ANALYSIS OF FINANCIAL RISK USING EXTREME VALUE THEORY

UNIVERSITY OF NAIROBI

SCHOOL OF MATHEMATICS

Submitted by:

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Declaration

I the undersigned declare that this project report is my original work and that to the best of my knowledge has not been presented for the award of a degree in any other University.

Signature ..... Date 18/08/2009

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This project report has been submitted for examination with my approval as a supervisor.

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Dr. Patrick G. O. Weke

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Abstract

Financial Risk management is about understanding large movements in the financial market. This study examines the modeling of extreme financial data using the methods of Extreme Value Theory. The two models are fitted to the NSE 20 Share Index and it emerges that the Peaks Over Threshold model gives a better fit to the data as opposed to the Block Maxima Model. The maximum likelihood method has been used to estimate the parameters of the extreme value models.The Extreme Value Theory based quantiles are used to estimate the Value-at-Risk,Expected shortfall and the Return level for the the data.

Key words: Extreme Value Theory, Generalized Extreme Value, Block Maxima Model, Generalized Pareto Distribution,Peaks Over Threshold Model, Value-at Risk,Expected shortfall.

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Chapter 1

Introduction

1.1 A brief History of Risk Management

Risk management is a practice that can be traced as far back as the ancient Babylon in 1800 BC where options were used as a means to provide cover against risk, though it is only until the twentieth century that a formula for valuation of derivatives was developed. Before the 1950s the success of an investment was based on its return. Due to the dynamic nature of the financial world, the theory of portfolio selection was established in the 1950's, where risk was measured using standard deviation.

The following decades saw several methods of risk management methodology evolving including the Sharpe ratio, the Capital Asset Pricing Method and the Asset Pricing Theory.

The Black-Scholes-Merton formula for valuation of derivatives appeared in 1973 and not only did it work but it changed the world of finance. Its importance was felt in 1978 when the crash of the American Stock market was blamed on the formula.

Technological developments, especially in information technology, also aided the growth of new risk management and investment products which was further aided by the deregulation of the 1980s and the abolition of the Bretton-Woods system of fixed exchange rates.

Regulation is an important part of the financial market and hence the Basel Committee of Banking Supervision was established in 1974 by the central bank governors of the Group of ten (G-10). This committee formulates broad supervisory standards and guidelines and recommends statements of best practice. The first Basel Accord on Banking Supervision was formulated in 1988 with its main emphasis on credit risk. The second Basel Accord adopts the three-pillar concept which seeks to achieve a more holistic approach to risk management by focusing on interaction between different risks categories.

Recent financial disasters and volatility of financial markets have emphasized the importance of effective risk management for financial institutions.

1.2 Financial Risk Management

Financial risk management is about understanding the large movements in the value of asset portfolios. Assessing the probability of rare and extreme events is an important issue in the risk management of financial portfolios. One of the goals of financial risk management is the accurate calculation of large potential losses due to extreme events such as stock market crashes. Managing extreme risks often requires estimating quantiles and tail probabilities beyond those observed in the data. The use of quantitative risk measures has become an essential management tool to be placed in parallel with models of return.

Extreme value theory provides the solid fundamentals needed for the statistical modeling of such events and the computation of extreme risk measures. Extreme Value theorem seeks to quantify the probabilistic behavior of unusually large losses and to develop the tools for managing extreme risks. Traditional parametric and non-parametric methods for estimating distributions and densities often give very poor fits to the extreme tails of the distribution.This is because they make inferences about the tail distribution after estimating the entire return distribution. The methods of extreme value theory focus on modeling the tail behavior of a loss distribution using only extreme values rather than all the data.The link between Extreme Value the-

orem and risk management is that Extreme Value theory methods fit extreme quantiles better than conventional approaches for heavy-tailed data.

1.3 Measures of Tail Risk

Value at Risk (VaR)

Value at Risk (VaR) summarizes the worst loss over a target horizon with a given level of confidence.It is an estimate of how much a given portfolio can lose at a given time period and confidence interval.VaR is a popular approach as it provides a single quantity that summarizes the overall market risk faced by an institution.

VaR was proposed by J.P. Morgan in 1994 and became a standard measure that financial analysts use to quantify risk.According to the Basle committee 1996 paper 'Amendment to the Capital Accord to incorporate Market Risks',in calculating the value-at-risk, a 99th percentile, one-tailed confidence interval is to be used.

Expected Shortfall(ES)

This is defined as the expected size of a loss that exceeds VaR.

1.4 Objectives of the Study

The main objective of this study will be to quantify the probability of unusually large losses and to develop tools for managing extreme risks. The specific objectives are;

- Construct Extreme Value Theorem Models for analysis of extreme risks.
- Quantify large financial losses using Extreme Value Theorem.
- Empirical comparison of the two models of extreme value.

1.5 Significance of the Study

The results of this study will contribute considerably to the financial market practitioners who would like to study the behavior of extreme movements in the market.

This study will also form good reference material for further research into extreme market risk movements.

1.6 Organization of Report

Chapter two examines related works that have been done with respect to extreme value theory and related studies.

Chapter three looks at the extreme value models, the Block Maxima Model and the Peaks Over Threshold Model,the various risk measures, value at risk and expected shortfall.

Chapter four discusses the exploratory data analysis and estimates the parameters of the Block Maxima Model and the Peaks Over Threshold Model. Conclusions and recommendations for further research are then made and a list of references given.

Review

Chapter 2

Literature Review

In recent years the banking industry has recognized the importance of operational risk in shaping the future of financial institutions. This recognition has led to an increased emphasis on the importance of sound operational risk management at financial institutions and to greater prominence of operational risk in banks. Several methods have been used to analyze Value at Risk (VaR) with almost the same conclusion, that extreme value theory (EVT) is the most suitable method to analyze the tail distribution.

Sarma(2002) analyses the tail behavior of the Nifty innovation distribution using extreme value theory and finds that the essential features of the innovation distribution is very different from the normal distribution and that the extreme value theory based generalized Pareto distribution model of tail estimation is able to capture these features of the innovation distribution and

gives a better fit as opposed to the normal distribution. The differences in the upper and lower tail behaviors necessitate different treatment of both.

Recently studies concerning stock market returns have been done. Gencay and Selcuk(2004) investigate the relative performance of Value-at-Risk (VaR) models with the daily stock market returns of nine different emerging markets. Well known modeling approaches such as the variance-covariance method, historical simulation and the extreme value theory (EVT) are used to generate VaR estimates and provide the tail forecasts of daily returns at the 0.999 percentile along with 95 percent confidence intervals for stress testing purposes. The results indicate that EVT based VaR estimates are more accurate at higher quantiles. It is concluded that the Generalized Pareto Distribution and the extreme value theory are an indispensable part of risk management in general and the VaR calculations in particular, in emerging markets.

A similar paper, Maghyereh and Al-Zoubi(2006), concludes that the return distributions of the Middle east and North African (MENA) markets are characterized by fat tails which implies that VaR measures relying on the normal distribution will underestimate VaR and suggests that the extreme value approach, by modeling the tails of the return distributions, are more relevant to measure VaR in most of the MENA.

Ahangarani(2005) estimates VaR using both the classical non parametric

methods and extreme value theory. The Monte Carlo simulations used in this case concludes that EVT has less bias to relative to the classical non parametric methods.

The issue of limited data and selection of a threshold are tackled in Plesko, (2006). In the case of minimal data where extreme risk measures are necessary it is concluded that EVT is the natural method of estimating the risk measures. Selection of a threshold is found to be a trade off between accuracy and variance. The paper analyses the quality and applicability of extreme value theorem for the estimation of high quantiles in the application of Operational Risk. In addition the Log normal tail fitting as an alternative to EVT is studied and the conclusion is that it is not yet able to yield satisfying results for quantile estimations and that EVT may be the only viable method to accurately estimate extreme quantiles. However it is found that expected EVT quantile estimators overestimate theoretical quantiles, which comes from the functional form of the quantile estimator.

Gilli and Kellezi(2006) uses both the Block Maxima Model and the Peaks Over Threshold Model to model tail-related risk measures such as Value-at-Risk, expected shortfall and return level concludes that POT method is more superior as it better exploits the information in the data sample.

Francois M. Longin(1999) uses an application of extreme value theory to compute the value at risk of a market position. and concludes that extreme

value method has three main advantages over classical methods. First, as the extreme value method is parametric, out-of-sample Value at Risk computations are possible for high probability values. Second, as the extreme value method does not assume a particular model for returns but lets the data speak for themselves to fit the distribution tails, the model risk is considerably reduced as opposed to the normal distribution or any given distribution. Third, as the extreme value method focuses on extreme events, the event risk is explicitly taken into account.

Alexander J. McNeil(1999)concentrates on the peaks-over-threshold (POT) model and emphasizes the generality of this approach.In addition he finds that whenever tails of probability distributions are of interest, it is natural to consider applying the theoretically supported methods of EVT as methods based around assumptions of normal distributions are likely to underestimate tail risk. Methods based on historical simulation can only provide very imprecise estimates of tail risk. EVT is the most scientific approach to an inherently difficult problem - predicting the size of a rare event. Review

Chapter 3

Methodology

Extreme Value Theory(EVT)

Extreme value theorem deals with the study of the asymptotic behavior of extreme (minima and maxima) observations of a random variable. By dealing with only extreme observations,EVT can provide a better treatment to the estimation of tail quantiles like Value at Risk (VaR) and does not require to make a priori assumption about the return distribution.Moreover, EVT based methods inherently incorporates separate estimation of the upper and the lower tails, and thereby emphasizes the necessity to treat both the tails separately due to possible existence of asymmetry in the return series. There are two main approaches to extreme values;the Block Maxima Model and the Threshold exceedances Model.The Block Maxima Model is

the traditional model of analyzing data it is however wasteful of data and has been superceeded by the threshold exceedances which utilizes data on extreme outcomes more efficiently.

3.1 Block Maxima Model

This approach utilizes the 'extremal types theorem' to model the distribution of extreme (largest or smallest) observations collected from non-overlapping blocks of fixed size from the data. Then the 'generalized extreme value' distribution is fitted to these block extrema. This distribution reflects the behavior of very high profits (in case of maxima)and very high losses (in case of minima) from the portfolio.

Definition 3.1 ***Limit Probabilities for Maxima:*** *Let* $X_1, X_2, ...$ *be a sequence of iid non-degenerate random variables with common distribution function* F *and let sample maxima be*

$M_1 = X_1, M_n = max(X_1, X_2, ..., X_n), n \geq 2$. *The corresponding minima is obtained from the maxima by using the identity;*

$min(X_1, X_2, ..., X_n) = -max(-X_1, -X_2, ..., -X_n)$.

The distribution function of the maximum M_n is:

$$P(M_n \leq x) = P(X_1 \leq x, ..., X_n \leq x) = F^n(x), \qquad x \in \mathbb{R}, n \in \mathbb{N}.$$

The behavior of M_n is related to the distribution function F in its tail near the right endpoint, denoted by;

$$x_F = sup\{x \in \mathbb{R} : F(x) < 1\}$$

Definition 3.2 ***Max-stable distribution****: A non-degenerate random variable X is called max-stable if it satisfies*

$$max(X_1, ...X_n) \stackrel{d}{=} c_n X + d_n \tag{3.1}$$

for iid $X, X_1, ..., X_n$*,appropriate constants* $c_n > 0$ *and* $d_n \in \mathbb{R}$ *and every* $n \geq 2$.

Remark 3.3 *The centring constant* d_n *and normalizing constant* $c_n > 0$ *will be referred to as norming constants.*

Equation 3.1 can be re-written as;

$$c_n^{-}1(M_n - d_n) \stackrel{d}{=} X \tag{3.2}$$

It is therefore concluded that every max-stable distribution is a limit distribution for for maxima of iid random variables.

Theorem 3.4 ***Limit property of max stable laws****: The class of max-stable distributions coincides with the class of all possible (non-degenerate) limit laws for(properly normalized) maxima of iid random variables.*

Proof. Assume that for appropriate norming constants,

$$\lim_{n\to\infty} F^n(c_n x + d_n) = H(x), \qquad x \in \mathbb{R}$$

for some non-degenerate distribution function H. We anticipate that the possible limits H are continuous functions on the whole of R. Then for every $k \in N$

$$\lim_{n\to\infty} F^{nk}(c_n x + d_n) = (\lim_{n\to\infty} F^n(c_n x + d_n))^k = H^k(x), \qquad x \in \mathbb{R}$$

and

$$\lim_{n\to\infty} F^{nk}(c_{nk} x + d_{nk}) = H(x), \qquad x \in \mathbb{R}$$

There exists constants $\tilde{c}_k > 0, \tilde{d}_k \in \mathbb{R}$ such that

$$\lim_{n\to\infty} c_{nk}/c_n = \tilde{c}_k$$

and

$$\lim_{n\to\infty} (d_{nk} - d_n)/c_n = \tilde{d}_k$$

and for iid random variables $Y_1, ...Y_K$ with df H,

$$max(Y_1, ...Y_K) \stackrel{d}{=} \tilde{c}_k Y_1 + \tilde{d}_k$$

■

Theorem 3.5 ***Fisher-Tippet(1928) Theorem***:*Also known as the 'Extremal Type Theorem'*

Let (X_n)*be a sequence of iid random variables, if there exists norming constants* $c_n > 0, d_n \in \mathbb{R}$ *and some non-degenerate df* H *such that*

$$c_n^{-}1(M_n - d_n) \xrightarrow{d} H$$

then H *belongs to one of the following dfs:*

Fréchet: $$\Phi_\alpha(x) = \begin{cases} 0, & x \leq 0 \\ \exp\{-x^{-\alpha}\}, & x > 0 \end{cases} \qquad \alpha > 0$$

Weibull: $$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha})\}, & x \leq 0 \\ 1, & x > 0 \end{cases} \qquad \alpha > 0$$

Gumbel: $$\Lambda(x) = \exp\{-e^{-x}\}, \qquad x \in \mathbb{R}$$

This theorem suggests that the asymptotic distribution of the maxima belongs to one of the three distributions above regardless of the original distribution of the observed data.Frechet and Weibull distributions attain the shape of a Gumbel distribution when the tail index parameter as α goes to ∞ and $-\infty$, respectively.

Definition 3.6 *The dfs* $\Phi_\alpha, \Psi_\alpha, \Lambda$,*are called standard extreme value distributions,the corresponding standard extremal random variables are;*

Fréchet: $M_n \stackrel{d}{=} n^{1/\alpha} X$

Weibull: $M_n \stackrel{d}{=} n^{-1/\alpha} X$

Gumbel: $M_n \stackrel{d}{=} X + \ln n$

The Gumbel distribution is a limit law for the thin-tailed distributions such as the normal or log-normal distributions. The Fréchet distribution is obtained as a limiting distribution for the fat-tailed distributions such as Student's t or the Stable Paretian distributions. The marginal distribution of a stationary garch process is also in the domain of attraction of the Fréchet family. Finally, the Weibull distribution is obtained when the distribution of returns has no tail.

3.1.1 Generalized Extreme Value Distribution

Jenkinson(1955) and Von Mises(1954)show that the three families of extreme distribution could be represented into a one parameter model,by taking the reparameterization where;

$\xi = 1/\alpha > 0$ corresponds to Fréchet distribution,

$\xi = 0$ corresponds to Gumbel distribution,

$\xi = -1/\alpha < 0$ corresponds to Weibull distribution,

This representation is known as the generalized extreme value distribution (GEV);

$$H_\xi(x) = \begin{cases} \exp\left\{-(1+\xi x)^{-1/\xi}\right\} & \text{if} \quad \xi \neq 0 \\ \exp\left\{-\exp\left\{-x\right\}\right\} & \text{if} \quad \xi = 0 \end{cases} \tag{3.3}$$

$$where \qquad 1 + \xi x > 0$$

The support of H_ξ is

$$x > -\xi^{-}1, \qquad \text{for} \qquad \xi > 0$$

$$x < -\xi^{-}1, \qquad \text{for} \qquad \xi < 0$$

$$x \in \mathbb{R}, \qquad \text{for} \qquad \xi = 0$$

$$H_\xi(x) = \exp\left\{-(1+\xi x)^{-1/\xi}\right\}, \qquad 1+\xi x > 0, \tag{3.4}$$

where H_ξ $, \xi \in \mathbb{R}$ describes the limit distribution of all normalized maxima. The parameter ξ, called the shape parameter, models the distribution tails. Each of the three extreme value distributions can be obtained as a special case of the GEV distribution. When $\xi > 0$, we get the Frèchet distribution, when $\xi < 0$ we get the Weibull distribution and $\xi = 0$ is the case of the Gumbel distribution. H_ξ is a standard generalized extreme value distribution.

Definition 3.7 ***Maximum domain of attraction****: The random variable X(the distribution function F of X, the distribution of X) belongs to the maximum domain of attraction of the extreme value distribution H if there exist constants* $c_n > 0, d_n \in \mathbb{R}$ *such that;*

$$c_n^{-}1(M - n - d_n) \xrightarrow{d} H$$

We write $X \in$ MDA(H) ,($F \in$ MDA(H)).

3.1.2 Parameter Estimation for Generalized Extreme Value Distribution

Introduce the location and scale parameters, $\mu \in \mathbb{R}, \beta > 0$ in the Generalized Extreme Value Distribution to get:

$$H_{\xi;\mu,\beta}(x) = \exp\left\{-(1+\xi\frac{x-\mu}{\beta})^{-1/\xi}\right\}, \qquad 1+\xi\frac{x-\mu}{\beta} > 0. \tag{3.5}$$

The extremal index θ consists of shape, scale and location parameters, ξ, β, μ respectively. Our data consists of a sample $X_1, ..., X_n$ iid from H_θ.

Assumption 3.8 *X_i has an exact extreme value distribution H_ξ.*

This assumption is not realistic and so at times the more tenable assumption that X_i are approximately H_ξ distributed is used.

The Generalized Extreme Value distribution can be fitted using various methods like the maximum likelihood method and the probability-weighted moments method . We will use the maximum likelihood method to estimate θ.

3.1.3 Maximum Likelihood Estimation of θ

Note that the extremal index θ consists of shape, scale and location parameters, ξ, β, μ respectively. Suppose we have data from an unknown underlying

distribution F which we suppose lies in the domain of attraction of an extreme value distribution H_ξ for some ξ and that the data are realizations of iid random variables. Divide the data into m blocks of size n. The true distribution of the n-block maximum M_n can be approximated for large enough n by a three parameter Generalized Extreme Value distribution, $H_{\xi;\mu,\beta}$. It will be assumed that the block size n is quite large so that the block maxima observations can be taken to be independent.

The number (m)and size (n) of the blocks are determined as a trade-off. A large value of n leads to a more accurate approximation of the block maxima distribution by a Generalized Extreme Value distribution and a low bias in the parameter estimates; a large value of m gives more block maxima data for the Maximum Likelihood Estimation and leads to a low variance in the parameter estimates.

Consider financial applications where daily return data are divided into yearly blocks and the maximum daily falls or rises within these blocks are analyzed. Denote the block maximum of the jth block by M_{nj}. Let $h_{\xi;\mu,\beta}$ be the density of the Generalized Extreme Value distribution. The likelihood function will be;

$$L(\xi, \mu, \beta; M_{ni}) = \prod_{i=1}^{m} h_{\xi,\mu,\beta}(M_{ni}) I_{\{1+\xi(M_{ni}-\mu/\beta>0\}}. \qquad (3.6)$$

Denote the log-likelihood by $\ell(\xi, \mu, \beta; M_{ni}) = \ln L(\xi, \mu, \beta; M_{ni})$.Then,

$$\begin{aligned}\ell(\xi, \mu, \beta; M_{ni}) &= \sum_{i=1}^{m} \ln h_{\xi,\mu,\beta}(M_{ni}) \\ &= -m \ln \beta - (1 + \frac{1}{\xi}) \sum_{i=1}^{m} \ln(1 + \xi \frac{M_{ni} - \mu}{\beta}) - \sum_{i=1}^{m} (1 + \xi \frac{M_{ni} - \mu}{\beta})^{-1/\xi}\end{aligned}$$

The above equation must be maximized subject to the parameter constraints that $\beta > 0$ and $1 + \xi(M_{ni} - \mu) > 0$ for all i and $\xi > -\frac{1}{2}$.

The fitted Generalized Extreme Value model can be used to analyze stress losses.There are two approaches;

Definition 3.9 (Return Level) *The return level estimation problem defines the frequency of occurrence of the stress event and estimates its magnitude.Let H denote the distribution function of the true distribution of the n-block maximum.The k n-block return level is* $r_{n,k} = q_{1-1/k}(H)$*i.e the (1-1/k) quantile of H.*

The return level can be estimated by;

$$\hat{r}_{n,k} = H^{-1}_{\xi,\mu,\beta}(1 - \frac{1}{k}) = \hat{\mu} + \frac{\hat{\beta}}{\hat{\xi}}((-\ln(1 - \frac{1}{k}))^{-\hat{\xi}} - 1) \tag{3.7}$$

Definition 3.10 (Return Period) *The return period estimation problem defines the size of the stress event and estimates the frequency of its occurrence.Let H denote the distribution function of the true distribution of the n-block maximum.*

The return period of the event $\{M_n > u\}$ is given by

$$k_{n,u} = 1/\bar{H}(u). \tag{3.8}$$

3.2 Threshold Exceedances Model

In this approach all data that are extreme due to the fact that they exceed a designated high level are used.

3.2.1 Generalized Pareto distribution (GPD)

The main distributional model for threshold exceedances is the generalized Pareto distribution.

Definition 3.11 *G_ξ is defined by*

$$G_\xi(x) = \begin{cases} 1 - (1+\xi x)^{-1/\xi} & \text{if} \quad \xi \neq 0 \\ 1 - e^{-x} & \text{if} \quad \xi = 0 \end{cases} \tag{3.9}$$

where

$$\begin{aligned} x \geq 0 \quad & \text{if} \quad \xi \geq 0 \\ 0 \leq x \leq -1/\xi \quad & \text{if} \quad \xi = 0 \end{aligned}$$

G_ξ is a standard generalized Pareto distribution.

Introduce the scale parameter β to get

$$G_{\xi,\beta}(x) = 1 - (1 + \xi\frac{x}{\beta})^{-1/\xi}, \qquad x \in D(\xi, \beta), \tag{3.10}$$

where

$$x \in D(\xi, \beta) = \begin{cases} [0, \infty) & \text{if} \quad \xi \geq 0 \\ [0, -\beta/\xi] & \text{if} \quad \xi < 0 \end{cases}$$

The Generalized Pareto Distribution(GPD), $G_{\xi,\beta}$, $\xi \in \mathbb{R}$, appears as the limit distribution of scaled excesses over high thresholds.

Definition 3.12 ***Mean excess function****: Let x be a random variable with distribution function(df) F and right endpoint x_F. For a fixed $u > x_F$,*

$$F_u(x) = P(X - u \leq x / X > u), \qquad x \geq 0$$

is the excess of the random variable X(of the df F)over the threshold u.

The function

$$e(u) = E(X - u / X > u), \quad 0 \leq u < x_F$$

is called the mean excess function of X.

Theorem 3.13 ***The Pickands-Balkema-de Haan Theorem***

Suppose that $X1, X2, ..., Xn$ are n independent realizations of a random variable X with a distribution function F(x). Let x_F be the finite or infinite right endpoint of the distribution F. The distribution function of the excesses over certain (high) threshold u is given by

$$F_u(x) = Pr\left\{ X - u \leq x / X > u \right\} = \frac{F(x+u) - F(u)}{1 - F(u)}$$

$$for \qquad 0 \leq x < x_F - u.$$

The Pickands-Balkema-de Haan theorem (Balkema and de Haan 1974; Pickands 1975) states that if the distribution function F$\in DA(H_\xi)$ then $\exists$ a positive measurable function $\beta(k)$ such that;

$$\lim_{u \to x_F} \sup_{0 \le x < x_F - u} \left| F_u(x) - G_{\xi,\beta(u)}(x) \right| = 0 \tag{3.11}$$

and vice versa, where $G_{\xi,\beta(u)}(x)$ denote the Generalized Pareto distribution. The above theorem states that as the threshold k becomes large, the distribution of the excesses over the threshold tends to the Generalized Pareto distribution, provided the underlying distribution F belongs to the domain of attraction of the Generalized Extreme Value distribution.

3.2.2 Parameter Estimation for GPD;Peaks-Over-Threshold Model

The Peaks-Over-Threshold (POT) Model, attempts to estimate the tails of the underlying return distribution, instead of modeling the distribution of extremes as in the Block Maxima Model approach.In the POT model, a certain threshold is identified to define the starting of the tail of the return distribution. Then the distribution of the 'excesses' over the threshold point is estimated.This approach utilizes the Pickands-Balkema-de Haan theorem to fit a generalized Pareto distribution to the excesses over specific thresholds. The POT Model uses the following assumptions;

Assumption 3.14

- *Exceedances occur at a homogeneous Poisson process in time.*
- *The corresponding excesses over u are independent and have a Generalized Pareto Distribution.*
- *Excesses and exceedances are independent of each other.*

Suppose $X_1, ..., X_n$ are iid with distribution function $F \in MDA(H_\xi)$ for some $\xi \in \mathbb{R}$. Choose a high threshold u and denote by

$$N_u = card\{i : i = 1, ..., n, X_i > u\}$$

the number of exceedances of u by $X_1, ..., X_n$.Denote the corresponding excesses by$Y_1, ..., Y_{N_u}$. The excess distribution function of X is given by;

$$F_u(y) = P(Y \leq x/X > u), \qquad y \geq 0.$$

This can be written as;

$$\bar{F}(u + y) = \bar{F}(u)\bar{F}_u(y). \tag{3.12}$$

Equation (3.10) is the limit result for $F_u(y)$ for an appropriate positive function β a function of u.For large u the following approximation is derived;

$$\bar{F}_u(y) \approx \bar{G}_{\xi,\beta(u)}(y) \tag{3.13}$$

ξ and $\beta(u)$ are estimated from the excess data and depend on u.

From equation (3.11) a natural estimator for $F(u)$ is given by the empirical distribution function

$$\widehat{(\bar{F}(u))} = \bar{F}_n(u) = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i>u\}} = \frac{N_u}{n}$$

Equation (3.12) leads to the estimator;

$$\widehat{(\bar{F}_u(y))} \approx \bar{G}_{\hat{\xi},\hat{\beta}}(y) \tag{3.14}$$

for appropriate $\hat{\xi} = \hat{\xi}_{N_u}$ and $\hat{\beta} = \hat{\beta}_{N_u}$. The resulting tail estimator will be;

$$\widehat{(\bar{F}(u+y))} = \frac{N_u}{n}(1+\hat{\xi}\frac{y}{\beta})^{\frac{-1}{\hat{\xi}}} \tag{3.15}$$

The quantile is estimated by inverting the tail estimator formula 3.14 to get;

$$\hat{x}_p = u + \frac{\beta}{\hat{\xi}}\left((\frac{n}{N_u}(1-p))^{-\hat{\xi}} - 1\right) \tag{3.16}$$

To calculate the relevant estimators the following input is required;

- reliable models for the point process of exceedances.
- a sufficiently high threshold u.
- estimators $\hat{\xi}$ and $\hat{\beta}$.

Remark 3.15 *Selection of the threshold u is a trade-off between bias and variance. A high value of u results in too few exceedances and hence high variance estimators. A low value of u results in biased estimators.*

For the Pickands-Balkema-de Haan theorem to hold the value of u should be sufficiently high. The Generalized Pareto distribution estimators are unbiased if and only if $k \to u$, i.e if the threshold is sufficiently high.

3.2.3 Maximum Likelihood Estimation

The data is iid with common distribution function F. Let F be Generalized Pareto Distribution with parameters ξ and β so that the density f is

$$f(x) = \frac{\xi}{\beta}\left(1 + \xi\frac{x}{\beta}\right)^{(-\frac{1}{\xi}-1)}, \qquad x \in D(\xi, \beta)$$

The log-likelihood function equals

$$\ell((\xi, \beta); M_{ni}) = -n \ln \beta - (\frac{1}{\xi} + 1)\sum_{i=1}^{n} \ln(1 + \frac{\xi}{\beta}M_{ni}) \tag{3.17}$$

The above equation must be maximized subject to the parameter constraints that $\xi > -\frac{1}{2}$.

3.2.4 Measures of Tail Risk

The Generalised Pareto distribution for excess losses is used to estimate the tail of the underlying loss distribution F and associated risk measures.

Definition 3.16 (Quantile function) *: The generalized inverse of the distribution function F*

$$F^{\leftarrow}(t) = \inf\{x \in \mathbb{R} : F(x) \geq t\} \qquad 0 < t < 1$$

is called the quantile function of the df F. The quantity $x_t = F^{\leftarrow}(t)$ defines the t-quantile of F.

Definition 3.17 ***Value-at-Risk****: This is the capital sufficient to cover, in most instances, losses from a portfolio over a holding period of a fixed number of days.*

Assumption 3.18 *Let F be a loss distribution with right endpoint x_F and assume that for some high threshold u we have $F_u(x) = G_{\xi,\beta}(x)$ for $0 \leq x < x_F - u$ and some $\xi \in \mathbb{R}$ and $\beta > 0$.*

We see that for $x \geq u$,

$$\begin{aligned}
\overline{F}(x) &= P(x > u)P(X > x/X > u) \\
&= \overline{F}(u)P(X - u > x - u/X > u) \\
&= \overline{F}(u)\overline{F}_u(x - u) \\
&= \overline{F}(u)(1 + \xi\frac{x - u}{\beta})^{-1/\xi}
\end{aligned}$$

This formula may be inverted to obtain a high quantile of the underlying distribution interpreted as VaR. For $\alpha \geq F(u)$, VaR is equal to

$$VaR_\alpha = q_\alpha(F) = u + \frac{\beta}{\xi}((\frac{1-\alpha}{\bar{F}(u)})^{-\xi} - 1).$$

Definition 3.19 ***Expected Shortfall****: This estimates the potential size of the loss exceeding VaR.*

Expected shortfall (ES) is equal to,

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_x(F)dx = \frac{VaR_\alpha}{1-\xi} + \frac{\beta - \xi\mu}{1-\xi}. \tag{3.18}$$

Chapter 4

Data Analysis

The data used in this study is the weekly prices of the NSE 20 Share Index for the period February 2002 to November 2008.The data has been analyzed using S-Plus Fin metrics module.

4.1 Exploratory Data Analysis

Figure (4.1) is an illustration of the actual weekly NSE 20-share index prices and the weekly percentage returns. Table (4.1) shows the basic statistics for the NSE 20 share Index.

Figure (4.2) shows the qq-plot of the returns which suggests that the tails of the return distribution are fatter than the normal distribution which suggests the Frechet family of generalized extreme value distributions with $\xi > 0$ for

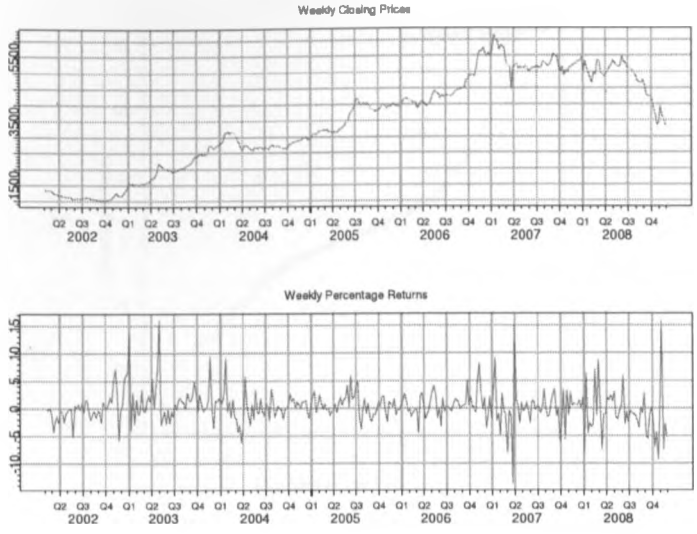
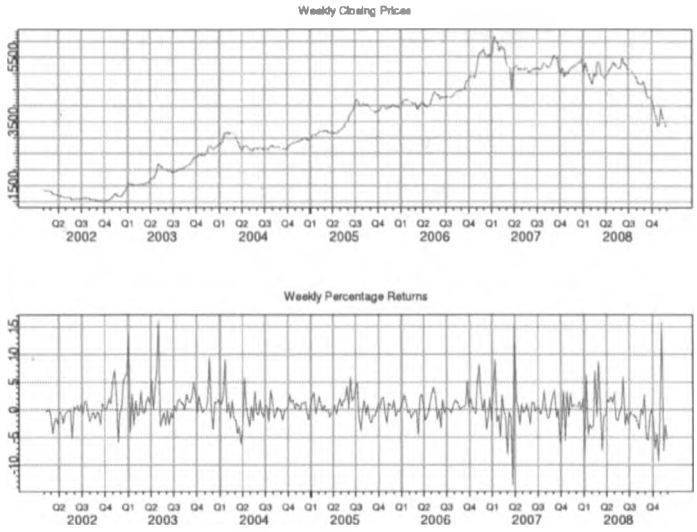


Figure 4.1: weekly closing prices and percentage returns on the NSE 20 Share Index.

Data	Index
Min	1008.790
1st quartile	2455.390
Median	3527.500
3rd quartile	4879.860
Max	6161.460

Table 4.1: Basic Statistics for NSE 20 Share Index



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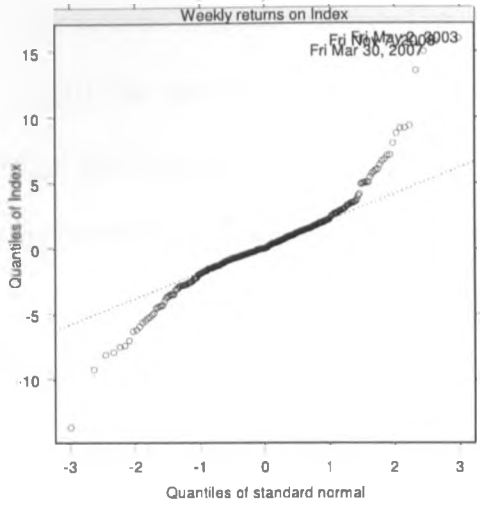


Figure 4.2: Normal Q-Q Plot.

the block maximum of negative returns.

4.2 Parameter Estimation

4.2.1 Block Maxima Model

The returns on the data has been organised into monthly blocks.

Figure (4.3) gives several graphical summaries of the monthly block maxima.The largest weekly negative return in a monthly block is 13.65 percent in 2007. The number of records is consistent with iid behavior.The his-

togram resembles a Frechet density. The qq-plot uses the Gumbel, H_0, as the reference distribution. For this distribution, the quantiles satisfy $H_0^{-1}(p) = -\ln(-\ln(p))$.The downward curve in the plot indicates a generalized extreme value distribution with $\xi > 0$.

We need to estimate the shape,scale and location parameters, ξ, β, μ, respec-

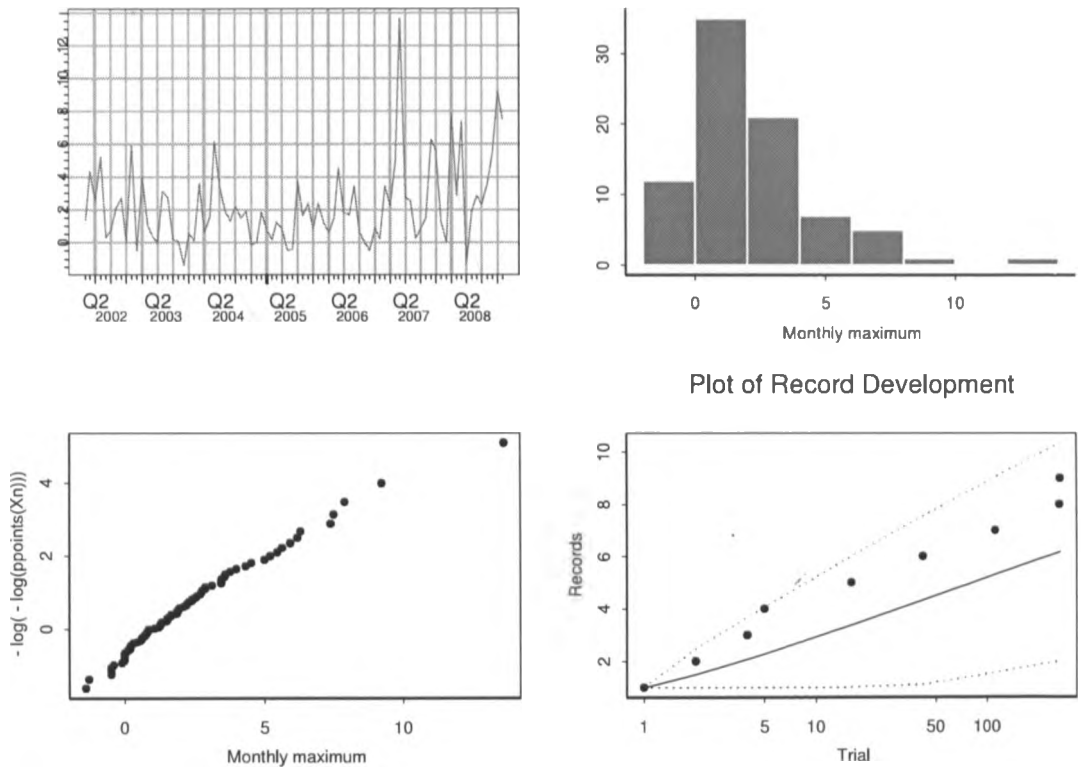


Figure 4.3: Monthly block maxima, histogram,Gumbel qq-plot and records summary for the NSE 20 share Index.

tively. The maximum likelihood estimators for these parameters using block maxima are computed using the monthly blocks from the weekly(negative) returns on NSE 20 share Index.Table (4.2) shows an estimate of these parameters, which form the extremal index θ,with $\xi = 0.139$, after fitting the model.

From the analysis we find that the probability that a new record maximum

parameter	value	Std. Error
ξ	1.39×10^{-1}	8×10^{-2}
β	1.62	1.55×10^{-1}
μ	1.08	2.0×10^{-1}

Table 4.2: Estimated Parameters for GEV distribution.

weekly negative return will be established during the next month is 0.52 percent.

To test the fit to the generalized extreme value we see from Figure (4.4) of residual plots, the scatter plot of the residuals, with a lowest estimate of trend, does not reveal any significant unmodeled trend in the data.The qq-plot, using the exponential distribution as the reference distribution,is linear and appears to validate the generalized extreme value distribution.

The estimate of the 12 month return level is 5.79 percent. The plot of

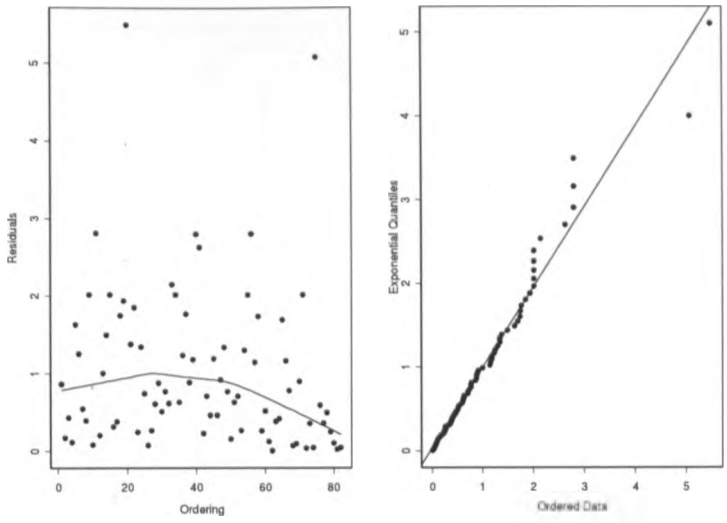


Figure 4.4: Residual plot for GEV.

this is seen in figure (4.5).

4.2.2 Generalized Pareto Distribution

The mean excess plot is used to determine the threshold (u) for the Generalized Pareto distribution.The mean excess plot for the NSE 20-share Index data is illustrated in figure (4.6). The mean excess plot for the NSE 20-share Index negative returns is linear in u for $u < 0.5$.This suggests that the threshold $u = 0.5$ may be appropriate for the GPD approximation to be valid.

Diagnostic plots for Generalized Pareto distribution fit to weekly negative returns on NSE 20-share Index are shown in figure (4.7) with Table (4.3)

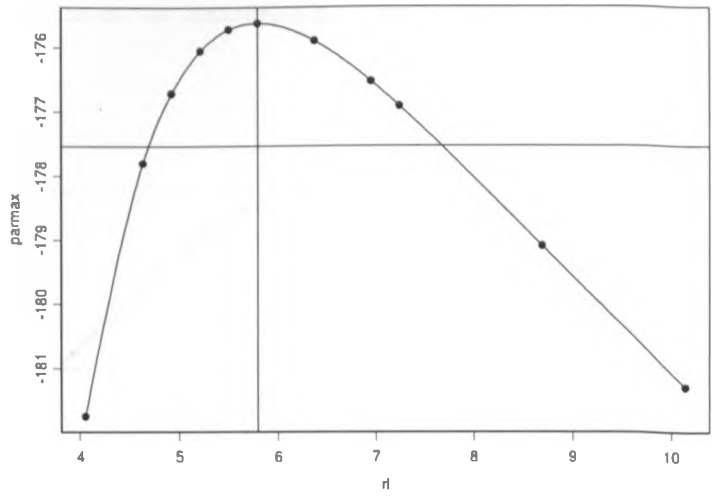


Figure 4.5: 12-month return level.

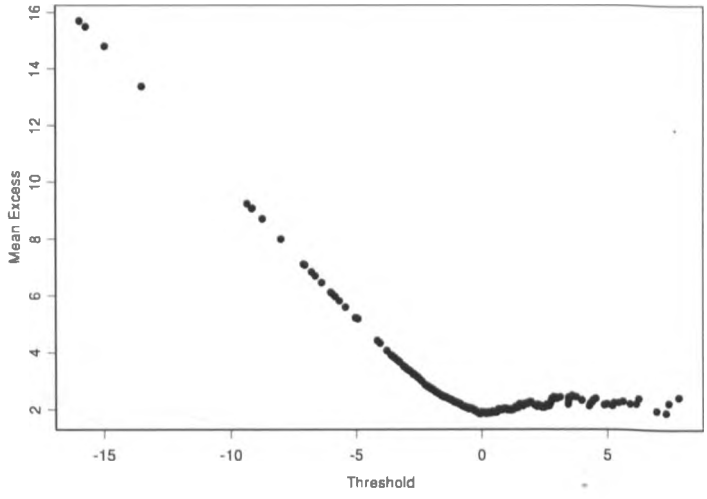


Figure 4.6: Mean excess plot.

showing an estimate of the parameters, which form the extremal index θ, and appears to fit the distribution of threshold excesses fairly well.

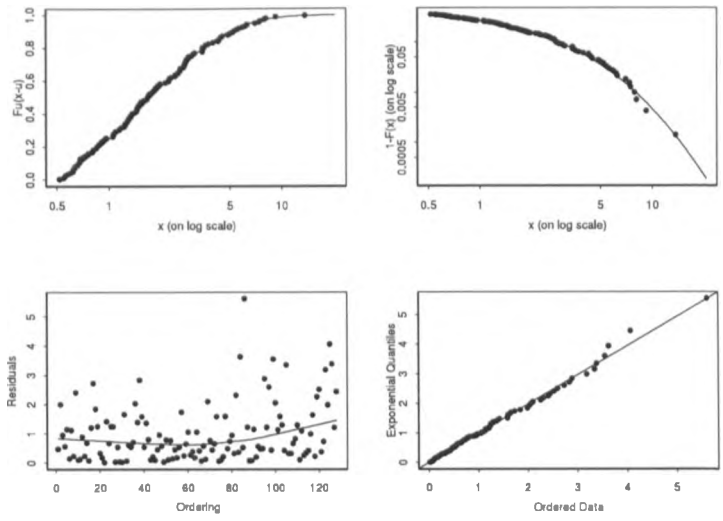


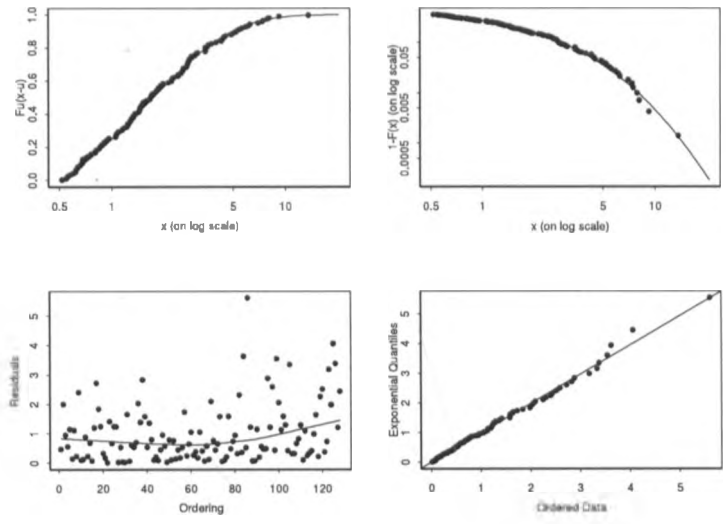
Figure 4.7: Diagnostic plots for GPD fit to NSE 20 Share Index.

Parameter	Value	Std. Error
ξ	1.07×10^{-1}	1.06×10^{-1}
β	1.712×10^{-1}	2.36×10^{-1}

Table 4.3: Parameter estimates for the GPD distribution.

The number of exceedances is 128 observations.Notice that $\xi = 0.107$ is fairly close to zero and indicates that the return distribution is not so heavy-

showing an estimate of the parameters, which form the extremal index θ, and appears to fit the distribution of threshold excesses fairly well.



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β	1.712×10^{-1}	2.36×10^{-1}

Table 4.3: Parameter estimates for the GPD distribution.

The number of exceedances is 128 observations.Notice that $\xi = 0.107$ is fairly close to zero and indicates that the return distribution is not so heavy-

tailed.This is in addition to this estimate being smaller than the generalized extreme value estimate $\xi = 0.139$ based on weekly data.

4.2.3 Measures of Tail risk

Table (4.4) shows the estimates of VaRq and ESq based on the generalized Pareto distribution approximations. With 5 percent probability the daily return could be as low as -4.3 percent and, given that the return is less than 4.3 percent, the average return value is -6.6 percent. Similarly, with 1 percent probability the daily return could be as low as -7.9 percent with an average return of -10.55 percent given that the return is less than -7.9 percent.

Note that the estimates of VaRq and ESq based on the normal distribution

p	Quantile	Shortfall
0.95	4.300	6.60
0.99	7.927	10.547

Table 4.4: GPD based risk measures estimates

are fairly close to the estimates based on the generalized Pareto distribution.

p	Quantile	Shortfall
0.95	4.953	6.290
0.99	7.133	8.217

Table 4.5: Risk measure estimates based on assumptions of normal distribution.

Chapter 5

Conclusions and Recommendations

In this chapter we give a conclusion of the findings and recommendations for further research.

5.1 Conclusions

We have illustrated how extreme value theory can be used to model tail-related measures such as Value at Risk, expected shortfall and return level applying it to the NSE 20 Share Index weekly returns. A comparison of the two models indicate that the "Peaks Over Threshold" model gives a better estimation to the extreme values as it focuses on modeling the tail behavior

using extreme values beyond a certain threshold as opposed to the Blocks Maxima Model which utilizes only the maximum losses in large blocks. In addition it is easier to calculate risk measures using the 'Peaks Over Threshold ' model as opposed to the Block Maxima Model. In addition normal quantiles are close to the Generalized Pareto distribution quantiles which implies that the risk is not underestimated.

5.2 Recommendations

More work can consider the use of non-parametric models, e.g the Hill method, to estimate tail distributions and densities and comparing this with the parametric models.

Further work could consider multivariate approach to extreme value analysis to give a complete picture of the market risk.

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Appendix

Data

Weekly closing prices for the NSE 20- Share Index for the period 1st February 2002 to 28th November 2008.

1340.31 1336.81 1332.54 1335.52 1317.78 1260.9 1232.87 1214.54 1183.1 1177.81 1168.88 1139.32 1129.01 1124.54 1124.82 1122.64 1064.24 1071.07 1071.45 1079.2 1075.77 1086.62 1079.27 1094.86 1111.5 1103.44 1080.12 1070.38 1066.08 1047.87 1043.38 1036.74 1008.79 1021.95 1025.61 1025.61 1032.75 1053.78 1064.86 1118.44 1197.86 1239.36 1166.01 1161.63 1167.18 1225.95 1298.86 1384.98 1572.12 1509.43 1554.07 1510.63 1536.75 1520.16 1507.96 1557.74 1549.68 1542.73 1564.2 1598.81 1617.35 1705.13 1705.13 1766.48 1886.36 2187.48 2119.13 2083.21 2074.67 2016.87 2015.34 1959.6 1948.73 1920.52 1938.22 1934.14 1963.21 2000.98 2027.54 2048.81 2047.58 2107.43 2139.7 2169.17 2218.03 2328.05 2398.22 2384.38 2445.39 2470.67 2457.21 2455.39

2458.21 2502.97 2736.98 2733.53 2635.03 2674.2 2705.46 2753.33 2788.98

2818.29 2893.12 3157.88 3138.37 3175.8 3125.57 3175.36 3129.18 3074.07

2939.31 2849.55 2673.84 2581.46 2727.73 2747.52 2707.6 2626.12 2644.8 2593.99

2680.75 2662.49 2639.83 2686.99 2669.34 2634.67 2657.76 2674.56 2614.95

2708.03 2757.28 2715.34 2700.15 2712.65 2710.73 2704.15 2652.64 2650.2

2648.71 2673.54 2745.83 2778.29 2829.65 2832.3 2868.26 2885.16 2885.16

2921.53 2954.09 2999.54 2979.13 2923.86 2945.58 3007.94 3102.16 3078.93

3092.24 3167.79 3198.06 3191.78 3219.37 3208.66 3211.64 3170.25 3155.01

3139.54 3163.95 3138.17 3165.19 3227.59 3242.89 3292.75 3353.51 3492.58

3506.05 3716.9 3780.08 3860.83 4006.27 4203.51 4142.8 3987.04 3982 4049.95

4012.42 4047.26 3980.37 3884.63 3847.17 3801.87 3791.57 3832.69 3843.49

3921.4 4008.76 3976.62 3880.88 3928.16 3951.59 3955.38 4022.02 3972.82

3963.26 3953.01 3973.04 4030.97 4140.66 4199.53 4173.5 4163.64 4100.22

4071 4062.56 4055.78 3872.21 3955.42 4067.41 4101.64 4025.3 3973.79 3960.19

4025.21 4149.14 4316.72 4411.81 4338.42 4339.47 4189.66 4272.43 4246.8

4260.49 4271.72 4272.6 4244.16 4271.68 4340.88 4407.54 4451.08 4451.08

4469.6 4490.84 4523.8 4750.8 4778.35 4879.86 4903.9 4906.49 4864.02 5106.65

5515.34 5654.46 5642.04 5752.57 5553.08 5477.86 5589.64 5487.73 5645.65

6161.46 6025.41 5961.61 5663.65 5817.04 5798.73 5732.67 5268.99 5245.62

5171.13 4465.09 5133.67 5215.2 5242.88 5099 5148.07 5116.02 5114.17 5167.34

5134.51 5001.77 5068.68 5137.45 5124.14 5146.73 5160.89 5117.37 5137.51

5317.73 5320.42 5240.83 5171.3 5234.7 5371.72 5560.23 5484.63 5491.27 5146.46

5005.89 5175.8 4884.75 5044.06 4980.49 5124.31 5147.62 5198.23 5234.54

5265.15 5339.8 5339.8 5444.83 5015.5 5335.23 5098.48 4967.88 4795.96 4657.6

4986.06 4924.35 5354.68 5354.68 4959.44 4907.29 4855.36 4951.73 5021.82

5141.62 5207.23 5364.72 5274.44 5170.55 5149.96 5175.83 5477.7 5320.28

5284.08 5152.03 5129.73 5081.34 5025.84 4963.46 4849.97 4676.9 4689.92

4648.78 4774.75 4541.93 4294.3 4244.83 4251.3 4174.84 3882.76 3716.32 3373.87

3386.65 3918.78 3625.59 3527.5 3341.47

S-Plus Script

```
Index

Index.ts

summary(Index)

Index08 = getReturns(Index.ts, type="discrete", percentage=T)

par(mfrow=c(2,1))

plot(Index.ts, main="Weekly Closing Prices")

plot(Index08, main="Weekly Percentage Returns")

qqPlot(seriesData(Index08))

boxplot(seriesData(Index08))

histPlot(seriesData(Index08))
```

qqPlot(Index08,strip.text="Weekly returns on Index",xlab="Quantiles of standard normal",ylab="Quantiles of Index")

autocorTest(Index08, lag.n=10, method="lb")

par(mfrow=c(2,1))

acf(seriesData(Index08))

acf(seriesData(Index08),type="partial")

MonthlyMax.Index = aggregateSeries(-Index08, by="months",FUN=max)

MonthlyMax.Index

Xn = sort(seriesData(MonthlyMax.Index))

Xn

par(mfrow=c(2,2))

plot(MonthlyMax.Index)

hist(seriesData(MonthlyMax.Index),xlab="Monthly maximum")

plot(Xn,-log(-log(ppoints(Xn))),xlab="Monthly maximum")

tmp = records(-Index08)

gev.fit.month = gev(-Index08, block="month")

gev.fit.month$n$

$gev.fit.month$par.ests

gev.fit.month$par.ses$

$par(mfrow = c(1, 2))$

$plot(gev.fit.month)$

$1-pgev(max(gev.fit.month$data),xi=gev.fit.month$par.ests["xi"], mu = gev.fit.month$par.ests["

$rlevel.month.12 = rlevel.gev(gev.fit.month, k.blocks = 12, type = "profile")$

$class(rlevel.month.12)$

$names(rlevel.month.12)$

$rlevel.month.12$rlevel

me.Index = meplot(-Index08)

par(mfrow=c(1,2)) qplot(-Index08, threshold=0.5, main="Index.ts returns

negative returns")

gpd.Index = gpd(-Index08, threshold=0.5)

gpd.Index$upper.thresh$

$gpd.Index$n.upper.exceed

gpd.Index$p.less.upper.thresh$

$gpd.Index$upper.par.ests

gpd.Index$upper.par.ses$

$par(mfrow = c(2, 2))$

$plot(gpd.Index)$

$riskmeasures(gpd.Index, c(0.95, 0.99))$

$Index.mu = mean(-Index08)$

$Index.sd = sqrt(var(-Index08))$

$var.95 = Index.mu + Index.sd * qnorm(0.95)$

$var.99 = Index.mu + Index.sd * qnorm(0.99)$

$var.95$

$var.99$

$z95 = (var.95 - Index.mu)/Index.sd$

$z99 = (var.99 - Index.mu)/Index.sd$

$es.95 = Index.mu + Index.sd * dnorm(z95)/(1 - pnorm(z95))$

$es.99 = Index.mu + Index.sd * dnorm(z99)/(1 - pnorm(z99))$

$es.95$

$es.99$