UNIVERSITY OF NAIROBI, SCHOOL OF MATHEMATICS
A MASTERS OF SCIENCE IN APPLIED MATHEMATICS PROJECT

TOPIC

APPLICATION OF THE NAVIER-STOKES EQUATIONS IN THE LOCALISATION OF ATHEROSCLEROSIS

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Declaration

This project as presented in this report is my original work and has not been presented for any other university award.

Signature: ____________________

Date: ________________________________

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This project has been submitted as part of fulfillment for Masters of Science in Applied Mathematics of the University of Nairobi with my approval as the supervisor.

Signature: ___________________

Date: ______________________________

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Contents

Abstract

Localization of atherosclerosis plaque has been a great problem for many centuries. Many researchers have done a lot of studies in blood flow with the motivations of understanding the localization of atherosclerosis in arteries. This project presents a mathematical modeling of the arterial blood flow which is derived from the Navier-Stokes equation and some assumptions. A system of non-linear partial differential equations for blood flow is obtained. Finite element method (FEM) is then adopted to solve the equations numerically. Apart from FEM, we will use the Galerkin stabilization method to solve the problem of oscillations of solutions at high Reynolds numbers. We will also use the method of artificial incompressibility and the Newton-Raphson method, to deal with the problems of incompressibility and the problem of non-linear terms respectively. The results obtained will help in explaining the localization of the atherosclerosis disease.

1. Introduction

In order to have a good understanding of this project, we need to have some basic knowledge of the cardiovascular system, the atherosclerosis disease and also the wall shear stress.

1.1. Cardiovascular system

Cardiovascular system is a complex system in human body consisting of the heart, blood and blood vessels. It's a very vital component of human body, since it helps in supplying human body with nutrients and oxygen and also helps in the removal of waste products; therefore without it human body cells will die due to lack of nutrients and also due to accumulation of waste products. In order to have a good understanding of cardiovascular system, we need to study how it works.

The circulatory system is divided into two parts; the low pressure pulmonary circulation, linking circulation and gas exchange in the lungs and the high pressure systematic circulation, providing oxygen and nutrients to body tissues.

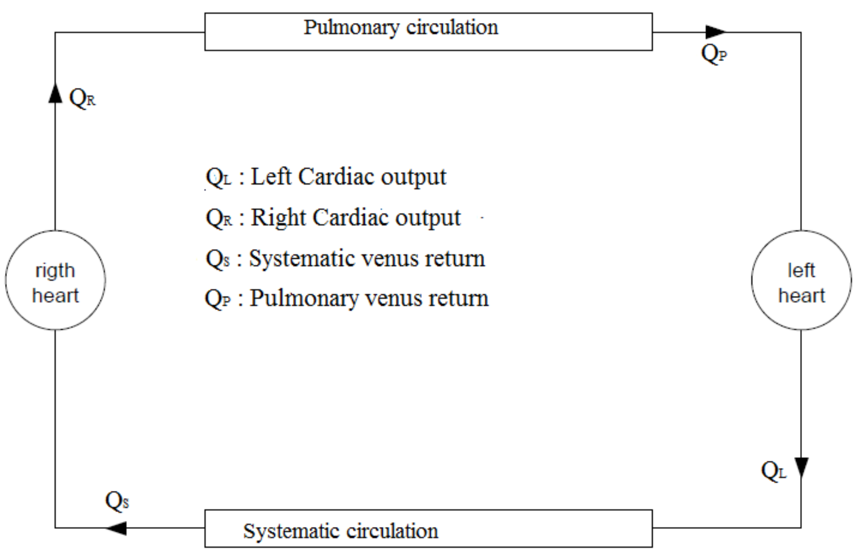


Figure 1: Circulatory system (Gohil, 2002)

All arteries except Pulmonary artery carry oxygenated blood from the heart to body tissues. After the blood has released oxygen to the tissues and taken up carbon dioxide, it's carried back to the body through veins, (Campbell, 1996, pp. 63-69). Therefore arteries carry and veins mainly carry oxygen and carbon dioxide respectively. This explains why atherosclerosis plaques are found mostly in arteries. As we will see later, this is because oxygen is involved in the formation of the disease and it's mainly found in arteries. In this project therefore we will only be concerned with arteries where the disease is most likely to occur.

Arteries are subdivided into large and medium arteries, arterioles and capillaries. Therefore since blood is not a homogenous fluid but is a suspension of particles in fluid called plasma, blood particles must be taken into account in small arteries, the arterioles and capillaries since their size become comparable to that of the vessel, hence blood is not homogenous in these arteries and have non standard behaviors, non-Newtonian. On the other hand, in large vessels the size of the plasma suspensions is very small compared to the radius of the vessels, hence fluid is homogeneous and shows standard behavior of a fluid, Newtonian, (Quarterson, 2002). In this project we will consider blood to be Newtonian since we will model blood flow through large vessels.

Carotid and coronary arteries supply blood to the brain and the heart respectively. These arteries are very vital to human beings and interference with blood flow them can result into stroke and heart attack if the carotid and coronary arteries are affected. These two diseases are among the leading diseases that causes death, for instance, stroke is the third common cause of death (Chatzizis, 2008) and is also responsible for 5 million permanently disabled people per year worldwide, (Cheng, 2006).

1.2.Atherosclerosis

This is a disease in which cholesterol and calcium builds up in the inner lining of the arteries forming a substance called plaque. Over time, the fat and calcium builds up narrows the artery and blocks blood flow through it.

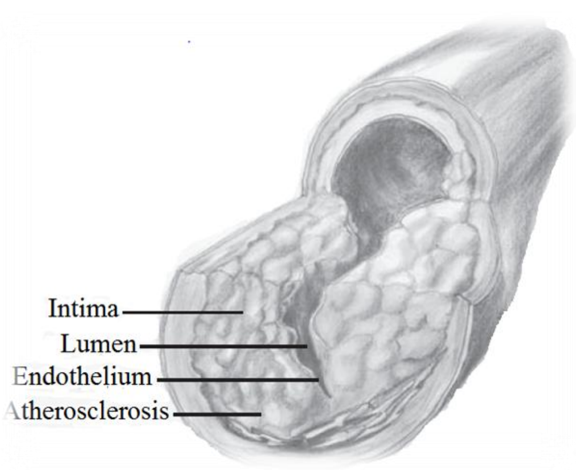


Figure 2: Atherosclerosis plaque

The origin of the atherosclerosis plaque, (Wake, 2005) is the excessive accumulation of low density lipo-proteins (LDL) particles in the arterial wall. According to (Calves, 2010), the first step in the atherosclerosis process is the accumulation of LDL cholesterol, in the intima, where part of it is oxidized and becomes pathological. In order to remove it, circulating immune cells (e.g. monocytes) are recruited.

Ones in the intima, the monocytes differentiate to become macrophages that ingest by phagocytosis the oxidized LDL. The ingestion of large amount of oxidized LDL transforms the fatty macrophages into form cells, it's these form cells which are responsible for the growth of sub-endothelial plaque which eventually emerges in the artery lumen. Increase of macrophage concentration also leads to the production of pro – inflammatory cytokines which contributes to recruitment of more macrophages.

Wall shear stress

Wall shear stress is the friction between the blood and the blood vessel walls. It depends on the velocity and the viscosity of the blood. The wall shear stress, τ, in Cartesian coordinates is given by;

$\tau = \mu \frac{du}{dy}$ and in cylindrical coordinates it's given by; $\tau = \mu \frac{du}{dr}$

μ is the coefficient of viscosity, du, dy and dr are small changes of velocity in x – direction , small changes of displacement in y and radial directions respectively.

In no-slip boundary conditions wall shear stress is highest near the walls and decreases to zero as you move towards the centre, while the velocity is highest at the centre and decreases as you move towards the walls. This is shown below.

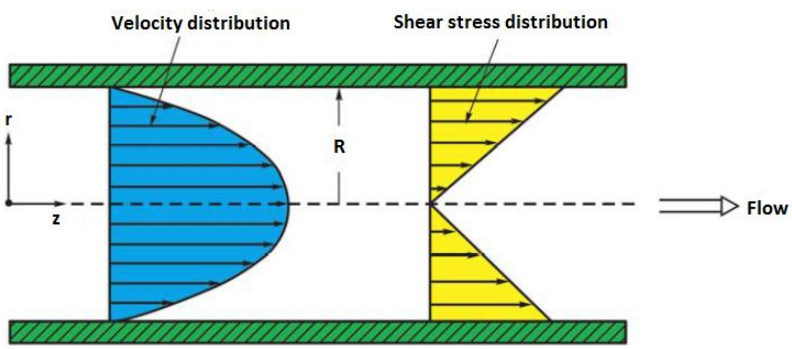


Figure 3: wall shear stress and velocity distribution

Low wall shear stress contributes to atherogenesis in the following way. Endothelial cells are highly aligned and elongated when exposed to directional shear stress. In areas of low shear stress, endothelium cells lose their alignment and organization leading to increased macromolecular permeability. This allows cholesterol particles to enter the vessel wall and once oxidized they stimulate inflammatory response and stimulate the migration and proliferation of smooth muscle cells into the growing lesion. Since the amount of materials entering the region is greater than those leaving, the atherosclerosis plague becomes larger.

1.3. Description of the problem and approach

Experimental results in vivo and in vitro have shown that atherosclerosis lesions develop in the vicinity of branching arteries and strong curvature. In these areas, low wall shear stress has been established. Measurement of wall shear in arteries has faced many challenges, which include difficulties in performing these experiments in living organisms and also since the wall shear stress is very small, the measuring instruments may not detect it, (Ritzen R. , 2012).

In this project to determine the wall shear stress distribution, we are going to derive the relationship between wall shear stress distribution and velocity distribution and then use the distribution of velocity to predict wall shear stress distribution.

To do this, we are going to solve the Navier-Stokes equations analytically and numerically and use the solutions obtained in predicting the wall shear stress distribution. This will solve the problems encountered in using the experimental methods.

1.4.Objectives of the project

The objectives of this project are;

- to convert the Navier – Stokes equations in Cartesian coordinates to different coordinate systems; the cylindrical, and spherical coordinate
- to solve the Navier stokes equations analytically and using Numerical methods.
- to apply the solutions of the Navier-stokes obtained in explaining the localization of the atherosclerosis.

2. Literature review

Several works have been done in this area. Some writers mainly concentrated on the solution of the Navier-Stokes which they did successfully. For instance, (Jiajan, 2010), was able to solve the non-linear two dimension Navier-Stokes equations. He successfully dealt with the non – linear problem by using the Newton – Raphson method which reduced algebraic equations to linear. In his work, the problem of instability was solved by using Galerkin stabilization method. And finally he dealt with the problem of incompressibility condition by adding an artificial term to the continuity equation. Although he successfully solved the Navier – Stokes equations, he did not apply the solutions obtained to blood flow problems.

Another person who did research in this area is (Kazakidi, 2008). He was able to give a good mathematical back-ground of atherosclerosis and its relationship with wall shear stress. He then solved the equations by finite element method.

Other researchers like (Gohil, 2002), used Cosmol Multiphysics software to simulate the distribution of wall shear stress at bifurcation points.

In this project, we have extended the mathematical background given by (Kazakidi, 2008) using the work done by (Jiajan, 2010). Furthermore in this project we derive the relationship between wall shear stress and velocity, and then predict the wall shear stress distribution by only considering the velocity distribution.

3. Mathematical model

In this section we are going to come up with equations that describe blood flow in blood vessels, the Navier Stokes, and continuity equations. We will first write the equations in different coordinates, and then solve them both analytically and by numerical methods. Since blood is very complex, for instance, the blood is non-Newtonian, the flow is turbulent and also it is unsteady, we will first simplify the problem by coming up with several assumptions.

3.1.Assumptions

The flow in arteries is very complex, and it may be very difficult to come up with the equation to describe it. y. To simplify the problem we make some assumptions;

1. The blood is Newtonian; it obeys Newton's laws of motion. This assumption is very important to this project because it will enable us to use the Navier-Stokes equations. This assumption is justified since we are going to base our model on large vessels where blood behaves as if it is Newtonian
2. The blood is incompressible. With this assumption, the balance of mass and the balance of momentum turn into Navier Stokes Equation as the governing equation of motion.

2.2.Equations of motion in various coordinates

In this section, we are going to look at the equations of motion in Cartesian, cylindrical and spherical coordinates.

2.2.1. Cartesian coordinates

In Cartesian coordinates, the Navier-Stokes and the continuity equations are given by;

(a) Continuity equation

The general form of equation of conservation of mass is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad (1)$$

In equation above, u, v and w are velocities in x, y and z- directions and ρ is the density.

The above equation is valid for steady and unsteady, compressible and incompressible fluid. In vector form, equation can be written in vector form as;

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{q}) = 0 \qquad (2)$$

Equation (1) is the first form of continuity equation

Here, $\mathbf{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ and $\boldsymbol{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$

The continuity equation can also be written in another form, we use product rule on the divergent term in equation (2) to get,

$$\frac{\partial \rho}{\partial t} + \mathbf{\nabla} \cdot (\rho \boldsymbol{q}) = \frac{\partial \rho}{\partial t} + \boldsymbol{q} \cdot \mathbf{\nabla}\rho + \rho \mathbf{\nabla} \cdot \boldsymbol{q} = 0 \tag{3}$$

In equation (3) above, the terms, $\frac{\partial \rho}{\partial t} + \boldsymbol{q} \cdot \mathbf{\nabla}\rho$ can be replaced by the material derivative, $\frac{D\rho}{Dt}$, therefore equation (2) will become,

$$\frac{D\rho}{Dt} + \rho \mathbf{\nabla}.\boldsymbol{q} = 0 \tag{4}$$

Equation four is the second form of continuity equation

There are two special cases for the continuity equation (2)

1. For Steady flow, the equation does not depend on time, therefore equations (1) and (2) becomes

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{5}$$

   Or in vector form

$$\mathbf{\nabla}.\, \rho\, \boldsymbol{q} = 0 \tag{6}$$

   this follows since by definition, ρ is not a function of time for steady flow, but could be a function of position.

2. For incompressible fluids the density, ρ is constant throughout the flow field so that the equations (1) and (2) become;

$$\mathbf{\nabla} \cdot \boldsymbol{q} = 0 \tag{7}$$

   Or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{8}$$

The above equation is a special form of equation (5) when density is not a function of position and it applies to both steady and unsteady flow of incompressible fluids; this is the equation we will use in this project since we will assume that blood is incompressible as stated in the assumptions.

(b) The Navier stokes equations

Navier-Stokes equations are the equations of conservation of linear momentum. The general form of the equations for incompressible flow of Newtonian (constant viscosity) fluid is given by;

$$\frac{\partial \boldsymbol{q}}{\partial t} = -(\boldsymbol{q}.\boldsymbol{\nabla})\boldsymbol{q} + \upsilon\nabla^2\boldsymbol{q} - \frac{1}{\rho}\nabla P + \boldsymbol{g} + \boldsymbol{f} \quad \textbf{(9)}$$

υ is kinetic viscosity (constant) and is given by $\upsilon = \frac{\mu}{\rho}$, ρ is density (constant), P is pressure and $\boldsymbol{g}$ is the gravitational force.

In the equation (9) above,

- $\frac{\partial \boldsymbol{q}}{\partial t}$ – Acceleration term
- $(\boldsymbol{q}.\boldsymbol{\nabla})\boldsymbol{q}$ **–** is the advection term; the force exerted on the particles of the fluid by other particles of the fluid surrounding it
- $\upsilon\nabla^2\boldsymbol{q}$ **–** velocity diffusion terms; describes how the fluid motion is damped, highly viscous fluid e.g. honey stick together while low viscous fluid flow freely, e.g. air
- $\boldsymbol{\nabla}P$ - pressure term, fluids flow in the direction of largest change in pressure

From equation (2), $\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ and $\boldsymbol{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$**.** Replacing this in equation (9) we obtain

$$\frac{\partial}{\partial t}(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) = -(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w).\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) +$$

$$\upsilon\nabla^2(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) - \frac{1}{\rho}\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)P + \rho(\mathbf{i}g_{\boldsymbol{x}} + \mathbf{j}g_{\boldsymbol{x}} + \mathbf{k}g_{\boldsymbol{x}}) + (\mathbf{i}f_{\boldsymbol{x}} + \mathbf{j}f_{\boldsymbol{x}} + \mathbf{k}f_{\boldsymbol{x}}) \quad (10)$$

In equation (10) the laplacian is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Replacing the laplacian above in equation (10), we obtain

$$\frac{\partial}{\partial t}(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) + \ \upsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) - \frac{1}{\rho}\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)P + (\boldsymbol{i}g_{\boldsymbol{x}} + \boldsymbol{j}g_{\boldsymbol{x}} + \boldsymbol{k}g_{\boldsymbol{x}}) + (\boldsymbol{i}f_{\boldsymbol{x}} + \boldsymbol{j}f_{\boldsymbol{x}} + \boldsymbol{k}f_{\boldsymbol{x}}) \quad (11)$$

Collecting coefficients of, $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ together, this leads to equations in x, y and z directions respectively as follows;

$$\frac{\partial u}{\partial t} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u + \upsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u + \frac{1}{\rho}\frac{\partial P}{\partial x} + g_{\boldsymbol{x}} + f_x \quad (12)$$

$$\frac{\partial v}{\partial t} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)v + \upsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)v + \frac{1}{\rho}\frac{\partial P}{\partial x} + g_{\boldsymbol{x}} + f_y$$

$$\frac{\partial w}{\partial t} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)w + \upsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)w + \frac{1}{\rho}\frac{\partial P}{\partial x} + g_{\boldsymbol{x}} + f_z$$

The above three equations are the Navier-Stokes equations in x, y and z components. In this project we will neglect the body forces. Therefore dropping the body forces and rearranging the equations we obtain;

x-component

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{13}$$

y-component

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{14}$$

z-component

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{15}$$

Here, υ is the coefficient of viscosity, ρ is the density of the fluid and $\mathbf{g}$ is the gravitational force

2.2.2. Equations of motion in cylindrical coordinates

In cylindrical coordinates, the coordinates r is the radial distance from the z axis, θ is the angle measured from a line parallel to the x- axis, z is the coordinates along the z- axis. The velocity components are the radial velocity u, the tangential velocity v, and the axial velocity w. Thus the velocity at some arbitrary point p can be expressed as

$$\mathbf{q}=u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z} \tag{16}$$

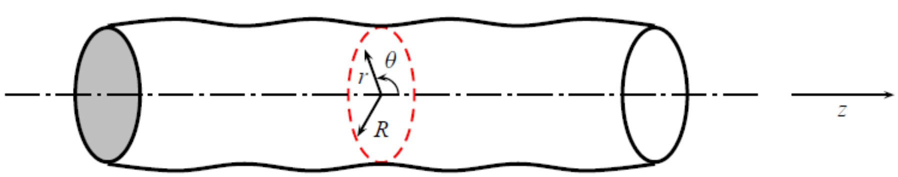


Figure 4: Shape of an artery

Cartesian coordinates can be expressed into cylindrical coordinates using the relations,

$$x=r\cos\theta,\quad y=r\sin\theta,\quad \text{and}\quad z=z \tag{17}$$

The above relations implies that

$$\begin{aligned} x &= x(r,\theta)\\ y &= y(r,\theta)\\ z &= z \end{aligned} \tag{18}$$

I.e. Cartesian coordinates can be expressed in terms of cylindrical coordinates, also

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \qquad r = \sqrt{x^2 + y^2}, \qquad z = z \tag{19}$$

This relations also implies that

$$\theta = \theta(x, y), \qquad r = r(x, y), \qquad z = z \tag{20}$$

Therefore equations (20) and (18) show the relationship between Cartesian coordinates and cylindrical coordinates. In general form, the relationship between Cartesian coordinates and any other coordinate system can be represented by,

$$\begin{aligned} x &= x(u_1, u_2, u_3) \qquad & u_1 &= u_1(x, y, z) \\ y &= y(u_1, u_2, u_3) \qquad & u_2 &= u_2(x, y, z) \\ z &= z(u_1, u_2, u_3) \qquad & u_3 &= u_3(x, y, z) \end{aligned}$$

In the above equation, u_1, u_2, u_3 are the curvilinear coordinates, they can be cylindrical coordinates or spherical coordinates. In cylindrical coordinates;

$$u_1 = u_r = r,\ u_2 = u_\theta = \theta \text{ and } \mathrm{u}_3 = u_z = z \tag{21}$$

To convert Cartesian coordinates to cylindrical coordinates, we first convert them into curvilinear coordinates before to cylindrical coordinates.

In this section we will transform the continuity and momentum equations from Cartesian to cylindrical coordinates. We will start by converting the continuity equation, and then followed by the momentum equations.

(a) Continuity equation

The continuity equation in vector form as shown in (7) is given by;

$$\nabla \cdot \boldsymbol{q} = 0$$

In order to understand how to convert the equation in curvilinear coordinates, we first need to first know several parameters which we are going to use.

Unit vectors (in curvilinear coordinates)

In curvilinear coordinates u_1, u_2 and u_3 the unit vectors, e_1, e_2 and e_3 are given by

$$\mathbf{e_i} = \frac{\partial \mathbf{r}/\partial u_i}{\left|\partial \mathbf{r}/\partial u_i\right|} = \frac{1}{h_i}\frac{\partial \mathbf{r}}{\partial u_i} \tag{22}$$

Where, $i = 1, 2, 3$

$\mathbf{r}$ is a position vector of any point in Cartesian coordinate system and is given by;

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{23}$$

$\partial\mathbf{r}/\partial u_i$ Is a vector in the direction of the tangent to the u_i - curve. In cylindrical coordinates, the unit vectors$\boldsymbol{e_1}$, $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$ are given as$\boldsymbol{e_r}$, $\boldsymbol{e_\theta}$ and$\boldsymbol{e_z}$.

Scale factors

We will take the curvilinear coordinates u_1, u_2 and u_3 to be orthogonal. From equation (22), the scale factors are h_i where $i = 1, 2, 3$ and are given by;

$$h_i = \left|\frac{\partial \mathbf{r}}{\partial u_i}\right| \tag{24}$$

In equation (24) above,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{25}$$

Replacing equation (25) into equation (24) we obtain the equation,

$$h_i = \left|\frac{\partial \tilde{r}}{\partial u_i}\right| = \left|\frac{\partial x}{\partial u_i}\mathbf{i} + \frac{\partial y}{\partial u_i}\mathbf{j} + \frac{\partial z}{\partial u_i}\mathbf{k}\right| \tag{26}$$

$i = 1, 2, 3$

From equation (26) above, we get,

$$\begin{aligned} h_i^2 &= \left(\frac{\partial \tilde{r}}{\partial u_i}\right)\left(\frac{\partial \tilde{r}}{\partial u_i}\right) = \left(\frac{\partial x}{\partial u_i}\mathbf{i} + \frac{\partial y}{\partial u_i}\mathbf{j} + \frac{\partial z}{\partial u_i}\mathbf{k}\right) \bullet \left(\frac{\partial x}{\partial u_i}\mathbf{i} + \frac{\partial y}{\partial u_i}\mathbf{j} + \frac{\partial z}{\partial u_i}\mathbf{k}\right) \\ &= \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2 \end{aligned} \tag{27}$$

Replacing equation (17) into (27), we obtain

$$h_i^2 = \left(\frac{\partial}{\partial u_i} r\cos\theta\right)^2 + \left(\frac{\partial}{\partial u_i} r\sin\theta\right)^2 + \left(\frac{\partial}{\partial u_i} z\right)^2 \tag{28}$$

$i = 1,\ 2 \text{ and } 3$

In cylindrical coordinates the scale factors; h_1, h_2 and h_2 and the coordinates u_1, u_2 and $u_3$are given by;

$$\begin{aligned} &h_r = h_1 \quad h_\theta = h_2 \text{ and } \mathrm{h}_z = h_3 \\ &u_1 = u_r = r,\ \ u_2 = u_\theta = \theta,\ \ u_3 = u_z = z \end{aligned} \tag{29}$$

Now replacing (29) into (28) above we get

$$\begin{aligned} h_r^2 &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial}{\partial r}(r\cos\theta)\right)^2 + \left(\frac{\partial}{\partial r}(r\sin\theta)\right)^2 + \left(\frac{\partial}{\partial r} z\right)^2 \\ &= \cos^2\theta + \sin^2\theta = 1 \end{aligned} \tag{30}$$

$$h_\theta^2 = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial}{\partial \theta}(r\cos\theta)\right)^2 + \left(\frac{\partial}{\partial \theta}(r\sin\theta)\right)^2 + \left(\frac{\partial}{\partial \theta}z\right)^2$$
$$= r^2\cos^2\theta + r^2\sin^2\theta = r^2 \quad (31)$$

$$h_z^2 = \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2 = \left(\frac{\partial}{\partial z}(r\cos\theta)\right)^2 + \left(\frac{\partial}{\partial z}(r\sin\theta)\right)^2 + \left(\frac{\partial}{\partial z}z\right)^2 = 1 \quad (32)$$

From the equations; (30), (31) and (32), the scale factors in cylindrical coordinates are;

$$h_r = 1, \quad h_\theta = r, \quad h_z = 1 \quad (33)$$

Now after the above explanations, we now express the continuity equation $\nabla\cdot\boldsymbol{q} = 0$ in cylindrical coordinates. We first express the divergence, $\nabla\cdot\boldsymbol{q}$ in curvilinear coordinates. Before we do this, we first find the value of Del operator, ∇ in curvilinear coordinates. To do this, we first find the value of $\nabla\emptyset$, where $\emptyset$ is any scalar function. In curvilinear coordinates, it will be written as

$$\nabla\emptyset = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3 \quad (34)$$

$\mathbf{e_1}$ $\mathbf{e_2}$ $\mathbf{e_3}$, are unit vectors along u_1, u_2, u_3 curves.

Let $\boldsymbol{r}$ be a position vector of a point $\boldsymbol{P}$ in Cartesian coordinates. Then $\boldsymbol{r}$ from (25) is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Using the relations (17) the equation above becomes

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + z\mathbf{k} \quad (35)$$

We need to express $d\emptyset$ in two different ways and compare the coefficients of du_1, du_2 and du_3 to obtain the values of, f_1, f_2, and f_3 in equation (34).

In the first expression,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)\bullet\left(\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz\right)$$
$$= \nabla\phi\bullet dr \quad (36)$$

Then using equation

$$\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$$

$$d\mathbf{r} = \frac{\partial\mathbf{r}}{\partial u_1}du_1 + \frac{\partial\mathbf{r}}{\partial u_2}du_2 + \frac{\partial\mathbf{r}}{\partial u_2}du_3 \quad (37)$$

From (22), equation (37) can be written as

$$d\mathbf{r} = h_1 du_1 \mathbf{e_1} + h_2 du_2 \mathbf{e_2} + h_3 du_3 \mathbf{e_3} \tag{38}$$

We therefore obtain

$$\begin{aligned} d\phi &= \nabla\phi . dr = \left(f_1\mathbf{e_1} + f_2\mathbf{e_2} + f_3\mathbf{e_3} \right).\left(h_1\mathbf{e_1}du_1 + h_2\mathbf{e_2}du_2 + h_3\mathbf{e_3}du_3 \right) \\ &= f_1 h_1 du_1 + f_2 h_2 du_2 + h_3 f_3 du_3 \end{aligned} \tag{39}$$

$d\phi$ can also be expressed as,

$$d\phi = \frac{\partial\phi}{\partial u_1}du_1 + \frac{\partial\phi}{\partial u_2}du_2 + \frac{\partial\phi}{\partial u_2}du_3 \tag{40}$$

Comparing equations (39) and equations (40), get the following,

$$f_1 = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1} \qquad f_2 = \frac{1}{h_2}\frac{\partial\phi}{\partial u_2} \qquad f_3 = \frac{1}{h_3}\frac{\partial\phi}{\partial u_3} \tag{41}$$

Replacing (41) in (34) we obtain,

$$\boldsymbol{\nabla}\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\mathbf{e_3} \tag{42}$$

Therefore, from equation (42)

$$\boldsymbol{\nabla} = \frac{1}{h_1}\frac{\partial}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial}{\partial u_3}\mathbf{e_3} \tag{43}$$

In cylindrical coordinates, equation (43) can be written as

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e_\theta} + \frac{\partial}{\partial z}\mathbf{e_z} \tag{44}$$

We will then proceed to find the value of $\boldsymbol{\nabla}\cdot\boldsymbol{q}$ in curvilinear coordinates. In curvilinear coordinates, we will take $\boldsymbol{q}$ to be equal to

$$\boldsymbol{q} = u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2} + u_3\boldsymbol{e_3}$$

$$\nabla\bullet\mathbf{q} = \nabla\bullet(u_1\mathbf{e_1} + u_2\mathbf{e_2} + u_3\mathbf{e_3}) = \nabla\bullet(u_1\mathbf{e_1}) + \nabla\bullet(u_2\mathbf{e_2}) + \nabla\bullet(u_3\mathbf{e_3}) \tag{45}$$

From, equation (42) above,

$$\nabla u_1 = \frac{1}{h_1}\frac{\partial u_1}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial u_1}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial u_1}{\partial u_3}\mathbf{e}_3 = \frac{\mathbf{e_1}}{h_1} \tag{46}$$

$$\nabla u_2 = \frac{1}{h_1}\frac{\partial u_2}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial u_2}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial u_3}{\partial u_3}\mathbf{e_3} = \frac{\mathbf{e_2}}{h_2} \tag{47}$$

$$\nabla u_3 = \frac{1}{h_1}\frac{\partial u_3}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial u_3}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial u_3}{\partial u_3}\mathbf{e_3} = \frac{\mathbf{e_3}}{h_3} \tag{48}$$

We then deal with each term on the right hand-side of equation (45), but first we derive certain relations. From equations (46), (47) and (48), we obtain;

$$\frac{\mathbf{e_1}}{h_2h_3} = \frac{\mathbf{e_2}\times\mathbf{e_3}}{h_2h_3} = \nabla u_2 \times \nabla u_3 \quad \Rightarrow \mathbf{e_1} = h_2h_3\nabla u_2\times\nabla u_3 \tag{49}$$

$$\frac{\mathbf{e_2}}{h_1h_3} = \frac{\mathbf{e_1}\times\mathbf{e_3}}{h_1h_3} = \nabla u_1 \times \nabla u_3 \quad \Rightarrow \mathbf{e_2} = h_1h_3\nabla u_1\times\nabla u_3 \tag{50}$$

$$\frac{\mathbf{e_3}}{h_1h_2} = \frac{\mathbf{e_1}\times\mathbf{e_2}}{h_1h_2} = \nabla u_1 \times \nabla u_2 \quad \Rightarrow \mathbf{e_3} = h_1h_2\nabla u_1\times\nabla u_2 \tag{51}$$

We will then express all the terms on the left hand-side in equation (45) in curvilinear coordinates, we will begin with the term, $\nabla\bullet(u\mathbf{e_1})$

$$\begin{aligned}\nabla\bullet(u_1\mathbf{e_1}) &= \nabla\bullet\left(h_2h_3u_1\nabla u_2\times\nabla u_3\right)\\ &= h_2h_3u_1\nabla\bullet\left(\nabla u_2\times\nabla u_3\right) + \left(\nabla u_2\times\nabla u_3\right).\nabla\left(h_2h_3u_1\right)\end{aligned} \tag{52}$$

But in equation (52) above,

$$\nabla\bullet\left(\nabla u_2\times\nabla u_3\right) = \nabla u_3\bullet\left(\nabla\times\nabla u_2\right) - \nabla u_2\bullet\left(\nabla\times\nabla u_3\right) = 0$$

Hence,

$$\nabla\bullet(u_1\mathbf{e_1}) = \left(\nabla u_2\times\nabla u_3\right).\nabla\left(h_2h_3u_1\right) \tag{53}$$

From equations, (47 – 49), equation (53) becomes

$$\nabla\bullet(u_1\mathbf{e_1}) = \frac{\mathbf{e_1}}{h_2h_3}.\nabla\left(h_2h_3u_1\right) \tag{54}$$

Using equation (42), we can express $\nabla(h_2h_3u_1)$ in the above equation as

$$\nabla\left(h_2h_3u_1\right) = \frac{1}{h_1}\frac{\partial\left(h_2h_3u_1\right)}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial\left(h_2h_3u_1\right)}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial\left(h_2h_3u_1\right)}{\partial u_3}\mathbf{e_3} \tag{55}$$

Now replacing equation (55) into (54) we obtain

$$\nabla\bullet(u_1\mathbf{e_1}) = \frac{\mathbf{e_1}}{h_2h_3}\bullet\left(\frac{1}{h_1}\frac{\partial}{\partial u_1}\left(h_2h_3u_1\right)\mathbf{e_1} + \frac{1}{h_2}\frac{\partial}{\partial u_2}\left(h_2h_3u_1\right)\mathbf{e_2} + \frac{1}{h_3}\frac{\partial}{\partial u_3}\left(h_2h_3u_1\right)\mathbf{e_3}\right) \tag{56}$$

Simplifying equation (56), we obtain,

$$\nabla\bullet(u_1\mathbf{e_1}) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}\left(h_2h_3u_1\right) \tag{57}$$

In a similar way the remaining two terms in equation (42) can be found to be

$$\nabla \bullet (u_2 \mathbf{e_2}) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_2} \left(h_1 h_3 u_2 \right) \tag{58}$$

$$\nabla \bullet (u_3 \mathbf{e_3}) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_3} \left(h_1 h_2 u_3 \right) \tag{59}$$

Replacing equations (57 – 59) in (45) we obtain

$$\nabla \bullet \mathbf{q} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} \left(h_2 h_3 u_1 \right) + \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_2} \left(h_1 h_3 u_1 \right) + \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_3} \left(h_1 h_2 u_1 \right) \tag{60}$$

Using equations (33) we can now express equation (60) in cylindrical coordinates as

$$\nabla \bullet \mathbf{q} = \frac{1}{r} \frac{\partial \left(r u_r \right)}{\partial r} + \frac{1}{r} \frac{\partial \left(u_\theta \right)}{\partial \theta} + \frac{\partial \left(u_z \right)}{\partial z} \tag{61}$$

(b) Momentum equation

The momentum equation in vector form as shown in equation (9)

$$\rho \left(\frac{\partial q}{\partial t} + \left(\mathbf{q}.\nabla \right) \mathbf{q} \right) = -\nabla p + p\mathbf{g} + \mu \nabla^2 \mathbf{q}$$

We want to express the above equation in cylindrical coordinates. Since we have already calculated the value of the grad, ∇ we only need to find the value of the Laplacian, ∇^2. To do this, we first find the value of $\nabla^2 f$, where f is any scalar function and then and then finally find the laplacian.

$$\nabla^2 f = \nabla \bullet \left(\nabla f \right) \tag{62}$$

We then let

$$\boldsymbol{F = \nabla f,} \text{ where } \boldsymbol{F} = F_1 \boldsymbol{e_1} + F_2 \boldsymbol{e_2} + F_3 \boldsymbol{e_3} \tag{63}$$

From equation (42),

$$F = \nabla f = \frac{\mathbf{e_1}}{h_1} \frac{\partial f}{\partial u_1} + \frac{\mathbf{e}_2}{h_2} \frac{\partial f}{\partial u_2} + \frac{\mathbf{e_3}}{h_3} \frac{\partial}{\partial u_3} \tag{64}$$

Comparing coefficients of $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$, in equations (63), (64), we obtain;

$$F_1 = \frac{1}{h_1} \frac{\partial f}{\partial u_1}, \; F_2 = \frac{1}{h_2} \frac{\partial f}{\partial u_2}, \; F_3 = \frac{1}{h_3} \frac{\partial f}{\partial u_3} \tag{65}$$

From equation (60), equation (62) can be written as,

$$\begin{aligned} \nabla^2 f &= \nabla \bullet (\nabla f) = \nabla \bullet \mathbf{F} = \nabla \bullet (F_1 \mathbf{e_1} + F_2 \mathbf{e_2} + F_3 \mathbf{e_3}) \\ &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(h_2 h_3 F_1 \right) + \frac{\partial}{\partial u_2} \left(h_1 h_3 F_2 \right) + \frac{\partial}{\partial u_3} \left(h_1 h_2 F_3 \right) \right] \end{aligned} \tag{66}$$

Equation (66) can be simplified to

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right] \tag{67}$$

Hence the laplacian in curvilinear coordinates is given by;

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right] \tag{68}$$

In cylindrical coordinates, equation (68) becomes

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2} \tag{69}$$

The gravitational force, $\boldsymbol{g}$ in cylindrical coordinates can be written as,

$$\mathbf{g} = g_r\mathbf{e_r} + g_\theta\mathbf{e_\theta} + g_z\mathbf{e_z} \tag{70}$$

Substituting, ∇, ∇^2 and $\mathbf{g}$ from equations; (42), (69) and (70) respectively, in equation (60),

$$\rho\left(\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}.\nabla)\mathbf{q}\right) = -\nabla p + p\mathbf{g} + \mu\nabla^2\mathbf{q}$$

We obtain;

$$\rho\left\{\frac{\partial}{\partial t}(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z})+\left[(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z}).\left(\frac{\partial}{\partial r}\mathbf{e_r}+\frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e_\theta}+\frac{\partial}{\partial z}\mathbf{e_z}\right)\bullet(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z})\right]\right\} =$$
$$-\left(\frac{\partial}{\partial r}\mathbf{e_r}+\frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e_\theta}+\frac{\partial}{\partial z}\mathbf{e_z}\right)p + p(g_r\mathbf{e_r}+g_\theta\mathbf{e_\theta}+g_z\mathbf{e_z})+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right](u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z}) \tag{71}$$

On simplifying equation (71), we obtain

$$\rho\frac{\partial}{\partial t}u_r\mathbf{e_r}+\rho\frac{\partial}{\partial t}u_\theta\mathbf{e_\theta}+\rho\frac{\partial}{\partial t}u_z\mathbf{e_z}+\rho\left(u_r\frac{\partial}{\partial r}\mathbf{e_r}+u_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e_\theta}+u_z\frac{\partial}{\partial z}\mathbf{e_z}\right)\bullet(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z})$$
$$= -\left(\frac{\partial p}{\partial r}\mathbf{e_r}+\frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e_\theta}+\frac{\partial p}{\partial z}\mathbf{e_z}\right)+(pg_r\mathbf{e_r}+pg_\theta\mathbf{e_\theta}+pg_z\mathbf{e_z})$$
$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right](u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_z\mathbf{e_z})$$

$$\rho\frac{\partial}{\partial t}u_r\mathbf{e_r}+\rho\frac{\partial}{\partial t}u_\theta\mathbf{e_\theta}+\rho\frac{\partial}{\partial t}u_z\mathbf{e_z}+\rho\left(u_r\frac{\partial}{\partial r}\mathbf{e_r}+u_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+u_z\frac{\partial}{\partial z}\mathbf{e_z}\right)u_r\mathbf{e_r}+\rho\left(u_r\frac{\partial}{\partial r}\mathbf{e_r}+u_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+u_z\frac{\partial}{\partial z}\mathbf{e_z}\right)u_\theta\mathbf{e_\theta}$$
$$+\rho\left(u_r\frac{\partial}{\partial r}\mathbf{e_r}+u_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+u_z\frac{\partial}{\partial z}\mathbf{e_z}\right)u_z\mathbf{e_z}=-\left(\frac{\partial p}{\partial r}\mathbf{e_r}+\frac{1}{r}\frac{\partial p}{\partial\theta}\mathbf{e_\theta}+\frac{\partial p}{\partial z}\mathbf{e_z}\right)+\left(pg_r\mathbf{e_r}+pg_\theta\mathbf{e_\theta}+pg_z\mathbf{e_z}\right)+$$
$$\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right]u_r\mathbf{e_r}+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right]u_\theta\mathbf{e_\theta}+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right]u_z\mathbf{e_z} \quad (73)$$

Now collecting terms that contain $\mathbf{e_r}$, we obtain;

$$\rho\frac{\partial}{\partial t}u_r+\rho\left(u_r\frac{\partial}{\partial r}+u_\theta\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}\right)u_r=-\frac{\partial p}{\partial r}+pg_r+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)u_r \quad (74)$$

While those that contain $\mathbf{e_\theta}$, are

$$\rho\frac{\partial}{\partial t}u_\theta+\rho\left(u_r\frac{\partial}{\partial r}+u_\theta\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}\right)u_\theta=-\frac{\partial p}{\partial\theta}+pg_\theta+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)u_\theta \quad (75)$$

And finally those containing e_z are

$$\rho\frac{\partial}{\partial t}u_z+\rho\left(u_r\frac{\partial}{\partial r}+u_\theta\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}\right)u_z=-\frac{\partial p}{\partial z}+pg_z+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)u_z \quad (76)$$

Therefore, equations; (74), (75) and (76) are the Navier – Stokes equation in cylindrical coordinates. On simplifying the three equations we obtain

r-component

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial r}+\rho q_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2 u_r}{\partial z^2}\right] \quad (77)$$

θ - component

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_r}{\partial z}\right)$$
$$=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\rho q_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right] \quad (78)$$

z- component

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}\right) \\ =-\frac{\partial p}{\partial z}+\rho q_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right] \tag{79}$$

2.2.3. Equations of motion in spherical coordinates

Finally, we shall convert the equations of motion in spherical coordinates. In this section however we will not convert the equations of motion in curvilinear coordinates since we have already done this in the previous section. We will simply use what we have already derived.

The coordinates of a point P in spherical coordinates is given by the ordered pair, r, θ and $\emptyset$.

- r is the distance from the origin to the point P
- θ is the angle between the x −axis and the line from the origin to the point P
- $\emptyset$ is the angle between the $z -$ axis and the line from the origin to the point P

The relationship between the cylindrical coordinates and the spherical coordinates is given by;

$$x=r\sin\phi\cos\theta,\quad y=r\sin\phi\sin\theta,\qquad z=r\cos\phi \tag{80}$$

Scale factors

In spherical coordinates, the scale factors are, h_r, h_θ and h_ϕ as shown in equation (26) are given by;

$$h_i^{\,2}=\left(\frac{\partial x}{\partial u_i}\right)^2+\left(\frac{\partial y}{\partial u_i}\right)^2+\left(\frac{\partial z}{\partial u_i}\right)^2 \tag{81}$$

Where $i=r,\ \theta$ and ϕ

$$h_r^{\,2}=\left(\frac{\partial x}{\partial u_r}\right)^2+\left(\frac{\partial y}{\partial u_r}\right)^2+\left(\frac{\partial z}{\partial u_r}\right)^2 \tag{82}$$

$$h_r^{\,2}=\left(\frac{\partial}{\partial r}r\sin\phi\cos\theta\right)^2+\left(\frac{\partial}{\partial r}r\sin\phi\sin\theta\right)^2+\left(\frac{\partial}{\partial r}r\cos\phi\right)^2 \\ =\sin^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta+\cos^2\phi=\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)+\cos^2\phi=1 \tag{83}$$

Similarly h_θ can be given by

$$h_\theta^{\,2}=\left(\frac{\partial x}{\partial u_\theta}\right)^2+\left(\frac{\partial y}{\partial u_\theta}\right)^2+\left(\frac{\partial z}{\partial u_\theta}\right)^2 \tag{84}$$

$$h_\theta{}^2 = \left(\frac{\partial}{\partial u_\theta} r\sin\phi\cos\theta\right)^2 + \left(\frac{\partial}{\partial u_\theta} r\sin\phi\sin\theta\right)^2 + \left(\frac{\partial}{\partial u_\theta} r\cos\phi\right)^2$$

$$= r^2\sin^2\phi\sin^2\theta + r^2\sin^2\phi\cos^2\theta = r^2\sin^2\phi\left(\sin^2\theta+\cos^2\theta\right) = r^2\sin^2\phi \tag{85}$$

Similarly, h_ϕ can be expressed as

$$h_\phi{}^2 = \left(\frac{\partial x}{\partial u_\phi}\right)^2 + \left(\frac{\partial y}{\partial u_\phi}\right)^2 + \left(\frac{\partial z}{\partial u_\phi}\right)^2 \tag{86}$$

$$h_\phi{}^2 = \left(\frac{\partial}{\partial u_\phi}\rho\sin\phi\cos\theta\right)^2 + \left(\frac{\partial}{\partial u_\phi}\rho\sin\phi\sin\theta\right)^2 + \left(\frac{\partial}{\partial u_\phi}\rho\cos\phi\right)^2$$

$$= r^2\cos^2\phi\cos^2\theta + r^2\cos^2\phi\sin^2\theta + r^2\sin^2\phi = r^2\cos^2\phi\left(\cos^2\theta+\sin^2\theta\right) + r^2\sin^2\phi \tag{87}$$

$$= r^2\left(\cos^2\phi+\sin^2\phi\right) = r^2$$

Therefore the scale factors in spherical coordinates are given by

$$h_r{}^2 = 1, \quad h_\theta = r\sin\phi, \text{ and } h_\phi = r \tag{88}$$

In spherical coordinates, the velocity vector $\boldsymbol{q}$ is given by,

$$q = u_r e_r + u_\theta e_\theta + u_\phi e_\phi \tag{89}$$

We will then express in spherical coordinates, the continuity equations and the momentum equation

(a) Continuity equation

We will express the continuity equation in equation (7) $\nabla.\boldsymbol{q} = 0$ in curvilinear coordinates. From equation (60), we have;

$$\nabla\bullet\mathbf{q} = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}\left(h_2h_3u_1\right) + \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_2}\left(h_1h_3u_1\right) + \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_3}\left(h_1h_2u_1\right)$$

In spherical coordinates,

$$u_1, u_2 \text{ and } u_3 \text{ are } u_r, u_\theta \text{ and } u_\phi \tag{90}$$

Using equations (93) and (95), we obtain;

$$\nabla\bullet\mathbf{q} = \frac{1}{r^2\sin\phi}\frac{\partial}{\partial r}\left(r^2\sin\phi u_r\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\theta}\left(ru_\theta\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(r\sin\phi u_\phi\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2u_r\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\theta}\left(ru_\theta\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(r\sin\phi u_\phi\right) \tag{91}$$

(b) Navier-Stokes equations

To express in spherical coordinates, the Navier-Stokes equation (60);

$$\rho\left(\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}.\nabla)\mathbf{q}\right)=-\nabla p+\rho\mathbf{g}+\mu\nabla^2\mathbf{q}$$

We first express the grad, laplacian, velocity and gravity in spherical coordinates.

In curvilinear coordinates, the laplacian from equation (67) is given by

$$\nabla^2=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right]$$

Replacing equation (92) and (94) into the above equation we obtain

$$\nabla^2=\frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(\rho^2\sin\phi\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right] \tag{92}$$

We can also express g as

$$\mathbf{g}=g_r\mathbf{e_r}+g_\theta\mathbf{e_\theta}+g_\phi\mathbf{e}_\phi \tag{93}$$

From equation (43), in curvilinear coordinates, grad is given by;

$$\nabla=\frac{1}{h_1}\frac{\partial}{\partial u_1}\mathbf{e_1}+\frac{1}{h_2}\frac{\partial}{\partial u_2}\mathbf{e_2}+\frac{1}{h_3}\frac{\partial}{\partial u_3}\mathbf{e_3}$$

In spherical coordinates the above equation is written as;

$$\nabla=\frac{\partial}{\partial r}\mathbf{e_r}+\frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+\frac{1}{r}\frac{\partial}{\partial\phi}\mathbf{e}_\phi \tag{94}$$

And finally we have;

$$\mathbf{q}=u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_\phi\mathbf{e}_\phi \tag{95}$$

Now replacing equations, (92), (94) and (95) into equation (60) we obtain,

$$\rho\left(\frac{\partial}{\partial t}\left(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_\phi\mathbf{e}_\phi\right)+\left(\left(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_\phi\mathbf{e}_\phi\right).\left(\frac{\partial}{\partial r}\mathbf{e_r}+\frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+\frac{1}{\rho}\frac{\partial}{\partial\phi}\mathbf{e}_\phi\right)\right)\left(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_\phi\mathbf{e}_\phi\right)\right)$$
$$=-\left(\frac{\partial}{\partial r}\mathbf{e_r}+\frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+\frac{1}{r}\frac{\partial}{\partial\phi}\mathbf{e}_\phi\right)p+\rho\left(g_r\mathbf{e_r}+g_\theta\mathbf{e_\theta}+g_\phi\mathbf{e}_\phi\right)+$$
$$\mu\left(\frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right]\right)\left(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_\phi\mathbf{e}_\phi\right) \tag{96}$$

Expanding equation (96) we obtain

$$\left(\rho\frac{\partial}{\partial t}u_r\mathbf{e_r}+\rho\frac{\partial}{\partial t}u_\theta\mathbf{e_\theta}+\rho\frac{\partial}{\partial t}u_\phi\mathbf{e}_\phi\right)+\rho\left(u_r\frac{\partial}{\partial r}\mathbf{e_r}+\frac{u_\theta}{r\sin\phi}\frac{\partial}{\partial\theta}\mathbf{e_\theta}+\frac{u_\phi}{\rho}\frac{\partial}{\partial\phi}\mathbf{e}_\phi\right)\bullet\left(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_\phi\mathbf{e_j}\right)$$
$$=-\left(\frac{\partial p}{\partial r}\mathbf{e_r}+\frac{1}{r\sin\phi}\frac{\partial p}{\partial\theta}\mathbf{e_\theta}+\frac{1}{r}\frac{\partial p}{\partial\phi}\mathbf{e}_\phi\right)+\left(\rho g_r\mathbf{e_r}+\rho g_\theta\mathbf{e_\theta}+\rho g_\phi\mathbf{e}_\phi\right)+ \quad (97)$$
$$\mu\left(\frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right]\right)\left(u_r\mathbf{e_r}+u_\theta\mathbf{e_\theta}+u_\phi\mathbf{e}_\phi\right)$$

We then write equation (97) in component form, we first collect all the terms with $\mathbf{e_r}$;

$$\rho\frac{\partial}{\partial t}u_r+\rho\left(u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r\sin\phi}\frac{\partial}{\partial\theta}+\frac{u_\phi}{r}\frac{\partial}{\partial\phi}\right)u_r=-\frac{\partial p}{\partial r}+\rho g_r+$$
$$\mu\left(\frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right]\right)u_r \quad (98)$$

Similarly terms containing $\mathbf{e_\theta}$ are,

$$\rho\frac{\partial}{\partial t}u_\theta e_\theta+\rho\left(u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r\sin\phi}\frac{\partial}{\partial\theta}+\frac{u_\phi}{\rho}\frac{\partial}{\partial\phi}\right)u_\theta=-\frac{1}{r\sin\phi}\frac{\partial p}{\partial\theta}+\rho g_\theta+$$
$$\mu\left(\frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right]\right)u_\theta \quad (99)$$

And finally the terms containing e_ϕ

$$\rho\frac{\partial}{\partial t}u_\phi+\rho\left(u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r\sin\phi}\frac{\partial}{\partial\theta}+\frac{u_\phi}{\rho}\frac{\partial}{\partial\phi}\right)u_\phi=-\frac{1}{r}\frac{\partial p}{\partial\phi}+\rho g_\phi$$
$$+\mu\left(\frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial}{\partial\phi}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right]\right)u_\phi \quad (100)$$

Equations; (98), (99) and (100) are the momentum equations in spherical coordinates.

2.3.Solution of the Navier-Stokes equations

These equations are very complex and it is very difficult to solve them analytically. This is because of their non-linear terms arising from the convective acceleration terms. However there are a few special cases for which the convective acceleration vanishes because of the nature of the geometry of the flow system, in such cases exact solutions are possible. In this project, we are going to first solve the equations analytically; this will help us have an idea of the distribution of velocity and wall shear stress and finally we solve using numerical methods.

2.3.1. Analytical solution of Navier Stokes equations

Analytical solution is obtained when governing Boundary Value Problem is integrated using the methods of classical differential equations. Full Navier – Stokes equations have no known general analytical solution due to its complexity. In order to solve the equations analytically we simplify them by making several assumptions about the fluid, the flow or the geometry of the problem. The assumptions are; we assume the flow is laminar, steady, incompressible and parallel between plates. By making these assumptions it's possible to obtain the analytical solutions. The continuity equation and the momentum equation for steady incompressible flow as shown in equations (1) (13), (14) and (15) are given by;

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{101}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{102}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{103}$$

Since the flow is constrained by flat parallel walls of the channel, no component of the velocity in y and z directions; this implies that $v=w=0$. Also the gradient in v and w are zero, i.e.

$$\frac{\partial v}{\partial y}=\frac{\partial w}{\partial y}=\frac{\partial v}{\partial z}=\frac{\partial w}{\partial z}=\frac{\partial w}{\partial x}=\frac{\partial v}{\partial x}=0 \text{ also; } \frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 w}{\partial y^2}=\frac{\partial^2 v}{\partial z^2}=\frac{\partial^2 w}{\partial z^2}=\frac{\partial^2 v}{\partial x^2}=0 \tag{104}$$

Therefore the equations above become;

$$\frac{\partial u}{\partial x}=0 \quad \text{this implies that } \frac{\partial^2 u}{\partial x^2}=0 \tag{105}$$

$$\frac{dp}{dx}=\mu\frac{d^2u}{dy^2}+\mu\frac{d^2u}{dx^2} \tag{106}$$

Using equation (106), equation (107) becomes;

$$\frac{dp}{dx}=\mu\frac{d^2u}{dy^2} \tag{107}$$

The above equation implies that u is not a function of x alone therefore, $\frac{dp}{dx}$ is constant .On integrating it twice we obtain the results;

$$\int \frac{d^2u}{dy^2}dy = \int \frac{1}{\mu}\frac{dp}{dx}dy$$

, where A is a constant

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + A$$

On integrating again we obtain

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}y^2 + Ay + B$$, B is another constant (108)

In the next step now we calculate the values of the constants of integration, A and B. To do this, we apply the no – slip boundary conditions at the walls. The boundaries conditions are;

At $y = -h,\ u = 0$ and also at $y = h,\ u = 0$ (109)

Applying the boundary conditions in equations (110) to equation (109), we obtain;

$0 = \frac{1}{2}\frac{1}{\mu}\frac{dp}{dx}h^2 + Ah + B,$ and $0 = \frac{1}{2}\frac{1}{\mu}\frac{dp}{dx}h^2 + Ah + B$. Solving these equations simultaneously, we obtain; A=0 and $B = -\frac{1}{2}\frac{h^2}{\mu}\frac{dp}{dx}$. Substituting for A and B in equation (110), we get,

$$u(y) = \frac{1}{2}\frac{1}{\mu}\frac{dp}{dx}y^2 - \frac{1}{2}\frac{h^2}{\mu}\frac{dp}{dx} = -\frac{1}{2}\frac{h^2}{\mu}\frac{dp}{dx}\left(1 - \left(\frac{y}{h}\right)^2\right) \quad (110)$$

Equation (111) can be used to calculate the velocity distribution in a pipe, in radial direction
Wall shear stress can be calculated from equation (111) by using the relation;

$$\tau = \mu\frac{du}{dy} \quad (111)$$

Substituting equation (111) into (112), we obtain;

$$\tau = \frac{dP}{dx}y \quad (112)$$

From equation (113) above we get that wall shear stress is zero at the centre where y= 0 and maximum on the walls, where y = h.

To have a clear picture on how velocity and wall shear stress is distributed in a straight pipe, we plot the equations (111) and (113) respectively.

Plot of velocity distribution in a straight pipe

Equation of velocity in a straight pipe is given by (111);

$u(y) = -\frac{1}{2}\frac{h^2}{\mu}\frac{dp}{dx}\left(1-\left(\frac{y}{h}\right)^2\right)$, in this equation we will take the radius of the pipe, h = 0.5 and the value of $\frac{1}{2}\frac{h^2}{\mu}\frac{dp}{dx}$ = 1, then the equation becomes;

$$u(y) = -\left(1-\left(\frac{y}{0.5}\right)^2\right) \quad (113)$$

We then plot the equation (111) using MATLAB

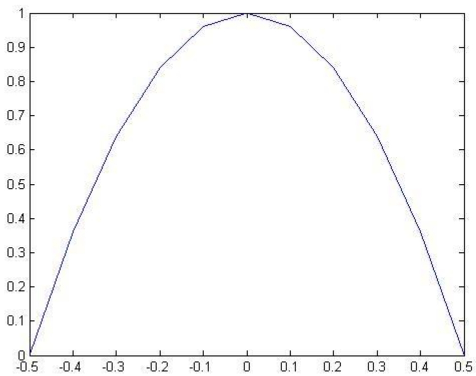


Figure 5: Velocity distribution in a straight pipe

To calculate wall shear stress, we replace equation (114) to equation (113) to obtain;

$$\tau = \mu\frac{du}{dy} = 4y \quad (114)$$

The plotting, we obtain the graph below,

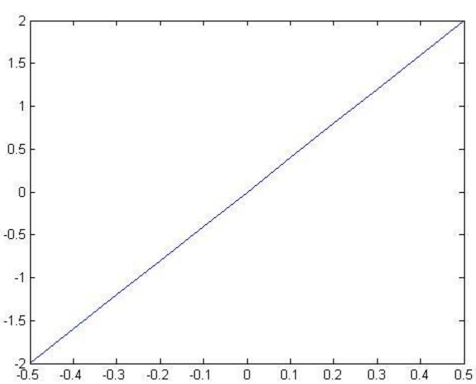


Figure 6: Stress distribution in a straight pipe

Velocity and wall shear stress distribution at bifurcation point.

From the above explanations, it's clear that, velocity is symmetric about the axis. However for flows at bifurcation points, this is not the case. The velocity curve tends to lean on one side more than the other, as shown below.

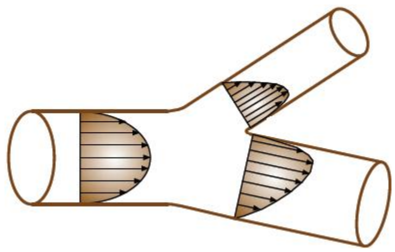


Figure 7: Velocity distribution at bifurcation point

From the figure above, as the fast moving flow arrives at the flow divider of the bifurcation, it is forced to follow one of the branches. Due to its high inertia, acting pressure gradient cannot displace it immediately into axial directions of the daughter branches and hence the flow moves next to the inner walls of the bifurcation (Kazakidi, 2008).

To explain this curve equation (112) can be modified as follows;

$$u(y) = -\frac{1}{2}\frac{h^2}{\mu}\frac{dp}{dx}\left(1 + \frac{y}{K} - \left(\frac{y}{h}\right)^2\right) \text{ where, } k > h \tag{115}$$

Taking;

$h = 0.5$, k=0.8 and $-\frac{1}{2}\frac{h^2}{\mu}\frac{dp}{dx} = 1,$ we will obtain,

$$\text{u(y)} = 1 + \frac{10}{8}y - (2y)^2$$

Plotting the equation above we obtain the graph given below

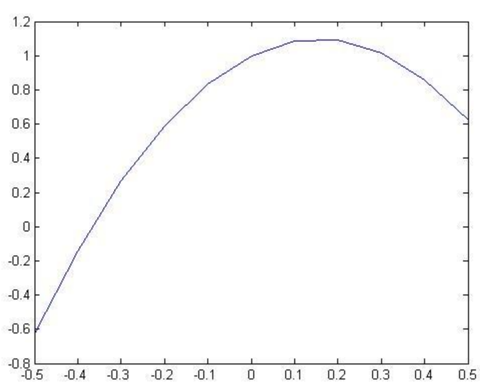


Figure 8: Velocity distribution at branch point

To get the wall shear stress distribution we differentiate the equation for velocity to get;

$$\tau = \frac{du}{dy} = \frac{10}{8} - 8y \qquad (116)$$

On plotting the above curve we obtain;

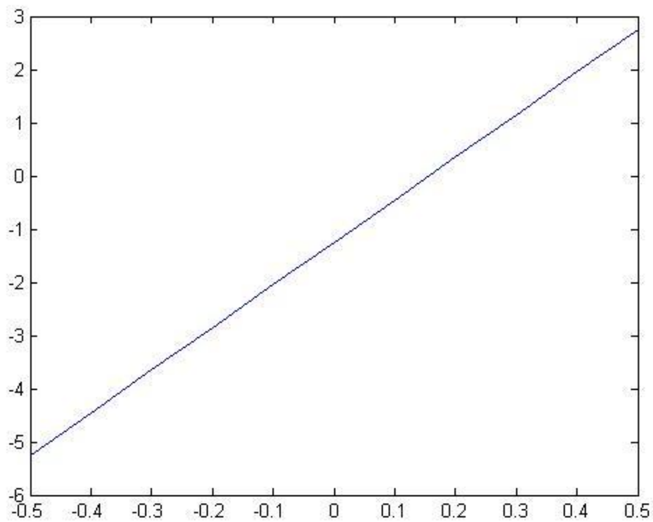


Figure 9: Stress distribution at branch point

The above equation (117) gives the wall shear stress on the outer wall. Clearly the wall shear stress in the equation (117) is less than that in equation (115). Hence wall shear stress is lower on the inner side of the wall at bifurcation point than on the straight tube.

2.3.2. Numerical methods

There are various numerical methods which can be used to solve them, finite element method, finite volume method, finite difference method and Lattice Boltzmann Method. All these methods can be used to solve the equations. However in this project, we are going to concentrate on only one method, the finite element method, since it's simple and able to solve complex geometries compared to other methods. Furthermore by this method we are able to include boundary conditions in the formulations. Before we look at how to solve the Navier-Stokes equations using the finite element method, we are going to have a brief introduction to what is FEM using a simple example.

Finite Element Method

Finite element method is a numerical procedure for solving the differential equations in of physics and engineering (J.Segerlind, 1976). In this method, the solution domain is divided into simply shaped regions or elements, then an approximate solution of the partial differential equation can be developed for each of these elements and finally, the total solution is then generated by linking together the individual solutions. To explain how this method works, we are going to consider one dimension and two dimension case.

Formulation of FEM for one dimension problems

To demonstrate how finite element method operates, we will consider the following one dimension steady – diffusion equation.

$$\mu \frac{dT}{dx} - k\frac{d^2T}{dx^2} = f \text{ in } \Omega \tag{117}$$

In the above equation, μ and k are the known constant velocity and diffusivity respectively, $f(x)$ is the known source function and T(x) is the unknown scalar which is to be determined. This second order ordinary differential equation is supported by two boundary conditions provided at two ends of the one dimensional domain. At the boundary, either the value of the unknown or the value of its first derivative or an equation involving both the unknown and the first derivative is specified.

In the first step in solving by FEM, we first divide the problem domain into a finite number of elements and try to obtain polynomial type approximate solution for each element. For one-dimensional case, the simplest polynomial is a linear polynomial or straight line as shown below.

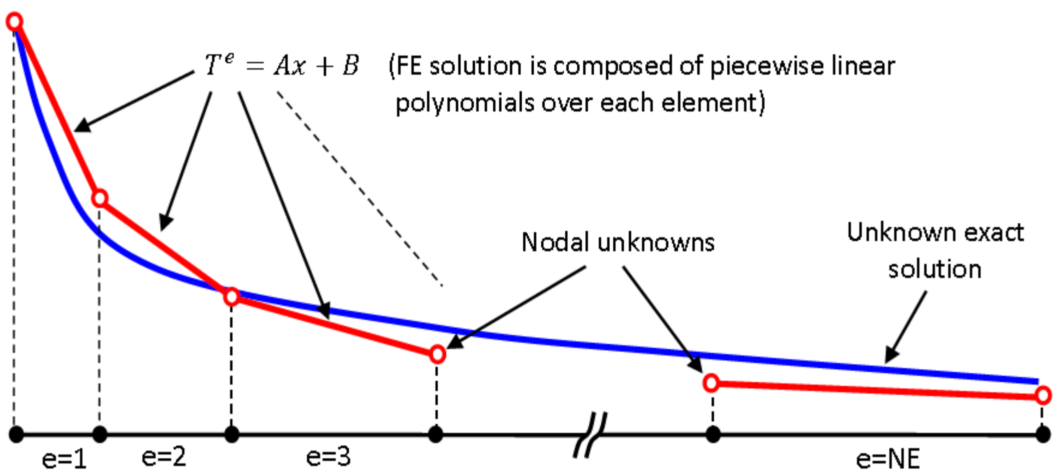


Figure 10: A linear polynomial element

In the figure above, the number of elements is NE and the number of nodes is NN = NE+1. The approximate solution we will use is C^0 continuous. In theory it is possible to use higher order continuous approximation such as C^1 but they are not preferred because of the complicated Mathematics they require especially for 2D and 3D problems.

The method of weighted residual and the weak form of the equation

The diffusion equation (123) together with boundary conditions is known as the strong form of the problem. FEM is a weighted residual type numerical method and it makes use of the weak form of the ODE. In order to obtain the weak for, we first obtain the residual; the residual of the DE is obtained by collecting all the terms on one side as follows;

$$R(x) = \mu \frac{dT}{dx} - k\frac{d^2T}{dx^2} - f \qquad (118)$$

It's important to note that for exact solution T(x), R(x) = 0. But for an approximate solution, the residual will not vanish. This implies that the smaller the residual the closer the approximate solution to the exact solution. The method of weighted residual tries to minimize the residual in a weighted integral sense as follows;

$$\int_\Omega w(x)R(x)dx, \text{ where, R}(x) \text{ is the residual and w(x) is weght function} \qquad (119)$$

Substituting the residual into this equation we obtain,

$$\int_\Omega \left(wu\frac{dT}{dx} - wk\frac{d^2T}{dx^2} - wf \right)dx = 0 \qquad (120)$$

Since we have NN unknowns, we will choose as many functions to obtain NN equations. Since we are using C^0 continuous solution in the above weighted residual statement, the second order derivative that appears in the diffusion term in equation (121) cannot be evaluated properly. In order to be able to work with a C^0 continuous approximation solution, we need to lower the differentiation requirement of the unknown in the weighted residual statement. This is done by applying integration by parts to the second term (diffusive term) of the equation (121). But before we do this we state the following theorem;

Theorem: Fundamental theorem of calculus, $\int_\Omega \frac{\partial u}{\partial y}dx = \int_\Gamma un_x d\Gamma$ ***where n_x is a unit a unit outward normal to the boundary,*** Γ of Ω.

Using the above theorem we write the diffusion term as

$$w\frac{d^2T}{dx^2} = -\frac{d}{dx}\left(wk\frac{dT}{dx} \right) + k\frac{dw}{dx}\frac{dT}{dx}, \ \Rightarrow \int_\Omega -w\frac{d^2T}{dx^2} = \int_\Omega k\frac{dw}{dx}\frac{dT}{dx} - \int_\Gamma wk\frac{dT}{dx}n_x d\Gamma \qquad (121)$$

The last term in the second part of equation above is called a boundary integral. This term is evaluated at the boundaries Γ of the problem domain Ω, where n_x is the x – component of the unit outward normal to the boundary.

It's now important to note that the integration by parts have lowered the differential order of the unknown T from 2 to 1 and increased the differential order of the of the weight function from 0 to 1. Integration by parts is not applied to the advection term, because it only contains a first order derivative of T.

On substituting the diffusion term back into the weighted equation (121), we obtain;

$$\int_{\Omega}\left(wu\frac{dT}{dx} + k\frac{dw}{dx}\frac{dT}{dx}\right)dx = \int_{\Omega} wfdx + \int_{\Gamma} wk\frac{dT}{dx}n_x d\Gamma \qquad (122)$$

We note that, the terms on the left hand side of the above equation now includes only first order derivatives of the unknown. This is called the weak form of the problem due to this lower differentiability requirement compared to the original weighted residual statement. The weak forms allow us to work with C^0 continuous approximate solutions.

Primary and secondary variables and boundary conditions

In finite element method formulation, the boundary term on the RHS of equation (123) is very important. This is because it can be used identify the primary and secondary variables of a problem. To do this, we separate the boundary term into two parts; the first part containing the weight function and possibly its derivatives and the second part contains the dependent variable and possibly its derivatives. In our case, the first part includes only w. The dependent variable of the problem T, expressed in the same form as this first part of the boundary term called the primary variable, for this problem, Primary variable is T.

Part two includes, $wk\frac{dT}{dx}n_x$, which is the secondary variable of the problem. Secondary variables always have important physical meaning such as the amount of heat flux that passes through the boundary in heat transfer. If secondary variable is provided at a boundary is known as a natural or Neumann boundary condition and if primary variable is provided at a boundary the problem is known as essential (Dirichlet) boundary condition. If both Secondary and Primary variables are provided, then the problem is called mixed boundary condition.

Essential BC (EBC): $T = T_0$

Natural BC (NBC): $k\frac{dT}{dx}n_x = q_0$

Mixed BC (MBC): $k\frac{dT}{dx}n_x = \alpha T + \beta$

Constructing approximate solution

The C^0 approximate solution is given by;

$$T_{app}(x) = \sum_{j=1}^{NN} T_j S_j(x) \tag{123}$$

Where T_{app} is the approximate solution, NN is the number of nodes in the FE mesh, T_j 's are the nodal unknown values that we are going to calculate at the end of the finite element solution and S_j 's are the shape functions that are used to construct the approximate solution.

Mesh function have compact support in that, they are non-zero only over the element which touch the node with which they are associated with, everywhere else, they are equal to zero.

We now substitute the approximate solution given in (124) into (123) to obtain

$$\int_\Omega \left(wu\left(\sum_{j=1}^{NN} T_j \frac{dS_j}{dx} \right) + k\frac{dw}{dx}\left(\sum_{j=1}^{NN} T_j \frac{dS_j}{dx} \right) \right) dx = \int_\Omega wfdx + \int_\Gamma wk(SV)d\Gamma \tag{124}$$

Since we have NN unknowns, we need to have NN equations; therefore we need to select the weight functions NN times. In using Galerkin finite element method, the weight functions are selected are same as the shape functions, i.e.

$$w(x) = s_i(x)$$

Using the above substitution equation (130) becomes;

$$\int_\Omega \left(S_i u\left(\sum_{j=1}^{NN} T_j \frac{dS_j}{dx} \right) + k\frac{dS_i}{dx}\left(\sum_{j=1}^{NN} T_j \frac{dS_j}{dx} \right) \right) dx = \int_\Omega S_i\, fdx + \int_\Gamma S_i(SV)d\Gamma \tag{125}$$

Putting the summation sign outside, we obtain,

$$\sum_{j=1}^{NN} \left(\left(\int_\Omega S_i u \frac{dS_i}{dx} + k\frac{dS_i}{dx}\frac{dS_j}{dx} \right) dx \right) T_j = \int_\Omega S_i\, fdx + \int_\Gamma S_i(SV)d\Gamma \qquad \text{i = 1,2,3,....} \tag{126}$$

To further simplify the equation we can use the following compact matrix notation,

[K]{T} = {F} + {B}

Which is known as the global equation system, {T} is the vector of nodal unknowns with NN entries, [K] is the global square stiffness matrix of the size NN x NN with entries given below.

$$K_{ij} = \left(\int_\Omega S_i u \frac{dS_i}{dx} + k\frac{dS_i}{dx}\frac{dS_j}{dx} \right) dx \tag{127}$$

{F} and {B} are the global force vector and boundary integral vector of the size NN x 1 with entries given as;

$$F_i = \int_\Omega S_i\, fdx \quad \text{and } \mathrm{B}_i = \int_\Gamma S_i(SV)d\Gamma \qquad \text{i = 1,2,3,....NN} \tag{128}$$

[K] and {F} integrals are evaluated over the whole domain, where as the boundary integral is evaluated only at the problem boundary.

We will calculate K_{ij}, F_i and B_i for each element and then assemble together all the element matrices, i.e.

$$[K] = \sum_{e=1}^{NE}\left[K^e\right] \text{ and } [\mathrm{F}] = \sum_{e=1}^{NE}\left[\mathrm{F}^e\right] \tag{129}$$

After assembling the elements we will obtain the results as follows. For illustrations, we pick 5 – nodes

$$[\mathrm{K}]=\begin{bmatrix} \mathrm{K}_{11}^{1} & \mathrm{K}_{12}^{1} & 0 & 0 & 0 \\ \mathrm{K}_{21}^{1} & \mathrm{K}_{22}^{1}+\mathrm{K}_{11}^{2} & \mathrm{K}_{12}^{2} & 0 & 0 \\ 0 & \mathrm{K}_{21}^{2} & \mathrm{K}_{22}^{2}+\mathrm{K}_{11}^{3} & \mathrm{K}_{12}^{3} & 0 \\ 0 & 0 & \mathrm{K}_{21}^{3} & \mathrm{K}_{22}^{3}+\mathrm{K}_{11}^{4} & \mathrm{K}_{12}^{4} \\ 0 & 0 & 0 & \mathrm{K}_{21}^{4} & \mathrm{K}_{22}^{4} \end{bmatrix}$$

$$\{\mathrm{F}\} = \begin{Bmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_2^2 + F_1^3 \\ F_2^3 + F_1^4 \\ F_2^4 \end{Bmatrix} \text{ and } \{B\} = \begin{Bmatrix} B_1^1 \\ B_2^1 + B_1^2 \\ B_2^2 + B_1^3 \\ B_2^3 + B_1^4 \\ B_2^4 \end{Bmatrix} = \begin{Bmatrix} B_1^1 \\ 0 \\ 0 \\ 0 \\ B_2^4 \end{Bmatrix}$$

On assembling the matrices together we obtain;

$$\begin{bmatrix} \mathrm{K}_{11} & \mathrm{K}_{12} & \mathrm{K}_{13} & \mathrm{K}_{14} & \mathrm{K}_{15} \\ \mathrm{K}_{21} & \mathrm{K}_{22} & \mathrm{K}_{23} & \mathrm{K}_{24} & \mathrm{K}_{25} \\ \mathrm{K}_{31} & \mathrm{K}_{32} & \mathrm{K}_{33} & \mathrm{K}_{34} & \mathrm{K}_{35} \\ \mathrm{K}_{41} & \mathrm{K}_{42} & \mathrm{K}_{43} & \mathrm{K}_{44} & \mathrm{K}_{45} \\ \mathrm{K}_{51} & \mathrm{K}_{52} & \mathrm{K}_{53} & \mathrm{K}_{54} & \mathrm{K}_{55} \end{bmatrix} \begin{Bmatrix} \mathrm{T}_1 \\ \mathrm{T}_2 \\ \mathrm{T}_3 \\ \mathrm{T}_4 \\ \mathrm{T}_5 \end{Bmatrix} = \begin{Bmatrix} \mathrm{F}_1 \\ \mathrm{F}_2 \\ \mathrm{F}_3 \\ \mathrm{F}_4 \\ \mathrm{F}_5 \end{Bmatrix} + \begin{Bmatrix} \mathrm{B}_1 \\ 0 \\ 0 \\ 0 \\ \mathrm{B}_5 \end{Bmatrix} \tag{130}$$

To demonstrate how this method operates, we will look at the example;

Solve the following using Galerkin FEM on the mesh of 5 elements

$$\mu \frac{dT}{dx} - k\frac{d^2T}{dx^2} = f \text{ in } 0 \le x \le 1, \text{ u = 3, k = 1 and f =1 wth}$$

$$\text{EBC T(0) = 0, T(1) = 0}$$

Solution

To begin with, we calculate the shape functions as follows;

$$S_1 = \frac{x - x_2}{x_1 - x_2} = \frac{1}{2}(1 - x) \quad \text{and} \quad S_2 = \frac{x - x_1}{x_2 - x_1} = \frac{1}{2}(1 + x)$$

Then using equation (128), we obtain;

$$K_{11}^{e} = \int_0^1 \left(\frac{1}{2}(1-x)(3)\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right) dx = \frac{7}{2}$$

$$K_{12}^{e} = \int_0^1 \left(\frac{1}{2}(1-x)(3)\left(\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\right) dx = -\frac{7}{2}$$

$$K_{21}^{e} = \int_0^1 \left(\frac{1}{2}(1-x)(3)\left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right) dx = -\frac{13}{2}$$

$$K_{22}^{e} = \int_0^1 \left(\frac{1}{2}(1+x)(3)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right) dx = \frac{13}{2}$$

We also use equation (129) to calculate F as follows;

$$F_1^{e} = \int_0^1 S_i\, f dx = \int_0^1 \frac{1}{2}(1-x)\, dx$$

$$= \left[\frac{1}{2}x - \frac{1}{4}x^2 \right]_{-1}^{1} = 1$$

, similarly, $F_2 = 1$

Therefore the system (131) becomes

$$\begin{bmatrix} \frac{7}{2} & -\frac{7}{2} & 0 & 0 & 0 & 0 \\ -\frac{13}{2} & \frac{13}{2}+\frac{7}{2} & -\frac{7}{2} & 0 & 0 & 0 \\ 0 & -\frac{13}{2} & \frac{13}{2}+\frac{7}{2} & -\frac{7}{2} & 0 & 0 \\ 0 & 0 & -\frac{13}{2} & \frac{13}{2}+\frac{7}{2} & -\frac{7}{2} & 0 \\ 0 & 0 & 0 & -\frac{13}{2} & \frac{13}{2}+\frac{7}{2} & -\frac{7}{2} \\ 0 & 0 & 0 & & -\frac{13}{2} & \frac{13}{2} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1+1 \\ 1+1 \\ 1+1 \\ 1+1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The last vector is zero because the boundary conditions are all zero.

Formulation of FEM for two dimension problems

This method is similar to the one dimension problem we have discussed above. We follow the same procedures however the only difference comes from the elements used. In 1D problem we used straight lines to approximate the solutions, however in 2D problems, we divide the solution domain into triangles with 3 nodes.

Finite Element Method for Nervier-Stokes Equations

In this section we are going to apply the Finite Element Method to solve the Navier Stokes equations. In the solution of NSE there are several problems that are encountered. We have problem of non-linear terms, the convection and the advection terms in the momentum equations, the problem of incompressibility and the problem of instability of solutions at high Reynolds number. We shall deal with this problem by using the Newton-Raphson method. In order to deal with the problem of incompressibility we will add the artificial compressibility term to the continuity equation and finally to stabilize the Navier stokes equations we will use the Galerkin least squares.

Solution of Navier-Stokes equations

$$\text{x - momentum } \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\text{-}\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{\partial P}{\partial x}-\rho f_x=0 \qquad (131)$$

$$\text{y - momentum } \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)\text{-}\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{\partial P}{\partial y}-\rho f_y=0 \qquad (132)$$

$$\text{z - momentum } \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)\text{-}\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{\partial P}{\partial z}-\rho f_z=0 \qquad (133)$$

$$\text{continuity } \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (134)$$

Expression of the Navier-Stokes equations in conservative form

The conservative form will enable us apply integration by parts easily to the higher order terms.

The continuity equation (135) above can be expressed as;

$$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z}$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 w}{\partial x\partial z} \qquad (135)$$

Also we can write it as;

$$\frac{\partial^2 v}{\partial y^2}=-\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 w}{\partial y\partial z} \qquad (136)$$

$$\frac{\partial^2 w}{\partial z^2}=-\frac{\partial^2 u}{\partial x\partial z}-\frac{\partial^2 v}{\partial y\partial z} \qquad (137)$$

We then express the momentum equations (6) and (7) in other forms,

Momentum equation in x – direction

The equation (132) above can be expressed as;

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\text{-}2\mu\frac{\partial^2 u}{\partial x^2}\text{-}\mu\left(-\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{\partial P}{\partial x}-\rho f_x=0 \qquad (138)$$

Replacing equation (136) into equation (139), we obtain;

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\text{-}2\mu\frac{\partial^2 u}{\partial x^2}\text{-}\mu\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{\partial P}{\partial x}-\rho f_x=0 \qquad (139)$$

We factorize the above equation to obtain;

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\text{-}\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}-P\right)\text{-}\mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)-\rho f_x=0 \qquad (140)$$

Momentum equation in y – direction

Similarly equation (133) can also be written in different forms. The equation above can be expressed as;

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{y}+w\frac{\partial v}{\partial z}\right)\text{-}2\mu\frac{\partial^2 v}{\partial y^2}\text{-}\mu\left(-\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{\partial P}{\partial y}-\rho f_y=0 \qquad (141)$$

Replacing equation (137) into equation (142), we obtain;

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{y}+w\frac{\partial v}{\partial z}\right)\text{-}2\mu\frac{\partial^2 v}{\partial y^2}\text{-}\mu\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{\partial P}{\partial y}-\rho f_y=0 \qquad (142)$$

We factorize the above equation to obtain;

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)\text{-}\frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}-P\right)\text{-}\mu\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 w}{\partial y\partial z}\right)-\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}\right)-\rho f_y=0 \qquad (143)$$

Momentum equation in z – direction

Similarly equation (134) can also be written in different forms. The equation above can be expressed as;

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)\text{-}2\mu\frac{\partial^2 w}{\partial z^2}\text{-}\mu\left(-\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{\partial P}{\partial w}-\rho f_z=0 \qquad (144)$$

Replacing equation (138) into equation (145), we obtain;

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)\text{-}2\mu\frac{\partial^2 w}{\partial z^2}\text{-}\mu\left(-\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 v}{\partial y\partial z}\right)+\frac{\partial P}{\partial w}-\rho f_z=0 \qquad (145)$$

We factorize the above equation to obtain;

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)-\frac{\partial}{\partial z}\left(2\mu\frac{\partial w}{\partial z}-P\right)-\mu\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)-\mu\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)-\rho f_z=0 \quad (146)$$

Therefore the new forms of the momentum equations are given by;

x - momentum

$$-\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}-p\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\rho f_x=0 \quad (147)$$

y - momentum

$$-\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}-p\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\rho f_y=0 \quad (148)$$

z - momentum

$$-\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial w}{\partial z}-p\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\rho f_z=0 \quad (149)$$

continuity $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ (150)

The above new form of the Stokes equations is more suitable to derive the weak form of the Stokes equations.

Construction of approximate solution

In this section we are going to come up with the approximate solutions to the above differential equations. the approximate solution to be chosen need to be C^0 continuous, i.e. only the zero order derivative, but not higher order, this is because higher order derivative makes the calculation complicated especially when dealing with 2D and 3D problems. The approximate solution for the above differential equations is given by; $\tilde{u},\ \tilde{v},\ \tilde{w}$ and $\tilde{p}$

$$\tilde{u}(x,y)=\sum_{j=1}^{n}\psi_i(x,y,z)u_i(t),\quad \tilde{v}(x,y)=\sum_{j=1}^{n}\psi_i(x,y,z)v_i(x,y)\quad \text{and } \tilde{w}(x,y)=\sum_{j=1}^{n}\psi_i(x,y,z)w_i(t)$$
$$\tilde{P}(x,y)=\sum_{j=1}^{n}\psi_i(x,y,z)\,p_i(t) \quad (151)$$

In this project for simplicity, we will use u, v, w and p in place of $\tilde{u},\ \tilde{v},\ \tilde{w}$ and $\tilde{p}$ respectively. Therefore (151) becomes;

$$u=\sum_{j=1}^{n}\psi_i(x,y,z)u_i(t),\quad v=\sum_{j=1}^{n}\psi_i(x,y,z)v_i(x,y)\quad \text{and } w=\sum_{j=1}^{n}\psi_i(x,y,z)w_i(t)$$
$$P=\sum_{j=1}^{n}\psi_i(x,y,z)\,p_i(t) \quad (152)$$

Since equations in (153) are approximations, when we replace them in equations (148 – 151), the right hand of the equations will not be equal to zero but to a residual R.

x - momentum

$$-\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}-p\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\rho f_x=R \quad (153)$$

y - momentum

$$-\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}-p\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\rho f_y=R \quad (154)$$

y - momentum

$$-\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial w}{\partial z}-p\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\rho f_z=R \quad (155)$$

continuity $\frac{\partial u}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial w}{\partial z}=R$ (156)

From the above equations, the residual is the difference between the exact solution and the approximate solution. Therefore, to come up with a good approximation, we need to minimize the residual, i.e. the smaller the residual, the more close the approximate solution to the exact solution.

To minimize the residual, there are several methods which include; the method of variation and weighted residual. In this project, we will use the method of weighted residual. To minimize the residual, this method uses the following weighted function;

$$\int_\Omega R(x,y)w(x,y)d\Omega=0 \quad (157)$$

Where W(x, y) is a weight function and R(x, y) is a residual function as shown in equations (154 – 157). As per the procedure of weighted residual approach, we will use two weight functions Q and w for continuity and momentum equations respectively.

Replacing equations (154 – 157) in (158) we obtain;

x - momentum

$$\int_v w\left[-\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}-p\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\right. \quad (158)$$

$$\left.\mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\rho f_x\right]dv=0$$

y - momentum

$$\int_v w\left[-\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}-p\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right. \quad (159)$$

$$\left.+\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\rho f_y\right]dv=0$$

z - momentum

$$\int_v w\left[-\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial w}{\partial z}-p\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\right. \quad (160)$$

$$\left.\mu\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\rho f_z\right]dv=0$$

$$\text{continuity} \quad \int_v Q\left[\frac{\partial u}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial w}{\partial z}\right]dv=0 \quad (161)$$

For the purpose of finite element method formulation, we use Galerkin criterion, where the weight functions are chosen to be the same as shape functions. Therefore in this case, we choose the weight functions as; in continuity equation, we replace the weight function Q_j is replaced by, ϕ_j while in momentum equation, w_i is replaced by, ψ_i. Replacing the weight functions above and the approximation functions (153) in equations (159 – 162), we obtain;

x-momentum

$$\int_v\left[\rho\psi\psi^T\frac{\partial\psi^T}{\partial t}\right]dv+\int_v\left[\rho\left(\psi\psi^T\frac{\partial\psi^T}{\partial x}u+\psi\psi^T\frac{\partial\psi^T}{\partial y}v+\psi\psi^T\frac{\partial\psi^T}{\partial z}w\right)u\right]dv$$

$$-\int_v\left[2\mu\psi\frac{\partial^2\psi^T}{\partial x^2}u+\mu\psi\frac{\partial^2\psi^T}{\partial y^2}u+\mu\psi\frac{\partial^2\psi^T}{\partial z^2}u+\mu\psi\frac{\partial^2\psi^T}{\partial x\partial y}v+\mu\psi\frac{\partial^2\psi^T}{\partial z\partial x}w\right]dv \quad (162)$$

$$+\int_v\left[\psi\frac{\partial\phi^T}{\partial x}p\right]dv=\int_v\left[\psi\rho f_x\right]dv$$

y - momentum

$$\int_v\left[\rho\left(\psi\psi^T\frac{\partial\psi^T}{\partial t}\right)\right]dv+\int_v\left[\rho\left(\psi\psi^T\frac{\partial\psi^T}{\partial x}u+\psi\psi^T\frac{\partial\psi^T}{\partial y}v+\psi\psi^T\frac{\partial\psi^T}{\partial z}w\right)v\right]dv$$

$$-\int_v\left[2\mu\psi\frac{\partial^2\psi^T}{\partial y^2}v+\mu\psi\frac{\partial^2\psi^T}{\partial x^2}v+\mu\psi\frac{\partial^2\psi^T}{\partial z^2}v+\mu\psi\frac{\partial^2\psi^T}{\partial x\partial y}u+\mu\psi\frac{\partial^2\psi^T}{\partial y\partial z}w\right]dv \quad (163)$$

$$+\int_v\left[\psi\frac{\partial\phi^T}{\partial y}p\right]dv=\int_v\left[\psi\rho f_y\right]dv$$

z - momentum

$$\int_v \left[\rho\psi\psi^T \frac{\partial \psi^T}{\partial t} \right] dv + \int_v \left[\rho \left(\psi\psi^T \frac{\partial \psi^T}{\partial x} u + \psi\psi^T \frac{\partial \psi^T}{\partial y} v + \psi\psi^T \frac{\partial \psi^T}{\partial z} w \right) w \right] dv$$
$$- \int_v \left[2\mu\psi \frac{\partial^2 \psi^T}{\partial z^2} w + \mu\psi \frac{\partial^2 \psi^T}{\partial x^2} w + \mu\psi \frac{\partial^2 \psi^T}{\partial y^2} w + \mu\psi \frac{\partial^2 \psi^T}{\partial x \partial z} u + \mu\psi \frac{\partial^2 \psi^T}{\partial y \partial z} v \right] dv \qquad (164)$$
$$+ \int_v \left[\psi \frac{\partial \phi^T}{\partial z} p \right] dv = \int_v \left[\psi\rho f_y \right] dv$$

continuity

$$\int_v \phi \left[u \frac{\partial \psi^T}{\partial x} + v \frac{\partial \psi^T}{\partial y} + w \frac{\partial \psi^T}{\partial z} \right] dv = 0 \qquad (165)$$

To ensure continuity of the field variable, in the next section we will choose appropriate shape functions. We will discuss this in the next chapter.

Integration by parts in three dimension is known as Gauss divergence theorem and is given by,

$$\int_v u \left(\nabla \bullet \mathbf{v} \right) dv = \int_A u \left(v \bullet \tilde{n} \right) dA - \int_A v \bullet \nabla u dv \qquad (166)$$

Applying equation (162) above, in equations (159 – 162) and re – arranging the terms, we obtain the weak form of 3 dimension Navier-Stokes equation as;

x-momentum

$$\int_v \rho \left[\psi\psi^T \frac{\partial \psi^T}{\partial t} \right] dv + \int_v \left[\rho \left(\psi \left(\psi^T u \right) \frac{\partial \psi^T}{\partial x} + \psi \left(\psi^T v \right) \frac{\partial \psi^T}{\partial y} + \psi \left(\psi^T w \right) \frac{\partial \psi^T}{\partial z} \right) u \right] dv$$
$$+ \int_v \left[2\mu \left[\frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial x} \right] + \mu \left(\frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial y} \right) + \mu \left(\frac{\partial \psi}{\partial z} \frac{\partial \psi^T}{\partial z} \right) \right] u dv \qquad (167)$$
$$+ \int_v \left[\mu \left[\frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial y} \right] v + \mu \left(\frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial z} \right) w \right] dv - \int_v \left[p \left(\frac{\partial \psi}{\partial x} \phi^T \right) \right] dv = \int_v \left[\psi\rho f_x \right] dv$$

y-momentum

$$\int_v \rho \left[\psi\psi^T \frac{\partial \psi^T}{\partial t} \right] dv + \int_v \rho \left(\psi \left(\psi^T u \right) \frac{\partial \psi^T}{\partial x} + \psi \left(\psi^T v \right) \frac{\partial \psi^T}{\partial y} + \psi \left(\psi^T w \right) \frac{\partial \psi^T}{\partial z} \right) v dv$$
$$+ \int_v \left[\mu \left[\frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial x} \right] + 2\mu \left(\frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial y} \right) + \mu \left(\frac{\partial \psi}{\partial z} \frac{\partial \psi^T}{\partial z} \right) \right] v dv \qquad (168)$$
$$+ \int_v \left[\mu \left[\frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial x} \right] u + \mu \left(\frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial z} \right) w \right] dv - \int_v \left[p \left(\frac{\partial \psi}{\partial y} \phi^T \right) \right] dv = \int_v \psi\rho f_y dv$$

z - momentum

$$\int_v \rho\left[\psi\psi^T \frac{\partial \psi^T}{\partial t}\right]dv+\int_v \rho\left(\psi\left(\psi^T u\right)\frac{\partial \psi^T}{\partial x}+\psi\left(\psi^T v\right)\frac{\partial \psi^T}{\partial y}+\psi\left(\psi^T w\right)\frac{\partial \psi^T}{\partial z}\right)wdv$$
$$+\int_v\left[\mu\left[\frac{\partial \psi}{\partial x}\frac{\partial \psi^T}{\partial x}\right]+\mu\left(\frac{\partial \psi}{\partial y}\frac{\partial \psi^T}{\partial y}\right)+2\mu\left(\frac{\partial \psi}{\partial z}\frac{\partial \psi^T}{\partial z}\right)\right]wdv \tag{169}$$
$$+\int_v\left[\mu\left[\frac{\partial \psi}{\partial z}\frac{\partial \psi^T}{\partial x}\right]u+\mu\left(\frac{\partial \psi}{\partial z}\frac{\partial \psi^T}{\partial y}\right)v\right]dv-\int_v\left[p\left(\frac{\partial \psi}{\partial z}\phi^T\right)\right]dv=\int_v \psi\rho f_z dv$$

continuity

$$\int_v \phi\left[\frac{\partial \psi^T}{\partial x}u+\frac{\partial \psi^T}{\partial y}v+\frac{\partial \psi^T}{\partial z}w\right]dv=0 \tag{170}$$

The above four equations are the weak form of the Navier – Stokes equations. To write the equations in matrix form we determine some coefficient matrix formulae as follows;

1. Mass matrix

$$M=\int_v \rho\psi\psi^T dv \tag{171}$$

2. Convective matrix

$$C(u,v,w)=\int_v \rho\left(\psi\left(\psi^T u\right)\frac{\partial \psi^T}{\partial x}+\psi\left(\psi^T v\right)\frac{\partial \psi^T}{\partial y}+\psi\left(\psi^T w\right)\frac{\partial \psi^T}{\partial z}\right)dv \tag{172}$$

3. Diffusive matrix

$$K_{ij}=\int_v \mu\left(\frac{\partial \psi}{\partial x_i}\frac{\partial \psi^T}{\partial x_j}\right)dv \tag{173}$$

4. Gradient matrix

$$Q_i=\int_v \frac{\partial \psi}{\partial x_i}\phi^T dv \tag{174}$$

5. Force vector

$$K_{ij}=\int_v \rho\psi f_i dv \tag{175}$$

Substituting the above coefficient matrix formulae in the above equations, we obtain the following matrix form of the weak statement;

$$\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} C(u,v,w) & 0 & 0 & 0 \\ 0 & C(u,v,w) & 0 & 0 \\ 0 & 0 & C(u,v,w) & 0 \\ 0 & 0 & 0 & C(u,v,w) \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ p \end{Bmatrix}$$

$$+ \begin{bmatrix} 2K_{11} + K_{22} + K_{33} & K_{13} & K_{13} & Q_1 \\ K_{21} & K_{11} + 2K_{22} + K_{33} & K_{23} & Q_2 \\ K_{31} & K_{32} & K_{11} + K_{22} + 2K_{33} & Q_3 \\ Q_1^T & Q_2^T & Q_3^T & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ p \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \end{Bmatrix} \tag{176}$$

From the above equation, we notice that;

1. There is a zero appearing in the diagonal of the mass matrix due to incompressibility constrain
2. The convection matrix contains non-linear terms; this can lead to unstable solution in convection dominated flows.

In order to solve the matrix above we start by addressing the two above problems; to do this we will use the methods of artificial compressibility and the Galerkin Stabilization method.

Artificial compressibility

The zero appearing on the diagonal is due to the incompressibility condition, $\nabla \bullet \mathbf{u} = \mathbf{0}$.The idea behind this technique is to convert an elliptical problem into hyperbolic by introducing an artificial term in the continuity equation. Using this technique the alternative form of the continuity equation (1) is given by;

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

The equation above is the artificial continuity equation,

ρ is the artificial density and β artificial compresibility such that $P = \frac{\rho}{\beta}$. The additional term in the equation above will not greatly affect the solutions since as calculation progresses and become independent of time the term will disappear.

The weak form of the above equation becomes;

$$\int_v Q\left(\beta \frac{\partial P}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dv = 0 \tag{177}$$

We define a new matrix for this artificial equation. From the above equation the pressure matrix is $M_p = \int_v \beta \phi \phi^T dv$. Therefore the matrix (173) becomes

$$\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M_p \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} C(u,v,w) & 0 & 0 & 0 \\ 0 & C(u,v,w) & 0 & 0 \\ 0 & 0 & C(u,v,w) & 0 \\ 0 & 0 & 0 & C(u,v,w) \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ p \end{Bmatrix}$$

$$+ \begin{bmatrix} 2K_{11} + K_{22} + K_{33} & K_{13} & K_{13} & -Q_1 \\ K_{21} & K_{11} + 2K_{22} + K_{33} & K_{23} & -Q_2 \\ K_{31} & K_{32} & K_{11} + K_{22} + 2K_{33} & -Q_3 \\ Q_1^T & Q_2^T & Q_3^T & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ p \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \end{Bmatrix} \quad (178)$$

This matrix is now not singular.

.

$$\int_v \phi \left(\beta \phi^T \frac{\partial P}{\partial t} + \frac{\partial \psi^T}{\partial x} u + \frac{\partial \psi^T}{\partial y} v + \frac{\partial \psi^T}{\partial z} \right) dv = 0 \quad (179)$$

Therefore $K_{ij}^{33} \neq 0, \;\; K_{ij}^{33} = \int_\Omega \tilde{s}_j \frac{\partial \tilde{s}_j}{\partial t} d\Omega$, the matrix will now not be singular

Taylor Galerkin stabilization technique

Incompressible Navier – Stokes equations has non – linear unsymmetrical convective terms. As the Reynold number increases in high velocity flows, these terms start dominating the flow field inducing oscillations in it thus making the solutions unstable.

This method involves writing higher order time derivative of Taylor series in terms of spatial derivatives from governing partial differential equations. Taylor Galerkin is applied only on convective term only in Navier – Stokes – equation. The derivation of the Taylor Galerkin method for Navier- Stokes equations is as shown in appendix A.

We will consider derivation of Taylor Galerkin technique for Navier – Stoke equation. To avoid tediousness during derivation, we consider the vector form of the Navier – Stokes equations. Although this method can be applied to the entire N-S-E, only convective terms are considered here as it's the only term which brings stability issues.

Taylor series expansion for n+1th time step is given by;

$$u^{n+1} = u^n + \triangle t \frac{\partial u^n}{\partial t} + \frac{\triangle t^2}{2} \frac{\partial^2 u^n}{\partial t^2} + . \;\; . \;\; . \quad (180)$$

The incompressible N-S-E which consists of only of convective terms are given by,

$$u^{n+1} = u^n + \vartriangle t \frac{\partial u^n}{\partial t} + \frac{\vartriangle t^2}{2} \frac{\partial^2 u^n}{\partial t^2} + . \ . \ . \tag{181}$$

The incompressible N-S-E consisting of only convective terms are given

$$\frac{\partial u}{\partial t} = -\mathbf{u} \bullet (\nabla \mathbf{u}) \tag{182}$$

The second derivative of u with respect to t is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(-\mathbf{u} \bullet \nabla)\mathbf{u} = -\mathbf{u} \bullet \nabla \left(\frac{\partial u}{\partial t} \right) = \mathbf{u} \bullet \nabla (\mathbf{u} . \nabla) \mathbf{u} \tag{183}$$

$$\frac{\partial^2 u}{\partial t^2} = \mathbf{u} \bullet \nabla (\mathbf{u} . \nabla) \mathbf{u} \tag{184}$$

Substituting (185) and (183) into (182), we obtain;

$$\frac{u^{n+1} - u^n}{\Delta t} = -\mathbf{u} \bullet \nabla \mathbf{u} + \frac{\Delta t}{2} \mathbf{u} . \nabla (\mathbf{u} \bullet \nabla) \mathbf{u} . \ . \ . \tag{185}$$

Comparing (186) and (183), the stabilization term obtained by Taylor Galerkin method is a strong form of Navier-Stokes equation. Thus we will obtain the weak form using the method of weighted residual and applying integration by parts (Gauss theorem), therefore we obtain;

$$\begin{aligned} -\frac{\Delta t w}{2} \int_v \mathbf{u} . \nabla (\mathbf{u} \bullet \nabla \mathbf{u}) &= - - \frac{\Delta t w}{2} \int_v (\mathbf{u} . \nabla \mathbf{u})(\nabla \bullet \mathbf{u}) \, dv \\ &= \int_v w \left[\left(u \frac{\partial \mathbf{u}}{\partial x} + v \frac{\partial \mathbf{u}}{\partial y} + w \frac{\partial \mathbf{u}}{\partial z} \right) \left(\frac{\partial \mathbf{v}}{\partial x} + \frac{\partial \mathbf{v}}{\partial y} + \frac{\partial \mathbf{v}}{\partial z} \right) \right] dv \end{aligned} \tag{186}$$

Thus the expression of Taylor Galerkin stabilization is given by;

$$\begin{aligned} \mathrm{K}_{\mathrm{Sec}}(u,v,w) = \frac{\Delta t w}{2} \int_v \psi \psi^T \Bigg[& uu \frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial x} + uv \frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial y} + uw \frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial z} + uv \frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial x} \\ & + vv \frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial y} + vw \frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial z} + uw \frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial z} + vw \frac{\partial \psi}{\partial z} \frac{\partial \psi^T}{\partial y} + ww \frac{\partial \psi}{\partial z} \frac{\partial \psi^T}{\partial z} \Bigg] dv \end{aligned} \tag{187}$$

Thus the stabilization matrix is written as

$$\begin{aligned} \mathrm{K}_{\mathrm{Sec}}(u,v,w) = \frac{\Delta t}{2} \int_v \psi \psi^T \Bigg[& uu \frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial x} + uv \frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial y} + uw \frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial z} + uv \frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial x} \\ & + vv \frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial y} + vw \frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial z} + uw \frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial z} + vw \frac{\partial \psi}{\partial z} \frac{\partial \psi^T}{\partial y} + ww \frac{\partial \psi}{\partial z} \frac{\partial \psi^T}{\partial z} \Bigg] dv \end{aligned} \tag{188}$$

Therefore the stabilization matrix become will become;

$$\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M_p \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} C(u,v,w) & 0 & 0 & 0 \\ 0 & C(u,v,w) & 0 & 0 \\ 0 & 0 & C(u,v,w) & 0 \\ 0 & 0 & 0 & C(u,v,w) \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ p \end{Bmatrix}$$

$$+ \begin{bmatrix} 2K_{11} + K_{22} + K_{33} & K_{13} & K_{13} & -Q_1 \\ K_{21} & K_{11} + 2K_{22} + K_{33} & K_{23} & -Q_2 \\ K_{31} & K_{32} & K_{11} + K_{22} + 2K_{33} & -Q_3 \\ Q_1^T & Q_2^T & Q_3^T & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ p \end{Bmatrix} \qquad (189)$$

$$\begin{bmatrix} K_{TG}(u,v,w) & 0 & 0 & 0 \\ 0 & K_{TG}(u,v,w) & 0 & 0 \\ 0 & 0 & K_{TG}(u,v,w) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ p \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \end{Bmatrix}$$

This is the final weak form of incompressible N-S-E which will be discretized . Now in the next section we concentrate on finding suitable interpolation functions, ϕ and ψ .

Choice of interpolation functions

Accuracy and the computational efficiency of solutions depend mostly on the choice of shape functions. The choice of interpolation function for pressure is constrained by the role of pressure. The pressure here serves only as a Lagrange multiplier that serves to enforce the incompressibility constraint on the velocity field, therefore in order to prevent an over-constrained system of discretized equations; interpolation used for pressure must be at least one order lower than that of velocity.

Therefore accurate solutions can only be obtained for both velocity and pressure fields by using unequal interpolation in such a way that the shape functions associated with velocity variables are one order higher than those associated with pressure. Both pressure and velocity need to be at least C^1 continuous. In this project, first order (linear) shape functions ϕ is selected for pressure and second order quadratic shape function ψ is selected for velocity variables.

To get the shape function, in this section we consider a tetrahedral element, however we will not derive the shape functions, and we shall just import them. The quadratic shape function ψ is obtained from 10 node tetrahedron as shown;

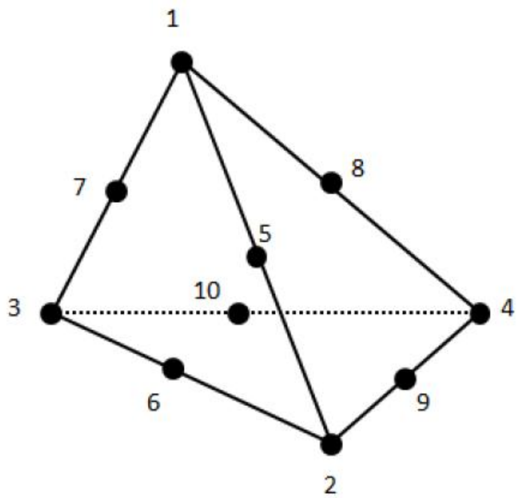


Figure 11: A ten node tetrahedron element

The quadratic coordinate L_i is given as;

$$\psi = \begin{Bmatrix} L_1(2l_1 - 1) \\ L_2(2l_2 - 1) \\ L_3(2l_3 - 1) \\ L_4(2l_4 - 1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_1L_3 \\ 4L_1L_4 \\ 4L_2L_4 \\ 4L_3L_4 \end{Bmatrix} \tag{190}$$

Similarly the linear shape functions ϕ is obtained from a 4 node tetra element as shown below,

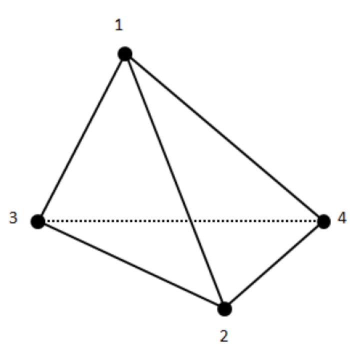


Figure 12: A four node tetra element

$$\phi = \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} \tag{191}$$

The natural coordinates L_1, L_2, L_3 and L_4 are functions of the Cartesian coordinates and are given as shown below;

$$L_1 = \left(a_1 + b_1 x + c_1 y + d_1 z\right)/6V, \qquad L_2 = \left(a_2 + b_2 x + c_2 y + d_2 z\right)/6V$$
$$L_3 = \left(a_3 + b_3 x + c_3 y + d_3 z\right)/6V, \qquad L_4 = \left(a_4 + b_4 x + c_4 y + d_4 z\right)/6V \tag{192}$$

The derivatives of the above functions are given by;

$$\frac{\partial L_i}{\partial x_j} = \frac{b_i}{6v}, \text{ where j = x, y, z and i = 1, 2, . . 4} \tag{193}$$

In the equations above the coefficients a_i, b_i, c_i and d_{id} are given by;

$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad b_1 = \begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}$$
$$C_1 = \begin{vmatrix} x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \\ x_4 & 1 & z_4 \end{vmatrix} \qquad d_1 = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} \tag{194}$$

Other constants can be calculated by using cyclic permutation of subscripts 1, 2, 3, and 4 is defined counter-clockwise.

The volume V of an element can be found by using

$$V = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} \tag{195}$$

Now after gathering the necessary information about the quadratic and linear shape functions we substituted in the coefficient matrices. Before we replace the shape functions into the coefficient matrices, we first transform them into matrices.

Transformation of interpolation functions

Form (191), the vector form of interpolation function can be transformed to a new matrix that consists of the coefficient matrix [A] and the vector form of matrix {R} as expressed below;

$$\psi = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}}_{[\mathrm{A}]} \underbrace{\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_1 L_2 \\ L_2 L_3 \\ L_1 L_3 \\ L_1 L_4 \\ L_2 L_4 \\ L_3 L_4 \end{Bmatrix}}_{\{R\}} \tag{196}$$

Coefficient matrices construction

1. Mass matrix $M = \int_v \rho \psi \psi^T dv$. We take the new matrix transformation into the mass matrix and integrate this form by using the exact integral formula of

   $$M = \int_v L_1^a L_2^b L_3^c dA = \frac{a!b!c!}{(a+b+c+2)!} 2A$$

2. The mass matrix M_P $M_p = \frac{1}{\beta} \int_v \phi \phi^T dv$

   From the above matrix we find that;

$$\phi\phi^T = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix} \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \end{bmatrix} = \begin{bmatrix} L_1^2 & L_1 L_2 & L_1 L_3 & L_1 L_4 \\ L_2 L_1 & L_2^2 & L_1 L_2 & L_2 L_4 \\ L_3 L_1 & L_3 L_2 & L_3^2 & L_3 L_4 \\ L_4 L_1 & L_4 L_2 & L_4 L_3 & L_4^2 \end{bmatrix} \tag{197}$$

we will let

$$G = \int_v \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 & L_1L_4 \\ L_2L_1 & L_2^2 & L_1L_2 & L_2L_4 \\ L_3L_1 & L_3L_2 & L_3^2 & L_3L_4 \\ L_4L_1 & L_4L_2 & L_4L_3 & L_4^2 \end{bmatrix} dv = \frac{A}{12}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \tag{198}$$

$$M_p = \frac{1}{\beta}\int_v \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 & L_1L_4 \\ L_2L_1 & L_2^2 & L_1L_2 & L_2L_4 \\ L_3L_1 & L_3L_2 & L_3^2 & L_3L_4 \\ L_4L_1 & L_4L_2 & L_4L_3 & L_4^2 \end{bmatrix} dv = \frac{1}{\beta}G \tag{199}$$

3. Diffusive matrix

$$K_{ij} = \int_v \mu\left(\frac{\partial\psi}{\partial x_i}\frac{\partial\psi^T}{\partial x_j}\right)dv \quad \text{i,j = x, y and z} \tag{200}$$

$$\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\begin{Bmatrix} L_1(2l_1-1) \\ L_2(2l_2-1) \\ L_3(2l_3-1) \\ L_4(2l_4-1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_1L_3 \\ 4L_1L_4 \\ 4L_2L_4 \\ 4L_3L_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}}_{[A]} \frac{\partial}{\partial x}\underbrace{\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_1L_2 \\ L_2L_3 \\ L_1L_3 \\ L_1L_4 \\ L_2L_4 \\ L_3L_4 \end{Bmatrix}}_{\{R\}}$$

$$\frac{\partial\psi}{\partial x} = [A]\frac{\partial}{\partial x}\underbrace{\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_1L_2 \\ L_2L_3 \\ L_1L_3 \\ L_1L_4 \\ L_2L_4 \\ L_3L_4 \end{Bmatrix}}_{\{R\}} = [A]\underbrace{\begin{Bmatrix} 2L_1b_1 \\ 2L_2b_2 \\ 2L_3b_3 \\ 2L_4b_4 \\ L_1b_2 + L_2b_1 \\ L_2b_3 + L_3b_2 \\ L_1b_3 + L_3b_1 \\ L_1b_4 + L_4b_1 \\ L_2b_4 + L_4b_1 \\ L_3b_4 + L_4b_3 \end{Bmatrix}}_{\{R\}} = [A]\underbrace{\begin{bmatrix} 2b_1 & 0 & 0 & 0 \\ 0 & 2b_2 & 0 & 0 \\ 0 & 0 & 2b_3 & 0 \\ 0 & 0 & 0 & 2b_4 \\ b_2 & b_1 & 0 & 0 \\ 0 & b_3 & b_2 & 0 \\ b_4 & 0 & b_1 & 0 \\ b_4 & 0 & 0 & b_1 \\ 0 & b_4 & 0 & b_2 \\ 0 & 0 & b_4 & b_3 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix}}_{H}$$

Therefore, $\frac{\partial\psi}{\partial x} = [A][B][H]$ (201)

Similarly,

$$\frac{\partial\psi}{\partial y} = [\mathrm{A}][\mathrm{C}][\mathrm{H}] \quad , \quad \frac{\partial\psi}{\partial x} = [\mathrm{A}][\mathrm{D}][\mathrm{H}] \tag{202}$$

Therefore equation (201) can be written as

$$K_{11} = \int_v \mu\left(\frac{\partial\psi}{\partial x}\frac{\partial\psi^T}{\partial x}\right)dv = \mu[\mathrm{A}][\mathrm{B}]\left[\mathrm{A}^{\mathrm{T}}\right]\left[\mathrm{B}^{\mathrm{T}}\right]\int_A [\mathrm{H}]\left[\mathrm{H}^{\mathrm{T}}\right]d\mathrm{A} \tag{203}$$

From equation (199) equation (204) can be written as

$$K_{11} = \mu[\mathrm{A}][\mathrm{B}][\mathrm{G}]\left[\mathrm{A}^{\mathrm{T}}\right]\left[\mathrm{B}^{\mathrm{T}}\right] \tag{204}$$

Similarly

$$\begin{aligned}
&K_{22} = \mu[\mathrm{A}][\mathrm{C}][\mathrm{G}]\left[\mathrm{C}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right] \quad K_{33} = \mu[\mathrm{A}][\mathrm{D}][\mathrm{G}]\left[\mathrm{D}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right] \quad K_{12} = \mu[\mathrm{A}][\mathrm{B}][\mathrm{G}]\left[\mathrm{C}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right]\\
&K_{32} = \mu[\mathrm{A}][\mathrm{D}][\mathrm{G}]\left[\mathrm{C}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right] \quad K_{23} = \mu[\mathrm{A}][\mathrm{C}][\mathrm{G}]\left[\mathrm{D}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right] \quad K_{13} = \mu[\mathrm{A}][\mathrm{B}][\mathrm{G}]\left[\mathrm{D}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right]\\
&K_{21} = \mu[\mathrm{A}][\mathrm{C}][\mathrm{G}]\left[\mathrm{A}^{\mathrm{T}}\right]\left[\mathrm{B}^{\mathrm{T}}\right] \quad K_{31} = \mu[\mathrm{A}][\mathrm{D}][\mathrm{G}]\left[\mathrm{A}^{\mathrm{T}}\right]\left[\mathrm{B}^{\mathrm{T}}\right]
\end{aligned} \tag{205}$$

4. The stabilized mass matrix

$$\begin{aligned}
\mathrm{K}_{\mathrm{Sec}}(u,v,w) = \frac{\Delta t}{2}\int_v \Bigg[& uu\frac{\partial\psi}{\partial x}\frac{\partial\psi^T}{\partial x} + uv\frac{\partial\psi}{\partial x}\frac{\partial\psi^T}{\partial y} + uw\frac{\partial\psi}{\partial x}\frac{\partial\psi^T}{\partial z} + uv\frac{\partial\psi}{\partial y}\frac{\partial\psi^T}{\partial x}\\
& + vv\frac{\partial\psi}{\partial y}\frac{\partial\psi^T}{\partial y} + vw\frac{\partial\psi}{\partial y}\frac{\partial\psi^T}{\partial z} + uw\frac{\partial\psi}{\partial x}\frac{\partial\psi^T}{\partial z} + vw\frac{\partial\psi}{\partial z}\frac{\partial\psi^T}{\partial y} + ww\frac{\partial\psi}{\partial z}\frac{\partial\psi^T}{\partial z}\Bigg]dv
\end{aligned} \tag{206}$$

$$\begin{aligned}
K_{se}(u,v) = \frac{\Delta t}{2}\Big[& uu[\mathrm{A}][\mathrm{B}][\mathrm{G}]\left[\mathrm{A}^{\mathrm{T}}\right]\left[\mathrm{B}^{\mathrm{T}}\right] + uv[\mathrm{A}][\mathrm{B}][\mathrm{G}]\left[\mathrm{C}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right] + uw[\mathrm{A}][\mathrm{B}][\mathrm{G}]\left[\mathrm{D}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right]\\
& + uv[\mathrm{A}][\mathrm{C}][\mathrm{G}]\left[\mathrm{A}^{\mathrm{T}}\right]\left[\mathrm{B}^{\mathrm{T}}\right] + vv[\mathrm{A}][\mathrm{C}][\mathrm{G}]\left[\mathrm{C}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right] + vw[\mathrm{A}][\mathrm{C}][\mathrm{G}]\left[\mathrm{D}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right]\\
& + wu[\mathrm{A}][\mathrm{D}][\mathrm{G}]\left[\mathrm{A}^{\mathrm{T}}\right]\left[\mathrm{B}^{\mathrm{T}}\right] + vw[\mathrm{A}][\mathrm{D}][\mathrm{G}]\left[\mathrm{C}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right] + ww[\mathrm{A}][\mathrm{D}][\mathrm{G}]\left[\mathrm{D}^{\mathrm{T}}\right]\left[\mathrm{A}^{\mathrm{T}}\right]\Big]
\end{aligned} \tag{207}$$

5. Stiffness gradient matrix, $Q_i = \int_v \frac{\partial\psi}{\partial x_i}\phi^T dv$

$$Q_1 = \int_v \frac{\partial \psi}{\partial x} \phi^T dv = [\mathrm{A}][\mathrm{B}]\int_V [\mathrm{H}][\mathrm{H}]^T dv = [\mathrm{A}][\mathrm{B}][\mathrm{G}]$$

$$Q_2 = \int_v \frac{\partial \psi}{\partial y} \phi^T dv = [\mathrm{A}][\mathrm{C}]\int_V [\mathrm{H}][\mathrm{H}]^T dv = [\mathrm{A}][\mathrm{C}][\mathrm{G}] \tag{208}$$

$$Q_3 = \int_v \frac{\partial \psi}{\partial z} \phi^T dv = [\mathrm{A}][\mathrm{D}]\int_V [\mathrm{H}][\mathrm{H}]^T dv = [\mathrm{A}][\mathrm{D}][\mathrm{G}]$$

Mass convective matrix

$$C(u,v,w) = \int_v \rho \left(\psi\left(\psi^T u\right)\frac{\partial \psi^T}{\partial x} + \psi\left(\psi^T v\right)\frac{\partial \psi^T}{\partial y} + \psi\left(\psi^T w\right)\frac{\partial \psi^T}{\partial z} \right) dv \tag{209}$$

The convective matrices are given by;

$$C_x(u) = \int_v \rho\psi\left(\psi^T u\right)\frac{\partial \psi^T}{\partial x} dv = [\mathrm{A}][\mathrm{A}]^{\mathrm{T}}[\mathrm{A}][\mathrm{B}]\int_V [\mathrm{R}][\mathrm{R}]^{\mathrm{T}}[v][\mathrm{H}]dv$$

$$C_y(v) = \int_v \rho\psi\left(\psi^T v\right)\frac{\partial \psi^T}{\partial y} dv = [\mathrm{A}][\mathrm{A}]^{\mathrm{T}}[\mathrm{A}][\mathrm{C}]\int_V [\mathrm{R}][\mathrm{R}]^{\mathrm{T}}[v][\mathrm{H}]dv \tag{210}$$

$$C_z(w) = \int_v \rho\psi\left(\psi^T w\right)\frac{\partial \psi^T}{\partial z} dv = [\mathrm{A}][\mathrm{A}]^{\mathrm{T}}[\mathrm{A}][\mathrm{D}]\int_V [\mathrm{R}][\mathrm{R}]^{\mathrm{T}}[w][\mathrm{H}]dv$$

Or

$$C_x(u) = \frac{2A}{540}[\mathrm{A}][\mathrm{A}]^{\mathrm{T}}[\mathrm{A}][\mathrm{B}][\mathrm{F}] \qquad C_y(v) = \frac{2A}{540}[\mathrm{A}][\mathrm{A}]^{\mathrm{T}}[\mathrm{A}][\mathrm{C}][\mathrm{F}]$$

$$C_z(w) = \frac{2A}{540}[\mathrm{A}][\mathrm{A}]^{\mathrm{T}}[\mathrm{A}][\mathrm{D}][\mathrm{F}] \tag{211}$$

Now this is the final general form of the finite element matrix formula.

Equation (190) still contains non-linear terms. To line arise the non-linear terms we use the Newton Raphson method on the non-linear term only,

$$\begin{bmatrix} \mathrm{C}+2\mathrm{K}_{11}+\mathrm{K}_{22}+\mathrm{K}_{33}+K_{se} & \mathrm{K}_{12} & \mathrm{K}_{13} & -Q_1 \\ \mathrm{K}_{21} & \mathrm{C}+\mathrm{K}_{11}+2\mathrm{K}_{22}+\mathrm{K}_{33}+K_{se} & \mathrm{K}_{23} & -Q_2 \\ \mathrm{K}_{31} & \mathrm{K}_{32} & \mathrm{C}+\mathrm{K}_{11}+\mathrm{K}_{22}+2\mathrm{K}_{33}+K_{se} & -Q_3 \\ Q_1^T & Q_2^T & Q_3^T & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ p \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \end{Bmatrix} \tag{212}$$

Or

$$\left[K(u)\right]\mathrm{U}=\left\{F\right\} \tag{213}$$

We now transform the equation above into Newton – Raphson form as,

$$R(u)=\left[K(u)\right]\mathrm{U}\text{-}\left\{F\right\}=0 \tag{214}$$

Newton-Raphson method is based on truncated Taylor series expansion of R(U) about the known solution U^n

$$0=R(U)+\frac{\partial R}{\partial U}\Delta U+0\left(\Delta U^2\right) \tag{215}$$

Where, $\Delta U=U^{n+1}-U^n$. Omitting the terms of order higher, we obtain,

$$R(U^n)=-\frac{\partial R}{\partial U}\left(U^{n+1}-U^n\right)\cong -J\left(U^n\right)\left(U^{n+1}-U^n\right) \tag{216}$$

Where J is the Jacobean

By solving equation (217), we get

$$U^{n+1}=U^n-J^{-1}\left(U^n\right)R\left(U^n\right) \tag{217}$$

Applying equation (214) to (216), we obtain,

$$\begin{aligned}
R_1 &= C_1(u)u+C_2(v)u+C_3(w)u+\left(2K_{11}+K_{22}+K_{33}\right)u+K_{12}v+K_{13}w-Q_1p-F_1\\
&\quad+\frac{\Delta t}{2}\Big[K_{s_{11}}(uu)u+K_{S_{12}}(uv)u+K_{S_{21}}(vu)u+K_{S_{22}}(vv)u\\
&\qquad+K_{S_{13}}(uw)u+K_{S_{23}}(vw)u+K_{S_{31}}(wu)u+K_{S_{32}}(wv)u+K_{S_{33}}(ww)u\Big]\\
R_2 &= C_1(u)v+C_2(v)v+C_3(w)v+\left(K_{11}+2K_{22}+K_{33}\right)v+K_{21}u+K_{23}w-Q_1p-F_2\\
&\quad+\frac{\Delta t}{2}\Big[K_{s_{11}}(uu)v+K_{S_{12}}(uv)v+K_{S_{21}}(vu)v+K_{S_{22}}(vv)v\\
&\qquad+K_{S_{13}}(uw)v+K_{S_{23}}(vw)v+K_{S_{31}}(wu)v+K_{S_{32}}(wv)v+K_{S_{33}}(ww)v\Big]\\
R_3 &= C_1(u)w+C_2(v)w+C_3(w)w+\left(K_{11}+K_{22}+2K_{33}\right)w+K_{31}u+K_{32}v-Q_1p-F_3\\
&\quad+\frac{\Delta t}{2}\Big[K_{s_{11}}(uu)w+K_{S_{12}}(uv)w+K_{S_{21}}(vu)w+K_{S_{22}}(vv)w\\
&\qquad+K_{S_{13}}(uw)w+K_{S_{23}}(vw)w+K_{S_{31}}(wu)w+K_{S_{32}}(wv)w+K_{S_{33}}(ww)w\Big]
\end{aligned} \tag{218}$$

The Jacobian in (218) is given by;

$$J = -\frac{\partial R}{\partial U} \tag{219}$$

Now to evaluate the above matrix, we substitute (220) into it. We get the derivative of the stiffness matrix in the Newton-Raphson form to be,

$$J = -\frac{\partial R}{\partial U} = -\begin{bmatrix} \frac{\partial R_1}{\partial u} & \frac{\partial R_1}{\partial v} & \frac{\partial R_1}{\partial w} & \frac{\partial R_1}{\partial p} \\ \frac{\partial R_2}{\partial u} & \frac{\partial R_2}{\partial v} & \frac{\partial R_2}{\partial w} & \frac{\partial R_2}{\partial p} \\ \frac{\partial R_3}{\partial u} & \frac{\partial R_3}{\partial v} & \frac{\partial R_3}{\partial w} & \frac{\partial R_3}{\partial p} \\ \frac{\partial R_4}{\partial u} & \frac{\partial R_4}{\partial v} & \frac{\partial R_4}{\partial w} & \frac{\partial R_4}{\partial p} \end{bmatrix} \tag{220}$$

$$\begin{aligned} \frac{\partial R_1}{\partial u} &= C_1(u) + C_2(v) + C_3(w) + (2K_{11} + K_{22} + K_{33}) \\ &\quad + \frac{\Delta t}{2}\Big[K_{s_{11}}(uu) + K_{S_{12}}(uv) + K_{S_{21}}(vu) + K_{S_{22}}(vv) \\ &\qquad + K_{S_{13}}(uw) + K_{S_{23}}(vw) + K_{S_{31}}(wu) + K_{S_{32}}(wv) + K_{S_{33}}(ww)\Big] \\ \frac{\partial R_2}{\partial v} &= C_1(u) + C_2(v) + C_3(w) + (K_{11} + 2K_{22} + K_{33}) \\ &\quad + \frac{\Delta t}{2}\Big[K_{s_{11}}(uu) + K_{S_{12}}(uv) + K_{S_{21}}(vu) + K_{S_{22}}(vv) \\ &\qquad + K_{S_{13}}(uw) + K_{S_{23}}(vw) + K_{S_{31}}(wu) + K_{S_{32}}(wv) + K_{S_{33}}(ww)\Big] \\ \frac{\partial R_3}{\partial w} &= C_1(u) + C_2(v) + C_3(w) + (K_{11} + K_{22} + 2K_{33}) \\ &\quad + \frac{\Delta t}{2}\Big[K_{s_{11}}(uu) + K_{S_{12}}(uv) + K_{S_{21}}(vu) + K_{S_{22}}(vv) \\ &\qquad + K_{S_{13}}(uw) + K_{S_{23}}(vw) + K_{S_{31}}(wu) + K_{S_{32}}(wv) + K_{S_{33}}(ww)\Big] \end{aligned} \tag{221}$$

Similarly we derive,

$$\frac{\partial R_1}{\partial v}, \frac{\partial R_3}{\partial v}, \frac{\partial R_4}{\partial v}, \frac{\partial R_2}{\partial u}, \frac{\partial R_3}{\partial u}, \frac{\partial R_4}{\partial u}, \frac{\partial R_1}{\partial w}, \frac{\partial R_2}{\partial w}, \frac{\partial R_3}{\partial w}, \frac{\partial R_1}{\partial p}, \frac{\partial R_2}{\partial p}, \frac{\partial R_3}{\partial p}$$

We now substitute (220) into (218) to get;

$$R\left(U^{n}\right)=-\begin{bmatrix}\frac{\partial R_1}{\partial u} & \frac{\partial R_1}{\partial v} & \frac{\partial R_1}{\partial w} & \frac{\partial R_1}{\partial p}\\ \frac{\partial R_2}{\partial u} & \frac{\partial R_2}{\partial v} & \frac{\partial R_2}{\partial w} & \frac{\partial R_2}{\partial p}\\ \frac{\partial R_3}{\partial u} & \frac{\partial R_3}{\partial v} & \frac{\partial R_3}{\partial w} & \frac{\partial R_3}{\partial p}\\ \frac{\partial R_4}{\partial u} & \frac{\partial R_4}{\partial v} & \frac{\partial R_4}{\partial w} & \frac{\partial R_4}{\partial p}\end{bmatrix}\left(U^{n+1}-U^{n}\right) \tag{222}$$

Thus (214) can be replaced by;

$$\begin{bmatrix}\frac{\partial R_1}{\partial u} & \frac{\partial R_1}{\partial v} & \frac{\partial R_1}{\partial w} & \frac{\partial R_1}{\partial p}\\ \frac{\partial R_2}{\partial u} & \frac{\partial R_2}{\partial v} & \frac{\partial R_2}{\partial w} & \frac{\partial R_2}{\partial p}\\ \frac{\partial R_3}{\partial u} & \frac{\partial R_3}{\partial v} & \frac{\partial R_3}{\partial w} & \frac{\partial R_3}{\partial p}\\ \frac{\partial R_4}{\partial u} & \frac{\partial R_4}{\partial v} & \frac{\partial R_4}{\partial w} & \frac{\partial R_4}{\partial p}\end{bmatrix}\begin{bmatrix}\Delta u^{n+1}\\ \Delta v^{n+1}\\ \Delta w^{n+1}\\ \Delta p^{n+1}\end{bmatrix}=\begin{bmatrix}R_1\\ R_2\\ R_3\\ R_4\end{bmatrix} \quad \text{where, } \begin{bmatrix}\Delta u^{n+1}\\ \Delta v^{n+1}\\ \Delta w^{n+1}\\ \Delta p^{n+1}\end{bmatrix}=\begin{bmatrix}u^{n+1}-u^{n}\\ v^{n+1}-v^{n}\\ w^{n+1}-w^{n}\\ p^{n+1}-p^{n}\end{bmatrix} \tag{223}$$

We substitute (225) into (190) to obtain

$$\begin{bmatrix}\mathrm{M} & 0 & 0 & 0\\ 0 & M & 0 & 0\\ 0 & 0 & M & 0\\ 0 & 0 & 0 & M_p\end{bmatrix}\begin{Bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\\ \dot{p}\end{Bmatrix}+\begin{bmatrix}\frac{\partial R_1}{\partial u} & \frac{\partial R_1}{\partial v} & \frac{\partial R_1}{\partial w} & \frac{\partial R_1}{\partial p}\\ \frac{\partial R_2}{\partial u} & \frac{\partial R_2}{\partial v} & \frac{\partial R_2}{\partial w} & \frac{\partial R_2}{\partial p}\\ \frac{\partial R_3}{\partial u} & \frac{\partial R_3}{\partial v} & \frac{\partial R_3}{\partial w} & \frac{\partial R_3}{\partial p}\\ \frac{\partial R_4}{\partial u} & \frac{\partial R_4}{\partial v} & \frac{\partial R_4}{\partial w} & \frac{\partial R_4}{\partial p}\end{bmatrix}\begin{bmatrix}\Delta u^{n+1}\\ \Delta v^{n+1}\\ \Delta w^{n+1}\\ \Delta p^{n+1}\end{bmatrix}=-\begin{bmatrix}R_1\\ R_2\\ R_3\\ R_4\end{bmatrix} \tag{224}$$

Or $\left[M\right]\dot{\mathbf{U}}+\left[K\mathbf{U}\right]\Delta\mathbf{U}=\left\{\mathbf{R}\right\}$ (225)

To derive the solution of the time approximation method using implicit method, we start by time derivative term in the above equation

$$\dot{\mathbf{U}}=\frac{\partial U}{\partial t}=\frac{U^{n+1}-U^{n}}{\Delta t} \tag{226}$$

Substituting (228) into (227) we obtain,

$$[M]U^{n+1} = \Delta t\left[R\left(U^{n+1}\right) - K\left(U^{n+1}\right)\left(U^{n+1}\right)\right] + [M]U^{n} \qquad (227)$$

Once we have obtained the final form of the solution of Navier-Stokes equation, we now apply it to flows through pipe.

Finite element solutions and programming

Now after obtaining the final form of Navier – Stokes equations, we use a matlab program to solve the equation. The Matlab program is given in appendix A. to test the Matlab program, we first apply it to lid driven cavity problem.

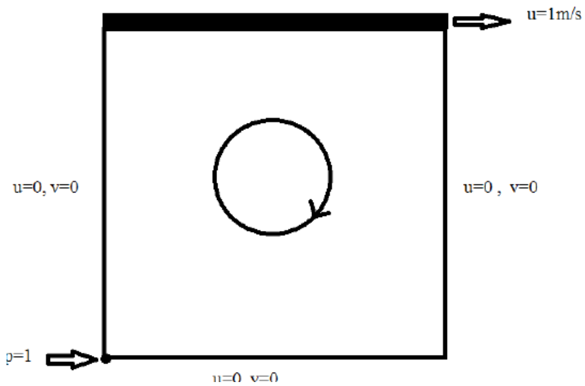


Figure 13: Lid driven cavity problem

4. Results and Conclusion

After applying the computer program developed to lid-driven the following results were obtained.

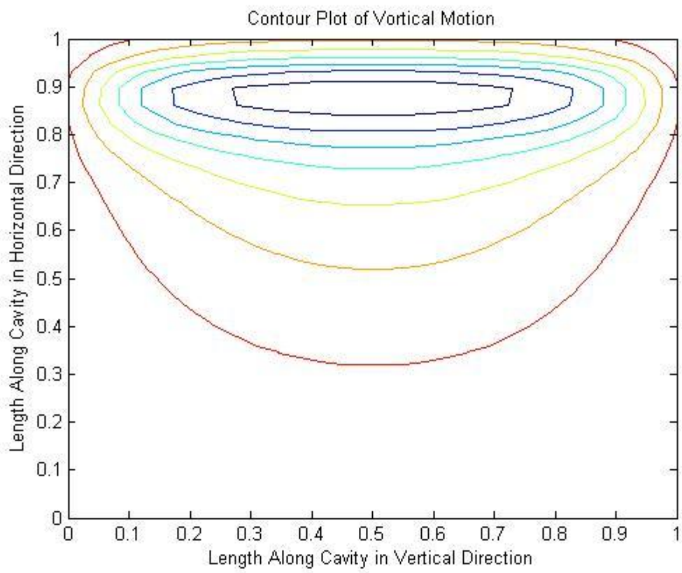


Figure 14: Solutions to lid driven cavity problem

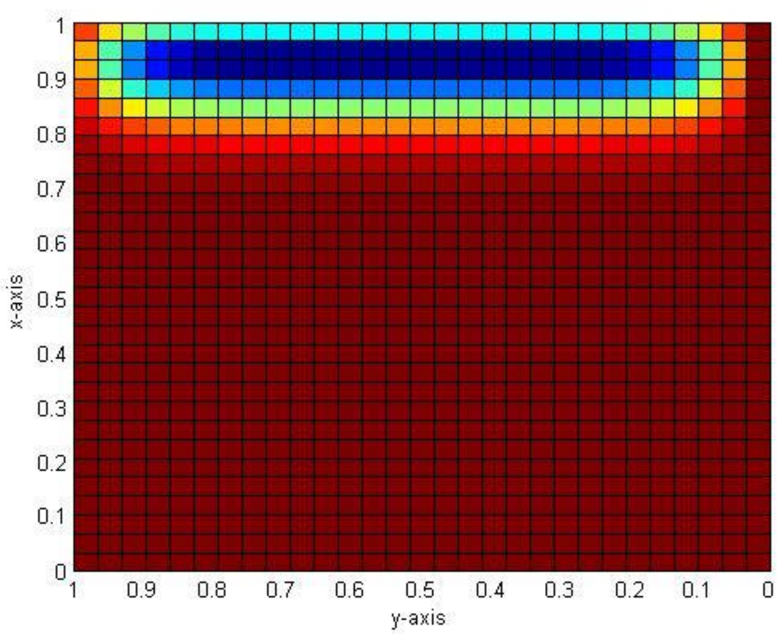
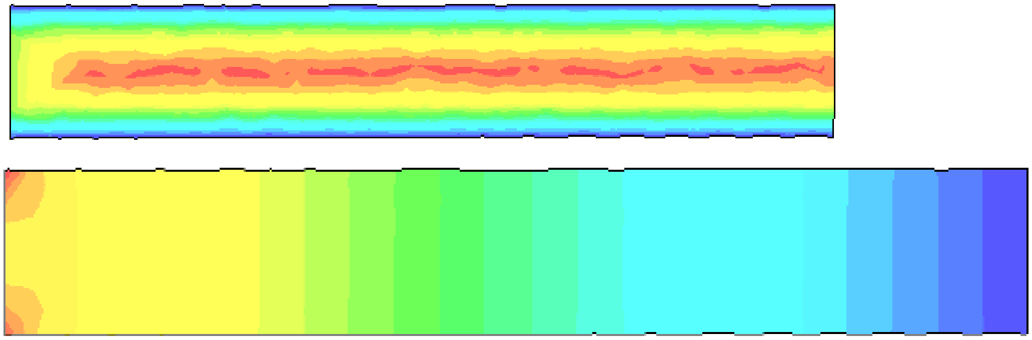
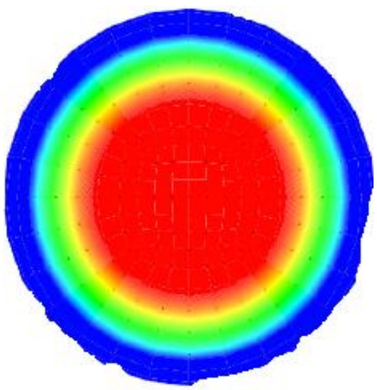


Figure 15: Solutions to lid driven cavity problem

Since these results are true according to those published, hence the matlab program developed is correct. After applying the Matlab program in appendix A to the above problem, the following results were obtained.



When the graph above is rotated laterally, we obtain



From the two figures above it can be seen that uniform inlet flow develops into a parabolic profile as it moves along the length.

5. Bibliography

Calves, V. (2010). *Cardiovascular system.* Paris.

Campbell, N. A. (1996). *Biology, human physiology.* University of California: Benjamin publishing company.

Chatzizis, Y. (2008). *Prediction of the localisation of high-risk cocronaary atherosclerotic plaques on the basis of low endothelial shear stress.* Harvard Medical School.

Cheng, C. (2006). *Atherosclerotic lesion size and vulnerability are determined by patterns of fluid shear stress.* University medical center Rotterdam, Netherland.

Gohil, T. (2002). SIMULATION OF OSCILLATORY FLOW IN AN AORTIC BIFURCATION USING FVM AND FEM. *International Journal for Numerical Methods in Fluids* , India.

Gunzburger, M. D. (2013). *Finite Element Methods.* Paris.

J.Segerlind, L. (1976). *Applied Finite Element Analysis.* New York: John Wiley and Sons.

Jiajan, W. (2010). *Solution of to incompressible Navier-Stokes equations by using finite element method.* University of texas.

Kazakidi, A. (2008). *Computational Studies of Blood.* University of London.

Kumar, D. D. (2008). *Fluid mechanics and fluid Power engineering.* India: S.K. Karataria and sons.

Nicolass, F. (2005). Numerial Analysis of carotid artery.

Organisation, W. H. (2011). *Cardiovascular diseases.*

Quarterson, A. (2002). *Mathematical modelling and Numerical simulation of cardiovascular system.* switzerland.

Ritzen, R. (. (2012). *Modeling of blood flow in coronary arteries with junctions.* Netherlands.

Ritzen, R. (2012). *Mathematical modeling of blood flow through a deformable thin-walled vessel.* Netherland.

Ruengsakulrach, P. *Wall shear stress and Atherosclerosis.* Thailand.

Wake, A. K. (2005). *Modeling of fluid mechanics in individual human carotid arteries.* Georgia.

WHO. (2002). *World health report.*

Xiu-YIng, K. (2008). *Blood flow at arterial bifurcation.* Beijing.

6. Table of figures

APPENDIX A MATLAB PROGRAM FOR N-S-E

Adapted from (Jiajan, 2010)

```
clc
clear all
Parameter input
Mxpoiv=625;
Mxpoip=169;
Mxele=288;
Mxfree=1;
Mxneq=2*Mxpoiv+Mxpoip; Number of equation in system
Npoiv=625; Number nodes of velocity components
Npoip=169; Number nodes of pressure
Nelem=288; Number of elements
Reynolds number (density*velocity*lenghth/viscosity)
den=1;
vis=0.01;
Artificial parameter
betha=8;
Matrix dimensions
Uvel=zeros(Mxpoiv,1);
Vvel=zeros(Mxpoiv,1);
Pres=zeros(Mxpoiv,1);
SysM=zeros(Mxneq,Mxneq); size of Mass matrix
SysK=zeros(Mxneq,Mxneq); size of stiffness matrix System
SysR=zeros(Mxneq,1);
SysN=zeros(Mxneq,1); size of force matrix system
Sol=zeros(Mxneq,1); size of result solution
Dsol=zeros(Mxneq,1); size of result solution
nodes=zeros(Mxele,6);
nodesf=zeros(Mxfree,4);
Ibcu=zeros(Mxpoiv,1); number of boundary for U
Ibcv=zeros(Mxpoiv,1); number of boundary for V
Ibcp=zeros(Mxpoiv,1); number of boundary for P
deltt=0.1; time step size for transient analysis
stime=0.0; initial time
ftime=0.0005; termination time
ntime=fix((ftime-stime)/deltt); number of time increment
Read input file (element connectivity)
fid = fopen('Mesh288elements.txt', 'r');
data1 = textscan (fid,;
nodes=[data1{1} data1{2} data1{3} data1{4} data1{5} data1{6} ];
Read input file ( Node co-ordinates)
fid = fopen('Meshnodes625.txt, r');
data2 = textscan(fid,;
X1=data2{1} ;
Y1=data2{2} ;
gcoord=[X1 Y1];
Plot mesh
```

```
figure(1)
clf
plot(X1,Y1,'.r','MarkerSize',10)
hold on
tri=nodes;
triplot(tri,X1,Y1);
Boundary condition
fid = fopen('625BC.txt', 'r');
data1 = textscan(fid, '
Ibcu=data1{1} ;
Ibcv=data1{2} ;
Ibcp=data1{3} ;
Uvel=data1{4} ;
Vvel=data1{5} ;
Pres=data1{6} ;
B.1.2 Loop for number of equations
Neq=2*Npoiv+Npoip;
Loop for initilize force vector
for i=1:Neq
SysR(i)=0;
end
Loop for initilize stiffness matrix
for i=1:Neq
for j=1:Neq
SysK(i,j)=0;
end
end
Loop for initilize mass matrix
for i=1:Neq
for j=1:Neq
SysM(i,j)=0;
end
end
B.1.3 Mass matrix construction
[SysM]=TRInew1(Npoiv,Nelem,den,vis,gcoord,nodes,SysM);
B.1.4 Initial guess value input
for i=1:Npoiv loop for velocity components
Sol(i)=0.0356;
Sol(i+Npoiv)=0.00032272;
end
for i=1:Npoip loop for pressure
Sol(i+Npoiv+Npoiv)=0.998;
end
B.1.5 Numerical programming codes "AC Backward Newton-Raphson Method"
SysK=SysM+deltt*SysK;
sum=zeros(ntime,1);
check=zeros(ntime,1);
for it=1:ntime iteration in each time plane
for iter=1:100 number of iterations
```

```
Newton-Raphson Non-linear matrix construction
[SysK,SysR]=TRI(Npoiv,Npoip,Nelem,Nfree,Neq,den,vis,
gcoord,nodes,SysK,SysR,Sol);
Backward different scheme
SysN=deltt*SysR+SysM*Sol;
apply boundary condition call ”ApplyBC55”
[SysK, SysN]=ApplyBC55(Npoiv,Npoip,Neq,Ibcu,Ibcv,Ibcp,
SysK,SysN,Uvel,Vvel,Pres);
Solve equations
Sol=inv(SysK)*SysN
iteration checking
for i=1:Npoip
Psol(i,it)=Sol(2*Npoiv+i);
Psol(i,it+1)=Sol(2*Npoiv+i);
end
Iteration=iter
end
Ntime=it
end
Convergence Checking
for it=1:ntime
for i=1:Npoip
sum(it)=sum(it)+(Psol(i,it+1)-Psol(i,it))/deltt;
check(it)=abs(1/betha*sum(it));
end
Check(it,1)=check(it);
if(abs(check(it))¡1e-6)
fprintf('Pressure converged in iterations');
break;
end
Ntime=it;
end
B.1.6 Output
plot the history of convergence
figure(2)
time=0:deltt:Ntime*deltt;
semilogy(time,(Check(:,1)),'-');
xlabel('Time')
ylabel('Pressure convergence')
title('Convergence history')
Print output
for i=1:Npoiv
Uvelocity(i)=Sol(i);
Vvelocity(i)=Sol(Npoiv+i);
end
for i=1:Npoip
Pressure(i)=Sol(i+2*Npoiv);
end
Velocity=[Uvelocity' Vvelocity']
```

```
Pressure=Pressure'
plot velocity vector
x=X1;
y=Y1;
scale=2;
u=Uvelocity';
v=Vvelocity';
figure(3)
quiver ( x, y, u, v, scale, 'b' );
axis equal
hold on
k = convhull ( x, y );
plot ( x(k), y(k), 'r' );
hold on
xmin = min ( x );
xmax = max ( x );
ymin = min ( y );
ymax = max ( y );
delta = 0.05 * max ( xmax - xmin, ymax - ymin );
plot ( [ xmin - delta,xmax + delta,xmax + delta,xmin - delta, xmin - delta ],
[ ymin - delta, ymin - delta,ymax + delta,ymax + delta,ymin - delta ],
'w' );
hold off
B.2 Sub programming "Functions"
B.2.1 Mass matrix Coefficient Function "TRInew1"
function [SysM]=TRInew1(Npoiv,Nelem,den,vis,gcoord,nodes,SysM)
A=zeros(6,6);B=zeros(6,3);C=zeros(6,3);G=zeros(3,3);
M11=zeros(6,6);M22=zeros(6,6);M33=zeros(3,3);Mele=zeros(15,15);
Rele=zeros(15,1);
Set up matrix [A]
for i=1:6
for j=1:6
A(i,j)=0;
end
end
A(1,1)=1;A(2,2)=1;A(3,3)=1;A(4,4)=4;A(5,5)=4;
A(6,6)=4;A(1,5)=-1;A(1,6)=-1;A(2,4)=-1;A(2,6)=-1;
A(3,4)=-1;A(3,5)=-1;
betha=8;
Anew=vis/den; kinematic viscosity
for iel=1:Nelem loop for the total number of elements
II=nodes(iel,1);
JJ=nodes(iel,2);
KK=nodes(iel,3);
LL=nodes(iel,4);
MM=nodes(iel,5);
84
NN=nodes(iel,6);
x1=gcoord(II,1); y1=gcoord(II,2);
```

```
x2=gcoord(JJ,1); y2=gcoord(JJ,2);
x3=gcoord(KK,1); y3=gcoord(KK,2);
Area=0.5*((x2*y3)+(x1*y2)+(x3*y1)-(x2*y1)-(x1*y3)-(x3*y2));
Area2=2*Area;
b1=(y2-y3)/Area2;
b2=(y3-y1)/Area2;
b3=(y1-y2)/Area2;
c1=(x3-x2)/Area2;
c2=(x1-x3)/Area2;
c3=(x2-x1)/Area2;
Compute matrix [B] [C] and [G]
for i=1:6
for j=1:3
B(i,j)=0;
C(i,j)=0;
end end
B(1,1)=2*b1;B(5,1)=b3;B(6,1)=b2;B(2,2)=2*b2;
B(4,2)=b3; B(6,2)=b1; B(3,3)=2*b3; B(4,3)=b2; B(5,3)=b1;
C(1,1)=2*c1; C(2,2)=2*c2; C(3,3)=2*c3; C(5,1)=c3;
C(5,3)=c1; C(6,1)=c2; C(6,2)=c1; C(4,2)=c3; C(4,3)=c2;
Fac=Area/12;
Fac2=2*Fac;
G(1,1)=Fac2; G(2,2)=Fac2; G(3,3)=Fac2; G(1,2)=Fac; G(1,3)=Fac;
G(2,1)=Fac; G(2,3)=Fac; G(3,1)=Fac; G(3,2)=Fac;
Set up mass matrix
D(1,1)=12; D(2,2)=12; D(3,3)=12; D(4,4)=2; D(5,5)=2; D(6,6)=2;
D(1,2)=2; D(1,3)=2; D(1,4)=1; D(1,5)=3; D(1,6)=3; D(2,3)=2; D(2,4)=3;
D(2,5)=1; D(2,6)=3; D(3,4)=3; D(3,5)=3; D(3,6)=1; D(4,5)=1;
D(4,6)=1; D(5,6)=1;
for i=1:6
for j=1:6
D(j,i)=D(i,j);
end
end
M11=2*Area/360*den*A*D*A';
M22=2*Area/360*den*A*D*A';
M33=G/betha;
Then the matrix (15x15) on LHS is
for i=1:15
for j=1:15
Mele(i,j)=0;
end
end
Final local mass matrix
for i=1:6
for j=1:6
Mele(i,j)=M11(i,j);
Mele(i+6,j+6)=M22(i,j);
end
```

```
end
for i=1:3
for j=1:3
Mele(i+12,j+12)=M33(i,j);
end
end
Assembly Global mass matrix
[SysM]=ASSEMBLEnew1(iel,nodes,Mele,SysM,Npoiv);
end
```

B.2.2 Assembly to Global mass matrix function "ASSEMBLEnew1"

```
function [SysM]=ASSEMBLEnew1(iel,nodes,Mele,SysM,Npoiv)
Contribution of coeficients assocoated with u,v velocities
for i=1:6
for j=1:6
II=nodes(iel,i);
JJ=nodes(iel,j);
k=i+6;
l=j+6;
KK=Npoiv+II;
LL=Npoiv+JJ;
SysM(II,JJ)=SysM(II,JJ)+Mele(i,j);
SysM(II,LL)=SysM(II,LL)+Mele(i,l);
SysM(KK,JJ)=SysM(KK,JJ)+Mele(k,j);
SysM(KK,LL)=SysM(KK,LL)+Mele(k,l);
end
end
Contribution of coeficients assocoated with P pressure
for i=1:3
for j=1:3
II=nodes(iel,i);
JJ=nodes(iel,j);
k=i+12;
l=j+12;
KK=2*Npoiv+II;
LL=2*Npoiv+JJ;
SysM(II,LL)=SysM(II,LL)+Mele(i,l);
SysM(KK,LL)=SysM(KK,LL)+Mele(k,l);
end
end
```

B.2.3 Local stiffness coefficient matrix function "TRI"

```
function [SysK,SysR]=TRI(Npoiv,Npoip,Nelem,Neq,den,vis,
gcoord,SysK,SysR,Sol);
A=zeros(6,6);B=zeros(6,3);C=zeros(6,3);G=zeros(3,3);F=zeros(6,6,3);
Uele=zeros(6,1);Vele=zeros(6,1);Pele=zeros(3,1);
Sxx=zeros(6,6);Sxy=zeros(6,6);Syx=zeros(6,6);Syy=zeros(6,6);
Qx=zeros(3,6);Qy=zeros(3,6);Qxt=zeros(6,3);Qyt=zeros(6,3);
ABGXUG=zeros(6,6);AGBXUG=zeros(6,6);AGBYVG=zeros(6,6);
ABGYVG=zeros(6,6);ABGXVG=zeros(6,6);ABGYUG=zeros(6,6);
Gxx=zeros(6,6);Gyy=zeros(6,6);Alx=zeros(6,6);Aly=zeros(6,6);
```

```
Akele=zeros(15,15);Rele=zeros(15,1);FX=zeros(6,1);FY=zeros(6,1);
FI=zeros(3,1);
Set up matrix [A]
for i=1:6
for j=1:6
A(i,j)=0;
end
end
A(1,1)=1; A(2,2)=1; A(3,3)=1; A(4,4)=4; A(5,5)=4;
A(6,6)=4; A(1,5)=-1; A(1,6)=-1; A(2,4)=-1; A(2,6)=-1;
A(3,4)=-1; A(3,5)=-1;
Anew=vis/den;
for iel=1:Nelem loop for the total number of elements
II=nodes(iel,1);
JJ=nodes(iel,2);
KK=nodes(iel,3);
LL=nodes(iel,4);
MM=nodes(iel,5);
NN=nodes(iel,6);
x1=gcoord(II,1); y1=gcoord(II,2);
x2=gcoord(JJ,1); y2=gcoord(JJ,2);
x3=gcoord(KK,1); y3=gcoord(KK,2);
Area=0.5*((x2*y3)+(x1*y2)+(x3*y1)-(x2*y1)-(x1*y3)-(x3*y2));
Area2=2*Area;
b1=(y2-y3)/Area2;
b2=(y3-y1)/Area2;
b3=(y1-y2)/Area2;
c1=(x3-x2)/Area2;
c2=(x1-x3)/Area2;
c3=(x2-x1)/Area2;
Compute matrix [B] [C] and [G]
for i=1:6
for j=1:3
B(i,j)=0;
C(i,j)=0;
end
end
B(1,1)=2*b1; B(5,1)=b3; B(6,1)=b2; B(2,2)=2*b2;
B(4,2)=b3; B(6,2)=b1; B(3,3)=2*b3; B(4,3)=b2;
B(5,3)=b1; C(1,1)=2*c1; C(2,2)=2*c2; C(3,3)=2*c3;
C(5,1)=c3; C(5,3)=c1; C(6,1)=c2; C(6,2)=c1;
C(4,2)=c3; C(4,3)=c2;
Set up [G] matrix
Fac=Area/12;
Fac2=2*Fac;
G(1,1)=Fac2; G(2,2)=Fac2; G(3,3)=Fac2; G(1,2)=Fac; G(1,3)=Fac;
G(2,1)=Fac; G(2,3)=Fac; G(3,1)=Fac; G(3,2)=Fac;
Set up matrix [F]
Factor=2*Area/5040;
```

```
F4=Factor*4;
F6=Factor*6; F12=Factor*12; F24=Factor*24; F120=Factor*120;
F(1,1,1)=F120; F(1,2,1)=F12; F(1,3,1)=F12; F(1,4,1)=F6;
F(1,5,1)=F24; F(1,6,1)=F24; F(2,2,1)=F24; F(2,3,1)=F4;
F(2,4,1)=F6; F(2,5,1)=F4; F(2,6,1)=F12; F(3,3,1)=F24;
F(3,4,1)=F6; F(3,5,1)=F12; F(3,6,1)=F4; F(4,4,1)=F4;
F(4,5,1)=F4; F(4,6,1)=F4; F(5,5,1)=F12; F(5,6,1)=F6; F(6,6,1)=F12;
for i=1:6
for j=1:6
F(j,i,1)=F(i,j,1);
end
end
F(1,1,2)=F24; F(1,2,2)=F12; F(1,3,2)=F4; F(1,4,2)=F4;
F(1,5,2)=F6; F(1,6,2)=F12; F(2,2,2)=F120; F(2,3,2)=F12;
F(2,4,2)=F24; F(2,5,2)=F6; F(2,6,2)=F24; F(3,3,2)=F24;
F(3,4,2)=F12; F(3,5,2)=F6; F(3,6,2)=F4; F(4,4,2)=F12;
F(4,5,2)=F4; F(4,6,2)=F6; F(5,5,2)=F4; F(5,6,2)=F4; F(6,6,2)=F12;
for i=1:6
for j=1:6
F(j,i,2)=F(i,j,2);
end
end
F(1,1,3)=F24; F(1,2,3)=F4; F(1,3,3)=F12; F(1,4,3)=F4;
F(1,5,3)=F12; F(1,6,3)=F6; F(2,2,3)=F24; F(2,3,3)=F12;
F(2,4,3)=F12; F(2,5,3)=F4; F(2,6,3)=F6; F(3,3,3)=F120;
F(3,4,3)=F24; F(3,5,3)=F24; F(3,6,3)=F6; F(4,4,3)=F12;
F(4,5,3)=F6; F(4,6,3)=F4; F(5,5,3)=F12; F(5,6,3)=F4; F(6,6,3)=F4;
for i=1:6
for j=1:6
F(j,i,3)=F(i,j,3);
end
end
Extract element nodal u v and p
Uele(1)=Sol(II);
Uele(2)=Sol(JJ);
Uele(3)=Sol(KK);
Uele(4)=Sol(LL);
Uele(5)=Sol(MM);
Uele(6)=Sol(NN);
Vele(1)=Sol(II+Npoiv);
Vele(2)=Sol(JJ+Npoiv);
Vele(3)=Sol(KK+Npoiv);
Vele(4)=Sol(LL+Npoiv);
Vele(5)=Sol(MM+Npoiv);
Vele(6)=Sol(NN+Npoiv);
Pele(1)=Sol(II+Npoiv+Npoiv);
Pele(2)=Sol(JJ+Npoiv+Npoiv);
Pele(3)=Sol(KK+Npoiv+Npoiv);
Compute [sxx] [sxy] [syx] [syy] matrices
```

```
for IA=1:6
for IB=1:6
K11=0;
K22=0;
K12=0;
K21=0;
for i=1:6
for j=1:3
for k=1:3
for l=1:6
K11=K11+A(IA,i)*B(i,j)*A(IB,l)*B(l,k)*G(j,k);
K22=K22+A(IA,i)*C(i,j)*A(IB,l)*C(l,k)*G(j,k);
K12=K12+A(IA,i)*C(i,j)*A(IB,l)*B(l,k)*G(j,k);
K21=K21+A(IA,i)*B(i,j)*A(IB,l)*C(l,k)*G(j,k);
end
end
end
end
Sxx(IA,IB)=2*Anew*K11+Anew*K22;
Sxy(IA,IB)=Anew*K12;
Syx(IA,IB)=Anew*K21;
Syy(IA,IB)=Anew*K11+2*Anew*K22;
end
end
Compute stabilize term matrices
for IA=1:6
for IB=1:6
Ks11uu=0;
Ks112uu=0;
Ks22vv=0;
Ks222vv=0;
Ks222vu=0;
Ks222uv=0;
Ks12uv=0;
Ks12uu=0;
Ks12vu=0;
Ks12vv=0;
Ks21vu=0;
Ks21uv=0;
Ks21uu=0;
Ks21vv=0;
for i=1:6
for j=1:3
for k=1:3
for l=1:6
Ks11uu=Ks11uu+A(IA,i)*B(i,j)*A(IB,l)*B(l,k)*G(j,k)*Uele(l)*Uele(i);
Ks112uu=Ks112uu+A(IA,i)*B(l,k)*A(IB,l)*B(i,j)*G(j,k)*2*Uele(l)*Uele(i);
Ks22vv=Ks22vv+A(IA,i)*C(i,j)*A(IB,l)*C(l,k)*G(j,k)*Vele(l)*Vele(i);
Ks222vv=Ks222vv+A(IA,i)*C(l,k)*A(IB,l)*C(i,j)*G(j,k)*2*Vele(l)*Vele(i);
```

```
Ks12uv=Ks12uv+A(IA,i)*C(i,j)*A(IB,l)*B(l,k)*G(j,k)*Uele(l)*Vele(i);
Ks12vu=Ks12vu+A(IA,i)*C(l,k)*A(IB,l)*B(i,j)*G(j,k)*Vele(l)*Uele(i);
Ks21vu=Ks21vu+A(IA,i)*B(i,j)*A(IB,l)*C(l,k)*G(j,k)*Vele(l)*Uele(i);
Ks222vu=Ks222vu+A(IA,i)*C(i,j)*A(IB,l)*C(l,k)*G(j,k)*2*Vele(l)*Uele(i);
Ks222uv=Ks222uv+A(IA,i)*C(i,j)*A(IB,l)*C(l,k)*G(j,k)*2*Uele(l)*Vele(i);
Ks12uu=Ks12uu+A(IA,i)*C(i,j)*A(IB,l)*B(l,k)*G(j,k)*Uele(l)*Uele(i);
Ks12vv=Ks12vv+A(IA,i)*C(i,j)*A(IB,l)*B(l,k)*G(j,k)*Vele(l)*Vele(i);
Ks21uv=Ks21uv+A(IA,i)*B(i,j)*A(IB,l)*C(l,k)*G(j,k)*Uele(l)*Vele(i);
Ks21uu=Ks21uu+A(IA,i)*B(i,j)*A(IB,l)*C(l,k)*G(j,k)*Uele(l)*Uele(i);
Ks21vv=Ks21vv+A(IA,i)*B(i,j)*A(IB,l)*C(l,k)*G(j,k)*Uele(l)*Uele(i);
end
end
end
end
KS11uu(IA,IB)=Ks11uu;
KS112uu(IA,IB)=Ks112uu;
KS22vv(IA,IB)=Ks22vv;
KS222vv(IA,IB)=Ks222vv;
KS222vu(IA,IB)=Ks222vu;
KS222uv(IA,IB)=Ks222uv;
KS12uv(IA,IB)=Ks12uv;
KS12uu(IA,IB)=Ks12uu;
KS12vu(IA,IB)=Ks12vu;
KS12vv(IA,IB)=Ks12vv;
KS21vu(IA,IB)=Ks21vu;
KS21uv(IA,IB)=Ks21uv;
KS21uu(IA,IB)=Ks21uu;
KS21vv(IA,IB)=Ks21vv;
end
end
for i=1:6
for j=1:6
Kse1(i,j)=deltt/2*(KS11uu(i,j)+KS112uu(i,j)+KS22vv(i,j)+
KS12uv(i,j)+KS12vu(i,j) +KS21vu(i,j)+KS21vu(i,j));
Kse2(i,j)=deltt/2*(KS11uu(i,j)+KS22vv(i,j)+
KS12uv(i,j)+KS12uv(i,j)+KS21vu(i,j)+KS21uv(i,j)+KS222vv(i,j));
Kse3(i,j)=deltt/2*(KS12uu(i,j)+KS21uu(i,j)+KS222vu(i,j));
Kse4(i,j)=deltt/2*(KS12vv(i,j)+KS21vv(i,j)+KS222uv(i,j));
end
end
Compute [Q1] and [Q2]
for IA=1:3
for IB=1:6
Cx=0;
Cy=0;
for i=1:6
for j=1:3
Cx=Cx+A(IB,i)*B(i,j)*G(j,IA);
Cy=Cy+A(IB,i)*C(i,j)*G(j,IA);
```

```
end
end
Qx(IA,IB)=Cx/den;
Qy(IA,IB)=Cy/den;
end end
Then the corresponding two matrices on the upper right
for IA=1:3
for IB=1:6
Qxt(IB,IA)= -Qx(IA,IB);
Qyt(IB,IA)= -Qy(IA,IB);
end
end
Compute all matrices associated with the inertia term
for IA=1:6
for IB=1:6
Cabgxug=0;
Cagbxug=0;
Cagbyvg=0;
Cabgyvg=0;
Cabgxvg=0;
Cabgyug=0;
for i=1:6
for j=1:6
for k=1:6
for l=1:6
for m=1:3
Cabgxug=Cabgxug+A(IA,i)*A(IB,j)*A(k,l)*B(l,m)*F(i,j,m)*Uele(k);
Cagbxug=Cagbxug+A(IA,i)*A(k,j)*A(IB,l)*B(l,m)*F(i,j,m)*Uele(k);
Cagbyvg=Cagbyvg+A(IA,i)*A(k,j)*A(IB,l)*C(l,m)*F(i,j,m)*Vele(k);
Cabgyvg=Cabgyvg+A(IA,i)*A(IB,j)*A(k,l)*C(l,m)*F(i,j,m)*Vele(k);
Cabgxvg=Cabgxvg+A(IA,i)*A(IB,j)*A(k,l)*B(l,m)*F(i,j,m)*Vele(k);
Cabgyug=Cabgyug+A(IA,i)*A(IB,j)*A(k,l)*C(l,m)*F(i,j,m)*Uele(k);
end
end
end
end
end
ABGXUG(IA,IB)=Cabgxug;
AGBXUG(IA,IB)=Cagbxug;
AGBYVG(IA,IB)=Cagbyvg;
ABGYVG(IA,IB)=Cabgyvg;
ABGXVG(IA,IB)=Cabgxvg;
ABGYUG(IA,IB)=Cabgyug;
end
end
Take in system equation
for i=1:6
for j=1:6
Gxx(i,j)=ABGXUG(i,j)+AGBXUG(i,j)+AGBYVG(i,j)+Sxx(i,j)+Kse1(i,j);
```

```
Gyy(i,j)=ABGYVG(i,j)+AGBYVG(i,j)+AGBXUG(i,j)+Syy(i,j)+Kse2(i,j);
Alx(i,j)=ABGXVG(i,j)+Sxy(i,j)+Kse3(i,j);
Aly(i,j)=ABGYUG(i,j)+Syx(i,j)+Kse4(i,j);
end
end
Then the matrix (15x15) on LHS is
for i=1:15
for j=1:15
Akele(i,j)=0;
end
end
for i=1:6
for j=1:6
Akele(i,j)=Gxx(i,j);
Akele(i+6,j+6)=Gyy(i,j);
Akele(i,j+6)=Aly(i,j);
Akele(i+6,j)=Alx(i,j);
end
for j=1:3
Akele(i,j+12)=Qxt(i,j);
Akele(i+6,j+12)=Qyt(i,j);
end
end
for i=1:3
for j=1:6
Akele(i+12,j)=Qx(i,j);
Akele(i+12,j+6)=Qy(i,j);
end
end
Begin computing the residuals on RHS of element equation
for i=1:6
Term1=0;
Term2=0;
Term3=0;
Term4=0;
Term5=0;
Term6=0;
Term7=0;
Term8=0;
Term9=0;
for j=1:6
Term1=Term1+ABGXUG(i,j)*Uele(j);
Term2=Term2+ABGYUG(i,j)*Vele(j);
Term4=Term4+Sxx(i,j)*Uele(j);
Term5=Term5+Sxy(i,j)*Vele(j);
Term6=Term6+KS11uu(i,j)*Uele(j);
Term7=Term7+KS12uv(i,j)*Uele(j);
Term8=Term8+KS21vu(i,j)*Uele(j);
Term9=Term9+KS22vv(i,j)*Uele(j);
```

```
end
for j=1:3
Term3=Term3+Qxt(i,j)*Pele(j);
end
FX(i)=Term1+Term2+Term3+Term4+Term5+
deltt/2*(Term6+Term7+Term8+Term9);
end
for i=1:6
Term1=0;
Term2=0;
Term3=0;
Term4=0;
Term5=0;
Term6=0;
Term7=0;
Term8=0;
Term9=0;
for j=1:6
Term1=Term1+ABGXVG(i,j)*Uele(j);
Term2=Term2+ABGYVG(i,j)*Vele(j);
Term4=Term4+Syx(i,j)*Uele(j);
Term5=Term5+Syy(i,j)*Vele(j);
Term6=Term6+KS11uu(i,j)*Vele(j);
Term7=Term7+KS12uv(i,j)*Vele(j);
Term8=Term8+KS21vu(i,j)*Vele(j);
Term9=Term9+KS22vv(i,j)*Vele(j);
end
for j=1:3
Term3=Term3+Qyt(i,j)*Pele(j);
end
FY(i)=Term1+Term2+Term3+Term4+Term5+
deltt/2*(Term6+Term7+Term8+Term9);
end
for i=1:3
Term1=0;
Term2=0;
for j=1:6
Term1=Term1+Qx(i,j)*Uele(j);
Term2=Term2+Qy(i,j)*Vele(j);
end
FI(i)=Term1+Term2;
end
Thus the residual vector on RHS of element equation is
for i=1:6
Rele(i)=-FX(i);
Rele(i+6)=-FY(i);
end
for i=1:3
Rele(i+12)=-FI(i);
```

```
end
[SysK SysR]=ASSEMBLE(iel,nodes,Akele,Rele,SysK,SysR,
Npoiv,Neq,Nelem);
end
B.2.4 Assembly to Global stiffness matrix function "ASSEMBLE"
function [SysK SysR]=ASSEMBLE(iel,nodes,Akele,Rele,
SysK,SysR,Npoiv,Neq,Nelem)
Contribution of coeficients assocoated with U, and V velocity
for i=1:6
for j=1:6
II=nodes(iel,i);
JJ=nodes(iel,j);
k=i+6;
l=j+6;
KK=Npoiv+II;
LL=Npoiv+JJ;
SysK(II,JJ)=SysK(II,JJ)+Akele(i,j);
SysK(II,LL)=SysK(II,LL)+Akele(i,l);
SysK(KK,JJ)=SysK(KK,JJ)+Akele(k,j);
SysK(KK,LL)=SysK(KK,LL)+Akele(k,l);
end
end
for i=1:6
for j=1:3
II=nodes(iel,i);
JJ=nodes(iel,j);
k=i+6;
l=j+12;
KK=Npoiv+II;
LL=2*Npoiv+JJ;
SysK(II,LL)=SysK(II,LL)+Akele(i,l);
SysK(KK,LL)=SysK(KK,LL)+Akele(k,l);
SysK(LL,II)=SysK(LL,II)+Akele(l,i);
SysK(LL,KK)=SysK(LL,KK)+Akele(l,k);
end
end
Assembly load vector system
Contribution of value with U, and V velocity
for i=1:6
II=nodes(iel,i);
k=i+6;
KK=Npoiv+II;
SysR(II)=SysR(II)+Rele(i);
SysR(KK)=SysR(KK)+Rele(k);
end
Contribution of value with P pressure
for i=1:3
II=nodes(iel,i);
k=i+12;
```

```
KK=2*Npoiv+II;
SysR(KK)=SysR(KK)+Rele(k);
end
Apply boundary conditions function ”ApplyBC55”
function [SysM, SysN]=ApplyBC55(Npoiv,Npoip,Neq,Ibcu,SysM,
SysN,Uvel,Vvel,Pres);
Apply boundary condition for nodal U-velocity
IEQ1=1;
IEQ2=Npoiv;
for IEQ=IEQ1:IEQ2
IEQU=IEQ;
if(Ibcu(IEQU) =0)
for IR=1:Neq
if(IR==IEQ)
SysN(IR)=SysN(IR)-SysM(IR,IEQ)*Uvel(IEQU);
SysM(IR,IEQ)=0;
end
end
for IC=1:Neq
SysM(IEQ,IC)=0;
end
SysM(IEQ,IEQ)=1;
SysN(IEQ)=Uvel(IEQU);
end
end
Apply boundary condition for nodal V-velocity
IEQ1=Npoiv+1;
IEQ2=2*Npoiv;
for IEQ=IEQ1:IEQ2
IEQV=IEQ-Npoiv;
if(Ibcv(IEQV) =0)
for IR=1:Neq
if(IR==IEQ)
SysN(IR)=SysN(IR)-SysM(IR,IEQ)*Vvel(IEQV);
SysM(IR,IEQ)=0;
end
end
for IC=1:Neq
SysM(IEQ,IC)=0;
end
SysM(IEQ,IEQ)=1;
SysN(IEQ)=Vvel(IEQV);
end
end
Apply boundary condition for nodal P-pressure
IEQ1=2*Npoiv+1;
IEQ2=Neq;
for IEQ=IEQ1:IEQ2
IEQP=IEQ-2*Npoiv;
```

```
if(Ibcp(IEQP) =0)
for IR=1:Neq
if(IR==IEQ)
SysN(IR)=SysN(IR)-SysM(IR,IEQ)*Pres(IEQP);
SysM(IR,IEQ)=0;
end
end
for IC=1:Neq
SysM(IEQ,IC)=0;
end
SysM(IEQ,IEQ)=1;
SysN(IEQ)=Pres(IEQP);
end
end
```