GAMMA AND RELATED DISTRIBUTIONS

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Declaration

I, Ayienda Kemunto Carolynne do hereby declare that this thesis is my original work and has not been submitted for a degree in any other University.

Sign: -- Date...............................

AYIENDA KEMUNTO CAROLYNNE

Sign: -- Date...............................

Supervisor: J.A.M OTTIENO

Dedication

I wish to dedicate this thesis to my beloved husband Ezekiel Onyonka and son Steve Michael.

To you all, I thank you for giving me the support needed to persue my academic dream.

Acknowledgements

I am grateful to the all-mighty God who has watched over me all my life and more so, during my challenging academic times. It is my prayer that he grants me more strength to achieve all my ambitions and aspirations in life.

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God bless you all.

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Abstract

The gamma distribution is one of the continuous distributions. Gamma distributions are very versatile and give useful presentations of many physical situations. They are perhaps the most applied statistical distribution in the area of reliability. Gamma distributions are of different types, 1, 2, 3, 4-parameters. They are applied in different fields among them finance, economics, hydrological and in civil engineering. In this study we have constructed different types of gamma distributions using transformation/change of variable and cumulative techniques and calculated their properties using moments, identified their special cases and calculated their properties too. We have also constructed gamma related distribution using transformation and cumulative techniques and most of these distributions are expressed using special functions, also we have used the gamma-generator and gamma exponetiated -generator to generate new family of distributions.

CHAPTER I: General Introduction

1.0 Introduction

Statistical distributions provide the foundation for the analysis of empirical data and for many statistical procedures. Empirical results can be sensitive to the degree to which distributional characteristics such as the mean, variance, skewness, and kurtosis of the data can be modeled by the assumed statistical distribution (James, B. et al., 1995). The Gamma distribution arises where one is concerned about the waiting time for a finite number of independent events to occur, assuming that events occur at a constant rate and chances that more than one event occurs in a small interval of time is negligible. Following the many ways of constructing distributions in this study we have use special functions approach. The gamma function, Beta function, Confluent Hypergeometric function and Bessel functions were mainly used.

A random variable X is said to have the two parameter (standard) gamma distribution if its distribution is given by

$$f\,(x) = \frac{x^{\alpha-1}e^{\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} \quad or \;\; f(x) = \frac{\beta^{\alpha}x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}\;, x\; >\; 0, \alpha\; > 0, \beta > 0$$

Where the parameter α is called the shape parameter, since it most influences the peakedness of the distribution (Stephenson *et al.*, 1999). The parameter β is called the inverse scale parameter, since most of its influence is on the spread of the distribution i.e. the standard deviation of the gamma distribution is proportional to $\frac{1}{\beta}$. Two-parameter gamma and two-parameter Weibull are the most popular distributions for analyzing any lifetime data (Rameshaw, D., 2001).

Gamma distributions are very versatile and give useful representation of many physical situations. They are perhaps the most applied statistical distributions in the area of reliability (Saralees, N. 2008). Also the Gamma distribution can be elaborated as the sum of a fixed number of exponential random variables under the condition that the shape and scale parameters have only positive values.

This distribution has applications in reliability and queuing theory, examples include the distribution of failure times of a component, the distribution between calibration of instrument which need re-calibration after a certain number of use and the distribution of waiting times of k customers who will arrive at a store.

The Gamma distribution can also be used to model the amounts of daily rainfall in a region (Das., 1955; Stephenson *et al.*, 1999). A gamma distribution was postulated because precipitation occurs only when water particles can form around dust of sufficient mass, and waiting the aspect implicit in the gamma distribution.

Gamma has a long history and it has several desirable properties. It has lots of applications in different fields other than lifetime distributions. The Gamma distribution has many sub-families e.g, exponential, Chi-Square, Weibull, Maxwell, Rayleigh among others. The Erlang distribution is equivalent to the Gamma distribution except for the imposed condition that the shape parameter must be a positive integer.

In this study, we have discussed the 1, 2, 3 and 4-parameter gamma probability density functions (pdf), calculated their properties, identified their special cases and calculated their properties too. We have also constructed the gamma related distributions using different techniques and some results are given using special function.

1.1 Literature Review

The two-parameter of a gamma represent the shape and scale parameters and because of the shape and scale parameters, it has quite a bit of flexibility to analyze any positive real data. It has increasing as well as decreasing failure rate depending the shape parameter, which gives an extra edge over exponential distribution which has only constant failure rate (Gauss, M. *et al*., 2012).

The most general form of the gamma distribution is the three parameter generalized gamma (GG) distribution (Stacy., 1962). The distribution is suitable for modeling data with different types of hazard rate functions; increasing, decreasing, bathtub shaped and unimodal, which makes it particularly useful for estimating individual hazard functions. The GG distribution has been used in several research areas such as engineering, hydrology and survival analysis. Gamma distributions are very versatile and give useful presentations of many physical situations. They are perhaps the most applied statistical distributions in of reliability (Saralees N., 2008).

The Generalized gamma (GG) model, having Weibull, gamma, exponential and Raleigh as special sub-models, among others, is a very popular distribution for modeling lifetime data and for modeling phenomenon with monotone failure rates.

The GG distribution plays a very important role in statistical inferential problems. When modeling monotone hazard rates, the Weibull distribution may be the initial choice because of its negatively and positively skewed density shapes. However, the Weibull distribution does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates, such as the bathtup-shaped and the unimodal failure rates, which are common in biological and reliability studies (Gauss, M. *et al*., 2011).

A new family of distributions generated by the gamma random variables have been recently introduced (Zografos and Balakrishnan., 2009). This family of distributions have their cumulative distribution function (cdf) given as

$$G(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{-log\bar{F}(x)} t^{\delta-1} e^{-t} dt, x \in R, \delta > 0 \tag{1.1}$$

Where $\bar{F}(x)$ is the survival function which is used to generate new distributions.

1.2 Applications

The Gamma distribution is extremely important in risk analysis modeling, with a number of different uses:

1.2.1 Poisson waiting time

The Gamma (α, β) distribution models the time required for α events to occur, given that the events occur randomly in a Poisson process with a mean time between events of β . For example, if we know that major flooding occurs in a town on average every six years, Gamma (4, 6) models how many years it will take before the next four floods occur.

1.2.2 Hydrological Analysis

In hydrology, the gamma distribution has the advantage of having only positive values, since hydrological variables such as rainfall and runoff are always positive (greater than zero) or equal to zero as a lower limit value (Markovic., 1965).

1.2.3 Reliability and queuing theory.

Examples include the distribution of failure times of components, the distribution of times between calibration of instruments which need re-calibration after a certain number of uses and the distribution of waiting times of k customers who will arrive at a store.

Problem statement

For practical purpose it was found necessary to extend one parameter gamma distribution to more parameters gamma distributions which are frequently used in practical applications.

Objective

The objective of the study was to construct the gamma distributions and gamma related distributions using transformation/change of variables approaches.

Specific Objectives

i). Construct gamma distributions.

ii). Identify their special cases.

iii). Study their moments.

iv). Construct gamma related distributions.

CHAPTER II: One Parameter Gamma Distribution

2.0 Introduction

Some researchers refer to one parameter gamma distribution as the standard gamma while others refer the two parameter gamma as the standard gamma (Nadarajah, S. 2008). In this chapter we shall consider only the one parameter gamma distribution case.

A random variable X is said to have the one parameter (standard) gamma distribution if its distribution is given by

$$f(x) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}\ , x > 0, \alpha > 0$$

Where the parameter α is called the shape parameter, since it most influences the peakedness of the distribution (Stephenson *et al*., 1999).

Before that we shall describe the Beta and Gamma functions and their properties. A family of probability density function that yields a wide variety of skewed distributional shape is the Gamma family. To define the family of Gamma distributions, we first need to introduce functions that play an important role in many branches of mathematics i.e the Gamma and Beta functions. The Gamma function is one of the most widely used special function encountered in advanced mathematics because it appears in almost every integral or series representation of other advanced mathematical functions while the beta function is a function of two variables that is often found in probability theory and mathematical statistics.

2.1 Gamma and Beta functions

2.1.1 Definitions

$$\mathrm{B}\ (\mathrm{a}, \mathrm{b}) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx \qquad (2.0)$$

$$where\ a, b > 0$$

A gamma function denoted by $\Gamma(\alpha)$ is a another special case function defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \qquad (2.1)$$

$\Gamma(k)$

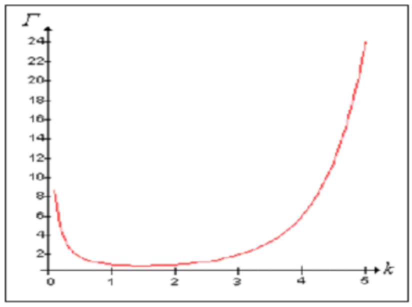

Figure 2.1: Graph of gamma function

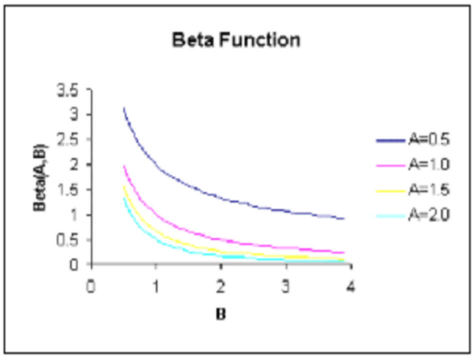

Figure 2.2: Graph of Beta function

2.1.2 Properties of the Beta and Gamma function

Property 2.1: $\boldsymbol{\Gamma(1) = 1}$

Proof n Put $\alpha = 1$

$$\Gamma(1) = \int_0^\infty e^{-t} dt$$

$$= -[e^{-t}]_0^\infty = -[0-1] = 1$$

Property 2.2 : $\boldsymbol{\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)}$

Proof: $\Gamma(\alpha+1) = \int_0^\infty e^{-t}\, t^{\alpha} dt$

Using integration by parts

$$Let\ u = t^\alpha \Rightarrow du = \alpha t^{\alpha-1} dt\ and\ dv = e^{-t} dt \Rightarrow v = -e^{-t}$$

Therefore

$$\Gamma(\alpha+1) = [-t^\alpha e^{-t}]_0^\infty + \alpha \int_0^\infty t^\alpha e^{-t} dt$$

$$= \alpha \int_0^\infty e^{-t}\, t^{\alpha-1} dt$$

$$= \alpha\,\Gamma(\alpha)\,;\ \alpha > 0 \quad \text{using } (2.2)$$

Property 2.3: $\boldsymbol{\Gamma(n+1) = n!}$ when n is a positive integer

Proof: from property (2.2) $\Gamma(n+1) = n\Gamma(n)$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

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$$= n(n-1)(n-2)\ldots[n-(n-1)]\Gamma(n(-n))$$

$$= n(n-1)(n-2)\ldots 3*2*\Gamma(1)$$

$$= n(n-1)(n-2)\ldots 3*2*1$$

$$= 1*2*3*4\ldots n$$

$$= n!$$

Property 2.4: $\boldsymbol{\Gamma(\alpha) = \int_0^\infty e^{-t^2}\, t^{2\alpha-1} dt}$

Proof $Let\ t = u^2 \Rightarrow dt = 2udu$

Therefore $\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1} dt$

$$= \int_0^\infty e^{-u^2}\, u^{2(\alpha-1)} 2udu$$

$$= \int_0^\infty e^{-t^2}\, t^{2\alpha-1} dt$$

Property 2.5: $\boldsymbol{B(a,b) = 2\int_0^{\frac{\pi}{2}} (\sin u)^{2a-1}\ (\cos u)^{2b-1} du}$

Proof Let $t = sin^2\theta \Rightarrow dt = 2sin\theta cos\theta d\theta$

Therefore $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$

$$= 2\int_0^{\frac{\pi}{2}} (sin^2\theta)^{a-1}(cos^2\theta)^{b-1} sin\theta cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (sin\theta)^{2a-1}(cos\theta)^{2b-1} d\theta$$

Property 2.6: $\boldsymbol{B(a,b) = \int_0^\infty \frac{u^{b-1}}{(1+u)^{a+b}} du}$

Proof Let t = $\frac{u}{1+u} \Rightarrow t + tu = u$

$$dt = \frac{(1+u)*1 - u*1}{(1+u)^2} = \frac{du}{(1+u)^2}, t = (1-t)u$$

$$u = \frac{t}{1-t}$$

$$When\ t = 0 \Rightarrow u = 0, t = 1 \Rightarrow \infty$$

Therefore $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$

$$= \int_0^\infty \left[\frac{u}{1+u}\right]^{a-1} (1 - \frac{u}{1+u})^{b-1} \frac{du}{(1+u)^2}$$

$$= \int_0^\infty \left[\frac{u}{1+u}\right]^{a-1} (\frac{1}{1+u})^{b-1} \frac{du}{(1+u)^2}$$

$$= \int_0^\infty \frac{u^{a-1}}{(1+u)^{a-1+b-1+2}} du = \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}} du$$

Property 2.7 : $\boldsymbol{B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}}$

Proof $\Gamma(a)\Gamma(b) = \left(\int_0^\infty x^{a-1}e^{-x}dx\right)\left(\int_0^\infty y^{b-1}e^{-y}dy\right)$

$$\Gamma(a)\Gamma(b) = \left(\int_0^\infty x^{a-1}e^{-x}dx\right)\left(\int_0^\infty y^{b-1}e^{-y}dy\right)$$

$$= \int_0^\infty \int_0^\infty e^{-(x+y)} x^{a-1}y^{b-1}dxdy$$

$Let\ x = s^2\ and\ y = t^2$

$\Rightarrow dx = 2sds\ and\ dy = 2tdt$

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty e^{-(s^2+t^2)}\, s^{2(a-1)} t^{2(b-1)} 2s2tdsdt$$

$$= 4\int_0^\infty \int_0^\infty e^{-(s^2+t^2)}\, s^{2a-2} t^{2b-2} stdsdt$$

$$= 4\int_0^\infty \int_0^\infty e^{-(s^2+t^2)}\, s^{2a-1} t^{2b-1} dsdt$$

$$Let\ s = rsin\theta\ and\ t = rcos\theta$$

$$|J| = \begin{vmatrix} \frac{ds}{d\theta} & \frac{dt}{d\theta} \\ \frac{ds}{dr} & \frac{dt}{dr} \end{vmatrix} = \begin{vmatrix} rcos\theta & -rsin\theta \\ \sin\theta & -cos\theta \end{vmatrix}$$

$$= rcos^2\theta + rsin^2\theta$$

$$= r$$

Therefore $\Gamma(a)\Gamma(b) = 4\int_0^\infty \int_0^\infty e^{-r^2}\, (rsin\theta)^{2a-1}(rcos\theta)^{2b-1} * |J| * d\theta dr$

$$= 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2}\, r^{2a-1}(sin\theta)^{2a-1} r^{2b-1}(cos\theta)^{2b-1} rd\theta dr$$

$$= \int_0^\infty \left[2\int_0^{\frac{\pi}{2}} (sin\theta)^{2a-1}(cos\theta)^{2b-1}\ \ d\theta \right] e^{-r^2} r^{2(a+b-1)} 2rdr$$

from property (2.4)

$$\Gamma(a)\Gamma(b) = \int_0^\infty B(a,b)\, e^{-r^2} r^{2(a+b-1)} 2rdr$$

$$= B(a,b)\int_0^\infty 2re^{-r^2} r^{2(a+b-1)} dr$$

$$put\ u = r^2 \Rightarrow du = 2rdr$$

Therefore $\Gamma(a)\Gamma(b) = B(a,b)\int_0^\infty e^{-u} u^{a+b-1} du$

$$= B(a,b)\Gamma(a+b)$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = B(a, b)$$

Property 2.8: $\Gamma\left(\boldsymbol{n} + \frac{\mathbf{1}}{\mathbf{2}}\right) = \frac{(\mathbf{2n})!\sqrt{\boldsymbol{\pi}}}{\boldsymbol{n}!\mathbf{2}^{\mathbf{2n}}}$

Proof Using $\Gamma(n) = (n-1)\Gamma(n-1)$

$$\Gamma\left(n + \frac{1}{2}\right) = \Gamma\left(\frac{2n+1}{2}\right) = \left(\frac{2n+1}{2} - 1\right) \Gamma\left(\frac{2n+1}{2} - 1\right)$$

$$= \left(\frac{2n-1}{2}\right) \Gamma\left(\frac{2n-1}{2}\right)$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right) \dots \left(\frac{2n-(2n-2)}{2}\right) *$$

$$\left(\frac{2n-(2n-1)}{2}\right)$$

$$= \frac{(2n-1)(2n-3)\dots(3)(1)(\sqrt{\pi})}{2^n}$$

$$= \frac{(2\text{n})!\ \sqrt{\pi}}{n!\,2^{2n}}$$

Corollary: 2.1 $\Gamma\left(\frac{\mathbf{1}}{\mathbf{2}}\right) = \sqrt{\boldsymbol{\pi}}$

Proof $\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = B(a, b) = 2\int_0^{\frac{\pi}{2}} (sin\theta)^{2a-1}(cos\theta)^{2b-1}\ d\theta$

$$\text{Put a} = \text{b} = \frac{1}{2} \Rightarrow \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)} = B\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} d\theta$$

$$= 2 * [\theta]_0^{\infty}$$

$$= \frac{\left(\Gamma(\frac{1}{2})\right)^2}{\Gamma(1)} = 2\int_0^{\frac{\pi}{2}} d\theta$$

$$= 2 * \frac{\pi}{2}$$

Thus $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi$

Therefore $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Corollary: 2.2 $\int_0^\infty e^{-t^2} dt = \sqrt{\pi}$

$$and \quad \int_0^\infty e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}$$

Proof : using property 2.4

$$\int_0^\infty e^{-t^2} t^{2\alpha-1} dt = \Gamma(\alpha) \;\; put\ \alpha = \frac{1}{2}$$

Therefore $\int_0^\infty e^{-t^2} dt = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Next let $t = \frac{z}{\sqrt{2}} \Longrightarrow dt = \frac{dz}{\sqrt{2}}$

Therefore $\int_0^\infty e^{-t^2} dt = \int_0^\infty e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2}}$

$$= \sqrt{\pi}$$

Thus $\int_0^\infty e^{-\frac{z^2}{2}} = \sqrt{2\pi}$

Property 2.9: $\int_0^\infty e^{-\beta x} x^\alpha dx = \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}\ ;\ \alpha > 0, \beta > 0$

Proof: Let $y = \beta x \Longrightarrow x = \frac{y}{\beta}$ and $dx = \frac{dy}{\beta}$

Therefore $\int_0^\infty e^{-\beta x} x^\alpha dx = \int_0^\infty e^{-y} \left(\frac{y}{\beta}\right)^\alpha \frac{dy}{\beta}$

$$= \frac{1}{\beta^{\alpha+1}} \int_0^\infty e^{-y} y^\alpha dy$$

$$= \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}\ ;\ \alpha, \beta > 0$$

Property 2.10: $(Legendre\ duplication\ formula)$

$$\sqrt{\mathbf{2v}} = \frac{\mathbf{2^{2v-1}}}{\sqrt{\boldsymbol{\pi}}}\ \boldsymbol{\Gamma}(\boldsymbol{v})\boldsymbol{\Gamma}\left(\boldsymbol{v}+\frac{\mathbf{1}}{\mathbf{2}}\right),\boldsymbol{v}>\mathbf{0}$$

Proof: from propery 2.7 $\quad B(\mu, v) = \frac{\Gamma(\mu)\Gamma(v)}{\Gamma(\mu+v)}$

Let $\mu = v$ therefore $B(v, v) = \frac{\Gamma(v)\Gamma(v)}{\Gamma(2v)}$

$$\int_0^1 t^{v-1}(1-t)^{v-1}dt = \frac{\Gamma(v)\Gamma(v)}{\Gamma(2v)}$$

$$i.e \int_0^1 (t(1-t))^{v-1}dt = \frac{\Gamma(v)\Gamma(v)}{\Gamma(2v)}$$

Make the subtistution $\quad t = \frac{1}{2}(1+s)$

therefore $\quad 2t = 1 + s \implies s = 2t - 1$

$$\frac{\Gamma(v)\Gamma(v)}{\Gamma(2v)} = \int_{-1}^{1} \left\{\frac{1}{2}(1+s)[1-\frac{1}{2}(1+s)\right\}^{v-1}\frac{ds}{2}$$

$$= \int_{-1}^{1} \left\{\frac{1}{2}(1+s)[1-(\frac{1}{2}-\frac{s}{2})\right\}^{v-1}\frac{ds}{2}$$

$$= (\frac{1}{4})^{v}\frac{1}{2}\int_{-1}^{1}\{(1+s)(1-s)\}^{v-1}ds$$

$$= (\frac{1}{4})^{v-1}\frac{1}{2}\int_{-1}^{1}(1-s^2)^{v-1}\ \ ds$$

$$= \int_{-1}^{1}(\frac{1}{2})^{2v-2}\frac{1}{2}(1-s^2)^{v-1}\ \ ds$$

$$= \int_{-1}^{1}\left(\frac{1}{2}\right)^{2v-1}(1-s^2)^{v-1}\ \ ds$$

Therefore $$\frac{\Gamma(v)\Gamma(v)}{\Gamma(2v)} = \int_{-1}^{1} \left(\frac{1}{2}\right)^{2v-1} (1-s^2)^{v-1} \quad ds$$

$$= 2\int_{-1}^{1} \left(\frac{1}{2}\right)^{2v-1} (1-s^2)^{v-1} \quad ds$$

Because of symmetry for $1 - s^2$

Let $$u = s^2 \Longrightarrow du = 2sds$$

Therefore $$\frac{du}{2s} = \frac{du}{2\sqrt{u}}$$

$$\int_0^1 (1-s^2)^{v-1} ds = \int_0^1 (1-u)^{v-1} \frac{du}{2u^{\frac{1}{2}}}$$

$$= \frac{1}{2}\int_0^1 \frac{(1-u)^{v-1}}{u^{\frac{1}{2}}} du$$

$$= \frac{1}{2}\int_0^1 u^{-\frac{1}{2}} \ (1-u)^{v-1} du$$

$$= \frac{1}{2}\int_0^1 u^{\frac{1}{2}-1} (1-u)^{v-1} \, du$$

$$= \frac{1}{2} B(\frac{1}{2}, v)$$

Therefore $$\frac{\Gamma(v)\Gamma(v)}{\Gamma(2v)} = 2\left(\frac{1}{2}\right)^{2v-1} \int_0^1 (1-s^2)^{v-1} \quad ds$$

$$= 2\left(\frac{1}{2}\right)^{2v-1} * \frac{1}{2} B\left(\frac{1}{2}, v\right)$$

$$= \left(\frac{1}{2}\right)^{2v-1} * B\left(\frac{1}{2}, v\right)$$

Hence $$\Gamma(2v) = \frac{\Gamma(v)\Gamma(v)}{\left(\frac{1}{2}\right)^{2v-1} B\left(\frac{1}{2}, v\right)} = \frac{2^{2v-1}\Gamma(v)\Gamma(v)}{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(v)}{\Gamma\left(\frac{1}{2}+v\right)}}$$

$$= \frac{2^{2v-1}\Gamma(v)\Gamma\left(v+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{2^{2v-1}\Gamma(v)\Gamma\left(v+\frac{1}{2}\right)}{\sqrt{\pi}}$$

Corollary: 2.3 when $v\ is\ a\ positive\ integer, then\ property\ 2.10\ becomes$

$$(2v-1)! = \frac{2^{2v-1}(v-1)!\,\Gamma\left(v+\frac{1}{2}\right)}{\sqrt{\pi}}$$

Therefore $\Gamma\left(v+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2v-1)!}{2^{2v-1}(v-1)!}$

$$= \frac{\sqrt{\pi}(2v-1)!}{2^{2v-1}(v-1)!}\frac{(2v)}{(2v)} * \frac{v}{(v-1)!\,v}$$

$$= \frac{\sqrt{\pi}(2v)!\,v}{2^{2v-1}v!} = \frac{(2v)!\sqrt{\pi}}{v!\,2^{2v}}$$

2.2 Derivation of One parameter Gamma Distribution

From definition (2.2) $\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx$

Therefore $1 = \int_0^\infty \frac{e^{-x}x^{\alpha-1}dx}{\Gamma(\alpha)}$

Hence $f(x) = \frac{e^{-x}x^{\alpha-1}}{\Gamma(\alpha)}\ ;\ x > 0\ ,$ (2.2)

which is the standard (one parameter) gamma pdf with parameter α.

If $\alpha = 1$, we have an exponential distribution with parameter 1.

If α is a positive integer, we have an Erlang distribution.

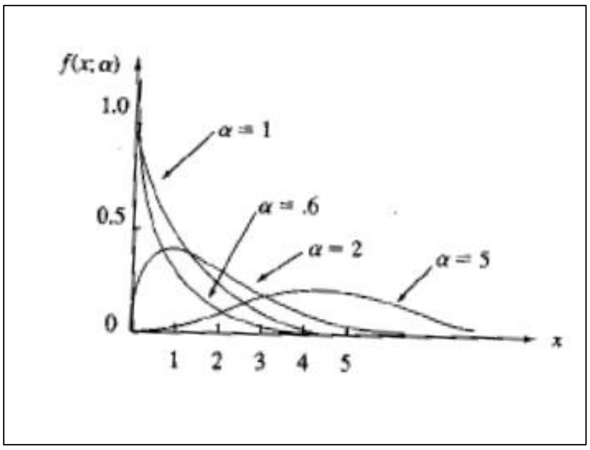

Figure 2.3: Graph of one parameter gamma

$$E(X^r) = \int_0^\infty x^r f(x)dx$$

$$= \int_0^\infty \frac{e^{-x}x^r x^{\alpha-1}dx}{\Gamma(\alpha)}$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-x}x^{\alpha+r-1}dx$$

$$= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}$$

$$E(X) = \mu = \alpha$$

$$E(X^2) = \frac{(\alpha+1)!}{(\alpha-1)!} = \frac{(\alpha+1)\alpha(\alpha-1)!}{(\alpha-1)!} = \alpha^2 + \alpha$$

$$E(X^3) = \frac{(\alpha+2)!}{(\alpha-1)!} = \frac{(\alpha+2)(\alpha+1)\alpha(\alpha-1)!}{(\alpha-1)!} = \alpha^3 + 3\alpha^2 + 2\alpha$$

$$E(X^4) = \frac{(\alpha+3)!}{(\alpha-1)!}$$

$$= \frac{(\alpha+3)(\alpha+2)(\alpha+1)\alpha(\alpha-1)!}{(\alpha-1)!}$$

$$= \alpha^4 + 6\alpha^3 + 11\alpha^2 + 6\alpha$$

$$VarX = E(X-\mu)^2$$

$$= E(X^2) - 2\,\mu\,E(X) + \mu^2$$

$$= \alpha^2 + \alpha - 2\,\alpha * \,\alpha + \alpha^2$$

$$= \ \alpha$$

Skewness: $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\alpha\,E(X^2) + 3\alpha^2\,E(X) - \alpha^3$$

$$= \ \alpha^3 + 3\alpha^2 + 2\alpha - 3\alpha\,(\alpha^2 + \alpha) + 3\alpha^2\,\alpha - \alpha^3$$

$$= 2\,\alpha$$

$$\gamma_1 = \frac{2\,\alpha}{\alpha^{\frac{3}{2}}} = \frac{2}{\sqrt{\alpha}}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\alpha\,E(X^3) + 6\alpha^2\,E(X^2) - 4\alpha^3 E(X) + \alpha^4$$

$$= \alpha^4 + 6\alpha^3 + 11\alpha^2 + 6\alpha - 4\alpha\,(\alpha^3 + 3\alpha^2 + 2\alpha) +$$

$$6\alpha^2\,(\alpha^2 + \alpha) - 4\alpha^3\alpha + \alpha^4$$

$$= 3\alpha^2 + 6\alpha$$

$$\gamma_2 = \frac{3\alpha^2 + 6\alpha}{\alpha^2} = 3 + \frac{6}{\alpha}$$

Laplace Transform $L_x(s) = E(e^{-sx})$

$$= \int_0^\infty \frac{e^{-sx}e^{-x}x^{\alpha-1}dx}{\Gamma(\alpha)}$$

$$= \int_0^\infty \frac{e^{-(s+1)x}x^{\alpha-1}dx}{\Gamma(\alpha)} \qquad (2.3)$$

Let $u = (s+1)x \Rightarrow du = (s+1)dx \text{ and it } \Rightarrow x = \frac{u}{(s+1)} \text{ replacing in } (2.3)$

$$L_x(s) = \int_0^\infty \frac{e^{-u}\left(\frac{u}{(s+1)}\right)^{\alpha-1}\frac{du}{s+1}}{\Gamma(\alpha)}$$

$$= \left(\frac{1}{(s+1)}\right)^\alpha \int_0^\infty \frac{e^{-u}u^{\alpha-1}du}{\Gamma(\alpha)}$$

$$= \left(\frac{1}{S+1}\right)^\alpha$$

2.3 Distributions related to one Parameter Gamma Distribution

Convolution

Let us take

$$f(x) = \frac{e^{-x}x^{\alpha-1}}{\Gamma(\alpha)} \text{ ; } x > 0 \ \alpha > 0 \qquad (2.4)$$

2.3.1 Sum of gamma indepedendet random variables

If $z = X + Y \text{ where } X \text{ and } Y \text{ are independent random variables}$

Suppose $X \sim Gamma(\mu) \text{ and } Y \sim Gamma(v)$

Using the Laplace transform technique,

$$L_x(s) = \left(\frac{1}{1+s}\right)^\mu \text{ and } L_y(s) = \left(\frac{1}{1+s}\right)^v$$

Therefore $L_z(s) = E(e^{-sz})$

$$= E\left(e^{-s(X+Y)}\right)$$

$$= E(e^{-sX})E(e^{-sY})$$

$$= L_x(s)L_y(s) = \left(\frac{1}{1+s}\right)^{\mu+v}$$

$$\text{Therefore } f(z) = \frac{e^{-x} z^{\mu+v-1}}{\Gamma(\mu+v)}; z > 0, \mu + v > 0 \qquad (2.5)$$

The gamma distribution is closed under convolution

Using the notion of convolution, then $f(z) = \int_0^z f_1(z-y) f_2(y) dy$

$$= \int_0^z \frac{e^{-(z-y)}(z-y)^{\mu-1}}{\Gamma(\mu)} * \frac{e^{-y} y^{v-1}}{\Gamma(v)} dy$$

$$= \frac{1}{\Gamma(\mu)\Gamma(v)} \int_0^z e^{-z} (z-y)^{\mu-1} y^{v-1} dy$$

$$put \qquad y = zt; dy = zdt \ and \ t = \frac{y}{z}$$

Therefore $f(z) = \frac{1}{\Gamma(\mu)\Gamma(v)} \int_0^1 e^{-z} (z - zt)^{\mu-1} (zt)^{v-1} z \, dt$

$$= \frac{1}{\Gamma(\mu)\Gamma(v)} \int_0^1 z^{\mu+v-1} e^{-z} (1-t)^{\mu-1} t^{v-1} \, dt$$

$$= \frac{z^{\mu+v-1} e^{-z}}{\Gamma(\mu)\Gamma(v)} \int_0^1 (1-t)^{\mu-1} t^{v-1} \, dt$$

$$= \frac{z^{\mu+v-1} e^{-z}}{\Gamma(\mu)\Gamma(v)} B(v, \mu)$$

$$= \frac{z^{\mu+v-1} e^{-z}}{\Gamma(\mu)\Gamma(v)} * \frac{\Gamma(\mu)\Gamma(v)}{\Gamma(\mu+v)}$$

$$= \frac{e^{-z} z^{\mu+v-1}}{\Gamma(\mu+v)}; \quad z > 0, \mu + v - 1 > 0 \qquad (2.6\ a)$$

Remark: Since f(z) is a pdf, then intergrating equation (2.5)

$$1 = \int_0^\infty \frac{z^{\mu+v-1} e^{-z}}{\Gamma(\mu)\Gamma(v)} B(v, \mu) dz$$

$$= \frac{1}{\Gamma(\mu)\Gamma(v)} B(v, \mu) \int_0^\infty z^{\mu+v-1} e^{-z} dz$$

$$= \frac{1}{\Gamma(\mu)\Gamma(v)} B(v, \mu) * \Gamma(\mu + v)$$

Alternatively Let $Z = X + Y$

$$F_{X+Y}(z) = Prob(Z \leq z) = Prob(X + Y \leq z)$$

$$= Prob(X \leq z - y, 0 < y < \infty)$$

$$= \int_0^\infty \int_0^{z-y} f_X(x) f_Y(y) dx dy$$

$$= \int_0^\infty F_X(z - y)\ f_Y(y) dy$$

$$f_{x+y}(z) = \int_0^\infty f_X(z - y) f_Y(y) dy$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha_1)} (z - y)^{\alpha_1 - 1} e^{-(z-y)} \frac{1}{\Gamma(\alpha_2)} y^{\alpha_2 - 1} e^{-y} dy$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (z - y)^{\alpha_1 - 1} e^{-(z-y)} y^{\alpha_2 - 1} e^{-y} dy$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (z - y)^{\alpha_1 - 1} e^{-z+y-y} y^{\alpha_2 - 1} dy$$

$$let\ y = zu \Longrightarrow dy = zdu$$

$$f_{x+y}(z) = \frac{e^{-z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (z - zu)^{\alpha_1 - 1} (zu)^{\alpha_2 - 1} zdu$$

$$= \frac{e^{-z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty z^{\alpha_1 + \alpha_2} (1 - u)^{\alpha_1 - 1} u^{\alpha_2 - 1} du$$

$$= \frac{e^{-z} z^{\alpha_1 + \alpha_2 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty u^{\alpha_2 - 1} (1 - u)^{\alpha_1 - 1} du$$

$$= \frac{e^{-z} z^{\alpha_1 + \alpha_2 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_2, \alpha_1)$$

$$= \frac{e^{-z} z^{\alpha_1 + \alpha_2 - 1}}{\Gamma(\alpha_1 + \alpha_2)}\ \ let\ \alpha_1 = \mu\ and\ \alpha_2 = v$$

$$Therefore\ \ f(x) = \frac{e^{-z} z^{\mu + v - 1}}{\Gamma(\mu + v)}\ \ , z > 0, \mu + v - 1 > 0 \qquad (2.6\ b)$$

2.3.2 Product of gamma distribution independent random variables

Derivation of the distribution of $Y = X_1X_2$ where X_1 and X_2 are mutually independent random variables and X_j has a gamma (α_j) distribution $(j = 1, 2)$

The joint density function of X_1 and X_2 is

$$f(x_1, x_{2)} = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} {x_1}^{\alpha_1 - 1} {x_2}^{\alpha_2 - 1} e^{-x_1 - x_2} \; ; \; x_{1,} x_2 > 0 \qquad (2.7)$$

Making the one-to-one transformation

$$y = x_1 x_2 \, , z = x_1 \implies x_1 = z \, and \, x_2 = \frac{y}{z}$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{} \\ \frac{1}{z} & -\frac{y}{z^2} \end{vmatrix} = \frac{1}{z}$$

The joint density function of $Y = X_1X_2$ and Z=X_1 is

$$f(y, z) = f_{x_1(y)} f_{x_2(z)*} |J|$$

$$= \frac{1}{\Gamma(\alpha_1)} {x_1}^{\alpha_1 - 1} {x_2}^{\alpha_2 - 1} e^{-x_1} \; \frac{1}{\Gamma(\alpha_2)} {x_2}^{\alpha_2 - 1} e^{-x_2}$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-z} z^{\alpha_1 - 1} e^{-\frac{y}{z}} \left(\frac{y}{z}\right)^{\alpha_2 - 1} \quad * \frac{1}{z}$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-z - \frac{y}{z}} \frac{z^{\alpha_1 - 1}}{z^{\alpha_2 - 1} * z} y^{\alpha_2 - 1}$$

Therefore $\qquad f(y) = \int_0^\infty f(y, z) dz$

$$= \frac{y^{\alpha_2 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty z^{\alpha_1 - \alpha_2 - 1} e^{-z - \frac{y}{z}} dz$$

Let $\qquad z = \sqrt{y}\, t \implies dz = \sqrt{y}\, dt$

Therefore $\qquad f(y) = \frac{y^{\alpha_2 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (\sqrt{y}\, t)^{\alpha_1 - \alpha_2 - 1} e^{-(\sqrt{y}\, t + \frac{\sqrt{y}}{t})} \sqrt{y}\, dt$

$$= \frac{y^{\alpha_2 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (\sqrt{y}\,)^{\alpha_1 - \alpha_2} t^{\alpha_1 - \alpha_2 - 1} e^{-\sqrt{y}(\, t + \frac{1}{t})} \; dt$$

$$= \frac{y^{\alpha_2 - 1} y^{\frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty t^{\alpha_1 - \alpha_2 - 1} e^{-\frac{2\sqrt{y}(t + \frac{1}{t})}{2}} \, dt$$

$$= \frac{y^{\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty t^{\alpha_1 - \alpha_2 - 1} e^{-\frac{2\sqrt{y}(t + \frac{1}{t})}{2}} \, dt$$

$$= \frac{y^{\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} 2K_{\alpha_1 - \alpha_2}\left(2\sqrt{y}\right)$$

where $K_v(\omega)$is the modified bessel function of the third kind.

$$f(y) = \frac{y^{\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} 2K_{\alpha_1 - \alpha_2}\left(2\sqrt{y}\right); y > 0 \qquad (2.8\ a)$$

Alternatively

$$Prob(Y \le y) = Prob(X_1 X_2 \le y) = Prob(X_1 \le \frac{y}{x_2}, 0 < x < \infty)$$

$$= \int_0^\infty \int_0^{\frac{y}{x_2}} f_1(x_1) f_2(x_2) \quad dx_1 dx_2$$

$$= \int_0^\infty \left(\int_0^{\frac{y}{x_2}} f_1(x_1) dx_1 \right) f_2(x_2) dx_2$$

$$H(y) = \int_0^\infty F\left(\frac{y}{x_2}\right) f_2(x_2) dx_2$$

$$h(y) = \frac{d}{dy} H(y)$$

$$= \int_0^\infty \frac{1}{x_2} f_1\left(\frac{y}{x_2}\right) f_2(x_2) dx_2$$

$$= \int_0^\infty \frac{1}{x_2} \left[\frac{1}{\Gamma(\alpha_1)} \left(\frac{y}{x_2}\right)^{\alpha_1 - 1} e^{-\frac{y}{x_2}} \right] \frac{1}{\Gamma(\alpha_2)} {x_2}^{\alpha_2 - 1} e^{-x_2} dx_2$$

$$h(y) = \int_0^\infty \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\frac{y}{x_2} - x_2} \frac{1}{x_2} * \left(\frac{y}{x_2}\right)^{\alpha_1 - 1} {x_2}^{\alpha_2 - 1} \, dx_2$$

$$= \frac{y^{\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty \frac{{x_2}^{\alpha_2-1}}{{x_2}^{\alpha_1}} * e^{-(x_2+\frac{y}{x_2})}\, dx_2$$

$$= \frac{y^{\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty {x_2}^{\alpha_2-\alpha_1-1} e^{-\left(x_2+\frac{y}{x_2}\right)}\, dx_2$$

Let $x_2 = \sqrt{y}\, z \implies \mathrm{d}x_2 = \sqrt{y}\, dz$

$$h(y) = \frac{y^{\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (\sqrt{y}\, z)^{\alpha_2-\alpha_1-1} e^{-\left((\sqrt{y}\, z)+\frac{y}{\sqrt{y}\, z)}\right)} \sqrt{y}\, dz$$

$$= \frac{y^{\alpha_2-1} y^{\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty z^{\alpha_1-\alpha_2-1} e^{-\frac{2\sqrt{y}(\,z+\frac{1}{z})}{2}}\, dz$$

$$= \frac{y^{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty z^{\alpha_2-\alpha_1-1} e^{-\frac{2\sqrt{y}(\,z+\frac{1}{z})}{2}}\, dz$$

$$= \frac{y^{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} 2K_{\alpha_2-\alpha_1}\left(2\sqrt{y}\right), y, \alpha_{1,}\alpha_2 > 0 \qquad (2.8\, b)$$

2.3.3 Quotient of gamma independent random variables by the change of variable technique

Case 1:

Derivation of the distribution of $Y_2 = \frac{X_1}{X_1+X_2}$ where X_1 *and* X_2 are mutually independent random variables and X_j has a gamma (α_j) distribution $(j = 1, 2)$

$$Let \quad Y_1 = X_1 + X_{2,}\, x_1 = y_1\, y_2 \;\; and\; Y_2 = \frac{X_1}{X_1 + X_2}, x_2 = y_1(1 - y_2)$$

$$|J| = \begin{vmatrix} \frac{\partial\, x_1}{\partial\, y_1} & \frac{\partial\, x_1}{\partial\, y_1} \\ \frac{\partial x_2}{\partial\, y_2} & \frac{\partial x_2}{\partial\, y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -y_1$$

$$f(y_1, y_2) = f(x_1, x_{2)*}|J| = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} {x_1}^{\alpha_1-1}{x_2}^{\alpha_2-1} e^{-(x_{1+}x_2)}\,; \qquad x_{1,}x_2 > 0$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (\, y_1\, y_2)^{\alpha_1-1}(\, y_1(1 - y_2))^{\alpha_2-1} e^{-\, y_1} * \; y_1$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \, {y_1}^{\alpha_1+\alpha_2-1} \, {y_2}^{\alpha_2-1}(1- \, y_2)^{\alpha_2-1}e^{-\,y_1}$$

Therefore $f(y_2) = \int_0^\infty f(y_1, y_2) dy_1$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \, {y_1}^{\alpha_1+\alpha_2-1} \, {y_2}^{\alpha_2-1}(1- \, y_2)^{\alpha_2-1}e^{-\,y_1} dy_1$$

$$= \frac{{y_2}^{\alpha_2-1}(1- \, y_2)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty \, e^{-\,y_1}{y_1}^{\alpha_1+\alpha_2-1} dy_1$$

$$= \frac{{y_2}^{\alpha_2-1}(1- \, y_2)^{\alpha_2-1}\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$$

$$= \frac{{y_2}^{\alpha_2-1}(1- \, y_2)^{\alpha_2-1}}{\mathrm{B}(\alpha_1, \alpha_2)} \; ; \; \alpha_1, \alpha_2 > 0, 0 < \, y_2 < 1 \qquad (2.9\, a)$$

which is Beta distribution of first kind.

Alternatively

Let $Z = \frac{X_1}{X_1+X_2}$

Prob $(Z \leq z) = Pro\left(\frac{X_1}{X_1+X_2} \leq z\right)$

$$= Prob(X_1 \leq z(x_1 + x_2))$$

$$= Prob((1-z)X_1 \leq zx_2)$$

$$= Prob(X_1 \leq \frac{zx_2}{1-z}, 0 < X_2 < \infty)$$

$$= \int_0^\infty \int_0^{\frac{zx_2}{1-z}} f(x_1)dx_1 f(x_2)d\,x_2$$

$$= \int_0^\infty F_{X_1}(\frac{zx_2}{1-z}) \; f(x_2)dx_2$$

$$h(z) = \int_0^\infty \frac{x_2}{(1-z)^2} f_{x_1}(\frac{zx_2}{1-z}) \; f(x_2)dx_2$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{x_2}{(1-z)^2} e^{-(\frac{zx_2}{1-z})} (\frac{zx_2}{1-z})^{\alpha_1-1} e^{-x_2} {x_2}^{\alpha_2-1} dx_2$$

$$= \frac{z^{\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)(1-z)^{\alpha_1+1}} \int_0^{\infty} {x_2}^{\alpha_1+\alpha_2-1} e^{-\frac{zx_2}{1-z}-x_2} \quad dx_2$$

$$= \frac{z^{\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)(1-z)^{\alpha_1+1}} \int_0^{\infty} {x_2}^{\alpha_1+\alpha_2-1} e^{-\frac{x_2}{1-z}} \quad dx_2$$

$$Let\ \ u = \frac{x_2}{1-z} \Longrightarrow x_2 = u(1-z), dx_2 = (1-z)du$$

$$h(z) = \frac{z^{\alpha_1-1}(1-z)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{\infty} u^{\alpha_1+\alpha_2-1} e^{-u} \quad du$$

$$= \frac{z^{\alpha_1-1}(1-z)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \Gamma(\alpha_1+\alpha_2)$$

$$= \frac{z^{\alpha_1-1}(1-z)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)} \quad let\ z = y_2$$

Therefore

$$f(y_2) = \frac{{y_2}^{\alpha_1-1}(1-y_2)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)}\ ;\ \alpha_{1,}\alpha_2 > 0, 0 < y_2 < 1 \qquad (2.9\ b)$$

which is a Beta distribution of first kind.

Case 2: Let $V = \frac{X_1}{X_2}$ and $U = X_1 + X_2$

Where $X_1 \sim \text{Gamma}(\alpha_1)$ and $X_2 \sim \text{Gamma}(\alpha_2)$

$$Thu\ \ x_1 = \frac{uv}{1+v}\ and\ x_2 = \frac{u}{1+v}$$

Derivation of the distribution of $V = \frac{X_1}{X_2}$ where X_1 and X_2 are mutually independent random variables and X_j has a gamma (α_j) distribution (j=1,2)

$$|J| = \begin{vmatrix} \frac{\partial\ x_1}{\partial u} & \frac{\partial\ x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{1+v} & \frac{u}{(1+v)^2} \\ \frac{v}{1+v} & -\frac{u}{(1+v)^2} \end{vmatrix} = \frac{-u}{(1+v)^2}$$

$$f(u,v) = f(x_1, x_{2)*}|J| = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} {x_1}^{\alpha_1-1}{x_2}^{\alpha_2-1}e^{-(x_{1+}x_{2)}} * \frac{u}{(1+v)^2}\ ; \qquad x_{1,}x_2 > 0$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(\frac{uv}{1+v}\right)^{\alpha_1-1}\left(\frac{u}{1+v}\right)^{\alpha_2-1} e^{-u} * \frac{u}{(1+v)^2}$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-u} u^{\alpha_1+\alpha_2-1} v^{\alpha_1-1} (1+v)^{-\alpha_1-\alpha_2} \tag{2.10}$$

Thus $\quad f(v) = \int_0^{\infty} f(u,v)du$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{\infty} e^{-u} u^{\alpha_1+\alpha_2-1} v^{\alpha_1-1} (1+v)^{-\alpha_1-\alpha_2} du$$

$$= \frac{v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{\infty} e^{-u} u^{\alpha_1+\alpha_2-1} du$$

$$= \frac{v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} * \Gamma(\alpha_1+\alpha_2)$$

$$= \frac{v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2}}{\mathrm{B}(\alpha_1,\alpha_2)} \;\; ; \; \alpha_1, \alpha_2 > 0, 0 < v < \infty \tag{2.11 a}$$

which is the Beta distribution of second kind.

Alternatively

Let $Z = \frac{X}{Y}$ $where\ X_1 = X\ and\ X_2 = Y$

$$F_{\frac{X}{Y}} = Prob\ (Z \le z) = Prob\ \left(\frac{X}{Y} \le z\right) = Prob(X \le zy)$$

$$= \int_0^{\infty} \int_0^{zy} f_x(x)\, f_y(y) dx dy = \int_0^{\infty} \left[\int_0^{zy} f_x(x) dx\right] f_y(y) dy$$

$$= \int_0^{\infty} F_{\frac{X}{Y}}(zy) f_y(y) dy$$

$$f_{\frac{x}{y}}(z) = \int_0^{\infty} y * f_x(zy) f_y(y) dy$$

$$= \int_0^{\infty} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (zy)^{\alpha_1-1} * y e^{-zy} e^{-y} y^{\alpha_2-1} dy$$

$$= \frac{z^{\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{\infty} y^{\alpha_1+\alpha_2-1} * e^{-zy-y} dy$$

Let $x = y(z+1) \Rightarrow y = \frac{x}{z+1} \Rightarrow dy = \frac{dx}{z+1}$

$$= \frac{z^{\alpha_1 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (\frac{x}{z+1})^{\alpha_1+\alpha_2-1} * e^{-x} \frac{dx}{z+1}$$

$$= \frac{z^{\alpha_1 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)(1+z)^{\alpha_1+\alpha_2}} \int_0^\infty (\frac{x}{z+1})^{\alpha_1+\alpha_2-1} * e^{-x} \frac{dx}{z+1}$$

$$= \frac{z^{\alpha_1 - 1}(1+z)^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty x^{\alpha_1+\alpha_2-1} * e^{-x} dx$$

$$= \frac{z^{\alpha_1 - 1}(1+z)^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \Gamma(\alpha_1 + \alpha_2), \qquad \text{let } z = v$$

$$= \frac{v^{\alpha_1 - 1}(1+v)^{-\alpha_1-\alpha_2}}{B(\alpha_1, \alpha_2)}; \ \alpha_1, \alpha_2 > 0, 0 < v < \infty \qquad (2.11\,b)$$

which is the Beta distribution of second kind

CHAPTER III: Two Parameter Gamma Distribution

3.0 Description

In this chapter the gamma distribution with two parameters is discussed in details including its derivation, properties and special cases. A random variable X is said to have the two parameter (standard) gamma distribution if its distribution is given by

$$f\,(x) = \frac{x^{\alpha-1}e^{\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} \quad or \;\; f(x) = \frac{\beta^{\alpha}x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)} \;, x\; > \;0, \alpha\; > 0, \beta > 0$$

Where the parameter α is called the shape parameter, since it most influences the peakedness of the distribution (Stephenson *et al*., 1999). The parameter β is called the inverse scale parameter, since most of its influence is on the spread of the distribution i.e. the standard deviation of the gamma distribution is proportional to $\frac{1}{\beta}$. Two-parameter gamma and two-parameter Weibull are the most popular distributions for analyzing any lifetime data (Rameshwar, D., 2001).

The two-parameter gamma distribution represents the scale and the shape parameters and because of the scale and shape parameters, it has quite a bit flexibility to analyze any positive real data. It has increasing as well as decreasing failure rate depending on the shape parameter, which gives an extra edge over exponential distribution, which has only constant failure (Rameshwar, D., 2001).

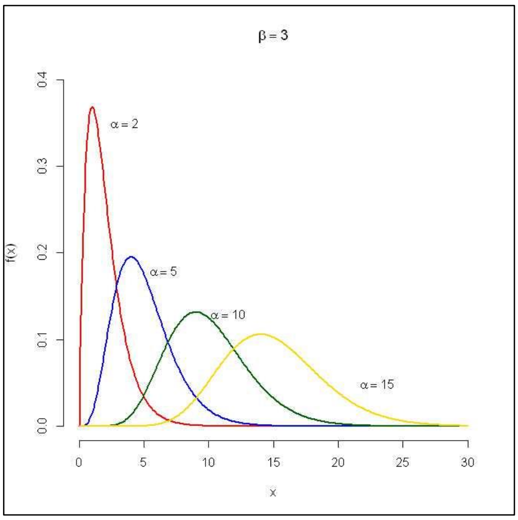

Figure 3.1: two parameter gamma - Beta fixed

The plots above in figure 3.1 illustrate, for example, that if the mean waiting time until the first event is β = 3, then we have a greater probability of our waiting time *X* being large if we are waiting for more events to occur (α = 15, say) than fewer (α = 2, say). It also makes sense that for fixed β, as α increases, the probability "moves to the right," as illustrated here with beta fixed at 3, and α increasing from 2 to 5 to 10 to 15:

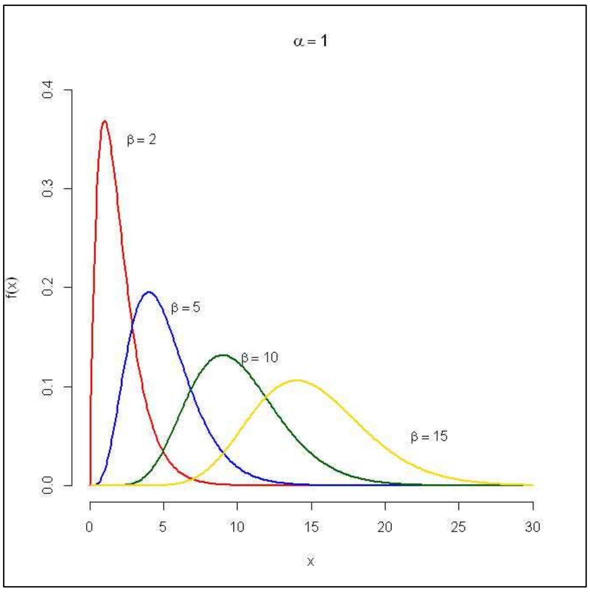

Figure 3.2: two parameter gamma - alpha fixed

The plots above in figure 3.2 illustrate, for example, that if we are waiting for α = 1 events to occur, we have a greater probability of our waiting time *X* being large if our mean waiting time until the first event is large (β= 15) than if it is small (β= 2).

3.1 Derivation of two parameter gamma distribution

Let $$g(y) = \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}\ ; x > 0, \alpha > 0 \qquad (3.0)$$

If $y = \frac{x}{\beta}$ then $\frac{dy}{dx} = \frac{1}{\beta}$ using the change of variables technique , the pdf of $x$is given by

$$f(x) = g(y) * |J| = \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)} * \frac{1}{\beta}$$

$$i.e\ f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}; x, \alpha, \beta > 0 \qquad (3.1)$$

which is a two parameter gamma distribution.

Alternatively, using cumulative distribution technique, let

Let $F(x) = prob(X \leq x)$

$= prob(\frac{X}{\beta} \leq \frac{x}{\beta})$

$= prob(Y \leq \frac{x}{\beta})$

$$= G_y(\frac{x}{\beta})$$

Therefore $f(x) = \frac{d}{dx}F(x) = \frac{1}{\beta}\ g(\frac{x}{\beta})$

$$f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} x, \alpha, \beta > 0 \qquad (3.2)$$

Paramerization of two parameter gamma distribution

Put $\theta = \frac{1}{\beta}$

Therefore $f(x) = \frac{\theta^{\alpha}}{\Gamma(\alpha)} e^{-\theta x} x^{\alpha-1} ; x > 0, \alpha, \theta > 0$ (3.3)

$$E(X^r) = \int_0^{\infty} \frac{x^r x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} dx$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^{\infty} x^{\alpha+r-1} e^{-\frac{x}{\beta}} dx \qquad (3.4)$$

$$Let\ u = \frac{x}{\beta} \Longrightarrow x = \beta u$$

$$du = \frac{dx}{\beta} \Longrightarrow \beta du = dx\ replacing\ in\ equation\ (3.4)$$

$$E(X^r) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^{\infty} (\beta u)^{\alpha+r-1} e^{-u} \beta du$$

$$= \frac{\beta^r}{\Gamma(\alpha)} \int_0^{\infty} u^{\alpha+r-1} e^{-u} du$$

$$= \frac{\beta^r \Gamma(r+\alpha)}{\Gamma(\alpha)} \tag{3.5}$$

$$E(X) = \beta\alpha$$

$$E(X^2) = \frac{\beta^2\Gamma(2+\alpha)}{\Gamma(\alpha)} = \frac{\beta^2(1+\alpha)!}{(\alpha-1)!}$$

$$= \frac{\beta^2(1+\alpha)\alpha(\alpha-1)!}{(\alpha-1)!}$$

$$= \beta^2\alpha^2 + \beta^2\alpha$$

$$E(X^{3)} = \frac{\beta^3\Gamma(3+\alpha)}{\Gamma(\alpha)} = \frac{\beta^3(2+\alpha)!}{(\alpha-1)!}$$

$$= \frac{\beta^3(2+\alpha)(1+\alpha)\alpha(\alpha-1)!}{(\alpha-1)!}$$

$$= \beta^3\alpha^3 + 3\beta^3\alpha^2 + 2\beta^3\alpha$$

$$E(X^{4)} = \frac{\beta^4\Gamma(4+\alpha)}{\Gamma(\alpha)} = \frac{\beta^4(3+\alpha)!}{(\alpha-1)!} = \frac{\beta^4(3+\alpha)(2+\alpha)(1+\alpha)\alpha(\alpha-1)!}{(\alpha-1)!}$$

$$= \beta^4\alpha^4 + 6\beta^4\alpha^3 + 11\beta^4\alpha^2 + 6\beta^4\alpha$$

$$Var\, X = \mu_2 = \beta^2\alpha^2 + \beta^2\alpha - (\beta\alpha)^2$$

$$= \beta^2\alpha$$

Skewness: $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\beta\alpha\, E(X^2) + 3\beta^2\alpha^2\, E(X) - \beta^3\alpha^3$$

$$= \beta^3\alpha^3 + 3\beta^3\alpha^2 + 2\beta^3\alpha - 3\beta\alpha(\beta^2\alpha^2 + \beta^2\alpha) + 3\beta^2\alpha^2(\beta\alpha) - \beta^3\alpha^3$$

$$= 2\,\beta^3\alpha$$

$$\gamma_1 = \frac{\mu_3}{\sigma^3} = \frac{2\,\beta^3\alpha}{\beta^2\alpha\sqrt{\beta^2\alpha}} = \frac{2}{\sqrt{\alpha}}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\beta\alpha\, E(X^2) + 6\beta^2\alpha^2\, E(X) - 4\beta^3\alpha^3 E(X) + \beta^4\alpha^4$$

$$= E(X^4) - 4\beta\alpha\, E(X^3) + 6\beta^2\alpha^2\, E(X^2) - 4\beta^3\alpha^3 E(X) + \beta^4\alpha^4$$

$$= \beta^4\alpha^4 + 6\beta^4\alpha^3 + 11\beta^4\alpha^2 + 6\beta^4\alpha - 4\beta\alpha(\beta^3\alpha^3 + 3\beta^3\alpha^2 + 2\beta^3\alpha) + 6\beta^2\alpha^2(\beta^2\alpha) - 4\beta^3\alpha^3(\beta\alpha) + \beta^4\alpha^4$$

$$= 3\beta^4\alpha^2 + 6\beta^4\alpha$$

$$\gamma_2 = \frac{\mu_4}{\sigma^4} = \frac{3\beta^4\alpha^2 + 6\beta^4\alpha}{(\beta^2\alpha)^2} = 3 + \frac{6}{\alpha}$$

Laplace Transform

$$L_x(s) = E(e^{-sx})$$

$$= \int_0^\infty e^{-sx} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)} dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty e^{-(s+\frac{1}{\beta})x} x^{\alpha-1}\, dx$$

$$put\ t = \left(s + \frac{1}{\beta}\right) x \Longrightarrow dt = \left(s + \frac{1}{\beta}\right) dx$$

Therefore

$$L_x(s) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty e^{-t} \left(\frac{t}{s+\frac{1}{\beta}}\right)^{\alpha-1} \frac{dt}{\left(s+\frac{1}{\beta}\right)}$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha) \left(s + \frac{1}{\beta}\right)^\alpha} \int_0^\infty e^{-t} t^{\alpha-1} dt$$

$$= \frac{1}{\left(s + \frac{1}{\beta}\right)^\alpha} \int_0^\infty \frac{e^{-t} t^{\alpha-1} dt}{\Gamma(\alpha)} = \left(\frac{1}{\beta s + 1}\right)^\alpha \qquad (3.6)$$

If $\beta = \frac{1}{\theta}$ $then\ L_x(s) = \left(\frac{1}{\frac{s}{\theta}+1}\right)^\alpha = \left(\frac{\theta}{\theta+s}\right)^\alpha$

3.2 Special Cases of two parameter gamma distribution

One parameter gamma distribution

$For\ \beta = 1$ in equation (3.2)

$$f(x) = \frac{e^{-x}x^{\alpha-1}}{\Gamma(\alpha)} \quad x > 0, \alpha > 0 \tag{3.7}$$

$$which\ is\ a\ gamma\ distribution\ with\ one - parameter\ \alpha.$$

Properties are discussed in an early section.

Exponential distribution

For $\alpha = 1\ and\ \beta = \theta\ in\ equation\ (3.2)$

$$f\ (x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}; \quad x > 0 \tag{3.8}$$

$$which\ is\ an\ exponential\ distribution\ with\ parameter\ \theta.$$

$$E(X^r) = \int_0^\infty \frac{1}{\theta} x^r e^{-\frac{x}{\theta}}\ dx$$

$$= \theta^r \Gamma(1 + r) \tag{3.9}$$

$$E(X) = \theta$$

$$E(X^{2)} = \theta^2 \Gamma(1 + 2) = \theta^2 * 2$$

$$E(X^{3)} = \theta^3 \Gamma(4) = \theta^3 * 3!$$

$$= 6\theta^3$$

$$E(X^{4)} = \theta^4 \Gamma(5) = \theta^4 * 4!$$

$$= 24\theta^4$$

$$Var\ X = \mu_2 = 2\theta^2 - \theta^2 = \theta^2$$

Skewness $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X - \mu)^3$$

$$= E(X^3) - 3\theta\ E(X^2) + 3\theta^2\ E(X) - \theta^3$$

$$= \theta^3 + 3\theta^3 \quad + 2\theta^3 - 3\theta(\theta^2 + \theta^2) + 3\theta^2(\theta) - \theta^3$$

$$= 2\ \theta^3$$

$$\gamma_1 = \frac{2\ \theta^3}{\theta^2\sqrt{\theta^2}} = 2$$

Kurtosis $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\theta\, E(X^3) + 6\theta^2\, E(X^2) - 4\theta^3 \quad E(X) + \theta^4$$

$$= \theta^4 + 6\theta^4 + 11\theta^4 + 6\theta^4 - 4\theta(\theta^3\alpha^3 + 3\theta^3\alpha^2 + 2\theta^3\alpha)$$

$$+ 6\theta^2\alpha^2(\theta^2\alpha) - 4\theta^3(\theta) + \theta^4$$

$$= 9\theta^4$$

$$\gamma_2 = \frac{3\theta^4 \quad + 6\theta^4}{(\theta^2)^2} = 9$$

Laplace Transform

subtituting $\beta = \theta, \alpha = 1$ *in equation* (3.6)

$$L_x(s) = \frac{1}{\theta s + 1}$$

Chi - Square distribution

For α = $\frac{v}{2}$ where v is the number of degrees of freedom β = 2, then equation (3.2) becomes;

$$f(x) = \frac{1}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)}\, x^{\frac{v-2}{2}} e^{-\frac{x}{2}}; \qquad x > 0 \tag{3.10}$$

which is a Chi − Square distribution

$$E(X^r) = \frac{2^r\Gamma\left(\frac{v}{2}+r\right)}{\Gamma\left(\frac{v}{2}\right)} \tag{3.11}$$

$$E(X) = \frac{2\Gamma(\frac{v}{2}+1)}{\Gamma\left(\frac{v}{2}\right)}$$

$$= v$$

$$E(X^{2)} = \frac{2^2\Gamma\left(\frac{v}{2}+2\right)}{\Gamma\left(\frac{v}{2}\right)} = \frac{2^2\left(\frac{v}{2}+1\right)!}{\left(\frac{v}{2}-1\right)!}$$

$$= v(v+2)$$

$$E(X^{3)} = \frac{2^3\Gamma\left(\frac{v}{2}+3\right)}{\Gamma\left(\frac{v}{2}\right)} = \frac{2^3\left(\frac{v}{2}+2\right)\left(\frac{v}{2}+1\right)\left(\frac{v}{2}\right)\left(\frac{v}{2}-1\right)!}{\left(\frac{v}{2}-1\right)!}$$

$$= v(v+2)(v+4)$$

$$E(X^{4)} = \frac{2^4\Gamma\left(\frac{v}{2}+4\right)}{\Gamma\left(\frac{v}{2}\right)} = \frac{2^4\left(\frac{v}{2}+3\right)\left(\frac{v}{2}+2\right)\left(\frac{v}{2}+1\right)\left(\frac{v}{2}\right)\left(\frac{v}{2}-1\right)!}{\left(\frac{v}{2}-1\right)!}$$

$$= v(v+2)(v+4)(v+6)$$

$$\text{Var X} = \mu_2 = E(X-\mu)^2 = 2v$$

Skewness $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3*2*\frac{v}{2}E(X^2) + 3*2^2(\frac{v}{2})^2\, E(X) - 2^3(\frac{v}{2})^3$$

$$= v(v+2)(v+4) - 3v*v(v+2) + 3*2^2(\frac{v}{2})^2\, v - 2^3(\frac{v}{2})^3$$

$$= 2^3 v = 8v$$

$$\gamma_1 = \frac{2*\ 2^3*\frac{v}{2}}{2^{2*}\frac{v}{2}\sqrt{2^{2*}\frac{v}{2}}} = \frac{2}{\sqrt{\frac{v}{2}}} = 2\sqrt{\frac{2}{v}}$$

Kurtosis $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\beta\alpha\, E(X^2) + 6\beta^2\alpha^2\, E(X) - 4\beta^3\alpha^3 E(X) + \beta^4\alpha^4$$

$$= 3*2^4*(\frac{v}{2})^2 \quad + 6*2^4*\frac{v}{2}$$

$$= 12v^2 + 48v = 12v(v+4)$$

$$\gamma_2 = \frac{3*2^4*\left(\frac{v}{2}\right)^2 + 6*2^4*\frac{v}{2}}{\left(2^2*\frac{v}{2}\right)^2}$$

$$= 3 + \frac{6}{\frac{v}{2}} = 3 + \frac{12}{v}$$

Laplace Transform

subtituting $\beta = 2, \alpha = \frac{v}{2}$ $in\ equation$ (3.6)

$$L_x(s) = \left(\frac{1}{2s+1}\right)^{\frac{v}{2}}$$

3.3 Distributions related to a two-parameter Gamma distribution

Convolution

Lets take $f(x)=\frac{\beta^{\alpha}e^{-\beta x}x^{\alpha-1}}{\Gamma(\alpha)}; x>0, \alpha>0, \beta>0$

3.3.1 Sum of two gamma independent random variables

If $Z=X+Y$ *where* X *and* Y *are independent random variables and* $X_1=X, X_2=Y$

Where $X\sim Gamma(\mu,\beta)$ *and* $Y\sim Gamma(v,\beta)$

Using the Laplace transform technique,

$$L_x(s)=\left(\frac{1}{\beta S+1}\right)^{\mu} \text{ and } L_y(s)=\left(\frac{1}{\beta S+1}\right)^{v}$$

Therefore $L_z(s)=E(e^{-sz})$

$$=E\left(e^{-s(x+y)}\right)$$

$$=E(e^{-sx})E(e^{-sy})$$

$$=L_x(s)L_y(s)=\left(\frac{1}{\beta S+1}\right)^{\mu+v}$$

Therefore $f(z)=\frac{\beta^{\mu+v}e^{-\beta x}z^{\mu+v-1}}{\Gamma(\mu+v)}; z>0, \mu+v>0, \beta>0$ (3.12)

The gamma distribution is closed under convolution

Using the notion of convolution, then $f(z)=\int_0^z f_1(z-y)f_2(y)dy$

$$=\int_0^z \frac{\beta^{\mu}e^{-\beta(z-y)}(z-y)^{\mu-1}}{\Gamma(\mu)} * \frac{\beta^{v}e^{-\beta y}y^{v-1}}{\Gamma(v)}dy$$

$$=\frac{\beta^{\mu+v}}{\Gamma(\mu)\Gamma(v)}\int_0^z e^{-\beta z}(z-y)^{\mu-1}y^{v-1}dy \text{ put } y=zt; dy=zdt \text{ and } t=\frac{y}{z}$$

Therefore $f(z)=\frac{\beta^{\mu+v}}{\Gamma(\mu)\Gamma(v)}\int_0^1 e^{-\beta z}(z-zt)^{\mu-1}(zt)^{v-1}z\,dt$

$$=\frac{\beta^{\mu+v}}{\Gamma(\mu)\Gamma(v)}\int_0^1 z^{\mu+v-1}e^{-\beta z}(1-t)^{\mu-1}t^{v-1}\,dt$$

$$=\frac{\beta^{\mu+v}z^{\mu+v-1}e^{-\beta z}}{\Gamma(\mu)\Gamma(v)}\int_0^1 (1-t)^{\mu-1}t^{v-1}\,dt$$

$$= \frac{\beta^{\mu+v} z^{\mu+v-1} e^{-\beta z}}{\Gamma(\mu)\Gamma(v)} B(v,\mu)$$

$$= \frac{\beta^{\mu+v} z^{\mu+v-1} e^{-\beta z}}{\Gamma(\mu)\Gamma(v)} * \frac{\Gamma(\mu)\Gamma(v)}{\Gamma(\mu+v)}$$

$$= \frac{\beta^{\mu+v} z^{\mu+v-1} e^{-\beta z}}{\Gamma(\mu)\Gamma(v)} B(v,\mu)\,; z > 0, \beta, \mu, v > 0 \qquad (3.13\ a)$$

Remark: Since $f(z)$ is a pdf, then intergrating equation (3.13 a)

$$1 = \int_0^\infty \frac{\beta^{\mu+v} z^{\mu+v-1} e^{-\beta z}}{\Gamma(\mu)\Gamma(v)} B(v,\mu) dz$$

$$= \frac{\beta^{\mu+v}}{\Gamma(\mu)\Gamma(v)}\ B(v,\mu) \int_0^\infty z^{\mu+v-1} e^{-\beta z} dz$$

$$= \frac{\beta^{\mu+v}}{\Gamma(\mu)\Gamma(v)}\ B(v,\mu) * \frac{\Gamma(\mu+v)}{\beta^{\mu+v}}$$

$$= \frac{B(v,\mu)\Gamma(\mu+v)}{\Gamma(\mu)\Gamma(v)} \quad therefore\ B(v,\mu) = \frac{\Gamma(\mu)\Gamma(v)}{\Gamma(\mu+v)}$$

Thus as a by-product we have found that $\mathrm{B}(v,\mu) = \frac{\Gamma(\mu)\Gamma(v)}{\Gamma(\mu+v)}$

Alternatively

Let $\quad Z = X + Y$

$F_{X+Y}(z) \quad = Prob(Z \le z) = Prob(X + Y \le z)$

$$= Prob(X \le z - y, 0 < y < \infty)$$

$$= \int_0^\infty \int_0^{z-y} f_X(x) f_Y(y) dx dy$$

$$= \int_0^\infty F_X(z-y)\ f_Y(y) dy$$

$$f_{x+y}(z) = \int_0^\infty f_X(z-y)\, f_Y(y) dy$$

$$= \int_0^\infty \frac{1}{\beta^{\alpha_1}\Gamma(\alpha_1)} (z-y)^{\alpha_1-1} e^{-\frac{1}{\beta}(z-y)} \frac{1}{\beta^{\alpha_2}\Gamma(\alpha_2)} y^{\alpha_2-1} e^{-\frac{1}{\beta}y} dy$$

$$= \frac{1}{\beta^{\alpha_1}\beta^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^\infty (z-y)^{\alpha_1-1}e^{-\frac{1}{\beta}(z-y)}\, y^{\alpha_2-1}e^{-\frac{1}{\beta}y}dy$$

$$= \frac{1}{\beta^{\alpha_1}\beta^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^\infty (z-y)^{\alpha_1-1}e^{-\frac{1}{\beta}(z-y+y)}\, y^{\alpha_2-1}dy$$

$$let\ y = zu \Longrightarrow dy = zdu$$

$$f_{x+y}(z)\ = \frac{e^{-\frac{z}{\beta}}}{\beta^{\alpha_1}\beta^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^\infty (z-zu)^{\alpha_1-1}\,(zu)^{\alpha_2-1}zdu$$

$$= \frac{e^{-\frac{z}{\beta}}}{\beta^{\alpha_1}\beta^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^\infty z^{\alpha_1+\alpha_2}(1-u)^{\alpha_1-1}\,u^{\alpha_2-1}du$$

$$= \frac{e^{-\frac{z}{\beta}}z^{\alpha_1+\alpha_2-1}}{\beta^{\alpha_1}\beta^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^\infty u^{\alpha_2-1}(1-u)^{\alpha_1-1}\,du$$

$$= \frac{e^{-\frac{z}{\beta}}z^{\alpha_1+\alpha_2-1}}{\beta^{\alpha_1}\beta^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}B(\alpha_2,\alpha_1)$$

$$= \frac{e^{-\frac{z}{\beta}}z^{\alpha_1+\alpha_{2-1}}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\ \ let\ \alpha_1 = \mu\ and\ \alpha_2 = v$$

$$Therefore\ \ f(x) = \frac{e^{-\frac{z}{\beta}}z^{\mu+v-1}}{\beta^{\mu+v}\Gamma(\mu+v)}\ \ ;\ z > 0, \beta, \mu+v-1 > 0$$

3.3.2 Product of gamma distributed random variables

Derivation of the distribution of $Y = X_1X_2$ where $X_1\ and\ X_2$ are mutually independent random variables and X_j has a gamma (α_j, β) distribution (j=1,2)

The joint density function of $X_1\ \ and\ X_2$ is

$$f(x_1,x_{2)} = \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}{x_1}^{\alpha_1-1}{x_2}^{\alpha_2-1}e^{-\frac{1}{\beta}(x_{1+}x_2)}\ ;\ x_{1,}x_2 > 0, \alpha_1, \alpha_2, \beta > 0$$

Making the one-to-one transformation

$$y = x_1x_2\,, z = x_1\ \Longrightarrow x_1 = z\ and\ x_2 = \frac{y}{z}$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & -\frac{1}{\frac{y}{z^2}} \\ \frac{1}{z} & \end{vmatrix} = \frac{1}{z}$$

The joint density function of $Y = X_1X_2$ and Z=X_1 is

$$f(y,z) = f_{x_1(y)}f_{x_2(z)*}|J|$$

$$= \frac{1}{\beta^{\alpha_1}\Gamma(\alpha_1)} {x_1}^{\alpha_1-1}{x_2}^{\alpha_2-1}e^{-\frac{x_1}{\beta}} \frac{1}{\beta^{\alpha_2}\Gamma(\alpha_2)} {x_2}^{\alpha_2-1}e^{-\frac{x_2}{\beta}}$$

$$= \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\frac{z}{\beta}} z^{\alpha_1-1} e^{-\frac{y}{\beta z}} \left(\frac{y}{z}\right)^{\alpha_2-1} * \frac{1}{z}$$

$$= \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\frac{z}{\beta}-\frac{y}{\beta z}} \frac{z^{\alpha_1-1}}{z^{\alpha_2-1} * z} y^{\alpha_2-1}$$

Therefore $\quad f(y) = \int_0^\infty f(y,z)dz$

$$= \frac{y^{\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty z^{\alpha_1-\alpha_2-1} e^{-\frac{z}{\beta}-\frac{y}{\beta z}} dz$$

Let $\quad z = \sqrt{y}\, t \Longrightarrow dz = \sqrt{y}\, dt$

Therefore $\quad f(y) = \frac{y^{\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (\sqrt{y}\, t)^{\alpha_1-\alpha_2-1} e^{-(\sqrt{y}\frac{t}{\beta}+\frac{\sqrt{y}}{\beta t})} \sqrt{y}\, dt$

$$= \frac{y^{\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (\sqrt{y}\,)^{\alpha_1-\alpha_2} t^{\alpha_1-\alpha_2-1} e^{-\frac{\sqrt{y}}{\beta}(\,t+\frac{1}{t})}\ dt$$

$$= \frac{y^{\alpha_2-1} y^{\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_2}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty t^{\alpha_1-\alpha_2-1} e^{-\frac{2\sqrt{y}(\,t+\frac{1}{t})}{2\beta}}\ dt$$

$$= \frac{y^{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty t^{\alpha_1-\alpha_2-1} e^{-\frac{2\sqrt{y}(\,t+\frac{1}{t})}{2\beta}}\ dt$$

$$= \frac{y^{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{2}{\beta} K_{\alpha_1-\alpha_2}(2\sqrt{y})$$

where $K_{\mathrm{v}}(\omega)$is the modified bessel function of the third kind.

$$f(y) = \frac{y^{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{2}{\beta} K_{\alpha_1-\alpha_2}\left(2\sqrt{y}\right) ; y, \alpha_1, \alpha_1, \beta > 0 \qquad (3.14\ a)$$

Alternatively

$$prob(Y \le y) = Prob(X_1X_2 \le y) = Prob(x_1 \le \frac{y}{x_2}\ , 0 < x < \infty)$$

$$= \int_0^\infty \int_0^{\frac{y}{x_2}} f_1(x_1)f_2(x_2) \quad dx_1dx_2$$

$$= \int_0^\infty \left[\int_0^{\frac{y}{x_2}} f_1(x_1)dx_1\right] f_2(x_2)dx_2$$

$$H(y) = \int_0^\infty F(\frac{y}{x_2})\, f_2(x_2)dx_2$$

$$h(y) = \frac{d}{dy}H(y)$$

$$= \int_0^\infty \frac{1}{x_2} f_1(\frac{y}{x_2}) f_2(x_2)dx_2$$

$$= \int_0^\infty \frac{1}{x_2}\left[\frac{1}{\beta^{\alpha_1}\Gamma(\alpha_1)}\left(\frac{y}{x_2}\right)^{\alpha_1-1} e^{-\frac{y}{\beta x_2}}\right]\frac{1}{\beta^{\alpha_2}\Gamma(\alpha_2)} {x_2}^{\alpha_2-1} e^{-\frac{x_2}{\beta}} dx_2$$

$$h(y) = \int_0^\infty \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\frac{y}{\beta x_2}-\frac{x_2}{\beta}} \frac{1}{x_2} * \left(\frac{y}{x_2}\right)^{\alpha_1-1} {x_2}^{\alpha_2-1}\, dx_2$$

$$= \frac{y^{\alpha_1-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty \frac{{x_2}^{\alpha_2-1}}{{x_2}^{\alpha_1}} * e^{-\frac{1}{\beta}(x_2+\frac{y}{x_2})}\, dx_2$$

Let $x_2 = \sqrt{y}\,\mathrm{z} \Longrightarrow dx_2 = \sqrt{y}\, dz$

$$h(y) = \frac{y^{\alpha_1-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty \frac{(\sqrt{y}\,\mathrm{z})^{\alpha_2-1}}{(\sqrt{y}\,\mathrm{z})^{\alpha_1}} * e^{-\frac{1}{\beta}(\sqrt{y}\,\mathrm{z}+\frac{y}{\sqrt{y}\,\mathrm{z}})} \sqrt{y}\, dz$$

$$= \frac{y^{\alpha_1-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (\sqrt{y}\,)^{\alpha_2-\alpha_1} z^{\alpha_2-\alpha_1-1} * e^{-\frac{2\sqrt{y}}{2\beta}(\,\mathrm{z}+\frac{1}{\mathrm{z}})}\ dz$$

$$= \frac{y^{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{2}{\beta} K_{\alpha_2-\alpha_1}(2\sqrt{y})$$

where $K_v(\omega)$is the modified bessel function of the third kind.

$$h(y) = \frac{y^{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\frac{2}{\beta}K_{\alpha_2-\alpha_1}\left(2\sqrt{y}\right) ; y > 0$$

But because of symmetry $K_v(\omega) = K_{-v}(\omega)$

$$K_{\alpha_2-\alpha_1}(\omega) = K_{-(\alpha_1-\alpha_2)}(\omega) = K_{\alpha_1-\alpha_2}(\omega)$$

Thus

$$h(y) = \frac{y^{\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\frac{2}{\beta}K_{\alpha_1-\alpha_2}\left(2\sqrt{y}\right) ; y > 0 \qquad (3.14\, b)$$

3.3.3 Quotient of gamma distribution independent random variables
Case 1:

Derivation of the distribution of $Y_2 = \frac{X_1}{X_1+X_2}$ where X_1 *and* X_2 are mutually independent random variables and X_j has a gamma (α_j, β) distribution (j=1,2)

$$Let \qquad Y_1 = X_1 + X_2 \quad and\ Y_2 = \frac{X_1}{X_1 + X_2}$$

$$thus\ \ x_1 = y_1\, y_2\ and\ x_2 = y_1(1 - y_2)$$

$$|J| = \begin{vmatrix} \frac{\partial\, x_1}{\partial\, y_1} & \frac{\partial\, x_1}{\partial\, y_1} \\ \frac{\partial x_2}{\partial\, y_2} & \frac{\partial x_2}{\partial\, y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -\, y_1$$

$$f(y_1, y_2) = f(x_1, x_{2)*}|J| = \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}{x_1}^{\alpha_1-1}{x_2}^{\alpha_2-1}e^{-\frac{1}{\beta}(x_{1+}x_{2)}} ;\quad x_{1,}x_2 > 0$$

$$= \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}(\, y_1\, y_2)^{\alpha_1-1}(\, y_1(1 - y_2))^{\alpha_2-1}e^{-\frac{1}{\beta}y_1} *\ y_1$$

$$= \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\ {y_1}^{\alpha_1+\alpha_2-1}\ {y_2}^{\alpha_2-1}(1 - y_2)^{\alpha_2-1}e^{-\frac{1}{\beta}y_1} \qquad (3.15)$$

Therefore $\quad f(y_2) = \int_0^\infty f(y_1, y_2) dy_1$

$$= \int_0^\infty \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \, {y_1}^{\alpha_1+\alpha_2-1} \, {y_2}^{\alpha_2-1}(1- \, y_2)^{\alpha_2-1} e^{-\frac{y_1}{\beta}} dy_1$$

Let $y = \frac{y_1}{\beta} \Longrightarrow \, y_1 = \beta y$

$\beta dy = d\, y_1$

$$= \frac{{y_2}^{\alpha_2-1}(1- \, y_2)^{\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty e^{-y} (\beta y)^{\alpha_1+\alpha_2-1} \beta dy$$

$$= \frac{\beta^{\alpha_1+\alpha_2} {y_2}^{\alpha_2-1}(1- \, y_2)^{\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty e^{-y} y^{\alpha_1+\alpha_2-1} dy$$

$$= \frac{{y_2}^{\alpha_2-1}(1- \, y_2)^{\alpha_2-1}\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$$

$$= \frac{{y_2}^{\alpha_2-1}(1- \, y_2)^{\alpha_2-1}}{\mathrm{B}(\alpha_1, \alpha_2)} \; ; 0 < v < 1 \,, \alpha_1, \alpha_2 > 0 \qquad (3.16\, a)$$

which is Beta distribution of first kind.

Alternatively

Let $\quad Z = \frac{X_1}{X_1+X_2}$

$$Prob\; (Z \le z) = \frac{X_1}{X_1 + X_2} \le z)$$

$$= Prob(X_1 \le z(x_1 + x_2)$$

$$= Prob((1-z)X_1 \le z x_2$$

$$= Prob(X_1 \le \frac{z x_2}{1-z}, 0 < X_2 < \infty)$$

$$= \int_0^\infty \int_0^{\frac{z x_2}{1-z}} f(x_1) dx_1 f(x_2) d\, x_2$$

$$= \int_0^\infty F_{X_1}(\frac{z x_2}{1-z}) \; f(x_2) dx_2$$

$$h(z) = \int_0^\infty \frac{x_2}{(1-z)^2} f_{x_1}(\frac{zx_2}{1-z})\ f(x_2)dx_2$$

$$= \int_0^\infty \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{x_2}{(1-z)^2} e^{-\frac{1}{\beta}(\frac{zx_2}{1-z})} (\frac{zx_2}{1-z})^{\alpha_1-1} e^{-\frac{1}{\beta}x_2} {x_2}^{\alpha_2-1} dx_2$$

$$= \frac{z^{\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)(1-z)^{\alpha_1+1}} \int_0^\infty {x_2}^{\alpha_1+\alpha_2-1}\, e^{-\frac{1}{\beta}\frac{zx_2}{1-z}-x_2} \quad dx_2$$

$$= \frac{z^{\alpha_1-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)(1-z)^{\alpha_1+1}} \int_0^\infty {x_2}^{\alpha_1+\alpha_2-1}\, e^{-\frac{1}{\beta}*\frac{x_2}{1-z}} dx_2$$

$$Let\ \ u = \frac{x_2}{\beta(1-z)} \Longrightarrow x_2 = \beta u(1-z), dx_2 = \beta(1-z)du$$

$$= \frac{z^{\alpha_1-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)(1-z)^{\alpha_1+1}} \int_0^\infty \left(\beta u(1-z)\right)^{\alpha_1+\alpha_2-1} e^{-u}\beta(1-z)du$$

$$= \frac{\beta^{\alpha_1+\alpha_2} z^{\alpha_1-1}(1-z)^{\alpha_1+\alpha_2}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)(1-z)^{\alpha_1+1}} \int_0^\infty u^{\alpha_1+\alpha_2-1}\, e^{-u} \quad du$$

$$= \frac{z^{\alpha_1-1}(1-z)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty u^{\alpha_1+\alpha_2-1}\, e^{-u} \quad du$$

$$= \frac{z^{\alpha_1-1}(1-z)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \Gamma(\alpha_1+\alpha_2)$$

$$= \frac{z^{\alpha_1-1}(1-z)^{\alpha_2-1}}{\mathrm{B}(\alpha_1+\alpha_2)} \quad let\ z = y_2$$

Therefore $f(x) = \frac{{y_2}^{\alpha_1-1}(1-y_2)^{\alpha_2-1}}{\mathrm{B}(\alpha_1+\alpha_2)}\ ;\ \alpha_{1,}\alpha_2 > 0, 0 < y_2 < 1$ (3.16 b)

which is a Beta distribution of first kind.

Case 2:

Let $X \sim Gamma(\alpha_1,\beta)\ \ and$ $\mathrm{Y} \sim Gamma(\alpha_2,\beta), u = x_1 + x_2, v = \frac{x_1}{x_2}$

$$x_1 = \frac{uv}{1+v}\ and\ x_2 = \frac{u}{1+v}$$

Derivation of the distribution of $V = \frac{X_1}{X_2}$ where X_1 *and* X_2 are mutually independent random variables and X_j has a gamma (α_j, β) distribution $(j = 1, 2)$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{1+v} & \frac{u}{(1+v)^2} \\ \frac{v}{1+v} & -\frac{u}{(1+v)^2} \end{vmatrix} = \frac{-u}{(1+v)^2}$$

$$f(u,v) = f(x_1, x_{2)*}|J| = \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} {x_1}^{\alpha_1-1}{x_2}^{\alpha_2-1}e^{-\frac{1}{\beta}(x_1+x_2)} * \frac{u}{(1+v)^2};\ x_{1,}x_2 > 0$$

$$= \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(\frac{uv}{1+v}\right)^{\alpha_1-1}\left(\frac{u}{1+v}\right)^{\alpha_2-1} e^{-\frac{u}{\beta}} * \frac{u}{(1+v)^2}$$

$$= \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\frac{u}{\beta}} u^{\alpha_1+\alpha_2-1} v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2}$$

Thus $\quad f(v) = \int_0^\infty f(u,v)du$

$$= \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty e^{-\frac{u}{\beta}} u^{\alpha_1+\alpha_2-1} v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2} du$$

$$= \frac{v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty e^{-\frac{u}{\beta}} u^{\alpha_1+\alpha_2-1} du$$

$$Let\ y = \frac{u}{\beta} \Longrightarrow u = \beta y$$

$$\beta dy = du$$

$$f(v) \quad = \frac{v^{\alpha_2-1}(1+v)^{-\alpha_1-\alpha_2}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty e^{-y}(\beta y)^{\alpha_1+\alpha_2-1}\beta dy$$

$$= \frac{\beta^{\alpha_1+\alpha_2} v^{\alpha_2-1}(1+v)^{-\alpha_1-\alpha_2}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty e^{-y} y^{\alpha_1+\alpha_2-1} dy$$

$$= \frac{v^{\alpha_2-1}(1+v)^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} * \Gamma(\alpha_1+\alpha_2)$$

$$= \frac{v^{\alpha_2-1}(1+v)^{-\alpha_1-\alpha_2}}{\mathrm{B}(\alpha_1,\alpha_2)} \quad 0 < v < \infty, \alpha_1, \alpha_2 > 0 \qquad (3.17\ a)$$

which is the Beta distribution of second kind.

Alternatively

Let $X_1 = X, X_2 = Y, Z = \frac{X}{Y}$

$$F_{\frac{X}{Y}} = Prob\ (Z \leq z) = Prob\ \left(\frac{X}{Y} \leq z\right) = Prob(X \leq zy)$$

$$= \int_0^\infty \int_0^{zy} f_x(x)\, f_y(y) dx dy = \int_0^\infty \left[\int_0^{zy} f_x(x) dx\right] f_y(y) dy$$

$$= \int_0^\infty F_{\frac{X}{Y}}(zy) f_y(y) dy$$

$$f_{\frac{x}{y}}(z) = \int_0^\infty y * f_x(zy) f_y(y) dy$$

$$= \int_0^\infty \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} (zy)^{\alpha_1-1} * ye^{-\frac{zy}{\beta}} e^{-\frac{y}{\beta}} y^{\alpha_2-1} dy$$

$$= \frac{z^{\alpha_1-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty y^{\alpha_1+\alpha_2-1} * e^{-\frac{zy}{\beta}-\frac{y}{\beta}} dy$$

Let $x = \frac{y}{\beta}(z+1) \Rightarrow y = \frac{\beta x}{z+1} \Rightarrow dy = \frac{\beta dx}{z+1}$

$$= \frac{z^{\alpha_1-1}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty \left(\frac{\beta x}{z+1}\right)^{\alpha_1+\alpha_2-1} * e^{-x} \frac{\beta dx}{z+1}$$

$$= \frac{z^{\alpha_1-1}\beta^{\alpha_1+\alpha_2}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)(1+z)^{\alpha_1+\alpha_2}} \int_0^\infty x^{\alpha_1+\alpha_2-1} * e^{-x} dx$$

$$= \frac{z^{\alpha_1-1}(1+z)^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty x^{\alpha_1+\alpha_2-1} * e^{-x} dx$$

$$= \frac{z^{\alpha_1-1}(1+z)^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \Gamma(\alpha_1+\alpha_2)$$

$$= \frac{z^{\alpha_1-1}(1+z)^{-\alpha_1-\alpha_2}}{B(\alpha_1, \alpha_2)};\ \alpha_1, \alpha_2, \beta > 0, 0 < z < \infty \qquad (3.17\ b)$$

which is the Beta distribution of second kind

3.3.4 Two-parameter related distributions when scale parameters are distinct Sum of independent gamma distributed random variable

$Let\ Z = X_1 + X_2$ where $X_1 and\ X_2$ are mutually independent each with probability density function of

$$f(x_i) = \frac{1}{\beta_i^{\alpha_i}\Gamma(\alpha_i)} x_i^{\alpha_i - 1} e^{-\frac{x_i}{\beta_i}} \quad x_i > 0, \alpha_i, \beta_i > 0, i = 1, 2$$

$$f(z) = \int_0^z f(z - x_2)\, f(x_2) dx_2$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2){\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}} \int_0^z (z - x_2)^{\alpha_1 - 1} e^{-\frac{(z - x_2)}{\beta_1}}\, {x_2}^{\alpha_2 - 1} e^{-\frac{x_2}{\beta_2}} dx_2$$

$$= \frac{e^{-\frac{z}{\beta_1}}}{\Gamma(\alpha_1)\Gamma(\alpha_2){\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}} \int_0^z (z - x_2)^{\alpha_1 - 1}\, {x_2}^{\alpha_2 - 1} e^{-\frac{x_2}{\beta_2} + \frac{x_2}{\beta_1}} dx_2$$

$$= \frac{e^{-\frac{z}{\beta_1}}}{\Gamma(\alpha_1)\Gamma(\alpha_2){\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}} \int_0^z (z - x_2)^{\alpha_1 - 1}\, {x_2}^{\alpha_2 - 1} e^{-x_2\left(\frac{1}{\beta_2} - \frac{1}{\beta_1}\right)} dx_2$$

$$let\ x_2 = zt \Longrightarrow dx_2 = zdt$$

$$f(z) = \frac{e^{-\frac{z}{\beta_1}}}{\Gamma(\alpha_1)\Gamma(\alpha_2){\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}} \int_0^z (z - zt)^{\alpha_1 - 1}\, {x_2}^{\alpha_2 - 1} e^{-zt\left(\frac{1}{\beta_2} - \frac{1}{\beta_1}\right)} zdt$$

$$= \frac{e^{-\frac{z}{\beta_1}} z^{\alpha_1 + \alpha_2 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2){\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}} \int_0^z t^{\alpha_2 - 1}(1 - t)^{\alpha_1 - 1}\, e^{-\left(\frac{z}{\beta_2} - \frac{z}{\beta_1}\right)t} dt$$

Recall $\ {}_1F_1(\alpha; \beta; -zx_1 - zx_2) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta - \alpha)} \int_0^1 t^{\alpha - 1}(1 - t)^{\beta - \alpha - 1}\, e^{-(zx_1 + zx_2)t} dt$

$$\frac{\Gamma(\alpha)\Gamma(\beta - \alpha)}{\Gamma(\beta)}\ {}_1F_1(\alpha; \beta; -zx_1 - zx_2) = \int_0^1 t^{\alpha - 1}(1 - t)^{\beta - \alpha - 1}\, e^{-(zx_1 + zx_2)t} dt$$

$$(Daya, K. et\ al., 2011)$$

Therefore

$$f(z) = \frac{e^{-\frac{z}{\beta_1}} z^{\alpha_1 + \alpha_2 - 1}\Gamma(\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2 + \alpha_1){\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}}\ {}_1F_1\left(\alpha_2; \alpha_2 + \alpha_1; -\left(\frac{z}{\beta_2} - \frac{z}{\beta_1}\right)\right),\ z, \alpha_1, \alpha_2, \beta > 0 \quad (3.18)$$

Quotient of independent gamma distributed random variable

Case 1

Derivation of the distribution of $Y_2 = \frac{X_1}{X_1 + X_2}$ $where\ X_1\ and\ X_2$ are mutually independent, and $let\ Y_1 = X_1 + X_2\ where\ X_i$ has a gamma (α_i, β_i) distribution with pdf

$$f(x_i) = \frac{1}{\beta_i^{\alpha_i}\Gamma(\alpha_i)} x_i^{\alpha_i - 1} e^{-\frac{x_i}{\beta_i}} \quad x_i > 0, \alpha_i, \beta_i > 0, i = 1, 2$$

Thus $x_1 = y_1 y_2\ and\ x_2 = y_1(1 - y_2)$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -y_1$$

$$f(y_1, y_2) = f(x_1, x_2) * |J|$$

$$= \frac{1}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} (y_1 y_2)^{\alpha_1 - 1}(y_1(1 - y_2))^{\alpha_2 - 1} e^{-\frac{y_1 y_2}{\beta_1}} e^{-\frac{y_1(1 - y_2)}{\beta_2}}$$

$$= \frac{1}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} {y_1}^{\alpha_1 + \alpha_2 - 1} {y_2}^{\alpha_1 - 1}(1 - y_2)^{\alpha_2 - 1} e^{-\frac{y_1(y_2(\beta_2 - \beta_1) + \beta_1}{\beta_1 \beta_2}}$$

$$f(y_2) = \int_0^\infty f(y_1, y_2)\, dy_1$$

Let $u = \frac{y_1(\beta_1 + y_2(\beta_2 - \beta_1)}{\beta_1\beta_2} \Longrightarrow y_1 = \frac{u\beta_1\beta_2}{\beta_1 + y_2(\beta_2 - \beta_1)}$

$$dy_1 = \frac{\beta_1\beta_2}{\beta_1 + y_2(\beta_2 - \beta_1)} du$$

$$f(y_2) = \frac{{y_2}^{\alpha_1 - 1}(1 - y_2)^{\alpha_2 - 1}}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty {y_1}^{\alpha_1 + \alpha_2 - 1} e^{-\frac{y_1(\beta_1 + y_2(\beta_2 - \beta_1)}{\beta_1\beta_2}}\, dy_1$$

$$= \frac{{y_2}^{\alpha_1 - 1}(1 - y_2)^{\alpha_2 - 1}}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty \left(\frac{u\beta_1\beta_2}{\beta_1 + y_2(\beta_2 - \beta_1)}\right)^{\alpha_1 + \alpha_2 - 1} e^{-u} \frac{\beta_1\beta_2}{\beta_1 + y_2(\beta_2 - \beta_1)} du$$

$$= \frac{{y_2}^{\alpha_1 - 1}(1 - y_2)^{\alpha_2 - 1}(\beta_1\beta_2)^{\alpha_1 + \alpha_2}}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)(\beta_1 + y_2(\beta_2 - \beta_1))^{\alpha_1 + \alpha_2}} \int_0^\infty u^{\alpha_1 + \alpha_2 - 1} e^{-u}\, du$$

$$= \frac{{y_2}^{\alpha_1 - 1}(1 - y_2)^{\alpha_2 - 1}(\beta_1\beta_2)^{\alpha_1 + \alpha_2}}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)\left(\beta_1 + y_2(\beta_2 - \beta_1)\right)^{\alpha_1 + \alpha_2}} \Gamma(\alpha_1 + \alpha_2)$$

$$= \frac{y_2^{\ \alpha_1-1}(1-y_2)^{\alpha_2-1}(\beta_1\beta_2)^{\alpha_1+\alpha_2}}{\mathrm{B}(\alpha_1,\alpha_2)\big(\beta_1+y_2\,(\beta_2-\beta_1)\big)^{\alpha_1+\alpha_2}}$$

$$= \frac{y_2^{\ \alpha_1-1}(1-y_2)^{\alpha_2-1}(\beta_1\beta_2)^{\alpha_1+\alpha_2}}{\mathrm{B}(\alpha_1,\alpha_2)\beta_1^{\ \alpha_1+\alpha_2}\left(1-y_2\left(1-\frac{\beta_2}{\beta_1}\right)\right)^{\alpha_1+\alpha_2}}$$

$$= \frac{\beta_1^{\ -\alpha_1}\beta_2^{\ \alpha_1}y_2^{\ \alpha_1-1}(1-y_2)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)\left(1-y_2\left(1-\frac{\beta_2}{\beta_1}\right)\right)^{\alpha_1+\alpha_2}}\ let\ \lambda = \frac{\beta_2}{\beta_1}$$

$$f(y_2) = \frac{\lambda^{\alpha_1}y_2^{\ \alpha_1-1}(1-y_2)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)\big(1-y_2\,(1-\lambda)\big)^{\alpha_1+\alpha_2}}\ ;\alpha_1,\alpha_2,\lambda > 0 \qquad (3.19\ a)$$

which is a generalized Beta distribution denoted by G3B.

Alternatively

Let $\quad Z = \frac{X_1}{X_1\ +X_2}$

Prob $(Z \le z) =$ Prob $\left(\frac{X_1}{X_1\ +X_2} \le z\right) =$ Prob $\big(X_1\ \le z(X_1\ + X_2)\big)$

$$= \text{Prob}\ \left(X_1\ \le \frac{zx_2}{1-z}\ ; 0 < x_2 < \infty\right) = \int_0^{\infty}\int_0^{\frac{zx_2}{1-z}} f_{X_1}\ (x_1) f_{X_2}\ (x_2)\, dx_1 dx_2$$

$$= \int_0^{\infty} \frac{x_2}{(1-z)^2} f_{X_1}\ \left(\frac{zx_2}{1-z}\right) f_{X_2}\ (x_2) dx_2$$

$$= \frac{1}{\beta_1^{\ \alpha_1}\beta_2^{\ \alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\infty} \frac{x_2}{(1-z)^2}\left(\frac{zx_2}{1-z}\right)^{\alpha_1-1} e^{-\frac{zx_2}{(1-z)\beta_1}} x_2^{\ \alpha_2-1} e^{-\frac{x_2}{\beta_2}}\, dx_2$$

$$= \frac{z^{\alpha_1-1}}{\beta_1^{\ \alpha_1}\beta_2^{\ \alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)(1-z)^{\alpha_1+1}}\int_0^{\infty} x_2^{\ \alpha_1+\alpha_2-1} e^{-\frac{zx_2}{(1-z)\beta_1}-\frac{x_2}{\beta_2}} dx_2$$

$$= \frac{z^{\alpha_1-1}}{\beta_1^{\ \alpha_1}\beta_2^{\ \alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)(1-z)^{\alpha_1+1}}\int_0^{\infty} x_2^{\ \alpha_1+\alpha_2-1} e^{-x_2\left(\frac{z(\beta_2-\beta_1)+\beta_1}{\beta_1\beta_2(1-z)}\right)} dx_2$$

Let $\quad u = x_2\left(\frac{z(\beta_2-\beta_1)+\beta_1}{\beta_1\beta_2(1-z)}\right) \Rightarrow x_2 = \frac{u\beta_1\beta_2(1-z)}{z(\beta_2-\beta_1)+\beta_1}\ thus\ dx_2 = \frac{\beta_1\beta_2(1-z)}{z(\beta_2-\beta_1)+\beta_1} du$

$$= \frac{z^{\alpha_1-1}}{\beta_1{}^{\alpha_1}\beta_2{}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)(1-z)^{\alpha_1+1}} \int_0^\infty \left(\frac{u\beta_1\beta_2(1-z)}{z(\beta_2-\beta_1)+\beta_1}\right)^{\alpha_1+\alpha_2-1} e^{-u} \frac{\beta_1\beta_2(1-z)}{z(\beta_2-\beta_1)+\beta_1} du$$

$$= \frac{z^{\alpha_1-1}(1-z)^{\alpha_2-1}\beta_1{}^{\alpha_2}\beta_2{}^{\alpha_1}}{\beta_1{}^{\alpha_1}\beta_2{}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)(\beta_1+z(\beta_2-\beta_1))^{\alpha_1+\alpha_2}} \int_0^\infty u^{\alpha_1+\alpha_2-1}e^{-u}du$$

$$= \frac{z^{\alpha_1-1}(1-z)^{\alpha_2-1}\beta_1{}^{\alpha_2}\beta_2{}^{\alpha_1}}{\mathrm{B}(\alpha_1,\alpha_2)\beta_1{}^{\alpha_1+\alpha_2}\left(1-z\left(1-\frac{\beta_2}{\beta_1}\right)\right)^{\alpha_1+\alpha_2}}$$

$$= \frac{\frac{\beta_2{}^{\alpha_1}}{\beta_1{}^{\alpha_1}} z^{\alpha_1-1}(1-z)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)\left(1-z\left(1-\frac{\beta_2}{\beta_1}\right)\right)^{\alpha_1+\alpha_2}}$$

$$= \frac{\left(\frac{\beta_2}{\beta_1}\right)^{\alpha_1} z^{\alpha_1-1}(1-z)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)\left(1-z\left(1-\frac{\beta_2}{\beta_1}\right)\right)^{\alpha_1+\alpha_2}} \quad let\ \lambda = \frac{\beta_2}{\beta_1}\ and\ z = y_2$$

Therefore

$$f(y_2) = \frac{\lambda^{\alpha_1} y_2{}^{\alpha_1-1}(1-y_2)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)\left(1-y_2(1-\lambda)\right)^{\alpha_1+\alpha_2}} \ ; y_2\ \alpha_1,\alpha_2,\lambda > 0 \qquad (3.19\ b)$$

Which is a generalized Beta distribution denoted by G3B.

Case 2, (Pham - Gia *et al.*, 1989)

Derivation of the distribution of $V = \frac{X_1}{X_2}$ $where\ X_1\ and\ X_2$ are mutually independent, and X_i has a gamma (α_i, β_i) distribution with pdf

$$f(x_i) = \frac{1}{\beta_i^{\alpha_i}\Gamma(\alpha_i)} x_i^{\alpha_i-1} e^{-\frac{x_i}{\beta_i}} \quad x_i > 0, \alpha_i, \beta_i > 0, i = 1, 2$$

Let $U = X_1 + X_2$

Thus

$$x_1 = \frac{uv}{1+v} \text{ and } x_2 = \frac{u}{1+v}$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{1+v} & \frac{u}{(1+v)^2} \\ \frac{v}{1+v} & -\frac{u}{(1+v)^2} \end{vmatrix} = \frac{-u}{(1+v)^2}$$

$$f(u,v) = f(x_1, x_{2)*}|J| = \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} {x_1}^{\alpha_1-1}{x_2}^{\alpha_2-1}e^{-\frac{1}{\beta}(x_1+x_2)} * \frac{u}{(1+v)^2};\ x_{1,}x_2 > 0$$

$$= \frac{1}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(\frac{uv}{1+v}\right)^{\alpha_1-1} e^{-\frac{uv}{(1+v)\beta_1}}\left(\frac{u}{1+v}\right)^{\alpha_2-1} e^{-\frac{u}{(1+v)\beta_2}} * \frac{u}{(1+v)^2}$$

$$= \frac{1}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1+\alpha_2-1}v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2}e^{-\frac{uv}{(1+v)\beta_1}-\frac{u}{(1+v)\beta_2}}$$

$$= \frac{1}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1+\alpha_2-1}v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2}e^{-u\left(\frac{\beta_2 v+\beta_1}{\beta_1\beta_2(1+v)}\right)}$$

Thus $f(v) = \int_0^\infty f(u,v)du$

$$= \int_0^\infty \frac{1}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1+\alpha_2-1}v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2}e^{-u\left(\frac{\beta_2 v+\beta_1}{\beta_1\beta_2(1+v)}\right)}du$$

$$= \frac{v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2}}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^\infty u^{\alpha_1+\alpha_2-1}e^{-u\left(\frac{\beta_2 v+\beta_1}{\beta_1\beta_2(1+v)}\right)}du$$

$$Let\ y = u\left(\frac{\beta_2 v+\beta_1}{\beta_1\beta_2(1+v)}\right) \Longrightarrow u = \frac{\beta_1\beta_2(1+v)y}{\beta_2 v+\beta_1}$$

$$\frac{\beta_1\beta_2(1+v)}{\beta_2 v+\beta_1}dy = du$$

$$= \frac{v^{\alpha_2-1}(1+v)^{-\alpha_1-\alpha_2}}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^\infty e^{-y}\left(\frac{\beta_1\beta_2(1+v)y}{\beta_2 v+\beta_1}\right)^{\alpha_1+\alpha_2-1}\frac{\beta_1\beta_2(1+v)}{\beta_2 v+\beta_1}dy$$

$$= \frac{(\beta_1\beta_2)^{\alpha_1+\alpha_2}v^{\alpha_2-1}}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)(\beta_1+\beta_2 v)^{\alpha_1+\alpha_2}}\int_0^\infty y^{\alpha_1+\alpha_2-1}e^{-y}dy$$

$$= \frac{{\beta_1}^{\alpha_2}{\beta_2}^{\alpha_1}v^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2){\beta_1}^{\alpha_1+\alpha_2}\left(1+\frac{\beta_2}{\beta_1}v\right)^{\alpha_1+\alpha_2}} * \Gamma(\alpha_1+\alpha_2)$$

$$= \frac{{\beta_1}^{-\alpha_1}{\beta_2}^{\alpha_1}v^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)\left(1+\frac{\beta_2}{\beta_1}v\right)^{\alpha_1+\alpha_2}} \; let\ \lambda = \frac{\beta_2}{\beta_1}$$

$$= \frac{\lambda^{\alpha_1}v^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)(1+\lambda v)^{\alpha_1+\alpha_2}} \; 0 < v < \infty, \alpha_1, \alpha_2 > 0 \; let\ v = z$$

Therefore

$$f(z) = \frac{\lambda^{\alpha_1}z^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)(1+\lambda z)^{\alpha_1+\alpha_2}} \; 0 < z < \infty, \alpha_1, \alpha_2, \lambda > 0 \qquad (3.20\ a)$$

Alternatively

Let $Z = \frac{X}{Y}$

$$F_{\frac{X}{Y}} = Prob\ (Z \le z) = Prob\ \left(\frac{X}{Y} \le z\right) = Prob(X \le zy)$$

$$= \int_0^\infty \int_0^{zy} f_x(x)\, f_y(y) dx dy = \int_0^\infty \left[\int_0^{zy} f_x(x)dx\right] f_y(y)dy$$

$$= \int_0^\infty F_{\frac{X}{Y}}(zy) f_y(y)dy$$

$$f_{\frac{x}{y}}(z) = \int_0^\infty y * f_x(zy) f_y(y)dy$$

$$= \int_0^\infty \frac{1}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}(zy)^{\alpha_1-1} * ye^{-\frac{zy}{\beta_1}}e^{-\frac{y}{\beta_2}}y^{\alpha_2-1}dy$$

$$= \frac{z^{\alpha_1-1}}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^\infty y^{\alpha_1+\alpha_2-1} * e^{-y(\frac{\beta_1+\beta_2 z}{\beta_1\beta_2})}dy$$

Let $\quad x = y(\frac{\beta_1+\beta_2 z}{\beta_1\beta_2}) \Longrightarrow y = \frac{\beta_1\beta_2 x}{\beta_1+\beta_2 z} \Longrightarrow dy = \frac{\beta_1\beta_2 dx}{\beta_1+\beta_2 z}$

$$= \frac{z^{\alpha_1 - 1}}{\beta_1{}^{\alpha_1}\beta_2{}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (\frac{\beta_1\beta_2 x}{\beta_1 + \beta_2 z})^{\alpha_1+\alpha_2-1} * e^{-x} \frac{\beta_1\beta_2 dx}{\beta_1 + \beta_2 z}$$

$$= \frac{z^{\alpha_1 - 1}(\beta_1\beta_2)^{\alpha_1+\alpha_2}}{\beta_1{}^{\alpha_1}\beta_2{}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)(\beta_1 + \beta_2 z)^{\alpha_1+\alpha_2}} \int_0^\infty x^{\alpha_1+\alpha_2-1} * e^{-x} dx$$

$$= \frac{\beta_1{}^{\alpha_2}\beta_2{}^{\alpha_1} z^{\alpha_1 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta_1{}^{\alpha_1+\alpha_2}(1 + \frac{\beta_2}{\beta_1} z)^{\alpha_1+\alpha_2}} \int_0^\infty x^{\alpha_1+\alpha_2-1} * e^{-x} dx$$

$$= \frac{\beta_1{}^{-\alpha_1}\beta_2{}^{\alpha_1} z^{\alpha_1 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\left(1 + \frac{\beta_2}{\beta_1} z\right)^{\alpha_1+\alpha_2}} \Gamma(\alpha_1 + \alpha_2) \;\; let\ \lambda = \frac{\beta_2}{\beta_1}$$

Therefore

$$f(z) = \frac{\lambda^{\alpha_1} z^{\alpha_1 - 1}}{B(\alpha_1, \alpha_2)(1 + \lambda z)^{\alpha_1+\alpha_2}};\ \alpha_1, \alpha_{2,}\ \lambda > 0, 0 < z < \infty \qquad (3.20\ b)$$

Product of independent gamma distributed random variable

Derivation of the distribution of $Z = X_1\ X_2\ where\ X_1\ and\ X_2$ are mutually independent, and X_i has a gamma (α_i, β_i) distribution with pdf

$f(x_i) = \frac{1}{\beta_i^{\alpha_i}\Gamma(\alpha_i)} x_i^{\alpha_i - 1} e^{-\frac{x_i}{\beta_i}} \quad x_i > 0, \alpha_i, \beta_i > 0, i = 1, 2$

Let $U = X_1 \implies X_2 = \frac{Z}{U}$

$$|J| = \begin{vmatrix} \frac{\partial\ x_1}{\partial z} & \frac{\partial\ x_1}{\partial u} \\ \frac{\partial x_2}{\partial z} & \frac{\partial x_2}{\partial u} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{u} & -\frac{z}{u^2} \end{vmatrix} = \frac{1}{u}$$

$$f(u, v) = f(x_1, x_{2)*}|J| = \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} x_1{}^{\alpha_1 - 1} x_2{}^{\alpha_2 - 1} e^{-\frac{1}{\beta}(x_{1+}x_2)} * \frac{1}{u};\ x_{1,}x_2 > 0$$

$$= \frac{1}{\beta_1{}^{\alpha_1}\beta_2{}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1 - 1} e^{-\frac{u}{\beta_1}} \left(\frac{z}{u}\right)^{\alpha_2 - 1} e^{-\frac{z}{\beta_2 u}} * \frac{1}{u}$$

$$= \frac{z^{\alpha_2 - 1}}{\beta_1{}^{\alpha_1}\beta_2{}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{-(\alpha_2 - \alpha_1 + 1} e^{-\frac{u}{\beta_1} - \frac{z}{\beta_2 u}},\ \ \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$$

Thus

$f(z) = \int_0^\infty f(u, z) du$

$$= \int_0^\infty \frac{z^{\alpha_2-1}}{\beta_1{}^{\alpha_1}\beta_2{}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{-(\alpha_2-\alpha_1+1}e^{-\frac{u}{\beta_1}-\frac{z}{\beta_2 u}}\, du$$

$$= \frac{z^{\alpha_2-1}}{\beta_1{}^{\alpha_1}\beta_2{}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty u^{-(\alpha_2-\alpha_1+1}e^{-\frac{u}{\beta_1}-\frac{z}{\beta_2 u}}\, du$$

Let $x = t = \frac{u}{\beta_1} \Longrightarrow u = \beta_1 t, du = \beta_1 d$

$f(z) = \frac{z^{\alpha_2-1}}{\beta_1{}^{\alpha_1}\beta_2{}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (\beta_1 t)^{-(\alpha_2-\alpha_1+1}e^{-t-\frac{z}{\beta_2\beta_1 t}}\, du$

Recall $\int_0^\infty t^{-(v+1)}e^{-t-\frac{z^2}{4t}}dt = 2\left(\frac{z}{2}\right)^{-v}K_v(z)$

Therefore

$$f(z) = \frac{2z^{\alpha_2-1}}{\beta_1{}^{\alpha_1}\beta_2{}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(\frac{z}{\beta_1\beta_2}\right)^{-\left(\frac{\alpha_2-\alpha_1}{2}\right)} K_{\alpha_2-\alpha_1}(2\left(\frac{z}{\beta_1\beta_2})^{\frac{1}{2}}\right) \qquad (3.21\ a)$$

$$z, \alpha_1, \alpha_1, \beta_1, \beta_2 > 0$$

Alternatively

Let $\quad Z = X_1X_2$

$F_{X_1X_2}(z) = Prob(Z \leq z) = Prob(X_1X_2 \leq z)$

$$= Prob(X_1 \leq \frac{z}{x_2}\ , 0 < x_2 < \infty)$$

$$= \int_0^\infty \int_0^{\frac{z}{x_2}} f_{X_1}(x) f_{X_2}(x_2) dx_1\, dx_2$$

$$= \int_0^\infty f_{X_2}(x_2) F_{X_1}(\frac{z}{x_2})\, dx_2$$

$$f_{x_1x_2}(z) = \int_0^\infty \frac{1}{x_2} f_{X_2}(x_2) f_{X_1}(\frac{z}{x_2})\, dx_2$$

$$= \int_0^\infty \frac{1}{\beta_1{}^{\alpha_1}\beta_2{}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} x_2{}^{\alpha_1-1} * \frac{1}{x_2} e^{-\frac{x_2}{\beta_1}} e^{-\frac{z}{\beta_2 x_2}} (\frac{z}{x_2})^{\alpha_2-1} dx_2$$

$$= \frac{z^{\alpha_2 - 1}}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty {x_2}^{\alpha_2 - \alpha_1 - 1} e^{-\frac{x_2}{\beta_1} - \frac{z}{\beta_2 x_2}} dx_2$$

Let $t = \frac{x_2}{\beta_1} \Longrightarrow x_2 = \beta_1 t, dx_2 = \beta_1 dt$

$$f(z) = \frac{z^{\alpha_2 - 1}}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (\beta_1 t)^{\alpha_2 - \alpha_1 - 1} e^{-t - \frac{z}{\beta_2 \beta_1 t}} \beta_1 dt$$

Recall $\int_0^\infty t^{-(v+1)} e^{-t - \frac{z^2}{4t}} dt = 2\left(\frac{z}{2}\right)^{-v} K_v(z)$

Therefore

$$f(z) = \frac{2z^{\alpha_2 - 1}}{{\beta_1}^{\alpha_1}{\beta_2}^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \left(\frac{z}{\beta_1\beta_2}\right)^{-\left(\frac{\alpha_2 - \alpha_1}{2}\right)} K_{\alpha_2 - \alpha_1}(2\left(\frac{z}{\beta_1\beta_2}\right)^{\frac{1}{2}});\ z, \alpha_1, \alpha_1, \beta_1, \beta_2 > 0 \quad (3.21\ b)$$

Two parameter gamma summary

Distribution Name	Parameters		Mean	Variance
	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$		
One parameter gamma	α	1	α	α
Exponential	1	θ	θ	θ^2
Chi-square	$\frac{v}{2}$	2	v	$2v$

CHAPTER IV: Stacy (three parameter) gamma distribution

4.0 Introduction

The most general form of the gamma distribution is the three parameter generalized gamma (GG) distribution (Stacy., 1962). The distribution is suitable for modeling data with different types of hazard rate functions; increasing, decreasing, bathtub shaped and unimodal, which makes it particularly useful for estimating individual hazard functions. The GG distribution has been used in several research areas such as engineering, hydrology and survival analysis. Gamma distributions are very versatile and give useful presentations of many physical situations. They are perhaps the most applied statistical distributions in of reliability (Saralees N., 2008).

The Generalized gamma (GG) model, having Weibull, gamma, exponential and Raleigh as special sub-models, among others, is a very popular distribution for modeling lifetime data and for modeling phenomenon with monotone failure rates.

The GG distribution plays a very important role in statistical inferential problems. When modeling monotone hazard rates, the Weibull distribution may be the initial choice because of its negatively and positively skewed density shapes. However, the Weibull distribution does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates, such as the bathtup-shaped and the unimodal failure rates, which are common in biological and reliability studies (Gauss, M. *et al*., 2011).

The three parameter gamma has different parameterization as used by different authors.

4.1 Derivation of Stacy (three parameter) gamma distribution

Case 1

$$let \quad f(y) = \frac{e^{-y} y^{\alpha-1}}{\Gamma(\alpha)}; y > 0, \alpha > 0 \tag{4.1}$$

Also let $Y = \left(\frac{X}{\theta}\right)^{\beta} \Rightarrow \frac{dy}{dx} = \beta \left(\frac{x}{\theta}\right)^{\beta-1} * \frac{1}{\theta}$

Thus $|J| = \left|\frac{dy}{dx}\right| = \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}$

Therefore $f(x) = f(y) * |J| = f(y) * \left|\frac{dy}{dx}\right|$

$$= \frac{e^{-y} y^{\alpha-1}}{\Gamma(\alpha)} * \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} \; but \; y = \left(\frac{x}{\theta}\right)^{\beta}$$

$$= \frac{e^{-\left(\frac{x}{\theta}\right)^{\beta}} \left(\frac{x}{\theta}\right)^{\beta(\alpha-1)}}{\Gamma(\alpha)} * \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}$$

$$= \frac{\beta}{\theta\Gamma(\alpha)}\left(\frac{x}{\theta}\right)^{\beta\alpha-1} e^{-\left(\frac{x}{\theta}\right)^{\beta}}; x > 0, \alpha, \theta, \beta > 0 \qquad (4.2)$$

Case 2

Method 1: Using cumulative distribution

Let $\quad Y = (\frac{X}{\beta})c$

$$G\,(x) = prob(X \leq x)$$

$$= prob(y \leq (\frac{x}{\beta})c)$$

$$= prob(y \leq \frac{x}{\beta})$$

$$G(x) = F((\frac{x}{\beta})c)$$

$$\frac{d\,G(x)}{dy} = \frac{d\,F((\frac{x}{\beta})^{c})}{dy}$$

$$g(x) = \frac{c}{\beta}\,(\frac{x}{\beta})^{c-1} f((\frac{x}{\beta})^{c})$$

$$= \frac{c}{\beta\Gamma(\alpha)}\, e^{-(\frac{x}{\beta})^{c}} (\frac{x}{\beta})^{c-1} ((\frac{x}{\beta})^{c})^{\alpha-1}$$

$$= \frac{c}{\beta\Gamma(\alpha)}\, e^{-\left(\frac{x}{\beta}\right)^{c}} \left(\frac{x}{\beta}\right)^{\alpha c-1} \; ; \; x, \alpha, \beta, c > 0 \qquad (4.3\ a)$$

which is a three-parameters gamma distribution

Method 2 : Using change of Variables

Let $\quad Y = \left(\frac{X}{\beta}\right)^c$

$$\frac{dy}{dx} = c\left(\frac{x}{\beta}\right)^{c-1} * \frac{1}{\beta}$$

$$= \frac{c}{\beta}\left(\frac{x}{\beta}\right)^{c-1}$$

$Then \quad g\,(x) = f(y) * |J|$

$$= \frac{e^{-y}}{\Gamma(\alpha)}\, y^{\alpha-1} \frac{c}{\beta} (\frac{y}{\beta})^{c-1} \;\; But\; y = \left(\frac{x}{\beta}\right)^c$$

$$= \frac{c}{\beta\Gamma(\alpha)}\, e^{-(\frac{y}{\beta})^c} (\frac{x}{\beta})^{\alpha c-1}, \alpha > 0, \beta > 0, c > 0 \qquad (4.3\, b)$$

which is a three-parameters gamma distribution

Case 3

Let the gamma distribution with two parameters be given by;

$$f(y) = \frac{e^{-\frac{y}{\beta}}}{\beta^\alpha \Gamma(\alpha)}\, y^{\alpha-1} \;\; ; \; y > 0, \alpha > 0, \beta > 0$$

Then let $\quad Y = X^c$

$$dy = cx^{c-1}dx$$

$$|J| = \left|\frac{dy}{dx}\right| = cx^{c-1}$$

$$f(x) = f(y) * \left|\frac{dy}{dx}\right|$$

$$= \frac{e^{-\frac{y}{\beta}}}{\beta^\alpha \Gamma(\alpha)}\, y^{\alpha-1} \; * \; cx^{c-1} \;\; but\; y = x^c$$

$$= \frac{e^{-\frac{x^c}{\beta}}}{\beta^\alpha \Gamma(\alpha)}\, x^{c(\alpha-1)} \;\; cx^{c-1}$$

$$= \frac{ce^{-\frac{x^c}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}\ x^{\alpha c-1}$$

$$Let\ \beta^{\frac{1}{c}} = b \implies \beta b^c$$

$$Thus \quad f\,(x) = \frac{ce^{-(\frac{x}{\beta})^c}}{\beta^{\alpha c}\Gamma(\alpha)}\ x^{\alpha c-1} \ \ ;\ x\ >\ 0,\ \ \alpha > 0, \beta > 0, c\ >\ 0 \qquad (4.4)$$

Case 4

Let $\quad f(z)\ =\ \frac{b^{\alpha}e^{-bz}}{\Gamma(\alpha)}\ z^{\alpha-1}\ \ ;\ \alpha, z, b > 0$

Also let $\quad z\ =\ x^c\ \implies\ dz\ =\ cx^{c-1}dx$

$$\frac{dz}{dx}\ =\ cx^{c-1}$$

Thus $\quad f(x)\ =\ f(z) * \left|\frac{dz}{dx}\right|$

$$= \frac{b^{\alpha}e^{-bz}}{\Gamma(\alpha)}\ z^{\alpha-1} *\ cx^{c-1} \qquad (4.5)$$

Also $\qquad \beta = b^{\frac{1}{c}}$ replacing in (4.5)

$$f(x) = \frac{1}{\Gamma(\alpha)}\beta^{\alpha c}e^{-(\beta x)^c}\ x^{\alpha c-1}\ \ ;\ x, c, \beta, \alpha > 0 \qquad (4.6)$$

which is a three-parameter gamma distribution.

For the purpose of this study we will use gamma distribution derived in case 1 above.

$$E(X^r) = \int_0^{\infty} x^r\,\frac{\beta}{\theta\Gamma(\alpha)}\left(\frac{x}{\theta}\right)^{\beta\alpha-1} e^{-\left(\frac{x}{\theta}\right)^{\beta}}dx$$

$$= \int_0^{\infty} \frac{\beta}{\theta^{\beta\alpha}\theta^{-r}\Gamma(\alpha)}\,x^{\beta\alpha+r-1}e^{-\left(\frac{x}{\theta}\right)^{\beta}}dx$$

$$= \int_0^\infty \frac{\beta}{\theta^{\beta(\alpha+\frac{r}{\beta})}\theta^{-r}\Gamma(\alpha)} \frac{\Gamma(\alpha+\frac{r}{\beta})}{\Gamma(\alpha+\frac{r}{\beta})} x^{\beta(\alpha+\frac{r}{\beta})-1} e^{-\left(\frac{x}{\theta}\right)^\beta} dx$$

$$= \frac{\theta^r \Gamma(\alpha+\frac{r}{\beta})}{\Gamma(\alpha)} \int_0^\infty \frac{\beta}{\theta^{\beta\left(\alpha+\frac{r}{\beta}\right)}} \frac{1}{\Gamma\left(\alpha+\frac{r}{\beta}\right)} x^{\beta\left(\alpha+\frac{r}{\beta}\right)-1} e^{-\left(\frac{x}{\theta}\right)^\beta} dx$$

$$= \frac{\theta^r \Gamma\left(\alpha+\frac{r}{\beta}\right)}{\Gamma(\alpha)}$$

$$E(X) = \frac{\theta \Gamma(\alpha+\frac{1}{\beta})}{\Gamma(\alpha)}$$

$$E(X^2) = \frac{\theta^2 \Gamma(\alpha+\frac{2}{\beta})}{\Gamma(\alpha)}$$

$$E(X^3) = \frac{\theta^3 \Gamma(\alpha+\frac{3}{\beta})}{\Gamma(\alpha)}$$

$$E(X^4) = \frac{\theta^4 \Gamma(\alpha+\frac{4}{\beta})}{\Gamma(\alpha)}$$

$$Var\ X = \frac{\theta^2 \Gamma(\alpha+\frac{2}{\beta})}{\Gamma(\alpha)} - \left(\frac{\theta \Gamma(\alpha+\frac{1}{\beta})}{\Gamma(\alpha)}\right)^2$$

$$= \frac{\theta^2}{\Gamma^2(\alpha)} \left[\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{\beta}\right) - \Gamma^2(\Gamma(\alpha+\frac{1}{\beta}))\right]$$

Skewness: $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$= \theta^3 \frac{\Gamma\left(\alpha+\frac{3}{\beta}\right)}{\Gamma(\alpha)} - 3\theta^3 \frac{\Gamma\left(\alpha+\frac{1}{\beta}\right)\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma^2(\alpha)} + 2\theta^3 \frac{\Gamma^3(\Gamma(\alpha+\frac{1}{\beta}))}{\Gamma^3(\alpha)}$$

$$\gamma_1 = \frac{\theta^3 \frac{\Gamma\left(\alpha+\frac{3}{\beta}\right)}{\Gamma(\alpha)} - 3\theta^3 \frac{\Gamma\left(\alpha+\frac{1}{\beta}\right)\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma^2(\alpha)} + 2\theta^3 \frac{\Gamma^3(\Gamma(\alpha+\frac{1}{\beta}))}{\Gamma^3(\alpha)}}{\left(\frac{\theta^2}{\Gamma^2(\alpha)}\left[\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{\beta}\right) - \Gamma^2(\Gamma(\alpha+\frac{1}{\beta}))\right]\right)^{\frac{3}{2}}}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4$$

$$= \theta^4 \frac{\Gamma\left(\alpha+\frac{4}{\beta}\right)}{\Gamma(\alpha)} - 4\theta^4 \frac{\Gamma\left(\alpha+\frac{1}{\beta}\right)\Gamma\left(\alpha+\frac{3}{\beta}\right)}{\Gamma^2(\alpha)} +$$

$$6\theta^4 \frac{\Gamma\left(\alpha+\frac{1}{\beta}\right)\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma^2(\alpha)} \quad - 3\theta^4 \frac{\Gamma^4(\Gamma(\alpha+\frac{1}{\beta}))}{\Gamma^4(\alpha)}$$

$$\gamma_2 \quad = \left(\frac{\theta^4 \frac{\Gamma\left(\alpha+\frac{4}{\beta}\right)}{\Gamma(\alpha)} - 4\theta^4 \frac{\Gamma\left(\alpha+\frac{1}{\beta}\right)\Gamma\left(\alpha+\frac{3}{\beta}\right)}{\Gamma^2(\alpha)} + 6\theta^4 \frac{\Gamma\left(\alpha+\frac{1}{\beta}\right)\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma^2(\alpha)} - 3\theta^4 \frac{\Gamma^4(\Gamma(\alpha+\frac{1}{\beta}))}{\Gamma^4(\alpha)}}{\left(\frac{\theta^2}{\Gamma^2(\alpha)}\left[\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{\beta}\right) - \Gamma^2(\Gamma(\alpha+\frac{1}{\beta}))\right]\right)^{2}} \right)$$

4.2 Special cases Stacy distribution

4.2.1 Two parameter gamma distribution (Pearson, K. 1893, Pearson, K. 1895, Johnson, N. 1994).

$\text{Stacy}(\theta, \alpha, 1) = \text{Gamma}(\alpha, \theta)$

$$f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\theta^\alpha \Gamma(\alpha)} \ ; \ \alpha, \theta > 0, x > 0 \qquad (4.7)$$

$$E(X^r) = \frac{\theta^r \Gamma(r+\alpha)}{\Gamma(\alpha)}$$

$$E(X) = \theta\alpha$$

$$E(X^2) = \frac{\theta^2 \Gamma(2+\alpha)}{\Gamma(\alpha)} = \frac{\theta^2 (1+\alpha)!}{(\alpha-1)!} = \frac{\theta^2 (1+\alpha)\alpha(\alpha-1)!}{(\alpha-1)!}$$

$$= \theta^2\alpha^2 + \theta^2\alpha$$

$$E(X^{3)} = \frac{\theta^3\Gamma(3+\alpha)}{\Gamma(\alpha)} = \frac{\theta^3(2+\alpha)!}{(\alpha-1)!} = \frac{\theta^3(2+\alpha)(1+\alpha)\alpha(\alpha-1)!}{(\alpha-1)!}$$

$$= \theta^3\alpha^3 + 3\theta^3\alpha^2 + 2\theta^3\alpha$$

$$E(X^{4)} = \frac{\theta^4\Gamma(4+\alpha)}{\Gamma(\alpha)} = \frac{\theta^4(3+\alpha)!}{(\alpha-1)!} = \frac{\theta^4(3+\alpha)(2+\alpha)(1+\alpha)\alpha(\alpha-1)!}{(\alpha-1)!}$$

$$= \theta^4\alpha^4 + 6\theta^4\alpha^3 + 11\theta^4\alpha^2 + 6\theta^4\alpha$$

$$Var\ X = \mu_2 = \theta^2\alpha^2 + \theta^2\alpha - (\theta\alpha)^2$$

$$= \theta^2\alpha$$

Skewness $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\theta\alpha\, E(X^2) + 3\theta^2\alpha^2\, E(X) - \theta^3\alpha^3$$

$$= \theta^3\alpha^3 + 3\theta^3\alpha^2 + 2\theta^3\alpha - 3\theta\alpha(\theta^2\alpha^2 + \theta^2\alpha) + 3\theta^2\alpha^2(\theta\alpha) - \theta^3\alpha^3$$

$$= 2\,\theta^3\alpha$$

$$\gamma_1 = \frac{2\,\theta^3\alpha}{\theta^2\alpha\sqrt{\theta^2\alpha}} = \frac{2}{\sqrt{\alpha}}$$

Kurtosis $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\theta\alpha\, E(X^2) + 6\theta^2\alpha^2\, E(X) - 4\theta^3\alpha^3 E(X) + \theta^4\alpha^4$$

$$= E(X^4) - 4\theta\alpha\, E(X^3) + 6\theta^2\alpha^2\, E(X^2) - 4\theta^3\alpha^3 E(X) + \theta^4\alpha^4$$

$$= \theta^4\alpha^4 + 6\theta^4\alpha^3 + 11\theta^4\alpha^2 + 6\theta^4\alpha - 4\theta\alpha(\theta^3\alpha^3 + 3\theta^3\alpha^2 + 2\theta^3\alpha) + 6\theta^2\alpha^2(\theta^2\alpha) - 4\theta^3\alpha^3(\theta\alpha) + \theta^4\alpha^4$$

$$= 3\theta^4\alpha^2 + 6\theta^4\alpha$$

$$\gamma_2 = \frac{3\theta^4\alpha^2 + 6\theta^4\alpha}{(\theta^2\alpha)^2} = 3 + \frac{6}{\alpha}$$

4.2.2 Half-Normal (semi-normal, positive definite normal, one-sided normal) distribution (Johnson, N. 1994).

$$\text{Stacy}\left(\sqrt{2\sigma^2}, \frac{1}{2}, 2\right) = \text{HalfNormal}(\sigma)$$

$$f(x) = \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \ , x, \sigma > 0 \qquad (4.8)$$

$$E(X^r) = \frac{(\sqrt{2\sigma^2})^r \Gamma(\frac{1}{2}+\frac{r}{2})}{\Gamma\left(\frac{1}{2}\right)}$$

$$E(X) = \frac{\sqrt{2\sigma^2}\Gamma(\frac{1}{2}+\frac{1}{2})}{\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{\sqrt{2\sigma^2}}{\sqrt{\pi}} = \sigma\sqrt{\frac{2}{\pi}}$$

$$E(X^{2)} = \frac{\left(\sqrt{2\sigma^2}\right)^2 \Gamma\left(\frac{1}{2}+\frac{2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\sigma^2 * 2 * \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$= \sigma^2$$

$$E(X^{3)} = \frac{\left(\sqrt{2\sigma^2}\right)^3 \Gamma\left(\frac{1}{2}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{2^{\frac{3}{2}}\sigma^3}{\sqrt{\pi}}$$

$$E(X^{4)} = \frac{\left(\sqrt{2\sigma^2}\right)^4 \Gamma\left(\frac{1}{2}+\frac{4}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{(2\sigma^2)^2\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{4\sigma^4 \frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$= 3\sigma^4$$

$$Var\ X = \mu_2 = E(X-\mu)^2 = \sigma^2 - \left(\sigma\sqrt{\frac{2}{\pi}}\right)^2$$

$$= \sigma^2(1 - \frac{2}{\pi})$$

Skewness: $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$= \frac{2^{\frac{3}{2}}\sigma^3}{\sqrt{\pi}} - 3\left(\sigma\sqrt{\frac{2}{\pi}}\right) * \sigma^2 + 2\left(\sigma\sqrt{\frac{2}{\pi}}\right)^3$$

$$= 2\sigma^3\sqrt{\frac{2}{\pi}} - 3\sigma^3\left(\sqrt{\frac{2}{\pi}}\right) + \frac{4\sigma^3}{\pi}\sqrt{\frac{2}{\pi}}$$

$$\gamma_1 = \frac{\sigma^3\sqrt{\frac{2}{\pi}}(\frac{4}{\pi} - 1)}{\left(\sigma^2(1 - \frac{2}{\pi})\right)^{\frac{3}{2}}} = \frac{\sqrt{2}(4 - \pi)}{(\pi - 2)^{\frac{3}{2}}}$$

Kurtosis $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X - \mu)^4$$

$$= E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= 3\sigma^4 - 4\left(\sigma\sqrt{\frac{2}{\pi}}\right) * 2\sigma^3\left(\sqrt{\frac{2}{\pi}}\right) + 6\sigma^2 * \frac{2}{\pi} * \sigma^3 - 3\sigma^4 * \frac{4}{\pi^2}$$

$$= 3\sigma^4 - \frac{4\sigma^4}{\pi} - 12\frac{\sigma^4}{\pi^2}$$

$$= \frac{\sigma^4}{\pi^2}(3\pi^2 - 4\pi - 12)$$

$$\gamma_2 = \frac{\frac{\sigma^4}{\pi^2}(3\pi^2 - 4\pi - 12)}{(\sigma^2(1 - \frac{2}{\pi}))^2} = \frac{(3\pi^2 - 4\pi - 12)}{(\pi - 2)^2}$$

4.2.3 Chi - Square distribution (Fisher, W.1924, Johnson, N. 1994).

$$\text{Stacy}\left(2, \frac{k}{2}, 1\right) = \text{ChiSqr(k)}$$

$$f(x) = \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)}\, x^{\frac{k-2}{2}} e^{-\frac{x}{2}}\ ,\ \ x > 0\,, k > 0 \qquad (4.9)$$

$$which\ is\ a\ Chi\ - Square\ distribution$$

$$E(X^r) = \frac{2^r\Gamma(\frac{k}{2}+r)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X) = \frac{2\Gamma(\frac{k}{2}+1)}{\Gamma\left(\frac{k}{2}\right)}$$

$$= k$$

$$E(X^{2)} = \frac{2^2\Gamma\left(\frac{k}{2}+2\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{2^2\left(\frac{k}{2}+1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= k(k+2)$$

$$E(X^{3)} = \frac{2^3\Gamma\left(\frac{k}{2}+3\right)}{\Gamma\left(\frac{v}{2}\right)} = \frac{2^3\left(\frac{k}{2}+2\right)\left(\frac{k}{2}+1\right)\left(\frac{k}{2}\right)\left(\frac{k}{2}-1\right)!}{\left(\frac{v}{2}-1\right)!}$$

$$= k(k+2)(k+4)$$

$$E(X^{4)} = \frac{2^4\Gamma\left(\frac{k}{2}+4\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{2^4\left(\frac{k}{2}+3\right)\left(\frac{k}{2}+2\right)\left(\frac{k}{2}+1\right)\left(\frac{k}{2}\right)\left(\frac{k}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= k(k+2)(k+4)(k+6)$$

Va X $= \mu_2 = \; = E(X-\mu)^2 \; = \; 2k$

Skewness: $\gamma_1 = \frac{\mu_3}{\sigma^3} = \frac{8k}{(2k)^{\frac{3}{2}}} = 2\sqrt{\frac{2}{k}}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$= k(k+2)(k+4) - 3k * k(k+2) + 2k^3$$

$$= \; 2^3 k = 8k$$

$$\gamma_1 = \frac{8k}{(2k)^{\frac{3}{2}}} = 2\sqrt{\frac{2}{k}}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4} = \frac{12k(k+4)}{(2k)^2} = 3 + \frac{12}{k}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= k(k+2)(k+4)(k+6) - 4k(k(k+2)(k+4)) + 6k^2\big(k(k+2)\big) - 3k^4$$

$$= 12k^2 + 48k = 12k(k+4)$$

4.2.4 Scaled Chi-Square distribution (Lee, P. 2009).

$$\text{Stacy}\left(2\sigma^2, \frac{k}{2}, 1\right) = \text{ScaledChiSqr}(\sigma, \text{k})$$

$$f(x) = \frac{1}{2\sigma^2\Gamma\left(\frac{k}{2}\right)}\left(\frac{x}{2\sigma^2}\right)^{k-1} exp\left\{-\frac{x^2}{2\delta^2}\right\} \;\; ; \; x, k > 0 \qquad (4.10)$$

$$E(X^r) = \frac{(2\sigma^2)^r\Gamma(\frac{k}{2}+r)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X) = \frac{2\sigma^2\Gamma(\frac{k}{2}+1)}{\Gamma\left(\frac{k}{2}\right)} = \frac{2\sigma^2\left(\frac{k+2}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= \frac{2\sigma^2\left(\frac{k}{2}\right)\left(\frac{k}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= \sigma^2 k$$

$$E(X^{2)} = \frac{(2\sigma^2)^2\Gamma\left(\frac{k}{2}+2\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{4\sigma^4\left(\frac{k+2}{2}\right)!}{\left(\frac{k}{2}-1\right)!} = \frac{4\sigma^4\left(\frac{k+2}{2}\right)\left(\frac{k}{2}\right)\left(\frac{k}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= 4\sigma^4\left(\frac{k+2}{2}\right)\left(\frac{k}{2}\right) = \sigma^4 k(k+2)$$

$$E(X^{3)} = \frac{(2\sigma^2)^3\Gamma\left(\frac{k}{2}+3\right)}{\Gamma\left(\frac{v}{2}\right)} = \frac{2^3\sigma^6\left(\frac{k+4}{2}\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= \frac{2^3\sigma^6\left(\frac{k+4}{2}\right)\left(\frac{k+2}{2}\right)\left(\frac{k}{2}\right)\left(\frac{k}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= \sigma^6 k(k+2)(k+4)$$

$$E(X^{4)} = \frac{(2\sigma^2)^4\Gamma\left(\frac{k}{2}+4\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{2^4\sigma^8\left(\frac{k+6}{2}\right)\left(\frac{k+4}{2}\right)\left(\frac{k+2}{2}\right)\left(\frac{k}{2}\right)\left(\frac{k}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= \sigma^8 k(k+2)(k+4)(k+6)$$

$$Var\ X = \quad \mu_2 = \quad E(X-\mu)^2 \ = \ 2\sigma^8 k$$

Skewness: $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$= \sigma^6 k(k+2)(k+4) - 3\sigma^6 k * k(k+2) + 2\sigma^6 k^3$$

$$= \ 2^3 k = 8\sigma^6 k$$

$$\gamma_1 = \frac{8\sigma^6 k}{(2\sigma^4 k)^{\frac{3}{2}}} = 2\sqrt{\frac{2}{k}}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= \sigma^8[k(k+2)(k+4)(k+6) - 4k(k(k+2)(k+4)) + 6k^2\big(k(k+2)\big)$$

$$-3k^4]$$

$$= 12\sigma^8 k^2 + 48\sigma^8 k = 12\sigma^8 k(k+4)$$

$$\gamma_2 = \frac{12\sigma^8 k(k+4)}{(2\sigma^4 k)^2} = 3 + \frac{12}{k}$$

4.2.5 Scaled Inverse Chi-Square (Gelman, A. et al., 2004).

Stacy ($\frac{1}{2\sigma^2}, \frac{k}{2}, -1) = ScaledInvChiSqr(\sigma, k)$

$$f(x/\sigma, m) = \frac{2\sigma^2}{\Gamma\left(\frac{k}{2}\right)}\left(\frac{1}{2\sigma^2 x}\right)^{\frac{k}{2}+1} exp\left\{-\frac{1}{2\sigma^2 x}\right\} \ ; x, k, \sigma > 0 \qquad (4.11)$$

$$E(X^r) = \frac{\left(\frac{1}{2\sigma^2}\right)^r \Gamma(\frac{k}{2}-r)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X) = \frac{\frac{1}{2\sigma^2}\Gamma(\frac{k}{2}-1)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\frac{1}{2\sigma^2}\left(\frac{k-2}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= \frac{\left(\frac{k-4}{2}\right)!}{2\sigma^2\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)!}$$

$$= \frac{1}{\sigma^2(k-2)}$$

$$E(X^2) = \frac{\left(\frac{1}{2\sigma^2}\right)^2 \Gamma\left(\frac{k}{2}-2\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\frac{1}{4\sigma^4}\left(\frac{k-4}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= \frac{\left(\frac{k-6}{2}\right)!}{4\sigma^4\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)\left(\frac{k-6}{2}\right)!}$$

$$= \frac{1}{\sigma^4(k-2)(k-4)}$$

$$E(X^3) = \frac{\left(\frac{1}{2\sigma^2}\right)^3 \Gamma\left(\frac{k}{2}-3\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\left(\frac{k-6}{2}-1\right)!}{8\sigma^6\left(\frac{k}{2}-1\right)!}$$

$$= \frac{\left(\frac{k-8}{2}\right)!}{8\sigma^6\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)\left(\frac{k-6}{2}\right)\left(\frac{k-8}{2}\right)!}$$

$$= \frac{1}{\sigma^6(k-2)(k-4)(k-6)}$$

$$E(X^4) = \frac{\left(\frac{1}{2\sigma^2}\right)^4 \Gamma\left(\frac{k}{2}-4\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\left(\frac{k-8}{2}-1\right)!}{16\sigma^8\left(\frac{k}{2}-1\right)!}$$

$$= \frac{\left(\frac{k-10}{2}\right)!}{16\sigma^8\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)\left(\frac{k-6}{2}\right)\left(\frac{k-8}{2}\right)\left(\frac{k-10}{2}\right)!}$$

$$= \frac{1}{\sigma^8(k-2)(k-4)(k-6)(k-8)}$$

$$Var\ X = \mu_2 = E(X-\mu)^2$$

$$= \frac{1}{\sigma^4(k-2)(k-4)} - \left(\frac{1}{\sigma^2(k-2)}\right)^2$$

$$= \frac{2}{\sigma^4(k-2)^2(k-4)}$$

Skewness $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$= \frac{1}{\sigma^6(k-2)(k-4)(k-6)} - 3\frac{1}{\sigma^2(k-2)} *$$

$$\frac{1}{\sigma^4(k-2)(k-4)} + 2\left(\frac{1}{\sigma^2(k-2)}\right)^3$$

$$= \frac{1}{\sigma^6(k-2)^3(k-4)(k-6)}\begin{bmatrix} (k-2)^2 - 3(k-2) \\ (k-6) + 2(k-4)(k-6) \end{bmatrix}$$

$$= \frac{1}{\sigma^6(k-2)^3(k-4)(k-6)}\begin{bmatrix} k^2 - 4k + 4 - 3k^2 + \\ 24k - 36 + 2k^2 - 20k + 48 \end{bmatrix}$$

$$= \frac{16}{\sigma^6(k-2)^3(k-4)(k-6)}$$

$$= \frac{\frac{16}{\sigma^6(k-2)^3(k-4)(k-6)}}{\left(\frac{2}{\sigma^4(k-2)^2(k-4)}\right)^{\frac{3}{2}}} = \frac{4\sqrt{2(k-4)}}{(k-6)}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= \frac{1}{\sigma^8(k-2)(k-4)(k-6)(k-8)} - 4\frac{1}{\sigma^2(k-2)} * \frac{1}{\sigma^6(k-2)(k-4)(k-6)} + 6(\frac{1}{\sigma^2(k-2)})^2\frac{1}{\sigma^4(k-2)(k-4)} - 3(\frac{1}{\sigma^2(k-2)})^4]$$

$$= \frac{\begin{matrix} k^3 - 6k^2 + 12k - 8 - 4(k^3 - 12k^2 + 36k - 32) + 6(k^3 - 16k^2 + 78k - 96) \\ -3k^3 + 54k^2 - 312k + 576 \end{matrix}}{(k-2)^4(k-2)(k-4)(k-8)}$$

$$= \frac{12k+60}{\sigma^8(k-2)^4(k-2)(k-4)(k-8)}$$

$$= \frac{12k+60}{\sigma^8(k-2)^4(k-4)(k-6)(k-8)} * \frac{\sigma^8(k-2)^4(k-4)^2}{4}$$

$$= \frac{(3k+15)(k-4)}{(k-6)(k-8)}$$

4.2.6: Inverse Chi-Square distribution (Gelman, A. et al., 2004).

Stacy $(\frac{1}{2}, \frac{k}{2}, -1) = InvChiSqr(\ k)$

$$f(x/\sigma, m) = \frac{2}{\Gamma\left(\frac{k}{2}\right)}\left(\frac{1}{2x}\right)^{\frac{k}{2}+1} exp\left\{-\frac{1}{2x}\right\} \ \ ; x, k > 0 \qquad (4.12)$$

$$E(X^r) = \frac{\left(\frac{1}{2}\right)^r \Gamma(\frac{k}{2}-r)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X) = \frac{\frac{1}{2}\Gamma(\frac{k}{2}-1)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\frac{1}{2}\left(\frac{k-2}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= \frac{\left(\frac{k-4}{2}\right)!}{2\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)!} = \frac{1}{(k-2)} \ ; k > 2$$

$$E(X^{2)} = \frac{\left(\frac{1}{2}\right)^2 \Gamma\left(\frac{k}{2}-2\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\frac{1}{4}\left(\frac{k-4}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= \frac{\left(\frac{k-6}{2}\right)!}{4\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)\left(\frac{k-6}{2}\right)!}$$

$$= \frac{1}{(k-2)(k-4)} \ ; k > 4$$

$$E(X^{3)} = \frac{\left(\frac{1}{2}\right)^3 \Gamma\left(\frac{k}{2}-3\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\left(\frac{k-6}{2}-1\right)!}{8\left(\frac{k}{2}-1\right)!}$$

$$= \frac{\left(\frac{k-8}{2}\right)!}{8\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)\left(\frac{k-6}{2}\right)\left(\frac{k-8}{2}\right)!}$$

$$= \frac{1}{(k-2)(k-4)(k-6)}$$

$$E(X^4) = \frac{\left(\frac{1}{2}\right)^4 \Gamma\left(\frac{k}{2}-4\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\left(\frac{k-8}{2}-1\right)!}{16\left(\frac{k}{2}-1\right)!}$$

$$= \frac{\left(\frac{k-10}{2}\right)!}{16\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)\left(\frac{k-6}{2}\right)\left(\frac{k-8}{2}\right)\left(\frac{k-10}{2}\right)!}$$

$$= \frac{1}{(k-2)(k-4)(k-6)(k-8)} \ ; k > 8$$

$$Var\ X = \ \mu_2 = E(X-\mu)^2$$

$$= \frac{1}{(k-2)(k-4)} - \left(\frac{1}{(k-2)}\right)^2$$

$$= \frac{2}{(k-2)^2(k-4)} \ ; k > 4$$

Skewness: $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$= \frac{1}{(k-2)(k-4)(k-6)} - 3\frac{1}{(k-2)} * \frac{1}{(k-2)(k-4)} + 2\left(\frac{1}{(k-2)}\right)^3$$

$$= \frac{1}{(k-2)^3(k-4)(k-6)}\begin{bmatrix}(k-2)^2 - 3(k-2)(k-6) + \\ 2(k-4)(k-6)\end{bmatrix}$$

$$= \frac{1}{(k-2)^3(k-4)(k-6)}\begin{bmatrix}k^2 - 4k + 4 - 3k^2 + 24k - 36 + \\ 2k^2 - 20k + 48\end{bmatrix}$$

$$= \frac{16}{(k-2)^3(k-4)(k-6)} \ ; k > 6$$

$$\gamma_1 = \frac{\frac{16}{(k-2)^3(k-4)(k-6)}}{\left(\frac{2}{(k-2)^2(k-4)}\right)^{\frac{3}{2}}}$$

$$= \frac{4\sqrt{2(k-4)}}{(k-6)}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= \frac{1}{(k-2)(k-4)(k-6)(k-8)} - 4\frac{1}{(k-2)} * \frac{1}{(k-2)(k-4)(k-6)}$$
$$+ 6(\frac{1}{(k-2)})^2 \frac{1}{\sigma^4(k-2)(k-4)} - 3(\frac{1}{(k-2)})^4]$$

$$= \frac{k^3 - 6k^2 + 12k - 8 - 4(k^3 - 12k^2 + 36k - 32) + 6(k^3 - 16k^2 + 78k - 96) - 3k^3 + 54k^2 - 312k + 576}{(k-2)^4(k-2)(k-4)(k-8)}$$

$$= \frac{12k + 60}{(k-2)^4(k-2)(k-4)(k-8)}$$

$$\gamma_2 = \frac{\frac{12k+60}{(k-2)^4(k-2)(k-4)(k-8)}}{\left(\frac{2}{(k-2)^2(k-4)}\right)^2}$$

$$= \frac{12k+60}{(k-2)^4(k-4)(k-6)(k-8)} * \frac{(k-2)^4(k-4)^2}{4}$$

$$= \frac{(3k+15)(k-4)}{(k-6)(k-8)}$$

4.2.7 Chi (ℵ) distribution (Johnson, N. 1994).

$$Stacy\left(\sqrt{2}, \frac{k}{2}, 2\right) = Chi(k)$$

For $\alpha = \frac{k}{2}$ where k is the number of degrees of freedom $\theta = \sqrt{2}$, $\beta = 2$ then equation (4.2) becomes

$$f(x) = \frac{\sqrt{2}}{\Gamma\left(\frac{k}{2}\right)}\left(\frac{x}{\sqrt{2}}\right)^{k-1} exp\left\{-\frac{x^2}{2}\right\}, x, k > 0 \tag{4.13}$$

$$E(X^r) = \frac{(\sqrt{2})^r\Gamma(\frac{k}{2}+\frac{r}{2})}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X) = \frac{\sqrt{2}\,\Gamma(\frac{k}{2}+\frac{1}{2})}{\Gamma\left(\frac{k}{2}\right)}$$

$$= \frac{\sqrt{2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X^{2)} = \frac{(\sqrt{2})^2\Gamma(\frac{k}{2}+\frac{2}{2})}{\Gamma\left(\frac{k}{2}\right)} = \frac{2\left(\frac{k+2}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= \frac{2\left(\frac{k}{2}\right)\left(\frac{k}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!} = k$$

$$E(X^{3)} = \frac{(\sqrt{2})^3\Gamma(\frac{k}{2}+\frac{3}{2})}{\Gamma\left(\frac{k}{2}\right)} = \frac{2(\sqrt{2})\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X^{4)} = \frac{(\sqrt{2})^4\Gamma(\frac{k}{2}+\frac{4}{2})}{\Gamma\left(\frac{k}{2}\right)} = \frac{4\Gamma\left(\frac{k+4}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$Var\ X = \mu_2 = \text{Variance} \quad = E(X-\mu)^2$$

$$= k - \left(\frac{\sqrt{2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^2$$

Skewness: $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$= \frac{2(\sqrt{\mathbf{2}})\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 3\frac{\sqrt{\mathbf{2}}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)} * k + 2\left(\frac{\sqrt{\mathbf{2}}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^3$$

$$\gamma_1 = \frac{\left(\frac{2(\sqrt{2})\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 3\frac{\sqrt{2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)} * k + 2\left(\frac{\sqrt{2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^3\right)}{\left(k - \left(\frac{\sqrt{2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^2\right)^{\frac{3}{2}}}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= \frac{4\Gamma\left(\frac{k+4}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 4\frac{\sqrt{2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \frac{* 2(\sqrt{2})\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} +$$

$$6\left(\frac{\sqrt{2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 * k - 3\left(\frac{\sqrt{2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^4$$

$$\gamma_2 = \frac{\frac{4\Gamma\left(\frac{k+4}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 4\frac{\sqrt{\mathbf{2}}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \frac{* 2(\sqrt{\mathbf{2}})\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} + 6\left(\frac{\sqrt{\mathbf{2}}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 * k - 3\left(\frac{\sqrt{\mathbf{2}}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^4}{\left(k - \left(\frac{\sqrt{\mathbf{2}}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^2\right)^2}$$

4.2.8 Scaled Chi-distribution (Gelman, N. 2004).

$$\text{Stacy}\left(\sqrt{2\delta^2}, \frac{k}{2}, 2\right) = \text{ScaledChi}(\sigma, \text{k})$$

For $= \frac{k}{2}$, $\theta = \sqrt{2}, \beta = 2$ then equation (4.2) becomes

$$f\left(\frac{x}{\delta}, k\right) = \frac{1}{\sqrt{2\delta^2}\Gamma\left(\frac{k}{2}\right)}\left(\frac{x}{\sqrt{2\delta^2}}\right)^{k-1} exp\left\{-\frac{x^2}{2\delta^2}\right\} \ ;x,k > 0 \qquad (4.14)$$

$$E(X^r) = \frac{\left(\sqrt{2\delta^2}\right)^r \Gamma\left(\frac{k}{2}+\frac{r}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X) = \frac{\sqrt{2\delta^2}\Gamma\left(\frac{\mathrm{k}}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{\mathrm{k}}{2}\right)}$$

$$= \frac{\sqrt{2\delta^2}\ \Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X^{2)} = \frac{(\sqrt{2\delta^2})^2\Gamma(\frac{k}{2}+\frac{2}{2})}{\Gamma\left(\frac{k}{2}\right)} = \frac{2\delta^2\left(\frac{k+2}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!}$$

$$= \frac{2\delta^2\left(\frac{k}{2}\right)\left(\frac{k}{2}-1\right)!}{\left(\frac{k}{2}-1\right)!} = \delta^2 k$$

$$E(X^{3)} = \frac{(\sqrt{2\delta^2})^3\Gamma(\frac{k}{2}+\frac{3}{2})}{\Gamma\left(\frac{k}{2}\right)} = \frac{2\delta^3(\sqrt{2})\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X^{4)} = \frac{(\sqrt{2\delta^2})^4\Gamma(\frac{k}{2}+\frac{4}{2})}{\Gamma\left(\frac{k}{2}\right)} = \frac{4\delta^4\Gamma\left(\frac{k+4}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$Var\ X = \mu_2 = E(X-\mu)^2 = \delta^2 k - \left(\frac{\sqrt{2\delta^2}\ \Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^2$$

Skewness: $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$= \frac{2\delta^3(\sqrt{2})\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 3\frac{\sqrt{2\delta^2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} * \delta^2 k +$$

$$2\left(\frac{\sqrt{2\delta^2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^3$$

$$\gamma_1 = \frac{\left(\frac{2\delta^3(\sqrt{2})\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 3\frac{\sqrt{2\delta^2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)} * \delta^2 k + 2\left(\frac{\sqrt{2\delta^2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^3\right)}{\left(\delta^2 k - \left(\frac{\sqrt{2\delta^2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^2\right)^{\frac{3}{2}}}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= \frac{4\delta^4\Gamma\left(\frac{k+4}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 4\frac{\sqrt{2\delta^2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\frac{2\delta^3(\sqrt{2})\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} +$$

$$6\left(\frac{\sqrt{2\delta^2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 * \delta^2 k - 3\left(\frac{\sqrt{2\delta^2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^4$$

$$\gamma_2 = \frac{\left(\begin{array}{c}\frac{4\delta^4\Gamma\left(\frac{k+4}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 4\frac{\sqrt{2\delta^2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\frac{2\delta^3(\sqrt{2})\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} + \\ 6\left(\frac{\sqrt{2\delta^2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 * \delta^2 k - \\ 3\left(\frac{\sqrt{2\delta^2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^4\end{array}\right)}{\left(\delta^2 k - \left(\frac{\sqrt{2\delta^2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^2\right)^2}$$

4.2.9 Rayleigh distribution (Strutt, J. 1880, Johnson, N. et al., 1994)

Stacy $(\sqrt{2\sigma^2}, 1, 2)$ = Rayleigh(σ)

Substituting $\theta = \sqrt{2\sigma^2}, \alpha = 1, \beta = 2$ equation (4.2) becomes

$$f(x/\sigma) = \frac{1}{\sigma^2} x exp\left\{-\frac{x^2}{2\sigma^2}\right\};\ \delta > 0 \tag{4.15}$$

$$E(X^r) = (\sqrt{2\sigma^2})^r \Gamma\left(1 + \frac{r}{2}\right)$$

$$E(X) = \sqrt{2\sigma^2}\Gamma\left(1 + \frac{1}{2}\right) = \sqrt{2\sigma^2}\Gamma\left(\frac{3}{2}\right)$$

$$= \sqrt{2\sigma^2}\ \left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \sigma\sqrt{\frac{\pi}{2}}$$

$$E(X^2) = (\sqrt{2\sigma^2})^2 \Gamma\left(1 + \frac{2}{2}\right)$$

$$= 2\sigma^2$$

$$E(X^3) = (\sqrt{2\sigma^2})^3 \Gamma\left(1 + \frac{3}{2}\right) = (\sqrt{2\sigma^2})^3 \Gamma\left(\frac{5}{2}\right)$$

$$= (\sqrt{2\sigma^2})^3 \frac{3}{2} * \frac{1}{2}\ \Gamma\left(\frac{1}{2}\right)$$

$$= 3\sigma^3\sqrt{\frac{\pi}{2}}$$

$$E(X^4) = (\sqrt{2\sigma^2})^4\Gamma\left(1+\frac{4}{2}\right) = (\sqrt{2\sigma^2})^3\Gamma(3)$$

$$= (2\sigma^2)^2 * 2$$

$$= 8\sigma^4$$

$$Var\ X = 2\sigma^2 - \left(\sigma\sqrt{\frac{\pi}{2}}\right)^2 = \sigma^2\left(\frac{4-\pi}{2}\right)$$

$$= \sigma^2\left(\frac{4-\pi}{2}\right)$$

Skewnes $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$= 3\sigma^3\sqrt{\frac{\pi}{2}} - 3\left(\sigma\sqrt{\frac{\pi}{2}}\right) * 2\sigma^2 + 2\left(\sigma\sqrt{\frac{\pi}{2}}\right)^3$$

$$= 3\sigma^3\sqrt{\frac{\pi}{2}} - 6\sigma^3\left(\sqrt{\frac{\pi}{2}}\right) + 2\sigma^3\frac{\pi}{2}\sqrt{\frac{\pi}{2}}$$

$$= \sigma^3\left(\sqrt{\frac{\pi}{2}}\right)(\pi - 3)$$

$$\gamma_1 = \frac{\sigma^3\left(\sqrt{\frac{\pi}{2}}\right)(\pi-3)}{(\sigma^2\left(\frac{4-\pi}{2}\right))^{\frac{3}{2}}} = \sqrt{\frac{\pi}{2}}\frac{(\pi-3)}{\left(\left(2-\frac{\pi}{2}\right)^{\frac{3}{2}}\right)}$$

Kurtosi $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= 8\sigma^4 - 4\,\sigma\sqrt{\frac{\pi}{2}} * 3\sigma^3\sqrt{\frac{\pi}{2}} + 6\left(\sigma\sqrt{\frac{\pi}{2}}\right)^2 * 2\sigma^2 - 3\left(\sigma\sqrt{\frac{\pi}{2}}\right)^4$$

$$= 8\sigma^4 - 12\sigma^4\left(\frac{\pi}{2}\right) + 12\sigma^4\left(\frac{\pi}{2}\right) - \frac{3\sigma^4\pi^2}{4}$$

$$= \sigma^4(8 - \frac{3\pi^2}{4})$$

$$\gamma_2 = \frac{\sigma^4(8 - \frac{3\pi^2}{4})}{(\sigma^2\left(\frac{4-\pi}{2}\right))^2} = \frac{(8 - \frac{3\pi^2}{4})}{\left(2 - \frac{\pi}{2}\right)^{\frac{3}{2}}}$$

4.2.10 Inverse Rayleigh distribution (Evans, M. et al., 2000).

$Stacy\ (\frac{1}{\sqrt{2\sigma^2}}, 1, -2) = InvRayleighn(\sigma)$

Substituting $\theta = \frac{1}{\sqrt{2\sigma^2}}, \alpha = 1, \beta = -2$ equation (3.2) becomes

$$f(x) = 2\sqrt{2\sigma^2}\left(\frac{1}{\sqrt{2\sigma^2}x}\right)^3 exp\left\{-\frac{1}{2\sigma^2x^2}\right\}, x, \sigma > 0 \quad (4.16)$$

$$E(X^r) = \left(\frac{1}{\sqrt{2\sigma^2}}\right)^r \Gamma\left(1 - \frac{r}{2}\right)$$

$$E(X) = \frac{\Gamma\left(1 - \frac{1}{2}\right)}{\sqrt{2\sigma^2}} = \frac{1}{\sigma}\sqrt{\frac{\pi}{2}}$$

$$E(X^2) = (\sqrt{2\sigma^2})^2\Gamma\left(1 - \frac{2}{2}\right)$$

$$-does\ not\ exist$$

$$E(X^3) - does\ not\ exist$$

$$E(X^4) - does\ not\ exist$$

$$Var\ X - does\ not\ exist$$

Skewness: $\gamma_1 - does\ not\ exist$

Kurtosis: $\gamma_2 - does\ not\ exist$

4.2.11 Maxwell (Maxwell-Boltzman. Maxwell speed) distribution (Maxwell, J. 1860, Abramowittz, M. 1965).

$Stacy\ (\sqrt{2\sigma^2}, \frac{3}{2}, 2) = Maxwell(\sigma)$

Substituting $\theta = \sqrt{2\sigma^2}, \alpha = \frac{3}{2}, \beta = 2$ equation (4.2) becomes

$$f(x/\sigma) = \frac{\sqrt{2}}{\sqrt{\pi}\sigma^3}x^2 exp\left\{-\frac{x^2}{2\sigma^2}\right\}; x, \sigma > 0 \quad (4.17)$$

$$E(X^r) = \frac{\left(\sqrt{2\sigma^2}\right)^r \Gamma\left(\frac{3}{2} + \frac{r}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

$$E(X) = \frac{\sqrt{2\sigma^2}\,\Gamma\left(\frac{3}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

$$= \frac{\sqrt{2\sigma^2}\,\Gamma(2)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{2^{\frac{3}{2}}\sigma}{\sqrt{\pi}} = 2\sigma\sqrt{\frac{2}{\pi}}$$

$$E(X^2) = \frac{(\sqrt{2\sigma^2})^2\,\Gamma\left(\frac{3}{2} + \frac{2}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

$$= \frac{2\sigma^2\,\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{2\sigma^2\,\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{2\sigma^2\,\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = 3\sigma^2$$

$$E(X^3) = \frac{(\sqrt{2\sigma^2})^3\,\Gamma\left(\frac{3}{2} + \frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{(\sqrt{2\sigma^2})^3\,\Gamma(3)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{(\sqrt{2\sigma^2})^3\,2}{\frac{1}{2}\sqrt{\pi}}$$

$$= 2^3\sigma^3\sqrt{\frac{2}{\pi}}$$

$$E(X^4) = \frac{(\sqrt{2\sigma^2})^4\,\Gamma\left(\frac{3}{2} + \frac{4}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{2^2\sigma^4\,\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

$$= \frac{2^2\sigma^4\,\frac{5}{2}\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{2^2\sigma^4\,\frac{5}{2}\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

$$= 15\sigma^4$$

$$Var\ X = 3\sigma^2 - \left(2\sigma\sqrt{\frac{2}{\pi}}\right)^2 = 3\sigma^2 - \frac{\mathbf{8}\sigma^2}{\pi}$$

$$= \sigma^2(3 - \frac{8}{\pi})$$

Skewness $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X^3) - 3\mu E(X^2) + 2\mu^3$$

$$= 8\delta^3\sqrt{\frac{2}{\pi}} - 3\left(2\sigma\sqrt{\frac{2}{\pi}}\right) * 3\sigma^2 + 2\left(2\sigma\sqrt{\frac{2}{\pi}}\right)^3$$

$$= 8\sigma^3\sqrt{\frac{2}{\pi}} - 18\sigma^3\left(\sqrt{\frac{2}{\pi}}\right) + \frac{32\sigma^3}{\pi}\left(\sqrt{\frac{2}{\pi}}\right)$$

$$= \frac{32\sigma^3}{\pi}\left(\sqrt{\frac{2}{\pi}}\right) - 10\sigma^3\left(\sqrt{\frac{2}{\pi}}\right)$$

$$= 2\sigma^3\left(\sqrt{\frac{2}{\pi}}\right)\left[\frac{16}{\pi} - 5\right]$$

$$\gamma_1 = \frac{2\sigma^3\left(\sqrt{\frac{2}{\pi}}\right)\left[\frac{16}{\pi} - 5\right]}{(\ \sigma^2(3 - \frac{\mathbf{8}}{\pi})\ )^{\frac{3}{2}}} = 2\sqrt{\frac{2}{\pi}}\frac{\left[\frac{16}{\pi} - 5\right]}{(3 - \frac{\mathbf{8}}{\pi})^{\frac{3}{2}}}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= 15\sigma^4 - 4 * 2\sigma\sqrt{\frac{2}{\pi}} * 8\sigma^3\sqrt{\frac{2}{\pi}} + 6\left(\sigma\sqrt{\frac{2}{\pi}}\right)^2 * 3\delta^2 - 3\left(\sigma\sqrt{\frac{2}{\pi}}\right)^4$$

$$= 15\sigma^4 - \frac{128\sigma^4}{\pi} + \frac{144\sigma^4}{\pi} - \frac{192\sigma^4}{\pi^2}$$

$$= 15\sigma^4 - \frac{16\sigma^4}{\pi} - \frac{192\sigma^4}{\pi^2}$$

$$= \sigma^4\left[15 - \frac{16}{\pi} - \frac{192}{\pi^2}\right]$$

$$\gamma_2 = \frac{\sigma^4\left[15 - \frac{16}{\pi} - \frac{192}{\pi^2}\right]}{(\sigma^2(3 - \frac{8}{\pi}))^2} = \frac{\left[15 - \frac{16}{\pi} - \frac{192}{\pi^2}\right]}{(3 - \frac{8}{\pi})^2}$$

4.2.12 Inverse Gamma (Johnson, *et al.*, 1994)

$Stacy(\theta, \alpha, -1) = InvGamma(\theta, \alpha)$

Substituting $\theta = \theta, \alpha = \alpha, \beta = -1$ equation (4.2) becomes

$$f(x/\theta, \alpha, \beta) = \frac{1}{\theta\Gamma(\alpha)}\left(\frac{\theta}{x}\right)^{\alpha+1} exp\left\{-\frac{\theta}{x}\right\} \; ; x > 0, \alpha, \theta > 0 \tag{4.18}$$

$$E(X^r) = \frac{\theta^r\Gamma(\alpha - r)}{\Gamma(\alpha)}$$

$$E(X) = \frac{\theta\Gamma(\alpha - 1)}{\Gamma(\alpha)} = \frac{\theta(\alpha - 2)!}{(\alpha - 1)!}$$

$$= \frac{\theta(\alpha - 2)!}{(\alpha - 1)(\alpha - 2)!} = \frac{\theta}{(\alpha - 1)}; \; \alpha > 1$$

$$E(X^2) = \frac{\theta^2\Gamma(\alpha - 2)}{\Gamma(\alpha)} = \frac{\theta^2(\alpha - 3)!}{(\alpha - 1)(\alpha - 2)(\alpha - 3)!}$$

$$= \frac{\theta^2}{(\alpha - 1)(\alpha - 2)}$$

$$E(X^{3)} = \frac{\theta^3\Gamma(\alpha - 3)}{\Gamma(\alpha)} = \frac{\theta^3(\alpha - 4)!}{(\alpha - 1)!}$$

$$= \frac{\theta^3(\alpha - 4)!}{(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)!} = \frac{\theta^3}{(\alpha - 1)(\alpha - 2)(\alpha - 3)}$$

$$E(X^{4)} = \frac{\theta^4\Gamma(\alpha - 4)}{\Gamma(\alpha)} = \frac{\theta^4(\alpha - 5)!}{(\alpha - 1)!}$$

$$= \frac{\theta^4(\alpha-5)!}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)!}$$

$$= \frac{\theta^4}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}$$

$$Var\,X \;= \frac{\theta^2}{(\alpha-1)(\alpha-2)} - \left(\frac{\theta}{(\alpha-1)}\right)^2$$

$$= \frac{\theta^2}{(\alpha-1)}\left(\frac{1}{(\alpha-2)} - \frac{1}{(\alpha-1)}\right)$$

$$= \frac{\theta^2}{(\alpha-1)^2(\alpha-2)}\ ;\ \alpha > 2$$

Skewness $\quad \gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\mu\, E(X^2) + 2\mu^3$$

$$= \frac{\theta^3}{(\alpha-1)(\alpha-2)(\alpha-3)} - 3\frac{\theta}{(\alpha-1)} * \frac{\theta^2}{(\alpha-1)(\alpha-2)} + 2\left(\frac{\theta}{(\alpha-1)}\right)^3$$

$$= \frac{\theta^3}{(\alpha-1)(\alpha-2)(\alpha-3)} - \frac{3\theta^3}{(\alpha-1)^2(\alpha-2)} + \frac{2\theta^3}{(\alpha-1)^3}$$

$$= \frac{\theta^3[(\alpha^2-2\alpha+1) - 3(\alpha^2-4\alpha+3) + 2(\alpha^2-5\alpha+6]}{(\alpha-1)^3(\alpha-2)(\alpha-3)}$$

$$= \frac{4\theta^3}{(\alpha-1)^3(\alpha-2)(\alpha-3)}$$

$$\gamma_1 = \frac{\mu_3}{\sigma^3} = \frac{\frac{4\theta^3}{(\alpha-1)^3(\alpha-2)(\alpha-3)}}{\left(\frac{\theta^2}{(\alpha-1)^2(\alpha-2)}\right)^{\frac{3}{2}}}$$

$$= \frac{4\sqrt{(\alpha-2)}}{(\alpha-3)}$$

Kurtosis $\quad \gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= \ E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= \frac{\theta^4}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)} - 4\frac{\theta}{(\alpha-1)}$$

$$* \frac{\theta^3}{(\alpha-1)(\alpha-2)(\alpha-3)} + 6\left(\frac{\theta}{(\alpha-1)}\right)^2 \frac{\theta^2}{(\alpha-1)(\alpha-2)} - 3\left(\frac{\theta}{(\alpha-1)}\right)^4$$

$$= \theta^4 \left(\frac{\begin{matrix} \alpha^3 - 3\alpha^2 - 1 - 4\alpha^3 + 24\alpha^2 - 36\alpha + 16 + \\ 6\alpha^3 - 48\alpha^2 + 114\alpha - 72 - 3\alpha^3 \\ +27\alpha^2 - 78\alpha + 72 \end{matrix}}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} \right)$$

$$= \theta^4 \left(\frac{3\alpha + 15}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} \right)$$

$$\gamma_2 = \frac{\theta^4 \left(\frac{3\alpha + 15}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} \right)}{\left(\frac{\theta^2}{(\alpha-1)^2(\alpha-2)} \right)^2}$$

$$= \theta^4 \left(\frac{3\alpha + 15}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} \right) * \frac{(\alpha-1)^4(\alpha-2)^2}{\theta^4}$$

$$= \frac{3(\alpha+5)(\alpha-2)}{(\alpha-3)(\alpha-4)};\ \alpha > 4$$

4.2.13 Inverse Exponential (Kleiber, *et al.*, 2003)
Stacy $(\theta, 1, -1) = InvExp(\theta)$

Substituting $\theta = \theta, \alpha = 1, \beta = -1$ equation (4.2) becomes

$$f(x/\theta) = \frac{\theta}{x^2}\, exp\left\{-\frac{\theta}{x}\right\}\ ;\ x > 0 \qquad (4.19)$$

$$E(X^r) = \theta^r \Gamma(1-r); k < 1$$

$$E(X) = \theta\, \Gamma(0) = \theta(-1)! = \infty$$

$Var\ X$ – does not exist

Skewness - does not exist

Kurtosis - does not exist

4.2.14 Inverse Chi distribution (Lee, P.M.2009)

Stacy$\left(\frac{1}{\sqrt{2}}, \frac{k}{2}, -2\right) = InvChi(k)$

Substituting $\theta = \frac{1}{\sqrt{2}}, \alpha = \frac{\mathrm{k}}{2}, \beta = -2$ equation (4.2) becomes

$$f(x) = \frac{2\sqrt{2}}{\Gamma\left(\frac{k}{2}\right)} \left(\frac{1}{\sqrt{2}x}\right)^{k+1} exp\left\{-\frac{1}{2x^2}\right\}, x, k > 0 \tag{4.20}$$

$$E(X^r) = \frac{\left(\frac{1}{\sqrt{2}}\right)^r \Gamma\left(\frac{k}{2} - \frac{r}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X) = \frac{\frac{1}{\sqrt{2}} \Gamma\left(\frac{k}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X^2) = \frac{\left(\frac{1}{\sqrt{2}}\right)^2 \Gamma\left(\frac{k}{2} - \frac{2}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$= \frac{\frac{1}{2} \Gamma\left(\frac{k-2}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$= \frac{\left(\frac{k-4}{2}\right)!}{2\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)!} = \frac{1}{k-2}$$

$$E(X^3) = \frac{\left(\frac{1}{\sqrt{2}}\right)^3 \Gamma\left(\frac{k}{2} - \frac{3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$= \frac{\left(\frac{1}{\sqrt{2}}\right)^3 \Gamma\left(\frac{k-3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X^4) = \frac{\left(\frac{1}{\sqrt{2}}\right)^4 \Gamma\left(\frac{k}{2}-\frac{4}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\Gamma\left(\frac{k-4}{2}\right)}{4\Gamma\left(\frac{k}{2}\right)}$$

$$= \frac{\left(\frac{k-6}{2}\right)!}{4\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)\left(\frac{k-6}{2}\right)!}$$

$$= \frac{1}{(k-2)(k-4)}$$

$$Var\ X = \frac{1}{k-2} - \left(\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2$$

Skewness $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\mu\, E(X^2) + 2\mu^3$$

$$= \frac{\left(\frac{1}{\sqrt{2}}\right)^3 \Gamma\left(\frac{k-3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 3\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} * \frac{1}{k-2} + 2\left(\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^3$$

$$\gamma_1 = \frac{\frac{\left(\frac{1}{\sqrt{2}}\right)^3 \Gamma\left(\frac{k-3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 3\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} * \frac{1}{k-2} + 2\left(\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^3}{\left(\frac{1}{k-2} - \left(\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2\right)^{\frac{3}{2}}}$$

Kurtosis $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= \frac{1}{(k-2)(k-4)} - 4\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} * \frac{\left(\frac{1}{\sqrt{2}}\right)^3 \Gamma\left(\frac{k-3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$+6\left(\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2\frac{1}{k-2}-3\left(\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^4$$

$$\gamma_2=\left(\frac{\frac{1}{(k-2)(k-4)}-4\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}*\frac{\left(\frac{1}{\sqrt{2}}\right)^3\Gamma\left(\frac{k-3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}+6\left(\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2\frac{1}{k-2}-3\left(\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^4}{\left(\frac{1}{k-2}-\left(\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2\right)^2}\right)$$

4.2.15 Scaled Inverse Chi distribution (Lee, P.M.2009)

Stacy$\left(\frac{1}{\sqrt{2\sigma^2}},\frac{k}{2},-2\right)=ScaledInvChi(\sigma,k)$

Substituting $\theta=\frac{1}{\sqrt{2\sigma^2}},\alpha=\frac{\mathrm{k}}{2},\beta=-2$ equation (4.2) becomes

$$f(x/\sigma,k)=\frac{2\sqrt{2\sigma^2}}{\Gamma\left(\frac{k}{2}\right)}\left(\frac{1}{\sqrt{2\sigma^2}x}\right)^{k+1}exp\left\{-\frac{1}{2\sigma^2x^2}\right\},x,k,\sigma>0 \tag{4.21}$$

$$E(X^r)=\frac{\left(\frac{1}{\sqrt{2\sigma^2}}\right)^r\Gamma\left(\frac{k}{2}-\frac{r}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X)=\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X^2)=\frac{\left(\frac{1}{\sqrt{2\sigma^2}}\right)^2\Gamma\left(\frac{k}{2}-\frac{2}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$= \frac{\frac{1}{2\sigma^2}\Gamma\left(\frac{k-2}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$= \frac{\left(\frac{k-4}{2}\right)!}{2\sigma^2\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)!} = \frac{1}{2\sigma^2(k-2)}$$

$$E(X^3) = \frac{\left(\frac{1}{\sqrt{2\sigma^2}}\right)^3\Gamma\left(\frac{k}{2}-\frac{3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$= \frac{\left(\frac{1}{\sqrt{2\sigma^2}}\right)^3\Gamma\left(\frac{k-3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$E(X^4) = \frac{\left(\frac{1}{\sqrt{2\sigma^2}}\right)^4\Gamma\left(\frac{k}{2}-\frac{4}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\Gamma\left(\frac{k-4}{2}\right)}{4\sigma^4\Gamma\left(\frac{k}{2}\right)}$$

$$= \frac{\left(\frac{k-6}{2}\right)!}{4\sigma^4\left(\frac{k-2}{2}\right)\left(\frac{k-4}{2}\right)\left(\frac{k-6}{2}\right)!}$$

$$= \frac{1}{4\sigma^4(k-2)(k-4)}$$

$$Var\ X = \frac{1}{2\sigma^2(k-2)} - \left(\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2$$

Skewness: $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X-\mu)^3$$

$$= E(X^3) - 3\mu\, E(X^2) + 2\mu^3$$

$$= \frac{\left(\frac{1}{\sqrt{2\sigma^2}}\right)^3\Gamma\left(\frac{k-3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 3\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} * \frac{1}{2\sigma^2(k-2)} + 2\left(\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^3$$

$$\gamma_1 = \frac{\frac{\left(\frac{1}{\sqrt{2\sigma^2}}\right)^3 \Gamma\left(\frac{k-3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - 3\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} * \frac{1}{2\sigma^2(k-2)} + 2\left(\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^3}{\left(\frac{1}{2\sigma^2(k-2)} - \left(\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2\right)^{\frac{3}{2}}}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= \frac{1}{4\sigma^4(k-2)(k-4)} - 4\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} * \frac{\left(\frac{1}{\sqrt{2\sigma^2}}\right)^3 \Gamma\left(\frac{k-3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$+ 6\left(\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 \frac{1}{2\sigma^2(k-2)} - 3\left(\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^4$$

$$\gamma_2 = \left(\frac{\frac{1}{4\sigma^4(k-2)(k-4)} - 4\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} * \frac{\left(\frac{1}{\sqrt{2\sigma^2}}\right)^3 \Gamma\left(\frac{k-3}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} + 6\left(\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 \frac{1}{2\sigma^2(k-2)} - 3\left(\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^4}{\left(\frac{1}{2\sigma^2(k-2)} - \left(\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2\right)^2}\right)$$

4.2.16 Wilson-Hilferty distribution (Wilson, *et al.*, *1931*, Johnson, *et al.*, 1994)

Stacy$(\theta, \alpha, 3) = WilsonHilferty(\theta, \alpha)$

$$f\left(\frac{x}{0}, \alpha\right) = \frac{3}{|\theta|\Gamma(\alpha)}\left(\frac{x}{\theta}\right)^{3\alpha-1} exp\left\{-\left(\frac{x}{\theta}\right)^{3}\right\} \; ; \; x, \alpha > 0 \qquad (4.22)$$

$$E(X^r) = \frac{\theta^r \Gamma\left(\alpha + \frac{r}{3}\right)}{\Gamma(\alpha)}$$

$$E(X) = \frac{\theta \Gamma\left(\alpha + \frac{1}{3}\right)}{\Gamma(\alpha)}$$

$$E(X^2) = \frac{\theta^2 \Gamma\left(\alpha + \frac{2}{3}\right)}{\Gamma(\alpha)}$$

$$E(X^3) = \frac{\theta^3 \Gamma(\alpha + 1)}{\Gamma(\alpha)}$$

$$E(X^4) = \frac{\theta^4 \Gamma\left(\alpha + \frac{4}{3}\right)}{\Gamma(\alpha)}$$

$$Var\, X = \frac{\theta^2 \Gamma\left(\alpha + \frac{2}{3}\right)}{\Gamma(\alpha)} - \left(\frac{\theta \Gamma\left(\alpha + \frac{1}{3}\right)}{\Gamma(\alpha)}\right)^2$$

$$= \theta^2 \left[\frac{\Gamma(\boldsymbol{\alpha})\Gamma\left(\boldsymbol{\alpha}+\frac{2}{3}\right) - (\Gamma\left(\boldsymbol{\alpha}+\frac{1}{3}\right))^2}{(\Gamma(\boldsymbol{\alpha}))^2}\right]$$

Skewness: $\gamma_1 = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = E(X - \mu)^3$$

$$= E(X^3) - 3\mu\, E(X^2) + 2\mu^3$$

$$= \frac{\theta^3 \Gamma(\alpha + 1)}{\Gamma(\alpha)} - 3\frac{\theta \Gamma\left(\alpha + \frac{1}{3}\right)}{\Gamma(\alpha)} * \frac{\theta^2 \Gamma\left(\alpha + \frac{2}{3}\right)}{\Gamma(\alpha)} + 2\left(\frac{\theta \Gamma\left(\alpha + \frac{1}{3}\right)}{\Gamma(\alpha)}\right)^3$$

$$\gamma_1 = \frac{\frac{\theta^3\Gamma(\alpha+1)}{\Gamma(\alpha)} - 3\frac{\theta\Gamma\left(\alpha+\frac{1}{3}\right)}{\Gamma(\alpha)} * \frac{\theta^2\Gamma\left(\alpha+\frac{2}{3}\right)}{\Gamma(\alpha)} + 2\left(\frac{\theta\Gamma\left(\alpha+\frac{1}{3}\right)}{\Gamma(\alpha)}\right)^3}{\left(\frac{\theta^2\Gamma\left(\alpha+\frac{2}{3}\right)}{\Gamma(\alpha)} - \left(\frac{\theta\Gamma\left(\alpha+\frac{1}{3}\right)}{\Gamma(\alpha)}\right)^2\right)^{\frac{3}{2}}}$$

Kurtosis: $\gamma_2 = \frac{\mu_4}{\sigma^4}$

$$\mu_4 = E(X-\mu)^4$$

$$= E(X^4) - 4\mu\, E(X^3) + 6\mu^2\, E(X^2) - 3\mu^4$$

$$= \frac{\theta^4\Gamma\left(\alpha+\frac{4}{3}\right)}{\Gamma(\alpha)} - 4\frac{\theta\Gamma\left(\alpha+\frac{1}{3}\right)}{\Gamma(\alpha)} * \frac{\theta^3\Gamma(\alpha+1)}{\Gamma(\alpha)} + 6\left(\frac{\theta\Gamma\left(\alpha+\frac{1}{3}\right)}{\Gamma(\alpha)}\right) *^2$$

$$\frac{\theta^2\Gamma\left(\alpha+\frac{2}{3}\right)}{\Gamma(\alpha)} - 3\left(\frac{\theta\Gamma\left(\alpha+\frac{1}{3}\right)}{\Gamma(\alpha)}\right)^2$$

$$\gamma_2 = \left(\frac{\frac{\theta^4\Gamma\left(\alpha+\frac{4}{3}\right)}{\Gamma(\alpha)} - 4\frac{\theta\Gamma\left(\alpha+\frac{1}{3}\right)}{\Gamma(\alpha)} * \frac{\theta^3\Gamma(\alpha+1)}{\Gamma(\alpha)} + 6\left(\frac{\theta\Gamma\left(\alpha+\frac{1}{3}\right)}{\Gamma(\alpha)}\right) *^2 \frac{\theta^2\Gamma\left(\alpha+\frac{2}{3}\right)}{\Gamma(\alpha)} - 3\left(\frac{\theta\Gamma\left(\alpha+\frac{1}{3}\right)}{\Gamma(\alpha)}\right)^2}{\left(\frac{\theta^2\Gamma\left(\alpha+\frac{2}{3}\right)}{\Gamma(\alpha)} - \left(\frac{\theta\Gamma\left(\alpha+\frac{1}{3}\right)}{\Gamma(\alpha)}\right)^2\right)^2}\right)$$

4.3 Different Parameterizations of Stacy Distribution

4.3.1 Stacy (1962) parameterization

$GG(\beta, \theta, \alpha) = GG(a, p, \frac{d}{p})$

$$f(x) = \frac{p}{a^d \Gamma\left(\frac{d}{p}\right)} e^{-\left(\frac{x}{a}\right)^p} x^{d-1} \; ; \; x > 0, p, a, d > 0 \qquad (4.23)$$

$$E(X^r) = \frac{a^r \Gamma\left(\frac{d+r}{p}\right)}{\Gamma\left(\frac{d}{p}\right)}$$

$$E(X) = \frac{a\,\Gamma\left(\frac{d+1}{p}\right)}{\Gamma\left(\frac{d}{p}\right)}$$

$$VarX = \frac{a^2}{\Gamma\left(\frac{d}{p}\right)}\left[\Gamma\left(\frac{d+2}{p}\right) - \Gamma\left(\frac{d-1}{p}\right)\right]$$

4.3.2 McDonald (1984) parameterization

$GG(\beta, \theta, \alpha) = GG(a, \beta,$

$$f(x) = \frac{a}{\beta^{ap}\Gamma(p)} e^{-(\frac{x}{\beta})^a} x^{ap-1} \; ; \; x > 0, \mathrm{p}, a, \beta > 0 \qquad (4.24)$$

$$E(X^r) = \frac{\beta^r \Gamma(p + \frac{r}{a})}{\Gamma(p)}$$

$$E(X) = \frac{\beta^a \Gamma(p + \frac{1}{a})}{\Gamma(p)}$$

$$Var\,X = \frac{\beta^{2a}\Gamma(p+\frac{1}{a})}{(\Gamma(p))^2}\left\{\Gamma(p)\Gamma\left(p + \frac{2}{a}\right) - \left(\Gamma(p + \frac{1}{a})\right)^2\right\}$$

4.3.3 Khodabin-Ahmahabadi (2010) parameterization

$G(\beta, \theta, \alpha) = GG(\tau, \lambda, \alpha)$

$$f(x) = \frac{\tau}{\lambda^{\alpha\tau}\,\Gamma(\alpha)} e^{-\left(\frac{x}{\lambda}\right)^\tau} x^{\tau\alpha-1} \; ; \; x \geq 0, \alpha, \tau, \lambda > 0 \qquad (4.25)$$

$$E(X^r) = \frac{\lambda^r}{\Gamma(\alpha)} \Gamma(\alpha + \frac{r}{\tau})$$

$$E(X) = \frac{\lambda}{\Gamma(\alpha)} \Gamma(\alpha + \frac{1}{\tau})$$

$$Var\ X = \lambda^{2\tau} \left[\frac{\Gamma(\alpha)\Gamma\left(\alpha + \frac{2}{\tau}\right) - (\Gamma(\alpha + \frac{1}{\tau}))^2}{(\Gamma(\alpha))^2} \right]$$

4.3.4 Taguchi (1980) parameterization

$GG(\beta, \theta, \alpha) = GG(h\alpha, \beta, \frac{1}{h})$

$$f(x) = \frac{h\alpha}{\beta^\alpha \Gamma\left(\frac{1}{h}\right)} e^{-\left(\frac{x}{\beta}\right)^{h\alpha}} x^{\alpha-1};\ x > 0,\ \ \frac{1}{h} \geq 1, \alpha, \beta > 0 \qquad (4.26)$$

$$E(X^r) = \frac{\beta^r \Gamma(\frac{1}{h} + \frac{r}{ha})}{\Gamma\left(\frac{1}{h}\right)}$$

$$E(X) = \frac{\beta \Gamma(\frac{1}{h} + \frac{1}{ha})}{\Gamma\left(\frac{1}{h}\right)}$$

$$Var\ X = \beta^2 \left[\frac{\Gamma\left(\frac{1}{h}\right) \Gamma\left(\alpha + \frac{2}{ha}\right) - (\Gamma(\frac{1}{h} + \frac{1}{ha}))^2}{(\Gamma(\frac{1}{h}))^2} \right]$$

Summary of Stacy (three-parameters) distribution

	Name of the distribution	Parameters			Mean	Variance
		θ	α	β		
1	Two parameter gamma	θ	α	1	$\theta\alpha$	$\theta^2\alpha$
2	Half-Normal	$\sqrt{2\sigma^2}$	$\frac{1}{2}$	2	$\sigma\sqrt{\frac{2}{\pi}}$	$\sigma^2(1-\frac{2}{\pi})$
3	Chi-Square	2	$\frac{k}{2}$	1	k	$2k$
4	Scaled Chi-Square	$2\sigma^2$	$\frac{k}{2}$	1	$\sigma^2 k$	$2\sigma^8 k$
5	Scaled Inverse Chi-Square	$\frac{1}{\sqrt{2\sigma^2}}$	$\frac{k}{2}$	-1	$\frac{1}{\sigma^2(k-2)}$	$\frac{2}{\sigma^4(k-2)^2(k-4)}$
6	Inverse Chi-Square	$\frac{1}{2}$	$\frac{k}{2}$	-1	$\frac{1}{(k-2)}$	$\frac{2}{(k-2)^2(k-4)}$
7	Chi	$\sqrt{2}$	$\frac{k}{2}$	2	$\frac{\sqrt{2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}$	$k-\left(\frac{\sqrt{2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^2$
8	Scaled Chi	$\sqrt{2\sigma^2}$	$\frac{k}{2}$	2	$\frac{\sqrt{2\sigma^2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}$	$\sigma^2 k-\left(\frac{\sqrt{2\sigma^2}\,\Gamma(\frac{k+1}{2})}{\Gamma\left(\frac{k}{2}\right)}\right)^2$
9	Rayleigh	$\sqrt{2\sigma^2}$	1	2	$\sigma\sqrt{\frac{\pi}{2}}$	$\sigma^2\left(\frac{4-\pi}{2}\right)$
10	Inverse Rayleigh	$\frac{1}{\sqrt{2\sigma^2}}$	1	-2	$\frac{1}{\sigma}\sqrt{\frac{\pi}{2}}$	∞
11	Maxwell	$\sqrt{2\sigma^2}$	$\frac{3}{2}$	2	$2\sigma\sqrt{\frac{2}{\pi}}$	$\sigma^2(3-\frac{8}{\pi})$

12	Inverse gamma	θ	α	1	$\frac{\theta}{(\alpha-1)}$	$\frac{\theta^2}{(\alpha-1)^2(\alpha-2)}$
13	Inverse Exponential	θ	1	1	∞	∞
14	Inverse Chi	$\frac{1}{\sqrt{2\sigma^2}}$	$\frac{k}{2}$	-2	$\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$	$\frac{1}{k-2}-\left(\frac{\frac{1}{\sqrt{2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2$
15	Scaled Inverse Chi	$\frac{1}{\sqrt{2\sigma^2}}$	$\frac{k}{2}$	-2	$\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$	$\frac{1}{2\sigma^2(k-2)}-\left(\frac{\frac{1}{\sqrt{2\sigma^2}}\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2$
16	Wilson-Hilferty	θ	α	3	$\frac{\theta\Gamma\left(\alpha+\frac{1}{3}\right)}{\Gamma(\alpha)}$	$\theta^2\left[\frac{\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{3}\right)-(\Gamma\left(\alpha+\frac{1}{3}\right))^2}{(\Gamma(\alpha))^2}\right]$

CHAPTER V: Stacy's Related Distributions

5.0 Introduction

In this chapter we will find the sum, power, product and quotient of Stacy independent distributed random variables by use of cumulative and change of variables techniques.

5.1 Sum of Stacy distributed random variables, by Stacy 1962

Let $X_i, \ldots, X_n$ be an independent set of random variables, X_i having frequency

$$f(x_i, a_i, d_i, p_i), i = 1, \ldots, n \tag{5.0}$$

Also let $Y = \sum_{i=1}^{n} X_i$ and cumulative distribution function of Y be given by $G_n(y)$ and the corresponding frequency function be given by $g_n(y)$

$$G_n(y) = \beta_n y^d \sum_{j=0}^{\infty} (-1)^j A_j \tag{5.1}$$

where $d = \sum_{i=1}^{n} d_i$ and $\beta_n = \prod_{i=1}^{n} \left[\frac{p_i}{{a_i}^{d_i}} \Gamma(\frac{d_i}{p_i}) \right]$,

$$A_j = \sum_{k_1+\cdots,k_n=j} \frac{y^{\sum_{i=1}^{n} {p_i}^{k_i}}}{\Gamma(d + \sum_{i=1}^{n} {p_i}^{k_i} + 1)} \prod_{i=1}^{n} \left[\frac{\Gamma(d_i + p_i k_i)}{k_i! {a_i}^{p_i k_i}} \right]$$

Differentiation of equation (5.1)

$$g_n(y) = \beta_n y^{d-1} \sum_{j=0}^{\infty} (-1)^j \overline{A}_j \tag{5.2}$$

where $\bar{A}_j = \sum_{k_1+\cdots,k_n=j} \frac{y^{\sum_{i=1}^{n} {p_i}^{k_i}}}{\Gamma(d + \sum_{i=1}^{n} {p_i}^{k_i})} \prod_{i=1}^{n} \left[\frac{\Gamma(d_i + p_i k_i)}{k_i! {a_i}^{p_i k_i}} \right]$

No general method have been found for easy evaluation of quantities $\bar{A}_j$.

5.2 Power transformation Stacy distributed random variables

Let X be a random variable with probability density function given by

$$f(x; a, d, p) = \frac{p}{a^a \Gamma(\frac{d}{p})} x^{d-1} e^{-\left(\frac{x}{a}\right)^p}; x, a, d, p > 0 \tag{5.3}$$

Let $Y = X^m \; where \; X \sim f(x; a, d, p)$ as shown in (5.3)

Therefore $x = y^{\frac{1}{m}} \Longrightarrow \frac{dx}{dy} = \frac{1}{m} y^{\frac{1}{m}-1}$

Thus $g(y) = \frac{p}{a^d \Gamma(\frac{d}{p})} e^{-\left(\frac{x}{a}\right)^p} x^{d-1} * \frac{1}{m} y^{\frac{1}{m}-1}$

$$= \frac{p}{ma^d\Gamma(\frac{d}{p})} e^{-\left(\frac{y^{\frac{1}{m}}}{a}\right)^p} (y^{\frac{1}{m}})^{d-1} * y^{\frac{1}{m}-1}$$

$$G(a^m, \frac{d}{m}, \frac{p}{m}) = \frac{p}{ma^d\Gamma(\frac{d}{p})} e^{-\frac{y^{\frac{p}{m}}}{a^d}} * y^{\frac{d}{m}-1}$$

$$= \frac{\frac{p}{m}}{a^d\Gamma(\frac{d}{p})} e^{-\left(\frac{y}{a^m}\right)^{\frac{p}{m}}} * y^{\frac{d}{m}-1} \quad ; y, d, p, m, a > 0 \qquad (5.4)$$

5.3 Product of Stacy distributed random variables

Derivation of the distribution of $U = XY$ where $X and\ Y$ are independently distributed with respective frequency functions $f(x;\ a_1, d_1, p)$ and $f(x;\ a_2, d_2, p)$. Let $X = Z$

Then $Y = \frac{U}{Z}$ and $X = Z$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{z} & \frac{-u}{z^2} \end{vmatrix} = \frac{1}{z}$$

$$f(t, w) = f(x, y)|J| = \frac{p}{{a_1}^{d_1}\Gamma(\frac{d_1}{p})} z^{d_1-1} e^{-\left(\frac{z}{a_1}\right)^p} * \frac{p}{{a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} \left(\frac{u}{z}\right)^{d_2-1} e^{-\left(\frac{u}{a_2 z}\right)^p} * \frac{1}{z}$$

$$= \frac{p^2}{{a_1}^{d_1}\Gamma(\frac{d_1}{p}){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} z^{d_1-1} e^{-\left(\frac{z}{a_1}\right)^p} \left(\frac{u}{z}\right)^{d_2-1} e^{-\left(\frac{u}{a_2 z}\right)^p} * \frac{1}{z}$$

$$f(u, z) = \frac{p^2 u^{d_2-1}}{{a_1}^{d_1}\Gamma\left(\frac{d_1}{p}\right){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} z^{d_1-d_2-1} e^{-\left(\frac{z}{a_1}\right)^p} e^{-\left(\frac{u}{a_2 z}\right)^p} ; d_1, d_2, a_1, a_{1,} p > 0$$

$$f(u) = \int_{\mathbf{0}}^{\infty} f(u, z) dz$$

$$= \frac{p^2 u^{d_2-1}}{{a_2}^{d_2}{a_1}^{d_1}\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)} \int_0^\infty z^{d_1-d_2-1} e^{-\left(\frac{z}{a_1}\right)^p} e^{-\left(\frac{u}{a_2 z}\right)^p} dz$$

Let $t = \left(\frac{z}{a_1}\right)^p \Longrightarrow z = a_1 t^{\frac{1}{p}}\ thus\ dz = \frac{a_1}{p} t^{\frac{1}{p}-1} dt$

$$= \frac{p u^{d_2-1}}{{a_2}^{d_2}{a_1}^{d_1}\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)} \int_0^\infty (a_1 t^{\frac{1}{p}})^{-(d_2-d_1+1)} e^{-t-\left(\frac{u}{a_2 a_1 t^{\frac{1}{p}}}\right)^p} \frac{a_1}{p} t^{\frac{1}{p}-1} dt$$

$$= \frac{pu^{d_2-1}}{(a_1a_2)^{d_2}\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)}\int_0^\infty t^{-(\frac{d_2}{p}-\frac{d_1}{p}+1)} \quad e^{-t-\frac{u^p}{a_1{}^p a_2{}^p t}}dt$$

Let $\quad \frac{z^2}{4t} = \frac{u^p}{a_1{}^p a_2{}^p t} \Longrightarrow z = 2(\frac{u}{a_1a_2})^{\frac{p}{2}}$

Therefore

$$f(u) = \frac{2pu^{d_2-1}}{(a_1a_2)^{d_2}\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)}\left[\left(\frac{u}{a_1a_2}\right)^{\frac{p}{2}}\right]^{-\left(\frac{d_2}{p}-\frac{d_1}{p}\right)} K_{\frac{d_2}{p}-\frac{d_1}{p}}\left(2\left(\frac{u}{a_1a_2}\right)^{\frac{p}{2}}\right), u > 0 \qquad (5.5\,a)$$

Alternatively

Let $U = XY$

$H(u) = Prob(U \leq u) = Prob(XY \leq u)$

$$= Prob(X \leq \frac{u}{y} \quad 0 < y < \infty)$$

$$= \int_0^\infty \int_0^{\frac{u}{y}} f_X(x)f_Y(y)dxdy$$

$$= \int_0^\infty F_X(\frac{u}{y})f_Y(y)dy$$

$$h(y) = \int_0^\infty \frac{1}{y} f_X(\frac{u}{y})f_Y(y)dy$$

$$= \frac{p^2}{a_1{}^{d_1}\Gamma(\frac{d_1}{p})a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)}\int_0^\infty \frac{1}{y}\left(\frac{u}{y}\right)^{d_2-1} y^{d_1-1}e^{-\left(\frac{u}{a_2y}\right)^p-\left(\frac{y}{a_1}\right)^p}dy$$

$$= \frac{p^2u^{d_2-1}}{a_1{}^{d_1}\Gamma(\frac{d_1}{p})a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)}\int_0^\infty y^{-(d_2-d_1+1}e^{-\left(\frac{u}{a_2y}\right)^p-\left(\frac{y}{a_1}\right)^p}dy$$

$Let\ t = \left(\frac{y}{a_1}\right)^p \Longrightarrow y = a_1t^{\frac{1}{p}}, dy = \frac{a_1}{p}t^{\frac{1}{p}-1}dt$

$$= \frac{a_1{}^{d_1+d_2}p^2u^{d_2-1}}{a_1{}^{d_1}\Gamma(\frac{d_1}{p})a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)}\int_0^\infty \left(t^{\frac{1}{p}}\right)^{-(d_2-d_1+1)} e^{-\left(\frac{u}{a_2a_1t^{\frac{1}{p}}}\right)^p-t} \frac{t^{\frac{1}{p}-1}}{p}dt$$

$$= \frac{pu^{d_2-1}}{(a_1a_2)^{d_2}\Gamma(\frac{d_1}{p})\Gamma\left(\frac{d_2}{p}\right)}\int_0^\infty t^{-(\frac{d_2}{p}-\frac{d_1}{p}+1)} \quad e^{-t-\frac{u^p}{a_1{}^pa_2{}^pt}}dt$$

Recall $\int_0^\infty t^{-(v+1)}e^{-t-\frac{z^2}{4t}}dt = 2\left(\frac{z}{2}\right)^{-v}K_v(z)$

Let $\frac{z^2}{4t} = \frac{u^p}{a_1{}^pa_2{}^pt} \Longrightarrow z = 2(\frac{u}{a_1a_2})^{\frac{p}{2}}$

Therefore

$$f(u) = \frac{2pu^{d_2-1}}{(a_1a_2)^{d_2}\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)}\left[\left(\frac{u}{a_1a_2}\right)^{\frac{p}{2}}\right]^{-\left(\frac{d_2}{p}-\frac{d_1}{p}\right)} K_{\frac{d_2}{p}-\frac{d_1}{p}}\left(2\left(\frac{u}{a_1a_2}\right)^{\frac{p}{2}}\right), u > 0 \qquad (5.5\,b)$$

5.4 Quotient of Stacy distributed random variables

Case 1

Derivation of the distribution of $T = \frac{X}{Y}$ where $X and\ Y$ are independently distributed with respective frequency functions $f(x;\ a_1, d_1, p)$ and $f(x;\ a_2, d_2, p)$

Let $T = \frac{X}{Y}$ and $W = X + Y$

$$x = \frac{tw}{t+1} \quad and\ y = \frac{w}{t+1}$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{w}{(t+1)^2} & \frac{t}{t+1} \\ \frac{-w}{(t+1)^2} & \frac{1}{t+1} \end{vmatrix} = \frac{w}{(t+1)^2}$$

$$f(t,w) = f(x,y)|J| = \frac{p}{a_1{}^{d_1}\Gamma(\frac{d_1}{p})}x^{d_1-1}e^{-\left(\frac{x}{a_1}\right)^p} * \frac{p}{a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)}y^{d_2-1}e^{-\left(\frac{y}{a_2}\right)^p} * \frac{w}{(t+1)^2}$$

$$= \frac{p^2}{a_1{}^{d_1}\Gamma(\frac{d_1}{p})a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)}x^{d_1-1}y^{d_2-1}e^{-\left(\frac{x}{a_1}\right)^p-\left(\frac{y}{a_2}\right)^p} * \frac{w}{(t+1)^2}$$

$$= \frac{p^2}{a_1{}^{d_1}\Gamma(\frac{d_1}{p})a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)}\left(\frac{tw}{t+1}\right)^{d_1-1}\left(\frac{w}{t+1}\right)^{d_2-1}e^{-\left(\frac{\frac{tw}{t+1}}{a_1}\right)^p-\left(\frac{\frac{w}{t+1}}{a_2}\right)^p} * \frac{w}{(t+1)^2}$$

$$= \frac{p^2}{a_1{}^{d_1}\Gamma(\frac{d_1}{p})a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)}\left(\frac{tw}{t+1}\right)^{d_1-1}\left(\frac{w}{t+1}\right)^{d_2-1}e^{-\left(\frac{tw}{(t+1)a_1}\right)^p-\left(\frac{w}{(t+1)a_2}\right)^p} * \frac{w}{(t+1)^2}$$

$$= \frac{p^2}{{a_1}^{d_1}\Gamma(\frac{d_1}{p}){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} \frac{t^{d_1-1}w^{d_1+d_2-1}}{(t+1)^{d_1+d_2}} e^{-\frac{w^p}{(t+1)^p}\left[\frac{t^p}{{a_1}^p}+\frac{1}{{a_2}^p}\right]}$$

$$= \frac{p^2}{{a_1}^{d_1}\Gamma(\frac{d_1}{p}){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} \frac{t^{d_1-1}w^{d_1+d_2-1}}{(t+1)^{d_1+d_2}} e^{-\frac{w^p}{(t+1)^p{a_1}^p{a_2}^p}[{a_2}^p t^p+{a_1}^p]}$$

$$f(t) = \int_0^\infty \frac{p^2}{{a_1}^{d_1}\Gamma(\frac{d_1}{p}){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} \frac{t^{d_1-1}w^{d_1+d_2-1}}{(t+1)^{d_1+d_2}} e^{-\frac{w^p}{(t+1)^p{a_1}^p{a_2}^p}[{a_2}^p t^p+{a_1}^p]} dw$$

$$= \frac{p^2}{{a_1}^{d_1}\Gamma(\frac{d_1}{p}){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} \frac{t^{d_1-1}}{(t+1)^{d_1+d_2}} \int_0^\infty w^{d_1+d_2-1} e^{-\frac{w^p}{(t+1)^p{a_1}^p{a_2}^p}[{a_2}^p t^p+{a_1}^p]} dw$$

Let $u = \frac{w^p}{(t+1)^p{a_1}^p{a_2}^p}[{a_2}^p t^p + {a_1}^p] \Rightarrow w = \left(\frac{u(t+1)^p{a_1}^p{a_2}^p}{[{a_2}^p t^p+{a_1}^p]}\right)^{\frac{1}{p}}$

$$dw = \frac{\frac{1}{p}u^{\frac{1}{p}-1}(t+1)a_1a_2}{([{a_2}^p t^p + {a_1}^p])^{\frac{1}{p}}} du$$

Therefore

$$f(t) = \frac{p^2}{{a_1}^{d_1}\Gamma\left(\frac{d_1}{p}\right){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} \frac{t^{d_1-1}}{(t+1)^{d_1+d_2}} \int_0^\infty \left(\frac{u^{\frac{1}{p}}(t+1)a_1a_2}{([{a_2}^p t^p + {a_1}^p])^{\frac{1}{p}}}\right)^{d_1+d_2-1}$$

$$* e^{-u} \frac{\frac{1}{p}u^{\frac{1}{p}-1}(t+1)a_1a_2}{([{a_2}^p t^p + {a_1}^p])^{\frac{1}{p}}} du$$

$$= \frac{p^2}{{a_1}^{d_1}\Gamma(\frac{d_1}{p}){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} \frac{t^{d_1-1}}{(t+1)^{d_1+d_2}\left(\left[1+\frac{{a_2}^p t^p}{{a_1}^p}\right]\right)^{\frac{d_1}{p}+\frac{d_2}{p}}} \int_0^\infty u^{\frac{d_1}{p}+\frac{d_2}{p}-\frac{1}{p}} (t+1)^{d_1+d_2}(a_1a_2)^{d_1+d_2}$$

$$* \frac{1}{p}u^{\frac{1}{p}-1}e^{-u}du$$

$$= \frac{pt^{d_1-1}(a_1a_2)^{d_1+d_2}}{{a_1}^{d_1}\Gamma(\frac{d_1}{p}){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)\left(\left[1+\frac{{a_2}^p t^p}{{a_1}^p}\right]\right)^{\frac{d_1}{p}+\frac{d_2}{p}}} \int_0^\infty u^{\frac{d_1}{p}+\frac{d_2}{p}-1} e^{-u}du$$

$$= \frac{p a_1{}^{-d_1} a_2{}^{d_1} t^{d_1-1}}{\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)\left(\left[1+\frac{a_2{}^p t^p}{a_1{}^p}\right]\right)^{\frac{d_1}{p}+\frac{d_2}{p}}} \Gamma(\frac{d_1}{p}+\frac{d_2}{p})$$

$$f(t) = \frac{p a_1{}^{-d_1} a_2{}^{d_1} t^{d_1-1}}{\mathrm{B}\left(\frac{d_1}{p},\frac{d_2}{p}\right)\left(\left[1+\frac{a_2{}^p t^p}{a_1{}^p}\right]\right)^{\frac{d_1}{p}+\frac{d_2}{p}}};\ t > 0, d_{1,} d_{2,} p, , a_1, a_2 > 0 \qquad (5.6\ a)$$

Alternatively

Let $H(t) = Prob\ (T \le t) = Prob\left(\frac{X}{Y} \le t\right) = Prob(X \le ty\ ; 0 < y < \infty)$

$$= \int_0^\infty F_X(ty) f_Y(y) dy$$

$$h(t) = \int_0^\infty y * f_X(ty) f_Y(y) dy$$

$$= \int_0^\infty y * \frac{p}{a_1{}^{d_1}\Gamma(\frac{d_1}{p})} (ty)^{d_1-1} e^{-\left(\frac{ty}{a_1}\right)^p} * \frac{p}{a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} y^{d_2-1} e^{-\left(\frac{y}{a_2}\right)^p} * dy$$

$$= \int_0^\infty \frac{p^2\ t^{d_1-1}}{a_1{}^{d_1}\Gamma(\frac{d_1}{p}) a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} y^{d_1+d_2-1} e^{-\left(\frac{ty}{a_1}\right)^p - \left(\frac{y}{a_2}\right)^p} dy$$

$$= \frac{p^2\ t^{d_1-1}}{a_1{}^{d_1}\Gamma(\frac{d_1}{p}) a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} \int_0^\infty y^{d_1+d_2-1} e^{-y^p\left(\frac{a_2{}^p t^p + a_1{}^p}{a_1{}^p a_2{}^p}\right)} dy$$

Let $u = y^p\left(\frac{a_2{}^p t^p + a_1{}^p}{a_1{}^p a_2{}^p}\right) \Longrightarrow y = \frac{u^{\frac{1}{p}} a_1 a_2}{(a_2{}^p t^p + a_1{}^p)^{\frac{1}{p}}}$

$$dy = \frac{1}{p} \frac{u^{\frac{1}{p}-1} a_1 a_2}{(a_2{}^p t^p + a_1{}^p)^{\frac{1}{p}}} du$$

$$= \frac{p^2\ t^{d_1-1}}{a_1{}^{d_1}\Gamma(\frac{d_1}{p}) a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} \int_0^\infty \left(\frac{u^{\frac{1}{p}} a_1 a_2}{(a_2{}^p t^p + a_1{}^p)^{\frac{1}{p}}}\right)^{d_1+d_2-1} e^{-u}\ \frac{1}{p} \frac{u^{\frac{1}{p}-1} a_1 a_2}{(a_2{}^p t^p + a_1{}^p)^{\frac{1}{p}}} du$$

$$= \frac{p\ t^{d_1-1} a_1{}^{d_2} a_2{}^{d_1}}{\Gamma(\frac{d_1}{p})\Gamma\left(\frac{d_2}{p}\right) a_1{}^{d_1+d_2}\left(1+\frac{a_2{}^p t^p}{a_1{}^p}\right)^{\frac{d_1}{p}+\frac{d_2}{p}}} \int_0^\infty u^{\frac{d_1}{p}+\frac{d_2}{p}-1} e^{-u}\ du$$

$$= \frac{p\, t^{d_1-1} a_1{}^{-d_1} a_2{}^{d_1}}{\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)\left(1+\frac{a_2{}^p t^p}{a_1{}^p}\right)^{\frac{d_1}{p}+\frac{d_2}{p}}}\Gamma\left(\frac{d_1}{p}+\frac{d_2}{p}\right)$$

$$f(t) = \frac{p\, a_1{}^{-d_1} a_2{}^{d_1} t^{d_1-1}}{\mathrm{B}\left(\frac{d_1}{p},\frac{d_2}{p}\right)\left(1+\frac{a_2{}^p t^p}{a_1{}^p}\right)^{\frac{d_1}{p}+\frac{d_2}{p}}} \;\; ; \; t > 0, d_1, d_2, a_1, a_2, p > 0 \qquad (5.6\,b)$$

Case 2

Derivation of the distribution of $Y_2 = \frac{X_1}{X_1 + X_2}$ $where\ X_1\ and\ X_2$ are mutually independent, and $let\ Y_1 = X_1 + X_2\ where\ X_i$ has a gamma (α_i, β_i) distribution with pdf

$$f(x_i) = \frac{p}{a_i^{d_i}\Gamma(\frac{d_i}{p})} x_i^{d_i-1} e^{-\left(\frac{x_i}{a_i}\right)^p} \quad x_i > 0, \frac{d_i}{p}, p, a_i > 0, i = 1, 2 \qquad (5.7)$$

Thus

$$x_1 = y_1 y_2\ and\ x_2 = y_1(1-y_2)$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = -y_1$$

$$f(y_1, y_2) = f(x_1, x_2) * |J|$$

$$= \frac{p^2}{a_1{}^{d_1}\Gamma(\frac{d_1}{p}) a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} (y_1 y_2)^{d_1-1} e^{-\left(\frac{y_1 y_2}{a_1}\right)^p} \left(y_1(1-y_2)\right)^{d_2-1} e^{-\left(\frac{y_1(1-y_2)}{a_2}\right)^p} * y_1$$

$$= \frac{p^2 y_2{}^{d_1-1}(1-y_2)^{d_2-1}}{a_1{}^{d_1}\Gamma(\frac{d_1}{p}) a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} y_1{}^{d_1+d_2-1} e^{-\left(\frac{y_1 y_2}{a_1}\right)^p - \left(\frac{y_1(1-y_2)}{a_2}\right)^p}$$

$$f(y_1, y_2) = \frac{p^2 y_2{}^{d_1-1}(1-y_2)^{d_2-1}}{a_1{}^{d_1}\Gamma(\frac{d_1}{p}) a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} y_1{}^{d_1+d_2-1} e^{-y_1{}^p\left(\frac{a_2{}^p y_2{}^p + a_1{}^p(1-y_2)^p}{a_1{}^p a_2{}^p}\right)}; \; y_{1,} y_2 > d_1, d_2, a_1, a_2, p > 0 \quad (5.8)$$

$$f(y_2) = \int_0^\infty f(y_1, y_2)\, dy_1$$

$$f(y_2) = \int_0^\infty \frac{p^2 y_2{}^{d_1-1}(1-y_2)^{d_2-1}}{a_1{}^{d_1}\Gamma(\frac{d_1}{p}) a_2{}^{d_2}\Gamma\left(\frac{d_2}{p}\right)} y_1{}^{d_1+d_2-1} e^{-y_1{}^p\left(\frac{a_2{}^p y_2{}^p + a_1{}^p(1-y_2)^p}{a_1{}^p a_2{}^p}\right)} dy_1$$

$$= \frac{p^2 {y_2}^{d_1 -1}(1-y_2)^{d_2-1}}{{a_1}^{d_1}\Gamma(\frac{d_1}{p}){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)}\int_0^\infty {y_1}^{d_1+d_2-1}e^{-{y_1}^p\left(\frac{{a_2}^p{y_2}^p+{a_1}^p(1-y_2)^p}{{a_1}^p{a_2}^p}\right)}dy_1$$

Let $u = {y_1}^p\left(\frac{{a_2}^p{y_2}^p+{a_1}^p(1-y_2)^p}{{a_1}^p{a_2}^p}\right) \Rightarrow y_1 = \left(\frac{u{a_1}^p{a_2}^p}{{a_2}^p{y_2}^p+{a_1}^p(1-y_2)^p}\right)^{\frac{1}{p}}$

$$dy_1 = \frac{u^{\frac{1}{p}-1}{a_1}^p{a_2}^p}{p({a_2}^p{y_2}^p+{a_1}^p(1-y_2)^p)^{\frac{1}{p}}}du$$

Therefore

$$f(y_2) = \frac{p^2 {y_2}^{d_1 -1}(1-y_2)^{d_2-1}}{{a_1}^{d_1}\Gamma\left(\frac{d_1}{p}\right){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)}\int_0^\infty \left(\left(\frac{u{a_1}^p{a_2}^p}{{a_2}^p{y_2}^p+{a_1}^p(1-y_2)^p}\right)^{\frac{1}{p}}\right)^{d_1+d_2-1} e^{-u}\frac{u^{\frac{1}{p}-1}{a_1}^p{a_2}^p}{p\left({a_2}^p{y_2}^p+{a_1}^p(1-y_2)^p\right)^{\frac{1}{p}}}du$$

$$= \frac{p{y_2}^{d_1-1}(1-y_2)^{d_2-1}(a_1a_2)^{d_1+d_2}}{{a_1}^{d_1}\Gamma\left(\frac{d_1}{p}\right){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)({a_2}^p{y_2}^p+{a_1}^p(1-y_2)^p)^{\frac{d_1}{p}+\frac{d_2}{p}}}\int_0^\infty u^{\frac{d_1}{p}+\frac{d_2}{p}-1}e^{-u}du$$

$$= \frac{p{y_2}^{d_1-1}(1-y_2)^{d_2-1}(a_1a_2)^{d_1+d_2}}{{a_1}^{d_1}\Gamma\left(\frac{d_1}{p}\right){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)({a_2}^p{y_2}^p+{a_1}^p(1-y_2)^p)^{\frac{d_1}{p}+\frac{d_2}{p}}}\Gamma(\frac{d_1}{p}+\frac{d_2}{p})$$

$$= \frac{p{y_2}^{d_1-1}(1-y_2)^{d_2-1}{a_1}^{d_2}{a_2}^{d_1}}{\mathrm{B}(\frac{d_1}{p},\frac{d_2}{p})({a_2}^p{y_2}^p+{a_1}^p(1-y_2)^p)^{\frac{d_1}{p}+\frac{d_2}{p}}} \quad let\, y_2 = t$$

$$f(t) = \frac{pt^{d_1-1}(1-t)^{d_2-1}{a_1}^{d_2}{a_2}^{d_1}}{\mathrm{B}(\frac{d_1}{p},\frac{d_2}{p})({a_2}^p t^p+{a_1}^p(1-t)^p)^{\frac{d_1}{p}+\frac{d_2}{p}}}; t>0, d_1, d_2, a_1, a_2, p > 0 \qquad (5.9\,a)$$

Alternatively

Let $T = \frac{X_1}{X_1 + X_2}$

$$H(t) = \quad Prob\,(T \le t) = Prob\left(\frac{X_1}{X_1+X_2} \le t\right)$$

$$= Prob(X_1 \le t(X_1 + X_2) = Prob(X_1 - X_1\, t \le tX_2)$$

$$= Prob\left(X_1 \le \frac{x_2 t}{1-t}\; 0 < x_2 < \infty\right) = \int_0^\infty \int_0^{\frac{x_2 t}{1-t}} f_{X_1}(x_1)\, f_{X_2}(x_2) dx_1 dx_2$$

$$= \int_0^\infty F_{X_1}(\frac{x_2 t}{1-t}) f_{X_2}(x_2) dx_2$$

$$= \int_0^\infty \frac{x_2}{(1-t)^2} f_{X_1}(\frac{x_2 t}{1-t}) f_{X_2}(x_2) dx_2$$

$$= \int_0^\infty \frac{p^2}{a_1^{d_1}\Gamma(\frac{d_1}{p}) a_2^{d_2}\Gamma\left(\frac{d_2}{p}\right)} \frac{x_2}{(1-t)^2}\left(\frac{x_2 t}{1-t}\right)^{d_1-1} e^{-\left(\frac{x_2 t}{(1-t)a_1}\right)^p} e^{-\left(\frac{x_2}{a_2}\right)^p} dx_2$$

$$= \frac{p^2\, t^{d_1-1}}{a_1^{d_1}\Gamma(\frac{d_1}{p}) a_2^{d_2}\Gamma\left(\frac{d_2}{p}\right)(1-t)^{d_1+1}} \int_0^\infty x_2^{d_1+d_2-1} e^{-x_2^p\left(\frac{a_2^p t^p + (1-t)^p a_1^p}{a_1^p a_2^p (1-t)^p}\right)} dx_2$$

Let $z = x_2^p\left(\frac{a_2^p t^p + (1-t)^p a_1^p}{a_1^p a_2^p (1-t)^p}\right) \Rightarrow x_2 = \left(\frac{z\, a_1^p a_2^p (1-t)^p}{a_2^p t^p + (1-t)^p a_1^p}\right)^{\frac{1}{p}}, dx_2 = \frac{z^{\frac{1}{p}-1} a_1 a_2 (1-t)}{p(a_2^p t^p + (1-t)^p a_1^p)^{\frac{1}{p}}} dz$

$$= \frac{p^2\, t^{d_1-1}}{a_1^{d_1}\Gamma(\frac{d_1}{p}) a_2^{d_2}\Gamma\left(\frac{d_2}{p}\right)(1-t)^{d_1+1}} \int_0^\infty \left(\left(\frac{z\, a_1^p a_2^p (1-t)^p}{a_2^p t^p + (1-t)^p a_1^p}\right)^{\frac{1}{p}}\right)^{d_1+d_2-1} e^{-z} \frac{z^{\frac{1}{p}-1} a_1 a_2 (1-t)}{p(a_2^p t^p + (1-t)^p a_1^p)^{\frac{1}{p}}} dz$$

$$= \frac{a_1^{d_2} a_2^{d_1} p\, t^{d_1-1} (1-t)^{d_1+d_2}}{\Gamma(\frac{d_1}{p})\Gamma\left(\frac{d_2}{p}\right)(1-t)^{d_1+1}(a_2^p t^p + (1-t)^p a_1^p)^{\frac{d_1}{p}+\frac{d_2}{p}}} \int_0^\infty z^{\frac{d_1}{p}+\frac{d_2}{p}-1} e^{-z} dz$$

$$f(t) = \frac{a_1^{d_2} a_2^{d_1} p\, t^{d_1-1}(1-t)^{d_2-1}}{\mathrm{B}(\frac{d_1}{p}, \frac{d_2}{p})(a_2^p t^p + (1-t)^p a_1^p)^{\frac{d_1}{p}+\frac{d_2}{p}}}; t > 0, d_1, d_2, a_1, a_2, p > 0 \qquad (5.9\, b)$$

Case 3: Quotient by Malik, 1967

Derivation of the distribution $W = \frac{X}{Y}, X \text{ and } Y$ are mutually independent, each with pdf $f(x; a_1, d_1, p)$ and $f(x; a_2, d_2, p)$ respectively;

$$f(x_i) = \frac{p}{a_i^{d_i}\Gamma(\frac{d_i}{p})} x_i^{d_i-1} e^{-\left(\frac{x_i}{a_i}\right)^p} \quad x_i > 0, \frac{d_i}{p}, p, a_i > 0, i = 1, 2 \qquad (5.10)$$

Let $U = \log X - \log Y$

Let the characteristic of U be defined by

$$\varphi(t) = \frac{p^2}{{a_1}^{d_1}\Gamma\left(\frac{d_1}{p}\right){a_2}^{d_2}\Gamma\left(\frac{d_2}{p}\right)}\int_0^{\infty} x^{d_1-1+it}e^{-\left(\frac{x}{a_1}\right)^p}dx\int_0^{\infty} y^{d_2-1+it}e^{-\left(\frac{y}{a_2}\right)^p}$$

$$= \frac{{a_1}^{it}{a_2}^{-it}\,\Gamma\left(\frac{d_1}{p}+\frac{it}{p}\right)\Gamma\left(\frac{d_2}{p}-\frac{it}{p}\right)}{\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)}$$

Therefore

$$f(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itu}{a_1}^{it}{a_2}^{-it}\,\Gamma\left(\frac{d_1}{p}+\frac{it}{p}\right)\Gamma\left(\frac{d_2}{p}-\frac{it}{p}\right)}{\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)}dt$$

Let $\quad \frac{d_2}{p}-\frac{it}{p} = -z$

$$f(u) = \frac{pe^{-d_2(u-\log\frac{a_1}{a_2})}}{\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)2\pi i}\int_{-\frac{d_2}{p}-i\infty}^{-\frac{d_2}{p}+i\infty} e^{-zp(u-\log\frac{a_1}{a_2})}\,\Gamma\left(\frac{d_1}{p}+\frac{d_2}{p}+z\right)\Gamma(-z)dz$$

$$= \frac{1}{2\pi i}\int_{-\frac{d_2}{p}-i\infty}^{-\frac{d_2}{p}+i\infty} e^{-zp(u-\log\frac{a_1}{a_2})}\,\Gamma\left(\frac{d_1}{p}+\frac{d_2}{p}+z\right)\Gamma(-z)dz = \Gamma\left(\frac{d_1}{p}+\frac{d_2}{p}\right)\left[1+e^{-p(u-\log\frac{a_1}{a_2})}\right]^{-\left(\frac{d_1}{p}+\frac{d_2}{p}\right)}$$

$$;\ u-\log\frac{a_1}{a_2}) > 0$$

$$f(u) = \frac{\Gamma\left(\frac{d_1}{p}+\frac{d_2}{p}\right)pe^{-d_2(u-\log\frac{a_1}{a_2})}}{\Gamma\left(\frac{d_1}{p}+\frac{d_2}{p}\right)\left[1+e^{-p(u-\log\frac{a_1}{a_2})}\right]^{\left(\frac{d_1}{p}+\frac{d_2}{p}\right)}};u-\log\frac{a_1}{a_2}) > 0 \tag{5.11}$$

But $\quad e^U = \frac{X}{Y} = W$

Therefore

$$f(w) = \frac{\Gamma\left(\frac{d_1}{p}+\frac{d_2}{p}\right)pe^{-d_2(\log\frac{a_1}{a_2})}w^{-d_2-1}}{\Gamma\left(\frac{d_1}{p}+\frac{d_2}{p}\right)\left[1+w^{-p}e^{-p(\log\frac{a_1}{a_2})}\right]^{\frac{d_1}{p}+\frac{d_2}{p}}}\quad ;\mathrm{w} > 0 \tag{5.12}$$

Chapter VI: Amoroso Distribution

6.0 Introduction

The Amoroso (generalized gamma, Stacy-Mihram) distribution is a four parameter, continuous, univariate, unimodal probability density, with semi-infinite range. The Amoroso distribution was originally developed to model lifetimes (Amoroso, L. 1925). It occurs as the Weibullization of the standard gamma distribution and, with Integer α, in extreme value statistics (Lee, P. 2009). The Amoroso distribution is itself a limiting form of various more general distributions, most notable the generalized beta and generalized beta prime distributions (McDonald, 1984).

A useful and important property of the Amoroso distribution is that many common and interesting probability distributions are special cases or limiting forms. In this chapter Amoroso distribution is derived, its properties calculated and special cases identified with their properties calculated too.

Let X be a random variable. If X follows a four-parameter generalized gamma distribution with parameters a, θ, α and β (a four-parameter distribution), the probability density function is given by Amoroso:

$$f(x/a,\theta,\alpha,\beta) = \frac{1}{\Gamma(\alpha)}\left|\frac{\beta}{\theta}\right|\left(\frac{x-a}{\theta}\right)^{\alpha\beta-1} exp\left\{-\left(\frac{x-a}{\theta}\right)^{\beta}\right\}; \qquad (6.0)$$

$$; > 0, x \geq a \; if \; \theta > 0 \quad and \; x \leq a, \theta < 0$$

Where α is the location parameter, θ is the scale parameter and α, β are the two shape parameters. The four-parameter Gamma distribution is extremely flexible. It is able to mimic several density function shapes. In addition it includes as special cases the Generalized Frechet, Generalized Weibull, Frechet, Weibull, Nakagami etc. Amoroso (four parameter) gamma distribution has many application and some are highlighted below;

- In finance, it is used to study the distribution of waiting times needed to reach a fixed level of return.
- Used in the study of health costs (Manning., 2005)
- In civil engineering, it is used as a flood frequency analysis model (Pham., 1995).
- In economics, it is used in various income distributions modeling (Pham., 1995, Kleiber., 2003).

6.1 Derivation of Amoroso distribution

Let $f(y) = \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}\ ;\ y > 0, \alpha > 0$ (6.1)

Also let $y = (\frac{x-\alpha}{\theta})^{\beta}$

$= \left|\frac{dy}{dx}\right| = \frac{\beta}{\theta}(\frac{x-\alpha}{\theta})^{\beta}$

Thus $f(x) = f(y) * \left|\frac{dy}{dx}\right|$

$$= \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} * \frac{\beta}{\theta}\left(\frac{x-\alpha}{\theta}\right)^{\beta-1}\ ;\ y, x > 0\ \alpha > 0 \qquad (6.2)$$

Replace $y = (\frac{x-\alpha}{\theta})^{\beta}$ *in equation (6.2)*

$$f(x) = \frac{1}{\Gamma(\alpha)}((\frac{x-\alpha}{\theta})^{\beta})^{\alpha-1} e^{-(\frac{x-\alpha}{\theta})^{\beta}} * \frac{\beta}{\theta}(\frac{x-\alpha}{\theta})^{\beta-1}$$

Therefore

$$f(x/a, \theta, \alpha, \beta) = \frac{1}{\Gamma(\alpha)}\left|\frac{\beta}{\theta}\right|\left(\frac{x-a}{\theta}\right)^{\alpha\beta-1} exp\left\{-\left(\frac{x-a}{\theta}\right)^{\beta}\right\};\ \alpha > 0, x \geq a\ if\ \theta > 0 \qquad (6.3)$$

$$E(X^r) = a^r + \frac{\theta^{r\Gamma\left(\alpha+\frac{r}{\beta}\right)}}{\Gamma(\alpha)}$$

$$E(X) = a + \frac{\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}$$

$$Var\ X = \theta^2\left[\frac{\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma(\alpha)} - \frac{\left[\Gamma\left(\alpha+\frac{1}{\beta}\right)\right]^2}{[\Gamma(\alpha)]^2}\right]$$

6.2 Special cases of Amoroso distribution

6.2.1 Stretched Exponential distribution (Laherrère *et al*, 1998**).**
Parameters: $a = 0, \alpha = 1$ and $\beta > 0$

$$f\left(\frac{x}{\theta}, \alpha, \beta\right) = \frac{\beta}{|\theta|}\left(\frac{x}{\theta}\right)^{\beta-1} exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\} \; ; \alpha > 0, x \geq a \; if \; \theta > 0 \qquad (6.4)$$

$$E(X^r) = \theta^r \Gamma\left(1 + \frac{r}{\beta}\right)$$

$$E(X) = \theta \Gamma\left(1 + \frac{1}{\beta}\right)$$

$$Var\,X = \theta^{\mathbf{2}}\left[\Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2\right]$$

$$Amoroso\ (x/0, \theta, 1, \beta) \equiv Stretched\ Exp\ (x/\theta, \alpha, \beta)$$

6.2.2 Pseudo-Weibull Distribution (Vodă., 1989)
Parameters: $a = 0, \; \alpha = 1 + \; 1/\beta$

$$f(x/a, \alpha, \beta) = \frac{1}{\Gamma\left(1+\frac{1}{\beta}\right)}\frac{\beta}{|\theta|}\left(\frac{x}{\theta}\right)^{\beta} exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\} \; ; \beta > 0 \qquad (6.5)$$

$$E(X^r) = \frac{\theta^r \Gamma\left(\left(1 + \frac{1}{\beta}\right) + \frac{r}{\beta}\right)}{\Gamma\left(1 + \frac{1}{\beta}\right)}$$

$$E(X) = \frac{\theta \Gamma\left(\left(1 + \frac{1}{\beta}\right) + \frac{1}{\beta}\right)}{\Gamma\left(1 + \frac{1}{\beta}\right)}$$

$$= \frac{\theta \Gamma\left(1 + \frac{2}{\beta}\right)}{\Gamma\left(1 + \frac{1}{\beta}\right)}$$

$$\mathrm{Var\ X} = \theta^2\left[\frac{\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1+\frac{3}{\beta}\right) - \left[\Gamma\left(1+\frac{2}{\beta}\right)\right]^2}{\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2}\right]$$

$$Amoroso\ (x/0, \theta, \alpha + \frac{1}{\beta}, \beta) \equiv Pseudo\text{-}Wei\ (x/\theta, \beta)$$

6.2.3 Standard Gamma (standard Amoroso) Distribution (Johnson *et al*, 1994).
Parameters: $a = 0, \theta = 1, \beta = 1$

$$f(x/\alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} exp\{-x\} \ ; \ \alpha > 0, x > 0 \qquad (6.6)$$

$$E(X^r) = \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)}$$

$$E(X) = \frac{\alpha(\alpha - 1)!}{(\alpha - 1)!} = \alpha$$

$$Var\,X = \alpha(\alpha + 1) - \alpha^2$$

$$= \alpha$$

Amoroso $(x/0, 1, \alpha, 1) \equiv G(x/\alpha)$

6.2.4 Pearson type III (Pearson 1895, Johnson *et al.*, 1994).
Parameters: $\beta = 1$

$$f\left(\frac{x}{a}, \theta, \beta\right) = \frac{1}{|\theta|\Gamma(\alpha)}\left(\frac{x - a}{\theta}\right)^{\alpha-1} exp\left\{-\left(\frac{x - a}{\theta}\right)\right\} \ ; \ \alpha > 0 \qquad (6.7)$$

$$E(X^r) = a^r + \frac{\theta^r \Gamma(\alpha + r)}{\Gamma(\alpha)}$$

$$E(X) = a + \frac{\theta\Gamma(\alpha + 1)}{\Gamma(\alpha)}$$

$$Var\,X = \theta^2 \left[\frac{\Gamma(\alpha + 2)\Gamma(\alpha) - [\Gamma(\alpha + 1)]^2}{[\Gamma(\alpha)]^2}\right]$$

Amoroso $(x/a, \theta, \alpha, 1) \equiv$ *Pearson III* $(x/a, \theta, \alpha)$

6.2.5 Exponential (Pearson type X, waiting time, negative exponential) distribution (Johnson *et al.*, 1994**).**

Parameters: $a = 0, \alpha = 1, \beta = 1$

$$f(x/\theta) = \frac{1}{\theta} exp\left\{-\frac{x}{\theta}\right\} \; ; \; x > 0 \tag{6.8}$$

$$E(X^r) = \theta^r \Gamma(1+r)$$

$$E(X) = \theta$$

$$Var\,X = \theta^2 \Gamma(1+2) - \theta^2$$

$$= \theta^2$$

Amoroso (x/$0, \theta, 1{,}1$) ≡ *Exp(x/*θ)

6.2.6 Shifted Exponential distribution (Johnson *et al.*, 1994**).**

Parameters: $\alpha = 1,\ \beta = 1$

$$f(x/a,\theta) = \frac{1}{|\theta|} exp\left\{-\left(\frac{x-a}{\theta}\right)\right\} \; ; \; x > 0 \tag{6.9}$$

$$E(X^r) = a^r + \theta^r \Gamma(1+r)$$

$$E(X) = a + \theta$$

$$\mathrm{Var\,X} = \theta^2 + 2\theta^2 - (\mathrm{a} + \theta)^2$$

$$= \theta^2 - 2\theta \mathrm{a}$$

$$= \theta(\theta - 2\mathrm{a})$$

Amoroso (x/$a, \theta, 1{,}1$) ≡ *ShiftedExp(x/*θ*)*

6.2.7 Nakagami (generalized normal, Nakagami-m) Distribution (Nakagami, M. 1960).

Parameters*:* $\alpha = \frac{m}{2}, \theta = \theta \;\; \beta = 2$

$$f(x/a,\theta,m) = \frac{1}{|\theta|\Gamma\left(\frac{m}{2}\right)}\left(\frac{x-a}{\theta}\right)^{k-1} exp\left\{-\left(\frac{x-a}{\theta}\right)^2\right\} \; ; \; \alpha > 0, x \geq a \, if \; > 0 \tag{6.10}$$

$$E(X^r) = a^r + \frac{\theta^r \Gamma\left(\frac{m}{2}+\frac{r}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}$$

$$E(X) = a + \frac{\theta\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$$Var\, X = \theta^2 \left[\frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{k}{2}\right)} - \frac{\left[\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)\right]^2}{\left[\Gamma\left(\frac{k}{2}\right)\right]^2} \right]$$

$$= \theta^2 \left[\frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+1\right) - \left[\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)\right]^2}{\left[\Gamma\left(\Gamma\left(\frac{m}{2}\right)\right)\right]^2} \right]$$

Amoroso (x/$a, \theta, \frac{m}{2}, 2$) ≡ Nak(x/$a, \theta, m$)

6.2.8 Pearson type V distribution (Pearson, K 1901).

Parameters: $\beta = -1$

$$f\left(\frac{x}{a}, \theta, \alpha\right) = \frac{1}{|\theta|\Gamma(\alpha)} \left(\frac{\theta}{x-a}\right)^{\alpha-1} exp\left\{-\frac{\theta}{x-a}\right\} \; ; \; x > 0 \tag{6.11}$$

$$E(X^r) = a^r + \frac{\theta^r \Gamma(\alpha - r)}{\Gamma(\alpha)}$$

$$E(X) = a + \frac{\theta\Gamma(\alpha-1)}{\Gamma(\alpha)} = a + \frac{\theta}{(\alpha-1)}$$

$$Var\, X = \theta^2 \left[\frac{\Gamma(\boldsymbol{\alpha})\Gamma(\boldsymbol{\alpha}-\mathbf{2}) - \left(\Gamma(\boldsymbol{\alpha}-\mathbf{1})\right)^{\mathbf{2}}}{\left(\Gamma(\boldsymbol{\alpha})\right)^{\mathbf{2}}} \right]$$

Amoroso (x/$a, \theta, \alpha, -1$) ≡ Pearson Type V(x/a, θ, α

6.2.9 Levy (van der Waals profile) distribution (Feller, W. 1971).

Parameters: a, $\theta = c$, $\alpha = \frac{1}{2}$, $\beta = -1$

$$f(x/a, c) = \sqrt{\frac{c}{2\pi}} \; \frac{1}{(x-a)^{\frac{3}{2}}} \; exp\left\{-\frac{c}{2(x-a)}\right\} \tag{6.12}$$

where a is the location parameter and c is the scale parameter.

$$E(X^r) = a^r + c^r\Gamma\left(\frac{1}{2} - r\right) \; ; \; r < \frac{1}{2}$$

$$E(X) = a + c\Gamma\left(\frac{1}{2} - 1\right) = \infty$$

$$Var\, X = \infty$$

Amoroso $(x/a, \frac{c}{2}, \frac{1}{2}, -1) \equiv$ Levy$(x/a, c)$

Extreme Order Statistics

6.2.10 Generalized Fisher-Tippett Distribution (Smirnov, N. 1949, Barndorff, O. 1963)
Parameters*:* $a, \theta = \omega, \alpha = n, \beta$

$$f\left(\frac{x}{a}, \omega, n, \beta\right) = \frac{n^n}{\Gamma(n)} \left|\frac{\beta}{\boldsymbol{\omega}}\right| \left(\frac{x-a}{\boldsymbol{\omega}}\right)^{n\beta-1} exp\left\{-n\left(\frac{x-a}{\boldsymbol{\omega}}\right)^{\beta}\right\} \tag{6.13}$$

$$E(X^r) = a^r + \frac{(\omega)^r \Gamma\left(n+\frac{r}{\beta}\right)}{\Gamma(n)}$$

$$E(X) = a + \frac{\omega\Gamma\left(n+\frac{1}{\boldsymbol{\beta}}\right)}{\Gamma(n)}$$

$$Var\, X = \omega^2 \left[\frac{\Gamma\left(n+\frac{2}{\beta}\right)}{\Gamma(n)} - \frac{\left[\Gamma\left(n+\frac{1}{\beta}\right)\right]^2}{[\Gamma(n)]^2}\right]$$

$$= \omega^2 \left[\frac{\Gamma\left(n+\frac{2}{\beta}\right)\Gamma(n) - \left[\Gamma\left(n+\frac{1}{\beta}\right)\right]^2}{[\Gamma(n)]^2}\right]$$

Amoroso (x/a, $\frac{\omega}{n^{\frac{1}{\beta}}}, n, \beta) \equiv$ *GenFisher Tippet(x/a,* $\omega, n, \beta)$

6.2.11 Fisher-Tippett (Generalized extreme Value, GEV, von Mises-Jenkinson, von Mises extreme value) Distribution (Fisher et al 1928, Von, M. 1936, Gumbel, E. 1958).
Parameters*:* $a, \theta = \omega, \ \alpha = 1, \beta$

$$f\left(\frac{x}{a}, \omega, n, \beta\right) = \left|\frac{\beta}{\boldsymbol{\omega}}\right| \left(\frac{x-a}{\boldsymbol{\omega}}\right)^{\beta-1} exp\left\{-\left(\frac{x-a}{\boldsymbol{\omega}}\right)^{\beta}\right\} \tag{6.14}$$

$$E(X^r) = a^r + (\omega)^r \Gamma\left(1+\frac{r}{\beta}\right)$$

$$E(X) = a + \omega\, \Gamma(1 + 1/\beta)$$

$$Var\, X = \omega^2 \Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2$$

Amoroso (x/a, $\omega, 1, \beta) \equiv$ *Fisher Tippet(x/a,* $\omega, \beta)$

6.2.12 Generalized Weibull Distribution (Smirnov, N. 1949, Barndorff, O. 1963)**.**
Parameters*:* $a, \theta = \omega,\ \alpha = n, \beta > 0$

$$f(x/a, \omega, n, \beta) = \frac{n^n}{\Gamma(n)} \frac{\beta}{|\omega|} \left(\frac{x-a}{\omega}\right)^{n\beta-1} exp\left\{-n\left(\frac{x-a}{\omega}\right)^{\beta}\right\} \tag{6.15}$$

$$E(X^r) = a^r + \frac{\omega^r \Gamma\left(n + \frac{r}{\beta}\right)}{\Gamma(n)}$$

$$E(X) = a + \frac{\omega \Gamma\left(n + \frac{1}{\beta}\right)}{\Gamma(n)}$$

$$Var\, X = \omega^2 \left[\frac{\Gamma\left(n + \frac{2}{\beta}\right)}{\Gamma(n)} - \frac{\left[\Gamma\left(n + \frac{1}{\beta}\right)\right]^2}{[\Gamma(n)]^2}\right]$$

$$= \omega^2 \left[\frac{\Gamma\left(n + \frac{2}{\beta}\right)\Gamma(n) - \left[\Gamma\left(n + \frac{1}{\beta}\right)\right]^2}{[\Gamma(n)]^2}\right]$$

Amoroso $(x/a, \frac{\omega}{n^{\frac{1}{\beta}}}, n, \beta) \equiv$ *GenFisher Tippet*$(x/a, \omega, n, \beta)$

6.2.13 Weibull (Fisher-Tippett type III, Gumbel type III, Rosin-Rammler,Rosin-Rammler-Weibull, extreme value type III, Weibull-Gnedenko) Distribution (Weibull, W. 1951, Johnson *et al.*, 1995).
Parameters: $a, \theta = \omega,\ \alpha = 1, \beta > 0$

$$f(x/a, \omega, \beta) = \left|\frac{\beta}{\omega}\right| \left(\frac{x-a}{\omega}\right)^{\beta-1} exp\left\{-\left(\frac{x-a}{\omega}\right)^{\beta}\right\} \tag{6.16}$$

$$E(X^r) = a^r + (\omega)^r \Gamma\left(1 + \frac{r}{\beta}\right)$$

$$E(X) = a + \omega\, \Gamma(1 + 1/\beta)$$

$$Var\, X = \omega^2 \Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2$$

Amoroso $(x/a, \omega, 1, \beta) \equiv$ *Wei*$(x/a, \omega, \beta)$

6.2.14 Generalized Fréchet Distribution (Smirnov, N. 1949, Barndorff, O. 1963)**.**

Parameters: $a, \theta = \omega,\ \alpha = n, \bar{\beta}$

$$f(x/a, \omega, n, \beta) = \frac{n^n}{\Gamma(n)} \frac{\bar{\beta}}{|\boldsymbol{\omega}|} \left(\frac{x-a}{\boldsymbol{\omega}}\right)^{-n\bar{\boldsymbol{\beta}}-1} exp\left\{-n\left(\frac{x-a}{\boldsymbol{\omega}}\right)^{-\bar{\boldsymbol{\beta}}}\right\} \tag{6.17}$$

$$E(X^r) = a^r + \frac{\omega^r \Gamma\left(n + \frac{r}{\bar{\beta}}\right)}{\Gamma(n)}$$

$$E(X) = a + \frac{\omega \Gamma\left(n + \frac{1}{\bar{\beta}}\right)}{\Gamma(n)}$$

$$Var\ X = \omega^2 \left[\frac{\Gamma\left(n + \frac{2}{\bar{\beta}}\right)}{\Gamma(n)} - \frac{\left[\Gamma\left(n + \frac{1}{\bar{\beta}}\right)\right]^2}{[\Gamma(n)]^2}\right]$$

$$= \omega^2 \left[\frac{\Gamma\left(n + \frac{2}{\bar{\beta}}\right)\Gamma(n) - \left[\Gamma\left(n + \frac{1}{\bar{\beta}}\right)\right]^2}{[\Gamma(n)]^2}\right]$$

Amoroso (x/a, $\frac{\omega}{n^{\frac{1}{\bar{\beta}}}}, n, -\bar{\beta}$*)* $\equiv$ *Gen- Fréchet (x/a,* $\omega, n, \bar{\beta}$ *)*

6.2.15 Fréchet (extreme value type II, Fisher-Tippett type II, Gumbel type II, inverse Weibull) Distribution (Fréchet, M. 1927, Gumbel, E. 1958).

Parameters: $a, \theta = \omega,\ \alpha = 1, \bar{\beta}$

$$f\left(\frac{x}{a}, \omega, \beta\right) = \frac{\bar{\boldsymbol{\beta}}}{|\omega|}\left(\frac{x-a}{\omega}\right)^{-\bar{\boldsymbol{\beta}}-1} exp\left\{-\left(\frac{x-a}{\omega}\right)^{-\bar{\boldsymbol{\beta}}}\right\} \tag{6.18}$$

$$E(X^r) = a^r + (\omega)^r \Gamma\left(1 + \frac{r}{\bar{\beta}}\right)$$

$$E(X) = a + \omega\,\Gamma\left(1 + \frac{1}{\bar{\beta}}\right)$$

$$Var\,X = \omega^2\Gamma\left(1 + \frac{2}{\bar{\beta}}\right) - \left[\Gamma\left(1 + \frac{1}{\bar{\beta}}\right)\right]^2$$

Amoroso $(x/a, \omega, 1, -\bar{\beta}) \equiv$ *Fréchet* $(x/a, \omega, \bar{\beta})$

Summary of Amoroso distribution

	Name of the distribution	Parameters				Mean	Variance
		a	θ	α	β		
1	Stretched Exponential	0	θ	1	$\beta > 0$	$\theta\Gamma\left(1+\frac{1}{\beta}\right)$	$\theta^2\left[\Gamma\left(1+\frac{2}{\beta}\right)-\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right]$
2	Pseudo-Weibull	0	θ	$1+\frac{1}{\beta}$	β	$\frac{\theta\Gamma\left(1+\frac{2}{\beta}\right)}{\Gamma\left(1+\frac{1}{\beta}\right)}$	$\theta^2\left[\frac{\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1+\frac{3}{\beta}\right)-\left[\Gamma\left(1+\frac{2}{\beta}\right.\right.}{\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2}\right.$
3	Standard gamma	0	1	α	1	α	α
4	Pearson type III	a	θ	α	1	$a + \frac{\theta\Gamma(\alpha+1)}{\Gamma(\alpha)}$	$\theta^2\left[\frac{\Gamma(\alpha+2)\Gamma(\alpha)-[\Gamma(\alpha+1)]^2}{[\Gamma(\alpha)]^2}\right]$
5	Exponential	0	θ	1	1		
6	Shifted Exponential	a	θ	1	1	θ	θ^2
7	Nakagami	a	θ	$\frac{m}{2}$	2	$a + \frac{\theta\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$	$\theta^2\left[\frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+1\right)-\left[\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)\right]^2}{\left[\Gamma\left(\Gamma\left(\frac{m}{2}\right)\right)\right]^2}\right]$
8	Pearson type V	a	θ	α	-1	$a + \frac{\theta\Gamma(\alpha-1)}{\Gamma(\alpha)}$	$\theta^2\left[\frac{\Gamma(\alpha)\Gamma(\alpha-2)-(\Gamma(\alpha-1))^2}{(\Gamma(\alpha))^2}\right]$
9	Levy	a	c	$\frac{1}{2}$	-1	∞	∞
10	Generalized Fisher-Tippet	a	ω	n	β	$a + \frac{\omega\Gamma\left(n+\frac{1}{\beta}\right)}{\Gamma(n)}$	$\omega^2\left[\frac{\Gamma\left(n+\frac{2}{\beta}\right)\Gamma(n)-\left[\Gamma\left(n+\frac{1}{\beta}\right)\right]^2}{[\Gamma(n)]^2}\right]$
11	Fisher-Tippet	a	ω	1	β	$a + \omega\,\Gamma(1 + 1/\beta)$	$\omega^2\Gamma\left(1+\frac{2}{\beta}\right)-\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2$

12	Generalized Weibull	a	ω	n	β>0	$a + \frac{\omega\Gamma\left(n+\frac{1}{\beta}\right)}{\Gamma(n)}$	$\omega^2\left[\frac{\Gamma\left(n+\frac{2}{\beta}\right)}{\Gamma(n)} - \frac{\left[\Gamma\left(n+\frac{1}{\beta}\right)\right]^2}{[\Gamma(n)]^2}\right]$
13	Weibull	a	ω	1	β>0	$a + \omega\,\Gamma(1 + 1/\beta)$	$\omega^2\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2$
14	Generalized Fréchet	a	ω	n	$\bar{\beta}$	$a + \omega\,\Gamma\left(1+\frac{1}{\bar{\beta}}\right)$	$\omega^2\left[\frac{\Gamma\left(n+\frac{2}{\bar{\beta}}\right)}{\Gamma(n)} - \frac{\left[\Gamma\left(n+\frac{1}{\bar{\beta}}\right)\right]^2}{[\Gamma(n)]^2}\right]$
15	Fréchet	a	ω	1	$\bar{\beta}$	$a + \omega\,\Gamma\left(1+\frac{1}{\bar{\beta}}\right)$	$\omega^2\Gamma\left(1+\frac{2}{\bar{\beta}}\right) - \left[\Gamma\left(1+\frac{1}{\bar{\beta}}\right)\right]^2$

CHAPTER VII: Unifying Ghitany and Nadarajah's Generalized gamma distributions.

7.0 Introduction

In this chapter the Generalized Gamma distribution as studied by Githany, 1998 and Nadarajah's 2011 is discussed and their probability density functions given by the use of integral presentation of Confluent Hypergeometric of second kind and by the use of Generalized gamma function.

7.1 Ghitany Generalized gamma distribution

Construction of Ghitany's Generalized Gamma Distribution

Let $$\Gamma_\lambda(m, \alpha n) = \int_0^\infty \frac{y^{m-1}e^{-y}}{(y+an)^\lambda}\,dy \qquad (7.0)$$

Divide both sides by $\Gamma_\lambda(m, \alpha n)$

$$1 = \int_0^\infty \frac{y^{m-1}e^{-y}dy}{(y+an)^\lambda \Gamma_\lambda(m, \alpha n)}$$

$$= \int_0^\infty \frac{y^{m-1}e^{-y}dy}{\alpha^\lambda(\frac{y}{\alpha}+n)^\lambda \Gamma_\lambda(m, \alpha n)} \qquad (7.1)$$

Let $x = \frac{y}{\alpha} \Rightarrow y = \alpha x,\ dy = \alpha dx$ *replacing in equation (7.1)*

$$1 = \int_0^\infty \frac{(\alpha x)^{m-1}e^{-\alpha x}\alpha \quad dx}{\alpha^\lambda(\frac{\alpha x}{\alpha}+n)^\lambda \Gamma_\lambda(m, \alpha n)}$$

$$= \int_0^\infty \frac{\alpha^{m-1}x^{m-1}e^{-\alpha x}\alpha \quad dx}{\alpha^\lambda(x+n)^\lambda \Gamma_\lambda(m, \alpha n)}$$

$$= \int_0^\infty \frac{\alpha^{m-\lambda}x^{m-1}e^{-\alpha x} \quad dx}{(x+n)^\lambda \Gamma_\lambda(m, \alpha n)}$$

Thus

$$f(x) = \frac{\alpha^{m-\lambda}x^{m-1}e^{-\alpha x}dx}{(x+n)^\lambda \Gamma_\lambda(m, \alpha n)} \ ; \ x \geq 0, \lambda, m, n, \alpha > 0 \qquad (7.2\ a)$$

Alternatively

Let $I = \int_0^\infty \frac{y^{m-1}e^{-y}}{(y+\alpha n)^\lambda} dy = \int_0^\infty \frac{y^{m-1}e^{-y}}{(\alpha n)^\lambda\left(1+\frac{y}{\alpha n}\right)^\lambda} dy$

$$let\ x = \frac{y}{\alpha n} \Longrightarrow \mathrm{y} = \alpha n x, dy = \alpha n dx$$

$$I = \int_0^\infty \frac{(\alpha n x)^{m-1}e^{-\alpha n x}}{(\alpha n)^\lambda(1+x)^\lambda} \alpha n dx = (\alpha n)^{\mathrm{m}-\lambda} \int_0^\infty x^{m-1}(1+x)^{-\lambda}e^{-\alpha n x}\, dx$$

$$= (\alpha n)^{\mathrm{m}-\lambda}\, \Gamma(m)\Psi(m, m+1-\lambda; \alpha n)$$

Therefore

$$f(x) = \frac{\alpha^{m-\lambda}x^{m-1}e^{-\alpha x}}{(x+n)^\lambda(\alpha n)^{\mathrm{m}-\lambda}\, \Gamma(m)\Psi(m, m+1-\lambda; \alpha n)}$$

$$= \frac{x^{m-1}e^{-\alpha x}}{(x+n)^\lambda n^{\mathrm{m}-\lambda}\, \Gamma(m)\Psi(m, m+1-\lambda; \alpha n)};\ x \geq 0, \lambda, m, n, \alpha > 0 \qquad (7.2\ b)$$

*b*y use of integral presentation of Confluent Hypergeometric function of second kind.

Also

$$f(x) = \frac{\alpha^{m-\lambda}x^{m-1}e^{-\alpha x}dx}{(x+n)^\lambda(\alpha n)^{\mathrm{m}-\lambda}\, \alpha n^{-a}\ {}_2F_0(a, \lambda;;\ -\alpha n^{-1})}\ ;\ x \geq 0, \lambda, m, n, \alpha > 0 \qquad (7.2\ c)$$

by using Generalized Hypergeometric function.

Properties using equation (7.2 a)

$$E(X^r) = \int_0^\infty \frac{\alpha^{m-\lambda}x^r x^{m-1}e^{-\alpha x}\ dx}{(x+n)^\lambda\Gamma_\lambda(m, \alpha n)}$$

$$= \frac{1}{\Gamma_\lambda(m,\alpha n)} \int_0^\infty \frac{\alpha^{m-\lambda}x^{m+r-1}e^{-\alpha x}\ dx}{(x+n)^\lambda} \qquad (7.3)$$

Let $u = \alpha x \Rightarrow x = \frac{u}{\alpha}$

$du = \alpha\, dx \Rightarrow \frac{du}{\alpha} = dx$ *replacing in equation (7.3)*

$$E(X^r) = \frac{1}{\Gamma_\lambda(m,\alpha n)} \int_0^\infty \frac{\alpha^{m-\lambda} \left(\frac{u}{\alpha}\right)^{m+r-1} e^{-u}}{(\frac{u}{\alpha}+n)^\lambda} \frac{du}{\alpha}$$

$$= \frac{\alpha^{m-\lambda}}{\Gamma_\lambda(m,\alpha n)} \int_0^\infty \frac{u^{m+r-1}\alpha^{-m-r+1-1}\alpha^\lambda e^{-u}\, du}{(u+\alpha n)^\lambda}$$

$$= \frac{\alpha^{-r}}{\Gamma_\lambda(m,\alpha n)} \int_0^\infty \frac{u^{m+r-1} e^{-u}\, du}{(u+\alpha n)^\lambda}$$

$$= \frac{\alpha^{-r}}{\Gamma_\lambda(m,\alpha n)} \Gamma_\lambda(m+r,\alpha n) \qquad (7.4)$$

$$E(X) = \frac{\alpha^{-}}{\Gamma_\lambda(m,\alpha n)} \Gamma_\lambda(m+1,\alpha n)$$

$$Var\, X = \frac{\alpha^{-2}}{\Gamma_\lambda(m,\alpha n)} \Gamma_\lambda(m+2,\alpha n) - \left[\frac{\alpha^{-}}{\Gamma_\lambda(m,\alpha n)} \Gamma_\lambda(m+1,\alpha n)\right]^2$$

$$= \alpha^{-2}\left[\frac{\Gamma_\lambda(m+2,\alpha n)}{\Gamma_\lambda(m,\alpha n)} - \left[\frac{\Gamma_\lambda(m+1,\alpha n)}{\Gamma_\lambda(m,\alpha n)}\right]^2\right]$$

7.2 Nadarajah's Generalized gamma distribution

Relationship between integral presentation of Confluent Hypergeometric function of second kind and Generalized Hypergeometric function.

$$\Psi(a,b;u) = \mathrm{u}^{-a}\ {}_2F_0(a, 1+a-b;;\ -u^{-1})$$

Proof

$$\text{Let } \Psi(a,b;u) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(1+t)^{b-a-1}e^{-ut}dt \tag{7.5}$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\sum_{k=0}^{\infty}\binom{b-a-1}{k}t^k e^{-ut}dt$$

$$= \sum_{k=0}^{\infty}\binom{b-a-1}{k}\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha+k-1}e^{-ut}dt$$

$$let\ y = ut \Rightarrow t = \frac{y}{u}, dt = \frac{dy}{u}$$

$$\Psi(a,b;u) = \sum_{k=0}^{\infty}\binom{b-a-1}{k}\frac{1}{\Gamma(\alpha)u^{\alpha+k}}\int_0^\infty y^{\alpha+k-1}e^{-y}dy$$

$$= \sum_{k=0}^{\infty}\binom{b-a-1}{k}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)u^{\alpha+k}}$$

$$= \frac{\mathrm{u}^{-a}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{(b-a-1)(b-a-1)..(b-a-k)\Gamma(\alpha+k)(\frac{1}{u})^k}{k!}$$

$$= \frac{\mathrm{u}^{-a}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{(b-a-1)(b-a-1)..(b-a-k)(\mathrm{a+k-1})\ldots(\mathrm{a+k-k}) * \Gamma(\alpha)((-u)^{-1})^k}{k!}$$

$$= \mathrm{u}^{-a}\sum_{k=0}^{\infty}(b-a-k)\ldots(b-a-1)a(a+1)\ldots(a+k-1)\frac{((-u)^{-1})^k}{k!}$$

$$= \mathrm{u}^{-a}\ {}_2F_0(a, 1+a-b;;\ -u^{-1}) \tag{7.6}$$

Thus $\Psi(a,b;u) = \mathrm{u}^{-a}\ {}_2F_0(a, 1+a-b;;\ -u^{-1})$ result which is consistent with formulae given by Abramowitz *et al.*, 1972, p.505)

Construction of Nadarajah's Generalized Gamma distribution

Nadarajah, 2008 considered,

$$I = \int_0^\infty x^{\alpha-1}(x+z)^{-\rho}e^{-\lambda \mathrm{x}}dx \tag{7.7}$$

$$= \int_0^\infty x^{\alpha-1}z^{-\rho}(1+\frac{x}{z})^{-\rho}e^{-\lambda \mathrm{x}}dx$$

Let $t = \frac{x}{z} \Longrightarrow x = zt, dx = zdt$

$$I = \int_0^\infty (zt)^{\alpha-1}z^{-\rho}(1+t)^{-\rho}e^{-\lambda zt}zdt$$

$$= z^{\alpha-\rho}\int_0^\infty t^{\alpha-1}(1+t)^{-\rho}e^{-\lambda zt}dt$$

$$= z^{\alpha-\rho}\,\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)$$

Therefore $\int_0^\infty x^{\alpha-1}(x+z)^{-\rho}e^{-\lambda \mathrm{x}}dx = z^{\alpha-\rho}\,\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)$

$$f(x) = \frac{x^{\alpha-1}(x+z)^{-\rho}e^{-\lambda \mathrm{x}}}{z^{\alpha-\rho}\,\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)}\ ; x > 0, \rho\ , \alpha, \lambda, z > 0 \tag{7.8}$$

by use integral presentation of Confluent Hypergeometric function of second kind.

Also

$$f(x) = \frac{x^{\alpha-1}(x+z)^{-\rho}e^{-\lambda \mathrm{x}}}{z^{\alpha-\rho}\,\Gamma(\alpha)(\lambda z)^{-\alpha}\ {}_2F_0(\alpha,\rho;\,;-(\lambda z)^{-1})}\ ; x > 0, \rho\ , \alpha, \lambda, z > 0 \tag{7.9}$$

by using Generalized Hypegeometric function.

Properties by use of equation (7.8)

Therefore $\int_0^\infty x^{\alpha-1}(x+z)^{-\rho}e^{-\lambda \mathrm{x}}dx\ = z^{\alpha-\rho}\,\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)$

$$E(X^r) = \int_0^\infty \frac{x^r x^{\alpha-1}(x+z)^{-\rho}e^{-\lambda \mathrm{x}}}{z^{\alpha-\rho}\,\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)}dx$$

$$= \int_0^\infty \frac{x^{r+\alpha-1}(x+z)^{-\rho}e^{-\lambda \mathrm{x}}dx}{z^{\alpha-\rho}\,\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)}$$

$$= \frac{1}{z^{\alpha-\rho}\,\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)}\int_0^\infty x^{r+\alpha-1}(x+z)^{-\rho}e^{-\lambda \mathrm{x}}dx$$

$$= \frac{z^{r+\alpha-\rho}\,\Gamma(r+\alpha)\Psi(\alpha+r,r+\alpha+1-\rho;\lambda z)}{z^{\alpha-\rho}\,\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)}$$

$$= \frac{z^{r}\,\Gamma(r+\alpha)\Psi(\alpha+r,r+\alpha+1-\rho;\lambda z)}{\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)}$$

$$E(X) = \frac{z\,\Gamma(1+\alpha)\Psi(1+\alpha,1+\alpha+1-\rho;\lambda z)}{\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)}$$

$$= \frac{z\,\alpha\,\Psi(1+\alpha,\ \ \alpha+2-\rho;\lambda z)}{\Psi(\alpha,\alpha+1-\rho;\lambda z)}$$

$$Var\,X = \frac{z^2\,\Gamma(2+\alpha)\Psi(\alpha+2,2+\alpha+1-\rho;\lambda z)}{\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)} - \left(\frac{z\,\alpha\,\Psi(1+\alpha,\ \ \alpha+2-\rho;\lambda z)}{\Psi(\alpha,\alpha+1-\rho;\lambda z)}\right)^2$$

$$= \frac{z^2\,\alpha(\alpha+1)\Gamma(2+\alpha)\Psi(\alpha+2,2+\alpha+1-\rho;\lambda z)}{\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)} - \left(\frac{z\,\alpha\,\Psi(1+\alpha,\ \ \alpha+2-\rho;\lambda z)}{\Psi(\alpha,\alpha+1-\rho;\lambda z)}\right)^2$$

7.3 Unified Ghitany, 1998 and Nadarajah, 2008 Generalized Gamma distribution.

By the by use integral presentation of Confluent Hypergeometric function of second kind.

Ghitanys' $GG(m,\alpha,n,\lambda\,)$ ≡Nadarajahs' $GG(\alpha,\lambda,z,\rho)$

$$f(x) = \frac{x^{m-1}e^{-\alpha x}}{(x+n)^{\lambda}n^{\mathrm{m}-\lambda}\,\Gamma(m)\Psi(m,m+1-\lambda;\alpha n)} = \frac{x^{\alpha-1}(x+z)^{-\rho}e^{-\lambda \mathrm{x}}}{z^{\alpha-\rho}\,\Gamma(\alpha)\Psi(\alpha,\alpha+1-\rho;\lambda z)}$$

CHAPTER VIII: Gamma Generalization using the generator Approach

8.0 Introduction

In the last few years, several ways of generating new probability distributions from classic ones were developed and discussed. Jones (2004) studied a distribution family that arises naturally from the distribution of order statistics. The beta-generated family proposed by Eugene et al. (2002) and further studied by Jones was discussed in Zografos and Balakrishnan (2009), who proposed a kind of gamma-generated family. Based on a baseline continuous distribution F(x) with survival function F(x) and density f(x), they defined the cumulative distribution function (cdf) and probability density function (pdf) as

$$H(x) = \frac{1}{\Gamma(\alpha)}\int_0^{-log\bar{F}(x)} t^{\delta-1}e^{-t}dt\ , t \in R, \delta > 0\ and\ h(x) = \frac{1}{\Gamma(\alpha)}[-log\bar{F}(x)]^{\delta-1}f(x) \quad (8.0)$$

respectively, where $\Gamma(.)$ is the gamma function.

For the purpose of this study we are going to define the cumulative distribution function (cdf) and probability density function (pdf) as

$$F(x) = \frac{1}{\Gamma(\alpha)}\int_0^{-logG(x)} t^{\alpha-1}e^{-t}\,dt, x \in R, \alpha > 0\ and\ f(x) = \frac{1}{\Gamma(\alpha)}[-log\bar{F}(x)]^{\alpha-1} * g(x) \qquad (8.1)$$

respectively where $\Gamma(.)$ is the gamma function.

8.1 Type I-Gamma 1 Generator

Let $\quad y = -log\ \bar{G}(x)$

where $\bar{G}(x) = 1 - G(x)$

$G(x)$ is a cumulative Distribution Function

then $\quad \bar{G}(x) = e^{-y}$, if $\bar{G}(x) = 0 \Rightarrow y = \infty\ and\ if\ \bar{G}(x) = 1 \Rightarrow y = 0$

Let $\quad w(t) = \frac{e^{-t}}{\Gamma(\alpha)}\ t^{\alpha-1}\ ; \alpha, t > 0$

Let us now consider *F(x)* given by;

$$F(x) = \int_{0}^{-log\bar{G}(x)} \frac{e^{-t}}{\Gamma(\alpha)} t^{\alpha-1}dt$$

$$F(x) = \frac{d \;\; F(x)}{dx} = \frac{d}{dx} \int_{0}^{-log\bar{G}(x)} \frac{e^{-t}}{\Gamma(\alpha)} t^{\alpha-1}dt$$

$$= \frac{1}{\Gamma(\alpha)} \frac{d}{dx} \int_{0}^{-log\bar{G}(x)} e^{-t}t^{\alpha-1}dt$$

$$= \frac{1}{\Gamma(\alpha)} d \int_{0}^{-log\bar{G}(x)} e^{-t}t^{\alpha-1}dt$$

$$= \frac{1}{\Gamma(\alpha)} e^{-log\bar{G}(x)}(-log\bar{G}(x))^{\alpha-1} \frac{d}{dx}(-log\bar{G}(x))$$

$$= \frac{\bar{G}(x)}{\Gamma(\alpha)}(-log\bar{G}(x))^{\alpha-1} * -\frac{1}{\bar{G}(x)} \frac{d}{dx}(\breve{G}(x))$$

$$= \frac{1}{\Gamma(\alpha)}(-log\bar{G}(x))^{\alpha-1} * -\frac{d}{dx}(1 - G(x)$$

$$= \frac{1}{\Gamma(\alpha)}(-log\bar{G}(x))^{\alpha-1} * \; g(x)\,; \quad -\infty < x < \infty, \alpha > 0 \qquad (8.2)$$

$$Type\ I - Gamma\ 1\ generator.$$

- **Type I – Gamma 1 Generated Distributions**

Gamma 1 - Exponential Distribution

Construction of Exponential Distributions

Let $X = -\frac{1}{\beta}\, ln\, Y$, where $y \sim U(0,1) \Rightarrow g(y) = 1$ therefore

$e^{-\beta x} = y \Rightarrow \left|\frac{dy}{dx}\right| = \left|-\beta e^{-\beta x}\right|$

Thus $g(x) = g(y) \left|\frac{dy}{dx}\right| = \beta e^{-\beta x}$; $x, \beta > 0$ (8.3)

which is an exponential distribution with parameter β.

Now let $g(x) = \beta e^{-\beta x}$; $x, \beta > 0$

Then $G(x) = 1 - e^{-\beta x} \Rightarrow \breve{G}(x) = e^{-\beta x}$

Therefore

$$F(x) = \frac{1}{\Gamma(\alpha)}(-log\breve{G}(x))^{\alpha-1} * \beta e^{-\beta x}$$

$$= \frac{1}{\Gamma(\alpha)}(-loge^{-\beta x})^{\alpha-1} * \beta e^{-\beta x}$$

$$= \frac{1}{\Gamma(\alpha)}(\beta x)^{\alpha-1} * \beta e^{-\beta x}$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1} \; ; \; x > 0, \alpha, \beta > 0 \quad (8.4)$$

which is a two-parameter gamma distribution.

Gamma 1 - Weibull Distribution

Construction of Weibull Distribution

Let $X = \left|\frac{Y}{\beta}\right|^{\tau}$ where $y \sim Exp(1) \Rightarrow g(y) = e^{-y}$

Therefore $\beta^{\tau} x = y^{\tau} \Rightarrow y = (\beta^{\tau} x)^{\frac{1}{\tau}}$

$$= \left|\frac{dy}{dx}\right| = \left|\frac{\beta}{\tau} x^{\frac{1}{\tau}-1}\right|$$

Thus $g(x) = g(y) * \left|\frac{dy}{dx}\right|$

$$= e^{-\beta x^{\frac{1}{\tau}}} * \frac{\beta}{\tau} x^{\frac{1}{\tau}-1}, let \; \frac{1}{\tau} = \theta$$

Therefore $g(x) = \beta\, \theta x^{\theta-1} e^{-\beta x^{\theta}}$; $x > 0, \beta, \theta > 0$ (8.5)

which is a Weibull distribution with parameters β, θ.

Now let $(x) = \beta\, \theta x^{\theta-1} e^{-\beta x^{\theta}}$ then $G(x) = 1 - e^{-\beta x^{\theta}} \Rightarrow \breve{G}(x) = e^{-\beta x^{\theta}}$

Therefore $f(x) = \frac{1}{\Gamma(\alpha)}(-log\breve{G}(x))^{\alpha-1} * g(x)$

$$= \frac{1}{\Gamma(\alpha)}(-loge^{-\beta x^{\theta}})^{\alpha-1} * \beta\, \theta x^{\theta-1} e^{-\beta x^{\theta}}$$

$$= \frac{1}{\Gamma(\alpha)}(\beta x^{\theta})^{\alpha-1} * \beta\, \theta x^{\theta-1} e^{-\beta x^{\theta}}$$

$$= \frac{\beta^{\alpha}\, \theta}{\Gamma(\alpha)} x^{\theta\alpha-1}\, e^{-\beta x^{\theta}} \quad ;\ x > 0, \alpha, \beta\ > 0 \qquad (8.6)$$

which a three-parameter gamma distribution.

8.2 Type I-Gamma 2 Generator

Let $F(X) = W(-log\, \bar{G}(x))$

$$(dF(X))/dx = d/dxW(-log\, G^{-}(x))\ d/dx(-logG^{-}(x))$$

$$= w(-log\, \bar{G}(x)) * -\frac{1}{\bar{G}(x)} \frac{d}{dx}(1 - G(x))$$

Therefore

$f(x) = w(-log\, \bar{G}(x)) * \frac{g(x)}{\bar{G}(x)}$ *where* $\bar{G}(x) = 1 - G(x)$ *and* $G(x)$ is the cumulative distribution function.

Let

$$w(t) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1}\, e^{-\beta t};\ t, \alpha, \beta > 0 \ \textit{then}$$

$$w(-log\, \bar{G}(x)) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}((-log\, \bar{G}(x))^{\alpha-1}\, e^{-\beta(-log\, \bar{G}(x))}$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)}((-log\, \bar{G}(x))^{\alpha-1}\, e^{log(\bar{G}(x))^{\beta}}$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)}((-log\, \bar{G}(x))^{\alpha-1}\, (\bar{G}(x))^{\beta}$$

Thus $f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}((-log\,\bar{G}(x))^{\alpha-1}\,(\,\bar{G}(x))^{\beta} * \frac{g(x)}{\bar{G}(x)}$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)}((-\log\bar{G}(x))^{\alpha-1}\left(\bar{G}(x)\right)^{\beta-1}g(x)\,;\; x > 0, \alpha, \beta > 0 \tag{8.7}$$

$w$hich is type I $-$ gamma 2 Generator.

- **Type I – Gamma 2 Generated Distributions**

Gamma 2 - Exponential Distribution

Let $g(x) = \beta e^{-\beta x}$ then $G(x) = 1 - e^{-\beta x} \Rightarrow \bar{G}(x) = e^{-\beta x}$

Therefore

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}((-log\,\bar{G}(x))^{\alpha-1}\,(\,\bar{G}(x))^{\beta-1}g(x)$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)}((-log\,e^{-\beta x})^{\alpha-1}\left(e^{-\beta x}\right)^{\beta-1}\beta e^{-\beta x}$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)}(\beta x)^{\alpha-1}\left(e^{-\beta x}\right)^{\beta-1}\beta e^{-\beta x}$$

$$= \frac{\beta^{2\alpha}}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\beta^2 x} \quad let\;\; \beta^2 = \theta$$

Then $f(x) = \frac{\theta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\theta x} \quad ;\; x > 0, \alpha, \theta > 0$ (8.8)

which is a two-parameter gamma distribution.

Gamma 2 - Weibull Distribution

let $g(x) = \beta\,\theta x^{\theta-1}e^{-\beta x^{\theta}}$ then $G(x) = 1 - e^{-\beta x^{\theta}} \Rightarrow \bar{G}(x) = e^{-\beta x^{\theta}}$

Therefore

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}((-log\,\bar{G}(x))^{\alpha-1}\left(\breve{G}(x)\right)^{\beta-1}g(x)$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)}((-log\,e^{-\beta x^{\theta}})^{\alpha-1}\left(e^{-\beta x^{\theta}}\right)^{\beta-1}\beta\,\theta x^{\theta-1}e^{-\beta x^{\theta}}$$

$$= \frac{\theta \beta^{2\alpha}}{\Gamma(\alpha)} x^{\theta\alpha-1} e^{-\beta^2 x^{\theta}} \quad let\, \beta^2 = \lambda$$

$$f(x) = \frac{\theta \lambda^{\alpha}}{\Gamma(\alpha)} x^{\theta\alpha-1} e^{-\lambda x^{\theta}} \; ; \; x, \lambda, \theta > 0 \tag{8.9}$$

which is a three-parameter gamma distribution.

8.3 Type II –Gamma 1 Generator

Let the Survival Distribution function be defined by;

$$S(x) = \int_0^{-log(G(x))} \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \, e^{-t} \; dt$$

$$i.e\; 1 - F(x) = \int_0^{-log(G(x))} \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \, e^{-t} \; dt$$

Therefore

$$-f(x) = \frac{d}{dx} \int_0^{-log(G(x))} \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \, e^{-t} \; dt$$

$$= \frac{1}{\Gamma(\alpha)} [-log(G(x)]^{\alpha-1} \, e^{-(-log(G(x))} \frac{d}{dx} [-log(G(x)]$$

$$-f(x) = \frac{(G(x)}{\Gamma(\alpha)} [-log(G(x)]^{\alpha-1} - \frac{g(x)}{(G(x)}$$

$$f(x) = \frac{1}{\Gamma(\alpha)} [-\, log(G(x)]]^{\alpha-1} g(x) \, ; \; -\infty < x < \infty, \alpha > 0 \tag{8.10}$$

Type II-gamma 1 generator

- **Type II - Gamma1 Generated Distributions**

Gamma1 - Gumbel Distribution

Construction of Gumbel Distribution

Let $y = e^{-x}$ $where\ y{\sim}Exp(1) \Rightarrow \frac{dy}{dx} = -e^{-x}$

Thus

$$g(y) = e^{-y},$$

$$g(x) = g(y)\left|\frac{dy}{dx}\right|$$

$$= e^{-y}e^{-x}\ but\ e^{-x}$$

$$= e^{-e^{-x}}e^{-x};\ -\infty < x < \infty \tag{8.11}$$

which is a Gumbel distribution

Now let $g(x) = e^{-e^{-x}}e^{-x}$

Then $G(x) = \int_{-\infty}^{x} e^{-e^{-y}}e^{-y}\,dy$, let $u = e^{-e^{-y}} \Rightarrow du = e^{-e^{-y}}e^{-y}\,dy$

Therefore $G(x) = \int_{-\infty}^{e^{-e^{-x}}} du = e^{-e^{-x}}$

Thus $f(x) = \frac{g(x)}{\Gamma(\alpha)}[-logG(x)]^{\alpha-1}$

$$= \frac{g(x)}{\Gamma(\alpha)}[-loge^{-e^{-x}}]^{\alpha-1}$$

$$= \frac{e^{-e^{-x}}e^{-x}}{\Gamma(\alpha)}(e^{-x})^{\alpha-1}$$

$$= \frac{e^{-e^{-x}}e^{-\alpha x}}{\Gamma(\alpha)};\ x, \alpha > 0 \tag{8.12}$$

Gamma 1 - Fréchet Distribution

Construction of Fréchet Distribution

Let $t = (\frac{\theta}{x})^{c}$ where t is an exponential with parameter 1.

Therefore $\frac{dt}{dx} = -c(\frac{\theta}{x})^{c-1}\frac{\theta}{x^2}$

$$g(x) = e^{-t}\left|\frac{dt}{dx}\right|$$

$$= \theta c e^{-(\frac{\theta}{x})^c} (\frac{\theta}{x})^{c-1} \frac{1}{x^2}$$

$$= \frac{c\theta^c e^{-(\frac{\theta}{x})^c}}{x^{c+1}} \; ; \; x \geq 0, c, \theta > 0 \qquad (8.13)$$

which is a Fréchet distribution.

Then $\quad G(x) = \int_o^x \frac{c\theta^c e^{-\left(\frac{\theta}{y}\right)^c}}{y^{c+1}} dy$

Let $\quad u = e^{-\left(\frac{\theta}{y}\right)^c}$

$$du = c\theta(\frac{\theta}{y})^{c-1} e^{-\left(\frac{\theta}{y}\right)^c} \frac{1}{y^2} dy$$

$$= c\theta^c \frac{1}{y^{c+1}} e^{-\left(\frac{\theta}{y}\right)^c}$$

Therefore $\quad G(x) = \int_o^{e^{-\left(\frac{\theta}{x}\right)^c}} du = e^{-\left(\frac{\theta}{x}\right)^c}$

Thus $\quad f(x) = \frac{g(x)}{\Gamma(\alpha)} [-logG(x)]^{\alpha-1}$

$$= \frac{[-\boldsymbol{log} e^{-\left(\frac{\theta}{x}\right)^c}]^{\boldsymbol{\alpha}-\mathbf{1}}}{\Gamma(\boldsymbol{\alpha})} \frac{c\theta^c e^{-(\frac{\theta}{x})^c}}{x^{c+1}}$$

$$= c\theta^{c\boldsymbol{\alpha}} \frac{e^{-(\frac{\theta}{x})^c}}{\Gamma(\boldsymbol{\alpha}) x^{c\boldsymbol{\alpha}+\mathbf{1}}} \; ; \; x, \alpha, c > 0 \qquad (8.14)$$

8.4 Type II-Gamma 2 Generator

Let the Survival Distribution function be defined by;

Let $\quad 1 - F(x) = S(x) = W(-log\, G(x))$

$$\frac{d(1-F(X))}{dx} = \frac{d}{dx} W(-log\, G(x)) \frac{d}{dx}(-logG(x))$$

$$-f(x) = w(-log\, G(x)) * -\frac{1}{G(x)} \frac{d}{dx}(G(x))$$

Thus

$f(x) = w(-log\, G(x)) * \frac{g(x)}{G(x)}$ where G(x) is the cumulative distribution function.

Let $w(t) = \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t};\ t,\alpha,\beta > 0\ then$

$$w(\text{-}log\ \breve{G}(x)) = \frac{\beta^\alpha}{\Gamma(\alpha)} ((-log\ G(x))^{\alpha-1} e^{-\beta(-log\ G(x))}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} ((-log\ G(x))^{\alpha-1} e^{log(G(x))^\beta}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} ((-\log G(x))^{\alpha-1} (G(x))^\beta$$

Thus

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} ((-log\ G(x))^{\alpha-1} (G(x))^\beta * \frac{g(x)}{G(x)}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} ((-\log G(x))^{\alpha-1} \left(G(x)\right)^{\beta-1} g(x)\ ;\ x > 0, \alpha, \beta > 0 \qquad (8.15)$$

which is type II − gamma 2 Generator.

- **Type II – Gamma 2 Generated Distributions**

Gamma 2 - Gumbel Distribution

Let $g(x) = e^{-e^{-x}} e^{-x};\ -\infty < x < \infty$

Then $G(x) = \int_{-\infty}^{x} e^{-e^{-y}} e^{-y}\, dy$, let $u = e^{-e^{-y}} \Rightarrow du = e^{-e^{-y}} e^{-y}\, dy$

$$Therefore\ \ G(x) = \int_{-\infty}^{e^{-e^{-x}}} du = e^{-e^{-x}}$$

$$Thus \qquad f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} (-log\, e^{-e^{-x}})^{\alpha-1} \left(e^{-e^{-x}}\right)^{\beta-1} * e^{-e^{-x}} e^{-x}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} (e^{-x})^{\alpha-1+1} \left(e^{-e^{-x}}\right)^{\beta-1+1} * e^{-e^{-x}}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\alpha x} e^{-\beta e^{-x}}\ ;\ \text{x}, \alpha, \beta > 0 \qquad (8.16)$$

Gamma 2 - Fréchet Distribution

Let $g(x) = \frac{c\theta^c e^{-(\frac{\theta}{x})^c}}{x^{c+1}}\ ;\ x \geq 0, c, \theta > 0$

Then $G(x) = \int_o^x \frac{c\theta^c e^{-\left(\frac{\theta}{y}\right)^c}}{y^{c+1}} dy$

$$Let\ u\ =\ e^{-\left(\frac{\theta}{y}\right)^c}$$

$$du\ =\ c\theta(\frac{\theta}{y})^{c-1}e^{-\left(\frac{\theta}{y}\right)^c}\ \frac{1}{y^2}\ dy$$

$$= c\theta^c\frac{1}{y^{c+1}}\ e^{-\left(\frac{\theta}{y}\right)^c}$$

$$Therefore\ G(x)\ =\ \int_o^{e^{-\left(\frac{\theta}{x}\right)^c}}\ du = e^{-\left(\frac{\theta}{x}\right)^c}$$

$$Thus\qquad f(x)\ =\ \frac{\beta^\alpha}{\Gamma(\alpha)}((-log\,G(x))^{\alpha-1}\ (\,G(x))^{\beta-1}$$

$$=\ \frac{\beta^\alpha}{\Gamma(\alpha)}((-log\,e^{-\left(\frac{\theta}{x}\right)^c})^{\alpha-1}\left(\,e^{-\left(\frac{\theta}{x}\right)^c}\right)^{\beta-1}\frac{c\theta^c e^{-(\frac{\theta}{x})^c}}{x^{c+1}}$$

$$=\ \frac{\beta^\alpha c\theta^c}{\Gamma(\alpha)}\left(\frac{\theta}{x}\right)^{c(\alpha-1)\,)}\ \left(\,e^{-\left(\frac{\theta}{x}\right)^c}\right)^\beta\frac{1}{x^{c+1}}$$

$$=\ \frac{c\beta^\alpha\theta^{\alpha c}}{\Gamma(\alpha)x^{\alpha c+1}}e^{-\frac{\beta\theta^c}{x^c}}\ =\frac{c(\beta\theta^c)^\alpha}{\Gamma(\alpha)x^{\alpha c+1}}e^{-\frac{\beta\theta^c}{x^c}}\ let\ \beta\theta^c=\lambda$$

$$Thus\ f(x) =\ \frac{c\lambda^\alpha}{\Gamma(\alpha)x^{\alpha c+1}}e^{-\frac{\lambda}{x^c}}\ ; \mathrm{x, c}, \alpha, \lambda > 0 \qquad (8.17)$$

CHAPTER IX: GAMMA EXPONENTIATED GENERATED DISTRIBUTIONS

9.0 Introduction

A new family of distributions, namely the exponentiated exponential distribution was introduced by Gupta *et al.* (1998). The family has two parameters (scale and shape) similar to the Weibull or gamma family. Properties of the distribution were studied by Gupta and Kundu (2001). They observed that many properties of the new family are similar to those of the Weibull or gamma family. Hence the distribution can be used as an alternative to a Weibull or gamma distribution. The two-parameter Weibull and Gamma distributions are the most popular distributions used for analyzing lifetime data.

Adding a parameter (a positive real number) to a cumulative distribution function (CDF) G by exponentiation produces a CDF which is richer and more flexible to modeling data. The exponentiated distribution (ED) $H(x) = [G(x)]^{\alpha}$ is quite different from G. The function $H(x) = [G(x)]^{\alpha}$ is known as: exponentiated distribution (ED) since G is exponentiated by α. It is also known as: proportional reversed hazard rate model (PRHRM) since reversed hazard rate function (RHRF) of H is defined by $\lambda_H^*(x) = \frac{d}{dx}[lnH(x)] = \frac{h(x)}{H(x)}$ where *h*(*x*)is the PDF corresponding to *H(x).*

$$\text{Hence} \quad \lambda_H^*(x) = \frac{\alpha\big(G(x)\big)^{\alpha-1} g(x)}{\big(G(x)\big)^{\alpha}} = \alpha\lambda_G^*(x) \qquad (9.0)$$

9.1 Type I-Gamma1 Exponentiated Generator

- **Type I-Gamma 1 Exponentiated Generated Distributions**

Let $F(x) = W[(-log\,\bar{G}(x))^{\alpha}]$

$$= W[(-\alpha log\,\bar{G}(x))]$$

$$\frac{dF(X)}{dx} = \frac{d}{dx}W[(-\alpha log\ \bar{G}(x))]$$

$$= w(-\alpha log\ \bar{G}(x)) * -\frac{\alpha}{\bar{G}(x)}\frac{d}{dx}(\bar{G}(x))$$

$$= w(-\alpha log\ \bar{G}(x)) * -\frac{\alpha}{\bar{G}(x)}\frac{d}{dx}(1 - G(x))$$

$$= w(-\alpha log\ \bar{G}(x)) * \frac{\alpha g(x)}{\bar{G}(x)}$$

If $$w(t) = \frac{t^{\sigma-1}e^{-t}}{\Gamma(\sigma)}\ ; x > 0, \sigma > 0$$

$$w(-\alpha log\ \bar{G}(x)) = \frac{(-\alpha log\ \bar{G}(x))^{\sigma-1}e^{-[(-\alpha log\ \bar{G}(x)]}}{\Gamma(\sigma)}$$

$$= \frac{(-\alpha log\ \bar{G}(x))^{\sigma-1}[\bar{G}(x)]^{\alpha}}{\Gamma(\sigma)}$$

Therefore $$f(x) = w(-\alpha log\ \bar{G}(x)) * \frac{\alpha g(x)}{\bar{G}(x)}$$

$$= \frac{\alpha^{\sigma}(-log\ \bar{G}(x))^{\sigma-1}[\bar{G}(x)]^{\alpha-1}\ g(x)}{\Gamma(\sigma)}; -\infty < x < \infty \tag{9.1}$$

Gamma 1- Exponentiated Generator.

Gamma 1 - Exponentiated Exponential Distribution

For Exponential distribution,

$$g(x) = \lambda e^{-\lambda x}$$

$$G(x) = 1 - e^{-\lambda x}\ and\ \bar{G}(x) = 1 - G(x) \Rightarrow \bar{G}(x) = e^{-\lambda x}$$

$$f(x) = \frac{\alpha^{\sigma}}{\Gamma(\sigma)}(-log\ \bar{G}(x))^{\sigma-1}[\bar{G}(x)]^{\alpha-1}\ g(x)$$

$$= \frac{\alpha^{\sigma}}{\Gamma(\sigma)}(-log\ e^{-\lambda x})^{\sigma-1}\left[e^{-\lambda x}\right]^{\alpha-1}\lambda e^{-\lambda x}$$

$$= \frac{\lambda^{\sigma}\ \alpha^{\sigma}}{\Gamma(\sigma)}x^{\sigma-1}\left[e^{-\lambda x}\right]^{\alpha-1+1} = \frac{(\lambda\alpha)^{\sigma}}{\Gamma(\sigma)}x^{\sigma-1}e^{-\alpha\lambda x}, let\ \lambda\alpha = \theta$$

Then $f(x) = \frac{\theta^{\sigma}}{\Gamma(\sigma)} x^{\sigma-1} e^{-\theta x}\ ;\ x > 0, \sigma, \theta > 0$ (9.2)

which is a two-parameter gamma distribution.

Gamma 1 - Exponentiated Weibull Distribution

For weibull distribution

let $g(x) = \beta\, \theta x^{\theta-1} e^{-\beta x^{\theta}},$ then $G(x) = 1 - e^{-\beta x^{\theta}} \Rightarrow \breve{G}(x) = e^{-\beta x^{\theta}}$

But $f(x) = \frac{\alpha^{\sigma}(-log\ \bar{G}(x))^{\sigma-1}[\bar{G}(x)]^{\alpha-1}\ g(x)}{\Gamma(\sigma)}$

$$= \frac{\alpha^{\sigma}}{\Gamma(\sigma)}\left(-\,log\ e^{-\beta x^{\theta}}\right)^{\sigma-1}\left[e^{-\beta x^{\theta}}\right]^{\alpha-1}\ \beta\, \theta x^{\theta-1} e^{-\beta x^{\theta}}$$

$$= \frac{\alpha^{\sigma}}{\Gamma(\sigma)}\left(\beta x^{\theta}\right)^{\sigma-1}\left[e^{-\beta x^{\theta}}\right]^{\alpha}\ \beta\, \theta x^{\theta-1}$$

$$= \frac{\theta(\beta\alpha)^{\sigma}}{\Gamma(\sigma)} x^{\theta\sigma-1} e^{-\alpha\beta x^{\theta}}; x, \sigma, \beta, \alpha > 0 \qquad (9.3)$$

which is a three-parameter gamma distribution /Stacy distribution.

9.2 Type I-gamma 2 Exponentiated Generator

Let $F(x) = W[(-log\ \bar{G}(x))^{\alpha}]$

$$= W[(-\alpha log\ \bar{G}(x))]$$

$$\frac{dF(X)}{dx} = \frac{d}{dx} W[(-\alpha log\ \bar{G}(x))]$$

$$= w(-\alpha log\ \bar{G}(x)) * -\frac{\alpha}{\bar{G}(x)}\frac{d}{dx}(\bar{G}(x))$$

$$= w(-\alpha log\ \bar{G}(x)) * -\frac{\alpha}{\bar{G}(x)}\frac{d}{dx}(1 - G(x))$$

$$= w(-\alpha log\ \bar{G}(x)) * \frac{\alpha g(x)}{\bar{G}(x)}$$

If $w(t) = \frac{\beta^{\sigma}}{\Gamma(\sigma)} e^{-\beta t} t^{\ \sigma-1}$; t>0, $\sigma, \beta > 0$

$$W(-\alpha log\ \bar{G}(x)) = \frac{\beta^{\sigma}}{\Gamma(\sigma)} e^{-\beta(-\alpha log\ \bar{G}(x))}(-\alpha log\ \bar{G}(x))^{\ \sigma-1}$$

$$= \frac{\beta^{\sigma}}{\Gamma(\sigma)} e^{\beta\alpha(log\ \bar{G}(x))}(-\alpha log\ \bar{G}(x))^{\ \sigma-1}$$

$$= \frac{\beta^{\sigma}}{\Gamma(\sigma)} (\bar{G}(x))^{\beta\alpha}(-\alpha log\ \bar{G}(x))^{\ \sigma-1}$$

Therefore $f(x) = w(-\alpha log\ \bar{G}(x)) * \frac{\alpha g(x)}{\bar{G}(x)}$

$$= \frac{\beta^{\sigma}}{\Gamma(\sigma)} \left(\bar{G}(x)\right)^{\beta\alpha}(-\alpha lo\ g\ \bar{G}(x))^{\sigma-1})) * \frac{\alpha g(x)}{\bar{G}(x)}$$

$$= \frac{\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)} \left(\bar{G}(x)\right)^{\beta\alpha-1}(-\log \bar{G}(x))^{\sigma-1}))g(x);\ -\infty < x < \infty, \alpha, \beta, \sigma > 0 \quad (9.4)$$

type I- Gamma 2 –Exponentiated Generator

- **Type I-Gamma 2 Exponentiated Generated Distributions**

Gamma 2 - Exponentiated Exponential Distribution

For Exponential distribution

$$g(x) = \lambda e^{-\lambda x}$$

$$G(x) = 1 - e^{-\lambda x}\ and\ \bar{G}(x) = 1 - G(x) \Rightarrow \bar{G}(x) = e^{-\lambda x}$$

$$f(x) = \frac{\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)} \left(\bar{G}(x)\right)^{\beta\alpha-1}(-lo\ g\ \bar{G}(x))^{\sigma-1})g(x)$$

$$= \frac{\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)} \left(e^{-\lambda x}\right)^{\beta\alpha-1}\left(-lo\ g\ e^{-\lambda x}\right)^{\sigma-1})\lambda e^{-\lambda x}$$

$$= \frac{\lambda^{\sigma}\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)} \left(e^{-\lambda x}\right)^{\beta\alpha} x^{\sigma-1}$$

$$= \frac{(\lambda\alpha\beta)^{\sigma}}{\Gamma(\sigma)} e^{-\lambda\beta\alpha x} x^{\sigma-1}\ \ let\ \ \lambda\alpha\beta = \theta$$

Then

$$f(x) = \frac{\theta^{\sigma}}{\Gamma(\sigma)} e^{-\theta x} x^{\sigma-1};\ x, \sigma, \theta > 0 \quad (9.5)$$

which is a two-parameter gamma distribution.

Gamma 2 - Exponentiated Weibull Distribution

For weibull distribution

Let $$g(x) = \beta\, \theta x^{\theta-1} e^{-\beta x^{\theta}},\ then\ G(x) = 1 - e^{-\beta x^{\theta}} \Rightarrow \bar{G}(x) = e^{-\beta x^{\theta}}$$

But $$f(x) = \frac{\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)}\left(\bar{G}(x)\right)^{\beta\alpha-1}(-lo\, g\, \bar{G}(x))^{\sigma-1})g(x)$$

$$= \frac{\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)}\left(e^{-\beta x^{\theta}}\right)^{\beta\alpha-1}\left(-lo\, g\, e^{-\beta x^{\theta}}\right)^{\sigma-1})\beta\, \theta x^{\theta-1}e^{-\beta x^{\theta}}$$

$$= \frac{\theta(\alpha\beta^{2})^{\sigma}}{\Gamma(\sigma)} e^{-\alpha\beta^{2}x^{\theta}} x^{\theta\sigma-1}\ Let\ \alpha\beta^{2} = \lambda$$

$$= \frac{\theta\lambda^{\sigma}}{\Gamma(\sigma)} e^{-\lambda x^{\theta}} x^{\theta\sigma-1}$$

9.3 A Class of Gamma Exponentiated Exponential Distributions.

For $$g(x) = \beta e^{-\beta x}\ \ and\ G(x) = 1 - e^{-\beta x} \Rightarrow \bar{G}(x) = e^{-\beta x}$$

Thus for $$f(x) = \frac{\alpha^{\sigma}}{\Gamma(\sigma)}\left(-lo\, g\, e^{-\beta x}\right)^{\sigma-1}\left[e^{-\beta x}\right]^{\alpha-1} \frac{d}{dx}\left(1 - e^{-\beta x}\right)$$

$$= \frac{\alpha^{\sigma}}{\Gamma(\sigma)}(\beta x)^{\sigma-1}\left[e^{-\beta x}\right]^{\alpha-1} \frac{d}{dx}\left(1 - e^{-\beta x}\right)\ replace\ x\ by\ \Phi(x)$$

$$= \frac{\alpha^{\sigma}}{\Gamma(\sigma)}(\beta\Phi(x))^{\sigma-1}\left[e^{-\beta\Phi(x)}\right]^{\alpha-1} \frac{d}{dx}\left(1 - e^{-\beta\Phi(x)}\right)$$

$$= \frac{\alpha^{\sigma}}{\Gamma(\sigma)}(\beta\Phi(x))^{\sigma-1}\left[e^{-\beta\Phi(x)}\right]^{\alpha-1} \beta\Phi'(x)e^{-\beta\Phi(x)}$$

$$= \frac{(\beta\alpha)^{\sigma}}{\Gamma(\sigma)}(\Phi(x))^{\sigma-1}e^{-\alpha\beta\Phi(x)}\,\Phi'(x) \qquad (9.6)$$

Special Cases

Case (i)

if $$\Phi(x) = x \Rightarrow \Phi'(x) = 1$$

Therefore $$f(x) = \frac{(\beta\alpha)^{\sigma}}{\Gamma(\sigma)}\left(\Phi(x)\right)^{\sigma-1}e^{-\alpha\beta\Phi(x)}\,\Phi'(x)$$

$$= \frac{(\alpha\beta)^{\sigma}}{\Gamma(\sigma)} e^{-\alpha\beta x} x^{\sigma-1}; x > 0, \alpha, \beta, \sigma > 0 \qquad (9.7)$$

Case (ii)

if $\Phi(x) = x^{\theta} \Rightarrow \Phi'(x) = \theta x^{\theta-1}$

Therefore

$$f(x) = \frac{(\beta\alpha)^{\sigma}}{\Gamma(\sigma)} (\Phi(x))^{\sigma-1} e^{-\alpha\beta\Phi(x)} \Phi'(x)$$

$$= \frac{(\alpha\beta)^{\sigma}}{\Gamma(\sigma)} e^{-\alpha\beta x^{\theta}} \left(x^{\theta}\right)^{\sigma-1} * \theta x^{\theta-1}$$

$$= \frac{\theta(\alpha\beta)^{\sigma}}{\Gamma(\sigma)} e^{-\alpha\beta x^{\theta}} x^{\theta\sigma-1} x > 0, \alpha, \beta, \sigma > 0 \qquad (9.8)$$

9.4 Type II-Gamma 1 Exponentiated Generator

Let the Survival Distribution function be defined by;

Let

$$1 - F(x) = W(-\alpha log\ G(x))$$

$$\frac{d(1-F(X))}{dx} = d/dx W(-\alpha log\ G(x))\ \ d/dx(-\alpha log G(x))$$

$$-f(x) = -w(-\alpha log\ G(x)) * \frac{\alpha g(x)}{G(x)}$$

$$f(x) = w(-\alpha log\ G(x)) * \frac{\alpha g(x)}{G(x)},$$

where G(x)is the cumulative distribution function.

Let $w(t) = \frac{e^{-t}}{\Gamma(\sigma)} t^{\sigma-1}\ ;\ t, \alpha > 0$ *then*

$$w(-\alpha log\ G(x)) = \frac{e^{-(-\alpha log\ G(x))}}{\Gamma(\sigma)} (-\alpha log\ G(x))^{\sigma-1}$$

$$= \frac{[G(x)]^{\alpha}}{\Gamma(\sigma)} [-\alpha log\ G(x)]^{\sigma-1}$$

Thus $f(x) = \frac{[G(x)]^{\alpha}}{\Gamma(\sigma)} [-\alpha log\ G(x)]^{\sigma-1} * \frac{\alpha g(x)}{G(x)}$

$$= \frac{\alpha^{\sigma}[G(x)]^{\alpha-1}}{\Gamma(\sigma)} [-\log G(x)]^{\sigma-1} g(x) \; ; \; -\infty < x < \infty, \alpha, \sigma > 0 \qquad (9.9)$$

- **Type II-Gamma 1 Exponentiated Generated Distributions**

Type II-Gamma 1 - Exponentiated Gumbel Distribution

$$Let\ f(x) = \frac{\alpha^{\sigma}[G(x)]^{\alpha-1}}{\Gamma(\sigma)} [-log\ G(x)]^{\sigma-1} g(x)$$

For Gumbel distribution let

Let $g(x) = e^{-e^{-x}}\ e^{-x}$; $-\infty < x < \infty$, then $G(x) = e^{-e^{-x}}$

Therefore $f(x) = \frac{\alpha^{\sigma}\left[\boldsymbol{e}^{-e^{-x}}\right]^{\alpha-1}}{\Gamma(\sigma)} \left[-log\ \boldsymbol{e}^{-e^{-x}}\right]^{\sigma-1} \boldsymbol{e}^{-e^{-x}}\ \boldsymbol{e}^{-x}$

$$= \frac{\alpha^{\sigma}\left[\boldsymbol{e}^{-e^{-x}}\right]^{\alpha-1}}{\Gamma(\sigma)} [\boldsymbol{e}^{-x}]^{\sigma-1} \boldsymbol{e}^{-e^{-x}}\ \boldsymbol{e}^{-x}$$

$$= \frac{\alpha^{\sigma}\left[\boldsymbol{e}^{-e^{-x}}\right]^{\alpha}}{\Gamma(\sigma)} [\boldsymbol{e}^{-x}]^{\sigma}$$

$$f(x) = \frac{\alpha^{\sigma} \boldsymbol{e}^{-\alpha e^{-x}}\ \boldsymbol{e}^{-\sigma x}}{\Gamma(\sigma)} \; ; \; x > 0, \sigma, \alpha > 0 \qquad (9.10)$$

Type II-Gamma 1 Exponentiated Fréchet Distribution

$$Let\ f(x) = \frac{\alpha^{\sigma}[G(x)]^{\alpha-1}}{\Gamma(\sigma)} [-log\ G(x)]^{\sigma-1} g(x)$$

For Fréchet distribution

$$g(x) = \frac{c\theta^{c} e^{-(\frac{\theta}{x})^{c}}}{x^{c+1}} \; ; \; x \geq 0, c, \theta > 0\ \ and\ G(x) = e^{-(\frac{\theta}{x})^{c}}$$

Thus $f(x) = \frac{\alpha^{\sigma}\left[e^{-(\frac{\theta}{x})^{c}}\right]^{\alpha-1}}{\Gamma(\sigma)} \left[-log\ \ e^{-(\frac{\theta}{x})^{c}}\right]^{\sigma-1} * \frac{c\theta^{c} e^{-(\frac{\theta}{x})^{c}}}{x^{c+1}}$

$$= \frac{c\theta^{c}\alpha^{\sigma}\left[e^{-(\frac{\theta}{x})^{c}}\right]^{\alpha}}{\Gamma(\sigma)} \left[(\frac{\theta}{x})^{c}\right]^{\sigma-1} * \frac{1}{x^{c+1}}$$

Therefore

$$= \frac{c\theta^{\sigma c}\alpha^{\sigma}\left[e^{-(\frac{\theta}{x})^{c}}\right]^{\alpha}}{\Gamma(\sigma)x^{\sigma c+1}}\ ;\ x,\sigma,\alpha,c > 0 \qquad (9.11)$$

9.5 Type II-Gamma 2 Exponentiated Generator

Let $1 - F(x) = S(x) = W(-\alpha log\ G(x))$

$$\frac{d(1-F(X))}{dx} = \frac{d}{dx}W(\text{-}\alpha log\ G(x))\frac{d}{dx}(-\alpha logG(x))$$

$$-f(x) = w(-\alpha log\ G(x)) * -\frac{\alpha g(x)}{G(x)}$$

Thus

$f(x) = w(-\alpha log\ G(x)) * \frac{\alpha g(x)}{G(x)}$ where G(x) is the cumulative distribution function.

If $\quad w(t) = \frac{\beta}{\Gamma(\sigma)}^{\sigma} e^{-\beta t}t^{\sigma-1}\ ;\ t,\sigma,\beta > 0$

$$w(-\alpha log\ G(x)) = \frac{\beta}{\Gamma(\sigma)}^{\sigma} e^{-\beta(-\alpha log\ G(x))}(-\alpha log\ G(x))^{\ \sigma-1}$$

$$= \frac{\beta}{\Gamma(\sigma)}^{\sigma} G(x)^{\alpha\beta}(-\alpha log\ G(x))^{\ \sigma-1}$$

Therefore

$$f(x) = \frac{\beta}{\Gamma(\sigma)}^{\sigma} G(x)^{\alpha\beta}(-\alpha \log G(x))^{\sigma-1} * \frac{\alpha g(x)}{G(x)}\ ;\ -\infty < x < \infty, \sigma,\alpha,\beta > 0 \qquad (9.12)$$

Which is type II-Gamma 2 Exponentiated Generator.

- **Type II-Gamma 2 Exponentiated Generated Distributions**

Gamma 2 - Exponentiated Gumbel Distribution

$$f(x) = \frac{\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)} (G(x))^{\alpha\beta}(-log\, G(x))^{\sigma-1} * \frac{g(x)}{G(x)}$$

For Gumbel distribution

Let $\quad g(x) = e^{-e^{-x}}\, e^{-x};\ -\infty < x < \infty$, then $G(x) = e^{-e^{-x}}$

Then $\quad f(x) = \frac{\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)} (e^{-e^{-x}})^{\alpha\beta}(-log\, e^{-e^{-x}})^{\sigma-1} * \frac{e^{-e^{-x}}\, e^{-x}}{e^{-e^{-x}}}$

$$= \frac{\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)} e^{-\alpha\beta e^{-x}}\, e^{-\sigma x}\ ;\ x, \sigma, \beta, \alpha > 0$$

$$f(x) = \frac{\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)} \left(G(x)\right)^{\alpha\beta-1}(-\log G(x))^{\sigma-1} * g(x)$$

Gamma 2-Exponentiated Fréchet Distribution

For Fréchet distribution

Let $\quad g(x) = \frac{c\theta^{c}e^{-(\frac{\theta}{x})^{c}}}{x^{c+1}}\ ;\ x \geq 0, c, \theta > 0$ and $G(x) = e^{-(\frac{\theta}{x})^{c}}$

Then $\quad f(x) = \frac{\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)} (\, e^{-(\frac{\theta}{x})^{c}})^{\alpha\beta-1}(-log\ e^{-(\frac{\theta}{x})^{c}})^{\sigma-1} * \frac{c\theta^{c}e^{-(\frac{\theta}{x})^{c}}}{x^{c+1}}$

$$= \frac{c\theta^{\sigma c}\alpha^{\sigma}\beta^{\sigma}}{\Gamma(\sigma)x^{\sigma c+1}} (\, e^{-(\frac{\theta}{x})^{c}})^{\alpha\beta}; \mathrm{x, c}, \alpha, \beta, \sigma > 0 \qquad (9.13)$$

9.6 A Class of Gamma Exponentiated Inverse Exponential Distributions

Let $\quad y = \frac{\theta}{x} \Rightarrow x = \frac{\theta}{y}$ where *x* is Exp (1) i.e

$$f(x) = e^{-x}\ ;\ x > 0,\ \ \frac{dx}{dy} = -\frac{\theta}{y^{2}}$$

$$g(y) = f(x) * \left|\frac{dx}{dy}\right| = e^{-x} * \frac{\theta}{y^{2}}$$

$$= \frac{\theta e^{-\frac{\theta}{y}}}{y^2}$$

$$G(x) = \int_0^x \frac{\theta e^{-\frac{\theta}{y}}}{y^2}\, dy \,, Let\ u = e^{-\frac{\theta}{y}} \Rightarrow du = \frac{\theta e^{-\frac{\theta}{y}}}{y^2}\, dy$$

Therefore $G(x) = \int_0^{e^{-\frac{\theta}{x}}} du = e^{-\frac{\theta}{x}}$

Then $g(x) = \frac{\theta e^{-\frac{\theta}{x}}}{x^2};\ x, \theta > 0$

Thus for $f(x) = \frac{\alpha^{\sigma}[G(x)]^{\alpha-1}}{\Gamma(\sigma)} [-log\ G(x)]^{\sigma-1} g(x)$

$$= \frac{\alpha^{\sigma}\left[e^{-\frac{\theta}{x}}\right]^{\alpha-1}}{\Gamma(\sigma)} \left[-log\ e^{-\frac{\theta}{x}}\right]^{\sigma-1} * \frac{d}{dx} e^{-\frac{\theta}{x}},$$

$$= \frac{\theta^{\sigma-1}\alpha^{\sigma}\left[e^{-\theta(\frac{1}{x})}\right]^{\alpha-1}}{\Gamma(\sigma)} \left[(\frac{1}{x})\right]^{\sigma-1} * \frac{d}{dx} e^{-\theta(\frac{1}{x})}\ ;\ replace\ \frac{1}{x}\ by\ \Phi(\frac{1}{x})$$

$$= \frac{\theta^{\sigma-1}\alpha^{\sigma}\left[e^{-\theta\Phi(\frac{1}{x})}\right]^{\alpha-1}}{\Gamma(\sigma)} \left[\Phi(\frac{1}{x})\right]^{\sigma-1} * \frac{d}{dx} e^{-\theta\Phi(\frac{1}{x})}$$

$$= \frac{(\alpha\theta)^{\sigma}}{\Gamma(\sigma)} e^{-\alpha\theta\Phi\left(\frac{1}{x}\right)} \left[\Phi\left(\frac{1}{x}\right)\right]^{\sigma-1} * (-\Phi'\left(\frac{1}{x}\right)$$

More generally, replace $\frac{1}{x}$ by $\frac{1}{\Phi(x)}$

Then $$f(x) = \frac{\theta^{\sigma-1}\alpha^{\sigma}\left[e^{-\theta\frac{1}{\Phi(x)}}\right]^{\alpha-1}}{\Gamma(\sigma)} \left[\frac{1}{\Phi(x)}\right]^{\sigma-1} * \frac{d}{dx} e^{-\theta(\frac{1}{\Phi(x)})}$$

$$= \frac{(\alpha\theta)^{\sigma} e^{-\frac{\alpha\theta}{\Phi(x)}}}{\Gamma(\sigma)} \left[\frac{1}{\Phi(x)}\right]^{\sigma-1} * -\frac{d}{dx}\left[\left(\frac{1}{\Phi(x)}\right)\right] \qquad (9.14)$$

Special Cases

Case (i)

For $f(x) = \frac{(\alpha\theta)^{\sigma} e^{-\alpha\theta\Phi\left(\frac{1}{x}\right)}}{\Gamma(\sigma)} \left[\Phi\left(\frac{1}{x}\right)\right]^{\sigma-1} * \left(-\Phi'\left(\frac{1}{x}\right)\right), if\ \Phi\left(\frac{1}{x}\right) = \frac{1}{x} \Rightarrow \Phi'\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

Then $f(x) = \frac{(\alpha\theta)^{\sigma} e^{-\alpha\theta\frac{1}{x}}}{\Gamma(\sigma)} \left[\frac{1}{x}\right]^{\sigma-1} * -\frac{d}{dx}(\frac{1}{x})$

$$= \frac{(\alpha\theta)^{\sigma} e^{-\alpha\theta\frac{1}{x}}}{\Gamma(\sigma)} \left[\frac{1}{x}\right]^{\sigma-1} * \frac{1}{x^2}$$

$$= \frac{(\alpha\boldsymbol{\theta})^{\sigma} e^{-\frac{\alpha\boldsymbol{\theta}}{x}}}{\Gamma(\sigma)\, x^{\sigma+1}} \ ;\ x > 0, \sigma, \theta, \alpha > 0 \qquad (9.15)$$

Case ii)

$$if\ \Phi\left(\frac{1}{x}\right) = (\frac{1}{x})^{c} \Rightarrow \Phi'\left(\frac{1}{x}\right) = -\frac{c}{x^{c+1}}$$

Thus $f(x) = \frac{(\alpha\theta)^{\sigma} e^{-\alpha\theta(\frac{1}{x})^{c}}}{\Gamma(\sigma)} \left[(\frac{1}{x})^{c}\right]^{\sigma-1} * (\frac{c}{x^{c+1}})$

$$= \frac{(\alpha\theta)^{\sigma} e^{-\frac{\alpha\theta}{x^{c}}}}{\Gamma(\sigma) x^{\sigma c+1}}$$

Special cases for

$$f(x) = \frac{(\alpha\theta)^{\sigma} e^{-\frac{\alpha\theta}{\Phi(x)}}}{\Gamma(\sigma)} \left[\frac{1}{\Phi(x)}\right]^{\sigma-1} * -\frac{d}{dx}[\left(\frac{1}{\Phi(x)}\right)]$$

Case i)

$$\Phi(x) = x$$

$$f(x) = \frac{(\alpha\theta)^{\sigma} e^{-\frac{\alpha\theta}{x}}}{\Gamma(\sigma)} \left[\frac{1}{x}\right]^{\sigma-1} * -\frac{d}{dx}[\left(\frac{1}{x}\right)]$$

$$= \frac{(\alpha\theta)^{\sigma}}{\Gamma(\sigma)\, x^{\sigma+1}}\, e^{-\frac{\alpha\theta}{x}}$$

Case (ii)

$$\Phi(x) = x^{c}$$

$$f(x) = \frac{(\alpha\theta)^{\sigma} e^{\frac{\theta\alpha}{x^{c}}}}{\Gamma(\sigma)} \left[\frac{1}{x^{c}}\right]^{\sigma-1} * -\frac{d}{dx}\left[\left(\frac{1}{x^{c}}\right)\right]$$

$$= \frac{(\alpha\theta)^{\sigma} e^{\frac{\theta\alpha}{x^{c}}}}{\Gamma(\sigma) x^{c\sigma-c}} * \frac{c}{x^{c+1}} = \frac{c(\alpha\theta)^{\sigma} e^{\frac{\theta\alpha}{x^{c}}}}{\Gamma(\sigma) x^{c\sigma+1}}$$

Case (iii)

$$\Phi(x) = e^{x}$$

$$f(x) = \frac{(\alpha\theta)^{\sigma} e^{-\frac{\alpha\theta}{e^{x}}}}{\Gamma(\sigma)} \left[\frac{1}{e^{x}}\right]^{\sigma-1} * -\frac{d}{dx}[\left(\frac{1}{e^{x}}\right)]$$

$$= \frac{(\alpha\theta)^{\sigma} e^{-\alpha\theta e^{-x}}}{\Gamma(\sigma)} * e^{-\sigma x}$$

10. Recommendations

We recommend further studies on the Gamma distribution Mixtures when;

i). Gamma is a mixed distribution

ii). Gamma is a mixing distribution

Also on the use of the Gamma distribution in the analysis of drought data in Kenya for drought monitoring to ending drought emergencies.

11. References

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