ON SOME CLASSES OF OPERATORS ON HILBERT SPACE

BY

YUNUS KHAN
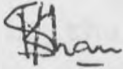
This dissertation is submitted in partial fulfilment
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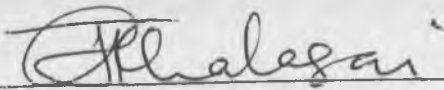
Contents

DECLARATION

This dissertation is my own work and has not been presented for a degree in any other University.

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This dissertation has been submitted for examination with my approval as the University Supervisor.

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(ii)

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Dedication

To my parents and my only brother Kalandar.

ON SOME CLASSES OF OPERATORS ON HILBERT SPACE

Notation

In this project, H will denote a Hilbert space with inner project denoted by $\langle \, , \rangle$ and T, A, B, X etc. will denote operators (i.e. bounded, linear transformations) on a Hilbert space H into itself or into another Hilbert space. Ker T, ran T, $\ker^{\perp}$ T, $\text{ran}^{\perp}$ T will denote the kernel of T, range of T, orthogonal complement of ker T, orthogonal complement of ran T, respectively. B(H) will denote the Banach algebra of all operators on H. If $A, B \in B(H)$ then [A,B] will denote AB-BA $\mathbb{C}$ and $\mathbb{R}$ will denote the fields of complex and real numbers, respectively.

Introduction

The study of normal operators has been very successful in the sense that a lot of interesting results has been obtained concerning these operators e.g. the classical Putnam - Fuglede theorem, which will be stated later. Many authors have defined new classes of operators by making them satisfy certain known properties of normal operators in the hope that some of the results which hold for normal operators, will also hold for these new classes of operators.

For example, spectraloid operators have been
defined using some properties of the sepectrum, spectal
radius, etc. of normal operators. Others have defined
other classes of operators by generalising the concept
of normality e.g. binormal operators [2]. By relaxing
the condition of normality we can also define other
classes of operators e.g. dominant operators [17] and
try to see which properties of normal operators will
still hold for these larger classes of operators.

In this project, we have studied spectraloid,
binormal and dominant operators. For each of these
three classes of operators, we have studied their
properties, subclasses and the properties of these sub-
classes. We have also looked at extensions of the
Putnam-Fuglede theorem and finally we have given
a problem which can lead to further research in this
area of operator theory.

Preliminary defintions and results.

(i)   The spectrum of an operator T, denoted by $\sigma(T)$,
      is defined by $\sigma(T) = \{\lambda\epsilon\ \mathbb{C} : \lambda I - T$ is not
      invertible$\}$ where I = identity operator.

(ii) The spectral radius of T, denoted by $r(T)$, is
      defined by $r(T) = \sup\ \{|\lambda| : \lambda\ \epsilon\ \sigma(T)\}$.

(iii) The numerical range of an operator T, denoted
      by $W(T)$, is defined by $W(T) = \{\langle Tf, f\rangle :$

$||\ f\ ||\ =\ 1\}.$

(iv)   The <u>numerical radius</u> of an operator T, denoted

by w(T), is defined by $w(T) = \sup \{|\lambda| : \lambda \epsilon\ W(T)\}$

(v)   A set A is said to be <u>convex</u> if for every

$x,y \epsilon\ A,\ tx + (1-t)y\ \ \epsilon\ A,$ where $0 \leq t \leq 1$

(vi)   The <u>convex hull</u> of a set M denoted by conv M,

is the intersection of all the convex sets which

contain M.   Note that conv M is itself convex

since it is the intersection of convex sets and

it is the smallest convex set which contains M.

(vii)   Two operators A and B are said to be <u>similar</u>

if there exists an invertible P   such that

$P^{-1}\ AP = B.$

The following result gives some properties of a

normal operator ([7]).

<u>Theorem A</u>

Let T be a normal operator. The following are

true:

(a) $r(T)\ \ =\ \ w(T)$

(b)   the closure of the numerical range of T   is the.

convex hull of its spectrum i.e $\overline{W(T)} = conv\ (\sigma(T)).$

(c) $w(T) = ||T||$

Using this result, we can now define spectra-
loid operators.

CHAPTER ONE

Spectraloid Operators

Definitions

(i)   Let T be any operator. T is said to be
      spectraloid if $r(T) = w(T)$.

(ii) Let T be any operator. T is said to be
      convexoid if the closure of the numerical
      range of T is the convex hull of its spectrum
      i.e. if $W(T) = \text{conv}(\sigma(T))$.

(iii) Let T be any operator. T is said to be
       normaloid if $w(T) = ||T||$

    The following result will be needed to show that
every normaloid operator is spectraloid.

Lemma 1.1

      If $w(A) = ||A||$, then $r(A) = ||A||$

Proof

      By multiplying by a suitable positive constant,
we can assume without loss of generality that
$||A|| = 1$. Since $w(A) = ||A||$, there exists a sequence
$\{f_n\}$ of unit vectors such that $|\langle Af_n, f_n \rangle| \to 1$.
Without loss of generality, by multiplying by a suitable
constant of modulus 1, we can assume that
$\langle Af_n, f_n \rangle \to 1$. Since $\langle Af_n, f_n \rangle \leq ||Af_n|| \leq 1$ and
$\langle Af_n, f_n \rangle \to 1$, it follows that $||Af_n|| \to 1$.

This implies that $\|Af_n - f_n\|^2 = \|Af_n\|^2 - 2\operatorname{Re}\langle Af_n, f_n\rangle$

$$+\ |\ \to\ 0$$

so that 1 is an approximate eigenvalue of A and

therefore $r(A) = 1$.

We can now prove that the class of spectraloid

operators contains both the convexoid and normaloid

operators.

Theorem 1.2

(a) Every normaloid operator is spectraloid

(b) Every convexoid operator is spectraloid

Proof

(a) Let T be a normaloid operator. Then $w(T) = \|T\|$.
By lemma 1.1, $r(T) = \|T\| = w(T)$ i.e
$r(T) = w(T)$. Hence T is spectraloid.

(b) For any operator T, we know that $\sigma(T) \subseteq \overline{W(T)}$
(see [7]). Thus we have $r(T) \leq w(T)$ for any T.
Now the closed disc with centre O and radius $r(T)$
includes $\sigma(T)$ and is convex. Hence if T is
convexoid then that disc contains $W(T) \Rightarrow w(T)$
$\leq r(T)$. Hence $r(T) = w(T)$ i.e T is spectraloid.

Remark

Since the class of spectraloid operators contains

both the convexoid and normaloid operators, it is interestin

to investigate whether the classes of convexoid and normaloid operators are related. In other words, we would like to know whether every normaloid operator is convexoid or whether every convexoid operator is normaloid. However we will show that these clases of operators are independent.

Example 1.1

We first give an example of a convexoid operator which is not normaloid. Let

$$M = \begin{pmatrix} O & O \\ 1 & O \end{pmatrix}$$

Let N be a normal operator whose spectrum is the closed disc D with centre O and radius $\frac{1}{2}$. If

$$A = \begin{pmatrix} M & O \\ O & N \end{pmatrix}$$

then $\sigma(A) = \{O\} \cup D = D$ and $W(A) = \text{conv}(W(M) \cup W(N))$

$$= D \quad \Rightarrow \quad A \text{ is convexoid.}$$

Since $\|A\| = 1$ (in fact $\|M\| = 1$), A is not normaloid.

Example 1.2

We now give an example of a normaloid operator which is not convexoid. Let M be as in example 1.1. Write

$$A = \begin{pmatrix} M & O \\ O & 1 \end{pmatrix}$$

Since $||A|| = 1$ and $W(A) = \text{conv}(D \cup \{1\})$ it follows

that $w(A) = 1$ and hence that A is normaloid since

$w(A) = ||A|| = 1$. However $\sigma(A) = \{0\} \cup \{1\}$ so that

$\text{conv } \sigma(A) = [0,1] \neq \overline{W(A)}$. Hence A is not convexoid.

Other classes of operators

(i) T is said to be <u>hyponormal</u> if $T^*T \geq TT^*$. This
means that $T^*T - TT^* \geq 0$ i.e. $\langle (T^*T-TT^*) x,x \rangle$
$\geq 0 \; \forall \; x \in H$.

(ii) T is said to be <u>paranormal</u> or an operator of
class (N) if

$$|| (T-zI)^{-1}|| = \frac{1}{d(z, \sigma(T))}$$

for all $z \notin \sigma(T)$. Note that we can talk of
$(T - zI)^{-1}$ since $z \notin \sigma(T)$. Also since $z \notin \sigma(T)$,
$d(z, \sigma(T))$ is not equal to zero. Equivalently,
T is said to be paranormal if $||Tx||^2 \leq ||T^2 x|| \, ||x||$
$\forall \; x \in H$, or T is paranormal if for all unit
vectors $x \in H$ (i.e. $||x|| = 1$) we have
$||Tx||^2 \leq ||T^2 x||$.

(iii) T is said to be <u>k-paranormal</u> or an operator of
class (N;K) if $||Tx||^k \leq ||T^k x||, ||x||^{k-1}$
$\forall \; x \in H$, where $k \geq 2$ is some integer. Equivalently
T is k-paranormal if $||Tx||^k \leq ||T^k x||$ for
$x \in H$ where $||x|| = 1$ and $k \geq 2$ is some integer.
Note that a 2-paranormal operator is paranormal.

We now show that the above classes of operators are, in fact, sub-classes of spectraloid operators.

Theorem 1.3

(a)   Every hypornormal operator is paranormal

(b)   Every paranormal operator is k-paranormal

(c)   Every k-paranormal operator is normaloid.

Proof

See the references cited in [11]

Remark

Since every normaloid operator is spectraloid it follows that the class of spectraloid operators contains the hyponormal, paranormal and the k-paranormal operators.

Theorem 1.4

Every paranormal operator is convexoid.

Proof.

To prove this theorem, we will need a result due to Orland ([10]). He proved that T is convexoid iff

$$||(T-zI)^{-1}|| \leq \frac{1}{d(z, \text{conv } \sigma(T))} \quad \text{for all } z \notin \text{conv } \sigma(T).$$

From this it follows that vevery paranormal operator is convexoid.

Remark

The converses of the above implications are not true

since there do exist counter examples, as the following
result shows.

Theorem 1.5

There exists an invertible paranormal operator

T   such that

1.   T   is not hyponormal

2. T^2 is not paranormal.

3. $\|T\| > r(T)$, and

4. T^{-1} is not paranormal.

Proof.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ Let   N   be a normal operator such

that $\sigma(N) = \overline{W(A)}$ and   let $T = A \oplus N$.   Then   T   is

paranormal.   We know that a hyponormal operator is

hyponormal on invariant subspaces.   Therefore since   A

is not hyponormal,   T   is not hyponormal.   Now $\overline{W(A)}$ is

the closed disc of radius $\frac{1}{2}$ about $z = 1$ and $\overline{W(A^2)}$

is the closed disc of radius 1   about $z = 1$.   Therefore

$$O \in W(A^2) \subseteq W(T^2) \quad \text{and}$$

$$O \notin \text{conv} \, (\sigma(T)^2) = \text{conv} \, \sigma(T^2)$$

Therefore conv $\sigma(T^2) \neq W(T^2)$ and so T^2 is not paranormal.

Let $x = \begin{bmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{bmatrix}$

Then $\|x\| = 1$ and $\|Ax\| = \dfrac{\sqrt{10}}{2}$,

Then $\|T\| \geq \|Ax\|$ $\frac{\sqrt{10}}{2} > \frac{3}{2} = r(T)$

Therefore $\|T\| > r(T)$.

If T^{-1} were paranormal, then

$$\|T\| = \|(T^{-1}-oI)^{-1}\| = \frac{1}{d(0,\sigma(T^{-1}))}$$

$$= r(T) \quad \text{contradiction. Hence } T^{-1}$$

is not paranormal.

We now look at some properties of paranormal operators.

Paranormal operators

We will use the following notation: If s is a compact subset of the complex number $\mathbb{C}$ and if $\varepsilon>0$, then let $s + (\varepsilon) = \{z: d(z,s) < \varepsilon\}$.

If S and S_n, $n = 1,2,3,\ldots$ are compact sets in $\mathbb{C}$, then the sequence $\{S_n\}$ approaches S, written $S_n \to S$, if for every $\varepsilon>0$ there exists a positive integer N such that, for $n \geq N$, $S_n \subseteq S + (\varepsilon)$ and $S \subseteq S_n + (\varepsilon)$. Note that it will always be assumed that $B(H)$, in this section, has the uniform operator (norm) topology.

To prove our next result, we will need the following lemma.

Lemma 1.6

If $T \in B(H)$ and $\varepsilon > 0$, then there exists $\delta > 0$ such that if $S \in B(H)$

and $||T-S|| < \delta$, then $\sigma(s) \subseteq \sigma(T) + (\varepsilon)$.

<u>Proof.</u>

See [7] , problem 103.

Theorem 1.7

If $\{T_n\}$ is a sequence of paranormal operators approaching the operator T in norm, then $\sigma(T_n) \to \sigma(T)$ as $n \to \infty$.

Proof

By lemma 1.6, we know that for each $\varepsilon > 0$ there exists a positive integer N such that, for $n \geq N$, $\sigma(T_n) \subseteq \sigma(T) + (\varepsilon)$. Therefore, to show that $\sigma(T_n) \to \sigma(T)$, it is enough to show that for each $\varepsilon > 0$, there exists a positive integer N such that $\sigma(T) \subseteq \sigma(T_n) + (\varepsilon)$, for all $n \geq N$. If this does not hold then without loss of generality we may assume that there exists $\varepsilon > 0$ and a sequence $\{z_n\} \subseteq \sigma(T)$ such that $d(z_n, \sigma(T_n)) \geq \varepsilon$ for all n. Since $\sigma(T)$ is compact, we may assume that $z_n \to z \in \sigma(T)$. If $|z_n - z| < \frac{\varepsilon}{2}$, then

$$d(z, \sigma(T_n)) \geq d(z_n, \sigma(T_n)) - |z - z_n|$$
$$\geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

Hence

$$|| (T_n - zI)^{-1} || = \frac{1}{d(z,\sigma(T_n))} \leq \frac{2}{\varepsilon} .$$

Now choose m so that

$$||(T_m - T)(T_m - zI)^{-1} || < 1. \quad \text{Then}$$

$I - (T_m - T)(T_m zI)^{-1}$ is invertible.

Let $A = (T_m - zI)^{-1}(I - (T_m - T)(T_m - zI)^{-1})^{-1}$. Then

$A(T - zI) = (T - zI) \ A = I$ so that $z \ \varepsilon \ \rho(T)$. contradiction.

CHAPTER TWO

Binormal Operators

Before we define what a binormal operator is, we first study the polar decomposition of an operator.

Definitions

An isometry is a linear transformation U (from a Hilbert space into itself or from one Hilbert space into another) such that $||Uf|| = ||f||$ for all f in the Hilbert space. A necessary and sufficient condition that a linear transformation U be an isometry is that $U*U = I$ (see $\begin{bmatrix} 7 \end{bmatrix}$ p. 69).

It is sometimes convenient to consider linear transformations U that act isometrically on a subset of a Hilbert space; this means that $||Uf|| = ||f||$ for all f in that subset.

A partial isometry is a linear transformation that is isometric on the orthogonal complement of its kernel. The orthogonal complement of the kernel of a partial isometry is called its initial space and the range is called the final space.

Theorem 2.1

A bounded linear transformation U is a partial isometry iff $U*U$ is a projection.

Proof.

Suppose that H and K are Hilbert spaces and suppose

that U is a partial isometry from H into K with initial space M. If E is the projection from H onto M, and if $f \in M$, then $\langle U^*Uf, f \rangle = ||Uf||^2 = ||f||^2 = \langle Ef, f \rangle$ If $f \perp M$, then

$$\langle U^*Uf, f \rangle = 0 = \langle Ef, f \rangle$$

$\Rightarrow \langle U^*Uf, f \rangle = \langle Ef, f \rangle$ for all f in H.

$\Rightarrow U^*U = E$.

Conversely, suppose that U is a bounded linear transformation from H into K such that U^*U is a projection with domain H and range M, say. It follows that

$$||Uf||^2 = \langle U^*Uf, f \rangle = \langle Ef, f \rangle$$
$$= ||Ef||^2 \qquad \text{for all } f.$$

Hence $||Uf|| = ||f||$ or $Uf = 0$ according as $f \in M$ or $f \perp M$.

Theorem 2.2

If A is a bounded linear transformation from a Hilbert space H to a Hilbert space K, then there exists a partial isometry U (from H to K) and there exists a positive operator P (on H) such that $A = UP$. The transformations U and P can be found so that $\ker U = \ker P$ and this additional condition uniquely determines them.

Proof

We first construct P. Since A^*A is a positive operator on H, it has a unique positive square root, call it P. Since

$$\| Pf \|^2 = \langle Pf, \, 'Pf \rangle = \langle P^2 f, \, f \rangle$$

$$= \langle A^*Af, \, f \rangle = \|Af\|^2$$

for all f in H, the equation $UPf = Af$ unambiguously defines a linear transformation U from the range R of P into the space K and that U is isometric on R. Since U is bounded on R it has a unique bounded extension to the closure $\bar{R}$ and a unique extension to a partial isometry from H to K with initial space $\bar{R}$. The equation $A = UP$ holds by construction. The kernel of a partial isometry is the orthogonal complement of its initial space and the orthogonal complement of the range of a Hermitian operator is its kernel. This implies that $\ker U = \ker P$ and this completes the existence proof.

To prove uniqueness, suppose that $A = UP$, where U is a partial isometry P is positive and $\ker U = \ker P$. It follows that $A^* = PU^*$ and hence that $A^*A = PU^*UP = PEP$ where E is the projection from H onto the initial space of U. Since that initial space is equal to $\ker^{\perp} U$ and hence to $\overline{\operatorname{ran} p}$, it follows that $EP = P$ and hence that $A^*A = P^2$. Since the equation $UPf = Af$ uniquely determines U for f in $\operatorname{ran} p$ and since $Uf = 0$ when f is in $\ker p$, it follows that U also is uniquely determined by the stated conditions.

Remark

The representation of A as the product of the unique
U and P satisfying the stated conditions is called the
polar decomposition of A or more accurately, the right-handed
polar decomposition of A.

Definition

In the polar decomposition of an operator, the operators
T*T and TT* frequently occur. We want to know for which
operators do T*T and TT* commute. Campbell [2] called
an operator T binormal   if   [T*T, TT*] = O   i.e
if (T*T)(TT*) = (TT*)(T*T).

The following result gives a simple characterization of
binormal operators.

Theorem 2.3

If T has the polar decomposition T = UP where
ker U = ker P, then, $[T*T, TT*] = O$ iff $[P,\ UPU*] = O.$

Proof

Note that U*U is the projection onto the range of P.
Thus U*UP = P since $P \geq O.$ Then $[T*T,\ TT*] = O$

$$\Leftrightarrow \left[P^2,\quad UP^2\ U*\right] \quad = O$$

$$\Leftrightarrow \left[P,\quad (UP^2\ U*)^{\frac{1}{2}}\right] = O$$

2. $\iff$ $\left[P, \ UPU* \right] \ = \ 0.$

Definition

An operator T is said to be quasinormal if $\left[T,T*T \right] = 0$
i.e. if $T(T*T)=(T*T)T$.

The next result shows that the class of binormal operators
contains the class of quasinormal operators.

Theorem 2.4

Every quasinormal operator T is binormal.

Proof

Since T is quasinormal, $T(T*T) = (T*T)T$.   Taking adjoints
we get $(T*T)T* = T*(T*T)$

Now $(T*T)(TT*) \ = \ (T*TT)T*$

$$= (TT* \ T) \ T*$$

$$= (TT*)(T*T).$$

Hence T is binormal.

Properties of binormal operators

Theorem 2.5

If $T \ \epsilon \ (BN) = \{T: \ \left[T*T,TT* \right] = 0 \ \}$ and   if α is any
complex number, then

1. $\alpha \ T \ \epsilon \ (BN)$

2. $T* \ \epsilon \ (BN)$,   and

$$= \ T^{*}x \ \varepsilon \ \ker \ (T^{2}) = \ker \ T$$

$$= \ TT^{*} \ x = 0$$

$$= \ x \ \varepsilon \ \ker \ (TT^{*}) = \ker(T^{*})$$

i.e   ker (T) $\subseteq$ ker (T*)

Corollary 2.9

If T is binormal and ascents of T and T* are both zero or one, then ker(T) = ker (T*).

Proof

From theorem 2.8, it follows that ker(T)$\subseteq$ ker(T*). Since T* is binormal and the ascent of T* is zero or one, we have that ker (T*) $\subseteq$ ker (T** = T).

$$= \ \ker \ (T^{*}) \ = \ \ker \ (T).$$

Independence of binormal operators.

We now show that (BN) is independent of several major classes of operatos. We know that if T is hyponormal, then it is normaloid and hence spectraloid. If T is any of these three classes of operators, then T need not satisfy the equation $\left[T^{*}T, TT^{*} \right] = 0$.

Example 2.1

If $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $|| \ T \ || = 1$, $w(T) = \frac{1}{2}$

r(T) = 0 and T is binormal. However, T is not spectraloid

3. $T^{-1} \varepsilon$ (BN) if it exists.

Proof

1. $(\alpha T)*(\alpha T)(\alpha T)(\alpha T)* = (\bar{\alpha}T*)(\alpha T)(\alpha T)(\bar{\alpha}T*)$

$$= |\alpha|^4 (T*T)(TT*)$$

$$= |\alpha|^4 (TT*)(T*T) \quad \text{(Since } T \varepsilon \text{ (BN))}$$

$$= (\alpha T)(\alpha T)^* (\alpha T)^* (\alpha T)$$

Hence $\alpha T \quad \varepsilon$ (BN).

2. Let $T^* = U.$ Then

$$(U*U)(UU*) = (TT*)(T*T)$$

$$= (T*T)(TT*)$$

$$= (UU*)(U*U)$$

Hence $U = T*$ is binormal.

3. If T is invertible, so is $T*$. Hence both $TT*$ and $T*T$ are invertible. Since $T \varepsilon$ (BN) we have

$$(T * T)(TT*) = (TT*) (T * T) \tag{a}$$

Let $T^{-1} = V.$ Taking the inverse of both sides of (a), we get

$$(TT*)^{-1} (T * T)^{-1} = (T * T)^{-1}(TT*)^{-1}$$

$$\Rightarrow T*^{-1} T^{-1} T^{-1}T*^{-1} = T^{-1} T*^{-1} T*^{-1} T^{-1}$$

$$\Rightarrow (T^{-1})* T^{-1} T^{-1}(T^{-1})* = T^{-1}(T^{-1})*(T^{-1})*T^{-1}$$

i.e. $(V*V)(VV*) = (VV*)(V*V)$

$$\Rightarrow V = T^{-1} \varepsilon \text{ (BN)}.$$

Remark

Note that (BN) is not closed under addition even when the operators commute.

For let $T_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and

$T_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Now $T_1 \in$ (BN) and

clearly $T_2 \in$ (BN). Let $T = T_1 + T_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Then $\left[T^*T, \ TT^* \right] = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

$\Rightarrow \quad T \notin$ (BN).

This example also shows that if $T \in$ (BN) then it does not imply that $\mu I + T \in$ (BN) for complex numbers μ. However, $\mu I + T$ will belong to (BN) if in addition T is normal as the next result shows.

Theorem 2.6

Suppose that $T \in$ (BN). The $T + \lambda I \in$ (BN) some complex $\lambda \neq 0$, iff T is normal.

Proof.

Suppose that $T \in$ (BN). Let $\lambda \neq 0$ be real. Then $\left[(T + \lambda I)^* (T + \lambda I), (T^* + \lambda I)(T + \lambda I)^* \right] = 0$ is equivalent to $\left[\left[T^*, T \right], T + T^* \right] = 0$

Thus if $T + \lambda I \in$ (BN) for some real $\lambda \neq 0$, then

$T + \lambda I \in$ (BN) for all real λ. But $0 \notin W(T + \lambda I)$ for λ

sufficiently large so T would be normal. The case when λ is

complex easily reduces to the one when λ is real.

Theorem 2.7

If T_1 and T_2 are doubly commutative binormal operators (i.e.

$T_1 T_2 = T_2 T_1$ and $T_1 T_2{}^* = T_2{}^* T_1$) then $T_1 T_2$ is binormal.

Proof

The proof is omitted since it is too lengthy.

Definition

The _ascent_ of an operator T is defined to be the smallest

positive integer n for which $\ker (T^n) = \ker (T^{n+k})$ for

all positive integers K.

Theorem 2.8

If T is binormal and ascent of T is zero or one, then

$\ker (T) \subseteq \ker(T^*)$

Proof.

$x \in \ker(T) \implies TT^*T^*Tx = 0$

$\qquad \implies T^*T\, TT^*x = 0$

$\qquad \implies TT^*x \in \ker(T^*T) = \ker(T)$

$\qquad \implies T^2 T^* x = 0.$

$$\Rightarrow T^*x \; \varepsilon \; \ker (T^2) = \ker T$$

$$\Rightarrow TT^* \; x = 0$$

$$\Rightarrow x \; \varepsilon \; \ker (TT^*) = \ker (T^*)$$

i.e $\ker (T) \; \subseteq \; \ker (T^*)$

Corollary 2.9.

If T is binormal and ascents of T and T* are both zero or one, then $\ker (T) = \ker (T^*)$

Proof.

From theorem 2.8, it follows that $\ker (T) \subseteq \ker (T^*)$. Since T* is binormal and the ascent of T* is zero or one, we have that $\ker (T^*) \subseteq \ker (T^{**} = T)$.

 $\Rightarrow \ker (T^*) = \ker (T)$.

Independence of binormal operators.

We now show that (BN) is independent of several major classes of operators. We know that if T is hyponormal, then it is normaloid and hence spectraloid. If T is any of these three classes of operators, then T need not satisfy the equation $\left[T^*T, \; TT^* \right] = 0.$

Example 2.1.

If $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,   then $\|T\| = 1, \quad w(T) = \tfrac{1}{2},$

$r(T) = 0$ and T is binormal. However, T is not spectraloid

since $r(T) \neq w(T)$. Also, T is not normaloid since $w(T) \neq || T ||$ and hence T is not hyponormal. Thus there are non-spectraloid operators in (EN).

Example 2.2

Let S be a weighted shift. Then $S*S$ and $SS*$ are both diagonal operators and hence commute. In particular, let T be the unilateral weighted shift with weight sequence $\{1, \frac{1}{2}, 1, 1, \ldots\}$. Then $r(T) = w(T) = || T || = 1$. T is not hyponormal since the weight sequence is not monotonic non-decreasing. Thus T is not hyponormal but it is binormal and normaloid.

One of the problems of Hilbert space theory is whether or not every operator has a non-trivial invariant subspace i.e. a subspace M of H such that $TM \subsetneq M$. The following theorem sheds some light on this problem in the case of binormal operators.

Theorem 2.10

If T is hyponormal and $[T*T, TT*] = 0$ then T has an invariant subspace.

Proof.

See Campbell [2].

Remark

If T is binormal, we have seen that $T*$ is also binormal. In theorem 2.10 we could therefore have assumed that T or $T*$

were hyponormal, that is, that T was semi-normal.

Finally, we note that if we intersect the classes of binormal and paranormal operators, we get the class of hyponormal operators as the next result shows.

Theorem 2.11

Suppose that $T \in (BN)$. If T is also paranormal, then it is hyponormal.

Proof

See Campbell [3].

CHAPTER THREE

Dominant Operators

Dominant Operators were first introduced by Stampfli and Wadhwa [17]. Duggal [5] has also studied this class of operators in trying to extend the Putnam-Fuglede theorem for larger classes of operators than normal operators.

Definition

An operator T is said to be dominant if $\text{ran}(T-\lambda) \subseteq \text{ran}(T-\lambda)^*$ for all $\lambda \in \sigma(T)$.

Remark.

Using a result of Douglas [4], which will be stated below, we can get an equivalent definition of dominant operators which is more commonly used.

Theorem A.

Let $A, B \in B(H)$. The following statements are equivalent:

1. $\text{ran } A \subseteq \text{ran } B$

2. $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$; and

3. there exists a bounded operator C on H so that $A = BC$.

Proof.

See Douglas [4].

From the equivalence of (1) and (2) in theorem A, we get that T is dominant iff $(T-\lambda)(T-\lambda)^* \leq \lambda^2 (T-\lambda)^*(T-\lambda)$.

Thus T is dominant iff for each $\lambda \in \sigma(T)$, there exists a constant $M_\lambda > o$ such that

$$||(T-\lambda)^*f|| \le M_\lambda ||(T-\lambda)f|| \quad \text{for all} \quad f \in H.$$

Remark.

If T is dominant, we have seen that there exists a constant M_λ such that $||(T-\lambda)^*f|| \le M_\lambda ||(T-\lambda)f||$ for all $f \in H$. A question which might arise is whether a single constant M will suffice for all λ's i.e. must the set of all M_λ's i.e $\{M_\lambda\}$, be bounded? The answer is negative since there exists an example of a dominant operator where the associated M_λ's of the dominant operator are not bounded. (See [17]).

Properties of dominant operators.

Theorem (3.1) (Generalised Putnam-Fuglede).

Let $T \in B(H)$ be dominant. Let N be normal. Assume $TW = WN$ where $W \in B(H)$ has dense range. Then T is normal and moreover W^*W commutes with N.

Proof.

See [17].

Corollary 3.2

Let $T \in B(H)$ be hyponormal. If T is similar to a normal operator, then T is normal.

Corollary 3.3

Let $T \in B(H)$ be dominant. Let $TW = WN$ where N

is normal and W is any non-zero operator in B(H).

Then T has a proper invariant subspace.

Proof.

See [17].

Definition

If A B(H) and M is a linear subspace of H, M is
reducing if AM $\subseteq$ M and AM$^\perp$ $\subseteq$ M$^\perp$.

Lemma 3.4

Let T ϵ B(H) be dominant. Let M $\subseteq$ H be an
invariant subspace for T. If T$|_M$ is normal,
then M reduces T. (i.e. M is a reducing
subspace of T)

Proof.

See [17].

Theorem 3.5

Let T ϵ B(H) be dominant. Let TB = BN where
B $\geq$ 0 and N is normal. Then T = T$_1$ $\oplus$ T$_2$ on
ker B $\oplus$ ker B$_\perp$ where T$_2$ is normal.

Proof.

See [17].

Theorem 3.6

Let T B(H) be dominant. If P(T) is normal
for some polynomial P, then T is normal.

Proof

See $[17]$.

Subclasses of dominant operators.

(i) Quasinormal operators.

Quasinormal operators were first introduced and studied by Brown $[1]$. He called an operator A such that $A(A*A) = (A*A)A$ an operator with property i.n.

We now show that the class of quasinormal operators contains the class of normal operators.

Theorem 3.7

Every normal operator T is quasinormal.

Proof.

Since T is normal, $TT* = T*T$. Now $T(T*T) = (TT*)T = (T*T)T$. $\Rightarrow$ T is quasinormal.

Remark.

The converse of theorem 3.7 is false for let T be an isometry. We know that a necessary and sufficient condition that a linear transformation be an isometry is that $T*T = I$. Then $T(T*T) = TI = (T*T)T = IT = T$ i.e. T commutes with $T*T$ $\Rightarrow$ T is quasinormal. If T is not unitary, $TT* \neq T*T = I$. i.e. T is not normal.

The following theorem characterizes quasinormal operators.

Theorem 3.8

The following are mutually equivalent conditions on an

arbitrary operator A:

1. A is quasinormal.

2. If A = UP is the polar decomposition of A,
then UP = PU.

3. A = VP = PV with P $\geq$ 0 and V isometric.

4. A = VB = BV with B self-adjoint and V
isometric.

Proof.

See [1].

Definition

Let A be a convex set. A point a$\in$A is called an
extreme point of A if it does not belong to the
interior of any segment in the set A.

Theorem 3.9

If T is quasinormal, then T^n, n$\in$N, is
quasinormal.

Proof.

See [14] .

Theorem 3.10

If T is quasinormal and satisfies any one of the
following conditions, then T is normal

(i)   ker(T*) $\subseteq$ ker (T).

(ii) the ascent of T* is 0 or 1.

(iii) 0 is an extreme point of W(T).

(iv)   ker $(T^*) = \{0\}$

(v) T^* is invertible

(vi) $T - \lambda I$ is quasinormal for all real λ.

Proof.

(i)   Since $H = \ker(T^*) \oplus \overline{T(H)}$, for any $x \in H$, $x = u + v$, $u \in \ker(T^*)$ and $v \in \overline{T(H)}$. Now the normality of T follows from the fact that

$$\|T^*x\| = \|T^*v\| = \|Tv\| = \|Tx\|$$

(ii) $x \in \ker(T^*) \Rightarrow T^*x = 0 \Rightarrow T^*TT^*x = 0$

$\Rightarrow T^*T^*Tx = 0 \Rightarrow Tx \in \ker(T^{*2}) = \ker(T^*) \Rightarrow$
$T^*Tx = 0 \Rightarrow x \in \ker(T^*T) = \ker(T)$ i.e.
$\ker(T^*) \subseteq \ker T$.

The result follows from (i)

(iii) T being hyponormal (to be proved later) and 0 being an extreme point of $W(T)$, by a result of Stampfli [16], $\ker(T) = \ker(T^*)$, which is the case (i).

(iv) $\ker(T^*) = \{0\} = \ker(T^*) \subseteq \ker(T)$

(v) T^* being invertible, $\ker(T^*) = \{0\}$
The result follows from (iv).

(vi) For $\lambda_0 \notin \sigma(T)$, $S = T - \lambda_0 I$ is an invertible quasinormal. Hence S is normal and consequently T is normal.

Theorem 3.11

Let $T = A + iB$ be a cartesian   decomposition

of T. Then T is quasinormal if AB commutes with A and B.

Proof.

T = A + iB where A and B are self-adjoint i.e. $A^* = A$, $B^* = B$.

$\therefore$ $T^* = A^* - iB^* = A - iB$.

$T(T^*T) = A(A^2+B^2) - B(AB-BA) + i\left[A(AB-BA) + B(A^2+B^2)\right]$.

$(T^*T)T = (A^2+B^2)A - AB^2 + BAB + i\left[A^2B + B^3 + ABA - BA^2\right]$

$T(T^*T) - (T^*T)T = 2i\left[BA^2 - ABA\right] + 2(AB^2 - BAB)$

Since $(AB)A = A(AB)$ and
$(AB)B = B(AB)$

$\Rightarrow$ $\left[(AB)\bar{A}\right]^* = \left[A(AB)\right]^*$

i.e. $ABA = BA^2$

$\Rightarrow$ $T(T^*T) - (T^*T)T = 0$

i.e. $T(T^*T) = (T^*T)T$.

$\Rightarrow$ T is quasinormal.

Corollary 3.12

Let T = A+iB and AB = C + iD.

Then T is quasinormal if both A and B commutes with each C and D.

Proof.

From theorem 3.11 above, T is quasinormal if AB commutes with A and B i.e. if

$$(AB)A = A(AB) \qquad (i)$$

$$(AB)B = B(AB) \qquad (ii)$$

From (i), we get

$$(C+iD)A = A(C+iD)$$

$$\Rightarrow \quad CA+iDA = AC + iAD$$

$$\Rightarrow \quad CA = AC \quad \text{and} \quad DA = AD.$$

From (ii), we get

$$(C+iD)B = B(C+iD)$$

$$\Rightarrow \quad CB + iDB = BC + iBD$$

$$\Rightarrow \quad CB = BC \quad \text{and} \quad DB = BD$$

$$\Rightarrow \quad A \text{ and } B \text{ commutes with both } C \text{ and } D.$$

(ii) <u>Subnormal operators</u>

Another way of weakening the hypotheses of normal operators is that of defining subnormal operators.

<u>Definition</u>.

An operator A on a Hilbert space H is <u>subnormal</u> if it has a normal extension. In other words, A is subnormal if there exists a normal operator B on a Hilbert space K such that H is a subspace of K, the subspace H is invariant under the operator B and the restriction of B to H coincides with A.

<u>Theorem 3.13</u>

Every quasinormal operator is subnormal.

Proof.

Let A be quasinormal. Then ker A reduces A since
ker A = ker A*A for every operator A and since
A* commutes with A*A(A being quasinormal) it follows
that ker A*A is invariant under A*. Thus every
quasinormal operator is the direct sum of 0 and an
operator with trivial kernel. Without loss of gene-
rality, we can assume that ker A = 0. If UP is
the polar docomposition of A, then U is an
isometry and UP = PU and U*P = PU*. Since
U is isometric, then if E is the projection UU*,
then

$$(I - E)U = U*(I-E) = 0.$$

We now construct a normal extension for A.
If A acts on H, then a normal extension B
can be constructed that acts on H ⊕ H (if H is
identified with H ⊕ 0, then H is a subspace
of H ⊕ H). An operator on H ⊕ H is given by a
two-by-two matrix whose entries are operators on
H. If in particular

$$V = \begin{pmatrix} U & I-E \\ 0 & U^* \end{pmatrix}$$

and

$$Q = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$$

then V is unitary, Q is positive, V and Q

commute and therefore

$$B = VQ = \begin{pmatrix} VP & (I-E)P \\ 0 & U*P \end{pmatrix}$$

is a normal extension of A.

Remark

The converse of theorem 3.13 is false for consider
a non-zero scalar added to the unilateral shift.
This is subnormal just like the unilateral shift but
if it were also quasinormal, then the unilateral
shift would be normal.

Definition

A normal extension B (on K) of a subnormal operator
A (on H) is _minimal_ if there is no reducing subspace
of B between H and K. In other words, B is
minimal over A if whenever M reduces B and
$H \subseteq M$, it follows that M = K.

Theorem 3.14

If B_1 and B_2 (on K_1 and K_2) are minimal normal
extensions of the subnormal operator A on H, then
there exists an isometry U from K_1 onto K_2
that carries B_1 onto B_2 (i.e. $UB_1 = B_2 U$)
and is equal to the identity on H.

Proof.

See [7] pp. 306-307.

Remark.

In view of the result of theorem 3.14 we can talk

of "the" minimal normal extension of a subnormal

operator.

Theorem 3.15

If A is subnormal and if B is its minimal normal

extension, then $\sigma(B) \subseteq \sigma(A)$.

Proof.

See $\boxed{7}$, p. 308.

(iii) Hyponormal operators.

If A (on H) is subnormal, with normal extension

B(on K) we want to find out the relation between A*

and B*. Consider the projection P from K onto H.

If f and g are in H, then

$$\langle A^*f, g \rangle = \langle f, Ag \rangle = \langle f, Bg \rangle$$
$$= \langle B^*f, g \rangle = \langle B^*f, Pg \rangle$$
$$= \langle PB^*f, g \rangle$$

Since the operator PB* on K leaves H invariant,

its restriction to H is an operator on H and this

restriction is equal to A*. We have A*f = PB*f

for every f in H. If f ε H, then

$$||A^*f|| = ||PB^*f|| \leq ||B^*f|| = ||Bf||$$

(since B is normal)

$$= ||Af||$$

i.e. $||A^*f|| \leq ||Af||$.

This is equivalent to the operator inequality
$AA^* \leq A^*A$.

Properties of hyponormal operators.

Theorem 3.16

Every subnormal operator is hyponormal.

Proof.

See the remarks following theorem 3.15.

Remark.

The converse of theorem 3.16 is false since there exists a hyponormal operator that is not subnormal (see [7], pp. 309-310).

Theorem 3.17

If A is hyponormal, then $||A^n|| = ||A||^n$
and so $||A|| = r(A)$.

Proof.

If $f \in H$ and $n \geq 1$, then

$$||A^n f||^2 = \langle A^n f, A^n f \rangle$$

$$= \langle A^* A^n f, A^{n-1} f \rangle$$

$$\leq ||A^* A^n f|| \ ||A^{n-1} f||$$

$$\leq ||A^{n+1} f|| \ ||A^{n-1} f||$$

Hence $||A^n||^2 \leq ||A^{n+1}|| \ ||A^{n-1}||$

Suppose $||A^k|| = ||A||^k$ for $1 \leq k \leq n$.

Then $\quad ||A||^{2n} = ||A^n||^2 \leq ||A^{n+1}|| \quad ||A^{n-1}||$

$$= ||A^{n+1}|| \quad ||A||^{n-1}$$

So $\quad ||A||^{n+1} \leq ||A^{n+1}|| \leq ||A||^{n+1}$

Since $\quad r(A) = \lim ||A^n||^{1/n} = ||A||$.

Theorem 3.18

If T is a hyponormal operator such that $\sigma(T)$ is a set of real numbers, then T is self-adjoint.

Proof.

See [15].

Definition.

A complex number λ is said to be an approximate proper value for the operator T if there exists a sequence x_n such that $||x_n|| = 1$ and $||(T-\lambda I) x_n|| \to 0$.

The approximate point spectrum of an operator T, denoted by $\pi(T)$, is the set of approximate proper values of T.

Theorem 3.19

Let T be a hyponormal operator and let

$\lambda_1, \lambda_2 \in \pi(T)$, $\lambda_1 \neq \lambda_2$. If x_n and y_n are the sequences of unit vectors of H such that

$||(T-\lambda_1 I) x_n|| \to 0$ and $||(T-\lambda_2 I)y_n|| \to 0$, then

$\langle x_n, y_n \rangle \to 0$

Proof.

See [15].

Theorem 3.20

If T is hyponormal, $\sigma(T^*) = \pi(T^*)$.

Proof.

See [15] .

Theorem 3.21

Let N be a hyponormal operator. If for an
arbitrary operator A, for which $0 \notin \overline{W(A)}$,
$AN = N^*A$, then N is self-adjoint.

Proof.

Since $0 \notin \overline{W(A)}$, A is invertible. Hence
$N = A^{-1}N^*A$ and it follows from theorem 3.20 and
the fact that if A and B are similar, then
$\pi(A) = \pi(B)$, that $\sigma(N) = \sigma(N^*) = \pi(N^*) = \pi(N)$.
By theorem 3.18, it is sufficient to prove that $\sigma(N)$
is real. Suppose on the contrary that there exists a $z \in \sigma(N)$ such that
$z \neq \bar{z}$. Since $z \in \sigma(N) = \pi(N)$, there exists a sequence x_n of unit
vectors such that $||(N^*-\bar{z}I)x_n|| \leq ||(N-zI)x_n|| \to 0$
Since $0 \notin \overline{W(A)}$, the relation

$$||(N^*-\bar{z}I)x_n|| = ||(ANA^{-1} - \bar{z}I)x_n|| = ||A(N-\bar{z}I)A^{-1}x_n|| \to 0$$

implies that $||(N-\bar{z}I)A^{-1}x_n|| \to 0$.
Hence $\langle x_n, A^{-1}x_n \rangle = \langle AA^{-1}x_n, A^{-1}x_n \rangle \to 0$ by
theorem 3.19. Put $y_n = A^{-1}x_n /||A^{-1}x_n||$, then

$||y_n|| = 1$ and $\langle Ay_n, y_n \rangle \to 0$ i.e. $0 \in \overline{W(A)}$,

a contradiction.

Corollary 3.22

Let N be a seminormal operator (i.e. N is such that either N or N* is hyponormal). If for an arbitrary operator A, for which $0 \notin \overline{W(A)}$, AN = N*A, then N is self-adjoint.

Proof.

Suppose that N* is hyponormal. By the proof of theorem 3.21, $0 \notin \overline{W(A)}$ implies $0 \notin \overline{W(A^{-1})}$. Now AN = N*A implies $A^{-1}N^* = NA^{-1}$ i.e. BM = M*B where M = N* is hyponormal and $0 \notin \overline{W(B)} = \overline{W(A^{-1})}$. Hence M = M* by theorem 3.21 i.e. N = N*.

Theorem 3.23

Let T be hyponormal; then $||Tx|| = ||T^*x||$ iff TT*x = T*Tx.

Proof.

See [16].

Theorem 3.24

Let T be hyponormal on H. Then W(T) has at most a countable number of extreme points.

Proof.

See [16] .

(iv) M-hyponormal operators

The notion of an M-hyponormal operator is due to Stampfli (unpublished) and is a generalisation of a hyponormal operator.

Definition

An operator T on H is called an __M-hyponormal__
operator if there exists a real number M such that
$||(T-zI)^*x|| \leq M||(T-zI)x||$ for all x $\in$ H and for
all z $\in$ $\mathbb{C}$.

Properties of M-hyponormal operators

Theorem 3.25

Every hyponormal operator is M-hyponormal

Proof.

Let T be hyponormal. Then $T^*T-TT^* \geq 0$ or equivalently
$||T^*x|| \leq ||Tx||$ for all x $\in$ H. Note that T is
hyponormal iff M = 1 where M is the one used in the
definition. Thus T is M-hyponormal.

Remark

The converse of theorem 3.25 is false since there exists
an example of an M-hyponormal operator which is not
hyponormal
(See $[18]$).

Proposition 3.26

T is an M-hyponormal operator iff $M^2(T-z)^*(T-z)-(T-z)(T-z^*)$
≥ 0 for all z $\in$ $\mathbb{C}$.

Proof.

See $[18]$.

Proposition 3.27

If T is an M-hyponormal operator, then
(i) $Tx = zx$ implies that $T^*x = \bar{z}x$

(ii) $\|(T^*-\bar{z})^{-1}x\| \leq M\|(T-z)^{-1}x\|$ for all z in $P(T)$.

Proof

See [18].

Proposition 3.28

Let T be an M-hyponormal operator.

(i)   If $(T-z)^n x = 0$ then $(T-z)x = 0$

(ii) If $Tx = z_1 x$ and $Ty = z_2 y$, $z_1 \neq z_2$
      then $\langle x, y \rangle = 0$.

(iii) If there exists a polynomial $p(z)$ such
that $P(T) = 0$, then T is normal.

(iv) If H is finite dimensional, then T is normal.

Proof.

See [18].

Theorem 3.29

Let T be an M-hyponormal operator which is similar
to a normal operator and such that the area of $\sigma(T)$ is
zero. Then T is a normal operator.

Proof.

See [18].

Theorem 3.30

Every M-hyponormal operator is dominant.

Proof.

The proof follows immediately from the definitions
of M-hyponormal and dominant operators.

Remark

The converse of theorem 3.30 is false since there

exists a dominant operator which is not M-hyponormal.

Theorem 3.31

If T is an M-hyponormal operator on H, then λT, $\lambda \neq 0$ and $T + \lambda I$ are also M-hyponormal operators for every scalar λ .

Proof.

See [12] .

Theorem 3.32

Let $\{T_n\}$ be a sequence of M-hyponormal operators converging uniformly to the M-hyponormal operator T, so that $\|T_n - T\| \to 0$ as $n \to \infty$. Then z_0 is in $\sigma(T)$ iff there exists a sequence $\{z_n\}$, z_n are in $\sigma(T_n)$, such that $z_n \to z_0$.

Proof.

See [12] .

Theorem 3.33

Let T be an M-hyponormal on H. Then there exists operators A, B and S which satisfy

(i) $B \geq A \geq 0$

(ii) $\|S\| \leq M$, $M \geq 1$.

(iii) $S^*AS = B$.

In this case, T can be expressed as

$$T = \frac{1}{M}(A^{\frac{1}{2}}S + \lambda I) \quad \text{for some complex } \lambda .$$

Proof.

See [12].

Theorem 3.34

If H is finite-dimensional and T is an
M-hyponormal operator on H, then T is normal.

Proof.

See [12].

Extensions of the Putnam-Fuglede theorem.

The Putnam-Fuglede theorem states that if A and B
are normal operators and if X is an operator such
that AX = XB then A*X = XB* (see [7]).

We will now look at some of the relaxations of
the hypotheses on A and B.

Theorem 3.35

If A and B* are subnormal and if X is an
operator such that AX = XB, then A*X = XB*.

Proof.

See [6].

The following theorem extends the Putnam-Fuglede
theorem to the case when A and B are M-hyponormal.

Theorem 3.36

If A and B are M-hyponormal and AX = XB*,
then A*X = XB.

Proof.

See [9].

Duggal [5] extended theorem 3.36 to the case when A is dominant and B* is M-hyponormal, as the next theorem shows.

Theorem 3.37

Let A be dominant, B* be M-hyponormal such that AX = XB. Then A*X = XB*.

Proof.

See [5].

Definition.

Let H and K be Hilbert spaces, $S \in B(H)$ and $T \in B(K)$. The commutator $C(S,T): B(K,H) \to B(K,H)$ of S and T is defined by $C(S,T)A = SA - AT$. $C^n(S,T)$ will mean n times application of $C(S,T)$.

Theorem 3.38

Let S and T* be hyponormal and assume $C^n(S,T)A = 0$ for some natural number n and some $A \in B(K,H)$. Then $C(S,T)A = C(S^*,T^*)A = 0$.

Proof.

See [13].

Conclusion

In all the three classes of operators that we have studied, there is still a lot of research

that can be carried out to determine other properties
of these operators.

There is still a lot to be done especially in the
extension of the Putnam-Fuglede theorem. One research
problem which can be tackled by those interested in
this field is to try to extend the work done by
Furuta [6] , Moore et al [9] Duggal [5] and Radjabalibour
[13] by trying to answer the following question:
Does the Putnam-Fuglede theorem hold if A and B
are both dominant?

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