MAXIMAL NUMERICAL RANGE OF OPERATORS IN HILBERT SPACES

A project submitted to the Department of Mathematics, University of Nairobi, in partial fulfilment of the degree of Master of Science in Pure Mathematics.

By

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DECLARATION

By Candidate:

This project is my original work and has not been presented for a degree in any other University.

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By Supervisor:

This project has been submitted for examination with my approval as University supervisor.

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INTRODUCTION

The maximal numerical range of a linear operator T, on a Hilbert space H, denoted by Wo(T), is a subset of the complex field **C**. It is a relatively new concept in operator theory, having been introduced only in 1970 by Stampfli. It owes part of its motivation to the usual numerical range, W(T). It is therefore important that the two concepts, that is, W(T) and Wo(T) be defined alongside one another so as to provide a good understanding of the latter.

Definition

Let T be a bounded linear operator on a complex Hilbert space H. Then,

(1) the numerical range of T is defined to be the set;

$$W(T) = \{ \langle Tx,x \rangle : x \in H, \ \|x\| = 1 \}$$

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(2) the maximal numerical range of T is defined to be the set;

$$W_0(T) = \{\lambda \in C : \langle Tx_n, x_n \rangle \to \lambda, \text{ where } \|x_n\| = 1 \text{ and } \|Tx_n\| \to \|T\|\}.$$

A lot of research has been done with the usual numerical range. Following is an outline of some of the research works in this area:

Donoghue(1957) studied the numerial range of a bounded operator. Stampfli(1966) looked at the extreme points of the numerical range of a hyponormal operator. Jointly with Williams, Stampfli(1968) went further and studied the growth conditions and the numerical range in a Banach algebra. Halmos(1967) in his book , " A Hilbert space problem book" , dedicated an entire chapter to the numerical range. A few years later, Bonsall and Duncan(1971) wrote a book on numerical ranges of operators on normed spaces and of elements of normed algebras. Embry(1971) used subsets associated with the numerical range to classify special operators. Dash(1973) considered tensor products and joint numerical range. Lancaster(1975) studied the boundary of the numerical range, he introduced the concept of essential numerical range and proved two results which indicate a set theoretic relationship between the boundary of the numerical range and the essential numerical range. These were to be re-visited later by Williams(1977), who tried to simplify the proofs. A study of the growth of numerical ranges of powers of Hilbert space operators was done by Shiu(1976). Khalagai(1979) in his M.Sc. thesis, did not only discuss the topological properties of the numerical range, but he also showed the bearing that the numerical range has got on properties of operators. He showed how some conditions can be imposed on the numerical range of a bounded operator in order to achieve normality, similarity, uniticity and self adjointness of the operator. He further showed how the conclusion of normality from classes of operators such as quasinormal, hyponormal, paranormal and quasihyponormal can be attained via the numerical range. Finally, he showed how the numerical range can be used to achieve the positivity of a product of operators.

On the contrary, however, the maximal numerical range has not enjoyed as much exposition. Stampfli(1970), though using it as a powerful tool in determining the norm of a derivation, seemed not to pay much interest to it, but instead treated it casually. It is nevertheless remarkable to note that Fong(1979) considered the essential maximal numerical range as an independent subject. The work of Sheth and Duggal(1984) is also worth being acknowledged. They initiated the study of maximal numerical range as a subject in itself. Other major contributors in this field are Khan(1988), who re-visited the idea of essential maximal numerical range; and Cho(1988), whose result about the joint maximal numerical range is of particular importance in this area.

One can clearly see from the foregoing that the maximal numerical range is still a virgin area of study as opposed to the usual numerical range. This project is therefore geared towards providing a curtain-raiser for future research in the area of maximal numerical range. To achieve this goal, the project has been divided into four chapters.

In chapter one, the topological properties of the maximal numerical range are established. Towards this end, a comparative study of the usual numerical range and the maximal numerical range is carried out. It is shown that W(T) and Wo(T) share the properties of convexity, boundedness, compactness and connectedness; and that Wo(T) has the additional property of closure which is not true in general for W(T). Some other properties for which conditions must be imposed on the operator T(and sometimes on H) so that they hold true for both W(T) and Wo(T) are also considered in this chapter.

Chapter two is concerned with the role played by the maximal numerical range in determining the norm of a derivation on the Banach algebra, B(H). An identity for the norm is derived using the maximal numerical range.

In chapter three, the essential maximal numerical range is discussed. This concept is used in deriving an analogous identity for the norm of a derivation in the quotient algebra, B(H)/K(H).

The last chapter focuses on the concepts of algebraic maximal numerical range and the joint maximal numerical range.

It is however, necessary that before proceeding with this discussion, some standard definitions and notation that will be adhered to herein be given.