o(t)-1)}$$

2.4.3 NEGATIVE BINOMIAL

Let Z be a negative binomial-distributed random variable with the probability function

$$P(z) = \binom{z-1}{k-1} p^{k} (1-p)^{z-k} \;, z = k, k + 1, \ldots, where\ k > 0\ and\ p > 0.$$

The unconditional survival function is given by

$$S(t) = E[e^{(-ZH(t))}]$$

$$S(t) = \sum_{Z=0}^{\infty} e^{-ZH(t)} * \binom{z-1}{k-1} p^{k} (1-p)^{z-k}$$

$$S(t) = \left(\frac{p}{q}\right)^{k} \sum_{Z=0}^{\infty} \binom{z-1}{k-1} \left(qe^{-H(t)}\right)^{Z}$$

$$S(t) = \left(\frac{p}{q}\right)^{k} \sum_{Z=0}^{\infty} \binom{z-1}{k-1} \left(qS_o(t)\right)^{Z}$$

$$Using\ (1-qS_o(t))^{-k} = \sum_{z=0}^{\infty} \binom{z-1}{k-1} (qS_o(t))^{Z-k}$$

$$S(t) = \left(\frac{p}{q}\right)^{k} * (qS_o(t))^{k} * (1-qS_o(t))^{-k}$$

$$S(t) = \left(\frac{pS_o(t)}{1-qS_o(t)}\right)^{k}$$

2.4.4 BINOMIAL DISTRIBUTION

Let Z be a binomial distributed random variable with the probability function

$$P(z) = \binom{n}{z} p^z (1-p)^{n-z}, z = 0,1,2\ldots n \; where \; p > 0.$$

The unconditional survival function is given by

$$S(t) = E[e^{(-ZH(t))}]$$

$$S(t) = \sum_{Z=0}^{n} e^{-ZH(t)} * \binom{n}{z} p^z (1-p)^{n-z}$$

$$S(t) = \sum_{Z=0}^{n} \binom{n}{z} (pe^{-H(t)})^Z q^{n-z}$$

$$S(t) = (q + pe^{-H(t)})^n$$

$S(t) = (q + pS_o(t))^n$ where $p > 0$ and $p + q = 1$

However, in this study frailty models with continuous distribution of 'frailty' z will be considered. Since, with continuous frailty it is possible to capture the finest change in unobserved heterogeneity. Unlike the discrete models, continuous models do not allow having zero risks.

CHAPTER 3

CONTINUOUS FRAILTY MODELS

GAMMA MIXTURES

3.1 GAMMA FRAILTY MODEL

Vaupel *et al.* (1979) suggest a Gamma distribution, due to its mathematical tractability. From a computational and analytical point of view, it fits well to failure data because it is easy to derive the closed form expressions of unconditional survival, cumulative density and hazard function. This is due to the simplicity of the Laplace transform.

Vaupel Approach (1979)

<u>Construction</u>

$Let\ Z \sim \Gamma(p, b)$

With shape parameter p and scale parameter b. The marginal density of Z is;

$$f(z) = \frac{b^p z^{p-1} e^{-bz}}{\Gamma(p)} \ ; z > 0, b > 0, p > 0$$

The Laplace transformation is given by;

$$L_Z(s) = \left(\frac{b}{b+s}\right)^p = \left(1 + \frac{s}{b}\right)^{-p}$$

This is required to integrate out the distribution of the unobserved frailty. Once the frailty is integrated out, accounting for unobserved heterogeneity is reduced to estimating the variance of the frailty term. $\delta^2 = \frac{1}{b}$

The mean frailty at birth is

$$E(z) = -L'(0) = p * (1 + \frac{s}{b})^{-p-1} * \frac{1}{b} \ @s = 0$$

$$= \frac{p}{b}$$

$$Variance; \ Var(z) = L''(0) - (L'(0))2$$

$$= -p(-p-1) * (1 + \frac{s}{b})^{-p-2} * \left(\frac{1}{b}\right)^2 - \left(\frac{p}{b}\right)^2 \ @s = 0$$

$$= \frac{p}{b^2}$$

$$Coefficient \ of \ variation; \ cv(z) = \frac{sd}{mean} = \frac{1}{\sqrt{p}}$$

The CV shows that p plays the role of measuring, in relative terms, the level of heterogeneity in population. If $p \to \infty$, then $cv(z) \to 0$, i.e. the population can be considered homogeneous; for small values of p, on the contrary, the value of $cv(z)$ is high, expressing a wide dispersion, i.e. heterogeneity in the population.

However, the coefficient of variation is constant and does not change with age. This is a unique property of the gamma distributed frailty, since other assumed forms of frailty usually exhibit a decreasing coefficient of variation. i.e. Inverse Gaussian distributed frailty.

The marginal survival function is given by;

$$S(x) = L_Z(H_0(x))$$

$$S(x) = (1 + \frac{H_0(x)}{b})^{-p}$$

$$f(x) = -h_o(t) \ L_Z'(H_0(x))$$

$$f(x) = h_o(t) \ (1 + \frac{H_0(x)}{b})^{-p-1} \frac{p}{b}$$

$$h(x) = \frac{f(x)}{s(x)} = h_o(t)\,(1 + \frac{H_0(x)}{b})^{-1}\frac{p}{b}$$

For purposes of identifiability assume the distribution of Z has mean normalized to one

(i.e. the standard mortality table describes an "average individual") and variance $\delta^2 = \frac{1}{b}$.

Let $p = b$ (i.e. one parametric gamma distribution). The hazard becomes,

$$h(x) = \frac{h_o(x)}{1 + \frac{H_o(x)}{b}}$$

$$h(x) = \frac{h_o(x)b}{b + H_o(x)}$$

CHOICE OF $h_o(x)$

3.1.1 Gompertz – Gamma Frailty Model

Benjamin Gompertz (1825) idea of "exponential aging", postulated that $h(x)$ satisfies the simple differential equation

$$\frac{dh(x)}{dx} = \beta h(x)$$

Solving this

$$\frac{dh(x)}{h(x)} = \beta dx$$

$$\int \frac{dh(x)}{h(x)} = \int \beta dx$$

$$Ln\, h(x) = \beta x + c$$

$$h(x) = \propto e^{\beta x}\ where\ \propto = e^{c}$$

In words, this implies that a person's probability of dying increases at a constant exponential rate as age increases. Gompertz' Law is often found to be quite accurate (at least as a first approximation) for ages over about 25 or 30.

Hence,

$h_o(x) = \propto e^{\beta x}$ ∝ > 0 represents baseline mortality and β > 0 is the rate of increase of mortality with age.

$$H_o(x) = \int_0^x h_o(x)\,dt$$

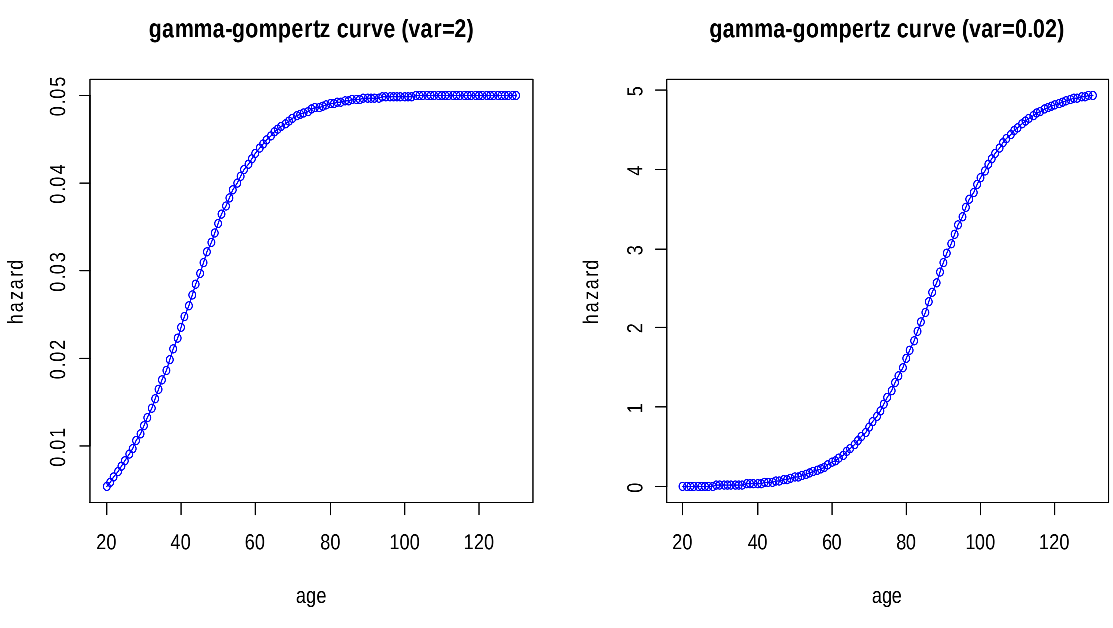
$$= \int_0^x \propto e^{\beta t} dt$$

$$H_o(x) = \frac{\propto}{\beta}(e^{\beta x} - 1)$$

Using the Gompertz model the hazard becomes

$$h(x) = \frac{\propto be^{\beta x}}{b + \frac{\propto}{\beta}(e^{\beta x} - 1)}$$

$$for\ b = 0.05\ \propto = 0.0008 \beta = 0.1\ h(x) has\ a\ log - logistic\ shape\ shown\ below$$



The hazard can be re-parametized as:

$$h(x) = \frac{\propto be^{\beta x}}{b - \frac{\propto}{\beta} * \{1 + \frac{\propto}{\beta b - \propto} e^{\beta x}\}}$$

$$h(x) = \frac{1}{b - \frac{\propto}{\beta}} * \frac{\propto be^{\beta x}}{1 + \frac{\propto}{\beta b - \propto} e^{\beta x}}$$

$$let\ \acute{\alpha} = \frac{\propto b}{b - \frac{\propto}{\beta}}, \delta = \frac{\propto}{\beta b - \propto}$$

Thus,

$$h(x) = \frac{\acute{\alpha} e^{\beta x}}{1 + \delta e^{\beta x}}$$

Has a logistic shape and belongs to the Perks family (1932)

3.1.2 Weibull-Gamma Frailty Model

Alternatively, $h_o(x)$ can be chosen to follow a weibull (λ, p) distribution with probability density function

$f(x) = \lambda p x^{p-1} exp(-\lambda x^p)\ where\ p > 0, \lambda > 0$ p is the shape parameter

The survival function is;

$$s(x) = \Pr(X > x)$$

$s(x) = \int_x^\infty \lambda p t^{p-1} exp(-\lambda t^p)\, dt$ ….. eqt1

$$let\ z = \lambda t^p\ \frac{dz}{dt} = \lambda p t^{p-1} \qquad substituting\ in\ eqn\ 1$$

$$s(x) = \int_{\lambda x^p}^\infty \exp(-z)\, dz$$

$$s(x) = exp(-\lambda x^p)$$

$$h(x) = \frac{f(x)}{s(x)} = \frac{\lambda p x^{p-1} exp(-\lambda x^p)}{exp(-\lambda x^p)}$$

$h_o(x) = \lambda p x^{p-1}$

$H_o(x) = \int_0^x h_o(t)\, dt$

$H_o(x) = \int_0^x \lambda p t^{p-1}\, dt$

$H_o(x) = \lambda x^p$

$if\ p > 1$ the hazard increases and if $p < 1$ the hazard decreases

The extreme value character of the Weibull distribution makes it appropriate for the distribution of individual time to death, because there are different causes of death which compete with each other.

Using Weibull as the baseline hazard, the hazard function for gamma;

$$h(x) = \frac{h_o(x)b}{b + H_o(x)}$$

$becomes$

$$h(x) = \frac{\lambda p x^{p-1} * b}{b + \lambda x^p}$$

Let $b = 20$ $p = 1.1$ $\lambda = 0.01$ the output is shown below

3.1.3 Exponential-Gamma Frailty Model

A special case of the weibull distribution is the exponential distribution when the shape parameter is one ($p = 1$)

The Weibull hazard is given by

$h_o(x) = \lambda p x^{p-1}$

Substituting $p = 1$

$h_o(x) = \lambda$

This is the hazard of an exponential distribution which is constant.

The cumulative hazard is given by;

$$H_o(x) = \int_0^x h_o(t)\,dt$$

$$= \int_0^x \lambda\,dt$$

$$H_o(x) = \lambda x$$

Using the Exponential distribution for the baseline hazard $h_o(x)$;

The hazard for the gamma distribution;

$$h(x) = \frac{h_o(x)b}{b + H_o(x)}$$

becomes

$$h(x) = \frac{\lambda b}{b + \lambda x}$$

3.1.4 Log-logistic Gamma Frailty Model

The probability density function for a log-logistic distribution is

$$f(x) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{x}{\propto}\right)^{\beta-1}}{\left(1 + \left(\frac{x}{\propto}\right)^{\beta}\right)^2} \quad x > 0\ \beta, \propto > 0$$

$$S(x) = Pr(X > x)$$

$$S(x) = \int_x^{\infty} \frac{(\frac{\beta}{\propto})(\frac{u}{\propto})^{\beta-1}}{(1 + (\frac{u}{\propto})^{\beta})^2}\,du$$

$$let\ y = (\frac{u}{\propto})^{\beta}\ \frac{dy}{du} = (\frac{\beta}{\propto})(\frac{u}{\propto})^{\beta-1}$$

$$S(x) = \int_{(\frac{x}{\propto})^{\beta}}^{\infty} \frac{dy}{(1+y)^2}$$

$$s(x) = \frac{1}{1 + (\frac{x}{\propto})^{\beta}}$$

$$h(x) = \frac{(\frac{\beta}{\propto})(\frac{x}{\propto})^{\beta-1}}{1 + (\frac{x}{\propto})^{\beta}}$$

Thus

$$h_o(t) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}}$$

The cumulative hazard is given by;

$$H_o(x) = \int_0^x \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}} dt$$

$$H_o(x) = ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}$$

The hazard for the gamma distribution;

$$h(x) = \frac{h_o(x)b}{b + H_o(x)}$$

becomes

$$h(x) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{x}{\propto}\right)^{\beta-1}}{1 + \left(\frac{x}{\propto}\right)^{\beta}} /\{1 + \frac{1}{b} ln\left(1 + \left(\frac{x}{\propto}\right)^{\beta}\right)\}$$

3.1.5 Log normal - Gamma Frailty Model

Using the log normal distribution with parameters μ and σ:

$$W = ln(x) \sim N(\mu, \sigma)$$

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi}} e^{\frac{-(lnx-\mu)^2}{2\sigma^2}}$$

$$F(x) = \Phi(\frac{lnx - \mu}{\sigma})$$

$$h(x) = \frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})}$$

$$h(x) = -\frac{d}{dx} \ln(1 - F(x))$$

$$H(x) = -ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))$$

Thus

$$h_o(x) = \frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})}$$

The cumulative hazard is given by;

$$H_o(x) = \int_0^x -\frac{d}{dx} \ln(1 - F(x))\, dx$$

$$H_o(x) = -ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))$$

The hazard for the gamma distribution;

$$h(x) = \frac{h_o(x)b}{b + Ho(x)}$$

$becomes$

$$h(x) = \frac{\frac{\emptyset(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})}}{1 - \frac{1}{b} ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))}$$

3.1.6 Exponential power - Gamma Frailty Model

Using the exponential power density with survival function;

$$S(t) = e^{1-e^{\lambda t^{\propto}}} \qquad \propto, \lambda > 0$$

$$h(t) = -\frac{d}{dt} ln\, s(t)$$

$$h(t) = -\frac{d}{dt}(1 - e^{\lambda t^{\propto}})$$

$$h_o(x) = \propto \lambda t^{\propto - 1} e^{\lambda t^{\propto}}$$

$$H(t) = -ln(S(t))$$

$$H(t) = -ln(e^{1-e^{\lambda t^{\propto}}})$$

$$H_o(x) = e^{\lambda t^{\propto}} - 1$$

The hazard for the gamma distribution;

$$h(x) = \frac{h_o(x)b}{b + H_o(x)}$$

becomes

$$h(x) = \frac{\propto \lambda x^{\propto - 1} e^{\lambda x^{\propto}}}{1 + (e^{\lambda x^{\propto}} - 1)/b}$$

3.1.7 Pareto - Gamma Frailty Model

Using the Pareto distribution with survival function;

$$S(t) = \frac{\lambda}{t} \qquad \propto > 0, \lambda > 0, t \geq \lambda$$

$$h(t) = -\frac{d}{dt} ln \frac{\lambda}{t}$$

$$h(t) = \frac{t^{-1}}{t}$$

$$h_o(t) = \frac{}{t}$$

$$H(t) = -ln(S(t))$$

$$H_o(x) = -ln(\frac{\lambda}{t})$$

The hazard for the gamma distribution;

$$h(x) = \frac{h_o(x)b}{b + H_o(x)}$$

becomes

$$h(x) = \frac{\bar{x}}{1 - ln(\frac{\lambda}{x})/b}$$

INVERSE-GAUSSIAN MIXTURES

3.2 INVERSE-GAUSSIAN FRAILTY MODEL

Alternative to the Gamma distribution is the Inverse Gaussian as a frailty distribution introduced by Hougaard (1984). When the inverse Gaussian is used, the variability of *Zx* decreases with age which can be justified by the fact that those with low frailty keep on living.

Hougaard Approach (1984)

Construction

$Let\ Z \sim IG(\mu,)$

The probability density function of Z is

$$f(z,\mu,) = \left(\frac{}{2\pi z^3}\right)^{1/2} exp\left\{-\frac{(z-\mu)^2}{2z\mu^2}\right\} \quad for\ z > 0 > 0\ \mu > 0$$

Substituting $= \frac{\mu^2}{\beta}$

$$f(z,\mu,\lambda) = \mu\left(\frac{1}{2\pi\beta z^3}\right)^{1/2} exp\left\{-\frac{(z-\mu)^2}{2z\beta}\right\} \quad \text{for } z > 0\ \beta > 0\ \mu > 0$$

The Laplace transform is given by;

$$L_Z(s) = \exp\{-\frac{\mu}{\beta}[(1 + 2\beta s)^{\frac{1}{2}} - 1]\}$$

Mean = $-L'_z(0) = \mu$

Variance = $L''_z(0) - \mu^2$

$= \mu\beta$

Coefficient of Variation = $\frac{\sqrt{\beta}}{\mu}$

For identifiability reasons the mean is normalized to one. i.e. $\mu = 1$ thus the variance

$\delta^2 = \beta$

The Laplace transform becomes

$$L_z(s) = \exp\left[\frac{1-(1+2s\delta^2)^{1/2}}{\delta^2}\right]$$

The marginal survival function is given by;

$$S(x) = L_Z(H_0(x))$$

$$= \exp\left[\frac{1-(1+2H_0(x)\delta^2)^{1/2}}{\delta^2}\right]$$

$$f(x) = -h_o(x)\, L_Z'(H_0(x))$$

$$= \frac{h_o(x)}{(1+2H_0(x)\delta^2)^{1/2}} \exp\left[\frac{1-(1+2H_0(x)\delta^2)^{1/2}}{\delta^2}\right]$$

$$h(x) = \frac{f(x)}{s(x)} = \frac{h_o(x)}{(1+2H_0(x)\delta^2)^{1/2}}$$

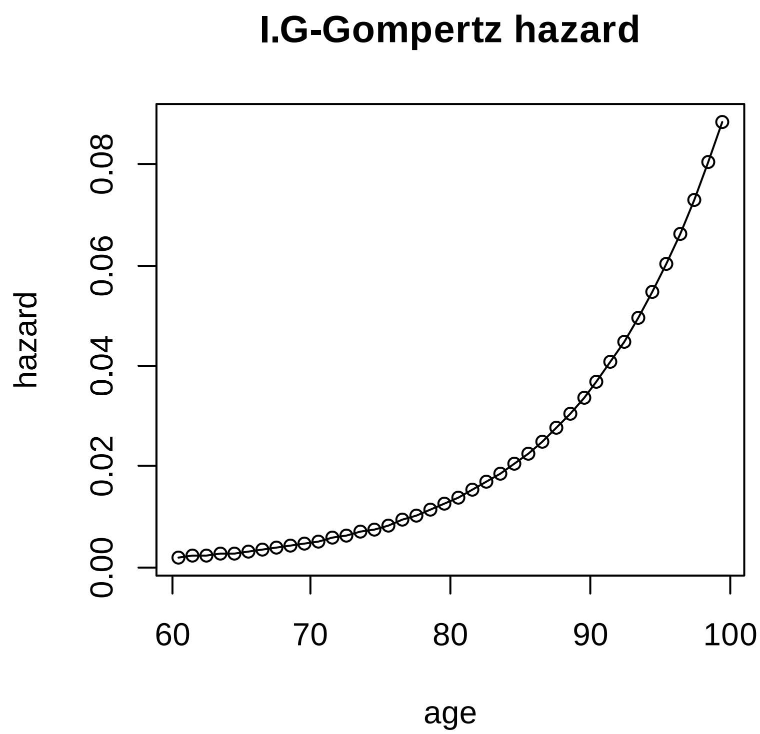
CHOICE OF $h_o(x)$

3.2.1 Gompertz – Inverse Gaussian Frailty Model

Using Gompertz assumption for the baseline mortality $h_o(x) = \propto e^{\beta x}$

The Inverse Gaussian hazard becomes

$$h(x) = \frac{\propto e^{\beta x}}{(1 + 2\frac{\propto}{\beta}(e^{\beta x} - 1)\delta^2)^{1/2}}$$



3.2.2 Weibull-Inverse Gaussian Frailty Model

Using Weibull distribution for $h_o(t)$

$$h_o(t) = \lambda p x^{p-1}$$

$$H_0(x) = \lambda x^p$$

$$h(x) = \frac{h_o(x)}{(1 + 2H_0(x)\delta^2)^{1/2}}$$

becomes,

$$h(x) = \frac{\lambda p x^{p-1}}{(1 + 2\,\lambda x^p \delta^2)^{1/2}}$$

3.2.3 Exponential Inverse-Gaussian Frailty Model

Using the Exponential distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \lambda$$

The cumulative hazard

$$H_o(t) = \lambda t$$

The hazard for the inverse Gaussian distribution;

$$h(t) = \frac{h_o(t)}{(1 + 2H_0(t)\delta^2)^{1/2}}$$

becomes,

$$h(t) = \frac{\lambda}{(1 + 2\,\lambda t\delta^2)^{1/2}}$$

3.2.4 Log-logistic Inverse-Gaussian Frailty model

Using the Log-logistic distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}}$$

The cumulative hazard

$$H_o(x) = ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}$$

The hazard for the inverse Gaussian distribution;

$$h(t) = \frac{h_o(t)}{(1 + 2H_0(t)\delta^2)^{1/2}}$$

becomes,

$$h(t) = \frac{\frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1+\left(\frac{t}{\propto}\right)^{\beta}}}{(1+2ln\{1+\left(\frac{t}{\propto}\right)^{\beta}\}\delta^2)^{1/2}}$$

3.2.5 Log normal – Inverse Gaussian Frailty Model

Using the Log normal distribution for the baseline hazard $h_o(t)$;

$$h_o(x) = \frac{f(x)}{1-\Phi(\frac{lnx-\mu}{\sigma})}$$

The cumulative hazard is given by;

$$H_o(x) = -ln(1-\ \Phi(\frac{lnx-\mu}{\sigma}))$$

The hazard for the inverse Gaussian distribution;

$$h(t) = \frac{h_o(t)}{(1+2H_0(t)\delta^2)^{1/2}}$$

$becomes$,

$$h(t) = \frac{\frac{f(x)}{1-\Phi(\frac{lnx-\mu}{\sigma})}}{(1-2ln(1-\ \Phi(\frac{lnx-\mu}{\sigma}))\delta^2)^{1/2}}$$

3.2.6 Exponential power – Inverse Gaussian Frailty Model

Using the exponential power distribution for the baseline hazard $h_o(t)$;

$$h_o(x) = \propto \lambda t^{\propto-1}e^{\lambda t^{\propto}}$$

The cumulative hazard is given by;

$$H_o(x) = e^{\lambda t^{\propto}} - 1$$

The hazard for the inverse Gaussian distribution;

$$h(t) = \frac{h_o(t)}{(1 + 2H_0(t)\delta^2)^{1/2}}$$

becomes,

$$h(t) = \frac{\propto \lambda t^{\propto - 1} e^{\lambda t^{\propto}}}{(1 + 2(e^{\lambda t^{\propto}} - 1)\delta^2)^{1/2}}$$

3.2.7 Pareto – Inverse Gaussian Frailty Model

Using the Pareto distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{}{t}$$

The cumulative hazard is given by;

$$H_o(t) = -ln(\frac{\lambda}{t})$$

The hazard for the inverse Gaussian distribution;

$$h(t) = \frac{h_o(t)}{(1 + 2H_0(t)\delta^2)^{1/2}}$$

becomes,

$$h(t) = \frac{\overline{t}}{(1 - 2ln(\frac{\lambda}{t})\delta^2)^{1/2}}$$

POWER VARIANCE FUNCTIONS

3.3 POWER VARIANCE FUNCTION FRAILTY MODEL

Tweedy (1984) suggested the family of power variance functions that includes the Gamma, Inverse Gaussian and Positive stable distributions and later derived independently by Hougaard (1986).

Tweedie Approach (1984)

Construction

The PVF model is a three parameter family denoted by $PVF(r, k, \lambda)$.

The Laplace transform is

$$L(s) = e^{-\frac{k}{r}\{(\lambda+s)^r - \lambda^r\}}$$

The marginal survival function;

$$S(x) = L_Z(H_0(x))$$

$$= e^{-\frac{k}{r}\{(\lambda+H_0(x))^r - \lambda^r\}}$$

$$f(x) = -h_o(x)\, L_Z{}'(H_0(x))$$

$$= h_o(x)k(\lambda + H_0(x))^{r-1} e^{-\frac{k}{r}\{(\lambda+H_0(x))^r - \lambda^r\}}$$

$$h(x) = \frac{f(x)}{s(x)} = h_o(x)k(\lambda + H_0(x))^{r-1}$$

For identifiability the mean is normalized to one i.e. E [Z] = $k\lambda^{r-1}$ = 1 this implies that

Var [Z] = $\delta^2 = k(1-r)\lambda^{r-2} = \frac{1-r}{\lambda}$

The resulting hazard becomes;

$$h(t) = \frac{h_o(t)}{(1 + \frac{\delta^2}{1-r} H_o(t))^{1-r}}$$

SPECIAL CASES

Case 1

For $r = 0$, the hazard

$$h(x) = \frac{h_o(t)}{(1 + \frac{\delta^2}{1-r} H_o(t))^{1-r}}$$

Becomes

$$h(x) = \frac{h_o(t)}{(1 + \delta^2 H_o(t))}$$

This is the hazard function for the gamma $\Gamma(k, \lambda)$ distribution.

Case 2

For $r = 0.5$,

$$h(x) = \frac{h_o(t)}{(1 + \frac{\delta^2}{1-r} H_o(t))^{1-r}}$$

Becomes

$$h(t) = \frac{h_o(t)}{(1 + 2 * \delta^2 H_o(t))^{1/2}}$$

This is the hazard function for the inverse Gaussian distribution.

Case 3

For $r = -1$,

$$h(x) = \frac{h_o(t)}{(1 + \frac{\delta^2}{1-r} H_o(t))^{1-r}}$$

Becomes

$$h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} * \delta^2 H_o(t))^2}$$

This is the hazard function for the Non-central Gamma distribution with shape parameter zero.

Case 4

When $\lambda = 0$,

$$h(x) = h_o(x)k\left(\lambda + H_0(x)\right)^{r-1}$$

Becomes

$$h(x) = h_o(x)k\left(H_0(x)\right)^{r-1}$$

This is the hazard function for the positive stable distribution.

POSITIVE STABLE MIXTURES

A random variable Z is said to have a stable distribution if it has the property that a linear combination of two independent copies of the variable has the same distribution.

3.4 POSITIVE STABLE FRAILTY MODEL

Hougaard (1986) introduced the Positive Stable model as a frailty distribution. Despite the fact that no closed form expressions exist for the probability density or the survival function, the Laplace transform has a very simple form.

Hougaard Approach (1986)

Construction

The density function of positive stable law can be represented using infinite series expansion as.

$$f(z,k,r) = -\frac{1}{\pi z}\sum_{c=1}^{\infty}\frac{\Gamma(cr+1)}{c!}(-kz^{-r})^{c}sin(rc\pi)$$

This distribution has an infinite mean.

The Laplace transform is a special case of the $PVF(r, k, \lambda)$ Laplace

$$L(s) = e^{-\frac{k}{r}\{(\lambda+s)^r - \lambda^r\}}$$

When $\lambda = 0$

$$L_Z(s) = e^{\frac{-ks^r}{r}}$$

For identifiability reasons let $k = r$

$$L_Z(s) = e^{-s^r} \quad 0 < r < 1$$

The marginal survival function;

$$S(x) = L_Z(H_0(x))$$

$$= e^{-H_0(x)^r}$$

$$f(x) = -h_o(x)\, L_Z{}'(H_0(x))$$

$$= rh_0(x)H_0(x)^{r-1} * e^{-H_0(x)^r}$$

The hazard function is

$$h(x) = \frac{f(x)}{s(x)} = rh_0(x)H_0(x)^{r-1}$$

The positive stable distribution is the only frailty distribution which preserves the proportional hazards assumption in the unconditional hazards after integrating out the frailty.

CHOICE OF $h_o(x)$

3.4.1 Gompertz –Positive Stable Frailty Model

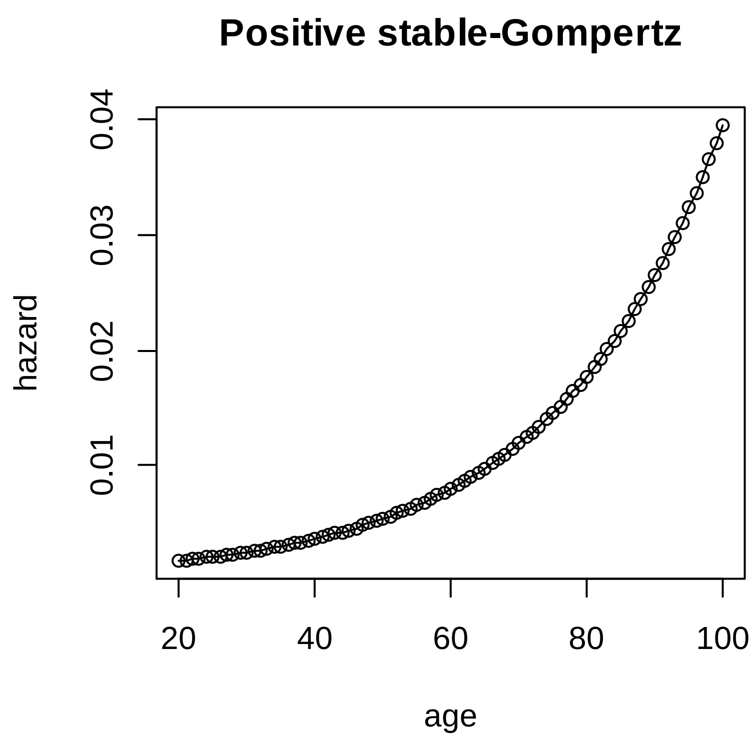
Using Gompertz assumption for the baseline mortality $h_o(x) = \propto e^{\beta x}$

The hazard function for Positive Stable

$h(x) = rh_0(x)H_0(x)^{r-1}$

Becomes,

$$h(x) = r \propto e^{\beta x}\left\{\frac{\propto}{\beta}\left(e^{\beta x} - 1\right)\right\}^{r-1}$$



The power variance family contains members whose relative frailty distribution in survivors becomes less homogeneous with time.

3.4.2 Weibull-Positive Stable Frailty Model

Using Weibull distribution for $h_o(t)$

$h_o(t) = \lambda p x^{p-1}$

$Ho(x) = \lambda x^p$

The hazard for the positive stable distribution;

$$h(x) = rh_0(x)H_0(x)^{r-1}$$

$becomes$,

$$h(x) = r\lambda p x^{p-1} (\lambda x^p)^{r-1}$$

3.4.3 Exponental-Positive Stable Frailty Model

Using the Exponential distribution for $h_o(t)$;

$$h_o(t) = \lambda$$

The cumulative hazard

$$H_o(x) = \lambda x$$

The hazard for the positive stable distribution;

$$h(x) = rh_0(x)H_0(x)^{r-1}$$

$becomes$,

$$h(x) = r\lambda (\lambda x)^{r-1}$$

3.4.4 Log-logistic Positive Stable Frailty Model

Using the Log-logistic distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}}$$

The cumulative hazard

$$H_o(x) = ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}$$

The hazard for the positive stable distribution;

$$h(x) = rh_0(x)H_0(x)^{r-1}$$

$becomes$,

$$h(x) = r\frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}} ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}^{r-1}$$

3.4.5 Lognormal-Positive Stable Frailty Model

Using the Log normal distribution for the baseline hazard $h_o(t)$;

$$h_o(x) = \frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})}$$

The cumulative hazard is given by;

$$H_o(x) = -ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))$$

The hazard for the positive stable distribution;

$$h(x) = rh_0(x)H_0(x)^{r-1}$$

$becomes$,

$$h(x) = r\frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})} * \{-ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))\}^{r-1}$$

3.4.6 Exponential power – Positive Stable Frailty Model

Using the exponential power distribution for the baseline hazard $h_o(t)$;

$$h_o(x) = \propto \lambda t^{\propto - 1} e^{\lambda t^{\propto}}$$

The cumulative hazard is given by;

$$H_o(x) = e^{\lambda t^{\propto}} - 1$$

The hazard for the positive stable distribution;

$$h(x) = rh_0(x)H_0(x)^{r-1}$$

$becomes$,

$$h(x) = r \propto \lambda t^{\propto -1} e^{\lambda t^{\propto}} * (e^{\lambda t^{\propto}} - 1)^{r-1}$$

3.4.7 Pareto – Positive Stable Frailty Model

Using the Pareto distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{}{t}$$

The cumulative hazard is given by;

$$H_o(t) = -ln(\frac{\lambda}{t})$$

The hazard for the positive stable distribution;

$$h(x) = rh_0(x)H_0(x)^{r-1}$$

$becomes$,

$$h(x) = \frac{r}{t}\{-ln\left(\frac{\lambda}{t}\right)\}^{r-1}$$

COMPOUND POISSON MIXTURES

3.5 COMPOUND POISSON FRAILTY MODEL

The compound Poisson distribution was introduced by Aalen (1988, 1992) as a frailty distribution.

Aalen (1988, 1992) approach

Construction

Using the Laplace transform obtained and $xi's \sim gamma(k, \lambda)$

$$L_Z(s) = e^{p(L_X(s)-1)}$$

$= e^{p((1+\frac{s}{\lambda})^{-k}-1)}$

$by\ reparametization\ substitute\ p = \dfrac{-k\lambda^r}{r}\ and\ k = -r$

$L_Z(s)= e^{\frac{-k\lambda^r}{r}*((1+\frac{s}{\lambda})^{r}-1)}$

$L_Z(s)= e^{\frac{-k}{r}*((\lambda+s)^r-\lambda^r)}$

For $r \geq 0$, the power variance function distribution (PVF) is obtained.

For $r < 0$ the compound Poisson distribution is obtained, these two subclasses are separated by the gamma distribution ($r = 0$).

For identifiability assume the mean frailty is normalized to one.

Mean

$L_Z{}'(s)= k(\lambda + s)^{r-1} * e^{\frac{-k}{r}*((\lambda+s)^r-\lambda^r)}$ @s=0

$L_Z{}'(0)= k\lambda^{r-1} = 1$

Variance

$L_Z{}''(s)$-$(L_Z{}'(0))^2$

$K(r-1)\ (\lambda + s)^{r-2} e^{\frac{-k}{r}*((\lambda+s)^r-\lambda^r)} + (k(\lambda + s)^{r-1})\text{^}2 * e^{\frac{-k}{r}*((\lambda+s)^r-\lambda^r)} - (k\lambda^{r-1})^2\ @s = 0$

$\delta^2 = K(r-1)\ \lambda^{r-2}$

$\delta^2 = \dfrac{(r-1)}{\lambda}$

The marginal survival function is given by

$$S(x) \ = \ L_Z(H_0(x))$$

$$S(x) \ = \ e^{\frac{-k}{r}*((\lambda+H_0(x))^r-\lambda^r)}$$

$$f(x) = -h_o(t) \ \ L_Z{}'(H_0(x))$$

$$f(x) = \ kh_o(t)(\lambda + H_0(x))^{r-1}e^{\frac{-k}{r}*((\lambda+H_0(x))^r-\lambda^r)}$$

$$h(x) = \ \frac{f(x)}{s(x)} = kh_o(t)(\lambda + H_0(x))^{r-1}$$

$$h(x) = \ h_o(t)\left(1 + \frac{\delta^2}{r-1}H_0(x)\right)^{r-1}$$

CHOICE OF $h_o(x)$

3.5.1 Gompertz –Compound Poisson Frailty Model

Using Gompertz assumption for the baseline mortality

$$h_o(x) \ = \propto e^{\beta x}$$

$$H_o(x) \ = \ \frac{\propto}{\beta}(e^{\beta x} - 1)$$

The hazard

$$h(x) = \ h_o(t)\left(1 + \frac{\delta^2}{r-1}H_0(x)\right)^{r-1}$$

$becomes$,

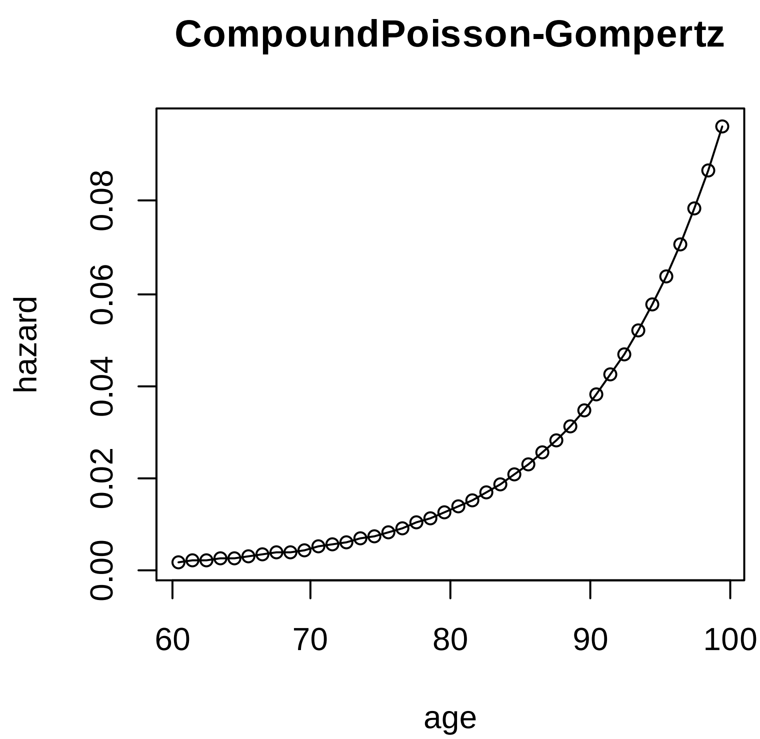
$$h(x) = \ \propto e^{\beta x} \ \left(1 + \frac{\delta^2}{r-1}\frac{\propto}{\beta}(e^{\beta x} - 1)\right)^{r-1}$$

RCODE

Using r=1.1

gcp=ho*((1+(0.05/0.1)*Ho)^0.1)

plot(x,gcp,main="CompoundPoisson-Gompertz",type="o", xlab="age",ylab="hazard")



3.5.2 Weibull –Compound Poisson Frailty Model

Using Weibull distribution for $h_o(t)$

$$h_o(t) = \lambda p x^{p-1}$$

$$Ho(x) = \lambda x^p$$

The hazard

$$h(x) = h_o(t)\left(1 + \frac{\delta^2}{r-1} H_0(x)\right)^{r-1}$$

Becomes,

$$h(x) = \lambda p x^{p-1}\left(1 + \frac{\delta^2}{r-1}\lambda x^p\right)^{r-1}$$

3.5.3 Exponential –Compound Poisson Frailty Model

Using the Exponential distribution for $h_o(t)$;

$$h_o(t) = \lambda$$

The cumulative hazard

$$H_o(x) = \lambda x$$

The hazard

$$h(x) = h_o(t)\left(1 + \frac{\delta^2}{r-1}H_0(x)\right)^{r-1}$$

Becomes

$$h(x) = \lambda\left(1 + \frac{\delta^2}{r-1}\lambda x\right)^{r-1}$$

3.5.4 Log-logistic Compound Poisson Frailty model

Using the Log-logistic distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}}$$

The cumulative hazard

$$H_o(x) = ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}$$

The hazard

$$h(x) = h_o(t)\left(1 + \frac{\delta^2}{r-1} H_0(x)\right)^{r-1}$$

Becomes

$$h(x) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1+\left(\frac{t}{\propto}\right)^{\beta}} \left(1 + \frac{\delta^2}{r-1} ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}\right)^{r-1}$$

3.5.5 Lognormal-Compound Poisson Frailty Model

Using the Log normal distribution for the baseline hazard $h_o(t)$;

$$h_o(x) = \frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})}$$

The cumulative hazard is given by;

$$H_o(x) = -ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))$$

The hazard

$$h(x) = h_o(t)\left(1 + \frac{\delta^2}{r-1} H_0(x)\right)^{r-1}$$

Becomes

$$h(x) = h_o(t)\left(1 - \frac{\delta^2}{r-1} ln(1 - \Phi(\frac{lnx-\mu}{\sigma}))\right)^{r-1}$$

3.5.6 Exponential Power – Compound Poisson Frailty Model

Using the exponential power distribution for the baseline hazard $h_o(t)$;

$$h_o(x) = \propto \lambda t^{\propto-1} e^{\lambda t^{\propto}}$$

The cumulative hazard is given by;

$$H_o(x) = e^{\lambda t^{\propto}} - 1$$

The hazard

$$h(x) = h_o(t)\left(1 + \frac{\delta^2}{r-1} H_0(x)\right)^{r-1}$$

Becomes

$$h(x) = \propto \lambda t^{\propto -1} e^{\lambda t^{\propto}}\left(1 + \frac{\delta^2}{r-1} * e^{\lambda t^{\propto}} - 1\right)^{r-1}$$

3.5.7 Pareto – Compound Poisson Frailty Model

Using the Pareto distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{}{t}$$

The cumulative hazard is given by;

$$H_o(t) = -ln(\frac{\lambda}{t})$$

The hazard

$$h(x) = h_o(t)\left(1 + \frac{\delta^2}{r-1} H_0(x)\right)^{r-1}$$

Becomes

$$h(x) = \frac{}{t}\left(1 - \frac{\delta^2}{r-1} ln(\frac{\lambda}{t})\right)^{r-1}$$

LOG-NORMAL MIXTURES

A log-normal distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed.

3.6 LOG-NORMAL FRAILTY MODEL

McGilchrist and Aisbett (1991) used the Log-normal frailty mixture to model multivariate dependence structures.

McGilchrist and Aisbett (1991) approach

Construction

Assuming normally distributed random effect W with $E[W] = 0$ and frailty $Z = e^W$

The hazard is given by

$$h(t) = h_o(t)e^W \qquad W \sim N(0, \delta^2)$$

If $Z = e^W$ then $Z \sim lognormal(0, \delta)$

The probability density function of Z is given by

$$f(z) = \frac{1}{z\delta\sqrt{2\pi}} e^{-\frac{(lnz)^2}{2\delta^2}} \quad z > 0$$

With scale parameter $\delta > 0$

$$E(Z) = exp(\delta^2/2) \;\; Var(Z) = exp(2\delta^2) - exp(\delta^2)$$

The Laplace transform has no explicit form but can be approximated i.e. using the LambertW function

$$L_Z(s) \approx \frac{1}{\sqrt{LW(s\delta^2 e^{\mu})}} e^{-\frac{LW^2(s\delta^2 e^{\mu}) + 2LW(s\delta^2 e^{\mu})}{2\delta^2}}$$

Using the notation $Ł_Z(s)$ for this approximation, the marginal survival function is given by

$$S(x) = Ł_Z\big(H_o(x)\big)$$

$$S(x) \approx \frac{1}{\sqrt{LW(H_o(x)\delta^2 e^{\mu})}} e^{-\frac{LW^2(H_o(x)\delta^2 e^{\mu}) + 2LW(H_o(x)\delta^2 e^{\mu})}{2\delta^2}}$$

$$f(x) = -S'(x) = -Ł_Z'\big(H_o(x)\big)$$

$$h(x) = \frac{f(x)}{s(x)}$$

$$h(x) = \frac{-Ł_Z'\big(H_o(x)\big)}{Ł_Z\big(H_o(x)\big)}$$

CHOICE OF $h_o(x)$

3.6.1 Gompertz –Log-normal Frailty Model

Using Gompertz assumption for the baseline mortality

$$h_o(x) = \propto e^{\beta x}$$

$$H_o(x) = \frac{\propto}{\beta}(e^{\beta x} - 1)$$

The hazard

$$h(x) = \frac{-Ł_Z'\big(H_o(x)\big)}{Ł_Z\big(H_o(x)\big)}$$

Becomes

$$h(x) = \frac{-Ł_Z'\left(\frac{\propto}{\beta}(e^{\beta x} - 1)\right)}{Ł_Z\left(\frac{\propto}{\beta}(e^{\beta x} - 1)\right)}$$

3.6.2 Weibull – Log-normal Frailty Model

Using Weibull distribution for $h_o(t)$

$$h_o(t) = \lambda p x^{p-1}$$

$$Ho(x) = \lambda x^p$$

The hazard

$$h(x) = \frac{-Ł_Z{}'\left(H_o(x)\right)}{Ł_Z\left(H_o(x)\right)}$$

Becomes

$$h(x) = \frac{-Ł_Z{}'(\lambda x^p)}{Ł_Z(\lambda x^p)}$$

This model has been used by Damgaard et.al (2002) for sire evaluation of longevity with and without genetic interpretations.

3.6.3 Exponential – Log-normal Frailty Model

Using the Exponential distribution for $h_o(t)$;

$$h_o(t) = \lambda$$

The cumulative hazard

$$H_o(x) = \lambda x$$

The hazard

$$h(x) = \frac{-Ł_Z{}'\left(H_o(x)\right)}{Ł_Z\left(H_o(x)\right)}$$

Becomes

$$h(x) = \frac{-Ł_Z{}'(\lambda x)}{Ł_Z(\lambda x)}$$

3.6.3 Log-logistic – Log-normal Frailty Model

Using the Log-logistic distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}}$$

The cumulative hazard

$$H_o(x) = ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}$$

The hazard

$$h(x) = \frac{-Ł_Z{}'\left(H_o(x)\right)}{Ł_Z\left(H_o(x)\right)}$$

Becomes

$$h(x) = \frac{-Ł_Z{}'\left(ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}\right)}{Ł_Z\left(ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}\right)}$$

OTHER FRAILTY MIXTURES

RECIPROCAL INVERSE GAUSSIAN MIXTURES

3.7 RECIPROCAL INVERSE-GAUSSIAN FRAILTY MODEL

Given that $Z = \frac{1}{X}$ where $X \sim IG(\mu,)$ then Z is said to be the reciprocal of the IG distribution. The probability density function is given by;

$$(z) = \left(\frac{}{2\pi z}\right)^{\frac{1}{2}} \exp\left\{-\frac{\mu^2}{2z}\left(1-\frac{z}{\mu}\right)^2\right\}$$

is the shape parameter and μ is the location parameter

The Laplace transform is given by

$$L_Z(s) = \left(1+\frac{2s}{}\right)^{-1/2} \exp\left\{\mu\left[1-\left(1+\frac{2s}{}\right)^{\frac{1}{2}}\right]\right\}$$

For identifiability the mean is normalized to one i.e. E [Z] = μ = 1

The Laplace becomes

$$L_Z(s) = \left(1+\frac{2s}{}\right)^{-1/2} \exp\left\{\left[1-\left(1+\frac{2s}{}\right)^{\frac{1}{2}}\right]\right\}$$

The marginal survival function is given by

$$S(x) = L_Z(H_0(x))$$

$$S(x) = \left(1+\frac{2H_0(x)}{}\right)^{-1/2} \exp\left\{\left[1-\left(1+\frac{2H_0(x)}{}\right)^{\frac{1}{2}}\right]\right\}$$

$$f(x) = -h_o(x)\ L_Z{}'(H_0(x))$$

$$f(x) = \left\{\frac{h_o(x)}{} * \left(1+\frac{2H_0(x)}{}\right)^{-\frac{3}{2}} + h_o(x) * \left(1+\frac{2H_0(x)}{}\right)^{-1}\right\} \exp\left\{\left[1-\left(1+\frac{2H_0(x)}{}\right)^{\frac{1}{2}}\right]\right\}$$

$$h(x) = \frac{f(x)}{s(x)}$$

$$h(x) = \frac{h_o(x)}{} * \left(1+\frac{2H_0(x)}{}\right)^{-1} + h_o(x) * \left(1+\frac{2H_0(x)}{}\right)^{-\frac{1}{2}}$$

CHOICE OF $h_o(x)$

3.7.1 Gompertz –Reciprocal Inverse-Gaussian Model

Using Gompertz assumption for the baseline mortality

$$h_o(x) = \propto e^{\beta x}$$

$$H_o(x) = \frac{\propto}{\beta}(e^{\beta x} - 1)$$

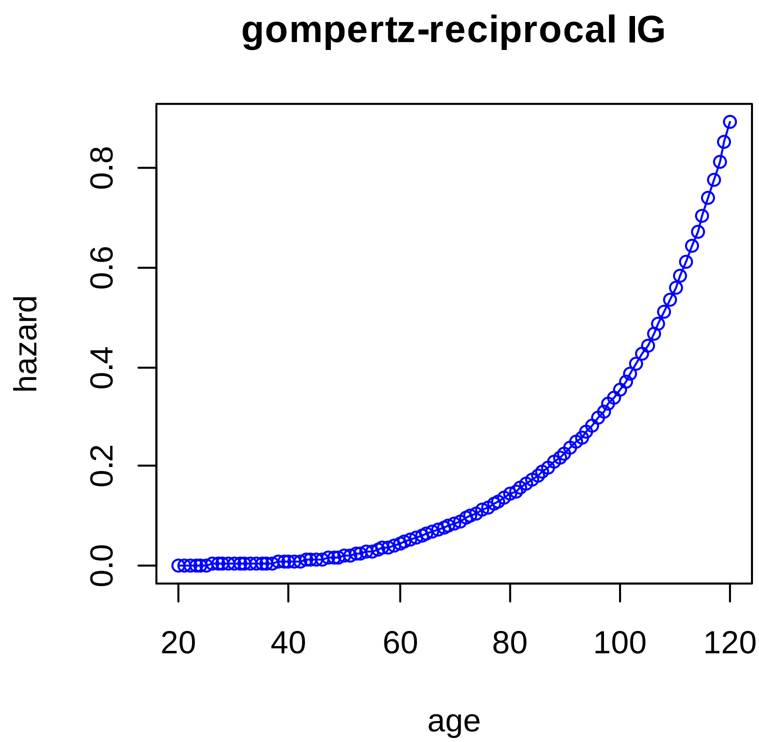
The hazard

$$h(x) = \frac{h_o(x)}{} * \left(1 + \frac{2H_0(x)}{}\right)^{-1} + h_o(x) * \left(1 + \frac{2H_0(x)}{}\right)^{-\frac{1}{2}}$$

Becomes

$$h(x) = \frac{\propto e^{\beta x}}{} * \left(1 + \frac{2\frac{\propto}{\beta}(e^{\beta x} - 1)}{}\right)^{-1} + \propto e^{\beta x} * \left(1 + \frac{2\frac{\propto}{\beta}(e^{\beta x} - 1)}{}\right)^{-\frac{1}{2}}$$

GRAPH



3.7.2 Weibull – Reciprocal Inverse-Gaussian Model

Using Weibull distribution for $h_o(t)$

$$h_o(t) \;=\; \lambda p x^{p-1}$$

$$H_0(x) \;= \lambda x^p$$

The hazard

$$h(x) \;=\; \frac{h_o(x)}{} * \left(1 + \frac{2H_0(x)}{}\right)^{-1} + h_o(x) * \left(1 + \frac{2H_0(x)}{}\right)^{-\frac{1}{2}}$$

Becomes

$$h(x) \;=\; \frac{\lambda p x^{p-1}}{} * \left(1 + \frac{2\lambda x^p}{}\right)^{-1} + \lambda p x^{p-1} * \left(1 + \frac{2\lambda x^p}{}\right)^{-\frac{1}{2}}$$

3.7.3 Exponential – Reciprocal Inverse-Gaussian Model

Using the Exponential distribution for $h_o(t)$;

$$h_o(t) \ = \ \lambda$$

The cumulative hazard

$$H_o(x) = \lambda x$$

The hazard function

$$h(x) = \frac{h_o(x)}{} * \left(1 + \frac{2H_0(x)}{}\right)^{-1} + h_o(x) * \left(1 + \frac{2H_0(x)}{}\right)^{-\frac{1}{2}}$$

Becomes

$$h(x) = \frac{\lambda}{} * \left(1 + \frac{2\lambda x}{}\right)^{-1} + \lambda * \left(1 + \frac{2\lambda x}{}\right)^{-\frac{1}{2}}$$

3.7.4 Log-logistic - Reciprocal Inverse-Gaussian Model

Using the Log-logistic distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}}$$

The cumulative hazard

$$H_o(x) = ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}$$

The hazard function

$$h(x) = \frac{h_o(t)}{} * \left(1 + \frac{2H_0(x)}{}\right)^{-1} + \frac{1}{} * \left(1 + \frac{2H_0(x)}{}\right)^{-\frac{1}{2}}$$

Becomes

$$h(x) = \frac{\frac{1}{-}\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1+\left(\frac{t}{\propto}\right)^{\beta}} * \left(1 + \frac{2ln\{1+\left(\frac{x}{\propto}\right)^{\beta}\}}{}\right)^{-1} + \frac{1}{-} * \left(1 + \frac{2ln\{1+\left(\frac{x}{\propto}\right)^{\beta}\}}{}\right)^{-\frac{1}{2}}$$

3.7.5 Lognormal - Reciprocal Inverse-Gaussian Model

Using the Log normal distribution for the baseline hazard $h_o(t)$;

$$h_o(x) = \frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})}$$

The cumulative hazard is given by;

$$H_o(x) = -ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))$$

The hazard function

$$h(x) = \frac{h_o(t)}{} * \left(1 + \frac{2H_0(x)}{}\right)^{-1} + \frac{1}{-} * \left(1 + \frac{2H_0(x)}{}\right)^{-\frac{1}{2}}$$

Becomes

$$h(x) = \frac{\frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})}}{} * \left(1 - \frac{2ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))}{}\right)^{-1} + \frac{1}{-} * \left(1 - \frac{2ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))}{}\right)^{-\frac{1}{2}}$$

3.7.6 Exponential power - Reciprocal Inverse-Gaussian Model

Using the exponential power distribution for the baseline hazard $h_o(t)$;

$$h_o(x) = \propto \lambda t^{\propto - 1} e^{\lambda t^{\propto}}$$

The cumulative hazard is given by;

$$H_o(x) = e^{\lambda t^{\propto}} - 1$$

The hazard

$$h(x) = \frac{h_o(t)}{} * \left(1 + \frac{2H_0(x)}{}\right)^{-1} + \frac{1}{} * \left(1 + \frac{2H_0(x)}{}\right)^{-\frac{1}{2}}$$

Becomes

$$h(x) = \frac{\propto \lambda t^{\propto-1} e^{\lambda t^{\propto}}}{} * \left(1 + \frac{2e^{\lambda t^{\propto}} - 1}{}\right)^{-1} + \frac{1}{} * \left(1 + \frac{2e^{\lambda t^{\propto}} - 1}{}\right)^{-\frac{1}{2}}$$

3.7.7 Pareto - Reciprocal Inverse-Gaussian Model

Using the Pareto distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{r}{t}$$

The cumulative hazard is given by;

$$H_o(t) = -ln(\frac{\lambda^r}{t^r})$$

The hazard

$$h(x) = \frac{h_o(t)}{} * \left(1 + \frac{2H_0(x)}{}\right)^{-1} + \frac{1}{} * \left(1 + \frac{2H_0(x)}{}\right)^{-\frac{1}{2}}$$

Becomes

$$h(x) = \frac{r}{t} * \left(1 - \frac{2ln(\frac{\lambda^r}{t^r})}{}\right)^{-1} + \frac{1}{} * \left(1 - \frac{2ln(\frac{\lambda^r}{t^r})}{}\right)^{-\frac{1}{2}}$$

INVERSE GAMMA MIXTURES

3.8 INVERSE GAMMA FRAILTY MODEL

The probability density function is given by

$$f(z, \propto, \beta) = \frac{\beta^{\propto}}{\Gamma(\propto)} z^{-\propto-1} exp\left(\frac{-\beta}{z}\right) \; for \; x > 0$$

β is the shape parameter and ∝ scale parameter

The Laplace transform is

$$L_Z(s) = \frac{2(\beta s)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4\beta s)$$

$K_{\propto}(.)$is the modified Bessel function of the II kind.

$$E(Z) = \frac{\beta}{\propto -1} \qquad Var(Z) = \frac{\beta^2}{(\propto -2)(\propto -1)^2}$$

For identifiability $E(Z) = \frac{\beta}{\propto -1} = 1$ and $\delta^2 = \frac{1}{\propto -2}$

The Laplace transform becomes

$$L_Z(s) = \frac{2\left(s(\propto -1)\right)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4s(\propto -1))$$

The marginal survival function

$$S(x) = L_Z(H_0(x))$$

$$S(x) = \frac{2(H_0(x)(\propto -1))^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4H_0(x)(\propto -1))$$

CHOICE OF $h_o(x)$

3.8.1 Gompertz –Inverse Gamma Frailty Model

Using Gompertz assumption for the baseline mortality

$$h_o(x) = \propto e^{\beta x}$$

$$H_o(x) = \frac{\propto}{\beta}(e^{\beta x} - 1)$$

The survival function becomes,

$$S(x) = \frac{2\left((\propto -1) \propto (e^{\beta x} - 1)\right)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4(\propto -1) \propto (e^{\beta x} - 1))$$

3.8.2 Weibull –Inverse Gamma Frailty Model

Using Weibull distribution for $h_o(t)$

$$h_o(t) = \lambda p x^{p-1}$$

$$H_0(x) = \lambda x^p$$

The survival function becomes

$$S(x) = \frac{2((\propto -1)\lambda x^p)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4(\propto -1)\lambda x^p)$$

3.8.3 Exponential –Inverse Gamma Frailty Model

Using the Exponential distribution for $h_o(t)$;

$$h_o(t) = \lambda$$

The cumulative hazard

$$H_o(x) = \lambda x$$

The survival function becomes

$$S(x) = \frac{2\left((\propto -1)\lambda x\right)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4(\propto -1)\lambda x)$$

3.8.4 Log-logistic - Inverse Gamma Frailty Model

Using the Log-logistic distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}}$$

The cumulative hazard

$$H_o(x) = ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}$$

The survival function becomes

$$S(x) = \frac{2\left((\propto -1)ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}\right)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4(\propto -1)ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\})$$

3.8.5 Lognormal - Inverse Gamma Frailty Model

Using the Log normal distribution for the baseline hazard $h_o(t)$;

$$h_o(x) = \frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})}$$

The cumulative hazard is given by;

$$H_o(x) = -ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))$$

The survival function becomes

$$S(x) = \frac{2\left(-(\propto -1)ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))\right)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(-4(\propto -1)ln(1 - \Phi(\frac{lnx - \mu}{\sigma})))$$

3.8.6 Exponential Power – Inverse Gamma Model

Using the exponential power distribution for the baseline hazard $h_o(t)$;

$h_o(x) = \propto \lambda t^{\propto-1} e^{\lambda t^{\propto}}$

The cumulative hazard is given by;

$H_o(x) = e^{\lambda t^{\propto}} - 1$

The survival function,

$$S(x) = \frac{2(\beta H_0(x))^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4\beta H_0(x))$$

Becomes

$$S(x) = \frac{2\left((\propto -1)(e^{\lambda t^{\propto}} - 1)\right)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4(\propto -1)(e^{\lambda t^{\propto}} - 1))$$

NON-CENTRAL GAMMA MIXTURES

3.9 NON-CENTRAL GAMMA FRAILTY MODEL

The probability density function for the non-central gamma distribution with Y being a mixing of the distributions of $X_1, X_2, \ldots, X_N$

Where $Xi's \sim Gamma(n, 1)\ and\ N \sim poisson(\lambda)$

Then the density function is a convolution with respective weights $\frac{e^{-\lambda}(\lambda)^i}{i!}$ i.e

$Y = X_1, X_2, \ldots, X_N$

$$Prob(Y = j) = \sum_{j=0}^{\infty} prob(X_1, X_2, \ldots, X_j | N = j) prob(N = j)$$

$$Prob(Y = j) = \sum_{j=0}^{\infty} \{\frac{X^{j-1} e^{-X}}{\Gamma(j)}\}^{*n} * \{\frac{\lambda^j e^{-\lambda}}{j!}\}$$

$$Prob(Y = j) = \sum_{j=0}^{\infty} \{\frac{X^{n+j-1} e^{-X}}{\Gamma(n+j)}\} * \{\frac{\lambda^j e^{-\lambda}}{j!}\}$$

$$f(x, n, \lambda) = \sum_{j=0}^{\infty} \{\frac{X^{n+j-1} e^{-X}}{\Gamma(n+j)}\} * \{\frac{\lambda^j e^{-\lambda}}{j!}\}$$

Where $\Gamma(n)$ is the central complete gamma function with $n > 0 \; \lambda > 0 \; x \geq 0$

The hazard function is a special case of the three parameter power variance function

when $r = -1$

$$h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} * \delta^2 H_o(t))^2}$$

CHOICE OF $h_o(x)$

3.9.1 Gompertz –Non central Gamma Frailty Model

Using Gompertz assumption for the baseline mortality

$$h_o(x) = \propto e^{\beta x}$$

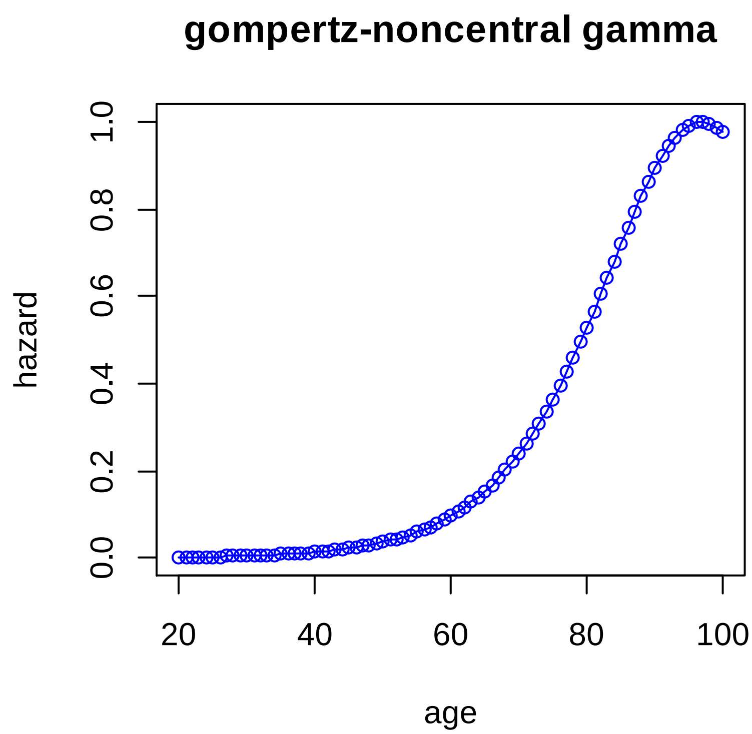
$$H_o(x) = \frac{\propto}{\beta}(e^{\beta x} - 1)$$

The hazard function

$$h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} * \delta^2 H_o(t))^2}$$

Becomes,

$$h(t) = \frac{\propto e^{\beta x}}{(1 + \frac{1}{2} * \delta^2 \frac{\propto}{\beta}(e^{\beta x} - 1))^2}$$



Justification

According to a study conducted by Olivieri (2001); mortality experience shows an increasing concentration of deaths around the mode of the curve of deaths and the mode moves towards older ages.

The Non-central gamma model can be used to explain the deceleration of the mortality rate at older ages as suggested above. The model further provides insights on the impact of omitted covariates and heterogeneity when estimating mortality rates for a heterogeneous population.

3.9.2 Weibull –Non central Gamma Frailty Model

Using Weibull distribution for $h_o(t)$

$h_o(t) = \lambda p x^{p-1}$

$H_0(x) = \lambda x^p$

The hazard function

$$h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} * \delta^2 H_o(t))^2}$$

Becomes,

$$h(t) = \frac{\lambda p x^{p-1}}{(1 + \frac{1}{2} * \delta^2 \lambda x^p)^2}$$

3.9.3 Exponential –Non central Gamma Frailty Model

Using the Exponential distribution for $h_o(t)$;

$h_o(t) = \lambda$

The cumulative hazard

$H_o(x) = \lambda x$

The hazard function

$$h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} * \delta^2 H_o(t))^2}$$

Becomes,

$$h(t) = \frac{\lambda}{(1 + \frac{1}{2} * \delta^2 \lambda x)^2}$$

3.9.4 Log-logistic – Non-Central Gamma Frailty Model

Using the Log-logistic distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}}$$

The cumulative hazard

$$H_o(x) = ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}$$

The hazard function

$$h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} * \delta^2 H_o(t))^2}$$

Becomes,

$$h(t) = \frac{\frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}}}{(1 + \frac{1}{2} * \delta^2 ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\})^2}$$

3.9.5 Lognormal – Non-Central Gamma Frailty Model

Using the Log normal distribution for the baseline hazard $h_o(t)$;

$$h_o(x) = \frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})}$$

The cumulative hazard is given by;

$$H_o(x) = -ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))$$

The hazard function

$$h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} * \delta^2 H_o(t))^2}$$

Becomes,

$$h(t) = \frac{\frac{f(x)}{1-\Phi(\frac{lnx-\mu}{\sigma})}}{(1-\frac{1}{2}*\delta^2 ln(1-\ \Phi(\frac{lnx-\mu}{\sigma})))^2}$$

3.9.6 Exponential Power – Non central Gamma Model

Using the exponential power distribution for the baseline hazard $h_o(t)$;

$$h_o(x) = \propto \lambda t^{\propto -1} e^{\lambda t^{\propto}}$$

The cumulative hazard is given by;

$$H_o(x) = e^{\lambda t^{\propto}} - 1$$

The hazard function

$$h(t) = \frac{h_o(t)}{(1+\frac{1}{2}*\delta^2 H_o(t))^2}$$

Becomes,

$$h(t) = \frac{\propto \lambda t^{\propto -1} e^{\lambda t^{\propto}}}{(1+\frac{1}{2}*\delta^2(e^{\lambda t^{\propto}}-1))^2}$$

3.9.7 Exponential Power – Non central Gamma Model

Using the Pareto distribution for the baseline hazard $h_o(t)$;

$$h_o(t) = \frac{r}{t}$$

The cumulative hazard is given by;

$$H_o(t) = -ln(\frac{\lambda^r}{t^r})$$

The hazard becomes,

$$h(t) = \frac{r}{t(1-\frac{1}{2}*\delta^2 ln(\frac{\lambda^r}{t^r}))^2}$$

CHAPTER 4

4.1 PARAMETER ESTIMATION

Model parameters are fixed quantitative values that characterize the model believed to reflect the real world. They have to be estimated either by statistical inference from observations or by expert opinion.

4.1.1CHOICE OF EXPLANATORY VARIABLES

In order to make comparisons between the Gamma-Gompertz model, the Inverse-Gaussian-Gompertz model and the Non-central Gamma-Gompertz model, it is necessary to estimate and fix the baseline model parameters using insurance based mortality rating. The baseline model has no underwriting.

The Gompertz parameters are estimated using simulation as shown in the R-CODE (appendix 1C)

4.1.2CHOICE OF THE INSUREDS LEVEL OF HETEROGENEITY

In Butt and Haberman (2002) an insurance application of frailty-based survival model is proposed. In particular, the authors discuss various choices and fit some models to two sets of life insurance mortality data. The obtained results suggest that when life annuities are referred to $\delta^2 = 1/b$ should fall in the range (0.025, 0.05).

Unless otherwise stated in this exercise the insured population will be considered to have heterogeneity level of $\delta^2 = 0.05$

4.1.3 SHARED FRAILTY MODEL

To show relevance of frailty models, the insured population can be grouped into two i.e. insured persons who have an above average life expectancy are clustered in one group and individual underwriting is only performed for impaired persons.

The hazard function for the j[th] insured in the group is defined as;

$h(t_j|Z) = Zh_o(t_j)exp(\beta' x_j) \;\; j = 1,..,k$

The joint survival function for the k individuals is given by

$$S(t_1, \dots, t_k) \; = \; pr(T_1 > t_1, \dots, T_k > t_k)$$

$$\mathrm{S}(t_1, \dots, t_k) \; = \; \int_0^\infty \prod_{j=1}^{k} \mathrm{pr}(T_i > t_i|Z)\, g(z)dz$$

Since $Z \sim Noncentral\ Gamma(b, \lambda)$ assuming shape parameter $b = 0$

$$\mathrm{S}(t_1, \dots, t_k) \; = \; e^{-\frac{\sum_{j=1}^{k} H_o(t)}{1+1/2\delta^2 \sum_{j=1}^{k} H_o(t)}}$$

CHAPTER 5

APPLICATIONS TO ACTUARIAL SCIENCE

The aims of this exercise are threefold:

- The first aim is to show that when heterogeneity is disregarded the expected residual lifetime is underestimated.
- Secondly, is that neglecting heterogeneity leads to an underestimation of the insurer's liability.
- Finally, is to show the relevance of the proposed non-central Gamma frailty mixture to reflect an insurer's mortality rating.

ILLUSTRATION

Consider three hypothetical insurers i.e. insurer x, y and z.

Insurer X assumes the population to be homogeneous and applies the KE 2001-2003 life tables.

Insurer Y assumes the population to be heterogeneous and uses frailty modeling to account for heterogeneity.

Finally, Insurer Z carries out underwriting and adjusts the rates to reflect safety loadings. In this case data from Jubilee insurance is considered.

APPROACH 1

The first approach is to use the KE 2001-2003 as the baseline hazard for the frailty model.

The KE life table was published by the Association of Kenya Insurers and is based on a study conducted between 2005 and 2007. The data compiled was supplied by 18 Kenya based insurance companies.

CASE 1: Inverse Gaussian Frailty

The inverse Gaussian frailty mixture is given by:

$$h(t) = \frac{h_o(t)}{(1 + 2 * \delta^2 H_o(t))^{1/2}}$$

The frailty model: $h(t|Z) = Z * h_o(t)$

When $Z = 1$ the hazard corresponds to an average individual, hence $h(t|1) = h_o(t)$.

Thus the baseline hazard $h_o(t)$ can be approximated using the standard life tables

CASE 2 : Non-central Gamma Frailty

The inverse Gaussian frailty mixture is given by:

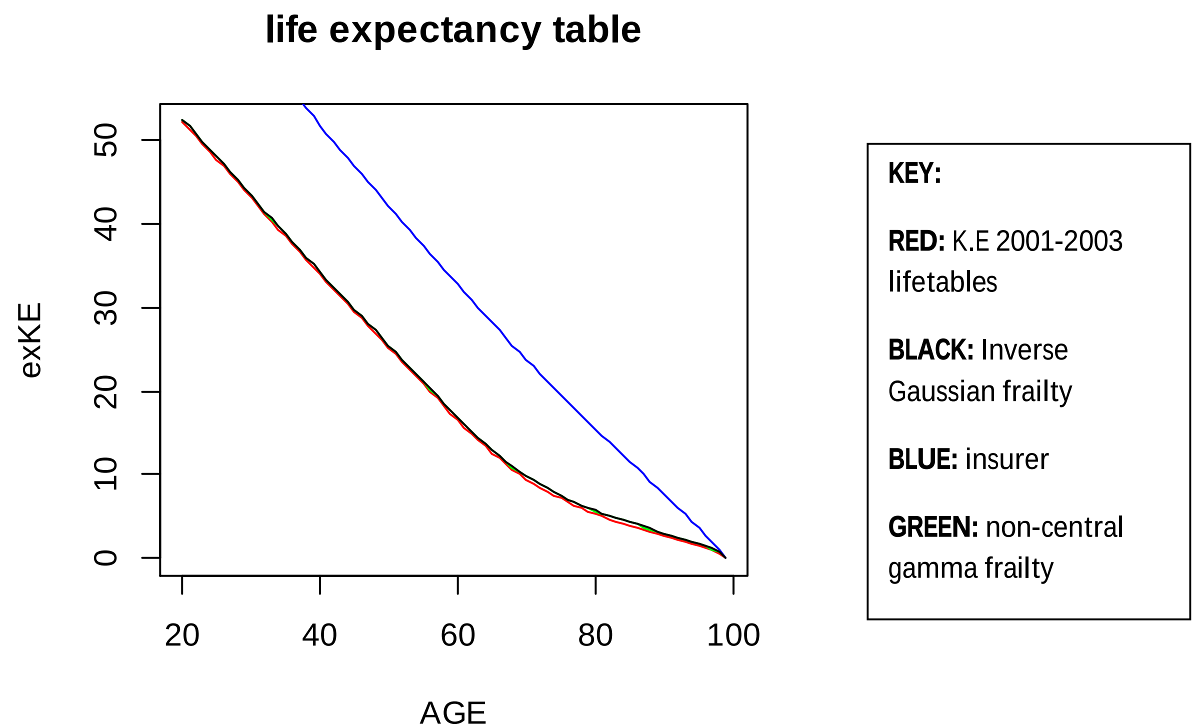
$$h(t) = \frac{h_o(t)}{(1 + 1/2 * \delta^2 H_o(t))^2}$$

Using similar assumptions for the baseline i.e.

$h_o(t) \sim K.E\ 2001 - 2003\ life\ table$

From TABLE 1.1 to 2.1 the output is represented in the graph below.

GRAPH



RESULTS

1. Ignoring heterogeneity leads to an underestimation of life expectancy.
2. The choice of frailty distribution does not have a significance impact on the life expectancy

APPROACH 2

Considering law based assumptions for the baseline hazard. i.e $H_o(t) \sim Gompertz$

Case 1: Non-central gamma gompertz model

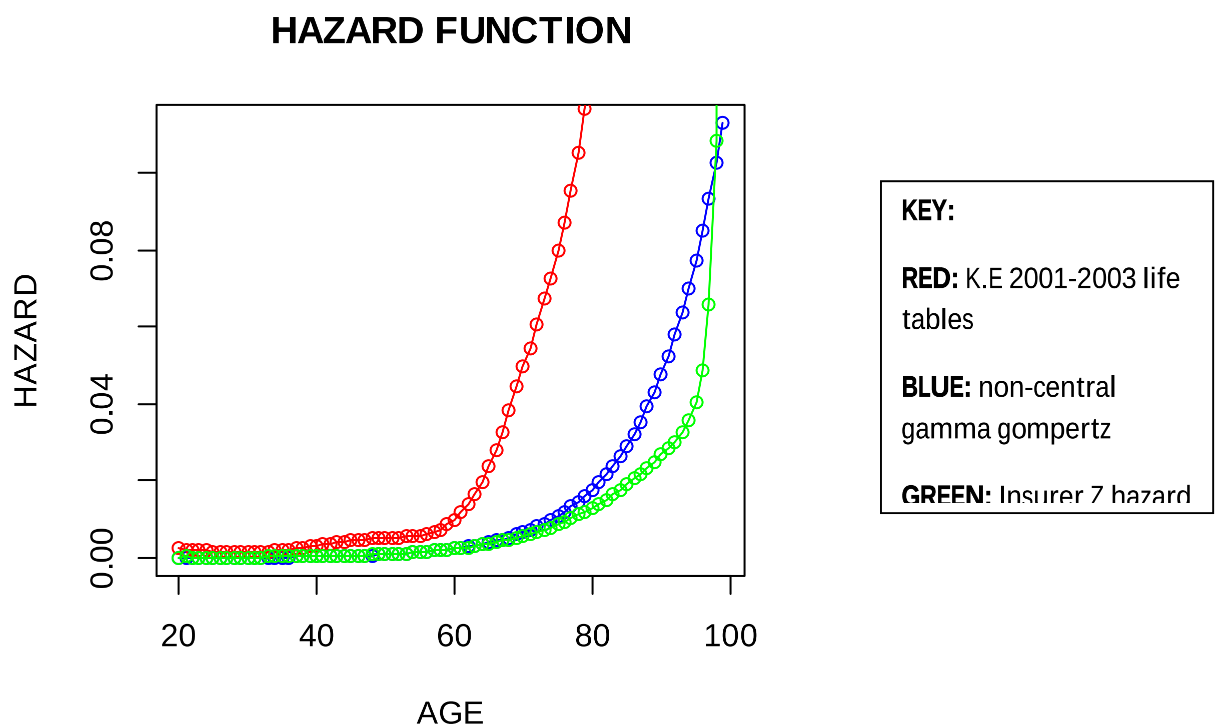
Considering $Z \sim Non - central\ Gamma\ gompertz$, $H_o(t) \sim Gompertz$

The model is given by:

$$h(x) = \frac{\propto e^{\beta x}}{(1 + \frac{1}{2} * \delta^2 \frac{\propto}{\beta}(e^{\beta x} - 1))^2}$$

From TABLE 2.1 the output is represented in the graph below.

GRAPH



RESULTS:

1. The results shows an underestimation of residual life time when heterogeneity is disregarded

5.1 PENSION SCHEME

Pension schemes are (essentially) deferred annuities whose benefits are payable on retirement.

5.1.1 Present Value

These are annuities which commence in *m* (say) years' time, provided that the annuitant is then active. Thus the present value of amount *b* payable for a future lifetime $T_{(x+t)}$

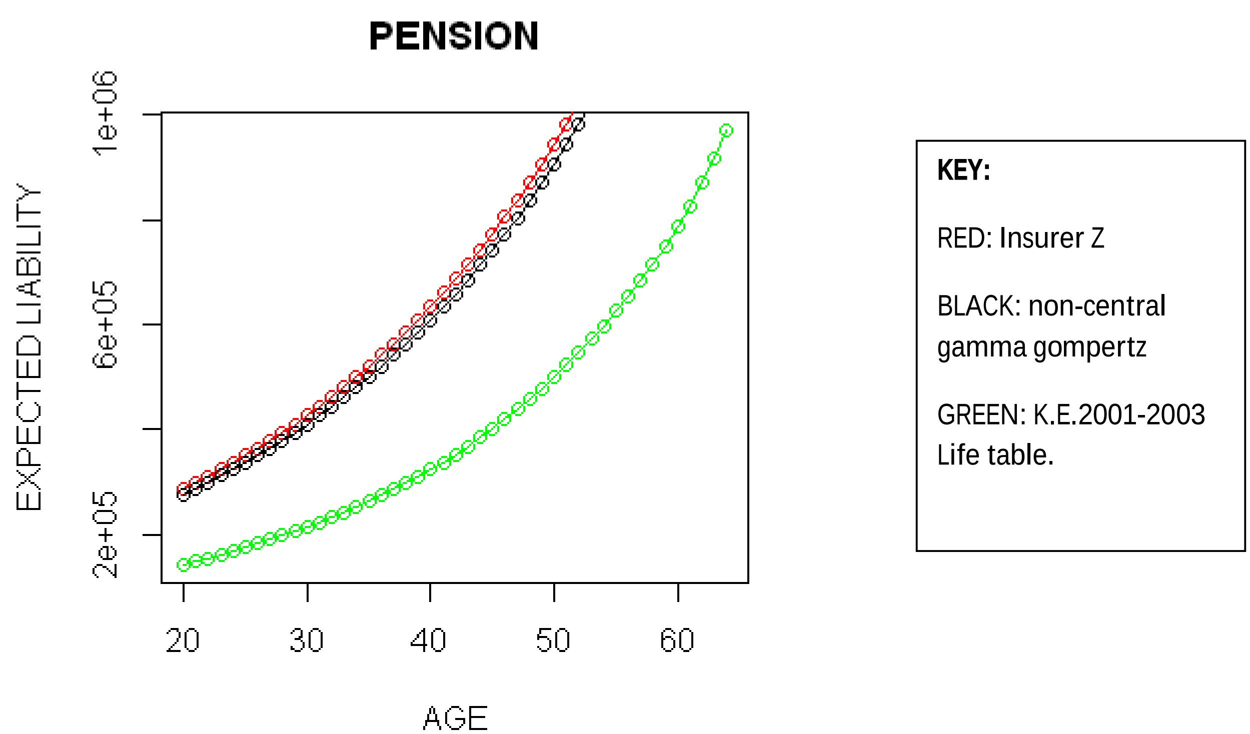
$$m|\bar{a}_x = \frac{D_{x+m}}{D_x} * a_{x+m}$$

Where $\frac{D_{x+m}}{D_x}$ is a pure endowment factor and a_{x+m} is an annuity factor at age x+m

For illustration purposes any safety loadings assigned by the insurer is not accounted for since the focus is on the effects of heterogeneity.

From TABLE 3.1 the output is represented in the graph below.

GRAPH



RESULTS

1. When heterogeneity is disregarded the expected liability is underestimated.
2. The non-central gamma frailty is a close estimate of the insurer liability.

CHAPTER 6

6.1 SUMMARY TABLE

MIXING DISTRIBUTION	BASELINE HAZARD	FRAILTY HAZARD
1. Gamma Distribution	Gompertz	$h(x) = \dfrac{\propto be^{\beta x}}{b + \frac{\propto}{\beta}(e^{\beta x} - 1)}$
	Weibull	$h(x) = \dfrac{\lambda p x^{p-1} * b}{b + \lambda x^{p}}$
	Exponential	$h(x) = \dfrac{\lambda b}{b + \lambda x}$
	Log-logistic	$h(x) = \dfrac{\left(\frac{\beta}{\propto}\right)\left(\frac{x}{\propto}\right)^{\beta-1}}{1 + \left(\frac{x}{\propto}\right)^{\beta}} / \{1 + \frac{1}{b} ln\left(1 + \left(\frac{x}{\propto}\right)^{\beta}\right)\}$
	Log normal	$h(x) = \dfrac{\dfrac{\emptyset(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})}}{1 - \frac{1}{b} ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))}$
	Exponential Power	$h(x) = \dfrac{\propto \lambda x^{\propto - 1} e^{\lambda x^{\propto}}}{1 + (e^{\lambda x^{\propto}} - 1)/b}$
	Pareto	$h(x) = \dfrac{\overline{x}}{1 - ln(\frac{\lambda}{x})/b}$
2. Inverse Gaussian Distribution	Gompertz	$h(x) = \dfrac{\propto e^{\beta x}}{(1 + 2\frac{\propto}{\beta}(e^{\beta x} - 1)\delta^{2})^{1/2}}$

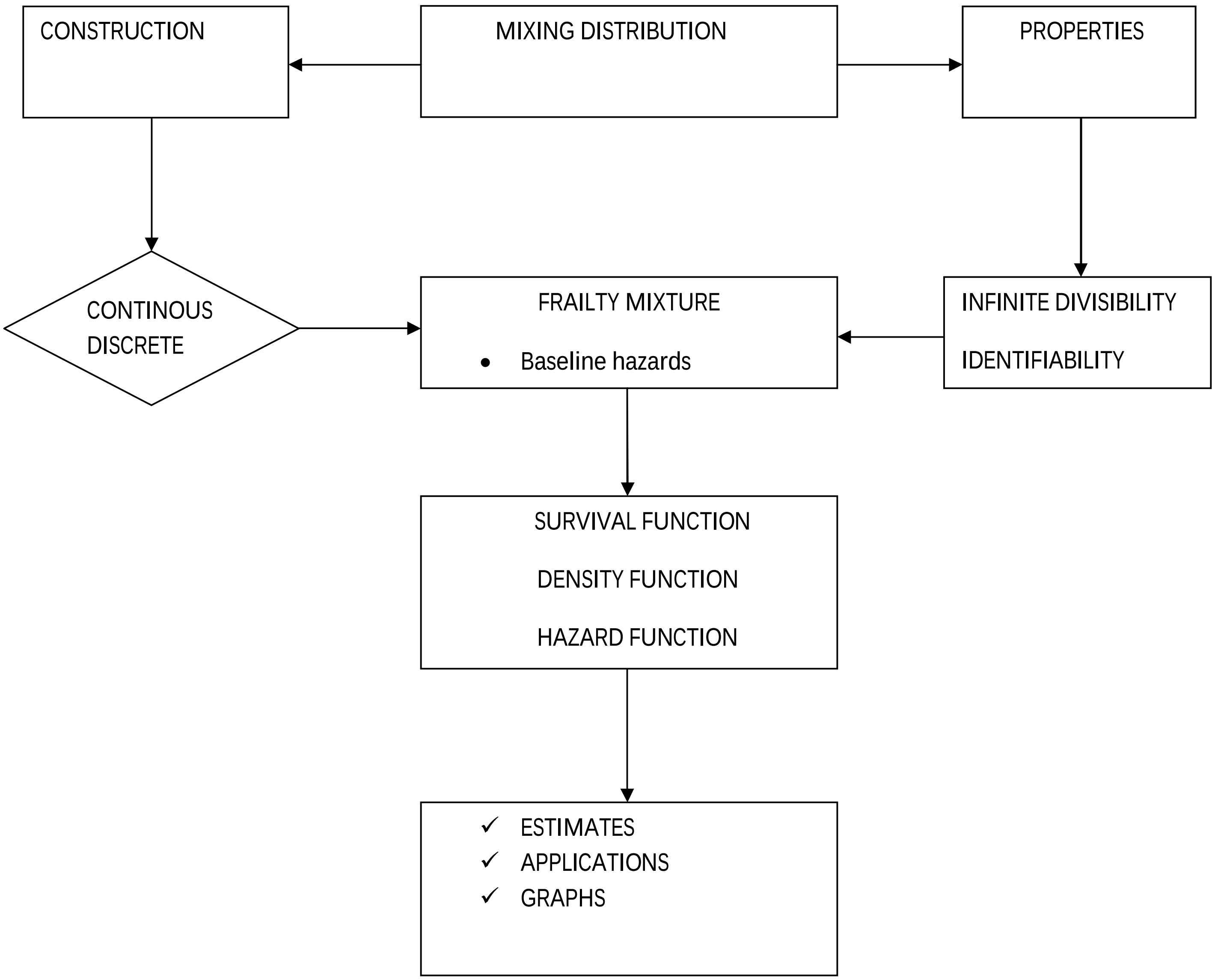
	Weibull	$h(x) = \frac{\lambda p x^{p-1}}{(1 + 2\,\lambda x^{p}\delta^{2})^{1/2}}$
	Exponential	$h(t) = \frac{\lambda}{(1 + 2\,\lambda t\delta^{2})^{1/2}}$
	Log-logistic	$h(t) = \frac{\frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}}}{(1 + 2ln\{1 + \left(\frac{t}{\propto}\right)^{\beta}\}\delta^{2})^{1/2}}$
	Log normal	$h(t) = \frac{\frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})}}{(1 - 2ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))\delta^{2})^{1/2}}$
	Exponential Power	$h(t) = \frac{\propto \lambda t^{\propto - 1} e^{\lambda t^{\propto}}}{(1 + 2(e^{\lambda t^{\propto}} - 1)\delta^{2})^{1/2}}$
	Pareto	$h(t) = \frac{\bar{t}}{(1 - 2ln(\frac{\lambda}{t})\delta^{2})^{1/2}}$
3. Positive Stable Distribution	Gompertz	$h(x) = r \propto e^{\beta x}\left\{\frac{\propto}{\beta}(e^{\beta x} - 1)\right\}^{r-1}$
	Weibull	$h(x) = r\lambda p x^{p-1}\,(\lambda x^{p})^{r-1}$
	Exponential	$h(x) = r\lambda\,(\lambda x)^{r-1}$
	Log-logistic	$h(x) = r\frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1 + \left(\frac{t}{\propto}\right)^{\beta}}\,ln\{1 + \left(\frac{x}{\propto}\right)^{\beta}\}^{r-1}$
	Log normal	$h(x) = r\frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})} * \{-ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))\}^{r-1}$

	Exponential Power	$h(x) = r \propto \lambda t^{\propto-1} e^{\lambda t^{\propto}} * (e^{\lambda t^{\propto}} - 1)^{r-1}$
	Pareto	$h(x) = \frac{r}{t}\{-ln\left(\frac{\lambda}{t}\right)\}^{r-1}$
4. Log-normal Distribution	Gompertz	$h(x) = \frac{-Ł_Z'\left(\frac{\propto}{\beta}(e^{\beta x}-1)\right)}{Ł_Z\left(\frac{\propto}{\beta}(e^{\beta x}-1)\right)}$
	Weibull	$h(x) = \frac{-Ł_Z'(\lambda x^p)}{Ł_Z(\lambda x^p)}$
	Exponential	$h(x) = \frac{-Ł_Z'(\lambda x)}{Ł_Z(\lambda x)}$
	Log-logistic	$h(x) = \frac{-Ł_Z'\left(ln\{1+\left(\frac{x}{\propto}\right)^{\beta}\}\right)}{Ł_Z\left(ln\{1+\left(\frac{x}{\propto}\right)^{\beta}\}\right)}$
5. Compound Poisson	Gompertz	$h(x) = \propto e^{\beta x} \left(1 + \frac{\delta^2}{r-1}\frac{\propto}{\beta}(e^{\beta x}-1)\right)^{r-1}$
	Weibull	$h(x) = \lambda p x^{p-1}\left(1 + \frac{\delta^2}{r-1}\lambda x^p\right)^{r-1}$
	Exponential	$h(x) = \lambda\left(1 + \frac{\delta^2}{r-1}\lambda x\right)^{r-1}$
	Log-logistic	$h(x) = \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1+\left(\frac{t}{\propto}\right)^{\beta}}\left(1 + \frac{\delta^2}{r-1}ln\{1+\left(\frac{x}{\propto}\right)^{\beta}\}\right)^{r-1}$
	Log normal	$h(x) = h_o(t)\left(1 - \frac{\delta^2}{r-1}ln(1 - \Phi(\frac{lnx-\mu}{\sigma}))\right)^{r-1}$
	Exponential Power	$h(x) = \propto \lambda t^{\propto-1} e^{\lambda t^{\propto}}\left(1 + \frac{\delta^2}{r-1} * e^{\lambda t^{\propto}} - 1\right)^{r-1}$
	Pareto	$h(x) = \frac{}{t}\left(1 - \frac{\delta^2}{r-1}ln(\frac{\lambda}{t})\right)^{r-1}$

6. Reciprocal Inverse Gaussian Distribution	Gompertz	$h(x) = \frac{\propto e^{\beta x}}{} * \left(1 + \frac{2\frac{\propto}{\beta}(e^{\beta x}-1)}{}\right)^{-1} + \propto e^{\beta x} * \left(1 + \frac{2\frac{\propto}{\beta}(e^{\beta x}-1)}{}\right)^{-\frac{1}{2}}$
	Weibull	$h(x) = \frac{\lambda p x^{p-1}}{} * \left(1 + \frac{2\lambda x^{p}}{}\right)^{-1} + \lambda p x^{p-1} * \left(1 + \frac{2\lambda x^{p}}{}\right)^{-\frac{1}{2}}$
	Log-logistic	$h(x) = \frac{1}{} \frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1+\left(\frac{t}{\propto}\right)^{\beta}} * \left(1 + \frac{2ln\{1+\left(\frac{x}{\propto}\right)^{\beta}\}}{}\right)^{-1} + \frac{1}{} * \left(1 + \frac{2ln\{1+\left(\frac{x}{\propto}\right)^{\beta}\}}{}\right)^{-\frac{1}{2}}$
	Log normal	$h(x) = \frac{\frac{f(x)}{1-\Phi(\frac{lnx-\mu}{\sigma})}}{} * \left(1 - \frac{2ln(1- \Phi(\frac{lnx-\mu}{\sigma}))}{}\right)^{-1} + \frac{1}{} * \left(1 - \frac{2ln(1- \Phi(\frac{lnx-\mu}{\sigma}))}{}\right)^{-\frac{1}{2}}$
	Exponential	$h(x) = \frac{\lambda}{} * \left(1 + \frac{2\lambda x}{}\right)^{-1} + \lambda * \left(1 + \frac{2\lambda x}{}\right)^{-\frac{1}{2}}$
	Exponential Power	$h(x) = \frac{\propto \lambda t^{\propto-1} e^{\lambda t^{\propto}}}{} * \left(1 + \frac{2e^{\lambda t^{\propto}}-1}{}\right)^{-1} + \frac{1}{} * \left(1 + \frac{2e^{\lambda t^{\propto}}-1}{}\right)^{-\frac{1}{2}}$
	Pareto	$h(x) = \frac{r}{t} * \left(1 - \frac{2ln(\frac{\lambda^{r}}{t^{r}})}{}\right)^{-1} + \frac{1}{} * \left(1 - \frac{2ln(\frac{\lambda^{r}}{t^{r}})}{}\right)^{-\frac{1}{2}}$
7. Inverse Gamma Distribution	Gompertz	$S(x) = \frac{2\left((\propto-1)\propto(e^{\beta x}-1)\right)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4(\propto-1)\propto(e^{\beta x}-1))$
	Weibull	$S(x) = \frac{2((\propto-1)\lambda x^{p})^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4(\propto-1)\lambda x^{p})$
	Exponential	$S(x) = \frac{2\left((\propto-1)\lambda x\right)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4(\propto-1)\lambda x)$
	Log-logistic	$S(x) = \frac{2\left((\propto-1)ln\{1+\left(\frac{x}{\propto}\right)^{\beta}\}\right)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4(\propto-1)ln\{1+\left(\frac{x}{\propto}\right)^{\beta}\})$

	Log normal	$S(x) = \frac{2\left(-(\propto -1)ln(1- \Phi(\frac{lnx-\mu}{\sigma}))\right)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(-4(\propto -1)ln(1- \Phi(\frac{lnx-\mu}{\sigma})))$
	Exponential Power	$S(x) = \frac{2\left((\propto -1)(e^{\lambda t^{\propto}}-1)\right)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(4(\propto -1)(e^{\lambda t^{\propto}}-1))$
8. Non-Central Gamma Distribution	Gompertz	$h(t) = \frac{\propto e^{\beta x}}{(1+\frac{1}{2} * \delta^2 \frac{\propto}{\beta}(e^{\beta x}-1))^2}$
	Weibull	$h(t) = \frac{\lambda p x^{p-1}}{(1+\frac{1}{2} * \delta^2 \lambda x^p)^2}$
	Exponential	$h(t) = \frac{\lambda}{(1+\frac{1}{2} * \delta^2 \lambda x)^2}$
	Log-logistic	$h(t) = \frac{\frac{\left(\frac{\beta}{\propto}\right)\left(\frac{t}{\propto}\right)^{\beta-1}}{1+\left(\frac{t}{\propto}\right)^{\beta}}}{(1+\frac{1}{2} * \delta^2 ln\{1+\left(\frac{x}{\propto}\right)^{\beta}\})^2}$
	Log normal	$h(t) = \frac{\frac{f(x)}{1-\Phi(\frac{lnx-\mu}{\sigma})}}{(1-\frac{1}{2} * \delta^2 ln(1- \Phi(\frac{lnx-\mu}{\sigma})))^2}$
	Exponential Power	$h(t) = \frac{\propto \lambda t^{\propto -1} e^{\lambda t^{\propto}}}{(1+\frac{1}{2} * \delta^2 (e^{\lambda t^{\propto}}-1))^2}$
	Pareto	$h(t) = \frac{r}{t(1-\frac{1}{2} * \delta^2 ln(\frac{\lambda^r}{t^r}))^2}$

6.2 MODEL FRAMEWORK



6.3 DISCUSSION

The conclusion to be reached from the analyses and discussions is that comparing the standard life tables with the Gamma-Gompertz and Inverse Gaussian model; shows an increase in the insurers expected liability when heterogeneity is considered. That is, assuming the insured to be homogeneous could lead to an underestimation of future liability.

Further, using Non-central Gamma model in estimating future liability by directly adjusting the A.K.I mortality tables shows an increase in longevity risk. The extent of heterogeneity of the insured group determines the level of risk.

A key point to note is that the non-central gamma frailty model as proposed gives better estimate of the insurer rates compared to the gamma frailty model with similar assumptions of the population level of heterogeneity. The correlation coefficient between the non-central gamma and insurers rates is also higher.

Thus, the non-central family distributions is recommended for further study as it gives better estimates for the insurer’s rating.

APPENDIX 1A

LAPLACE TRANSFORMS

1.1 Gamma Distribution

$Let\ Z \sim \Gamma(p, b)$ be gamma distributed with shape parameter p and scale parameter b. The probability density function is given by;

$$f(z) = \frac{b^p z^{p-1} e^{-bz}}{\Gamma(p)}$$

$$L_Z(s) = E[e^{-sZ}]$$

$$= \int_0^\infty e^{-sZ} \frac{b^p z^{p-1} e^{-bz}}{\Gamma(p)} dz$$

$$= \frac{b^p}{\Gamma(p)} \int_0^\infty e^{-Z*(s+b)} Z^{p-1} dz$$

$$let\ y = Z * (s + b)\ ;\ z = \frac{y}{s + b}\ ;\ dz = \frac{dy}{s + b}$$

$$= \frac{b^p}{\Gamma(p)} \int_0^\infty e^{-y} (\frac{y}{s+b})^{p-1} \frac{dy}{s+b}$$

$$= \frac{b^p}{\Gamma(p)(s+b)^p} \int_0^\infty e^{-y} (y)^{p-1}\, dy$$

$$L_Z(s) = (\frac{b}{b+s})^p = (1 + \frac{s}{b})^{-p}$$

1.2 Inverse-Gaussian distribution

Willmot (1986) derived the Laplace from a Generalized Inverse-Gaussian mixing distribution

$$(z) = \frac{\mu^{-\propto} z^{\propto-1} e^{-(z^2+\mu^2)/2\beta z}}{2K_\propto(\mu\beta^{-1})}$$

Where $K_\propto(.)$ is the modified Bessel function of the III kind

Substituting $\propto = -\frac{1}{2}$

$$(z) = \frac{\mu^{1/2} z^{-3/2} e^{-(z^2+\mu^2)/2\beta z}}{2K_{1/2}(\mu\beta^{-1})}$$

$$(z) \ = \mu(2\pi\beta z^3)^{-1/2} e^{-(z-\mu)^2/2\beta z}$$

The Laplace transform is

$$L_Z(s) = \exp\{-\frac{\mu}{\beta}(1 + 2\beta s)^{1/2} - 1\}$$

1.3 Reciprocal Inverse-Gaussian distribution

$$(z) = (\frac{}{2\pi\lambda})^{1/2} exp(-\frac{\mu^2}{2\lambda}(1 - \frac{\lambda}{\mu})^2\}$$

The Laplace transform is given by

$$L_Z(s) = (1 + \frac{2s}{})^{-1/2} \exp\{\mu[1 - \left(1 + \frac{2s}{}\right)^{\frac{1}{2}}]\}$$

1.4 Compound Poisson distribution

Let N be a Poisson random variable with parameter p > 0 and let $X_i, i = 1, 2, \ldots$ be i.i.d. random variables, independent of N.

Z ~ Compound Poisson Distribution (CPD) defined as

$$Z = \sum_{i=1}^{N} X_i$$

If $E(X)$ and $V(X)$ are the common mean and variance of the random variables $X_i, i = 1, 2, \ldots$ then, the moments of Z are given by

$$E(Z) = p * E(X)$$

$$V(Z) = p * [V(X) + [E(X)]^2]$$

The Laplace transform is given by

$$L_Z(s) = E[e^{-sZ}]$$

$$= E\{E\left[e^{-s(x_1+x_2+..+x_N)}\middle|N = n\right]\}$$

$$= E\{E[(e^{-sx} * e^{-sx} * \dots * e^{-sx})]$$

$$= E\{[E(e^{-sx})]^n\}$$

$$L_Z(s) = F(L_x(s))$$

$$L_Z(s) = e^{p(L_x(s)-1)}$$

1.5 Power Variance Functions

The PVF model is a three parameter family denoted by $PVF(r, k, \lambda)$.

The Laplace transform is

$$L(s) = e^{-\frac{k}{r}\{(\lambda+s)^r-\lambda^r\}}$$

Special Case (λ=0)

The Laplace transform becomes

$$L(s) = e^{-\frac{k}{r}*s^r}$$

This is the Laplace for a Positive Stable distribution.

1.6 Log-Normal distribution

If W is a random variable with a normal distribution, then $Z = exp(W)$ has a log-normal distribution.

$$f(w) = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{(w-\mu)^2}{2\delta^2}} \quad -\infty < w < \infty$$

The probability density function Z is given by

$$f(z) = \frac{1}{z\delta\sqrt{2\pi}} e^{-\frac{(lnz-\mu)^2}{2\delta^2}} \quad z > 0$$

Where $\delta > 0$ is the scale and $-\infty < \mu < \infty$ is the location parameter

The mean and variance are

$$E[Z] = e^{\mu+\delta/2}$$

$$Var[Z] = e^{2\mu+\delta} * e^{\delta^2-1}$$

$CV = (e^{\delta^2-1})^{1/2}$ Only depends on δ

An approximation of the Laplace transform is obtained by

$$L_z(s) = E[e^{-sz}]$$

$$= \int_0^\infty e^{-sz} f(z)dz$$

$$L_Z(s) = \int_0^\infty \frac{1}{z\delta\sqrt{2\pi}} e^{-sz-\frac{(lnz-\mu)^2}{2\delta^2}} dz$$

No closed form of the Laplace transform is known, however several approximation methods are possible. i.e.

The Initiative Approach

Consider for $k = 0, 1, 2, 3 \ldots$

E $[Z^K e^{-SZ}] = \int_0^{\infty} \frac{Z^{K-1}}{\delta\sqrt{2\pi}} e^{-SZ-\frac{(lnZ-\mu)^2}{2\delta^2}} dz$

Using change of variable

$w = logz$

$z = e^w$

$\frac{dz}{dw} = e^w$

E $[e^{wk-Se^w}] = \int_0^{\infty} \frac{e^{w(K-1)}}{\delta\sqrt{2\pi}} e^{-se^w-\frac{(w-\mu)^2}{2\delta^2}} dw * e^w$

$$= \int_{-\infty}^{\infty} \frac{1}{\delta\sqrt{2\pi}} e^{-se^w+kw-\frac{(w-\mu)^2}{2\delta^2}} dw$$

Replacing the expression - $-se^w + kw - \frac{(w-\mu)^2}{2\delta^2}$ by a by a Taylor approximation of second order around the value k that maximizes this expression. That is

- $se^k[1 + (w-k) + \frac{(w-k)^2}{2}] + kw - \frac{(w-\mu)^2}{2\delta^2}$

Thus the resulting integral can be explicitly obtained

$$= \int_{-\infty}^{\infty} \frac{1}{\delta\sqrt{2\pi}} e^{-se^k[1+(w-k)+\frac{(w-k)^2}{2}]+kw-\frac{(w-\mu)^2}{2\delta^2}} dx$$

The difficulty in the lognormal case is the explicit calculation of the value

$k = -LW(s\delta^2 e^{k\delta^2+\mu} + k\delta^2 + \mu)$

Where the function $LW[-e^{-1}, \infty]$ known as the LambertW, is the inverse of

$f\ (W)\ =\ We^{W}$:

In particular, with $k\ =0$

$$L_Z(s)\ \approx \frac{1}{\sqrt{LW(\delta^2 \mathrm{e}^{\mu})}}\, e^{-\frac{LW^2(\delta^2 \mathrm{e}^{\mu})+2LW(\delta^2 \mathrm{e}^{\mu})}{2\delta^2}}$$

1.7 Inverse Gamma Distribution

The probability density function is given by

$$f(z,\propto,\beta)\ = \frac{\beta^{\propto}}{\Gamma(\propto)} z^{-\propto-1} exp\left(\frac{-\beta}{z}\right)\ for\ x > 0$$

Where β is the shape parameter and ∝ scale parameter

The Laplace transform is given by

$$L_Z(s)\ = \frac{2(-\beta s)^{\frac{\propto}{2}}}{\Gamma(\propto)} K_{\propto}(-4\beta s)$$

Where $K_{\propto}(.)$is the modified Bessel function of the II kind.

Construction

The density function for the Gamma distribution is

$$f(x)\ = \frac{x^{(k-1)} e^{(-x/)}}{{}^{k}\Gamma(k)}$$

Define the transformation $Z\ =\ g(x)\ =\ \frac{1}{x}$

$$f(z)\ =\ f(g^{-1}(z))|\frac{d}{dz} g^{-1}(z)|$$

$$f(z)\ =\ \frac{(\frac{1}{z})^{(k-1)} e^{(-1/z)}\,\frac{1}{z^2}}{{}^{k}\Gamma(k)}$$

$$f(z)\ =\ \frac{(z)^{(-k-1)} e^{(-1/z)}}{{}^{k}\Gamma(k)}$$

Replacing k = $\propto$, $^{-1}$ = β

$$f(z) = \frac{\beta^{\propto}}{\Gamma(\propto)} z^{-\propto-1} exp\,(\frac{-\beta}{z})$$

1.8 Non Central gamma distribution

Let λ > 0, let N; X_1, X_2,.., X_N be independent random variables such that N is Poisson distributed with mean λ. Define Y=$X_1 + \cdots + X_N$ with the convention that Y = 0 if N = 0: then the Laplace transform is

L_Y(s) = E(e^{-sY})

L_Y(s) = E[$e^{-s(X_1+\cdots+X_n)}$]

L_Y(s) = $E\{(e^{-sX})^n\}$ since X_i $are\ IID$

L_Y(s) = $E\{(L_X(s))^n\}$ = F ($L_X(s)$)

L_Y(s) = $e^{-\lambda(1-L_X(s))}$ where L_X(s) is the Laplace tranform of X

Y being a mixing of the distributions of $X_1 + \cdots + X_N$ with respective weights $\frac{e^{-\lambda}(\lambda)^n}{n!}$ leads to the convolution;

$$f(x,b,\lambda) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}(\lambda)^n}{n!} [\frac{e^{-x}x^{b+n-1}}{\Gamma(b+n)}]$$

The Laplace transform is given by

L_X(s) = E (e^{-sx})

L_X(s) = $\int_0^{\infty} e^{-sx} f(x,b,\lambda)dx$

$L_X(s) = \frac{1}{(1+\lambda s)^b} e^{-\frac{sa(\lambda)^2}{1+\lambda s}}$ with a being the non-centrality parameter, b is the shape parameter and λ is the scale parameter

Using shape parameter $b = 0$

$$L_Y(s) = e^{-\frac{sa(\lambda)^2}{1+\lambda s}}$$

For identifiability the mean is normalized to one

Mean = $a(\lambda)^2$ = 1

Var = $2a(\lambda)^3 : \delta^2 = 2\lambda : \frac{1}{2} * \delta^2 = \lambda$

$$L_Y(s) = e^{-\frac{s}{1+\frac{1}{2}*\delta^2 s}}$$

The marginal survival function is given by;

$$S(x) = L_Y(H_o(t)) = e^{-\frac{H_o(t)}{1+1/2\delta^2 H_o(t)}}$$

$$S(x) = e^{-\frac{H_o(t)}{1+1/2\delta^2 H_o(t)}}$$

$$f(x) = \frac{h_o(t)}{(1 + 1/2\delta^2 H_o(t))\text{^}2} * e^{-\frac{H_o(t)}{1+1/2\delta^2 H_o(t)}}$$

The hazard function is given by;

$$h(t) = \frac{-s'(x)}{s(x)} = \frac{h_o(t)}{(1 + \frac{1}{2} * \delta^2 H_o(t))^2}$$

APPENDIX 1B

The Multiplicative Model using Taylor's Series

The multiplicative model can be described using Taylor's series expansion as shown

Let $h(t, z)$ be an individual hazard.

$$h_t(z) = \lim_{t\to 0} \frac{p(Tx < t|Zx = z)}{t}$$

By Taylor series expansion;

$$h(t; Z) = h(t; 0) + Zh'(t; 0) + \frac{Z^2}{2!}h''(t; 0) + \frac{Z^3}{3!}h'''(t; 0) + \cdots$$

$$h(t; Z) = h(t; 0) + Zh'(t; 0) + o(Z)$$

The o(Z) denotes the terms containing Z of higher order than one. By omitting these terms we get

$$h(t; Z) = h(t; 0) + Zh'(t; 0)$$

Making the natural assumption that zero frailty (susceptibility) yields zero mortality it holds that

$$h(t; Z) = Zh_o(t)$$

$$where\ h_o(t) = h'(t; 0) = \frac{d}{dz}\ h(t; z)\ @z = 0$$

Thus the underlying hazard $h_o(t)$ is a partial derivative of the individual hazard with respect to frailty taken at point $Z = 0$. this is the non-frailty hazard.

This shows that a multiplicative frailty model is a rather simplified view of how heterogeneity may act.

APPENDIX 1C

R-CODES

R-Code 1

r-code

```
alpha = 4.419e-06

beta=0.1

ho=alpha*(exp(beta*x))

Ho=(alpha/beta)*(exp(beta*x)-1)

gig=ho/sqrt(1+2*Ho*0.05)

plot(x,gig,main="I.G-Gompertz hazard",type="o",xlab="age",ylab="hazard")
```

R-code 2

Using r=1.1, λ=0.01

```
wcp=(1.1*0.01*age^0.1)*((1+0.5*0.01*age^1.1)^0.1)

plot(age,wcp,main="CompoundPoisson-Weibull",type="o", xlab="age",ylab="hazard")
```

R-code 3

```
grig=(ho/1.1)/(1+2*Ho/1.1)+ho/sqrt(1+2*Ho/1.1)

 plot(age,grig,main="gompertz-reciprocal IG",type="o",col="blue")
```

R-code 4

```
psgf=0.4*alpha*exp(beta*x)*(((alpha/beta)*(exp(beta*x)-1))^(0.4-1))

plot(x,psgf,main="Positive stable-Gompertz",type="o", xlab="age",ylab="hazard")
```

R-code 5

```
alpha=0.00004 #initial estimate

beta=0.1 #fixed for estimation

ho=alpha*(exp(beta*x))

qho=ho/(1+0.5*ho)

jubileeqx /((1-0.5* jubileeqx)*( exp(beta*x)))#inverse

median(jubileeqx /((1-0.5* jubileeqx)*( exp(beta*x))))

4.419414e-06 #alpha new estimate
```

R-code 4

```
alpha= 0.00025

beta=0.1

age=20:100

ho=alpha*(exp(beta*age))

Ho=(alpha/beta)*(exp(beta*age)-1)

gngf=ho/((1+0.05*Ho/2)^2)

plot(age,gngf,main="gompertz-non-centralgamma",type="o", xlab="age",ylab="hazard")
```

APPENDIX 1D

TABLES

TABLE1.0

K.E 2001-2003 Male mortality rates						
Age (x)	l_x	d_x	p_x	q_x	μ_x	e_x
20	100,000	239	0.99761	0.00239	0.00239	52.29
21	99,761	222	0.99778	0.00222	0.00223	51.41
22	99,539	206	0.99793	0.00207	0.00207	50.53
23	99,333	191	0.99808	0.00192	0.00193	49.63
24	99,142	178	0.99820	0.00180	0.00180	48.73
25	98,964	167	0.99831	0.00169	0.00169	47.81
26	98,797	158	0.99840	0.00160	0.00160	46.89
27	98,639	151	0.99847	0.00153	0.00153	45.97
28	98,489	146	0.99852	0.00148	0.00148	45.04
29	98,343	144	0.99854	0.00146	0.00146	44.11
30	98,199	144	0.99853	0.00147	0.00147	43.17
31	98,055	148	0.99849	0.00151	0.00151	42.23
32	97,907	154	0.99842	0.00158	0.00158	41.30
33	97,753	164	0.99832	0.00168	0.00168	40.36
34	97,589	178	0.99818	0.00182	0.00182	39.43
35	97,411	194	0.99800	0.00200	0.00200	38.50
36	97,216	213	0.99781	0.00219	0.00220	37.58
37	97,003	234	0.99758	0.00242	0.00242	36.66
38	96,769	257	0.99735	0.00265	0.00266	35.75
39	96,512	280	0.99709	0.00291	0.00291	34.85
40	96,232	304	0.99684	0.00316	0.00317	33.95
41	95,927	328	0.99658	0.00342	0.00343	33.06
42	95,599	352	0.99632	0.00368	0.00369	32.17
43	95247	375	0.99607	0.00393	0.00394	31.29
44	94872	396	0.99583	0.00417	0.00418	30.41
45	94476	415	0.99561	0.00439	0.00440	29.54
46	94061	432	0.99541	0.00459	0.00461	28.67
47	93629	446	0.99523	0.00477	0.00478	27.80
48	93183	458	0.99509	0.00491	0.00492	26.94
49	92725	465	0.99498	0.00502	0.00503	26.07
50	92260	469	0.99492	0.00508	0.00509	25.20
51	91791	478	0.99479	0.00521	0.00522	24.33
52	91313	485	0.99469	0.00531	0.00532	23.46
53	90828	495	0.99455	0.00545	0.00546	22.58

54	90333	506	0.99440	0.00560	0.00562	21.71
55	89827	526	0.99414	0.00586	0.00588	20.83
56	89301	558	0.99375	0.00625	0.00627	19.95
57	88743	600	0.99324	0.00676	0.00678	19.08
58	88143	654	0.99258	0.00742	0.00745	18.21
59	87489	745	0.99149	0.00851	0.00855	17.34
60	86744	863	0.99006	0.00994	0.00999	16.49
61	85882	1008	0.98826	0.01174	0.01181	15.66
62	84874	1182	0.98607	0.01393	0.01403	14.84
63	83692	1386	0.98344	0.01656	0.01670	14.05
64	82306	1619	0.98033	0.01967	0.01987	13.29
65	80686	1881	0.97669	0.02331	0.02358	12.55
66	78806	2167	0.97250	0.02750	0.02789	11.85
67	76639	2476	0.96770	0.03230	0.03283	11.19
68	74163	2798	0.96227	0.03773	0.03846	10.56
69	71365	3123	0.95624	0.04376	0.04475	9.98
70	68242	3303	0.95160	0.04840	0.04961	9.43
71	64939	3442	0.94700	0.05300	0.05446	8.91
72	61497	3628	0.94100	0.05900	0.06081	8.41
73	57869	3761	0.93500	0.06500	0.06721	7.94
74	54107	3788	0.93000	0.07000	0.07257	7.49
75	50320	3849	0.92350	0.07650	0.07958	7.06
76	46470	3876	0.91660	0.08340	0.08708	6.64
77	42595	3876	0.90900	0.09100	0.09541	6.24
78	38719	3872	0.90000	0.10000	0.10536	5.87
79	34847	3833	0.89000	0.11000	0.11653	5.52
80	31014	3660	0.88200	0.11800	0.12556	5.20
81	27354	3447	0.87400	0.12600	0.13467	4.90
82	23907	3251	0.86400	0.13600	0.14618	4.61
83	20656	2974	0.85600	0.14400	0.15549	4.33
84	17682	2723	0.84600	0.15400	0.16724	4.06
85	14959	2453	0.83600	0.16400	0.17913	3.80
86	12505	2151	0.82800	0.17200	0.18874	3.55
87	10354	1905	0.81600	0.18400	0.20334	3.28
88	8449	1665	0.80300	0.19700	0.21940	3.02
89	6785	1452	0.78600	0.21400	0.24080	2.77
90	5333	1237	0.76800	0.23200	0.26397	2.52
91	4096	1028	0.74900	0.25100	0.28902	2.28
92	3068	841	0.72569	0.27431	0.32064	2.04
93	2226	672	0.69799	0.30201	0.35956	1.81
94	1554	516	0.66785	0.33215	0.40369	1.60

95	1038	379	0.63517	0.36483	0.45387	1.39
96	659	264	0.59981	0.40019	0.51113	1.20
97	395	173	0.56163	0.43837	0.57691	0.99
98	222	106	0.52054	0.47946	0.65289	0.77
99	116	61	0.47638	0.52362	0.74154	0.48
100	55	55	0.00000	1.00000		0.00

TABLE1.1

KENYAN LIFETABLE - MALES					BENEFIT:
					SUM ASSURED:
Age (x)	$\bar{a}_x$	$\bar{A}_x$	P(Ā)	$\bar{a}_{x:65-x}$	$_{65-x\|}\bar{a}_x$
20	19.34732	10339.4	534.4101	18.47529	87203.66
21	19.28485	10636.9	551.5675	18.36702	91783.20
22	19.2159	10965.23	570.6334	18.25003	96587.18
23	19.14035	11325	591.6824	18.12408	101626.70
24	19.05812	11716.56	614.7806	17.98898	106913.84
25	18.96918	12140.07	639.9889	17.84457	112461.63
26	18.87355	12595.45	667.36	17.69071	118284.25
27	18.77126	13082.57	696.9466	17.52729	124397.00
28	18.66237	13601.07	728.7967	17.35421	130816.49
29	18.54701	14150.42	762.9488	17.1714	137560.89
30	18.42533	14729.84	799.4343	16.97883	144650.13
31	18.29753	15338.42	838.2783	16.77647	152105.93
32	18.16385	15974.99	879.4939	16.56433	159952.28
33	18.02454	16638.39	923.0967	16.34238	168215.17
34	17.87991	17327.1	969.0822	16.11067	176923.51
35	17.73029	18039.54	1017.442	15.8692	186108.95
36	17.57594	18774.54	1068.195	15.61789	195805.04
37	17.41702	19531.31	1121.392	15.35655	206047.36
38	17.25363	20309.37	1177.107	15.08489	216873.69
39	17.08572	21108.91	1235.471	14.80249	228323.34
40	16.91321	21930.39	1296.642	14.50883	240437.98
41	16.73588	22774.84	1360.839	14.20327	253260.94
42	16.55342	23643.68	1428.326	13.88505	266837.63
43	16.36542	24538.92	1499.437	13.55327	281214.96
44	16.17137	25463	1574.574	13.20695	296441.60
45	15.97063	26418.87	1654.215	12.84496	312567.72

46	15.7625	27410	1738.938	12.46605	329644.57
47	15.54613	28440.34	1829.416	12.06888	347724.59
48	15.32055	29514.51	1926.465	11.65195	366859.99
49	15.08471	30637.53	2031.032	11.21368	387103.68
50	14.83745	31815.01	2144.238	10.75237	408507.95
51	14.57744	33053.12	2267.416	10.26621	431123.45
52	14.30491	34350.88	2401.335	9.754407	455050.43
53	14.01867	35713.96	2547.6	9.21513	480353.61
54	13.71844	37143.62	2707.57	8.647083	507135.22
55	13.40349	38643.38	2883.084	8.048579	535490.69
56	13.07435	40210.71	3075.543	7.418551	565579.54
57	12.73171	41842.33	3286.467	6.755773	597593.47
58	12.37602	43536.06	3517.774	6.058587	631743.72
59	12.008	45288.58	3771.535	5.325099	668289.62
60	11.63145	47081.66	4047.789	4.55418	707726.85
61	11.24895	48903.07	4347.345	3.743182	750576.97
62	10.86296	50741.14	4671.024	2.888309	797464.89
63	10.47609	52583.38	5019.373	1.984433	849165.28
64	10.09108	54416.75	5392.56	1.024695	906638.41
65	9.710743	56227.88	5790.275	0	
66	9.337991	58002.89	6211.495		
67	8.975823	59727.5	6654.264		
68	8.627356	61386.86	7115.373		
69	8.295777	62965.81	7590.104		
70	7.984024	64450.34	8072.413		
71	7.678957	65903.05	8582.291		
72	7.378009	67336.13	9126.598		
73	7.089247	68711.19	9692.312		
74	6.810459	70038.75	10284		
75	6.532313	71363.25	10924.65		
76	6.262046	72650.24	11601.68		
77	5.999584	73900.06	12317.53		
78	5.746571	75104.88	13069.51		
79	5.508855	76236.86	13838.97		
80	5.290301	77277.6	14607.41		
81	5.078103	78288.06	15416.79		
82	4.869655	79280.68	16280.55		
83	4.672693	80218.59	17167.53		
84	4.474766	81161.1	18137.51		
85	4.282002	82079.02	19168.37		
86	4.091115	82988.01	20284.94		

87	3.888576	83952.48	21589.52		
88	3.68514	84921.22	23044.23		
89	3.478787	85903.85	24693.62		
90	3.278369	86858.22	26494.34		
91	3.081194	87797.16	28494.52		
92	2.882946	88741.19	30781.42		
93	2.68871	89666.13	33349.13		
94	2.503225	90549.39	36173.09		
95	2.324548	91400.23	39319.57		
96	2.148795	92237.15	42925.05		
97	1.967783	93099.11	47311.68		
98	1.763161	94073.51	53355.04		
99	1.489594	95376.2	64028.3		
100	1.024695	97590.01	95238.1		

TABLE2.0

insurer x male mortality rates						
Age (x)	lx	dx	p_x	q_x	μ_x	$e_{x(\text{insurer x})}$
20	100,000	23.12	0.999769	0.000231	0.000231	71.61
21	99,977	23.31	0.999767	0.000233	0.000233	70.63
22	99,954	23.01	0.99977	0.00023	0.00023	69.64
23	99,931	22.60	0.999774	0.000226	0.000226	68.66
24	99,908	22.10	0.999779	0.000221	0.000221	67.68
25	99,886	21.40	0.999786	0.000214	0.000214	66.69
26	99,864	20.79	0.999792	0.000208	0.000208	65.70
27	99,844	20.59	0.999794	0.000206	0.000206	64.72
28	99,823	20.38	0.999796	0.000204	0.000204	63.73
29	99,803	20.58	0.999794	0.000206	0.000206	62.74
30	99,782	20.97	0.99979	0.00021	0.00021	61.76
31	99,761	21.67	0.999783	0.000217	0.000217	60.77
32	99,739	22.36	0.999776	0.000224	0.000224	59.78
33	99,717	23.46	0.999765	0.000235	0.000235	58.80
34	99,694	24.75	0.999752	0.000248	0.000248	57.81
35	99,669	26.24	0.999737	0.000263	0.000263	56.83
36	99,643	28.04	0.999719	0.000281	0.000281	55.84
37	99,615	30.32	0.999696	0.000304	0.000304	54.86
38	99,584	32.91	0.99967	0.000331	0.000331	53.87
39	99,551	35.90	0.999639	0.000361	0.000361	52.89
40	99,515	39.28	0.999605	0.000395	0.000395	51.91

41	99,476	43.26	0.999565	0.000435	0.000435	50.93
42	99,433	47.33	0.999524	0.000476	0.000476	49.95
43	99,386	52.01	0.999477	0.000523	0.000523	48.98
44	99,334	56.87	0.999428	0.000573	0.000573	48.00
45	99,277	62.43	0.999371	0.000629	0.000629	47.03
46	99,214	67.97	0.999315	0.000685	0.000685	46.06
47	99,146	74.01	0.999254	0.000747	0.000747	45.09
48	99,072	80.35	0.999189	0.000811	0.000811	44.12
49	98,992	87.47	0.999116	0.000884	0.000884	43.16
50	98,905	94.97	0.99904	0.00096	0.000961	42.20
51	98,810	103.65	0.998951	0.001049	0.00105	41.24
52	98,706	113.31	0.998852	0.001148	0.001149	40.28
53	98,593	124.35	0.998739	0.001261	0.001262	39.33
54	98,468	136.75	0.998611	0.001389	0.00139	38.38
55	98,331	149.92	0.998475	0.001525	0.001526	37.43
56	98,182	164.13	0.998328	0.001672	0.001673	36.49
57	98,017	178.80	0.998176	0.001824	0.001826	35.55
58	97,839	194.41	0.998013	0.001987	0.001989	34.61
59	97,644	210.54	0.997844	0.002156	0.002159	33.68
60	97,434	228.38	0.997656	0.002344	0.002347	32.76
61	97,205	248.43	0.997444	0.002556	0.002559	31.83
62	96,957	270.95	0.997206	0.002795	0.002798	30.91
63	96,686	296.41	0.996934	0.003066	0.00307	30.00
64	96,389	324.39	0.996635	0.003365	0.003371	29.09
65	96,065	354.56	0.996309	0.003691	0.003698	28.19
66	95,711	385.77	0.995969	0.004031	0.004039	27.30
67	95,325	418.30	0.995612	0.004388	0.004398	26.41
68	94,906	451.51	0.995243	0.004757	0.004769	25.52
69	94,455	486.62	0.994848	0.005152	0.005165	24.65
70	93,968	525.19	0.994411	0.005589	0.005605	23.77
71	93,443	567.70	0.993925	0.006075	0.006094	22.91
72	92,875	615.85	0.993369	0.006631	0.006653	22.05
73	92,260	669.79	0.99274	0.00726	0.007286	21.19
74	91,590	728.08	0.992051	0.007949	0.007981	20.35
75	90,862	792.04	0.991283	0.008717	0.008755	19.51
76	90,070	856.93	0.990486	0.009514	0.00956	18.68
77	89,213	921.30	0.989673	0.010327	0.010381	17.86
78	88,291	983.81	0.988857	0.011143	0.011205	17.05
79	87,308	1046.47	0.988014	0.011986	0.012058	16.24
80	86,261	1112.29	0.987106	0.012894	0.012978	15.44
81	85,149	1182.91	0.986108	0.013892	0.01399	14.64

82	83,966	1260.41	0.984989	0.015011	0.015125	13.85
83	82,706	1344.95	0.983738	0.016262	0.016396	13.06
84	81,361	1432.09	0.982398	0.017602	0.017759	12.27
85	79,929	1517.72	0.981012	0.018988	0.019171	11.49
86	78,411	1597.82	0.979623	0.020378	0.020588	10.72
87	76,813	1670.53	0.978252	0.021748	0.021988	9.94
88	75,142	1748.12	0.976736	0.023264	0.023539	9.16
89	73,394	1818.84	0.975218	0.024782	0.025094	8.38
90	71,576	1884.14	0.973676	0.026324	0.026676	7.59
91	69,691	1945.94	0.972078	0.027922	0.028319	6.80
92	67,745	2008.49	0.970352	0.029648	0.030096	5.99
93	65,737	2097.40	0.968094	0.031906	0.032426	5.17
94	63,640	2220.15	0.965114	0.034886	0.035509	4.34
95	61,419	2418.76	0.960619	0.039381	0.040178	3.50
96	59,001	2806.75	0.952429	0.047572	0.04874	2.64
97	56,194	3578.39	0.936321	0.063679	0.065797	1.78
98	52,615	5400.04	0.897368	0.102632	0.108289	0.90
99	47,215	41869.77	0.113219	0.886781	2.178432	0.00

TABLE2.1

INSURER X ANNUITIES				**SUM ASSURED:**	**100000**
				BENEFIT:	**100000**
Age (x)	$\bar{a}_x$	$\bar{A}_x$	**P(Ā)**	$\bar{a}_{x:65-x}$	${}_{65-x}\vert\bar{a}_x$
20	20.71344	3723.547	179.7648	19.02748567	168595.6
21	20.67796	3886.932	187.9746	18.90730149	177066.3
22	20.64075	4058.33	196.6174	18.78111649	185963
23	20.6016	4238.633	205.743	18.64853538	195306.1
24	20.5604	4428.388	215.3844	18.5092191	205117.8
25	20.51703	4628.165	225.5768	18.36281208	215421.4
26	20.47133	4838.661	236.3628	18.20892321	226240.9
27	20.42322	5060.313	247.7725	18.04719696	237602.4
28	20.37265	5293.291	259.8234	17.87731328	249534
29	20.31951	5538.162	272.554	17.69886322	262064.2
30	20.26373	5795.136	285.9857	17.51148742	275224.2
31	20.20523	6064.629	300.1514	17.31477153	289046.1
32	20.14394	6346.982	315.0815	17.1082962	303564.4
33	20.07971	6642.847	330.8239	16.89156827	318814.1
34	20.01247	6952.514	347.409	16.66413792	334833.6
35	19.94212	7276.503	364.8812	16.42549344	351662.5

36	19.86853	7615.353	383.2873	16.17509718	369342.9
37	19.79159	7969.529	402.6724	15.91239996	387919.2
38	19.71124	8339.352	423.0759	15.63684998	407439.2
39	19.62736	8725.337	444.5497	15.34783511	427952.6
40	19.53985	9127.945	467.1452	15.04472217	449512.3
41	19.44858	9547.666	490.9183	14.72684113	472174.3
42	19.3535	9984.828	515.9184	14.39351309	495998.7
43	19.25441	10440.26	542.2271	14.04394383	521046.7
44	19.15122	10914.37	569.9044	13.67736856	547385.5
45	19.04376	11407.95	599.0389	13.29291735	575084
46	18.93192	11921.41	629.6991	12.88973846	604218.1
47	18.81548	12455.81	661.9984	12.46683658	634864
48	18.69428	13011.82	696.0326	12.02322392	667105.2
49	18.56812	13590.34	731.9177	11.55782868	701029
50	18.43689	14191.85	769.7531	11.06957136	736731.4
51	18.30037	14817.28	809.6709	10.55725717	774311.5
52	18.15851	15466.88	851.7702	10.01970087	813880.8
53	18.01118	16141.12	896.172	9.455611124	855557
54	17.85834	16840.17	942.9864	8.86364156	899469.4
55	17.6999	17564.26	992.3364	8.242340771	945756.3
56	17.5357	18314.17	1044.393	7.590099845	994560.4
57	17.36559	19090.49	1099.329	6.905218461	1046037
58	17.1893	19894.38	1157.37	6.185833751	1100346
59	17.00662	20726.68	1218.742	5.429984989	1157664
60	16.81729	21588.62	1283.716	4.63554958	1218174
61	16.62118	22480.56	1352.525	3.800305143	1282087
62	16.41827	23402.51	1425.395	2.921857962	1349641
63	16.20855	24354.34	1502.562	1.997603333	1421094
64	15.99207	25335.59	1584.259	1.024695077	1496738
65	15.76881	26346.19	1670.778	0	
66	15.53868	27386.38	1762.465		
67	15.30135	28457.38	1859.795		
68	15.05656	29560.31	1963.284		
69	14.80389	30696.88	2073.569		
70	14.54308	31867.99	2191.282		
71	14.27408	33073.54	2317.035		
72	13.99689	34313.14	2451.483		
73	13.71172	35585.3	2595.246		
74	13.4188	36888.46	2749.013		
75	13.11809	38222.16	2913.699		
76	12.80973	39585.1	3090.238		

77	12.49314	40979.33	3280.146		
78	12.16752	42408.04	3485.347		
79	11.83181	43875.55	3708.27		
80	11.48513	45385.11	3951.64		
81	11.12693	46938.32	4218.442		
82	10.75679	48535.98	4512.126		
83	10.37443	50177.83	4836.685		
84	9.979503	51863.78	5197.03		
85	9.571015	53596.72	5599.899		
86	9.147329	55382.45	6054.494		
87	8.706176	57229.7	6573.46		
88	8.244864	59149.05	7174.047		
89	7.761748	61145.13	7877.752		
90	7.253663	63229.96	8716.969		
91	6.71724	65416.07	9738.534		
92	6.148862	67716.49	11012.85		
93	5.544764	70143.94	12650.48		
94	4.902491	72701.36	14829.47		
95	4.218867	75391.78	17870.15		
96	3.491375	78205.84	22399.72		
97	2.719379	81099.53	29822.81		
98	1.900436	83976.91	44188.22		
99	1.024695	86540.97	84455.34		

TABLE3.0

NON-CENTRAL GAMMA MODEL: h(x)=h₀(x)/(1+0.5*δ^2*H₀(x))^2						
h₀(x) ~ Gompertz (ά,β)						
the resulting model: : h(x)=άexp(βx)/(1+0.5*0.05*δ^2*(ά/β)*(exp(βx)-1)))^2						
Age (x)	**lx**	**dx**	**qx**	**px**	**h(x)**	**ex (NON-CENTRAL GAMMA FRAILTY)**
20	100000	4.444218	4.44422E-05	0.999956	4.44E-05	69.95
21	99995.56	4.911379	4.9116E-05	0.999951	4.91E-05	68.95
22	99990.64	5.427619	5.42813E-05	0.999946	5.43E-05	67.96
23	99985.22	5.998086	5.99897E-05	0.99994	6E-05	66.96
24	99979.22	6.628471	6.62985E-05	0.999934	6.63E-05	65.96
25	99972.59	7.325057	7.32707E-05	0.999927	7.33E-05	64.97
26	99965.27	8.094784	8.0976E-05	0.999919	8.1E-05	63.97
27	99957.17	8.94532	8.94915E-05	0.999911	8.95E-05	62.98
28	99948.23	9.885129	9.89025E-05	0.999901	9.89E-05	61.98

29	99938.34	10.92356	0.000109303	0.999891	0.000109	60.99
30	99927.42	12.07095	0.000120797	0.999879	0.000121	60.00
31	99915.35	13.33868	0.0001335	0.999867	0.000134	59.00
32	99902.01	14.73934	0.000147538	0.999852	0.000148	58.01
33	99887.27	16.28684	0.000163052	0.999837	0.000163	57.02
34	99870.98	17.9965	0.000180197	0.99982	0.00018	56.03
35	99852.98	19.88524	0.000199145	0.999801	0.000199	55.04
36	99833.1	21.97176	0.000220085	0.99978	0.00022	54.05
37	99811.13	24.27664	0.000243226	0.999757	0.000243	53.06
38	99786.85	26.82262	0.000268799	0.999731	0.000269	52.08
39	99760.03	29.63478	0.000297061	0.999703	0.000297	51.09
40	99730.39	32.74074	0.000328293	0.999672	0.000328	50.10
41	99697.65	36.17099	0.000362807	0.999637	0.000363	49.12
42	99661.48	39.9591	0.000400948	0.999599	0.000401	48.14
43	99621.52	44.14206	0.000443098	0.999557	0.000443	47.16
44	99577.38	48.76064	0.000489676	0.99951	0.00049	46.18
45	99528.62	53.85967	0.000541148	0.999459	0.000541	45.20
46	99474.76	59.48855	0.000598027	0.999402	0.000598	44.23
47	99415.27	65.70156	0.00066088	0.999339	0.000661	43.25
48	99349.57	72.55843	0.000730335	0.99927	0.000731	42.28
49	99277.01	80.12474	0.000807083	0.999193	0.000807	41.31
50	99196.89	88.47255	0.000891888	0.999108	0.000892	40.35
51	99108.41	97.68092	0.000985597	0.999014	0.000986	39.38
52	99010.73	107.8365	0.00108914	0.998911	0.00109	38.42
53	98902.9	119.0344	0.001203548	0.998796	0.001204	37.46
54	98783.86	131.3783	0.001329958	0.99867	0.001331	36.51
55	98652.48	144.9821	0.001469625	0.99853	0.001471	35.56
56	98507.5	159.9698	0.001623935	0.998376	0.001625	34.61
57	98347.53	176.4767	0.00179442	0.998206	0.001796	33.66
58	98171.06	194.6502	0.001982766	0.998017	0.001985	32.72
59	97976.4	214.6504	0.002190838	0.997809	0.002193	31.79
60	97761.75	236.6511	0.002420693	0.997579	0.002424	30.86
61	97525.1	260.8404	0.002674598	0.997325	0.002678	29.93
62	97264.26	287.4213	0.002955056	0.997045	0.002959	29.01
63	96976.84	316.6125	0.003264826	0.996735	0.00327	28.10
64	96660.23	348.6487	0.003606951	0.996393	0.003613	27.19
65	96311.58	383.7808	0.003984784	0.996015	0.003993	26.29
66	95927.8	422.2761	0.00440202	0.995598	0.004412	25.40
67	95505.52	464.4176	0.00486273	0.995137	0.004875	24.51
68	95041.11	510.5035	0.005371397	0.994629	0.005386	23.63
69	94530.6	560.8458	0.005932955	0.994067	0.005951	22.76

70	93969.76	615.7681	0.006552833	0.993447	0.006574	21.89
71	93353.99	675.603	0.007237002	0.992763	0.007263	21.04
72	92678.39	740.6883	0.007992028	0.992008	0.008024	20.19
73	91937.7	811.3613	0.008825121	0.991175	0.008864	19.35
74	91126.34	887.9531	0.009744199	0.990256	0.009792	18.52
75	90238.38	970.7797	0.010757947	0.989242	0.010816	17.71
76	89267.6	1060.132	0.011875889	0.988124	0.011947	16.90
77	88207.47	1156.264	0.013108454	0.986892	0.013195	16.10
78	87051.21	1259.375	0.014467057	0.985533	0.014573	15.32
79	85791.83	1369.596	0.015964172	0.984036	0.016093	14.54
80	84422.24	1486.964	0.01761342	0.982387	0.01777	13.78
81	82935.27	1611.403	0.019429652	0.98057	0.019621	13.02
82	81323.87	1742.692	0.021429033	0.978571	0.021662	12.28
83	79581.18	1880.434	0.023629136	0.976371	0.023913	11.55
84	77700.74	2024.029	0.026049024	0.973951	0.026394	10.83
85	75676.71	2172.628	0.028709334	0.971291	0.02913	10.12
86	73504.09	2325.107	0.031632355	0.968368	0.032143	9.42
87	71178.98	2480.025	0.0348421	0.965158	0.035464	8.73
88	68698.95	2635.591	0.038364361	0.961636	0.03912	8.04
89	66063.36	2789.641	0.042226749	0.957773	0.043144	7.36
90	63273.72	2939.616	0.046458712	0.953541	0.047573	6.69
91	60334.11	3082.561	0.051091524	0.948908	0.052443	6.01
92	57251.54	3215.146	0.056158237	0.943842	0.057797	5.34
93	54036.4	3333.699	0.06169359	0.938306	0.063679	4.66
94	50702.7	3434.289	0.067733858	0.932266	0.070137	3.96
95	47268.41	3512.83	0.074316644	0.925683	0.077223	3.25
96	43755.58	3565.23	0.081480583	0.918519	0.084992	2.51
97	40190.35	3587.59	0.089264971	0.910735	0.093503	1.73
98	36602.76	3576.429	0.097709277	0.902291	0.102819	0.90
99	33026.33	3528.948	0.106852552	0.893147	0.113004	0.00

TABLE3.1

NON-CENTRAL GAMMA ANNUITIES				**SUM ASSURED:**	**100000**
				BENEFIT:	**100000**
Age (x)	$\bar{a}_x$	$\bar{A}_x$	**P(Ā)**	$\bar{a}_{x:65-x}$	${}_{65-x}|\bar{a}_x$

20	20.68897	10914.76	527.5642	19.05790536	163106
21	20.6484	11250.06	544.8393	18.93571234	171269
22	20.6059	11617.19	563.7799	18.80749188	179841.2
23	20.56139	12016.13	584.4027	18.67295023	188843.5
24	20.51476	12446.58	606.7138	18.53177963	198297.6
25	20.46592	12908.16	630.715	18.38365759	208226.3
26	20.41478	13400.26	656.4	18.22824623	218653.6
27	20.36124	13921.99	683.7497	18.06519156	229604.9
28	20.30519	14472.2	712.7339	17.89412268	241106.7
29	20.24652	15049.5	743.3128	17.71465101	253187.1
30	20.18512	15652.17	775.4311	17.52636941	265875.5
31	20.12088	16278.42	809.0311	17.32885133	279203
32	20.05367	16925.98	844.0338	17.1216498	293202.3
33	19.98338	17592.4	880.3518	16.90429649	307907.9
34	19.90986	18275.64	917.9188	16.67630059	323356
35	19.833	18974	956.6883	16.43714772	339585
36	19.75265	19686.12	996.6316	16.1862987	356635.3
37	19.66868	20411.31	1037.757	15.92318826	374549.5
38	19.58095	21149.18	1080.09	15.64722365	393372.6
39	19.48931	21899.97	1123.691	15.35778316	413152.3
40	19.3936	22664.33	1168.65	15.0542145	433938.8
41	19.29369	23443.57	1215.09	14.73583306	455785.4
42	19.1894	24239.53	1263.173	14.40192	478748.3
43	19.08059	25054.59	1313.093	14.05172018	502887.4
44	18.9671	25891.79	1365.09	13.6844399	528265.8
45	18.84875	26754.72	1419.443	13.29924438	554950.9
46	18.72539	27647.81	1476.487	12.89525501	583013.9
47	18.59686	28575.98	1536.603	12.47154624	612530.9
48	18.46297	29544.91	1600.225	12.02714222	643582.8
49	18.32357	30561.08	1667.856	11.56101291	676255.8
50	18.17849	31621.04	1739.476	11.0720698	710642.2
51	18.02756	32730.42	1815.576	10.55916105	746840.4
52	17.87063	33888.53	1896.326	10.02106601	784956.1
53	17.70751	35097.66	1982.077	9.456488914	825102.5
54	17.53807	36350.92	2072.687	8.864051839	867401.6
55	17.36213	37639.57	2167.912	8.242286492	911984.6
56	17.17956	38955.68	2267.561	7.589624887	958993.2
57	16.9902	40288.57	2371.284	6.904388582	1008581
58	16.79391	41601.91	2477.202	6.184776265	1060913
59	16.59057	42866.65	2583.796	5.428849386	1116172
60	16.38006	44052.36	2689.39	4.634515498	1174554

61	16.16225	45125.17	2792.01	3.799508885	1236274
62	15.93706	46048.29	2889.384	2.921367983	1301569
63	15.70439	46783.1	2978.982	1.997409006	1370698
64	15.46417	47289.58	3058.009	1.024695077	1443947
65	15.21633	47527.83	3123.474	0	
66	14.96084	47459.56	3172.253		
67	14.69765	47050.82	3201.249		
68	14.42675	46277.68	3207.768		
69	14.14816	45273.68	3199.971		
70	13.86188	44068	3179.079		
71	13.56795	42593.97	3139.308		
72	13.26642	40890.83	3082.279		
73	12.95737	39059.88	3014.491		
74	12.64087	37049.39	2930.921		
75	12.317	34883.7	2832.159		
76	11.98586	32576.84	2717.938		
77	11.64755	30118.88	2585.855		
78	11.30215	27532.81	2436.068		
79	10.94974	24962.79	2279.761		
80	10.59037	22452.75	2120.111		
81	10.22403	19980.84	1954.301		
82	9.850702	17647.69	1791.516		
83	9.470246	15429.75	1629.287		
84	9.082438	13358.1	1470.761		
85	8.686916	11488.68	1322.527		
86	8.283135	9763.678	1178.742		
87	7.870319	8190.526	1040.685		
88	7.447388	6744.838	905.665		
89	7.012872	5445.609	776.5162		
90	6.564795	4305.173	655.797		
91	6.100528	3311.535	542.8276		
92	5.616584	2461.001	438.1669		
93	5.10836	1759.137	344.3644		
94	4.569774	1203.415	263.3423		
95	3.99278	783.585	196.2505		
96	3.36669	482.9503	143.4496		
97	2.677237	280.6508	104.8285		
98	1.905241	0	0		
99	1.024695	0	0		

COMPARISON GRAPHS

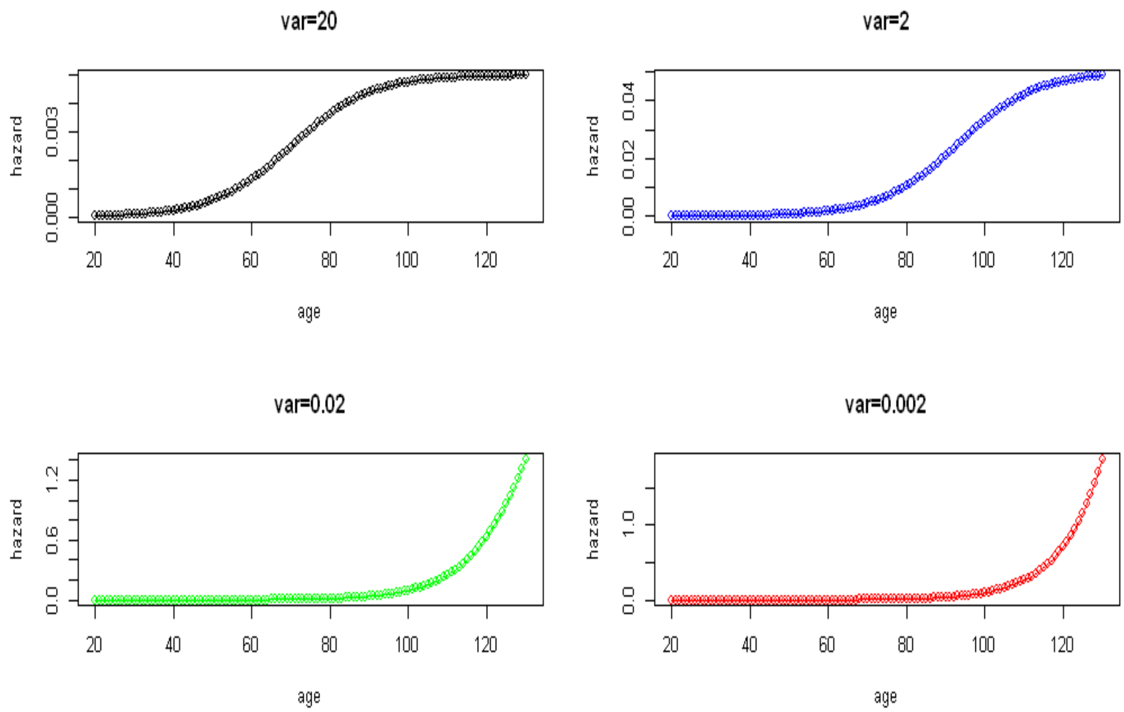
1.1 Heterogeneity effect in the Gamma-Gompertz model:

Graphs for the Gamma-Gompertz model with different values of b is shown

When $\propto\ = 8 * 10^{-7}\ \beta\ = 0.15$

$$h(x)\ = \frac{\propto\ be^{\beta x}}{b\ + \frac{\propto}{\beta}(e^{\beta x} - 1)}$$

b = 0.05 (black) b = 0.5 (blue) b = 50 (green) b = 500 (red)



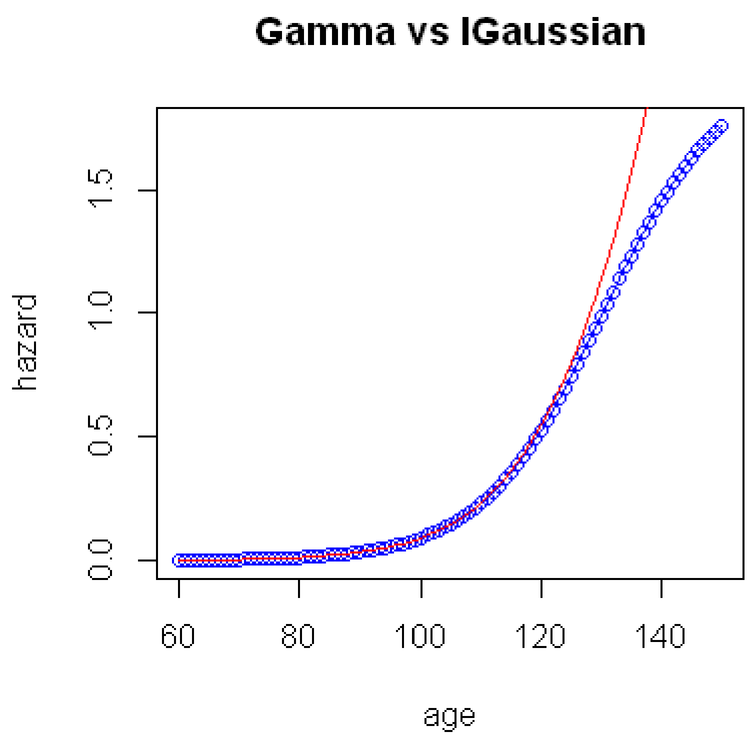
RESULTS

If b is low, then heterogeneity is strong; the opposite occurs if b is high.

Higher variance implies heterogeneity and lower variance homogeneity

1.2 Comparing the Gamma-Gompertz and Inverse Gaussian-Gompertz

Comparisons can be made with similar parameter estimators for both models.



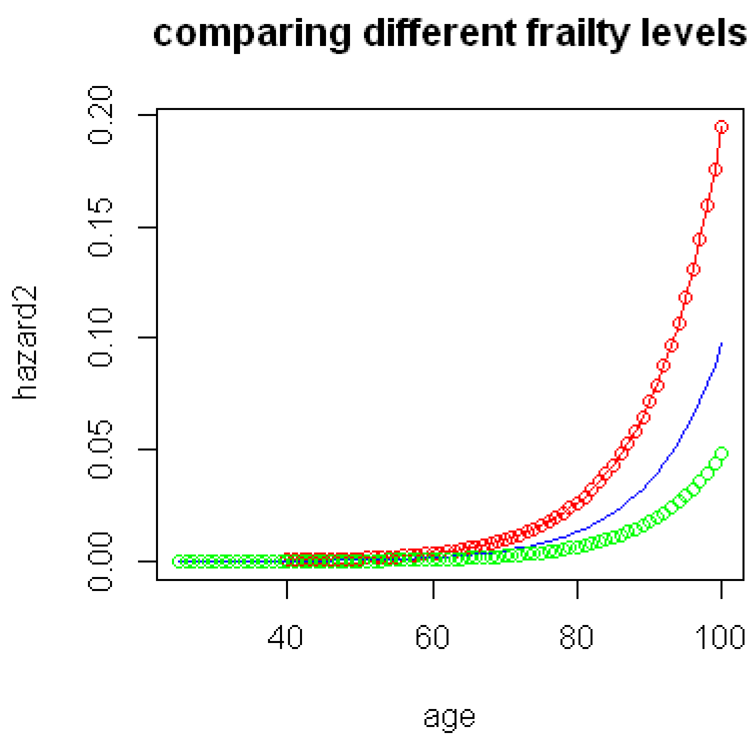
RESULTS

The output shows that the relative frailty distribution among survivors is independent of age for the Gamma, but becomes more homogeneous with time for the Inverse Gaussian. (i.e. concentration of deaths at small age interval)

1.3 Comparing different frailty levels i.e.

$z = 1\ (average), z = 2(more\ frail), z = 0.5(less\ frail)$

The frailty model is given by: $h(t) = Z * h_o(t)$, assuming $h_o(t) \sim gompertz(\propto, \beta)$



RESULTS

The output shows that for higher levels of frailty the probability of dying is higher compared to lower frailty levels. i.e. more frail individuals are likely to die earlier that the less frail ones.

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