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A MATHEMATICAL MODEL FOR THE CONTROL OF MALARIA WITH TEMPORARY IMMUNITY

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SUPERVISORS

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A Dissertation submitted in partial fulfillment for the award of a Master of Science degree in Applied mathematics.

NAIROBI

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June 24, 2013

DECLARATION

I the undersigned, do declare that this project is my original work and to the best of my knowledge has not been presented for the award of degree in any other university.

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Name : OCHOMBA WYCLIFF NYANG'ERA

Reg. No : I56/68820/2011

Date : _____

Declaration By Supervisors:

This project report has been submitted for a examination with my approval as supervisor:

Signature : _____

Name : Dr. Thomas Onyango

Date : _____

Signature : _____

Name : Dr. Nelson Owour

Date : _____

STATEMENT

This dissertation has been submitted in partial fulfillment of the requirements for a Master of Science degree at the University of Nairobi and is deposited in the University Library to be made available to borrowers under the rules of the Library.

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Above all, I thank God for His guidance and abundant favours upon me. To God is the glory.

DEDICATION

In memory of my late grandparents, Mr. and Mrs. Martin
and my late uncles Samuel, Haran and Caleb.

ABSTRACT

Malaria is an infectious disease transmitted between humans through mosquito bites that kill about two million people a year. Many infectious diseases including malaria are preventable, yet they remain endemic in many countries like Kenya due to lack of proper, adequate and timely control policies. The main goal of this project is to develop a mathematical model for the control of malaria. It has been shown that the model has unique disease-free and endemic equilibria.

A mathematical model for malaria is developed using ordinary differential equations. We analyze the existence and stability of disease-free and endemic malaria (malaria persisting in the population) equilibria. Key to our analysis is the definition of a reproductive number R_0 (the number of the new infections caused by one individual in an otherwise fully susceptible population) through the duration of the infectious period.

The methods for controlling any infectious disease include a rapid reduction in both the infected and susceptible populations as well as a rapid reduction in the susceptible class if a cure is available. For diseases of malaria where there are no vaccines, it is still possible to reduce the susceptible group through a variety of control measures.

The disease-free equilibrium is locally asymptotically stable, if $R_0 < 1$, and we also note that when $R_0 > 1$, the disease-free equilibrium is unstable and the endemic equilibrium is stable. Numerical simulations show that recoveries and temporary immunity keep the populations at oscillation patterns and eventually converge to a steady state.

Further simulation of the model clearly shows that, with proper combination of treatment and concerted effort aimed at prevention, malaria could be eliminated from our society. In fact, effective treatment offered to about fifty percent of the infected population together with about fifty percent prevention rate is all that is required to eliminate the diseases.

Contents

CHAPTER 1

1.INTRODUCTION

1.1Background

Malaria is the common name for diseases caused by single-celled parasites of the genus Plasmodium. Among the parasites of the genus Plasmodium four species have been identified which can cause disease in humans. These include: *Plasmodium falciparum*, *Plasmodium vivax*, *Plasmodium malaria* and *Plasmodium ovale*. Of these, *Plasmodium falciparum* is of greatest risk to non-immune humans. The *Plasmodium falciparum* variety of parasites account for 80% of cases and 90% of deaths (Kakkilaya, 2003).

Malaria remains arguably the greatest menace of our society in terms of morbidity and mortality and the occurrence of malaria in our part of the world correlates with poverty, ignorance and social deprivations in the community. An accurate knowledge of the incidence of malaria in endemic areas would be necessary towards the planning and development of effective preventive measures against the deadly scourge of malaria.

Malaria is spread by the bite of an infected female mosquito, of the genus anopheles each time the infected insect takes a blood meal. The symptoms in an infected human include bouts of fever, headache, vomiting flu-like, anemia (destroying red blood cell) and malaria can kill by clogging the capillaries that carry blood to the brain (cerebral malaria) or other vital organs. On the average the incubation period of *Plasmodium falciparum* is about 12 days in humans.

Malaria is endemic to tropical areas where the climatic and weather conditions allow continuous breeding of the mosquito. Malaria is one of the most important parasitic and infectious diseases especially in tropical and subtropical areas caused by protozoan parasites of the genus plasmodium. Malaria, affecting mainly children and pregnant women is one of the greatest menaces of our society in terms of morbidity and mortality and the occurrence of malaria in our part of the world correlates with poverty and ignorance (Perandin, 2003).

Malaria is a major public health problem in the world. The WHO estimates that in tropical countries among the 500 million cases of malaria infection, one million deaths occur annually. Malaria parasites are transmitted by female anopheles mosquitoes. Four species of plasmodium (P) causes human malaria. Among these, *P. falciparum* is responsible for most of the mortality *P. Vivax* causes considerable morbidity and *P. malariae* and *P. ovale*, are less prevalent around the world(Aslan and Seyrek, 2007).

This group of human pathogenic Plasmodium species is usually referred to as plasmodium . The parasites multiply within red blood cells, causing symptoms that include symptoms of anaemia,

as well as other general symptoms such as fever, chills, nausea, flu-like illness, and in severe cases, coma and death (Deressa et al, 2000). It is a disease that can be treated in just 48 hours, yet it can cause fatal complications if the diagnosis and treatment are delayed.

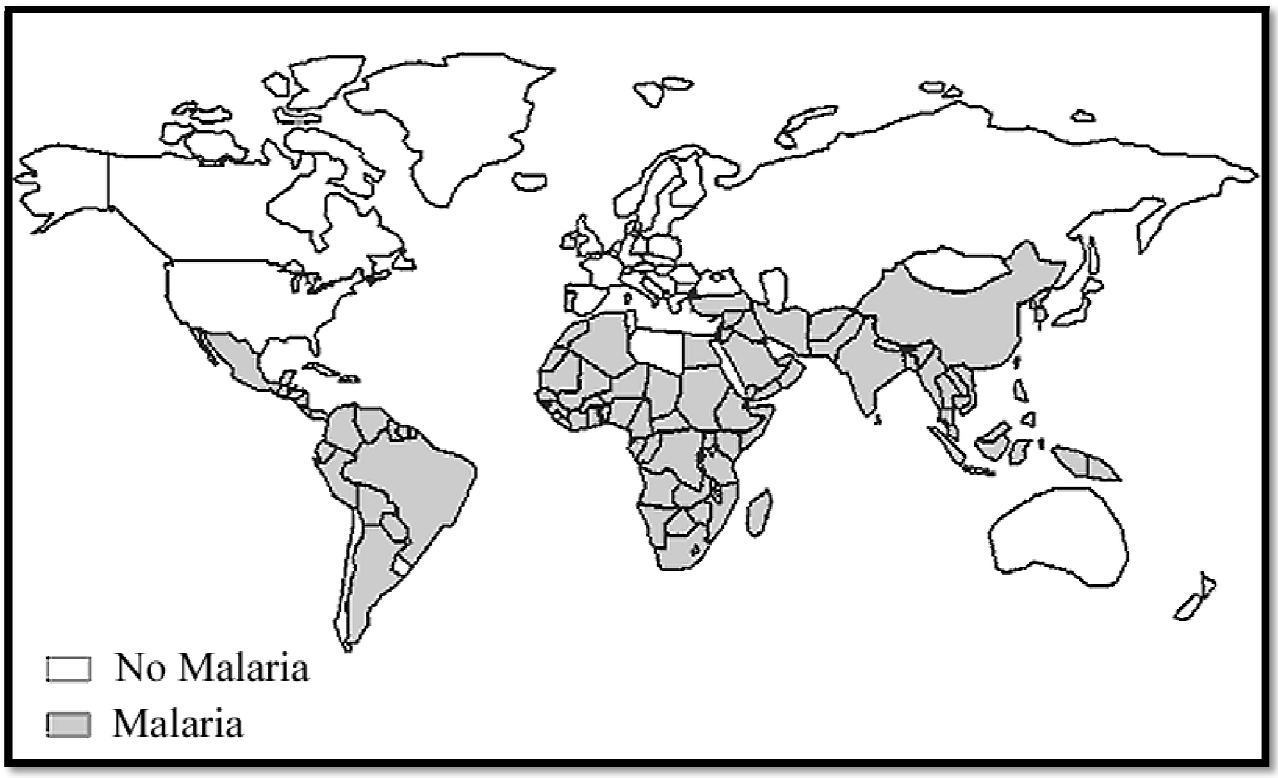


Figure1.1:Countries with endemic malaria transmission (WHO, 2000).

Malaria has been a significant factor in virtually all of the military campaigns involving the United States. In World War II and the Vietnam War, more personnel time was lost due to malaria than to bullets. The discovery that malaria was transmitted by mosquitoes unleashed a flurry of ambitious public Understanding health measures designed to stamp out malaria. These measures were targeted at both the larval stages (which thrive in still waters, such as swamps) and adult stages of the insect. In some areas, such as the southern United States, draining swamps and changing the way land was used was somewhat successful in eliminating mosquitoes.

The pace of the battle accelerated rapidly when the insecticide DDT and the drug chloroquine were introduced during World War II. DDT was remarkably effective and could be sprayed on the walls of houses where adult *Anopheles* mosquitoes rested after feeding. Chloroquine has been a highly effective medicine for preventing and treating malaria. In the mid-1950s, the World Health Organization (WHO) launched a massive worldwide campaign to eliminate malaria. At the beginning, the WHO program, which combined insecticide spraying and drug treatment, had many successes, some spectacular. In some areas, malaria was conquered completely, benefiting more than 600 million people, and was sharply curbed in the homelands of 300 million others.

Difficulties soon developed, however. Some stumbling blocks were administrative, others financial. Even worse, nature intervened. More and more strains of *Anopheles* mosquitoes developed resistance to DDT and other insecticides, and the environmental impact of DDT was recognized. Meanwhile, the *Plasmodium* parasite became resistant to chloroquine, the mainstay of antimalarial drug treatment in humans. Researchers estimate that infection rates increased by 40 percent between 1970 and 1997 in sub-Saharan Africa.

To cope with this dangerous resurgence, public health workers carefully select prevention methods best suited to a particular environment or area. In addition to medicines and insecticides, they are making efforts to control mosquitoes, by draining swampy areas and filling them with dirt, as well as using window screens, mosquito netting, and insect repellents. At the same time, scientists are intensively researching ways to develop better weapons against malaria, including : sophisticated techniques for tracking disease transmission worldwide , more effective ways of treating malaria , new ways(some quite ingenious) to control transmission of malaria by mosquitoes ,a vaccine for blocking malaria’s development and spread.

1.2 History of the mathematical modeling of malaria

It is important to establish the transmission dynamics of an epidemic in order to understand and predict it. Mathematical models are particularly helpful as experimental tools with which to evaluate and compare control procedures and preventive strategies, and to investigate the relative effects of various sociological, biological and environmental factors on the spread of diseases.

These models have played a very important role in the history and development of vector-host epidemiology. Several authors have used mathematical models to analyze the transmission and spread of malaria. Mathematical models of malaria transmission that include both mosquito and human populations have been reviewed and discussed in detail by various authors.

Nedelman (1985) did some further work on malaria model of Dietz et al (1974), and showed that the "vaccination" rate depends on a pseudo equilibrium approximation to the differential equation describing the mosquito dynamics in the malaria model. Nedelman surveys various data sets to statistically approximate parameters such as inoculation rates, rates of recovery and loss of immunity in humans, human-biting rates of mosquitoes and infectivity and susceptibility of humans and mosquitoes.

Dietz *et al* (1974) proposed a model with two different classes of humans: one without immunity to malaria and one class with some immunity. As the non-immune class falls sick, some people recover with immunity. The immune class can get infected, but does not fall clinically ill and cannot be infectious. The model also included super infection, a phenomenon usually associated with macro parasites.

Yang (2000) describes a compartmental model where humans follow an SEIRS-type (with more than one immune class for humans) pattern and mosquitoes follow a Susceptible-Exposed-Infectious (SEI) pattern. Yang (2000) defines a reproductive number, R_0 for this model and shows, through linear stability analysis, that the disease-free equilibrium is stable for $R_0 < 1$. He also derived an expression for an endemic equilibrium that is biologically relevant only when $R_0 > 1$. He used numerical simulations to support his proposition that for $R_0 > 1$, the disease-free equilibrium is unstable and the endemic equilibrium is stable.

The model for malaria transmission that we modified is an extension of the equations introduced by Tunwiine *et al* (2007).

1.3 Life cycle of malaria parasite

The human malaria parasite has a complex life cycle that requires both a human host and an insect host. In *Anopheles* mosquitoes, *Plasmodium* reproduces sexually (by merging the parasite's sex cells). In people, the parasite reproduces asexually (by cell division), first in liver cells and then, repeatedly, in red blood cells (RBCs).

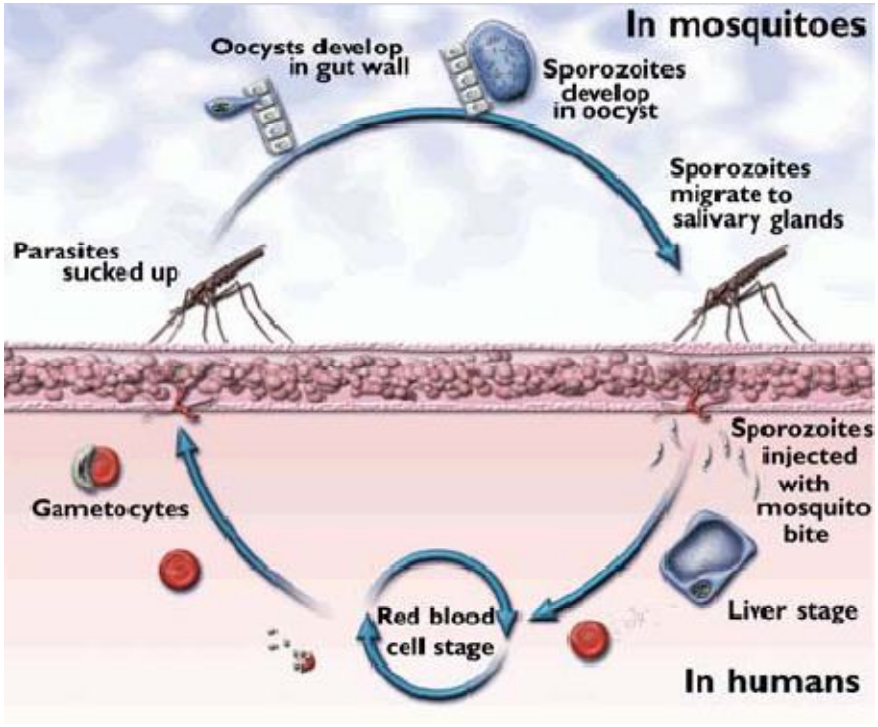


Figure1.2:The life cycle of malaria parasite.

When an infected female *Anopheles* mosquito bites a human, it takes in blood. At the same time, it injects saliva that contains the infectious form of the parasite, the *sporozoite*, into a person's bloodstream [1]The thread-like *sporozoite* then invades a liver cell[2] . There, during the next week or two (depending on the *Plasmodium* species), each *sporozoite* develops into a *schizont*, a structure that contains thousands of tiny rounded *merozoites* (another stage of the parasite). When the *schizont* matures, it ruptures and releases the *merozoites* into the bloodstream[3] .

Alternatively, some *P. vivax* and *P. ovale sporozoites* turn into *hypnozoites*, a form that can remain dormant in the liver for months or years. If they become active again, the *hypnozoites* develop into *schizonts* that then cause relapses in infected people.

Merozoites released from the liver upon rupture of *schizonts* rapidly invade RBCs, where they grow by consuming hemoglobin[4] . Within the RBC, most *merozoites* go through another round of asexual reproduction, again forming *schizonts* filled with yet more *merozoites*. When the

schizont matures, the cell ruptures and *merozoites* burst out. The newly released merozoites invade other RBCs, and the infection continues its cycle until it is brought under control, either by medicine or the body's immune system defenses.

The *Plasmodium* parasite completes its life cycle through the mosquito when some of the *merozoites* that penetrate RBCs do not develop asexually into *schizonts*, but instead change into male and female sexual forms known as *gametocytes*[4] . These circulate in the person's bloodstream, awaiting the arrival of a blood-sucking female *Anopheles* mosquito[5] .

When a female mosquito bites an infected person, it sucks up *gametocytes* along with blood. Once in the mosquito's stomach, the g*ametocytes* develop into sperm-like male gametes or large, egg-like female gametes[6] . Fertilization produces an *oocyst* filled with infectious *sporozoites*[7] . When the *oocyst* matures, it ruptures and the thread-like *sporozoites* migrate, by the thousands, to the mosquito's salivary (saliva-producing) glands[8] . The cycle starts over again when the mosquito bites its next victim[9] .

1.4 Life cycle of the mosquito

All mosquitoes must have water in which to complete their life cycle. This water can range in quality from melted snow water to sewage effluent and it can be in any container imaginable. The type of water in which the mosquito larvae is found can be an aid to the identification of which species it may be. Also, the adult mosquitoes show a very distinct preference for the types of sources in which to lay their eggs. They lay their eggs in such places such as tree holes that periodically hold water, tide water pools in salt marshes, sewage effluent ponds, irrigated pastures, rain water ponds, etc. Each species therefore has unique environmental requirements for the maintenance of its life cycle.

The feeding habits of mosquitoes are quite unique in that it is only the adult females that bite man and other animals. The male mosquitoes feed only on plant juices. Some female mosquitoes prefer to feed on only one type of animal or they can feed on a variety of animals. Female mosquitoes feed on man, domesticated animals, such as cattle, horses, goats, etc; all types of birds including chickens; all types of wild animals including deer, rabbits; and they also feed on snakes, lizards, frogs, and toads. Most female mosquitoes have to feed on an animal and get a sufficient blood meal before she can develop eggs. If they do not get this blood meal, then they will die without laying viable eggs.

However, some species of mosquitoes have developed the means to lay viable eggs without getting a blood meal. The flight habits of mosquitoes depend again on the species with which we are dealing. Most domestic species remain fairly close to their point of origin while some species known for their migration habits are often an annoyance far from their breeding place. The flight range for females is usually longer than that of males. Many times wind is a factor in the

dispersal or migration of mosquitoes. Most mosquitoes stay within a mile or two of their source. However, some have been recorded as far as 75 miles from their breeding source.

The length of life of the adult mosquito usually depends on several factors: temperature, humidity, sex of the mosquito and time of year. Most males live a very short time, about a week; and females live about a month depending on the above factors.

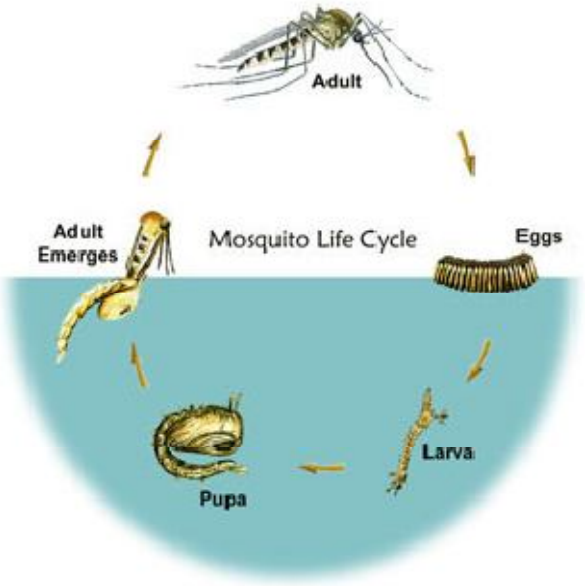


Figure 1.3:The life cycle of a mosquito

The mosquito goes through four separate and distinct stages of its life cycle and they are as follows: Egg, Larva, pupa, and adult. Each of these stages can be easily recognized by their special appearance.

Egg : Eggs are laid one at a time and they float on the surface of the water. Most eggs hatch into larvae within 48 hours.

Larva : The larva (larvae - plural) live in the water and come to the surface to breathe. They shed their skin four times growing larger after each molting. Most larvae have siphon tubes for breathing and hang from the water surface. Anopheles larvae do not have a siphon and they lay parallel to the water
surface. The larva feed on micro-organisms and organic matter in the water. On the fourth molt the larva changes into a pupa.

Pupa: The pupal stage is a resting, non-feeding stage. This is the time the mosquito turns into an adult. It takes about two days before the adult is fully developed. When development is complete, the pupal skin splits and the mosquito emerges as an adult.

Adult: The newly emerged adult rests on the surface of the water for a short time to allow itself to dry and all its parts to harden. Also, the wings have to spread out and dry properly before it can fly.
The egg, larvae and pupae stages depend on temperature and species characteristics as to how long it takes for development. Also, some species have naturally adapted to go through their entire life cycle in as little as four days or as long as one month.

1.5 Transmission of the disease

The malaria parasite typically is transmitted to people by mosquitoes belonging to the genus *Anopheles*. In rare cases, a person may contract malaria through contaminated blood, or a fetus may become infected by its mother during pregnancy. Because the malaria parasite is found in RBCs, malaria can also be transmitted through blood transfusion, organ transplant, or the shared use of needles or syringes contaminated with blood. Malaria also may be transmitted from a mother to her fetus before or during delivery (“congenital” malaria).

1.6 Rationale of the study

In the Tunwiine model, humans follow an SIRS-like pattern and mosquitoes follow a SI pattern, similar to that described by Yang (2000) but with only one immune class for humans. Humans move from the susceptible to the infected class at some probability when they come into contact with an infectious mosquito, as in conventional SIRS models.

However, infectious people can then recover with, or without, a gain in immunity; and either return to the susceptible class, or move to the recovered class. A new feature of this model is that although individuals in the recovered class are assumed to be “immune”, in the sense that they do not suffer from serious illness and do not contract clinical malaria, they still have low levels of Plasmodium in their blood stream and can pass the infection to susceptible mosquitoes.

After some period of time, these recovered individuals return to the susceptible class. Susceptible mosquitoes get infected and move to the infected class, at some probability when they come into contact with either infectious humans or recovered humans (albeit at a much lower probability). Both humans and mosquitoes leave the population through a density dependent natural death rate. This allows the model to account for changing human and mosquito populations. Variations in mosquito populations are crucial to the dynamics of malaria, population models do not account for this.

The model also includes human disease induced death as mortality for malaria in areas of high transmission can be high, especially in infants. In the modified model, we aim to capture some of the more important aspects of this epidemiology while still keeping it mathematically tractable. One of the major important factors that we include in the existing model is vaccination in order to determine its impact as a control measure for the spread of malaria.

CHAPTER 2

2.MODEL DESCRIPTION AND FORMULATION

2.1 Model formulation

As in Tumwiine et al. [24], the human population is divided into three epidemiological lasses that include the susceptible class S_H ,infective class I_H and immune class R_H.The mosquito population is divided into two epidemiological classes that include the susceptible class S_V and infective class I_V.

The vector population does not include immune class [4,12] as mosquitoes never recover from infection; that is, their infective period ends with their death due to their relatively short life-cycle. There is no vertical transmission and all the newborns are susceptible with a per capita birth rate λ_h. The infected human individuals recover at a constant rate v to join the susceptible. The infected individuals acquire immunity at constant rate r and may die due to the disease at a rate δ. The natural per capita death rate is assumed to be the same constant μ_h for all humans. The mosquito population has λ_v and μ_v as the natural per capita birth and mortality rates respectively. The infected female mosquito bite humans at a rate a.

The fraction of the bites that successfully infect humans is b and c is the fraction of bites that infect mosquitoes when they bite infected humans. The incidence term is of the standard form with the terms $\frac{abS_H I_V}{N_H}$ denoting the rate at which the human hosts S_H get infected by infected mosquitoes I_V and $\frac{acS_V I_H}{N_H}$ for the rate at which the susceptible mosquitoes S_V are infected by the infected human hosts I_H .The rate of infection of human host S_H by infected vector I_V is dependent on the total number of humans N_H available per infected vector [21].

The parameter a is the average number of bites per mosquito per day. This rate depends on a number of factors, in particular, climatic ones, but for simplicity in this paper we assume a to be a constant.

The parameter ($0 \leq \sigma \leq 1$) determines the degree of partial protection for the recovered individuals given by a primary infection : $\sigma = 0$ implies complete protection, and $\sigma = 1$ implies no protection. The above description leads to the compartmental diagram in figure 1.5.

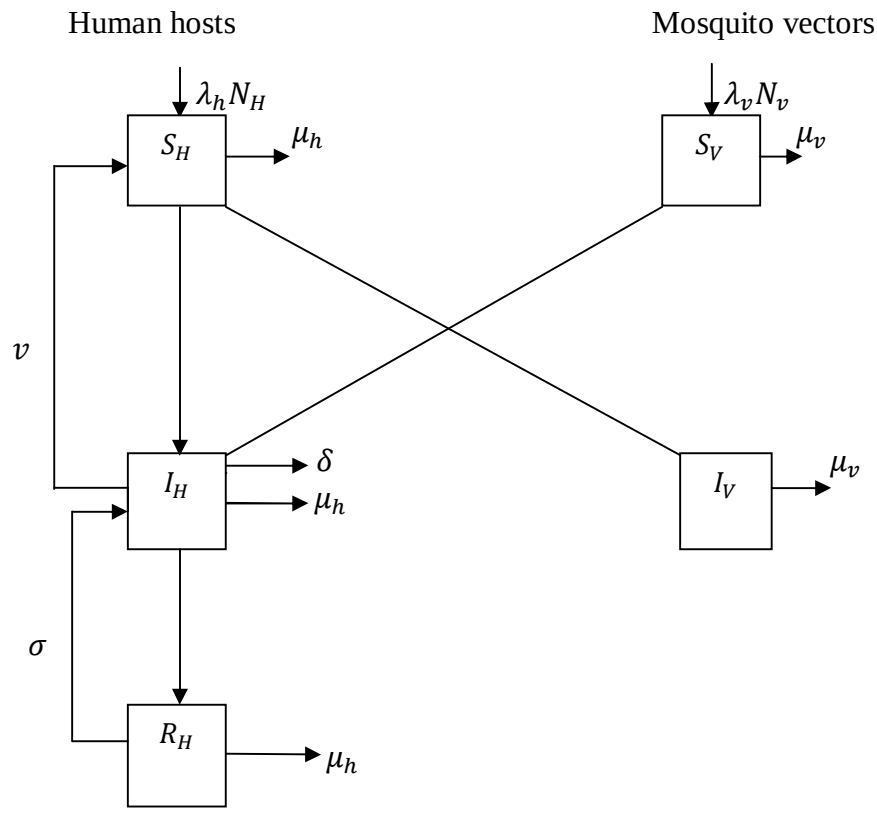


Fig 1.4: The host-vector dynamics of malaria transmission with temporary immunity

From the compartmental diagram figure1.4 above, we have the following set of equations for the dynamics of the model:

$$\begin{aligned}
\frac{dS_H}{dt} &= \lambda_h N_H - \frac{abS_H I_V}{N_H} + vI_H - \mu_h S_H \\
\frac{dI_H}{dt} &= \frac{abS_H I_V}{N_H} + \frac{\sigma abR_H I_V}{N_H} - (r + v + \delta + \mu_h) I_H \\
\frac{dR_H}{dt} &= rI_H - \frac{\sigma abR_H I_V}{N_H} - \mu_h R_H \\
\frac{dS_V}{dt} &= \lambda_v N_V - \frac{acS_V I_H}{N_H} - \mu_v I_V \\
\frac{dI_V}{dt} &= \frac{acS_V I_H}{N_H} - \mu_v I_V
\end{aligned} \tag{2.1.1}$$

with total population sizes $S_H + I_H + R_H = N_H$ and $S_V + I_V = N_V$

We assume that all infected human who recovered are moved to the recovered class and vaccinated human have temporary immunity that expires over time and again become susceptible, hence by including a vaccine parameter "α", the above model leads to the modified model:

$$\begin{aligned}
\frac{dS_H}{dt} &= \lambda_h N_H - \frac{abS_H I_V}{N_H} - \alpha S_H - \mu_h S_H \\
\frac{dI_H}{dt} &= \frac{abS_H I_V}{N_H} + \frac{\sigma abR_H I_V}{N_H} - (r + \delta + \mu_h) I_H \\
\frac{dR_H}{dt} &= rI_H - \frac{\sigma abR_H I_V}{N_H} - \mu_h R_H + \alpha S_H \\
\frac{dS_V}{dt} &= \lambda_v N_V - \frac{acS_V I_H}{N_H} - \mu_v S_V \\
\frac{dI_V}{dt} &= \frac{acS_V I_H}{N_H} - \mu_v I_V
\end{aligned} \tag{2.1.2}$$

2.2 Model analysis

2.2.1 Transformation of the system

The equations are obtained by differentiating each proportion with respect to time t. The proportions for the system are:

$s_h = \frac{S_H}{N_H}$, $i_h = \frac{I_H}{N_H}$, $r_h = \frac{R_H}{N_H}$, $s_v = \frac{S_V}{N_V}$, $i_v = \frac{I_V}{N_V}$ in the classes S_H, I_H, R_H, S_V and I_V of populations respectively and $m = \frac{N_V}{N_H}$ is the female vector-host ratio defined as the number of female mosquitoes per human host [2,7,23]. Scaling each of the new variables with respect to time gives the following system of equations:

$$\frac{dS_h}{dt} = \frac{d}{dt}\left(\frac{S_H}{N_H}\right) = \frac{1}{N_H}\left(\frac{dS_H}{dt} - \frac{S_H}{N_H}\frac{dN_H}{dt}\right)$$
$$= \frac{1}{N_H}\left(\lambda_h N_H - \frac{abS_H I_V}{N_H} - \alpha S_H - \mu_h S_H - s_h\left[\left(\lambda_h N_H - \frac{abS_H I_V}{N_H} - \alpha S_H - \mu_h S_H\right) + \left(\frac{abS_H I_V}{N_H} + \frac{\sigma abR_H I_V}{N_H} - \beta_1 I_H\right) + (rI_H - \frac{\sigma abR_H I_V}{N_H} - \mu_h R_H + \alpha S_H)\right]\right)$$

$$\frac{dS_h}{dt} = \lambda_h - abmi_v s_h - \alpha s_h - \mu_h s_h - \lambda_h s_h + abmi_v s_h^2 + \alpha s_h^2 + \mu_h s_h^2 - abmi_v s_h^2 - \sigma abmi_v s_h r_h - \beta_1 s_h i_h - r s_h i_h + \sigma abmi_v s_h r_h + \mu_h s_h r_h - \alpha s_h^2$$

$$\frac{dS_h}{dt} = \lambda_h - abmi_v s_h - \alpha s_h - \mu_h s_h - \lambda_h s_h + \mu_h s_h^2 + \beta_1 s_h i_h - r s_h i_h + \mu_h s_h r_h$$

$$= \lambda_h - abmi_v s_h - \alpha s_h + (r + \delta + \mu_h) s_h i_h - \mu_h s_h - \lambda_h s_h + \mu_h s_h^2 - r s_h i_h + \mu_h s_h r_h$$
$$= \lambda_h - abmi_v s_h - \alpha s_h + \delta s_h i_h + r s_h i_h - \mu_h s_h - \lambda_h s_h + \mu_h s_h^2 + \mu_h s_h i_h - r s_h i_h + \mu_h s_h r_h$$

$$= (1 - s_h)\lambda_h - abmi_v s_h - \alpha s_h + \delta s_h i_h - \mu_h s_h + \mu_h s_h(s_h + i_h + r_h)$$

$$\frac{dS_h}{dt} = (1 - s_h)\lambda_h - abmi_v s_h - \alpha s_h + \delta s_h i_h \text{ since } s_h + i_h + r_h = 1 \qquad (2.2.1)$$

For $i_h = \frac{I_H}{N_H}$, we have $\frac{di_h}{dt} = \frac{d}{dt}\left(\frac{I_H}{N_H}\right) = \frac{1}{N_H}\left(\frac{dI_H}{dt} - \frac{I_H}{N_H}\frac{dN_H}{dt}\right)$

$$\frac{di_h}{dt} = \frac{1}{N_H}\left(\frac{abS_H I_V}{N_H} + \frac{\sigma abR_H I_V}{N_H} - (r + \delta + \mu_h) I_H - \frac{I_H}{N_H}\left[\left(\frac{abS_H I_V}{N_H} + \frac{\sigma abR_H I_V}{N_H} - (r + \delta + \mu_h) I_H\right) + \left(\lambda_h N_H - \frac{abS_H I_V}{N_H} - \alpha S_H - \mu_h S_H\right) + (rI_H - \frac{\sigma abR_H I_V}{N_H} - \mu_h R_H + \alpha S_H)\right]\right)$$

$$\frac{di_h}{dt} = abms_h i_v + \sigma abmr_h i_v - (r+\delta+\mu_h)i_h - abms_h i_h i_v - \sigma abmr_h i_h i_v + (r+\delta+\mu_h)i_h^2 - \lambda_h i_h + abms_h i_h i_v + \alpha s_h i_h + \mu_h s_h i_h - r i_h^2 + \sigma abmr_h i_h i_v + \mu_h r_h i_h - \alpha s_h i_h$$

$$\frac{di_h}{dt} = abms_h i_v + \sigma abmr_h i_v - (r+\delta+\mu_h)i_h + (r+\delta)i_h^2 - \lambda_h i_h - r i_h^2 + \mu_h i_h^2 + \mu_h s_h i_h + \mu_h r_h i_h$$

$$= abms_h i_v + \sigma abmr_h i_v - (r+\delta+\mu_h)i_h + r i_h^2 + \delta i_h^2 - \lambda_h i_h - r i_h^2 + \mu_h i_h (s_h + i_h + r_h)$$

$$= abms_h i_v + \sigma abmr_h i_v - (r+\delta+\mu_h)i_h + \delta i_h^2 - \lambda_h i_h + \mu_h i_h$$

$$= abms_h i_v + \sigma abmr_h i_v - r i_h - \delta i_h - \mu_h i_h + \delta i_h^2 - \lambda_h i_h + \mu_h i_h$$

$$\frac{di_h}{dt} = abms_h i_v + \sigma abmr_h i_v - \beta_1 i_h + \delta i_h^2 \quad \text{where, } \beta_1 = \lambda_h + r + \delta \tag{2.2.2}$$

For $r_h = \frac{R_H}{N_H}$, we have $\frac{dr_h}{dt} = \frac{d}{dt}\left(\frac{R_H}{N_H}\right) = \frac{1}{N_H}\left(\frac{dR_H}{dt} - \frac{R_H}{N_H}\frac{dN_H}{dt}\right)$

$$\frac{dr_h}{dt} = \frac{1}{N_H}\left(rI_H - \frac{\sigma ab R_H I_V}{N_H} - \mu_h R_H + \alpha S_H - \frac{R_H}{N_H}\left[\left(rI_H - \frac{\sigma ab R_H I_V}{N_H} - \mu_h R_H + \alpha S_H\right) + \left(\lambda_h N_H - \frac{ab S_H I_V}{N_H} - \alpha S_H - \mu_h S_H\right) + \left(\frac{ab S_H I_V}{N_H} + \frac{\sigma ab R_H I_V}{N_H} - \beta_1 I_H\right)\right]\right)$$

$$\frac{dr_h}{dt} = r i_h - \sigma abmr_h i_v - \mu_h r_h + \alpha s_h - r i_h r_h + \sigma abmr_h^2 i_v + \mu_h r_h^2 - \alpha s_h r_h - \lambda_h r_h + abms_h i_v r_h + \alpha s_h r_h + \mu_h s_h r_h - abms_h i_v r_h - \sigma abmr_h^2 i_v + \beta_1 i_h r_h$$

$$= r i_h - \sigma abmr_h i_v - \mu_h r_h + \alpha s_h - r i_h r_h + \mu_h r_h^2 - \lambda_h r_h + \mu_h s_h r_h + \beta_1 i_h r_h$$

$$= r i_h - \sigma abmr_h i_v - \mu_h r_h + \alpha s_h - r i_h r_h + \mu_h r_h^2 - \lambda_h r_h + \mu_h s_h r_h + (r+\delta+\mu_h) i_h r_h$$

$$= r i_h - \sigma abmr_h i_v - \mu_h r_h + \alpha s_h - r i_h r_h + \mu_h r_h^2 - \lambda_h r_h + \mu_h s_h r_h + r i_h r_h + \delta i_h r_h + \mu_h i_h r_h$$

$$= r i_h - \lambda_h r_h - \sigma abmr_h i_v + \delta i_h r_h + \alpha s_h - \mu_h r_h + \mu_h r_h (s_h + i_h + r_h)$$

$$\frac{dr_h}{dt} = r i_h - \lambda_h r_h - \sigma abmr_h i_v + \delta i_h r_h + \alpha s_h \tag{2.2.3}$$

For $s_v = \frac{S_V}{N_H}$, we have $\frac{ds_v}{dt} = \frac{d}{dt}\left(\frac{s_v}{N_v}\right) = \frac{1}{N_V}\left(\frac{ds_v}{dt} - \frac{s_v}{N_v}\frac{dN_v}{dt}\right)$

$$\frac{ds_v}{dt} = \frac{1}{N_V}\left(\lambda_v N_V - \frac{acS_V I_H}{N_H} - \mu_v S_V - \frac{s_v}{N_v}\left[\left(\lambda_v N_V - \frac{acS_V I_H}{N_H} - \mu_v S_V\right) + \left(\frac{acS_V I_H}{N_H} - \mu_v I_V\right)\right]\right)$$

$$\frac{ds_v}{dt} = \lambda_v - acs_v i_h - \mu_v s_v - \lambda_v s_v + acs_v^2 i_h + \mu_v s_v s_v - acs_v^2 i_h + \mu_v s_v i_v$$

$$\frac{ds_v}{dt} = \lambda_v - acs_v i_h - \mu_v s_v - \lambda_v s_v + \mu_v s_v s_v + \mu_v s_v i_v$$

$$\frac{ds_v}{dt} = \lambda_v - acs_v i_h - \mu_v s_v - \lambda_v s_v + \mu_v s_v (s_v + i_v)$$

$$\frac{ds_v}{dt} = \lambda_v - \lambda_v s_v - acs_v i_h - \mu_v s_v + \mu_v s_v$$

$$\frac{ds_v}{dt} = \lambda_v(1 - s_v) - acs_v i_h \quad (2.2.4)$$

For $i_v = \frac{I_V}{N_V}$, we have $\frac{di_v}{dt} = \frac{d}{dt}\left(\frac{I_v}{N_v}\right) = \frac{1}{N_V}\left(\frac{dI_v}{dt} - \frac{I_v}{N_v}\frac{dN_v}{dt}\right)$

$$\frac{di_v}{dt} = \frac{1}{N_V}\left(\frac{acS_V I_H}{N_H} - \mu_v I_V - \frac{I_v}{N_v}\left[\left(\frac{acS_V I_H}{N_H} - \mu_v I_V\right) + (\lambda_v N_V - \frac{acS_V I_H}{N_H} - \mu_v S_V)\right]\right)$$

$$\frac{di_v}{dt} = acs_v i_h - \mu_v i_v - acs_v i_h i_v + \mu_v i_v^2 - \lambda_v i_v + acs_v i_h i_v + \mu_v s_v i_v$$

$$\frac{di_v}{dt} = acs_v i_h - \lambda_v i_v - \mu_v i_v + \mu_v i_v(s_v + i_v)$$

$$\frac{di_v}{dt} = acs_v i_h - \lambda_v i_v - \mu_v i_v + \mu_v i_v \quad (2.2.5)$$

Thus the system reduces to ;

$$\frac{ds_h}{dt} = (1 - s_h)\lambda_h - abmi_v s_h - \alpha s_h + \delta s_h i_h ,$$
$$\frac{di_h}{dt} = abms_h i_v + \sigma abmr_h i_v - \beta_1 i_h + \delta i_h^2 ,$$
$$\frac{dr_h}{dt} = ri_h - \lambda_h r_h - \sigma abmr_h i_v + \delta i_h r_h + \alpha s_h ,$$
$$\frac{ds_v}{dt} = \lambda_v(1 - s_v) - acs_v i_h ,$$
$$\frac{di_v}{dt} = acs_v i_h - \lambda_v i_v \quad (2.2.6)$$

2.2.1 Existence and positivity of solutions

All state variables are assumed to be positive since the model is dealing with population. Invariant region is obtained by the following lemma.

Lemma 2.2.1

The solutions of the system are contained in the region $\Gamma \in \mathbb{R}^5$ and $\Gamma_h \cup \Gamma_v \subset \mathbb{R}_+^3 \times \mathbb{R}_+^2$ (Mtisi et al,2008). We first show that the feasible solutions are uniformly bounded in proper subsets $\Gamma \in \mathbb{R}_+^5$. Let $\{(s_h), (i_h), (r_h), (s_v), (i_v)\} \in \mathbb{R}^5$ be any solution of the system given by $N_h = s_h + i_h + r_h$ and $N_v = s_v + i_v$ with non-negative initial conditions.

Proof

In differential forms ,we write

$$\frac{dN_h}{dt} = \frac{ds_h}{dt} + \frac{di_h}{dt} + \frac{dr_h}{dt} = \frac{ds_h}{dt} = (1 - s_h)\lambda_h - abmi_v s_h - \alpha s_h + \delta s_h i_h + abms_h i_v + \sigma abmi_h i_v - \lambda_h i_h - ri_h - \delta i_h + \delta i_h^2 + ri_h - \lambda_h r_h - \sigma abmi_h i_v + \delta i_h r_h + \alpha s_h$$
$$= \lambda_h - \lambda_h s_h - \lambda_h i_h - \lambda_h r_h - \delta i_h + \delta s_h i_h + \delta i_h^2 + \delta i_h r_h$$
$$= \lambda_h - \lambda_h(s_h + i_h + r_h) - \delta i_h + \delta i_h(s_h + i_h + r_h)$$

$\frac{dN_h}{dt} = \lambda_h - \lambda_h N_h - \delta i_h + \delta i_h N_h$

since $s_h + i_h + r_h = N_h$

$$\frac{dN_h}{dt} + (\lambda_h - \delta i_h)N_h = \lambda_h - \delta i_h$$

which is a first order differential equation

To solve the above differential equation above, we make use of the integrating factor which is given by $p(x) = e^{(\lambda_h - \delta i_h)t}$

Multiplying all through the differential equation by the integrating factor yields

$d(N_h e^{(\lambda_h - \delta i_h)t}) = \lambda_h e^{(\lambda_h - \delta i_h)t} dt$

and integrating both sides with respect to t yields

$N_h e^{(\lambda_h - \delta i_h)t} = e^{(\lambda_h - \delta i_h)t} + c$

$N_h = 1 + ce^{-(\lambda_h - \delta i_h)t}$ (2.2.7)

Applying initial conditions : that is when t=0

$N_h(0) = 1 + c$ or

$N_h(0) - 1 = c$.

Thus $N_h = 1 + (N_h(0) - 1)e^{-(\lambda_h - \delta i_h)t}$. $N_h \longrightarrow 1$ as t$\longrightarrow \infty$.

(2.2.8)

And

$$\frac{dN_v}{dt} = \frac{dS_v}{dt} + \frac{dI_v}{dt}$$

$=\lambda_v(1 - s_v) - \lambda_v i_v$

$=\lambda_v - \lambda_v N_v$

$\frac{dN_v}{dt} + \lambda_v N_v = \lambda_v$ (2.2.9)

Since it is the first order linear differential equation we use an integrating factor. Integrating factor $If = e^{(\int \lambda_v dt)} = e^{\lambda_v t}$

$d(N_v e^{\lambda_v t}) \leq \lambda_v e^{\lambda_v t} dt$

$N_v e^{\lambda_v t} \leq e^{\lambda_v t} + c$

$N_v \leq 1 + ce^{-\lambda_v t}$

$N_v(0) \leq 1 + c$ or

$N_v(0) - 1 \leq c$.

Thus $N_v \leq 1 + (N_v(0) - 1)e^{-\lambda_v t}$. $N_v \longrightarrow 1$ as t$\longrightarrow \infty$. (2.2.10)

Hence the host population size $N_h \longrightarrow 1$ as t$\longrightarrow \infty$. For vector total population size $N_v \longrightarrow 1$ as t$\longrightarrow \infty$. Thus the feasible region for the model system is given by

$\Gamma = \{((s_h), (i_h), (r_h), (s_v), (i_v)) \in \mathbb{R}_+^5; s_h, i_h, r_h, s_v, i_v \geq 0, s_h + i_h + r_h = 1 \text{and } s_v + i_v = 1\}$

which is positively invariant set for the model system. Hence the model is well-posed and biologically meaningful. The feasible region for $\mathbb{R}_+^5$(the positive orthant $\mathbb{R}^5$). Thus the system is well-posed.

2.3Positivity of solutions

For the system to be epidemiologically meaningful and well posed we need to prove that all the state variables are non-negative $\forall t \geq 0$.

Lemma 2.3

Let $\{(s_h(0)), (i_h(0)), (r_h(0)), (s_v(0)), (i_v(0)) \geq 0\} \in \Gamma$. Then the solution set $\{(s_h(t)), (i_h(t)), (r_h(t)), (s_v(t)), (i_v(t))\}$ of the model system is positive for all $t \geq 0$

Proof

From the first equation of system(2.2.6) ,we have

$\frac{ds_h}{dt} = (1 - s_h)\lambda_h - abms_h i_v - \alpha s_h + \delta s_h i_h$

$\frac{ds_h}{dt} = \lambda_h + \delta s_h i_h - (\lambda_h + abmi_v + \alpha)s_h$

.Thus,

$\frac{ds_h}{dt} \geq -(\lambda_h + abmi_v + \alpha)s_h$

I t then follows that,

$\frac{ds_h}{s_h} \geq -(\lambda_h + abmi_v + \alpha)dt$

after separating variables .

Integrating yields

$s_h(t) \geq s_h(0)e^{-(\int(abmi_v)dt + (\lambda_h + \alpha)t)}$

Therefore,

$s_h(t) \geq s_h(0)e^{-(\int(abmi_v)dt + (\lambda_h + \alpha)t} > 0$ iff $(\int(abmi_v)dt + (\lambda_h + \alpha)t) > 0$ (2.3.1)

For the second equation of the model(2.2.6), we have

$\frac{di_h}{dt} = abms_h i_v + \sigma abmr_h i_v - \lambda_h i_h - ri_h - \delta i_h + \delta i_h^2$

or $\frac{di_h}{dt} = abms_h i_v + \sigma abmr_h i_v + \delta i_h^2 - (\gamma + \delta + \lambda_h)i_h$

Thus,

$\frac{di_h}{dt} \geq -(\gamma + \delta + \lambda_h)i_h$

Separating the variables we have,

$\frac{di_h}{i_h} \geq -(\gamma + \delta + \lambda_h)dt$

and integrating yields,

$i_h(t) \geq (i_h(0)e^{-(\gamma+\delta+\lambda_h)t}$.

Therefore $i_h(t) \geq (i_h(0)e^{-(\gamma+\delta+\lambda_h)t} > 0$ iff $(\gamma + \delta + \lambda_h) > 0$ (2.3.2)

For the third equation of the model system (2.2,6), we have

$\frac{dr_h}{dt} = ri_h - \lambda_h r_h - \sigma abmr_h i_v + \delta i_h r_h + \alpha s_h$ or

$\frac{dr_h}{dt} = ri_h - \lambda_h r_h - \sigma abmr_h i_v + \delta i_h r_h + \alpha s_h$or

$\frac{dr_h}{dt} = ri_h + \delta i_h r_h + \alpha s_h - \lambda_h r_h - \sigma abmr_h i_v$ or

$\frac{dr_h}{dt} = ri_h + \delta i_h r_h + \alpha s_h - (\lambda_h + \sigma abmr_h i_v)r_h$.

Thus,

$\frac{dr_h}{dt} \geq -(\sigma abmi_h i_v + \lambda_h)r_h$.

Separating the variables we have,

$\frac{dr_h}{r_h} \geq -(\sigma abmi_h i_v + \lambda_h)dt$

and after integration of the above equation yields

$r_h(t) \geq r_h(0)e^{-[\int(\sigma abmi_h i_v)dt + \lambda_h t]}$

Thus, $r_h(t) \geq r_h(0)e^{-[\int(\sigma abmi_h i_v)dt + \lambda_h t]} > 0$ if and only if $[\int(\sigma abmi_h i_v)\,dt + \lambda_h t] > 0$ (2.2.3)

For the fourth equation of the model system(2.2.6), we have

$\frac{ds_v}{dt} = \lambda_v(1 - s_v) - acs_v i_h$ or

$\frac{ds_v}{dt} = \lambda_v - \lambda_v s_v - acs_v i_h$ or

$\frac{ds_v}{dt} = \lambda_v - (\lambda_v + aci_h)s_v$.

Thus,

$\frac{ds_v}{dt} \geq -(\lambda_v + aci_h)s_v$

Separating variables we have,

$\frac{ds_v}{s_v} \geq -(\lambda_v + aci_h)dt$

and after integrating we have

$s_v(t) \geq s_v(0)e^{-(\lambda_v + aci_h)t}$.

Therefore, $s_v(t) \geq s_v(0)e^{-(\lambda_v + aci_h)t} > 0$ if and only if $(\lambda_v + aci_h) > 0$ (2.3.4)

For the fifth equation of the system(2.2.6), we have

$\frac{di_v}{dt} = acs_v i_h - \lambda_v i_v$

Thus,

$\frac{di_v}{dt} \geq -\lambda_v i_v$

Separating the variables we get

$\frac{di_v}{i_v} \geq -\lambda_v dt$

and after integrating the above equation we get

$i_v(t) \geq i_v(0)e^{-\lambda_v t}$

where,

$i_v(t) \geq i_v(0)e^{-\lambda_v t} > 0$ if and only if $\lambda_v > 0$ (2.3.5)

2.4 Equilibrium states

The equilibria are obtained by equating the right hand side of the system below to zero and solving for the state variables in terms of i_h^* for easy analysis of the model. The resulting equilibria is given by $E = (s_h^*, i_h^*, r_h^*, s_v^*, i_v^*)$ and from it we can work out the two equilibrium points.

$$\begin{aligned}
\frac{ds_h}{dt} &= (1 - s_h)\lambda_h - abmi_v s_h - \alpha s_h + \delta s_h i_h \\
\frac{di_h}{dt} &= abms_h i_v + \sigma abmr_h i_v - \beta_1 i_h + \delta i_h^2 \\
\frac{dr_h}{dt} &= ri_h - \lambda_h r_h - \sigma abmr_h i_v + \delta i_h r_h + \alpha s_h \\
\frac{ds_v}{dt} &= \lambda_v(1 - s_v) - acs_v i_h \\
\frac{di_v}{dt} &= acs_v i_h - \lambda_v i_v
\end{aligned} \tag{2.4.1}$$

We set ,

$\lambda_v(1 - s_v^*) - acs_v^* i_h^* = 0$ or

$\lambda_v - \lambda_v s_v^* - acs_v^* i_h^* = 0$ or

$\lambda_v - s_v^*(\lambda_v + aci_h^*) = 0$ or

$$s_v^* = \frac{\lambda_v}{\lambda_v + aci_h^*} \tag{2.4.2}$$

$acs_v^* i_h^* - \lambda_v i_v^* = 0$ or

$$i_v^* = \frac{acs_v^* i_h^*}{\lambda_v} = \frac{\frac{\lambda_v aci_h^*}{\lambda_v + aci_h^*}}{\lambda_v} = \frac{aci_h^*}{\lambda_v + aci_h^*}$$

$$i_v^* = \frac{aci_h^*}{\lambda_v + aci_h^*} \tag{2.4.3}$$

From the equation

$\lambda_h - \lambda_h s_h^* - abmi_v^* s_h - \alpha s_h^* + \delta s_h^* i_h^* = 0$ or

$\lambda_h - (\lambda_h + abmi_v^* + \alpha - \delta i_h^*)s_h^* = 0$ or

$$s_h^* = \frac{\lambda_h}{\lambda_h + abmi_v^* + \alpha - \delta i_h^*} = \frac{\lambda_h}{\lambda_h + abm\left(\frac{aci_h^*}{\lambda_v + aci_h^*}\right) + \alpha - \delta i_h^*}$$

$$= \frac{\lambda_h}{\frac{a^2 bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)}{\lambda_v + aci_h^*}}$$

$$s_h^* = \frac{\lambda_h(\lambda_v + aci_h^*)}{a^2 bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)} \tag{2.4.4}$$

$ri_h^* + \alpha s_h^* - r_h^*(\lambda_h + \sigma abmi_v^* - \delta i_h^*) = 0$ or

$$r_h^* = \frac{ri_h^* + \alpha s_h^*}{\lambda_h + \sigma abmi_v^* - \delta i_h^*} \text{ or}$$

$$r_h^* = \frac{ri_h^* + \alpha\left[\frac{\lambda_h(\lambda_v + aci_h^*)}{a^2bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)}\right]}{\lambda_h + \sigma abm\left[\frac{aci_h^*}{\lambda_v + aci_h^*}\right] - \delta i_h^*}$$

$$= \frac{\frac{[a^2bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)]ri_h^* + \lambda_h\alpha(\lambda_v + aci_h^*)}{a^2bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)}}{\frac{\sigma a^2bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h - \delta i_h^*)}{\lambda_v + aci_h^*}}$$

$$r_h^* = \frac{\{[a^2bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)]ri_h^* + \lambda_h\alpha(\lambda_v + aci_h^*)\}(\lambda_v + aci_h^*)}{[a^2bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)][\sigma a^2bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h - \delta i_h^*)]} \quad (2.4.5)$$

2.4 Existence of disease free equilibrium point(DFE)

The system is analyzed to determine the equilibrium point of the system and their stabilities. Let $E(s_h^*, i_h^*, r_h^*, s_v^*, i_v^*)$ be the equilibrium point of the DFE model which is obtained by setting $\frac{ds_h^*}{dt} = \frac{di_h^*}{dt} = \frac{dr_h^*}{dt} = \frac{ds_v^*}{dt} = \frac{di_v^*}{dt} = 0$

Therefore, in the absence of infection, that is, when $i_h^* = i_v^* = \alpha = 0$, the model system has steady state E_0 called the disease free equilibrium . When we substitute $i_h^* = i_v^* = r_h^* = \alpha = 0$ in the system, the system reduces to $(1 - s_h^*)\lambda_h = 0$ or $\lambda_h s_h^* = \lambda_h$ hence $s_h^* = 1$,also $(1 - s_v^*)\lambda_v = 0$ or $\lambda_v s_v^* = \lambda_v$ hence $s_v^* = 1$.Thus the disease free equilibrium point of the model is given by $E_0 = (s_h^*, i_h^*, r_h^*, s_v^*, i_v^*) = (1,0,0,1,0)$

At the disease free point, the susceptible populations is equal to the total population respectively , that is, $N_h = s_h^*$ and $N_v = s_v^*$. Thus the disease free equilibrium exists when $\lambda_h > 0$ and $\lambda_v > 0$.

2.5 Basic reproduction number

In order to assess the stability of disease free equilibrium(DFE) and the endemic equilibrium(EEP),the computation of the basic reproductive number R_0 is required. R_0 represents the average number of secondary infections that a single infectious host can generate in a totally susceptible population of hosts and vectors. In other words, to know how many infectious individuals are generated by a single infective introduced into a susceptible population. We now write the equations of the system beginning with the infective and use the next generation matrix to determine the basic reproductive number.

2.5.1 Next generation matrix

To determine the basic reproduction number, we consider the system of differential equations:

$$\begin{aligned}
\frac{di_h}{dt} &= abms_h i_v + \sigma abmr_h i_v - \lambda_h i_h - r i_h - \delta i_h + \delta i_h^2 , \\
\frac{di_v}{dt} &= acs_v i_h - \lambda_v i_v , \\
\frac{dr_h}{dt} &= r i_h - \lambda_h r_h - \sigma abmr_h i_v + \delta i_h r_h + \alpha s_h , \\
\frac{ds_h}{dt} &= (1 - s_h)\lambda_h - abmi_v s_h - \alpha s_h + \delta s_h i_h , \\
\frac{ds_v}{dt} &= \lambda_v (1 - s_v) - acs_v i_h
\end{aligned} \tag{2.5.1}$$

R_0 is obtained by taking the largest(dominant) eigen-value(spectral radius) of $\left[\frac{\partial F_i E_0}{\partial x_j}\right].\left[\frac{\partial V_i E_0}{\partial x_j}\right]^{-1}$

where ,

F_i is the rate of appearance of new infections in compartment i
V_i^+ is the transfer of individuals into compartment i
V_i^- is the transfer of individuals out of compartment i
E_0 is the disease free equilibrium point

$$E_0 = (s_h^*, i_h^*, r_h^*, s_v^*, i_v^*) = (1,0,0,1,0) .$$

The new infected compartment are i_h and i_v.

$$\text{Therefore,} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \delta i_h^{*2} \\ acs_v^* i_h^* \end{bmatrix}, F = \begin{bmatrix} \frac{\partial f_1(E_0)}{\partial i_h^*} & \frac{\partial f_1(E_0)}{\partial i_v^*} \\ \frac{\partial f_2(E_0)}{\partial i_h^*} & \frac{\partial f_2(E_0)}{\partial i_v^*} \end{bmatrix} = \begin{bmatrix} 2\delta i_h^* & 0 \\ acs_v^* & 0 \end{bmatrix}. \tag{2.5.2}$$

$$\text{At } E_0 = (s_h^*, i_h^*, r_h^*, s_v^*, i_v^*) = (1,0,0,1,0), F = \begin{bmatrix} 0 & 0 \\ ac & 0 \end{bmatrix}. \tag{2.5.3}$$

$$\text{Also } \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -(r + \lambda_h + \delta) i_h^* - abms_h^* i_v^* - \sigma abmr_h^* i_v^* \\ \lambda_v i_v^* \end{bmatrix}.$$

By linearizing at disease free equilibrium point we have

$$V = \begin{bmatrix} \frac{\partial V_1(E_0)}{\partial i_h^*} & \frac{\partial V_1(E_0)}{\partial i_v^*} \\ \frac{\partial V_2(E_0)}{\partial i_h^*} & \frac{\partial V_2(E_0)}{\partial i_v^*} \end{bmatrix} = \begin{bmatrix} (r + \lambda_h + \delta) & -abms_h^* - \sigma abmr_h^* \\ 0 & \lambda_v \end{bmatrix}$$

The jacobian matrix of V evaluated at E_0 is given by

$$V = \begin{bmatrix} (\lambda_h + r + \delta) & -abm \\ 0 & \lambda_v \end{bmatrix} \quad (2.5.4)$$

The inverse of the jacobian matrix above is given by

$$V^{-1} = \frac{1}{\lambda_v(\lambda_h+r+\delta)} \begin{bmatrix} (\lambda_h + r + \delta) & -abm \\ 0 & \lambda_v \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(\lambda_h+r+\delta)} & \frac{abm}{\lambda_v(\lambda_h+r+\delta)} \\ 0 & \frac{1}{\lambda_v} \end{bmatrix} \quad (2.5.5)$$

The product of F and V^{-1} ,

$$F\,V^{-1} = \begin{bmatrix} 0 & 0 \\ ac & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(\lambda_h+r+\delta)} & \frac{abm}{\lambda_v(\lambda_h+r+\delta)} \\ 0 & \frac{1}{\lambda_v} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{ac}{(\lambda_h+r+\delta)} & \frac{a^2bcm}{\lambda_v(\lambda_h+r+\delta)} \end{bmatrix} \quad (2.5.6)$$

Hence we develop a matrix

$$A = |FV^{-1} - IP| = 0$$

$$A = \begin{bmatrix} -P & 0 \\ \frac{ac}{(\lambda_h+r+\delta)} & \frac{a^2bcm}{\lambda_v(\lambda_h+r+\delta)} - P \end{bmatrix} \quad (2.5.7)$$

Then the eigenvalues (P) of matrix A are given by

$$P\left(\frac{a^2bcm}{\lambda_v(\lambda_h+r+\delta)} - P\right) = 0 \quad (2.5.8)$$

Thus $P = 0$ or $P = \frac{a^2bcm}{\lambda_v(\lambda_h+r+\delta)}$ (2.5.9)

Hence the reproduction number R_0 from (2.5.9) is the dominant eigenvalue of A ,which is

$$R_0 = \frac{a^2bcm}{\lambda_v(\lambda_h+r+\delta)} \quad (2.5.10)$$

2.6 Local stability

The local stability of the disease free equilibrium point is obtained by the following lemma.

Lemma 2.6.1

If $R_0 < 1$, the disease free equilibrium point E_0 of the model is locally assymptotically stable, and is unstable if $R_0 > 1$.The local stability of DFE equilibrium point E_0 is established by linearizing system (2.2.6) around a DFE, then

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial s_h} & \frac{\partial f_1}{\partial i_h} & \frac{\partial f_1}{\partial r_h} & \frac{\partial f_1}{\partial s_v} & \frac{\partial f_1}{\partial i_v} \\ \frac{\partial f_2}{\partial s_h} & \frac{\partial f_2}{\partial i_h} & \frac{\partial f_2}{\partial r_h} & \frac{\partial f_2}{\partial s_v} & \frac{\partial f_2}{\partial i_v} \\ \frac{\partial f_3}{\partial s_h} & \frac{\partial f_3}{\partial i_h} & \frac{\partial f_3}{\partial r_h} & \frac{\partial f_3}{\partial s_v} & \frac{\partial f_3}{\partial i_v} \\ \frac{\partial f_4}{\partial s_h} & \frac{\partial f_4}{\partial i_h} & \frac{\partial f_4}{\partial r_h} & \frac{\partial f_4}{\partial s_v} & \frac{\partial f_4}{\partial i_v} \\ \frac{\partial f_5}{\partial s_h} & \frac{\partial f_5}{\partial i_h} & \frac{\partial f_5}{\partial r_h} & \frac{\partial f_5}{\partial s_v} & \frac{\partial f_5}{\partial i_v} \end{bmatrix}$$

$$J_E = \begin{bmatrix} -(\lambda_h + abmi_v + \alpha - \delta i_h) & \delta s_h & 0 & 0 & -abms_h \\ abmi_v & -(\lambda_h + r + \delta) + 2\delta i_h & \sigma abmi_v & 0 & abms_h + \sigma abmr_h \\ \alpha & r + \delta r_h & \delta i_h - (\lambda_h + \sigma abmi_v) & 0 & \sigma abmr_h \\ 0 & -acs_v & 0 & -(\lambda_v + aci_h) & 0 \\ 0 & acs_v & 0 & aci_h & -\lambda_v \end{bmatrix}$$

At the DFE point $E_0 = (s_h, i_h, r_h, s_v, i_v) = (1,0,0,1,0)$,hence the jacobian matrix simplifies to

$$J_{E_0} = \begin{bmatrix} -(\lambda_h + \alpha) & \delta & 0 & 0 & -abm \\ 0 & -(r + \lambda_h + \delta) & 0 & 0 & abm \\ \alpha & r & -\lambda_h & 0 & 0 \\ 0 & -ac & 0 & -\lambda_v & 0 \\ 0 & ac & 0 & 0 & -\lambda_v \end{bmatrix}$$

We observe matrix that the matrix J_{E_0} has negative eigenvalues

$-(\lambda_h + \alpha), -\lambda_h, -\lambda_v$ (2.6.1)

and the remaining two can be obtained from the block matrix given by

$$B = \begin{bmatrix} -(r + \lambda_h + \delta) & abm \\ ac & -\lambda_v \end{bmatrix} \quad (2.6.2)$$

whose trace and determinant are given by

Trace $B = -(\lambda_h + \lambda_v + r + \delta)$ (2.6.3)

Det $B = \lambda_v(\lambda_h + r + \delta) - a^2bcm = \lambda_v(\lambda_h + r + \delta)(1 - R_0) > 0$ if $R_0 < 1$ (2.6.4)

where

$$R_0 = \frac{a^2bcm}{\lambda_v(\lambda_h+r+\delta)} \quad (2.6.5)$$

Thus E_0 is locally and asymptotically stable if and only if $R_0 < 1$ and unstable if $R_0 > 1$.

The quantity R_0 is the basic reproduction number of the new infections produced by one infected individual introduced in an otherwise susceptible population. It is useful quantity in the study of a disease as it sets the threshold for its establishment. If $R_0 < 1$,the disease free equilibrium is locally unstable.

Alternatively, the characteristic equation of the above matrix is

$$(\lambda_h + \alpha + \xi)(\lambda_h + \xi)(\lambda_v + \xi)[-(\lambda_h + r + \delta + \xi)(\lambda_v + \xi) + a^2bcm] = 0 \text{ or}$$

$$(\lambda_h + \alpha + \xi)(\lambda_h + \xi)(\lambda_v + \xi)[-\lambda_v(\lambda_h + r + \delta) - \lambda_v\xi - (\lambda_h + r + \delta)\xi + \xi^2) + a^2bcm] = 0$$

$$(\lambda_h + \alpha + \xi)(\lambda_h + \xi)(\lambda_v + \xi)[\xi^2 - (\lambda_v + \lambda_h + r + \delta)\xi - \lambda_v(\lambda_h + r + \delta) + a^2bcm] = 0$$

$$(\lambda_h + \alpha + \xi)(\lambda_h + \xi)(\lambda_v + \xi)\left[\xi^2 - (\lambda_v + \lambda_h + r + \delta)\xi - \lambda_v(\lambda_h + r + \delta)[1 - \frac{a^2bcm}{\lambda_v(\lambda_h+r+\delta)}]\right] = 0$$

$$(\lambda_h + \alpha + \xi)(\lambda_h + \xi)(\lambda_v + \xi)[\xi^2 - (\lambda_v + \lambda_h + r + \delta)\xi - \lambda_v(\lambda_h + r + \delta)(1 - R_0)] = 0 \quad (2.6.6)$$

where,

$$R_0 = \frac{a^2bcm}{\lambda_v(\lambda_h+r+\delta)} \quad (2.6.7)$$

There are five eigenvalues corresponding to the characteristic equation above.

Three of the eigenvalues $\xi_1 = -(\lambda_h + \alpha)$, $\xi_2 = -\lambda_h$, $\xi_3 = -\lambda_v$ (2.6.8)

have negative real parts.

The other two eigenvalues can be obtained from the quadratic equation

$$\xi^2 - (\lambda_v + \lambda_h + r + \delta)\xi - \lambda_v(\lambda_h + r + \delta)(1 - R_0) = 0 \quad (2.6.9)$$

Applying the Routh-Hurwitz criteria for a quadratic polynomial. It is easy to see that both the coefficients of (2.6.6) are positive if and only if $R_0 < 1$. Thus, all roots of (2.6.6) are with negative real parts if $R_0 < 1$, and one of its roots is with positive real part if $R_0 > 1$. Therefore, the disease-free equilibrium(DFE) E_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. Thus, we have the following result;

Theorem 2.6.2

The uninfected equilibrium E_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$ in Γ.

2.7 The endemic equilibrium point

The endemic equilibrium point of the model is obtained by setting

$$\frac{ds_h}{dt} = \frac{di_h}{dt} = \frac{dr_h}{dt} = \frac{ds_v}{dt} = \frac{di_v}{dt} = 0 \qquad (2.7.1)$$

Then by solving the given system each equilibrium point is expressed in terms of i_h^* at steady states and this yields,

$E_1 = (s_h^*, i_h^*, r_h^*, s_v^*, i_v^*)$

as an endemic equilibrium point where,

$\lambda_v(1 - s_v^*) - acs_v^* i_h^* = 0$ or

$\lambda_v - \lambda_v s_v^* - acs_v^* i_h^* = 0$ or

$\lambda_v - s_v^*(\lambda_v - aci_h^*) = 0$ or

$$s_v^* = \frac{\lambda_v}{\lambda_v - aci_h^*} \qquad (2.7.2)$$

$acs_v^* i_h^* - \lambda_v i_v^* = 0$ or

$$i_v^* = \frac{acs_v^* i_h^*}{\lambda_v} = \frac{\frac{\lambda_v aci_h^*}{\lambda_v - aci_h^*}}{\lambda_v} = \frac{aci_h^*}{\lambda_v - aci_h^*}$$

$$i_v^* = \frac{aci_h^*}{\lambda_v - aci_h^*} \qquad (2.7.3)$$

$\lambda_h - \lambda_h s_h^* - abmi_v^* s_h - \alpha s_h^* + \delta s_h^* i_h^* = 0$ or

$\lambda_h - (\lambda_h + abmi_v^* + \alpha - \delta i_h^*)s_h^* = 0$ or

$$s_h^* = \frac{\lambda_h}{\lambda_h + abmi_v^* + \alpha - \delta i_h^*}$$

$$= \frac{\lambda_h}{\lambda_h + abm\left(\frac{aci_h^*}{\lambda_v - aci_h^*}\right) + \alpha - \delta i_h^*}$$

$$= \frac{\lambda_h}{\frac{a^2 bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)}{\lambda_v - aci_h^*}}$$

$$s_h^* = \frac{\lambda_h(\lambda_v - aci_h^*)}{a^2 bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)} \qquad (2.7.4)$$

$ri_h^* + \alpha s_h^* - r_h^*(\lambda_h + \sigma abmi_v^* - \delta i_h^*) = 0$ or

$$r_h^* = \frac{ri_h^* + \alpha s_h^*}{\lambda_h + \sigma abmi_v^* - \delta i_h^*} \text{ or}$$

$$r_h^* = \frac{ri_h^* + \alpha\left[\frac{\lambda_h(\lambda_v - aci_h^*)}{a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)}\right]}{\lambda_h + \sigma abm\left[\frac{aci_h^*}{\lambda_v - aci_h^*}\right] - \delta i_h^*}$$

$$= \frac{\frac{[a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)]ri_h^* + \lambda_h\alpha(\lambda_v - aci_h^*)}{a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)}}{\frac{\sigma a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h - \delta i_h^*)}{\lambda_v - aci_h^*}}$$

$$r_h^* = \frac{\{[a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)]ri_h^* + \lambda_h\alpha(\lambda_v - aci_h^*)\}(\lambda_v - aci_h^*)}{[a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)][\sigma a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h - \delta i_h^*)]}$$

$$r_h^* = \frac{\{[a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)]ri_h^* + \alpha\lambda_h(\lambda_v - aci_h^*)\}[\lambda_v - aci_h^*]}{[a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)][\sigma a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h - \delta i_h^*)]} \quad (2.7.5)$$

$$ri_h^* + \alpha\left[\frac{\lambda_h(\lambda_v - aci_h^*)}{a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)}\right] - r_h^*(\lambda_h + \sigma abm\left[\frac{aci_h^*}{\lambda_v - aci_h^*}\right] - \delta i_h^*) = 0$$

$$\frac{[a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)]ri_h^* + \alpha\lambda_h(\lambda_v - aci_h^*)}{a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)} - \frac{r_h^*[\sigma a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h - \delta i_h^*)]}{\lambda_v - aci_h^*} = 0 \quad (2.7.6)$$

$$\frac{[a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)]ri_h^* + \alpha\lambda_h(\lambda_v - aci_h^*)}{a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)} = \frac{r_h^*[\sigma a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h - \delta i_h^*)]}{\lambda_v - aci_h^*}$$

$$r_h^* = \frac{\{[a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)]ri_h^* + \alpha\lambda_h(\lambda_v - aci_h^*)\}[\lambda_v - aci_h^*]}{[a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h + \alpha - \delta i_h^*)][\sigma a^2bcmi_h^* + (\lambda_v - aci_h^*)(\lambda_h - \delta i_h^*)]} \quad (2.7.9)$$

Let $r_h = 1 - s_h - i_h$ and $s_v = 1 - i_v$, the model system (2.2.6) can reduce to a 3-dimensional system given by equations (2.7.10),(2.7.13),and (2.7.14) as shown below

$$\frac{dS_h}{dt} = (1 - s_h)\lambda_h - abmi_v s_h - \alpha s_h + \delta s_h i_h \quad (2.7.10)$$

$$\frac{di_h}{dt} = abms_h i_v + \sigma abmr_h i_v - \beta_2 i_h^* + \delta i_h^2 \quad (2.7.11)$$

$$\frac{di_h}{dt} = abms_h i_v + \sigma abmi_v(1 - s_h - i_h) - \beta_2 i_h^* + \delta i_h^2 \quad (2.7.12)$$

$$\frac{di_h}{dt} = abms_h i_v + \sigma abmi_v - \sigma abmi_v s_h - \sigma abmi_v i_h - \beta_2 i_h^* + \delta i_h^2 \quad (2.7.13)$$

$$\frac{di_v}{dt} = acs_v i_h - \lambda_v i_v \quad (2.7.14)$$

We can now make the substitutions,

$r_h = 1 - s_h - i_h$ and $s_v = 1 - i_v$

in each of the equations (2.7.10),(2.7.13),and (2.7.14) to obtain the three dimensional system given below by equations (2.7.15)-(2.7.17)

$$\frac{di_v}{dt} = aci_h(1 - i_v) - \lambda_v i_v$$

$$\frac{di_v}{dt} = aci_h - aci_h i_v - \lambda_v i_v \quad (2.7.15)$$

That is,

$$\frac{dS_h}{dt} = (1 - s_h)\lambda_h - abmi_v s_h - \alpha s_h + \delta s_h i_h \quad (2.7.16)$$

$$\frac{di_h}{dt} = abms_h i_v + \sigma abmi_v - \sigma abmi_v s_h - \sigma abmi_v i_h - \beta_2 i_h + \delta i_h^2 \quad (2.7.17)$$

Equating the right hand side of the equations (2.7.15)-(2.7.17) of the above system to zero yields

$$(1 - s_h^*)\lambda_h - abmi_v^* s_h^* - \alpha s_h^* + \delta s_h^* i_h^* = 0 \quad (2.7.18)$$

$$abms_h^* i_v^* + \sigma abmi_v^* - \sigma abmi_v^* s_h^* - \sigma abmi_v^* i_h^* - \beta_2 i_h^* + \delta {i_h^*}^2 = 0 \quad (2.7.19)$$

$$aci_h^* - aci_h^* i_v^* - \lambda_v i_v^* = 0 \quad (2.7.20)$$

Expressing equation (2.7.18) and equation (2.7.20) in terms of i_h^* we get

$$s_h^* = \frac{\lambda_h}{\lambda_h + abmi_v^* + \alpha - \delta i_h^*}$$

$$= \frac{\lambda_h}{\lambda_h + abm\left(\frac{aci_h^*}{\lambda_v + aci_h^*}\right) + \alpha - \delta i_h^*}$$

$$= \frac{\lambda_h}{\frac{a^2 bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)}{\lambda_v + aci_h^*}}$$

$$= \frac{\lambda_h(\lambda_v + aci_h^*)}{a^2 bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)}$$

$$s_h^* = \frac{\lambda_h(\lambda_v + aci_h^*)}{a^2 bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)} \quad (2.7.21)$$

$$i_v^* = \frac{aci_h^*}{\lambda_v + aci_h^*} \quad (2.7.22)$$

Substituting the equations (2.7.21) and (2.7.22) into equation (2.7.19) we get,

$$abm\left[\frac{\lambda_h(\lambda_v + aci_h^*)}{a^2 bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)}\right]\left[\frac{aci_h^*}{\lambda_v + aci_h^*}\right] + \sigma abm\left[\frac{aci_h^*}{\lambda_v + aci_h^*}\right] -$$
$$\sigma abm\left[\frac{\lambda_h(\lambda_v + aci_h^*)}{a^2 bcmi_h^* + (\lambda_v + aci_h^*)(\lambda_h + \alpha - \delta i_h^*)}\right]\left[\frac{aci_h^*}{\lambda_v + aci_h^*}\right] - \sigma abm\left[\frac{aci_h^*}{\lambda_v + aci_h^*}\right] i_h^* - \beta_2 i_h^* + \delta {i_h^*}^2 = 0 \quad (2.7.23)$$

$$\left[\frac{\lambda_h a^2 bcm(\lambda_v+aci_h^*)i_h^*}{[a^2 bcmi_h^*+(\lambda_v+aci_h^*)(\lambda_h+\alpha-\delta i_h^*)](\lambda_v+aci_h^*)}\right]+\left[\frac{\sigma a^2 bcmi_h^*}{\lambda_v+aci_h^*}\right]-\left[\frac{\lambda_h\sigma a^2 bcm(\lambda_v+aci_h^*)i_h^*}{[a^2 bcmi_h^*+(\lambda_v+aci_h^*)(\lambda_h+\alpha-\delta i_h^*)](\lambda_v+aci_h^*)}\right]-\left[\frac{\sigma a^2 bcmi_h^{*2}}{\lambda_v+aci_h^*}\right]-\beta_2+\delta i_h^*=0 \quad (2.7.24)$$

$$\left[\frac{\lambda_h a^2 bcm(\lambda_v+aci_h^*)}{[a^2 bcmi_h^*+(\lambda_v+aci_h^*)(\lambda_h+\alpha-\delta i_h^*)](\lambda_v+aci_h^*)}\right]+\left[\frac{\sigma a^2 bcm}{\lambda_v+aci_h^*}\right]-\left[\frac{\lambda_h\sigma a^2 bcm(\lambda_v+aci_h^*)}{[a^2 bcmi_h^*+(\lambda_v+aci_h^*)(\lambda_h+\alpha-\delta i_h^*)](\lambda_v+aci_h^*)}\right]-\left[\frac{\sigma a^2 bcmi_h^*}{\lambda_v+aci_h^*}\right]-\beta_2+\delta i_h^*=0 \quad (2.7.25)$$

$$\lambda_h a^2 bmc(\lambda_v+aci_h^*)-\lambda_h\sigma a^2 bcm(\lambda_v+aci_h^*)+\{[a^2 bcmi_h^*+(\lambda_v+aci_h^*)(\lambda_h+\alpha-\delta i_h^*)]\}[\sigma a^2 bcm-\sigma a^2 bcmi_h^*]-[a^2 bcmi_h^*+(\lambda_v+aci_h^*)(\lambda_h+\alpha-\delta i_h^*)](\lambda_v+aci_h^*)\beta_2+[a^2 bcmi_h^*+(\lambda_v+aci_h^*)(\lambda_h+\alpha-\delta i_h^*)](\lambda_v+aci_h^*)\delta i_h^*=0 \quad (2.7.26)$$

$$\lambda_v\lambda_h a^2 bcm+\lambda_h a^2 bcmaci_h^*-\lambda_v\lambda_h\sigma a^2 bcm-\lambda_h\sigma a^3 bc^2 mi_h^*+\{\sigma a^2 bcm[a^2 bcmi_h^*+\lambda_h\lambda_v+\lambda_v\alpha-\lambda_v\delta i_h^*+\lambda_h aci_h^*+\alpha aci_h^*-\delta aci_h^{*2}]-\sigma a^2 bcmi_h^*[a^2 bcmi_h^*+\lambda_h\lambda_v+\lambda_v\alpha-\lambda_v\delta i_h^*+\lambda_h aci_h^*+\alpha aci_h^*-\delta aci_h^{*2}]\}-[a^2 bcmi_h^*+\lambda_h\lambda_v+\lambda_v\alpha-\lambda_v\delta i_h^*+\lambda_h aci_h^*+\alpha aci_h^*-\delta aci_h^{*2}](\lambda_v+aci_h^*)\beta_2+[a^2 bcmi_h^*+\lambda_h\lambda_v+\lambda_v\alpha-\lambda_v\delta i_h^*+\lambda_h aci_h^*+\alpha aci_h^*-\delta aci_h^{*2}](\lambda_v+aci_h^*)\delta i_h^*=0 \quad (2.7.27)$$

$$\lambda_v\lambda_h a^2 bcm+\lambda_h a^2 bcmaci_h^*-\lambda_v\lambda_h\sigma a^2 bcm-\lambda_h\sigma a^3 bc^2 mi_h^*+\{[(\sigma a^4 b^2 c^2 m^2 i_h^*+\lambda_h\lambda_v\sigma a^2 bcm+\lambda_v\alpha\sigma a^2 bcm-\lambda_v\sigma a^2 bcm\delta i_h^*+\lambda_h\sigma a^3 bc^2 mi_h^*+\alpha\sigma a^3 bc^2 mi_h^*-\delta\sigma a^3 bc^2 mi_h^{*2})+(-\sigma a^4 b^2 c^2 m^2 i_h^{*2}-\lambda_h\lambda_v\sigma a^2 bcmi_h^*-\lambda_v\alpha\sigma a^2 bcmi_h^*+\lambda_v\delta\sigma a^2 bcmi_h^{*2}-\lambda_h\sigma a^3 bc^2 mi_h^{*2}-\alpha\sigma a^3 bc^2 mi_h^{*2}+\delta a^3 bc^2 mi_h^{*3})]\}+[(-\lambda_v\beta_2 a^2 bcmi_h^*-\lambda_h\lambda_v^2\beta_2-\lambda_v^2\beta_2\alpha+\lambda_v\beta_2\delta i_h^*-\lambda_h\lambda_v\beta_2 aci_h^*-\lambda_v\beta_2\alpha aci_h^*+\lambda_v\beta_2\delta aci_h^{*2})+(-\beta_2 a^3 bc^2 mi_h^{*2}-\lambda_h\lambda_v\beta_2 aci_h^*-\lambda_v\alpha\beta_2 aci_h^*+\lambda_v\delta\beta_2 aci_h^{*2}-\lambda_h\beta_2 a^2 c^2 i_h^{*2}-\alpha\beta_2 a^2 c^2 i_h^{*2}+\delta\beta_2 a^2 c^2 i_h^{*3})]+[(\lambda_v\delta a^2 bcmi_h^{*2}+\lambda_h\lambda_v^2\delta i_h^*+\lambda_v^2\alpha\delta i_h^*-\lambda_v^2\delta^2 i_h^{*2}+\lambda_h\lambda_v\delta aci_h^{*2}+\lambda_v\delta\alpha aci_h^{*2}-\delta^2\lambda_v aci_h^{*3})+(a^3 bc^2 m\delta i_h^{*3}+\lambda_h\lambda_v ac\delta i_h^{*2}+\lambda_v\alpha ac\delta i_h^{*2}-\lambda_v\delta aci_h^{*3}+\lambda_h a^2 c^2\delta i_h^{*3}+\delta\alpha a^2 c^2 i_h^{*3}-\delta^2 a^2 c^2 i_h^{*4})]=0 \quad (2.7.28)$$

$$[\delta^2 a^2 c^2]i_h^{*4}-[\delta\alpha a^2 c^2+\lambda_h a^2 c^2\delta-\lambda_v\delta ac+\delta a^3 bc^2 m-\delta^2\lambda_v ac+\delta\beta_2 a^2 c^2+\delta a^3 bc^2 m]i_h^{*3}-[\lambda_v\alpha ac\delta+\lambda_h\lambda_v ac\delta+\lambda_v\delta\alpha ac+\lambda_h\lambda_v\delta ac+\lambda_v\delta a^2 bcm+\lambda_v\delta\beta_2 ac+\lambda_v\beta_2\delta ac+\lambda_v\delta\sigma a^2 bcm-\lambda_v^2\delta^2-\alpha\beta_2 a^2 c^2-\beta_2 a^3 bc^2 m-\lambda_h\sigma a^3 bc^2 m-\alpha\sigma a^3 bc^2 m-\lambda_h\beta_2 a^2 c^2-\sigma a^4 b^2 c^2 m^2-\delta\sigma a^3 bc^2 m]i_h^{*2}-[\lambda_v^2\alpha\delta+\lambda_h\lambda_v^2\delta+\lambda_v\beta_2\delta+\alpha\sigma a^3 bc^2 m+\lambda_h\sigma a^3 bc^2 m+\sigma a^4 b^2 c^2 m^2+\lambda_h a^2 bcmac-\lambda_v\sigma a^2 bcm\delta-\lambda_v\beta_2 a^2 bcm-\lambda_h\lambda_v\sigma a^2 bcm-\lambda_v\alpha\sigma a^2 bcm-\lambda_v\alpha\beta_2 ac-\lambda_h\lambda_v\beta_2 ac-\lambda_v\beta_2\alpha ac-\lambda_h\lambda_v\beta_2 ac-\lambda_h\sigma a^3 bc^2 m]i_h^*-[\lambda_v\lambda_h a^2 bcm+\lambda_h\lambda_v\sigma a^2 bcm+\lambda_v\alpha\sigma a^2 bcm-\lambda_v\lambda_h\sigma a^2 bcm-\lambda_h\lambda_v^2\beta_2-\lambda_v^2\beta_2\alpha]=0 \quad (2.7.29)$$

For easy analysis of the above polynomial we let each of the coefficients of i_h^* to be equal to some constants say A, B, C, D and E so as to obtain the polynomial

$$f(i_h^*) = Ai_h^{*4} + Bi_h^{*3} + Ci_h^{*2} + Di_h^* + E \tag{2.7.30}$$

or $$Ai_h^{*4} + Bi_h^{*3} + Ci_h^{*2} + Di_h^* + E = 0 \tag{2.7.31}$$

where

$$A = \delta^2 a^2 c^2, \tag{2.7.32}$$

$$B = -[\delta\alpha a^2 c^2 + \lambda_h a^2 c^2 \delta - \lambda_v \delta ac + \delta a^3 bc^2 m - \delta^2 \lambda_v ac + \delta\beta_2 a^2 c^2 + \delta a^3 bc^2 m],$$

$$= -[\delta ac[ac(+\lambda_h + \alpha + \beta_2) - \lambda_v \delta(\delta + 1)] + 2\delta a^3 bc^2 m]$$

$$= -\delta ac\{[ac(+\lambda_h + \alpha + \beta_2) - \lambda_v \delta(\delta + 1)] + 2a^2 bcm\} \tag{2.7.33}$$

$$C = -[\lambda_v \alpha ac\delta + \lambda_h \lambda_v ac\delta + \lambda_v \delta\alpha ac + \lambda_h \lambda_v \delta ac + \lambda_v \delta a^2 bcm + \lambda_v \delta\beta_2 ac + \lambda_v \beta_2 \delta ac + \lambda_v \delta\sigma a^2 bcm - \lambda_v^2 \delta^2 - \alpha\beta_2 a^2 c^2 - \beta_2 a^3 bc^2 m - \lambda_h \sigma a^3 bc^2 m - \alpha\sigma a^3 bc^2 m - \lambda_h \beta_2 a^2 c^2 - \sigma a^4 b^2 c^2 m^2 - \delta\sigma a^3 bc^2 m],$$

$$= -[ac(\lambda_v \delta[2(\lambda_h + \alpha + \beta_2) - \lambda_v \delta] - \beta_2 ac[\alpha + \lambda_h]) + \lambda_v \delta a^2 bcm[\sigma + 1] - a^3 bc^2 m(\beta_2 + \sigma[\lambda_h + \alpha + \delta + abm])]$$

$$= a^3 bc^2 m(\beta_2 + \sigma[\lambda_h + \alpha + \delta + abm]) - ac(\lambda_v \delta[2(\lambda_h + \alpha + \beta_2) - \lambda_v \delta] - \beta_2 ac[\alpha + \lambda_h]) - \lambda_v \delta a^2 bcm[\sigma + 1] \tag{2.7.34}$$

$$D = -[\lambda_v^2 \alpha\delta + \lambda_h \lambda_v^2 \delta + \lambda_v \beta_2 \delta + \alpha\sigma a^3 bc^2 m + \lambda_h \sigma a^3 bc^2 m + \sigma a^4 b^2 c^2 m^2 + \lambda_h a^3 bc^2 m - \lambda_v \sigma\delta a^2 bcm - \lambda_v \beta_2 a^2 bcm - \lambda_h \lambda_v \sigma a^2 bcm - \lambda_v \alpha\sigma a^2 bcm - \lambda_v \alpha\beta_2 ac - \lambda_h \lambda_v \beta_2 ac - \lambda_v \beta_2 \alpha ac - \lambda_h \lambda_v \beta_2 ac - \lambda_h \sigma a^3 bc^2 m],$$

$$= [+\lambda_v\{2\beta_2 ac(\alpha + \lambda_h) + a^2 bcm(\sigma[\lambda_h + \delta + \alpha] + \beta_2)\} - (\lambda_v \delta[\lambda_v(\alpha + \lambda_h) + \beta_2] + a^3 bc^2 m\{\lambda_h + \sigma[\alpha + abm]\})]$$

$$= +\lambda_v\{2\beta_2 ac(\alpha + \lambda_h) + a^2 bcm(\sigma[\lambda_h + \delta + \alpha] + \beta_2)\} - (\lambda_v \delta[\lambda_v(\alpha + \lambda_h) + \beta_2] + a^3 bc^2 m\{\lambda_h + \sigma[\alpha + abm]\}) \tag{2.7.35}$$

$$E = -[\lambda_v \lambda_h a^2 bcm + \lambda_h \lambda_v \sigma a^2 bcm + \lambda_v \alpha\sigma a^2 bcm - \lambda_v \lambda_h \sigma a^2 bcm - \lambda_h \lambda_v^2 \beta_2 - \lambda_v^2 \beta_2 \alpha]$$

$$= -\lambda_v \alpha\sigma a^2 bcm + \lambda_v^2 \beta_2 \alpha + \lambda_h \lambda_v^2 \beta_2 \left(1 - \frac{a^2 bcm}{\lambda_v \beta_2}\right)$$

$$= \lambda_v \alpha(\lambda_v \beta_2 - \sigma a^2 bcm) + \lambda_h \lambda_v^2 \beta_2 (1 - R_0) \tag{2.7.36}$$

The value of E in the above equation can only be greater than zero if $R_0 < 1$ where

$$R_0 = \frac{a^2 bmc}{\lambda_v[\lambda_h + r + \delta]}$$

It then follows that $A > 0$ .Further,$E > 0$ whenever $R_0 < 1$. The number of possible real roots of (2.7.31) depends on the signs of B, C and D. This can be analyzed by Descartes' rule of signs on the quartic.

2.8 Descartes rule of signs

Descartes' rule of signs is a method that can be used to determine the number of positive or negative roots of a polynomial.

Let $p(x) = \sum_{i=1}^{m} a_i x^i$ be a polynomial with real coefficients such that $a_m \neq 0$.

Define v to be the number of variations in sign of the sequence of coefficients $a_m \dots \dots \dots a_{n-1}$. By, 'variations in sign', we mean the number of values of n such that the sign of a_n differs from the sign of a_{n-1} , as n ranges from m down to 1.

For example, consider a polynomial $p(x) = x^2 - 4x + 4$. The coefficients are 1,-4,4, so there are 2 variations in sign (since the sign of 1 differs from that of -4, which in turn differs from that of 4.)

Then the number of positive real roots of $p(x)$ is $v - 2N$ for some integer N satisfying $0 \leq N \leq \frac{v}{2}$. The number N represents the number of irreducible factors of degree 2 in the factorization of $p(x)$. Thus $N = 0$ if it is known that $p(x)$ splits over the numbers. The number of negative roots of $p(x)$ may be obtained by the same method by applying the rule of signs to $p(x)$.

2.8.1 History of the method

This result is believed to have been first described by Rene Descartes in his 1637 work La Geometrie. In 1828, Carl Friedrich Gauss improved the rule by proving that when there are fewer roots of polynomials than there are variations of sign, the parity of the difference between the two is even.

The various possibilities for the roots of $f(i_h^*)$ are tabulated in the table below

Table 1: Number of possible positive real roots of $f(i_h^*)$ for $R_0>1$ and $R_0<1$.

Cases	1	2	3	4	5	R_0	No. of sign change	No. of positive real roots
1	+	+	+	+	+	$R_0<1$	0	0
	+	+	+	+	-	$R_0>1$	1	1
2	+	-	-	-	+	$R_0<1$	2	0,2
	+	-	-	-	-	$R_0>1$	1	1
3	+	+	-	-	+	$R_0<1$	2	0,2
	+	+	-	-	-	$R_0>1$	1	1
4	+	-	+	-	+	$R_0<1$	4	0,2,4
	+	-	+	-	-	$R_0>1$	3	1,3
5	+	-	-	+	+	$R_0<1$	2	0,2
	+	-	-	+	-	$R_0>1$	3	1,3
6	+	+	+	-	+	$R_0<1$	2	0,2
	+	+	+	-	-	$R_0>1$	1	1
7	+	+	-	+	+	$R_0<1$	2	0,2
	+	+	-	+	-	$R_0>1$	3	1,3
8	+	-	+	+	+	$R_0<1$	2	0,2
	+	-	+	+	-	$R_0>1$	3	1,3

Theorem 2.8.1

The system has a unique endemic equilibrium E^* if $R_0>1$ and Cases 1,2,3 and 6 are satisfied; it could have more than one endemic equilibrium if $R_0>1$and Cases 4, 5, 7, and 8 are satisfied; it could have 2 or more endemic equilibria if $R_0<1$ and Cases 2-8 are satisfied.

2.8.2 Stability of endemic equilibrium point

The existence of multiple endemic equilibria when $R_0<1$ (is shown in Table 1). Table 1 suggests the possibility of backward bifurcation [22-24], where the stable DFE coexists with a stable endemic equilibrium, when the reproduction number is less than unity. Thus, the occurrence of a backward bifurcation has an important implication for epidemiological control measures, since an epidemic may persist at steady state even if $R_0<1$ .This is explored below by using Centre Manifold Theory [25] .Now, we shall establish the conditions on parameter values that cause a backward bifurcation to occur in system (4), based on the use of Center Manifold theory, of the paper in Castillo-Chavez and Song [25].

Theorem 2.8.2

Let one consider the following general system of ordinary differential equations with a parameter φ: $f(x,\varphi), f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n(\mathbb{R} \times \mathbb{R})$.

Without loss of generality, it is assumed that $x = 0$ is an equilibrium point of the system for all values of the parameter . We assume that:

(A1) $A = D_x f(0,0)$ is the linearized matrix of system around the equilibrium point $x = 0$ with φ evaluated at zero is simple eigenvalues of A have negative real parts.

(A2) Matrix A has non-negative right eigenvalue w and a left eigenvalue v corresponding to the zero eigenvalue. Let f_k be the k^{th} component of f and

$$a_1 = \sum_{i,j,k=1}^{5} v_k w_i w_j \frac{\partial^2 f_k(0,0)}{\partial x_i \partial x_j}$$

$$b_1 = \sum_{i,k=1}^{5} v_k w_i \frac{\partial^2 f_k(0,0)}{\partial x_i \partial b}$$

(2.8.1)

The local dynamics of the system around 0 are totally determined by a_1 and b_1

(i) In the case where $a_1 > 0, b_1 > 0$,0ne has that when $\varphi < 0$ with φ close to zero, $x = 0$ is unstable; when $0 \leq \varphi \leq 1$, $x = 0$ is unstable and there exists a negative and locally stable equilibrium;

(ii) In case, where $a_1 < 0$, $b_1 < 0$, one has that when $\varphi < 0$ with $|\varphi|$ close to zero, $x = 0$ is locally asymptotically stable and there exists a positive unstable equilibrium; when $0 \leq \varphi \leq 1$, $x = 0$ is locally asymptotically stable and there exist a positive unstable equilibrium ;

(iii) In the case where $a_1 > 0, b_1 < 0$, one has that when $\varphi < 0$ with $|\varphi|$ close to zero ,$x = 0$ is unstable and there exists a locally asymtotically stable negative equilibrium; when $0 \leq \varphi \leq 1$, $x = 0$ is unstable and a positive unstable equilibrium appears

(iv) In the case where $a_1 < 0$, $b_1 > 0$, one has that when $\varphi < 0$ changes from negative to positive , $x = 0$ changes its stability from stable to unstable . Correspondingly a_1, negative unstable equilibrium becomes positive and locally asymptotically stable. Particularly, if $a_1 > 0$ and $b_1 > 0$, then a backward bifurcation occurs at $\varphi = 0$

To apply the stable manifold theorem, the following simplification and change of variables are made on the model

First, we let, $x_1 = s_h, x_2 = i_h, x_3 = r_h, x_4 = s_v , x_5 = i_v$ (2.8.2)

so that $N_h = x_1 + x_2 + x_3$ and $N_v = x_4 + x_5$ (2.8.3)

Further, by using the vector notation, $X = (s_h, i_h, r_h, s_v, i_v)^T$,the system can be written in the form $\frac{dx}{dt} = (f_1, f_2, f_3, f_4, f_5)^T$ as follows:

$$\frac{dx_1}{dt} = f_1 = (1 - x_1)\lambda_h - abmx_5x_1 - \alpha x_1 + \delta x_1x_2 \quad (2.8.4)$$

$$\frac{dx_2}{dt} = f_2 = abmx_5x_1 + \sigma abmx_3x_5 - \beta_2 x_2 + \delta x_2^2 \quad (2.8.5)$$

$$\frac{dx_3}{dt} = f_3 = rx_2 - \lambda_h x_3 - \sigma abmx_3x_5 + \delta x_2x_3 + \alpha x_1 \quad (2.8.6)$$

$$\frac{dx_4}{dt} = f_4 = \lambda_v(1 - x_4) - acx_4x_2 \quad (2.8.7)$$

$$\frac{dx_5}{dt} = f_5 = acx_4x_2 - \lambda_v x_5 \quad (2.8.8)$$

where,

$$\beta_2 = \lambda_h + r + \delta$$

The jacobian matrix evaluated at disease free equilibrium $E_0 = (1,0,0,1,0)$ with $b = b^*$ is

$$\begin{bmatrix} -\lambda_h - abmx_5 - \alpha + \delta x_2 & \delta x_1 & 0 & 0 & -abmx_1 \\ \sigma abmx_5 & -\beta_2 + 2\delta x_2 & \sigma abmx_5 & 0 & \sigma abmx_3 + abmx_1 \\ \alpha & r + \delta x_3 & -\lambda_h - \sigma abmx_5 + \delta x_2 & 0 & -\sigma abmx_3 \\ 0 & -acx_4 & 0 & -\lambda_v - acx_2 & 0 \\ 0 & acx_4 & 0 & acx_2 & -\lambda_v \end{bmatrix}$$

At DFE $x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1, x_5 = 0$

$$J_{DFE} = \begin{bmatrix} -(\lambda_h + \alpha) & \delta & 0 & 0 & -abm \\ 0 & -\beta_2 & 0 & 0 & abm \\ \alpha & r & -\lambda_h & 0 & 0 \\ 0 & -ac & 0 & -\lambda_v & 0 \\ 0 & ac & 0 & 0 & -\lambda_v \end{bmatrix} \quad (2.8.9)$$

Choosing b as a bifurcation parameter and solving for $b = b^*$ when $R_0 = 1$ gives

$R_0 = \frac{a^2bcm}{\lambda_v(\lambda_h+r+\delta)}$ or

$1 = \frac{a^2b^*cm}{\lambda_v(\lambda_h+r+\delta)}$ or

$$b^* = \frac{\lambda_v(\lambda_h+r+\delta)}{a^2cm} \quad (2.8.10)$$

It can be easily seen that the jacobian J of the linearized system has a simple zero eigenvalue and all other eigen-values have negative real parts. Hence the centre manifold theory can be used to analyze the dynamics of the system. For the case when $R_0 = 1$, it can be shown that the jacobian matrix J has a right eigenvector (corresponding to zero eigen-value) given by

$w = (w_1, w_2, w_3, w_4, w_5)^T$,

where,

$$\begin{bmatrix} -\lambda_h & \delta & 0 & 0 & -abm \\ 0 & -\beta_2 & 0 & 0 & abm \\ 0 & r & -\lambda_h & 0 & 0 \\ 0 & -ac & 0 & -\lambda_v & 0 \\ 0 & ac & 0 & 0 & -\lambda_v \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-\lambda_h w_1 + \delta w_2 - abmw_5 = 0 \quad (2.8.11)$$

$$-\beta_2 w_2 + abmw_5 = 0 \quad (2.8.12)$$

$$rw_2 - \lambda_h w_3 = 0 \quad (2.8.13)$$

$$-acw_2 - \lambda_v w_4 = 0 \quad (2.8.14)$$

$$acw_2 - \lambda_v w_5 = 0 \quad (2.8.15)$$

Using equation (2.8.11)
$-\beta_2 w_2 + abmw_5 = 0$,

we have $w_2 = \frac{abm}{\beta_2} w_5$ (2.8.16)

From equation (2.8.11)
$-\lambda_h w_1 + \delta w_2 - abmw_5 = 0$,

we have $w_1 = \frac{\delta w_2 - abmw_5}{\lambda_h}$ or

$w_1 = \frac{\delta(\frac{abm}{\beta_2} w_5) - abmw_5}{\lambda_h}$ or

$w_1 = \frac{\frac{abm(\delta - \beta_2)}{\beta_2} w_5}{\lambda_h}$ or

$w_1 = \frac{abm(\delta - \beta_2)}{\lambda_h \beta_2} w_5$ (2.8.17)

From equation (2.8.14)

$-acw_2 - \lambda_v w_4 = 0$

we have $w_4 = -\frac{ac}{\lambda_v} w_2$ or

$w_4 = -\frac{ac}{\lambda_v}(\frac{abm}{\beta_2})w_5$ or

$$w_4 = -\frac{a^2bcm}{\lambda_v \beta_2} w_5 \quad (2.8.18)$$

From equation (2.8.13)

$rw_2 - \lambda_h w_3 = 0$

we have $w_3 = \frac{rw_2}{\lambda_h}$ or

$w_3 = \frac{r\left(\frac{abm}{\beta_2} w_5\right)}{\lambda_h}$ or

$$w_3 = \left[\frac{\frac{\lambda_h \, rabm}{\beta_2}}{\lambda_h}\right] w_5$$

$$w_3 = \left[\frac{rabm}{\beta_2}\right] w_5 \quad (2.8.19)$$

From equation (2.8.15)

$acw_2 - \lambda_v w_5 = 0$ or

$w_5 = \frac{ac}{\lambda_v} w_2 = \frac{ac}{\lambda_v}\left[\frac{abm}{\beta_2}\right] w_5$ or

$w_5 = \frac{a^2bcm}{\lambda_v \beta_2} w_5$ or

$$w_5 = w_5 > 0 \quad (2.8.20)$$

since

$R_0 = \frac{a^2bcm}{\lambda_v \beta_2} = 1$

where, $\beta_2 = \lambda_h + r + \delta$

Similarly, the components of the left eigenvector of J (corresponding to the zero eigenvalue) denoted by

$$v = (v_1, v_2, v_3, v_4, v_5)^T \tag{2.8.21}$$

where,

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix} \begin{bmatrix} -\lambda_h & \delta & 0 & 0 & -abm \\ 0 & -\beta_2 & 0 & 0 & abm \\ \alpha & r & -\lambda_h & 0 & 0 \\ 0 & -ac & 0 & -\lambda_v & 0 \\ 0 & ac & 0 & 0 & -\lambda_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.8.22}$$

Expanding the system yields the system of equations

$$\begin{gathered} -\lambda_h v_1 = 0 \\ \delta v_1 - \beta_2 v_2 + r v_3 - ac v_4 + ac v_5 = 0 \\ -\lambda_h v_3 = 0 \\ -\lambda_v v_4 = 0 \\ -abm v_1 + abm v_2 - \lambda_v v_5 = 0 \end{gathered} \tag{2.8.23}$$

Solving the system (2.8.23), we obtain,

$$v_1 = 0, \tag{2.8.24}$$

$$v_3 = 0, \tag{2.8.25}$$

$$v_4 = 0, \tag{2.8.26}$$

From,
$\delta v_1 - \beta_2 v_2 + r v_3 - ac v_4 + ac v_5 = 0$, we have, $-\beta_2 v_2 + ac v_5 = 0$ implying

$$v_2 = \frac{ac}{\beta_2} v_5 \tag{2.8.27}$$

Again, from
$-abm v_1 + abm v_2 - \lambda_v v_5 = 0$,

we have,

$abm v_2 - \lambda_v v_5 = 0$ or

$v_5 = \frac{abm}{\lambda_v} v_2$ or

$v_5 = \frac{abm}{\lambda_v}\left[\frac{ac}{\beta_2}\right] v_5$ or

$$v_5 = \frac{a^2bcm}{\lambda_v\beta_2} v_5 \text{ or } v_5 = v_5 > 0 \tag{2.8.28}$$

since $R_0 = \frac{a^2bcm}{\lambda_v\beta_2} = 1$

where $\beta_2 = \lambda_h + r + \delta$

Computation of a_1: for transformed system ,the associated non-zero partial derivatives of f(evaluated at DFE) which we need in the computation of a_1 are given by

$$\frac{\partial^2 f_2(0,0)}{\partial x_5 \partial x_1} = \frac{\partial^2 f_2(0,0)}{\partial x_1 \partial x_5} = abm, \tag{2.8.29}$$

$$\frac{\partial^2 f_2(0,0)}{\partial x_3 \partial x_5} = \frac{\partial^2 f_2(0,0)}{\partial x_5 \partial x_3} = \sigma abm, \tag{2.8.30}$$

$$\frac{\partial^2 f_2(0,0)}{\partial x_2^2} = 2\delta\ , \tag{2.8.31}$$

$$\frac{\partial^2 f_5(0,0)}{\partial x_4 \partial x_2} = \frac{\partial^2 f_5(0,0)}{\partial x_2 \partial x_4} = ac \tag{2.8.32}$$

From

$$a_1 = \sum_{i,j,k}^{5} v_k w_i w_j \frac{\partial^2 f_k(0,0)}{\partial x_i \partial x_j}$$

it then follows that

$$a_1 = v_2 \sum_{i,j=1}^{5} w_i w_j \frac{\partial^2 f_2(0,0)}{\partial x_i \partial x_j} + v_5 \sum_{i,j=1}^{5} w_i w_j \frac{\partial^2 f_5(0,0)}{\partial x_i \partial x_j} \tag{2.8.33}$$

$$a_1 = 2v_2[w_1 w_5(abm) + w_3 w_5(\sigma abm) + w_2 w_2(\delta)] + 2v_5[w_2 w_4(ac)] \tag{2.8.34}$$

substituting

$w_1 = \frac{abm(\delta-\beta_2)}{\beta_2\lambda_h} w_5, w_2 = \frac{abm}{\beta_2} w_5, w_3 = \left[\frac{rabm}{\beta_2}\right] w_5, w_4 = -\frac{a^2bcm}{\lambda_v\beta_2} w_5,$

$w_5 = w_5 > 0$ and

$v_1 = 0\ , v_2 = \frac{ac}{\beta_2} v_5,\ v_3 = 0, v_4 = 0, v_5 = v_5 > 0$ in the equation above we get

$$a_1 = 2v_2[w_1w_5(abm) + w_3w_5(\sigma abm) + w_2w_2(\delta)] + 2v_5[w_2w_4(ac)] \quad (2.8.35)$$

$$a_1 = 2v_5w_5^2\left[\frac{ac}{\beta_2}\right]\left[abm\left(\frac{abm(\delta-\beta_2)}{\beta_2}\right) + \sigma abm\left[\frac{rabm}{\lambda_h\beta_2}\right] + \delta\left(\frac{abm}{\beta_2}\right)^2\right] + 2v_5w_5^2\left[ac\left(\frac{abm}{\beta_2}\right)\left(-\frac{a^2bcm}{\lambda_v\beta_2}\right)\right]$$

$$a_1 = 2v_5w_5^2\left[\frac{ac}{\beta_2}\right]\left\{\frac{a^2b^2m^2r}{\lambda_h\beta_2} + \frac{\sigma a^2b^2m^2[(\delta-\beta_2)]}{\beta_2\lambda_h} + \frac{\delta a^2b^2m^2}{\beta_2^2} - abm\left[\frac{a^2bcm}{\lambda_v\beta_2}\right]\right\}$$

$$a_1 = \left[\frac{2a^2bcmv_5w_5^2}{\lambda_h\lambda_v\beta_2^3}\right]\{\lambda_v\beta_2rabm + \sigma\lambda_v\beta_2abm(\delta-\beta_2) + \lambda_v\lambda_h\delta abm - \lambda_h\beta_2(a^2bcm)\}$$

$$= \left[\frac{2R_0w_5^2}{\lambda_h\beta_2}\right]\{\lambda_v\beta_2rabm + \sigma\lambda_v\beta_2abm(\delta-\beta_2) - \lambda_h\beta_2(a^2bcm) + \lambda_v\lambda_h\delta abm\}$$

$$= -\left[\frac{2abm}{\lambda_h\beta_2}\right]\{\sigma\lambda_v\beta_2^2 + \lambda_h\beta_2ac - \lambda_v(\beta_2r + \lambda_h\delta + \delta\sigma\beta_2)\} \quad (2.8.36)$$

If we let

$$g = \sigma\lambda_v\beta_2^2 + \lambda_h\beta_2ac \text{ and} \quad (2.8.37)$$

$$s = \lambda_v(\beta_2r + \lambda_h\delta + \delta\sigma\beta_2), \quad (2.8.38)$$

we then have $a_1 < 0$ if and only if $g > s$

For the sign of b_1, it can be shown that the associated non-vanishing partial derivatives of f are given by:

$$b_1 = \sum_{i,k=1}^{5} v_kw_i\frac{\partial^2f_k(0,0)}{\partial x_i\partial b}$$

$$b_1 = \sum_{i=1}^{5} v_2w_i\frac{\partial^2f_2(0,0)}{\partial x_i\partial b} + \sum_{i=1}^{5} v_5w_i\frac{\partial^2f_5(0,0)}{\partial x_i\partial b} \quad (2.8.39)$$

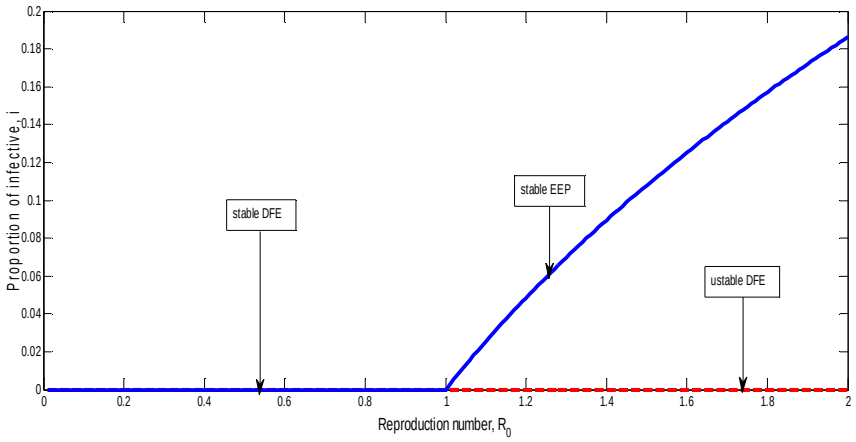
$$b_1 = v_2w_1\frac{\partial^2f_2(0,0)}{\partial x_1\partial b} + v_2w_5\frac{\partial^2f_2(0,0)}{\partial x_5\partial b} + v_2w_3\frac{\partial^2f_2(0,0)}{\partial x_3\partial b} + v_2w_5\frac{\partial^2f_2(0,0)}{\partial x_5\partial b} + v_5w_2\frac{\partial^2f_5(0,0)}{\partial x_2\partial b} \quad (2.8.40)$$

$$b_1 = v_2w_5\frac{\partial^2f_2(0,0)}{\partial x_5\partial b} = v_2w_5am > 0 \quad (2.8.41)$$

Thus $b_1 > 0$ (2.8.42)

Since $a_1 < 0$ and $b_1 > 0$ holds as stated in Theorem 2.8.2 above, then model (2.2.6) undergoes a forward bifurcation at $R_0 = 1$ and has a negative unstable endemic equilibrium point(EEP) which becomes positive and locally asymptotically stable when $\alpha = 0$.

Figure : The Diagram of Forward Bifurcation when
$\lambda_h = 0.89, \lambda_v = 0.35, a = 0.29, c = 0.75, m = 0.358, r = 0.00019, \delta = 0.333, \sigma = 0.01, \alpha = 0$



It is observed that as R_0 decreases to one, the disease also decreases due to the acquired immunity to malaria which develops gradually due to continuous exposure to infections. The disease free equilibrium (DFE) occurs when $R_0 < 1$. When$R_0 > 1$, it should be noted that the diseases continue to exist in population due to re-infection of the individuals who lose immunity, which implies that the disease can invade the population and persist at an alarming rate.

CHAPTER 3

3.NUMERICAL ANALYSIS OF THE MODEL

3.1 The model parameter estimates

The following parameter estimates we used in numerical simulation of the model.

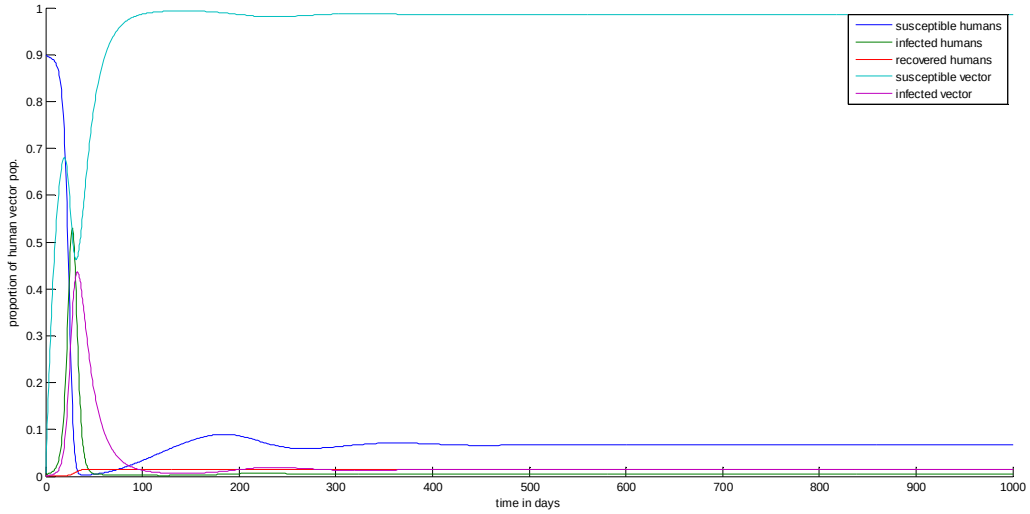
Parameter	Description	Value	Reference
a	the average daily biting rate on man by a single mosquito	0.29/day	[14,18]
b	the proportion of bites on man that produce an infection	0.75	[18]
c	the probability that a mosquito becomes infectious	0.75	[18]
r	the rate of recovery of human hosts from the disease	0.00019/day	[5]
δ	the per capita death rate due to the disease	0.333	[25]
λ_h	the per capita natural birth rate of humans	0.0015875/day	[8]
λ_v	the per capita natural birth rate of the mosquitoes	0.071/day	[8]
μ_h	the per capita natural death rate of the humans	0.00004/day	[5]
μ_v	the per capita natural death rate of the mosquitoes	0.05/day	[19]

3.2 Numerical analysis of the model

In this section, we present the numerical analysis of the model. The parameter values in table 1 are used in the simulations to illustrate the behavior of the model. We observe that in the early stages of the epidemic, there is a high prevalence of malaria because of a large proportion of infected mosquito vectors that results in a significant decrease in the number of susceptible human hosts. As the proportions of infected humans and infected mosquitoes decrease and remain at low level values, we observe a dramatic increase in the immune class.. In the absence of infected human hosts and mosquito vectors, the proportion of the immune class decreases as a result of immunity loss and this leads to an increase in the human susceptibles. We eventually have a higher proportion of immune humans compared to the proportion of susceptible humans. There are damped oscillations of the proportions until an endemic equilibrium level is eventually reached and this converges to a steady state that is asymptotically stable. These numerical results support the results earlier obtained analytically that the endemic equilibrium is stable.

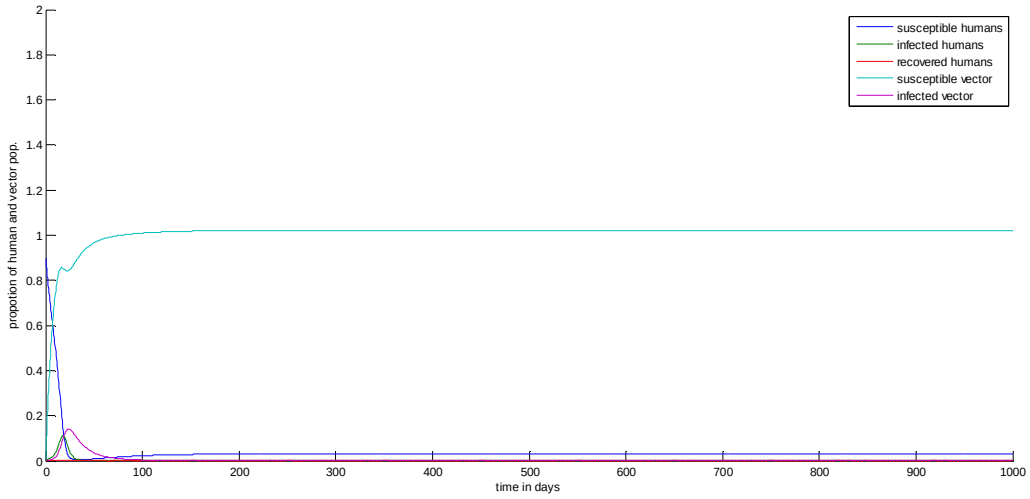
Also, since the is no vaccine for malaria and we hope to have one soon, it was observed from the simulations that the disease can be controlled by simply vaccinating about fifty percent of the total population.

Figure 3.2.1:Graph of human and vector population against time



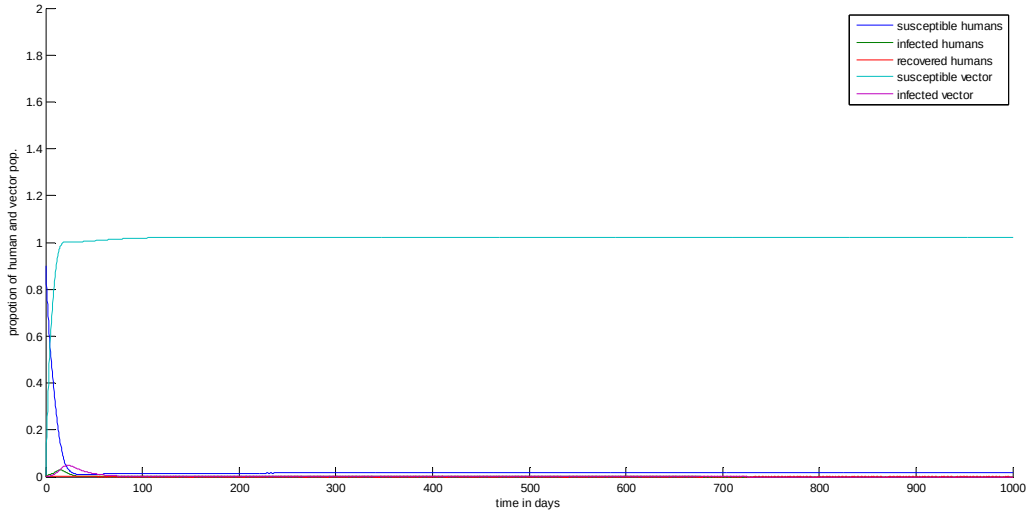
$\lambda_h = 0.89, \lambda_v = 0.35, a = 0.29, c = 0.75, m = 0.358, r = 0.00019, \delta = 0.333, \sigma = 0, \alpha = 0$

Figure 3.2.2: A graph of human and vector population when 5% of human population is vaccinated



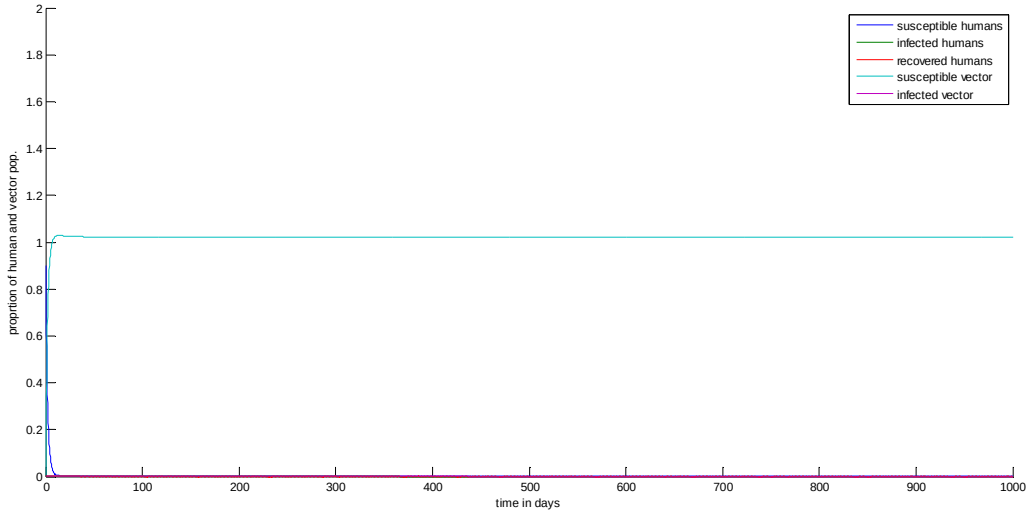
$\lambda_h = 0.89, \lambda_v = 0.35, a = 0.29, c = 0.75, m = 0.358, r = 0.00019, \delta = 0.333, \sigma = 0, \alpha = 0.05$

Figure 3.2.3: A graph of human and vector populations when 10% of human population is vaccinated



$\lambda_h = 0.89, \lambda_v = 0.35, a = 0.29, c = 0.75, m = 0.358, r = 0.00019, \delta = 0.333, \sigma = 0, \alpha = 0.1$

Figure 3.3.4 :A graph of human and vector population when 50% of human population is vaccinated



$\lambda_h = 0.89, \lambda_v = 0.35, a = 0.29, c = 0.75, m = 0.358, r = 0.00019, \delta = 0.333, \sigma = 0, \alpha = 0.5$

CHAPTER 4

4.DISCUSSION AND RECOMMENDATIONS

4.1 Discussion

We proposed a model with standard incidence for the dynamics of malaria within human hosts and mosquito vectors in which the reservoir of the susceptible human hosts is refilled by individuals who lose their immunity to the disease and newborns. The model was then reformulated in terms of the proportions of the classes of the respective populations.

Model analysis and simulations were carried out. Two equilibria points were obtained and their stability analyzed. It was established that for the basic reproduction number, $R_0 \leq 1$, the disease-free equilibrium is locally asymptotically stable so that the disease always dies out, and if $R_0 > 1$, the disease free equilibrium point is unstable while the endemic equilibrium emerges as a unique equilibrium point, re-invasion is always possible and the disease never dies out.

Thus, a threshold population size is necessary for the perpetuation of the disease. These may be based on the parameters of the threshold quantity, R_0.We notice that in order to reduce the basic reproduction number below 1, intervention strategies need to be focused on treatment and reduction on the contact between mosquito vector and human host. Thus, there is need for effective drugs, treated bed nets and insecticides that would reduce the mosquito population. Since malaria induced immunity is not everlasting, it remains a major obstacle to eradicate the disease even if individuals are protected. Numerical analysis revealed that the endemic equilibrium converges to a steady state.

We observe that there is a strong relationship between the proportion of infected mosquitoes and infected humans in the same locality in a way that arise in the proportion of infected mosquitoes results in an increase in the proportion of infected humans. Therefore, control efforts aimed at lowering the infectivity of infected individuals to the mosquito vector will contribute greatly to the lowering of the malaria transmission and this will eventually lower the prevalence of malaria and the incidence of the disease in that locality. This can be achieved by prompt provision of effective anti-malarial drugs to reduce transmission and morbidity. Thus, from the model, it is noted that recurrent and temporary immunity leads to oscillatory pattern in all the populations of the model. Also, from the model, it was observed that the disease can be controlled by vaccinating approximately fifty percent of the total population.

4.2 Recommendations

There is need to assess the global stability of this model through construction of appropriate Lyapunov function in order to evaluate the long time effect of partial immunity to re-infection.

We further recommend the investigation of the effect of the re-infection parameter σ and the transmission probability from an infectious human to a susceptible vector b on the associated backward bifurcation region, as a function of the average life span of mosquitoes ($1/\mu_v$).

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