AN ACTUARIAL MULTI-STATE MODELLING OF LONG TERM CARE INSURANCE PRODUCTS – A CASE STUDY OF THE KENYAN INSURANCE INDUSTRY

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AN ACTUARIAL RESEARCH PROJECT SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF SCIENCE IN ACTUARIAL SCIENCE, SCHOOL OF MATHEMATICS, UNIVERSITY OF NAIROBI

DECLARATION

This project is my original work and has not been presented for a degree to any other University

Signed.. **Date…………………………………**

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This research project has been submitted for examination with our approval as the University Supervisors:

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DEDICATION

This thesis is dedicated in honour of my late Dad Wilson Cherutich Kibwalei and Mum Linah Soti Kibwalei through whose efforts and struggles enabled me to pursue my education.

To my wife Emmy and my lovely children Collins and Purity whose love, dedication, patience and support enabled me to overcome all the obstacles and challenges throughout my masters studies.

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I wish to recognize the various Heads of ICT, Benefits and Operations Departments of the National Hospital Insurance Fund who made extra efforts in making the LTC data available for this academic project under the guidance of the Managing Director, Mr. Kirgotty. They helped me access all study materials and data such as claim limits without which, the exercise would have been incomplete.

I thank my family too who gave me their greatest support as I went missing from their social tables in order to complete the study successfully. I was also inspired by my class mates in many spheres of life whom I acknowledge for their untiring support. It is difficult to account for all the supporting people but assure them that their effort in assisting me is well recognized. Without your confidence in me, this is academic project would not have been a success.

ABSTRACT

This project illustrates how the mathematics of Markov Stochastic Processes can be used, through the framework of multiple state models, in the actuarial modeling of certain types of Long Term Insurance. Such situations arise when benefits are payable upon a change in the status of the insured or while the insured maintains a given status. Examples include long term medical care insurance, life insurance, annuities, pensions, and disability income insurance.

This markov stochastic processes considered both the time-continuous and time-discrete cases, with constant forces of transition being assumed. However, the project focused on calculations of expected present values of payment streams, and in particular, on calculations of net single and annual premiums of long term medical care stand lone and LTC rider benefits insurance products for the National Hospital Insurance Fund from which the entire data for the four state markov model were derived.

This project is structured as follows: chapter one describes the introduction and background of the use of markov stochastic processes in modeling insurance products, statement of the problem, the research objectives, significance and rationale/justification of the study. Chapter two describes the two state, three state, and four state markov models and processes using Chapman-Kolmogorov equations, Kolmogorov forward differential equations and theoretical and empirical literature reviewed based on similar studies by other actuarial scientists on multi state LTC pricing and reserving models. Chapter three describes the methodology with particular emphasis on the four state markov model, the estimation of the maximum likelihood estimates, parametric graduation of transition rates and probabilities, the calculations of premiums using the equivalence principle, and reserves calculations using Thiele's differential equations. Chapter four outlines the applications of the four-state multiple state markov model comprising healthy, outpatient and in patient sicknesses and death states within a continuous time-discrete state stochastic process framework with the actuarial pricing and reserving calculations of streams of benefit payments for the National Hospital Insurance Fund of Kenya. Finally, chapter five outlines the summary of research findings, conclusions, and recommendations based on the study findings and discusses opportunities for further research.

DEFINITIONS AND INTERPRETATIONS

(I) **"Agreement"** This Agreement and all or any Annexes and endorsements hereto

(II) **Commencement** The date on which an eligible employee or eligible dependant becomes a Member on or after the effective date.

(III) **Civil Servants** A public official who is a member of the civil service employed by the Government of Kenya and is actively in service.

(IV) **Disciplined Services** The Kenya Police, the Administration Police, the Prisons Service and the National Youth Service.

(V) **Customary and Reasonable Charges** Means charges for medical care made by a service provider which shall be considered by NHIF to be customary and reasonable to the extent that they do not exceed the general level of charges being made by other service providers of similar standing in the locality where the charge is incurred when providing like or comparable treatment, services or supplies to individuals of the same sex and of comparable age, for a similar disease or injury.

(VI) **Dependant** Means a declared legal spouse of the Member and / or unmarried child or legally adopted child who relies on a Member for support, provided always that such children are aged below 21 years of age at the date of enrollment or 25 years if enrolled into full time formal education.

(VII) **Effective Date** The date that this medical insurance cover commences as shown on the contract data page.

(VIII) **Eligible Employee** An employee is eligible for membership under this cover upon entering Full Time Active Service of the GOK.

(IX) **Eligible dependant** A dependant will be eligible if s/he is declared as a Dependant by the Principal Member at the commencement of the Medical Cover.

(X) **Employer** The GOK as indicated in the Contract Data Page.

(XI) **Exclusion** Category of treatment, conditions, activities and their related or consequential expenses that are excluded from this contract for which NHIF shall not be liable.

(XII) **Full Time Active Service** An employee (other than a temporary employee) is considered to be in Full Time Active service on any day if the employee is performing or is capable of performing, in the customary manner, all of the regular duties of employment.

(XIII) **Government** Government of the Republic of Kenya.

(XIV) **General Patient** A member or dependant who has been admitted to a hospital and has been assigned a standard ward bed and is receiving treatment under the care of the hospital's panel of physicians.

(XV) **Hospital / Facility** Means an institution, which is legally licensed as a health care provider and is recognized by NHIF.

(XVI) **In patient.** A member who has been admitted to a hospital, is assigned a bed and given diagnostic tests or receives treatment for a disease or injury.

(XVII) **In Force** The cover is in effect for the medical benefits specified in the Annex.

(XVIII) **In-Patient Treatment** Treatment which requires admission in and stay in a hospital or day care surgery.

(XIX) **Limit of Indemnity** This is NHIF's liability as limited in events and amount to the limits and sub limits specified in the Annex as applying to each item or type of cover provided. The overall maximum limit stated thereon is the maximum amount recoverable under this contract as a whole by any Member during any one period of insurance and in total respect of any one covered claimer event.

(XX) **Member** An eligible employee who has completed a membership application form or whose name is on the list given by the employer.

(XXI) **Optical Service** Eye care, eye examination, eye follow up, care and prescription of glasses.

(XXII) **Out Patient Treatment** Treatment that does not require admission and stay in hospital or day care.

(XXIII) **Period of Insurance** The period from the Effective date to the renewal date and each twelve-month period, or any such period as may be agreed between the parties, from the renewal date thereafter.

(XXIV) **Physician** Means a qualified medical practitioner licensed by the competent medical authorities of the country in which treatment is provided and who in rendering such treatment is practicing within the scope of his or her licensing and training.

(XXV) **Proportion of Expenses covered** As indicated on the contract data page

(XXVI) **Insured** Means Civil Servants and Disciplined Services

(XXVIII)KEPI Kenya Expanded Programme on Immunization

LIST OF FIGURES

LIST OF TABLES

TABLE OF CONTENTS

1.0 CHAPTER ONE: INTRODUCTION AND BACKROUND

1.1 Introduction

In the 2012 Unites States of America Presidential elections, one of the key campaign issues was the US national health service financing in which the current US President Barrack Obama sponsored a bill, known in US political parlance as Obamacare, in the US Congress. This legislation provides for universal health care financing for all Americans irrespective of age, gender, race and income. In the last one month the US Federal Government has been shut down with the US Congress refusing to raise the US debt ceiling and American private insurance companies and private medical firms lobbying aggressively against "Obamacare" legislation.

In Kenya, the last coalition government attempted to bring about a similar initiative through the Ministry of Health and the National Hospital Insurance Fund but there was a lot of resistance from the public and trade unions like the Central Organization of Trade Unions. Their main concerns revolved around the issues of the correct pricing in terms of premium rates and the corruption and governance criteria of the funds. At the time of writing this project, some former officials of N.H.I.F. are facing court cases over corruption allegations. This project is therefore an actuarial attempt to scientifically calculate the premium rates and reserves using the multi-state stochastic markov processes in continuous time-discrete state space.

In probability theory, a Markov chain is a stochastic process that refers to the sequence (or chain) of states such a process moves through (Baier, *et al*, 1999). The changes of state of the system are called transitions, and the probabilities associated with various state-changes are called transition probabilities. The process is characterized by a state space, a transition matrix describing the probabilities of particular transitions and an initial state or initial distribution across the state space. By convention, it is assumed that all possible states and transitions have been included in the definition of the processes, so there is always a next state and the process goes on forever.

A discrete-time random process involves a system which is in a certain state at each step, with the state changing randomly between steps(Markov, A. 2006). The steps are often thought of as moments in time, but they can equally well refer to physical distance or any other discrete measurement; formally, the steps are the integers or natural numbers, and the random process is a mapping of these to states.

The Markov property states that the conditional probability distribution for the system at the next step (and in fact at all future steps) depends only on the current state of the system, and not additionally on the state of the system at previous steps. Since the system changes randomly, it is generally impossible to predict with certainty the state of a Markov chain at a given point in the future(Markov,A, 1906). However, the statistical properties of the system's future can be predicted. In many applications, it is these statistical properties that are important.

A continuous-time Markov chain(Markov,A, 1906) is a mathematical model which takes values in some finite or countable set and for which the time spent in each state takes non-negative real values and has an exponential distribution. It is a random process with the Markov property which means that future behavior of the model (both remaining time in current state and next state) depends only on the current state of the model and not on historical behavior. The model is a continuous-time version of the Markov chain model, named because the output from such a process is a sequence (or chain) of states. Some traditional problems in actuarial mathematics are conveniently viewed in terms of multistate processes. It is assumed that, at any time, an individual is in one of a number of states. These properties of Markov chains can be used to model Long Term Care insurance products since the defined states of the insured lives such as healthy, moderately sick(outpatient), severely sick(inpatient), and dead. The individual's presence in a given state or transition from one state to another may have some financial impact. The main task in this project then is to quantify this impact, usually by estimating the expected value of future cash flows.

The simplest situation involves only two states: "alive" and "dead." As shown in Figure 2.1, an individual may make only one transition. For a simple life annuity, benefits are payable while the annuitant is in state 1 and cease upon transition to state 2. In the case of a whole life insurance policy, premiums are payable while the insured is in state 1, and the death benefit is paid at the time of transition to state 2. Approaches to calculating actuarial values in these cases are simple and well-known (Bowers et al.

2004). A more complicated situation arises for processes with additional states. Figure 2.7 illustrates the three-state process commonly used to describe the state of an individual insured under a disability income policy. In this case, premiums are payable while the insured is in state 1, and benefits are payable while the insured is in state 2 (usually after a waiting period). Actuarial calculations for this example are more difficult because the individual can make repeat visits to each of states 1 and 2. For this reason it is often assumed that transitions from state 2 to state 1, that is recoveries, are not possible. A multistate model provides an intuitively pleasing description of the possible outcomes in numerous other areas. In examining a long-term-care system, we can represent the several levels of care available as states of a multistate model. Ongoing costs could then be associated with each state. We could use a multistate process in a life insurance context to describe the movement of individuals among various risk categories such as smoking status and blood pressure grouping(Norris J.R. 1997). Pension plans can also be modeled within a multi-state framework. In the simplest case, states would be required for working plan members, retirees, and those who have died. A more complicated model might require a disabled state and three retired states that reflect the status of a joint and last survivor annuity.

1.2 Background

Many authors have used multistate models to analyze actuarial problems. Much of this work has drawn on the theory of stochastic processes to obtain new results of interest and to generalize results of more traditional methods. Such models are most tractable when it is assumed that the process satisfies the Markov property. Under this assumption, generalizations of a number of standard results from life contingencies can be done (Hoem 1988). The expected value and variance of the loss function in a Markov model setting has been modeled before(Pittacco et al 19993). The stochastic properties of the profit earned on an insurance policy were examined by Habberman (1999), who also analyzed the distribution of surplus. Tolley and Manton(2004) proposed models for morbidity and mortality that include various risk factors in the model state space. In modeling the mortality of individuals infected with the HIV virus, Panjer and Ramsay used a Markov process with states that represent the stages of infection. Waters (1984) discusses the development of formulas for probabilities and the estimation of parameters in a Markov model. The use of more general stochastic models has been considered by among others Hoem(1969), Hoem(1973) and Aalen (1987), Ramsay (1984), Seal (2000), Jones (1999)and Waters (1994).

In Kenya and other developing economies, most insurers offer separate insurance policies which provide financial support to policy-holders upon sickness, disability or death of the policyholder. The most common traditional life insurance products are the term life insurance policy, and the endowment policies. Another important type of insurance products are the long-term care annuity products including the disability insurance and the elder care insurance products, which are crucial to the social security system in an economy.

In the literature of life insurance and long-term care insurance, studies have been done on the valuation of insurance policies or portfolios of such products. For example, Beekman (1990) presented a premium calculation procedure for long term care insurance by studying the random variable of first time loss of independence of Activities of Daily Living (ADL). The data used was based on the result of a survey to the non-institutionalized elderly people in Massachusetts in 1974. Parker (1997) introduced two cash flow approaches to evaluate the average risk per policyholder for a traditional term life and endowment insurance portfolio and decomposed the total riskiness into the insurance risk and the investment risk by conditioning on the survivorship and the interest rate, respectively. An Ornstein-Uhlenbeck process was applied to model the interest rate.

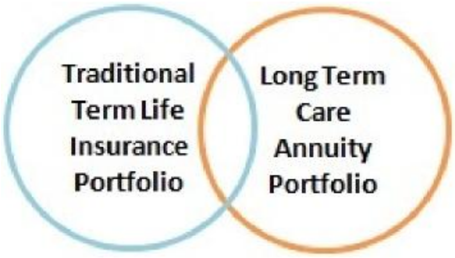


Figure 1.1: The Pools for ABC Insurance Company

In general, the insurance risk (also called the mortality risk) of an insurance portfolio results from mortality, disability or sickness. The average insurance risk per policy tends to zero as the number of contracts in the insurance portfolio goes to infinity under independent mortality assumption. Therefore, the insurance risk can be managed by risk pooling within the insurance company. The insurance risk for an insurance portfolio with a large size is relatively small compared to the investment risk which comes from the fluctuations and the correlations of the periodic interest rates according to Parker (1997). Marceau and Gaillardetz (1999) presented the reserve calculation for a life insurance portfolio under stochastic mortalities and AR(1) interest rate assumptions. However, it is not accurate to evaluate the insurance risk of the insurance company by separately evaluating the risk of each insurance portfolio consisting of only one type of insurance product. There might be some policyholders who are insured simultaneously by different types of insurance products (Figure 1.1). Therefore, independent mortality assumption does not hold in these cases.

In considering an insurance portfolio one key issue that needs to be addressed for valuation purposes is the methodology for calculating the transition probabilities including the assumptions of the forces of transition. This has a huge impact on the valuation results. A few papers discussing the health insurance cases have been published since 1980s. Waters (1984) gave the basic concepts of the transition probabilities, the forces of transition and their relationships. Ramsay (1984) studied the ruin probability of the surplus of a sickness insurance contract which pays the benefit only if the duration of the sickness exceeds certain period. Waters (1990) illustrated a method of calculating the moment of benefit payments for a sickness insurance contract by introducing the semi-Markov chain to model the transition process. In Jones (1994), a Markov chain model was presented to calculate the transition matrix of a multi-state insurance contracts consisting of three states with one-direction transitions only (e.g. healthy, permanently disabled and deceased). Constant and piecewise forces of transition were assumed in the paper. Levikson and Mizrahi (1994) priced a long term care contract with three different care levels assuming that the policyholders health status could only either remain unchanged or deteriorated.

1.3 Statement of the Problem

The Long Term Care (LTC) system in Kenya and other developing economies is largely unfunded and characterized by an absence of risk pooling or a sophisticated user pays mechanism. The system, therefore, stands somewhat isolated from many of its counterparts overseas which combine private funding mechanisms such as private LTC insurance with their respective State and publicly funded welfare programs. With the exception of a limited number of accident compensation policies where LTC is insured if attributable to accidents. Kenyan insurers do not currently engage in any form of LTC insurance business. As such, the task of pricing and reserving for private LTC insurance contracts for introduction into the Kenyan market is made difficult due to a lack of historical experience, adequate data and consensus on appropriate modeling methodology and assumptions.

The primary objective of this project, therefore, is to develop and test a multiple state model for pricing and reserving LTC insurance using currently available Kenyan data. In Leung (2004), a discrete time multiple state model was developed for projecting the needs and costs of LTC in Australia. In this project, I relax the assumption of discrete time and model the underlying process in a continuous time Markov framework. The purpose of this project is to enable calculation of transition intensities for application in Thiele's differential equation for pricing and reserving. This project concentrates on a generalization of Thiele's differential equation of multi-state markov model in designing different types of long term care (LTC) insurances including a whole life stand-alone LTC policy and LTC rider policy cover. The modeling framework and results presented in this project may be used as a starting point for the development of LTC insurance policies in Kenya.

1.4 Research Objectives

In this project, the general objective is to use the multistate markov framework to model LTC insurance products in the Kenyan insurance industry.

In particular, the project intends to achieve the following specific objectives:

1. Estimate the transition probabilities of the t-year probability ${}_t p_x^{ab}$ of a life aged (x) making a transition from state *a* to state *b* of a four state Markov model using Chapman-Kolmogorov and Forward Kolmogorov differential equations.

2. Calculate and graduate the maximum likelihood markov transition intensities or forces of transition μ_t using the Compertz-Mekahem methods of graduation

3. Calculate benefit premiums for a set of illustrative hypothetical LTC insurance products including a whole life stand-alone LTC policy and a LTC rider cover using the equivalence principle/equation of value

4. Calculate the reserves of an illustrative hypothetical LTC whole life stand-alone policy using Thiele's differential equations

1.5 Justification of the Study

Current premium rates for the various LTC products varies from one insurer to another. This is due to a lack of standard premium rate and its corresponding incident rate for the various products sold. Some insurers charge higher premiums which results in the low uptake rates of the product whereas others charge relatively low premium which also results in the company running at a loss. This calls for the need to find a rigorous and accurate formula for costing an LTC benefit product. This has necessitated the determination of an accurate pricing formula. Pricing LTC products is a key objective of any insurance industry that sells such a product. The project which considers the various premiums and incidence rates for different age groups will serve as a platform for insurers to know which group of people are most likely to suffer one condition than the other. The idea of knowing that one must insure against conditions of not being able to carry out one's normal duties, by virtue of suffering a major illness makes the project very justifiable.

1.6 Significance of the Study

It is very significant for an insurance company to value or price its products correctly such that such premium calculation is as accurate as possible. This indeed is true especially for Long Term Care products whose impact on the insured and the underwriter lasts for a long duration. Pricing of premium for the purchase of any insurance forms the basics of any quality insurance in any insurance company. Thus the data from the survey would therefore be of immense importance to the various insurance companies in the country. These include the National Hospital Insurance

Fund from which the current data is obtained, Britam, Panafric Life, Lion, Pacis, C.I.C., Old Mutual, among others. The correct pricing and reserving of LTC products will increase their sales and thereby increase the premium income of insurers. The regulatory bodies such as Insurance Regulatory Authority(I.R.A.) and professional insurance associations such as The Actuarial Society of Kenya(T.AS.K.), and Association of Kenyan Insurers(A.K.I.) will benefit immensely from an accurate valuation of LTC products in Kenya. At the end of the project, the various insurance companies will have a better and scientific means of pricing and reserving the LTC products.

2.0 CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

In this chapter the focus will be on the markov models in continuous time and discrete state space two state, three and four state markov models. The chapter will begin with the derivation and solution of the Kolmogorov forward and the Kolmogorov backward equations by using the Chapman-Kolmogorov equations. Then focus will shift to various versions of two, three and four state models with emphasis of the derivation and solution of transition intensities/forces, μ_{x+t}, from one state to another, say, from healthy to outpatient sickness using the Chapman-Kolmogorov equations and Kolmogorov forward equations.

The three state process is commonly used to describe the state of an individual insured under a disability income policy. In this case, premiums are payable while the insured is in state 1, and benefits are payable while the insured is in state 2 (usually after a waiting period). We will derive the probabilities of transition from one state to another via the forces of intensities.

In this project, the four state multi state models from alive/healthy through out-patient sickness, in-patient sickness all the way to death with recovery from either of the two cases of sickness will be used to model the medical data from N.H.I.F. in order to calculate premiums and reserves. The focus will be on deriving the probabilities of transition from one state to another via the forces of intensities.

2.2 Chapman-Kolmogorov Equations

A Markov chain $(X_t)_{t\in T}$ is called homogeneous, if it is time homogenous i.e. the following equation holds for all s,t$\in S$ such that $P[X_s = i] > 0 \ and \ P[X_t = i] > 0$ (Scott W.F., 1999)

$$P[X_{s+h} = j | X_S = i] = P[X_{t+h} = j | X_t = i \qquad (2.1)$$

For homogeneous Markov chain we use the notation :

$$p_{ij}(h) \coloneqq p_{ij}(s, s+h)$$
$$P(h) \coloneqq P(s, s+h \qquad (2.2)$$

Remarks that can be made are that a homogeneous Markov chain is characterized by the fact that the transition probabilities and therefore also the transition matrices, only depend on the size of the time increment and that for homogeneous Markov chain one can simplify the Chapman – Kolmogorov equations to the semi group property.

$$P(s+t) = P(s) \times P(t) \tag{2.3}$$

The semi – group property is popular in many different areas e.g. in quantum mechanics. The mapping. $P : T \to M_n(R), t \mapsto P(t)$ Defines a one parameter semi – group.

2.3 Kolmogorov Forward and Backward Differential Equations

In the following section, we will only consider Markov chains on a finite state space (Lecture notes, UoN, 2012). Thus point wise convergence and uniform convergence. Will coincide on S. This enables us to give some of the proofs in a simpler form.

Definition: Let $(X_t)_{t \in T}$ be a Markov chain with finite state space S and T $\subset R$ *for* $N \subset S$ we define.

$$p_{jN}(s.t) = \sum_{k \in N} p_{jk}(s,t) \tag{2.4}$$

Definition: Let. $(X_t)_{t \in T}$ be a Markov chain in continuous time with finite state space S $(X_t)_{t \in T}$ is called regular if

$$\mu_i(t) = \lim_{\Delta t \downarrow 0} \frac{1 - p_{it}(t, t+\Delta t)}{\Delta t} \; for \;\; all \;\; i \in S$$

$$\mu_{ij}(t) = \lim_{\Delta t \downarrow 0} \frac{p_{ij}(t, t+\Delta t)}{\Delta t} \; for \;\; all \;\; i + j \in S \tag{2.5}$$

are well defined and continuous with respect to t.

The functions $\mu_i(t)$ *and* $\mu_{i\,j}(k)$ are called transition rates of the Markov chain . Furthermore we define $\mu_{i\,i}$ by $\mu_{ii}(t) = -\mu_i(t) \;\; for \;\; all \;\; i \in S$

Remark that can be made is that the insurance model and the regularity of the Markov chain is used to derive the differential equations which are satisfied by the mathematical reserve corresponding to the policy.

Now, one can understand the transition rates as derivatives of the transition probabilities. For example we get for $i \neq j$

$$\mu_{ij}(t) = \lim_{\Delta t \downarrow 0} \frac{p_{ij}(t, t+\Delta t)}{\Delta t}$$
$$= \lim_{\Delta t \downarrow 0} \frac{p_{ij}(t, t+\Delta t - p_{ij}(t,t)}{\Delta t}$$
$$= \frac{d}{ds} p_{ij}(t,s)\Big|_{s-t}$$

Hence, $\mu_i(t)\,dt$ can be understood as the infinitesimal transition rate from *i to j* in the time interval [*t*,*t*+*dt*]can be understood as the infinitesimal probability of leaving state *i* in the corresponding time interval . Let us define

$$\Lambda(t) = \begin{pmatrix} \mu_{11}(t) & \mu_{12}(t) & \mu_{13}(t) & \cdots\cdots\cdots & \mu_1 n(t) \\ \mu_{21}(t) & \mu_{22}(t) & \mu_{23}(t) & \cdots\cdots\cdots & \mu_{2n}(t) \\ ; & : & ; & ; & ; \\ \mu_{n1}(t) & \mu_{n2}(t) & \mu_{n3}(t) & \cdots\cdots\cdots & \mu_{nn}(t) \end{pmatrix} \quad (2.6)$$

(Scott W.F., 1999)

In a sense . A generates the behavior of the Markov chain. That is the homogeneous Markov chain the following equation holds.

$$\Lambda(0) = \lim_{\Delta \to 0} \frac{P(\Delta t) - 1}{\Delta t} \quad (2.7)$$

$\Lambda = \Lambda(0)$ is called the generator of one parameter semi group. We can reconstruct P(t) by

$$P(t) = \exp(t\Lambda) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Lambda^n \quad (2.8)$$

Based on the transition rates we can prove Kolomogorov differential equations. These connect the partial derivatives of p_{ij} *with* μ

Theorem (Kolmogorov):

Let $\left(X_t\right)_{t\in T}$ be a regular Markov chain on a finite state space S. Then the following statements hold. : (Backward differential equations)

$$\frac{d}{ds}p_{i,j}(s,t)=\mu_1(s)p_{i\,j}(s,t)-\sum_{k\neq i}\mu_{ik}(s)p_{kj}(s,t)$$
$$\frac{d}{ds}P(s,t)=-\Lambda(s)P(s,t)$$

The Kolmogorov Forward differential equations are

$$\frac{d}{dt}p_{i\,j}(s,t)=-p_{ij}(s,t)\mu_j(t)+\sum_{k\neq j}p_{i\,k}(s,t)\mu_{kj}(t)$$
$$\frac{d}{ds}P(s,t)=P(s,t)\Lambda(t)$$

Proof:

The major part of the proof is based on the equations of Chapman and Kolmogorov.

We will prove the matrix version of the statement. This will help to highlight the key properties. Let $\Delta s>0$ *and set* $\xi:=s+\Delta s$

`

$$\frac{p(\xi,t)-P(s,t)}{\Delta s}=\frac{1}{\Delta s}(P(\xi,t)-P(s,\xi)P(\xi,t)$$
$$=(\frac{1}{\Delta s}(1-P(s,\xi)\times P(\xi,t)$$
$$\to-\Lambda(s)P(s,t)\ for\ \Delta s\downarrow 0$$

Where we used the Chapman – Kolmogorov equation and the continuity of the matrix multiplication.

Analogous one can prove the forward differential equation. Let $\Delta t.0$

$$\frac{P(s,t+\Delta t)-P(s,t)}{\Delta t}=\frac{1}{\Delta t}(P(s,t)P(t,t+\Delta t)-P(s,t)$$
$$=P(s,t)\times\frac{1}{\Delta t}(P(t,t+\Delta t)-1)$$
$$\to P(s,t)\Lambda(t)\ for\ \Delta t\downarrow 0$$

An important remark is that the primary application of Kolmogorov's difficult equations is to calculate the transition probabilities p_{ij} based on the rates μ.

Now the definition that we let $(X_t)_{t \in T}$ be a regular Markov chain on a finite state space S. We denote the conditional probability to stay during the interval [s,t] in *j* by

$$\bar{p}_{jj}(s,t) = P\left[\bigcap_{\xi \in |s,t|} \{X_\xi = j\} \middle| X_s = j \right]$$

where s, t $\in R$, s $\leq$ t and $j \in S$

In setting of a life insurance, this probability can for example be used to calculate the probability that the insured survives 5 years. The following theorem illustrates how this probability can be calculated based on the transition rates.

Theorem: Let $(X_t)_{t \in T}$ be a regular Markov chain (Scott W.F., 1999).

$$\bar{p}_{jj}(s,t) - \exp\left(- \sum_{k \neq j} \int_8^t \mu_{jk}(\tau) d\tau \right)$$

Holds for $s \leq t, \ if \ P[X_s = j] > 0$

Proof: We define $K_j(s,t) \ by \ K_j(s,t) = \bigcap_{\xi \in [s,t]} \{X_s = j\} Let \ \Delta t > 0.$ We have P|A $\cap B|C|$ =P[B|C]P[A|B $\cap$ C] and thus

$$\bar{p}_{jj} \text{ (s,t)+} \Delta t) = P[K_j(s,t) \cap K_j(t, t+\Delta t)| j \qquad (2.9)$$

$$= P[K_j(s,t)|X_s = j] P[K_j(t,t+\Delta t)|X_s = j \cap K_j(s,t)$$

$$= P[K_j(s,t)|X_s = j] P[K_j(t,t+\Delta t)|X_t - j]$$

$$= \quad P_{jj}(s,t)| P[K_j(t,t+\Delta t)|X_t = j]$$

Where we used the Markov property and the relation $\{X_s = j\} \cap K_j(s,t) = \{X_t = j\} \cap K_j(s,t).$

The previous equation yields.

$$P_{jj}(s,t+\Delta t)-p_{jj}(s,t)=\bar{p}_{jj}(s,t)\times(1-P[K_j(t,t+\Delta t)|X_t=j]$$

$$=-\bar{p}_{jj}(s,t)\times\left(\sum_{k\neq j}p_{jk}(t,t+\Delta t)+o(\Delta t)\right) \qquad (2.10)$$

where we used that the rates μ are well defined. Now taking the limit we get the differential equation

$$\frac{d}{dt}p_{jj}(s,t)=-\bar{p}_{jj}(s,t)\times\sum_{k\neq j}\mu_{jk}(t) \qquad (2.11)$$

Solving this equation with the boundary condition $p_{jj}(s,\bar{s})=-1$ yields the statement of the theorem.

2.4 A One-Direction Two-State Markov Model

This is a one direction two state markov chain process from a state of being alive to dead state estimated using the force of transition (Scott W.F., 1999).

The probability that alive at a given age will be dead at any subsequent age is governed by the age –dependent transition intensity$\mu_{x+t}(t \geq 0)$, in a way made precise by assumption 2 below.

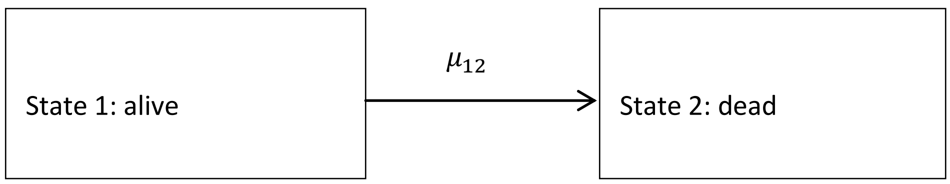
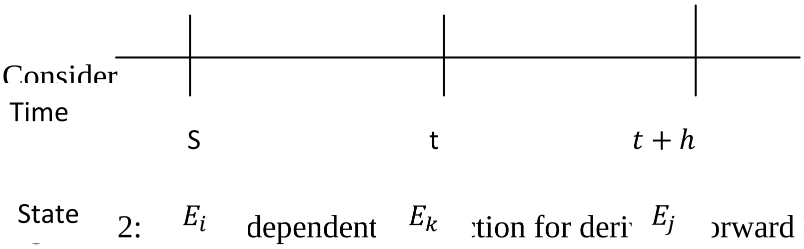


Figure 2.1: A one-way two-state markov model

The states E_i, E_k and $E_j=\mu_{12}$ are experienced at times s, t, and t+h as shown in the timeline in figure 2.2 below.

Consider



2: dependent tion for deri orward Kolmogorov equation

There are three assumptions underlying the simple two-state model:

Assumption 1

The probabilities that a life at any given age will be found in either state at any subsequent age depend only on the ages involved and on the state currently occupied. This is the Markov assumption. So, past events do not affect the probability of a future event. In particular, the past history of an individual –for example m current state of health, spells of sickness, occupation –is excluded from the model.

Assumption 2

For a short time interval of length dt

$$_{dt}q_{x+t} = \mu_{x+t}dt + 0(dt)(t \geq 0) \quad (2.12)$$

In other words, the probability of dying in a very short time interval dt is equal to the transitions intensity multiplied by the time interval, plus a small correction term. This is equivalent to assuming$_{dt}q_{x+t} \approx \mu_{x+t}dt$

Remember that a function $g(t)$ is said to be $'0(t)' if$ $\lim_{t\to 0}\frac{g(t)}{t} = 0$, in other words if $g(t)$ tends to zero “faster” than t itself. Where we are not concerned about the precise form of$g(t)$, we can use the term $0(t)$ in an equation to denote any function that is$0(t)$.

For the purpose of inference, we restrict our attention to ages between x and $x + 1$ and introduce a further assumption.

Assumption 3

$$\mu_{x+t}\ is\ a\ constant\ \mu\ for\ 0 \leq t < 1.$$

Our investigation will consist of many observations of small segments of lifetimes i.e. single years of age. Assumption 3 simplifies the model by treating the transition intensity a constant for all individuals aged x last birthday. This does not mean that we believe that the transition intensity will increase by a discrete step when an individual reaches age$x + 1$, although this is a consequence of the assumption.

Now by the Chapman –Kolmogorov equation:

$$p_{ij}(s, t+h) = \sum_k p_{ik}(s,t) p_{kj}\,(t, t+h)$$

For the two state one direction markov model

$$p_{11}(s, t+h) = p_{11}(s,t)p_{11}(t, t+h) + p_{12}(s,t)p_{21}(t, t+h)$$

$$= p_{11}(s,t)[1 - \mu_{12}h + 0(h)] + p_{12}(s,t).0$$

$$\therefore p_{11}(s, t+h)p_{11}(s,t) = [-\mu_{12}h + 0(h)]p_{11}(s,t) + p_{12}(s,t).0$$

$$\therefore \frac{d}{dt}p(s,t) = \lim_{h\to 0}\frac{p_{11}(s,t+h) - p_{11}(s,t)}{h} = -\mu_{12}p_{11}(s,t) + 0.p_{12}(s,t)$$

$$p_{12}(s, t+h) = p_{11}(s,t)p_{12}(t, t+h) + p_{12}(s,t)p_{21}(t, t+h)$$

$$= p_{11}(s,t)[\mu_{12}h + 0(h)] + p_{12}(s,t)[-\mu_{21}h + 0(h)]$$

$$\therefore \frac{d}{dt}p_{12}(s,t) = \lim_{h\to 0}\frac{p_{12}(s,t+h) - p_{12}(s,t)}{h}$$

$$= \mu_{12}p_{11}(s,t) - \mu_{21}p_{12}(s,t)$$

$$= \mu_{12}p_{11}(s,t) + 0.p_{12}(s,t) \qquad (2.13)$$

Thus Kolmogorov forward differential equations are:

$$\therefore p'_{11}(s,t) = -\mu_{12}p_{11}(s,t) + 0.p_{12}(s,t)$$

$$Initial\ condition p_{11}(s,t) = 1$$

$$p'_{12}(s,t) = \mu_{12}p_{11}(s,t) + 0.p_{12}(s,t) \quad (2.14)$$

In the matrix form we have:

Therefore $\therefore [p'_{11}(s,t), p'_{12}(s,t)] = [p_{11}(s,t), p_{12}(s,t)]\begin{bmatrix} -\mu_{12} & \mu_{12} \\ 0 & 0 \end{bmatrix}$

$$p'(t) = p(t)Q \quad (2.15)$$

2.5 A Two Directional Two State Markov Model

This is a two directional tow state markov model in which there is transition by the lives being studied from healthy state to the sick state $\mu_{12}$and backwards through the force of transition μ_{21} (Lecture notes, UoN 2012).

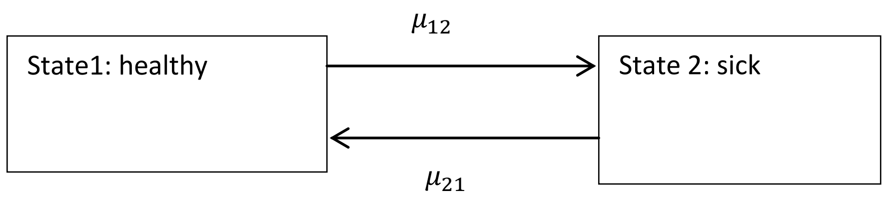


Figure 2.3: A two-directional two-state markov model

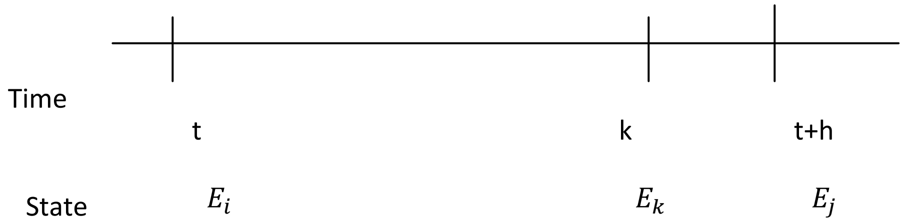


Figure 2.4: A time dependent transaction for deriving forward Kolmogorov equation

By Chapman –Kolmogorov equation

$$p_{ij}(s,t+h) = \sum_k p_{ik}(s,t)p_{kj}(t,t+h)$$

$$= \sum_{k=1}^{2} p_{ik}(s,t)\, p_{kj}(t,t+h)$$

$$p_{11}(s,t+h) = \sum_{k=1}^{2} p_{ik}(s,t)\, p_{ki}(t,t+h)$$

$$= p_{11}(s,t)p_{11}(t,t+h) + p_{12}(s,t)p_{21}(t,t+h)$$

$$= p_{11}(s,t)[1-\mu_{12}h + 0(h)] + p_{12}(s,t)[\mu_{12}h + 0(h)]$$

$$\therefore p_{11}(s,t+h) - p_{11}(s,t) = p_{11}(s,t)[-\mu_{12}h + 0(h)] + p_{12}(s,t)[\mu_{21}h + 0(h)]$$
(2.16)

$$\therefore \frac{d}{dt}p_{11}(s,t) = \lim_{h\to 0}\frac{p_{11}(s,t+h) - p_{11}(s,t)}{h}$$

$$\therefore p'_{11}(s,t) = p_{11}(s,t)(-\mu_{12}) + p_{12}(s,t).\mu_{21}$$

$$p_{12}(s,t+h) = p_{11}(s,t)p_{12}(t,t+h) + p_{12}(s,t)p_{22}(t,t+h)$$

$$= p_{11}(s,t)[\mu_{12}h + 0(h)] + p_{12}(s,t)[1-\mu_{21}h + 0(h)]$$

$$p_{12}(s,t+h) - p_{12}(s,t) = p_{11}(s,t)[\mu_{12}h + 0(h)] + p_{12}(s,t)[-\mu_{21}h + 0(h)]$$
(2.17)

$$\therefore \frac{d}{dt}p_{12}(s,t) = \lim_{h\to 0}\frac{p_{12}(s,t+h) - p_{12}(s,t)}{h}$$

$$p'_{12}(s,t) = p_{11}(s,t).\mu_{12} + p_{12}(s,t).-\mu_{21}$$

$$p_{21}(s,t+h) = p_{21}(s,t)p_{11}(t,t+h) + p_{22}(s,t)p_{22}(t,t+h)$$

$$= p_{21}(s,t)[1-\mu_{12}h + 0(h)] + p_{22}(s,t)[\mu_{21}h + 0(h)]$$

$$p_{21}(s,t+h) - p_{21}(s,t) = p_{21}(s,t)[-\mu_{12}h + 0(h)] + p_{22}(s,t)[\mu_{21}h + 0(h)]$$

$$\therefore p'_{21}(s,t) = p_{21}(s,t)[-\mu_{12}] + p_{22}(s,t).\mu_{21} \qquad (2.18)$$

$$p_{22}(s,t+h) = p_{21}(s,t)p_{12}(t,t+h) + p_{22}(s,t)p_{22}(t,t+h)$$

$$= p_{21}(s,t)[\mu_{12}h + 0(h)] + p_{22}(s,t)[1-\mu_{21}h + 0(h)]$$

$\therefore p'_{22}(s,t) = p_{21}(s,t)\mu_{12} + p_{22}(s,t)(-\mu_{21})$ (6)

In the matrix form:

$$\begin{bmatrix} p'_{11}(s,t) & p'_{12}(s,t) \\ p'_{21}(s,t) & p'_{22}(s,t) \end{bmatrix} = \begin{bmatrix} p_{11}(s,t) & p_{12}(s,t) \\ p_{21}(s,t) & p_{22}(s,t) \end{bmatrix} \begin{bmatrix} -\mu_{12} & \mu_{12} \\ \mu_{21} & -\mu_{21} \end{bmatrix}$$

In the compact form:

$P'(s,t) = P(s,t)Q$ Where $Q = \begin{bmatrix} -\mu_{12} & \mu_{12} \\ \mu_{21} & -\mu_{21} \end{bmatrix}$(22)

The solution to (6) is: $P(s,t) = e^{tQ+c} = Ke^{tQ}$

The initial condition is obtained by letting t= 1 $\therefore P(\,s,s) = I$

$and\ P(s,s) = Ke^{tQ} \quad I = Ke^{tQ}$

$= K\left[I + \sum_{k=1}^{\infty} \frac{t^k Q^k}{k!}\right] = KI + K\sum_{k=1}^{\infty} \frac{t^k Q^k}{k!}$

Comparing we have

$I = KI \rightarrow K = I$ and

$0 = K\sum_{K=1}^{\infty} \frac{t^k Q^k}{k!} \qquad P(s,t) = e^{tQ}$(23)

The eigen-values of Q are obtained by solving the equation:

$$|Q - \lambda I| = 0; i.e \begin{vmatrix} -\mu_{12} - \lambda & -\mu_{12} \\ -\mu_{21} & -\mu_{21} - \lambda \end{vmatrix} = 0$$

$$\therefore (\mu_{12} + \lambda) - \mu_{12}\mu_{21} = 0$$

$$\therefore \mu_{12}\mu_{21} + (\mu_{12} + \mu_{21})\lambda + \lambda^2 - \mu_{12}\mu_{21} = 0$$

$$\therefore (\mu_{12} + \mu_{21})\lambda + \lambda^2 = 0$$

$$\rightarrow \lambda = 0\ and\ \lambda = -(\mu_{12} + \mu_{21})$$

$$\therefore\ \lambda_1 = -(\mu_{12} + \mu_{21})\ and\ \lambda_2 = 0$$

The corresponding Eigen –vectors are:

$$for\ \lambda_1 = -(\mu_{12} + \mu_{21})\ we\ have:$$

$$\begin{bmatrix} -\mu_{12} & \mu_{12} & x_1 \\ \mu_{21} & -\mu_{21} & x_2 \end{bmatrix} = -(\mu_{12} + \mu_{21}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\therefore -\mu_{12}x_1 + \mu_{12}x_2 = -(\mu_{12} + \mu_{21})x_1 = -\mu_{21}x_1 - \mu_{21}x_1$

$\therefore \mu_{12}x_2 = -\mu_{21}x_1 \therefore x_2 = -\frac{\mu_{21}}{\mu_{12}}x_1(10)$

$for\ \lambda_2 = 0\ ,\ \begin{pmatrix} -\mu_{12} & \mu_{12} \\ \mu_{21} & -\mu_{21} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \therefore -\mu_{12}x_1 + \mu_{12}x_2 = 0 \therefore -x_1 + x_2 = 0$

$\therefore x_1 = x_2(24) \rightarrow x_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix}$

Thus the matrix based on the eigenvectors is:

$$U = (x_{-1}x_{-2}) = \begin{pmatrix} 1 & 1 \\ -\frac{\mu_{21}}{\mu_{12}} & 1 \end{pmatrix} \rightarrow U^{-1} = \begin{pmatrix} 1 & \frac{\mu_{21}}{\mu_{12}} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1+\frac{\mu_{21}}{\mu_{12}}} \end{pmatrix} = \frac{\mu_{12}}{\mu_{12}+\mu_{21}} \begin{pmatrix} 1 & \mu_{21} \\ -1 & \mu_{12} \end{pmatrix} \tag{2.19}$$

From (2.18),

$P(s,t) = e^{tQ} = 1 + \frac{tQ}{1!} + \frac{(tQ)^2}{2!} = 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} Q^k$

Since Q has distinct eigen- values, it can be expressed as:

$$Q = UDU^{-1} \rightarrow Q^k = UD^kU^{-1}$$

$$\therefore P(s,t) = 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} UD^kU^{-1} = 1 + U \left[\sum_{k=1}^{\infty} \frac{(ED)^k}{k!} \right] U^{-1} \tag{2.20}$$

Where

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \therefore (tD)^k = \begin{pmatrix} (\lambda_1 t)^k & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore \sum_{k=1}^{\infty} \frac{(tD)^k}{k!} = \begin{pmatrix} \sum_{k=1}^{\infty} \frac{(\lambda_1 t)^k}{k!} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} - 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore $U \sum_{k=1}^{\infty} \frac{(tD)^k}{k!} U = 1$ (2.21)

$$= \begin{pmatrix} 1 & 1 \\ -\frac{\mu_{21}}{\mu_{12}} & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} - 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \mu_{21} \\ -1 & \mu_{12} \end{pmatrix} \frac{\mu_{12}}{\mu_{12} + \mu_{21}}$$

$$= \begin{pmatrix} e^{\lambda_1 t} - 1 & 0 \\ -\frac{\mu_{21}}{\mu_{12}}\left(e^{\lambda_1 t} - 1\right) & 0 \end{pmatrix} \begin{pmatrix} 1 & \mu_{21} \\ -1 & \mu_{12} \end{pmatrix} \frac{\mu_{12}}{\mu_{12} + \mu_{21}}$$

$$= \begin{pmatrix} e^{\lambda_1 t} - 1 & \mu_{21}\left(e^{\lambda_1 t} - 1\right) \\ -\frac{\mu_{21}}{\mu_{12}}\left(e^{\lambda_1 t} - 1\right) & -\frac{\mu_{21}^2}{\mu_{12}}\left(e^{\lambda_1 t} - 1\right) \end{pmatrix} \frac{\mu_{12}}{\mu_{12} + \mu_{21}}$$

$$\therefore P(s,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e^{\lambda_1 t} - 1 & \mu_{21}\left(e^{\lambda_1 t} - 1\right) \\ -\frac{\mu_{21}}{\mu_{12}}\left(e^{\lambda_1 t} - 1\right) & \frac{\mu_{21}^2}{\mu_{12}}\left(e^{\lambda_1 t} - 1\right) \end{pmatrix} \frac{\mu_{12}}{\mu_{12} + \mu_{21}}$$

$$\therefore p_{11}(s,t) = 1 + \left(e^{\lambda_1 t} - 1\right) \frac{\mu_{12}}{\mu_{12} + \mu_{21}} = \frac{\mu_{12} + \mu_{21} + \mu_{12} e^{\lambda_1 t} - \mu_{12}}{\mu_{12} + \mu_{21}}$$

$$= \frac{1}{\mu_{12} + \mu_{21}} \left[\mu_{21} + \mu_{12} e^{-(\mu_{12} + \mu_{21})} t\right]$$

$$p_{21}(s,t) = \frac{-\mu_{21}}{\mu_{12}} \left(e^{\lambda_1 t} - 1\right) \frac{\mu_{12}}{\mu_{12} + \mu_{21}} = \frac{-\mu_{12}}{\mu_{12} + \mu_{21}} \left(e^{\lambda_1 t} - 1\right)$$

$$= \frac{\mu_{12}}{\mu_{12} + \mu_{21}} \left[1 - e^{-(\mu_{12} + \mu_{21})t}\right]$$

$$p'_{11}(s,t) = p_{11}(s,t)(-\mu_{21}) + p_{12}(s,t)\mu_{21} \qquad (2.22)$$

But

$p_{11}(s,t) + p_{12}(s,t) = 1$ and $p_{12}(s,t) = 1 - p_{11}(s,t)$

$$\therefore p'_{11}(s,t) = p_{11}(s,t)(\mu_{12}) + [1 - p_{11}(s,t)]\mu_{21} = -\mu_{12} p_{11}(s,t) + \mu_{21} - \mu_{21} p_{11}(s,t)$$

$$= (\mu_{12} + \mu_{21}) p_{11}(s,t) + \mu_{21}$$

$$\therefore p'_{11}(s,t) + (\mu_{12} + \mu_{21}) p_{11}(s,t) = \mu_{21} \qquad (2.23)$$

Where the Integrating factor $= e^{\int (\mu_{12} + \mu_{21}) dt} = e^{(\mu_{12} + \mu_{21})t}$

$$\therefore \frac{d}{dt} e^{(\mu_{12} + \mu_{21})t} p_{11}(s,t) = \mu_{21} e^{(\mu_{12} + \mu_{21})t}$$

$$\int d\, e^{(\mu_{12}+\mu_{21})t} p_{11}(s,t) = \int \mu_{12}\, e^{(\mu_{12}+\mu_{21})t}\, dt$$

$$e^{(\mu_{12}+\mu_{21})t} p_{11}(s,t) = \frac{\mu_{12}}{\mu_{12}+\mu_{21}} e^{(\mu_{12}+\mu_{21})t} + C$$

When t =j then we have: $e^{((\mu_{12}+\mu_{21})s} p_{11}(s,s) = \frac{\mu_{21}}{\mu_{12}+\mu_{21}} e^{((\mu_{12}+\mu_{21})s}$

$$e^{((\mu_{12}+\mu_{21})s}.1 = \frac{\mu_{21}}{\mu_{12}+\mu_{21}} e^{((\mu_{12}+\mu_{21})s}$$

$\therefore \left[1 - \frac{\mu_{21}}{\mu_{12}+\mu_{21}}\right] e^{((\mu_{12}+\mu_{21})s} = C$

$$\left[\frac{\mu_{12}+\mu_{21}-\mu_{21}}{\mu_{12}+\mu_{21}}\right] e^{((\mu_{12}+\mu_{21})s} = C$$

$C = \frac{\mu_{12}}{\mu_{12}+\mu_{21}} e^{((\mu_{12}+\mu_{21})s}$

$\therefore e^{((\mu_{12}+\mu_{21})t} p_{11}(s,t) = \frac{\mu_{21}}{\mu_{12}+\mu_{21}} e^{((\mu_{12}+\mu_{21})t} + \frac{\mu_{12}}{\mu_{12}+\mu_{21}} e^{((\mu_{12}+\mu_{21})s}$

$\therefore p_{11}(s,t) = \frac{\mu_{21}}{\mu_{12}+\mu_{21}} + \frac{\mu_{12}}{\mu_{12}+\mu_{21}} e^{-((\mu_{12}+\mu_{21})(t-s)}$ (2.24)

From (2.18)

$$p'_{21}(s,t) = -\mu_{12} p_{21}(s,t) + \mu_{21} p_{22}(s,t)$$

$$= -\mu_{12} p_{21}(s,t) + \mu_{21}[1 - p_{21}(s,t)] = -\mu_{12} p_{21}(s,t) + \mu_{21} - p_{21}(s,t)$$

$$\therefore p'_{21}(s,t) + (\mu_{12}+\mu_{21}) p_{21}(s,t) = \mu_{21}$$

$$integrating\ factor\ = e^{((\mu_{12}+\mu_{21})t}$$

$$\therefore \frac{d}{dt} e^{((\mu_{12}+\mu_{21})t} p_{21}(s,t) = \mu_{21} e^{((\mu_{12}+\mu_{21})t}$$

$e^{((\mu_{12}+\mu_{21})t} p_{21}(s,t) = \frac{\mu_{21}}{\mu_{12}+\mu_{21}} e^{((\mu_{12}+\mu_{21})t} + C\ \ when\ t = j, \pi p_{21}(s,s) = 0$

$$\therefore 0 = \frac{\mu_{21}}{\mu_{12}+\mu_{21}} e^{((\mu_{12}+\mu_{21})s} + C$$

$$\therefore C = -\frac{\mu_{21}}{\mu_{12}+\mu_{21}} e^{((\mu_{12}+\mu_{21})s}$$

$$\therefore e^{((\mu_{12}+\mu_{21})t}p_{21}(s,t) = \frac{\mu_{21}}{\mu_{12}+\mu_{21}}e^{((\mu_{12}+\mu_{21})t} - \frac{\mu_{21}}{\mu_{12}+\mu_{21}}e^{((\mu_{12}+\mu_{21})}$$

$$\therefore p_{21}(s,t) = \frac{\mu_{21}}{\mu_{12}+\mu_{21}} - \frac{\mu_{21}}{\mu_{12}+\mu_{21}}e^{-((\mu_{12}+\mu_{21})(t-s)} \quad (2.25)$$

2.6 A Two-State One-Direction Markov Model

This is a markov process with two states in one direction without the possibility of returning back to the initial condition. The interest is to estimate the transition forces from the state of being active to dead $\mu_{01}$and from active to retired μ_{02}.(Scott W.F., 1999).

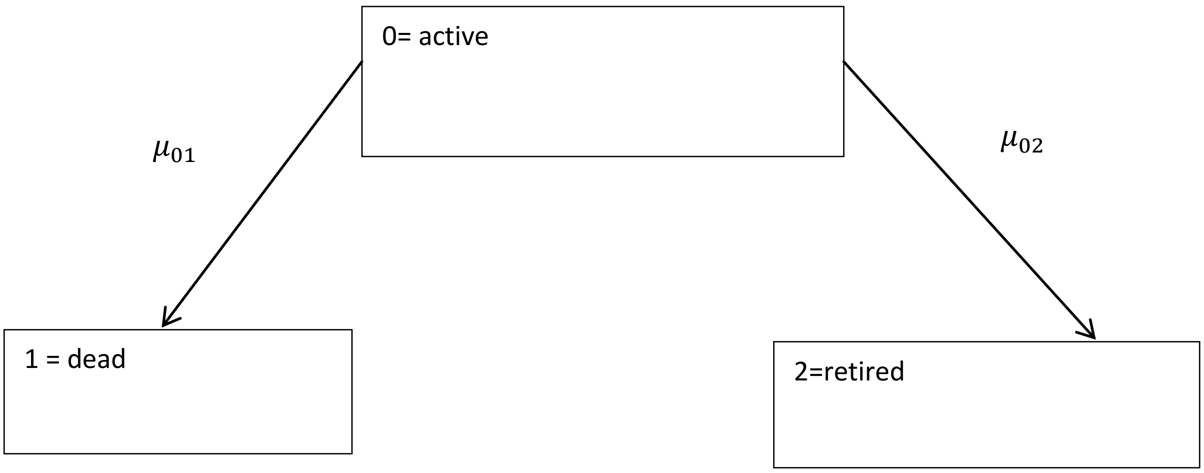


Figure 2.5: A two-state one directional markov model

Note that the term active is usually applied to individuals in employment, in order to differentiate them from individuals who are healthy but who have retired.

We need to derive the following two equations:

$$tP_x{}^{01} = \frac{\mu^{01}}{\mu^{01}+\mu^{02}}\left[1 - e^{-(\mu^{01}+\mu^{02})t}\right] \quad (2.26)$$

$$tP_x{}^{01} = \frac{\mu^{02}}{\mu^{01}+\mu^{02}}\left[1 - e^{-(\mu^{01}+\mu^{02})t}\right]$$
(2.27)

Again by the Chapman-Kolmogorov equation

$$p_{ij}(s,t+h) = \sum_k p_{ik}(s,t)p_{kj}(t,t+h) \therefore p_{00}(s,t+h) = \sum_{k=0}^{2} p_{0k}(s,t)p_{k0}(t,t+h)$$

$$= p_{00}(s,t)p_{00}(t,t+h) + p_{01}(s,t)p_{10}(t,t+h) + p_{01}(s,t).0 + p_{02}(s,t)p_{20}(t,t+h)$$

$$= p_{00}(s,t)[1-(\mu_{01}+\mu_{02})h - 0(h)]$$

$$\therefore p_{00}(s,t+h)p_{00}(s,t) = p_{00}(s,t)[-(\mu_{01}+\mu_{02})h - 0(h)]$$

$$\therefore \frac{d}{dt}p_{00}(s,t) = \lim_{h\to 0}\frac{p_{00}(s,t+h)-p_{00}(s,t)}{h} = -(\mu_{01}+\mu_{02})p_{00}(s,t)$$

$$\therefore \frac{1}{p_{00}(s,t)}\frac{d}{dt}p_{00}(s,t) = -(\mu_{01}+\mu_{02})$$

$$= p_{00}(s,t) = e^{-(\mu_{01}+\mu_{02})t} + C$$

But $p_{00}(s,s) = 1$ and $p_{00}(s,s) = e^{C}$ $c = 0 \therefore p_{00}(s,s) = e^{-(\mu_{01}+\mu_{02})t}$

$$p_{01}(s,t+h) - p_{01}(s,t)p_{01}(t,t+h) + p_{01}(s,t)p_{11}(t,t+h)$$
$$= p_{00}(s,t)p_{01}(t,t+h) + p_{01}(s,t).1$$

$$\frac{d}{dt}p_{01}(s,t) = p_{00}(s,t).\mu_{01} = \mu_{01}e^{-(\mu_{01}+\mu_{02})t}$$

$$\therefore p_{01}(s,t) = \frac{\mu_{01}}{(\mu_{01}+\mu_{02})}e^{-(\mu_{01}+\mu_{02})t} + Cwhen\ \ s=t$$

$p_{01}(s,t) = 0$ and $p_{01}(s,t) = \frac{\mu_{01}}{-(\mu_{01}+\mu_{02})} + C$

$$C = \frac{\mu_{01}}{(\mu_{01}+\mu_{02})} \therefore p_{01}(s,t) = \frac{\mu_{01}}{(\mu_{01}+\mu_{02})} - \frac{\mu_{01}}{(\mu_{01}+\mu_{02})}e^{-(\mu_{01}+\mu_{02})t}$$

$$= \frac{\mu_{01}}{(\mu_{01}+\mu_{02})}\left[1-e^{-(\mu_{01}+\mu_{02})t}\right]$$

Replacing 1 by 2 and vice versa we have:

$p_{02}(s,t) = \frac{\mu_{02}}{(\mu_{01}+\mu_{02})}\left[1-e^{-(\mu_{01}+\mu_{02})t}\right]$as required to show. (2.28)

The term in brackets is the probability of having left the active state and the fraction gives the conditional probability of each decrement having occurred , given that have of them has occurred.

$$p'_{00}(s,t)p'_{0j}(s,t) = [p_{00}(s,t)p_{0j}(s,t)]\begin{bmatrix}(\mu_{01}+\mu_{02}) & \mu_{02}\\ 0 & 0\end{bmatrix} \text{where} j \neq 0 \qquad (2.29)$$

2.7 A Two State Markov Model With Three Decrements

This is a one directional three state markov model with the forces of decrements of retirement, withdrawal, and death represented by transition forces μ_{12}, μ_{13} and μ_{14} respectively (Scott W.F., 1999).

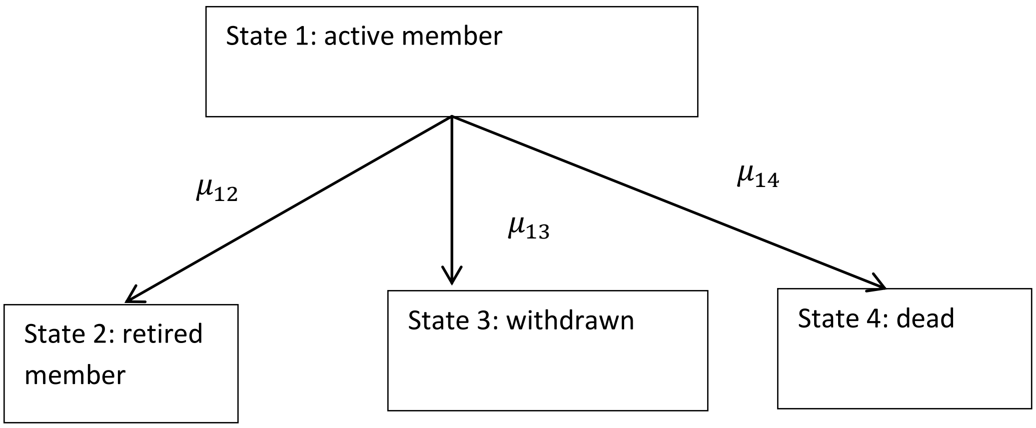


Figure 2.6: A two state markov model with three forces of decrement

By the Chapman Kolmogorov equation:

$$p_{ij}(s,t+h) = \sum_k p_{ik}(s,t)p_{kj}(t,t+h)$$

$$\therefore p_{11}(s,t+h) = \sum_k p_{ik}(s,t)p_{ki}(t,t+h)$$

$$= p_{11}(s,t)p_{11}(t,t+h) + p_{12}(s,t)p_{21}(t,t+h) + p_{13}(s,t)p_{31}(t,t+h) + p_{14}(s,t)p_{41}(t,t+h)$$

$$= p_{11}(s,t)\left[1 - \sum_{i=2}^{4} \mu_k h + 0(h)\right] + p_{12}(s,t).0 + p_{13}(s,t).0$$
$$+ p_{14}(s,t).0$$

$$\therefore p_{11}(s,t+h) - p_{11}(s,t) = p_{11}(s,t)\left[-\sum_{j=2}^{4} \mu_{ij} h + o(h)\right]$$

$$\therefore \frac{d}{dt} p_{11}(s,t) = \lim_{h\to 0} \frac{p_{11}(s,t+h) - p_{11}(s,t)}{h}$$

$\therefore p'_{11}(s,t) = -p_{11}(s,t)\sum_{j=2}^{4}\mu_{ij}and\frac{1}{p_{11}(s,t)}\frac{d}{dt}p_{11}(s,t) = -\sum_{j=2}^{4}\mu_{ij}$

$p_{11}(s,t) = \exp\{-\int\sum_{j=2}^{4}\mu_{ij}(tdt) + C\}andp_{11}(s,t) = \exp\left\{-\int_0^t\sum_{j=2}^{4}\mu_{ij}(t)dt\right\}$

(2.30)

Next we have $p_{12}(s,t+h) = \sum_k p_{1k}(s,t)\, p_{k2}(t,t+h)$

Hence $= p_{11}(s,t)p_{12}(t,t+h) + p_{12}(s,t)p_{22}(t,t+h) + p_{13}(s,t)p_{32}(t,t+h) + p_{14}(s,t)p_{42}(t,t+h)$

$$= p_{11}(s,t)[\mu_{12}^{t} h + 0(h)] + p_{12}(s,t)[1] + p_{13}(s,t).0 + p_{14}(s,t).0$$

$$\therefore p_{12}(s,t+h) - p_{12}(s,t) = p_{11}(s,t)[\mu_{12}(t)h + o(h)]$$

$\therefore p'_{12}(s,t) = p_{11}(s,t).\mu_{12}(t)and\therefore p_{12}(s,t) = \int_0^t p_{11}(s,t)\mu_{12}(t)dt$ (2.31)

In general

$$p_{ij}(s,t) = \int_0^t p_{11}(s,t)\mu_{12}(t)dt \;\; for\ j \neq i if\ \mu_{12}(t) = \mu_{ij} \;\; constant\ for\ 0 \leq t \leq 1$$

$p_{11}(s,t) = \exp\{-t\sum_{j=2}^{4}\mu_{ij}\}$ and $p_{ij}(s,t) = \mu_{ij}\int_0^t p_{11}(s,t)\,dt$

$$= \mu_{ij}\int_0^t \exp\left\{-t\sum_{j=2}^{4}\mu_{ij}\right\}dt = \mu_{ij}\int_0^t e^{-t\sum_{j=2}^{4}\mu_{ij}}\,dt = \frac{\mu_{ij}}{\sum_{j=2}^{4}\mu_{ij}}\left|e^{-t\sum_{j=2}^{4}\mu_{ij}}\right|$$

$$= \frac{\mu_{ij}}{\sum_{j=2}^{4} \mu_{ij}} \left\{ \left| e^{-t \sum_{j=2}^{4} \mu_{ij}} \right| - 1 \right\} = \frac{\mu_{ij}}{\sum_{j=2}^{4} \mu_{ij}} \left\{ 1 - \left| e^{-t \sum_{j=2}^{4} \mu_{ij}} \right| \right\}$$

$$= \frac{\mu_{ij}}{\sum_{j=2}^{4} \mu_{ij}} \{1 - \exp[-t \textstyle\sum_{j=2}^{4} \mu_{ij}]\} \qquad (2.32)$$

$$= p_{01}(t+h) = p_{00}(t)p_{01}(h) + p_{01}(t)p_{11}(h) + p_{02}(t)p_{21}(h)$$

$$= p_{00}(t)[\mu_h h + 0(h)] + p$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{and} A^{-1} = \frac{Cofactor\ A}{|A|} = \frac{Minor\ of\ transport\ A}{|A|}$$

$$= Minor \frac{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}{|A|} = \frac{\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}{|A|} \qquad (2.33)$$

2.8 A Three State Markov Model Calculation of Transition Probabilities

A life may be in the healthy state or the sick state on a number of separate occasions before making the one-way transition to the dead state. Alternatively, a life may pass from the able/healthy state to the dead state without ever having been in the sick state(Lecture Notes UoN, 2012).

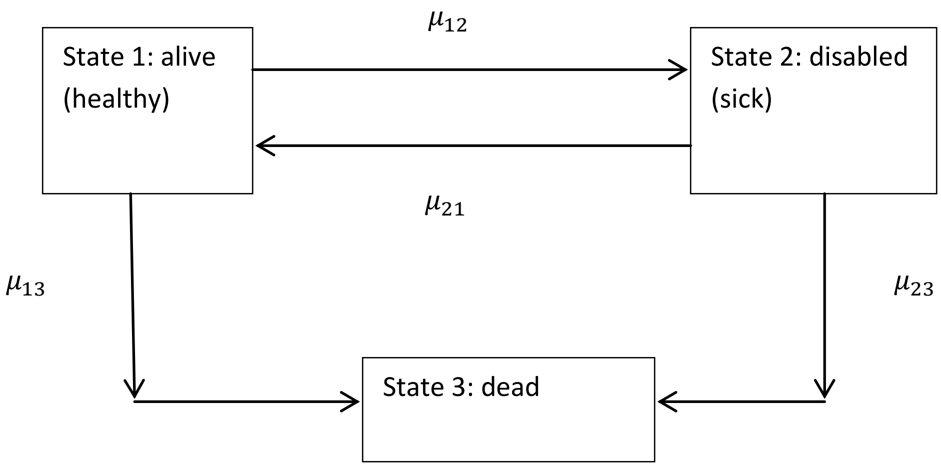


Figure 2.7: A three state markov sickness model

Now, by the Chapman-Kolmogorov equation

$$p_{1j}(s,t+h) = \sum_k p_{ik}(s,t)\, p_{kj}(t,t+h)$$

$$Let\ i = 1\ and\ \ j = 1\ , t = 1,2,3$$

$$\therefore p_{11}(s,t+h)$$
$$= p_{11}(s,t)p_{11}(t,t+h) + p_{12}(s,t)p_{21}(t,t+h) + p_{13}(s,t)p_{31}(t,t+h)$$

$$= p_{11}(s,t)[1-(\mu_{12}+\mu_{13})h + 0(h)] + p_{12}(s,t)[\mu_{21}h + 0(h)] + 0.p_{13}$$

$$\therefore p'_{11}(s,t) = \lim_{h\to 0}\frac{p_{11}(s,t+h) - p_{11}(s,t)}{h}$$

$$p_{11}(s,t) = -(\mu_{12}+\mu_{13})p_{11}(s,t) + p_{12}(s,t) + \mu_{21}p_{12}(s,t) + 0.p_{13}(s,t)$$

$$p_{12}(s,t+h) = p_{11}(s,t)p_{12}(t,t+h) + p_{12}(s,t)p_{22}(t,t+h) + p_{13}(s,t)p_{32}(t,t+h)$$

$$= p_{11}(s,t)[\mu_{12}h + 0(h)] + p_{12}(s,t)[1-(\mu_{21}+\mu_{23})h + 0(h)] + p_{13}(s,t)[0]$$

$$\therefore p'_{12}(s,t) = \mu_{12}p_{11}(s,t) - (\mu_{21}+\mu_{23})p_{12}(s,t) + 0.p_{13}(s,t)$$

$$finally\ \ let\ i = 1, j = 3\ and\ k = 1,2,3$$

$$p_{13}(s,t+h) = p_{11}(s,t)p_{13}(t,t+h) + p_{12}(s,t)p_{23}(t,t+h) + p_{13}(s,t)p_{33}(t,t+h)$$

$$\therefore p'_{13}(s,t) = \mu_{13}p_{11}(s,t) + \mu_{23}p_{12}(s,t) + 0.p_{13}(s,t)$$

$$[p'_{11}(s,t)p'_{12}(s,t)p'_{13}(s,t)] = [p_{11}(s,t)p_{12}(s,t)p_{13}(s,t)]Q \qquad (2.34)$$

Where

$$Q = \begin{bmatrix} -(\mu_{12}+\mu_{13}) & \mu_{12} & \mu_{13} \\ \mu_{21} & -(\mu_{21}+\mu_{23}) & \mu_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

2.9 A Four State Markov Model – Version I

In the second type of the four state markov process, a life move from healthy to a sick and infected state. Then death can occur as a result of the infection or death could also occur from causes not related with the disease infection(Lecture Notes UoN, 2012).

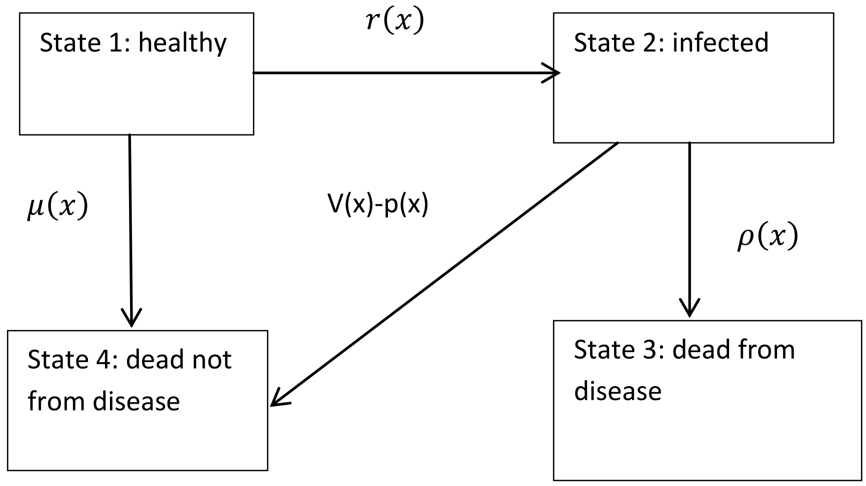


Figure 2.8: A four state markov model

From the Chapman-Kolmogorov equation

$$p_{ij}(s,t+h) = \sum_{k=1}^{4} p_{ik}(s,t\,)p_{kj}(t,t+h)$$

$$p_{11}(s,t+h) = p_{11}(s,t)p_{11}(t,t+h) + p_{12}(s,t)p_{21}(t,t+h) + p_{13}(s,t)p_{31}(t,t+h) + p_{14}(s,t)p_{41}(t,t+h)$$

$$= p_{11}(s,t)\{1 - (r(x) + \mu_1 x)h + 0(h)\} + p_{12}(s,t).\,0 + p_{13}(s,t).\,0 + p_{14}(s,t)\{\mu(x)h + 0(h)\}$$

$$\therefore \frac{d}{dt}p_{11}(s,t) = p_{11}(s,t)\big\{1 - \big(\gamma(x) + \mu(x)\big)\big\} + p_{14}(s,t).\,0$$

$$= p_{11}(s,t)\big[-\big(\gamma(x) + \mu(x)\big)\big]$$

$$p_{12}(s,t+h) = p_{11}(s,t)p_{12}(t,t+h) + p_{12}(s,t)p_{22}(t,t+h) + p_{13}(s,t)p_{32}(t,t+h) + p_{14}(s,t)p_{42}(t,t+h)$$

$$= p_{11}(s,t)\{\gamma(x)h + 0(h)\} + p_{12}(s,t)\{1 - v(x)h + 0(h)\} + p_{13}(s,t).\,0 + p_{14}(s,t)\{0\}$$

$$\therefore p_{12}(s,t+h) - p_{12}(s,t) = p_{11}(s,t)\{\gamma(x)h + 0(h)\} + p_{12}(s,t)\{-\gamma(x)h + 0(h)\}$$

$$\therefore \frac{d}{dt}p_{12}(s,t) = p_{11}(s,t).\gamma(x) + p_{12}(s,t)[-v(x)]$$

$$p_{14}(s,t+h) = p_{11}(s,t)p_{14}(t,t+h) + p_{12}(s,t)p_{24}(t,t+h) + p_{13}(s,t)p_{34}(t,t+h) + p_{14}(s,t)p_{44}(t,t+h)$$

$$= p_{11}(s,t)\{\mu(x)h + 0(h)\} + p_{12}(s,t)\{\big(v(x) - p(x)\big)h + 0(h)\} + p_{13}(s,t)\{0\} + p_{13}(s,t)\{1\}$$

$$\therefore p_{14}(s,t+h) - p_{14}(s,t) = p_{11}(s,t)[\mu(x)h + 0(h)] + p_{12}(s,t)\{[v(x) - p(x)]h + 0(h)\}$$

$$\therefore p_{14}(s,t) = p_{11}(s,t).\mu(x) + p_{12}(s,t)[v(x) - p(x)]$$

$$p_{23}(s,t+h) = p_{21}(s,t)p_{13}(t,t+h) + p_{22}(s,t)p_{23}(t,t+h) + p_{23}(s,t)p_{33}(t,t+h) + p_{24}(s,t)p_{43}(t,t+h)$$

$$\therefore p_{23}(s,t+h) = p_{21}(s,t)\{0\} + p_{22}(s,t)\{p(x)h + 0(h)\} + p_{23}(s,t)\{1\} + p_{24}(s,t)p_{43}\{0\}$$

$$= p_{22}(s,t)\{p(x)h + 0(h)\} + p_{23}(s,t)$$

$$\therefore p_{23}(s,t+h) - p_{23}(s,t) = p_{22}(s,t)\{p(x)h + 0(h)\}$$

$$\therefore \frac{d}{dt}p_{23}(s,t) = p_{22}(s,t).p(x)$$

Next

$$p_{24}(s,t+h) = p_{21}(s,t)p_{14}(t,t+h) + p_{22}(s,t)p_{24}(t,t+h) + p_{23}(s,t)p_{34}(t,t+h) + p_{24}(s,t)p_{44}(t,t+h)$$

$$= p_{21}(s,t).\mu(x) + p_{22}(s,t)\{[v(x) - p(x)]h + 0(h)\} + p_{23}(s,t).0 + p_{24}(s,t).1$$

$$\therefore p_{24}(s,t+h) - p_{24}(s,t) = p_{22}(s,t)\{[v(x) - p(x)]h + 0(h)\}$$

$$\frac{d}{dt}p_{24}(s,t) = p_{22}(s,t)[v(x) - p(x)]p_{21}(s,t).\mu(x) \qquad (3.41)$$

$$= \begin{bmatrix} p'_{11}(s,t) & p'_{12}(s,t) & p'_{13}(s,t) & p'_{14}(s,t) \\ p'_{21}(s,t) & p'_{22}(s,t) & p'_{23}(s,t) & p'_{24}(s,t) \\ p'_{31}(s,t) & p'_{32}(s,t) & p'_{33}(s,t) & p'_{34}(s,t) \\ p'_{41}(s,t) & p'_{42}(s,t) & p'_{43}(s,t) & p'_{44}(s,t) \end{bmatrix}$$

$$= \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \begin{bmatrix} -\big(r(x)+\mu(x)\big) & r(x) & 0 & \mu(x) \\ 0 & -v(x) & p(x) & v(x)-p(x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$p_{13}(s,t+h) = p_{11}(s,t)p_{13}(t,t+h) + p_{12}(s,t)p_{23}(t,t+h) + p_{13}(s,t)p_{33}(t,t+h) + p_{14}(s,t)p_{43}(t,t+h)$$

$$= 0 + p_{12}(s,t)[p(x)h + 0(h)] + p_{13}(s,t).1 + p_{14}(s,t).1$$

$$\therefore \frac{d}{dt}p_{13}(s,t) = p_{12}(s,t)p(x) \qquad (3.42)$$

All these models from the two state to the four state model version I are reviewed as a prelude to the four state model discussed in the methodology chapter three with healthy, out patient, in patient and death states which is the actual model that was applied in his study to the medical data obtained from e N.H.F. in Kenya.

2.11 Empirical Review

The use of Markov chains in life contingencies and their extension has been proposed by several authors, in both the time-continuous case and the time-discrete case; for example, see Amsler (1968), Amsler (1988), Haberman (1983), Haberman (1984), Hoem (1969), Hoem (1988), Jones (1993), Jones (1994), Wilkie (1988). The earliest paper, by Walker (1990), provides a brief introduction to the issues surrounding LTC insurance pricing and provides specimen net single and annual renewable premiums for a LTC benefit using illustrative morbidity rates for males, females and couples. Walsh and de Ravin (1995) perform similar calculations based on data sourced from the 1993 ABS survey of Disability, Ageing and Carers and calculated premium rates directly from prevalence rate data. The mathematical methodologies are not detailed in their respective papers, but it is clear that in both papers, calculations are based on an inception-annuity approach framework.

There has been a great deal of research which makes use of Multiple State models to provide a powerful tool for application in many areas of Actuarial Science, particularly in the Actuarial Assessment of sickness insurance and disability income benefits.

The early history of these models has been described by Seal (1977) and Daw (1979) in some detail and our purpose here is merely to outline the key historical developments in terms of the theory and its practical applications to insurance problems. The problem is the following:

Given two states A and B such that individuals in state A have mutually exclusive probabilities, possibly dependent on the time spent in state A, and the possibility of leaving state A because if (i) death or (ii) passage to state B, then what is the probability of an individual passing to state B and dying there within a given period?

Bernoullis state A consisted of individuals who had never had smallpox, while state B comprised those who had contracted smallpox and would either die from it, almost immediately, or survive and no longer be suffering from that disease. In solving this problem, Bernoulli started with Edmund Halleys (Breslau) life table and effectively produced the first double decrement life table with one of the related single decrement tables as well as considering the efficacy of inoculation and deriving a mathematical model of the behaviour of smallpox. During the next 50 years, there were a number of contributions from other authors on the subject, including Jean dAlembert and Jean Trembley.

Lambert (1772) explained how numerical data could be used to study Bernoullis problem and laid the practical foundations for the double decrement model and life table. He obtained an approximate formula for the rate of mortality and thereby setting down a practical connection between the double decrement model and the underlying single decrement models. Despite this progress by the early 1 800s there were two outstanding problems, namely (i) deriving accurate practical formulae for application to numerical data, linking the discrete and continuous cases; and (ii) obtaining exact results in a convenient form (d¶Alembert had derived an exact results in terms of an integral that was difficult to evaluate). These problems were attacked successfuly and independently by Cournot (1843) and Makeham (1867). They were the first to set down the fundamental relations of multiple decrement models: in modern notation:

for k = 1, 2,…, m

$$(a\mu)_x = \sum_{k=1}^{m} \mu_x^k \qquad (3.43)$$

Makeham (1867) also contains an analysis of the ”partial forces of mortality for different causes of death, suggesting an interpretation of his well –known formula for the aggregate force of mortality to represent separate contributions from *m* + *n* causes of death. Makeham went on to use connection between forces of decrement to interpret the prior development of the theory; he demonstrated that the earlier results of Bernoulli and d¶Alembert satisfied this addictive law for the forces of decrement and this multiplicative law for the probabilities (or corresponding functions).

In an internal report in 1875 (which was not placed in the public domain) on the invalidity and widows pension scheme for railway officials, Karup described the properties and use of singe decrement probabilities and forces of decrement in the context of an illness-death model (with no recoveries permitted), i.e. the ”independent or pure¶ probabilities of mortality and disablement. Hamza (1900) represents an important development by providing a systematic approach to disability benefits in both the continuous and discrete cases. Hamza’s paper is significant, setting down a notation which has been widely adopted in the following decades and which forms the basis for the notation we have utilized.

Pasquier took a dramatic step forward by providing a rigorous, mathematical discussion of the invalidity or sickness process with the introduction of a three state-death model in which recoveries were permitted. He derived the full differential equations for the transition probabilities and showed that these lead to a second-order differential of Riccati type which he then solved for the case of constant forces of transition. Du Pasquier work is very significant, presenting an early application of Markov Chains, and laying the foundations for modern actuarial applications to disability insurance, long-term care insurance and critical illness.

Despite the interest and importance of these problems to actuaries and the consistent contribution made to the actuarial literature since the mid-nineteenth century, these contributions have essentially been rediscovered and renamed as the Theory of competing risk by Neyman (1950) and Fix and Neyman (1951), and other statistical workers.

Applications of semi-Markov models to actuarial (and demographic) problems was done by Hoem (1972); the first application of semi-Markov processes to disability benefits appears in Janssen (1966). As far as disability benefits are concerned, the mathematics of Markov and semi-Markov chains provides both a powerful modelling tool and a unifying point of view, from which several calculation techniques and conventional procedures can be seen in a new light (seeHaberman (1988), Waters (1984), Waters (1989), Continuous Mortality Investigation Bureau (1991), Pitacco (1995)).

2.12 Critical Review

A range of methodologies have been applied to pricing LTC insurance including inception annuity approaches (Gatenby 1991) or risk renewal approaches (see Beekman 1989). Though an explicit and systematic use of the mathematics of multiple state (Markov and semi-Markov) models dates back to the end of the 1960s, it must be stressed that the basic mathematics of what we now call a Markov chain model were developed during the eighteenth century (see Seal (1977)); seminal contributions by D. Bernoulli and P.S. de Laplace demonstrate this for the time-continuous case. Moreover, the well known paper of Hamza (1900) provides the actuarial literature with the first systematic approach for disability benefits, in both the continuous and discrete case. Keyfitz and Rogers provide a method for determining transition probabilities under a Markov process. The approach was developed by assuming that forces of transition are constant within age intervals of a fixed length. Transition probability matrices were then calculated re-cursively for time periods that are multiples of this age interval.

Multiple state models are prevalent in the actuarial literature in areas including Life Insurance (see Pitacco 1995), Permanent Health Insurance (PHI) in the UK (see Waters 1984, Sansom and Waters 1988, Haberman 1993, Renshaw and Haberman 1998, Cordeiro (2001) and Disability Income Insurance (see Haberman and Pitacco 1999). It is therefore unsurprising that the suitability of multiple state modeling for LTC insurance has been well recognized and consequently applied. For instance, Levikson and Mizrahi (1994) consider an `upper triangular' (UT) multiple state model in the general Markovian framework where three care levels are considered and the insured life proceeds through the deteriorating stages of ADL failure until death. Premium calculation is subsequently performed via a representation of the discounted value of future benefits in a particular care level as a random variable. Similar frameworks have been studied by Alegre et al (2002), who also consider a LTC system with no recoveries and premium calculations derived by calculating annuity values in discrete time for a life in a LTC claiming state. Moreover, the valuation of LTC annuities to price LTC insurance in continuous time has been discussed by Pitacco (1993) and Czado and Rudolph (2002).

The chosen methodology in this project is a multiple state modeling approach within a continuous time Markov framework with premiums and reserves calculated by means of applying generalizations of Thiele's differential equations. This choice is motivated by the benefits of multiple state modeling being an accurate representation of the underlying insurance process, a greater degree of flexibility and scope for scenario testing and the ease of monitoring actual experience against expected at a practical level (Gatenby and Ward 1994, Robinson 1996 and Society of Actuaries Long-Term Care Insurance Valuation Methods Task Force 1995).

Despite the wide range of methodologies considered abroad, only limited literature concerning pricing LTC insurance contracts in Kenya, especially the use of multi state modeling within a markov framework and application of Chapman-Kolmogorov and Thiele’s differential equations, has been published. This project therefore intends to fill this research gap.

CHAPTER THREE: METHODOLOGY

3.1 Introduction

This chapter focuses on the estimation of probabilities of transition from one state to another using the type II four state markov model which involves the healthy, out patent sickness , in patient sickness and death. Then the parametric estimation of the maximum likelihood estimates of the forces of transition , μ, from one state to another, the general case, then the properties of the maximum likelihood estimators, the alternative derivation of the maximum likelihood estimates. The parametric graduation methods of Compertz and Mekahem with their logarithmic modification including the Perks formula with be outlined. The use of the equivalence principle in calculating premiums for some hypothetical Long Term Care insurance products will be described. Finally, the use Thiele's differential equation in calculating reserves will be explained.

3.2 A Four State Markov Model – Version II

Just like in the case of the three state markov process, in the four state process, a life may be in the healthy state, outpatient or in patient sick state on a number of separate occasions before making the one-way transition to the dead state(Lecture Notes UoN, 2012). Alternatively, a life may pass from the able/healthy state to the dead state without ever having been in the sick state.

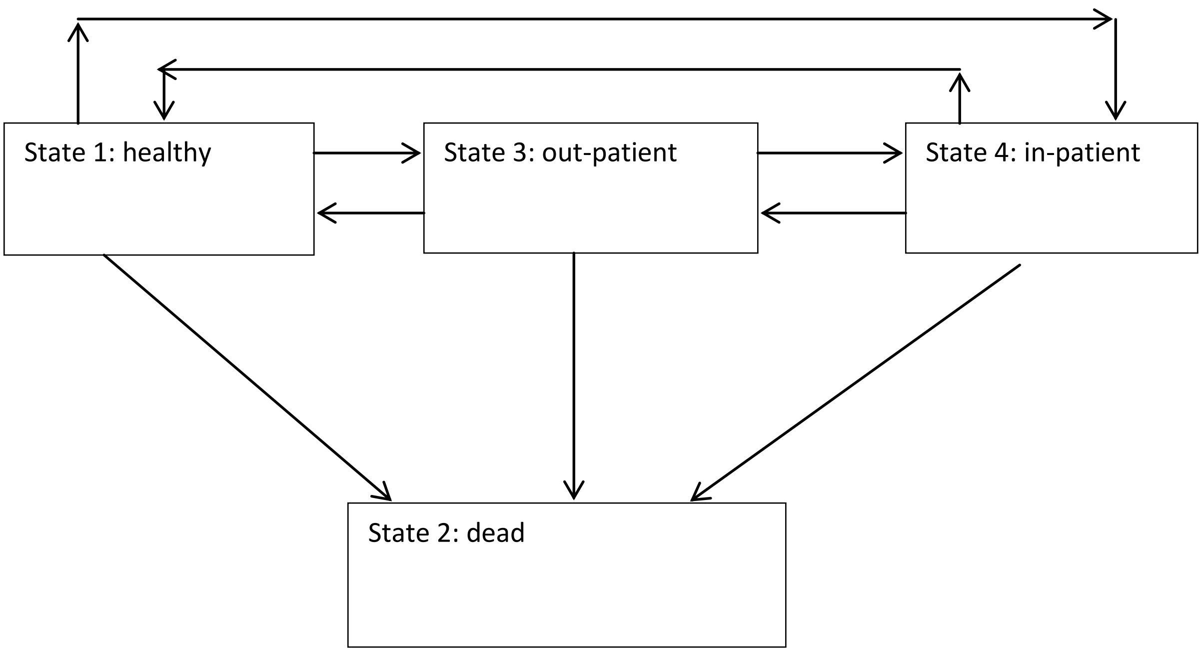


Figure 3.1: A four state markov model

By Chapman –Kolmogorov equation

$$p_{ij}(s,t) = \sum_k p_{ik}(s,t)p_{kj}\,(t,t+h)$$

In this case we shall use:

$$p_{ik}(s,t+h) = \sum_{k=1}^{j} p_{ik}(s,t)p_{kj}\,(t,t+1)\ for\ \ j = 1,2,3,\dots \qquad (3.35)$$

$$p_{11}(s,t+h) = p_{11}(s,t)p_{11}(t,t+h) + p_{12}(s,t)p_{21}(t,t+h) + p_{13}(s,t)p_{31}(t,t+h) + p_{14}(s,t)p_{41}(t,t+h)$$

(3.36)

$$= p_{11}(s,t)\{1-[\mu_{12}+\mu_{13}+\mu_{14}]h + 0(h)\} + p_{12}(s,t)\{0\} + p_{13}(s,t)\{\mu_{31}h + 0(h)\} + p_{14}(s,t)\{\mu_{41}h + 0(h)\}$$

$$\therefore \frac{d}{dt}p_{11}(s,t) = p_{11}(s,t)\{-(\mu_{12}+\mu_{13}+\mu_{14})\} + p_{12}(s,t)0 + p_{13}(s,t)\{\mu_{31}\} + p_{14}(s,t)\mu_{41}$$

$$p_{12}(s,t+h) = p_{11}(s,t)p_{12}(t,t+h) + p_{12}(s,t)p_{22}(t,t+h) + p_{13}(s,t)p_{32}(t,t+h) + p_{14}(s,t)p_{42}(t,t+h)$$

(3.37)

$$= p_{11}(s,t)[\mu_{12}h + 0(h)] + p_{12}(s,t)\{1\} + p_{13}(s,t)\{\mu_{32}h + 0(h)\} + p_{14}(s,t)\{\mu_{42}h + 0(h)i\}$$

$$\therefore \frac{d}{dt}p_{12}(s,t) = p_{11}(s,t)\mu_{12} + p_{12}(s,t).0 + p_{13}(s,t)\mu_{32} + p_{14}(s,t)\mu_{12}$$

$$p_{13}(s,t+h) = p_{11}(s,t)p_{13}(t,t+h) + p_{12}(s,t)p_{23}(t,t+h) + p_{13}(s,t)p_{33}(t,t+h) + p_{14}(s,t)p_{43}(t,t+h)$$

$$= p_{11}(s,t)[\mu_{13}h + 0(h)] + p_{12}(s,t)[0] + p_{13}(s,t)[\mu_{31}+\mu_{32}+\mu_{33}] + 0(h) + p_{14}(s,t)[\mu_{43}h + 0(h)]$$

(3.38)

$$\therefore \frac{d}{dt}p_{13}(s,t) = p_{11}(s,t)\mu_{13} + p_{12}(s,t).0 + p_{13}[\mu_{31} + \mu_{32} + \mu_{34}] + p_{14}(s,t)\mu_{43}$$

$$p_{14}(s,t+h) = p_{11}(s,t)p_{14}(t,t+h) + p_{12}(s,t)p_{24}(t,t+h) + p_{13}(s,t)p_{34}(t,t+h) + p_{14}(s,t)p_{44}(t,t+h)$$

$$= p_{11}(s,t)[\mu_{14}h + 0(h)] + p_{12}(s,t).0 + p_{13}(s,t)[\mu_{34}h + 0(h)] + p_{14}(s,t)[1 - \{\mu_{41} + \mu_{42} + \mu_{43}\}h + 0(h)]$$

(3.39)

$$\frac{d}{dt}p_{14}(s,t) = p_{11}(s,t)\mu_{14} + p_{12}(s,t).0 + p_{13}(s,t)\mu_{34} + p_{14}(s,t)[-(\mu_{41} + \mu_{42} + \mu_{43})]$$

$$\therefore p'(s,t) = p(s,t)Q \qquad (3.40)$$

Where

$$\boldsymbol{Q} = \begin{bmatrix} -((\boldsymbol{\mu_{12}} + \boldsymbol{\mu_{13}} + \boldsymbol{\mu_{14}}) & \boldsymbol{\mu_{12}} & \boldsymbol{\mu_{13}} & \boldsymbol{\mu_{14}} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\mu_{31}} & \boldsymbol{\mu_{32}} & -(\boldsymbol{\mu_{31}} + \boldsymbol{\mu_{32}} + \boldsymbol{\mu_{34}}) & \boldsymbol{\mu_{34}} \\ \boldsymbol{\mu_{41}} & \boldsymbol{\mu_{42}} & \boldsymbol{\mu_{43}} & -(\boldsymbol{\mu_{41}} + \boldsymbol{\mu_{42}} + \boldsymbol{\mu_{43}}) \end{bmatrix}$$

3.3 The four state model data suitability

In this study the analysis of the frequency of attendance data of patients who are members of the National Hospital Insurance Fund will be modeled using the four state markov model version II. This is because the data which are records of all the members and their ages (including the principal civil servants members and all their legal dependants), the out patient, in patient attendants and deaths on a monthly basis for the entire 2012 year. These data fits the four state version II model better because all the death records are captured within the healthy and sickness states within the model. This is unlike the four state model version I where some deaths could be attributed to other exogenous causes such as accidents and murders.

3.4 Estimation Using Markovian Models

We observe n independent and identically distributed lives obeying this model (Lecture notes, UoN, 2012). Let T_i be the actual death of the i^{th} life. In other words $T_i \dots\dots .. T_n$ are independent and identically distributed random variables with μ_t. Suppose that subject i enters the experiment at

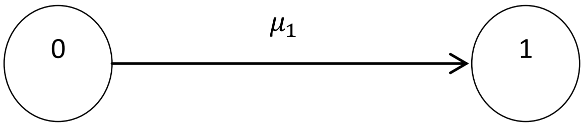


Figure 3.2: Force of transition/intensity

Time a_i and leaves at time b_i if he has not died by this time(Scott W.F., 1991). Let us make the simplifying assumption that $\mu_t = \mu, a \leq t \leq b$, where $a := inf_{a \leq i \leq n} a_{i,} b := sup_{1 \leq i \leq n} b_i$. We wish to estimate the one and only unknown parameter of the model, namely μ.

First of all, we know that$T_i > a_i$. So it is best to consider each of the variables T_i' which in distribution, equals $T_i - a_i$ given that $T_i > a_i$. What do we observe? We observe that random variables $X_i = T_i' \Lambda\, l_i$and$\delta_i = 1(T_i \leq l_i)$ where $l_i := b_i - a_i$.

Note that$P(T_i' \in dx) = \mu dx\ for\ 0 < x < l_i$, by assumption. Note also that $X_i$is neither discrete nor continuous; it is a mixture. It is easy to find the joint distribution of(X_i, δ_i) . Clearly,X_i has a density on $(0, l_i)$ and a discrete mass at l_i.(recall the notation δ_c for a mass of size 1 at the point c) . Indeed,

$$P(X_i \in dx, d_i = 0) = P(l_i \in dx, T_i > l_i) = e^{-\mu l_i} \delta_{li}(dx) = e^{-\mu x} \mu^0 \delta_{li}(dx)$$

$$P(X_i \in dx, d_i = 1) = P(T_i \in dx, T_i > l_i) = e^{-\mu l_i} \mu(dx) = e^{-\mu x} \mu^1(dx) \qquad (3.1)$$

We can compactify this as

$$P(X_1 \in dx, \delta_i = \theta) = e^{-\mu x} \mu^\theta [(1-\theta)\delta_{li}(dx) + \theta dx], x \geq 0, \theta \in \{0,1\}$$

Since we have assumed independence, we can write the joint probability as

$$P(X_1 \in dx_1, \delta_1 = \theta, \dots\dots, X_n \in dx_n, \delta_n = \theta_n) = \prod_{i=1}^{n} P(X_i \in dx, \delta_i = \theta) \qquad (3.2)$$

$= \prod_{i=1}^{n} e^{-\mu x i} \mu^{\theta_i} [(1-\theta_i)\delta_{li}(dx_i) + \theta_i dx_i] = e^{-\mu \sum_{i=1}^{n} \theta_i} \prod_{i=1}^{n} [(1-\theta_i)\delta_{li}(dx_i) + \theta_i dx_i]$

Here the variables $x_{1,} \ldots., x_n$ take the values in $[0, l_i], \ldots, [0, l_n]$,respectively, while the $\theta_{1,} \ldots., \theta_n$ take values 0 or 1 each. Thus, the likelihood corresponding to the observations $X_{1,} \ldots\ldots, X_n, \delta_1 \ldots\ldots, \delta_n$ is $L(X, \delta; \mu) = e^{-\mu \sum_{i=1}^{n} x_i} \mu^{\sum_{i=1}^{n} d_i}$

The maximum likelihood estimation $\hat{\mu}$ is defined by $L(X, \delta, \hat{\mu}) = \max L(X, \delta; \mu)$,

And is easily found to be

$$\hat{\mu} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} X_i} \quad /\boldsymbol{E_x^c} = \int_{\boldsymbol{K}}^{\boldsymbol{K+N+1}} \boldsymbol{P_{X,t}\, dt} \cong \sum_{\boldsymbol{t=K}}^{\boldsymbol{K+N}} \frac{\boldsymbol{1}}{\boldsymbol{2}} \left(\boldsymbol{P_{x,t}} + \boldsymbol{P_{x,t+1}}\right) \tag{3.3}$$

Note that $\hat{\mu}$ is a veritable statistic for it is just a function of the observations.

3.5 The general case of Markovian maximum likelihood estimates

When in state, the chain remains there for an exponentially distributed amount of time (sojourn time) with a particular parameter(Scott W.F., 1991).

$$\mu(x) = |q(x,x)| = \sum_{y \neq x} q(x,y)$$

At the end of this time, the chain jumps to some other state y with probability $\frac{q(x,y)}{\mu(x)}$.an observation consists of a sequence of states Y_0, Y_1, ... and the corresponding sojourn time'sσ_0, σ_1 .... The likelihood of the observation is

$$L = \mu(Y_0) e^{-\mu(Y_0)\sigma_0} \frac{q(Y_0, Y_1)}{\mu(Y_0)} \times \mu(Y_1) e^{-\mu(Y_1)\sigma_1} \frac{q(Y_1, Y_2)}{\mu(Y_1)} \times \ldots..$$

$$= e^{-\mu(Y_0)\sigma_0} q(Y_0, Y_1) \times e^{-\mu(Y_1)\sigma_1} q(Y_1, Y_2) \times \ldots. = e^{-\sum_x \mu(x)} W(x) \prod_{x \neq y} q(x,y)^{N(x,y)} \tag{3.4}$$

Where $W(x) := \sum_{i \geq 0} \sigma_i \, 1(Y_i = x)$ and $N(x,y) := \sum_{i \geq 0} 1(Y_i = x, Y_{i+1} = y)$

Are the total sojourn time in state x and the total number of jumps from state x to state$y$, respectively, for the sequence of our observations

Two things may happen when we observe a Markov chain, either we stop observing at some predetermined time or the Markov chain reaches an absorbing state and, a fortiori, we must stop the observation . For instance, in the Markov chain figure 4.8, there are three absorbing states:a, b, c and two transient ones: 1, 2. if we start with $Y_0 = 1$ we may be

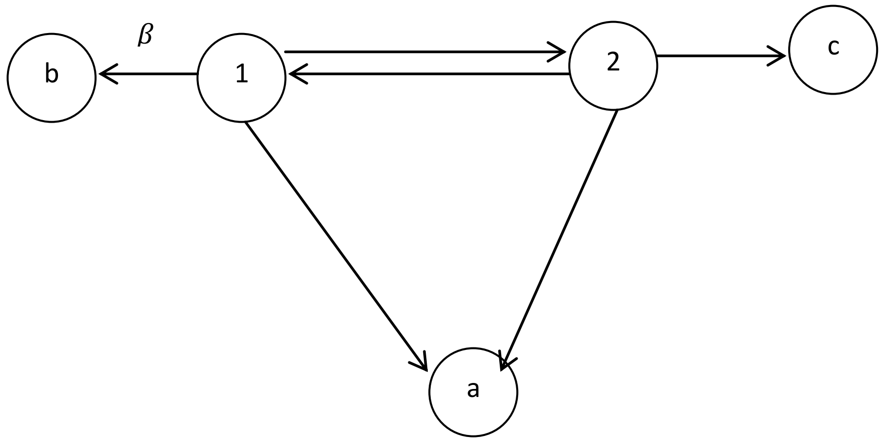


Figure 3.3: Three Absorbing States

Lucky and observe that 1,2,1,2,1,2,1,2,1,2,1,2……

For long time, at some point we will get bored and stop, or we may observe either of the

Following trajectories $1\ 2, \dots, 1, a$, $1\ 2, \dots 1, 2, a$, $1\ 2, \dots, 1, b$ and $1\ 2, \dots, 1, 2, c$

The likelihoods in each case are

$$\begin{array}{lllll} e^{-(\lambda_1+\alpha_1+\beta)W(1)} & {\lambda_1}^{N(1,2)} & e^{-(\lambda_2+2+\gamma)W(2)} & {\lambda_2}^{N(2,1)} & {\alpha_1}^{N(1,a)} \\ e^{-(\lambda_1+\alpha_1+\beta)W(1)} & {\lambda_1}^{N(1,2)} & e^{-(\lambda_2+2+\gamma)W(2)} & {\lambda_2}^{N(2,1)} & {\alpha_1}^{N(2,a)} \\ e^{-(\lambda_1+\alpha_1+\beta)W(1)} & {\lambda_1}^{N(1,2)} & e^{-(\lambda_2+2+\gamma)W(2)} & {\lambda_2}^{N(2,1)} & {\beta_1}^{N(1,b)} \\ e^{-(\lambda_1+\alpha_1+\beta)W(1)} & {\lambda_1}^{N(1,2)} & e^{-(\lambda_2+2+\gamma)W(2)} & {\lambda_2}^{N(2,1)} & {\gamma_1}^{N(2,c)} \end{array}$$

The quantities$N(1, a), N(2, a), N(2, c)$ take values 0 or 1 and amongst them, only one is 1. So we can summarize the likelihood in

$$L = e^{-(\lambda_1+\alpha_1+\beta)W(1)}{\lambda_1}^{N(1,2)}e^{-(\lambda_2+2+\gamma)W(2)}{\lambda_2}^{N(2,1)}{\alpha_1}^{N(1,a)}{\alpha_1}^{N(2,a)}{\beta_1}^{N(1,b)}{\gamma_1}^{N(2,c)}$$

(3.5)

This is an expression valid for all trajectories. Clearly, where as it may be possible to estimate λ_1, λ_2 with the observation of one trajectory alone, this is not the case with the remaining parameters; we need independent trials of the Marko chain. Thus, if we run the Markov chain $n$times, independently trials of to time, a moment of reflection shows the form of likelihood remains the same as in the last display, provided we interpret $W(x)$ as the total sojourn time in state x over all observations. Same interpretation holds for the quantities $N(x,y)$(Scott W. F., 1991).

Thus the general expressions are valid, in the same manner, for n independent trials. To do this, recall that $\mu(x) = \sum_{y:y\neq x} q(x,y)$ and write the log likelihood as $\log L = \sum_{x,y:x\neq y}[-W(x)q(x,y)\log q(x,y)]$

So, for a fixed pair of distinct statesx, y $\frac{\delta \log L}{\delta q(x,y)} = -W(x) + \frac{N(x,y)}{q(x,y)}$

Setting this equal to zero obtains the MLE estimator $\hat{q}(x,y) = \frac{N(x,y)}{W(x)}$ (3.6)

3.6 MLE estimators of the Markov chain rates

Define a random variable D_i as follows

$$D_i = \begin{cases} 1 & if\ the\ i^{th}\ life\ is\ observed\ to\ die \\ 0 & if\ the\ i^{th}\ life\ is\ not\ observed\ to\ die \end{cases}$$

D_i is an exmaple of an indicator random variable ; it indicates the occurence of death (Lecture Notes UoN, 1999).

So D_i is the extra random variable that completes our model. Now lets define random variable T_i as follows:$x + T_i =$ the age at which observation of the i^{th} life ends .

Notice D_i and T_i are not independent, since: $D_i = 0 \leftrightarrow T_i = b_i$i.e if no death has been observed , the life must have survived to $x + b_i$.

$D_i = 1 \leftrightarrow a_1 < T_i < b_i$i.e an observed death must have occurred between $x + a_i\ and\ x + b_i$.

Another definition is that $V_i = T_i - a_i$ where V_i is called the waiting time, it has a mixed distribution, with a probability mass at the point $b_i - a_i$.

Again, the pair (D_i, V_i) comprises a statistic, meaning that the outcome of our observation is a sample (d_i, v_i) drawn from the distribution of (D_i, V_i).

Let $f_i(d_i, v_i)$ be the joint distribution(D_i, V_i).

It is easily written down by considering the two cases $D_i = 0$ and $D_i = 1$

If $D_i = 0$ no death has been observed and the life is known to have survived for the period of $(b_i - a_i)$ $from$ $x + a_i$ to $x + b_i$

If $D_i = 1$, the life is known to have survived for the period $v_i(0 < v_i < b_i - a_i)$ from $x + a_i$ to $x + a_i + v_i$ before dying at age $x + a_i + v_i$

Therefore, $f_i(d_i, v_i)$ has a distribution that is specified by the following expression which is a combination of a probability mass (corresponding to$d_i = 0$) and a probability density (corresponding to $d_i = 1$) (Scott W.F., 1991).

$$f_i(d_i, v_i) = \begin{cases} b_i - a_i P_x + a_i & (d_i = 0) \\ v_i P_x + a_i . \mu_{x+a_i+v_i} & (d_i = 1) \end{cases} \tag{3.7}$$

$$= \begin{cases} \exp\left[- \int_0^{b_i - a_i} \mu_{x+a_i+t}\, dt \right] & (d_i = 0) \\ \exp\left[- \int_0^{v_i} \mu_{x+a_i+t}\, dt \right] \mu_{x+a_i+v_i} & (d_i = 1) \end{cases}$$

$$= \exp\left[- \int_0^{v_i} \mu_{x+a_i+t}\, dt \right] \mu_{x+}^{d_i} a_i + v_i \tag{3.8}$$

Now assume that μ_{x+t} is a constant μ for $0 \leq t < 1$ (this is the first time we have needed assumption 3) and $f_i(d_i, v_i)$ takes on the simple form:

$$f_i(d_i, v_i) = e^{-\mu v_i} \mu^{d_i}$$

We can then write down an expression for the joint probability function, provided that we can assume that the lifetimes of all the lives involved are statistically independent.

The joint function of all the (D_i, V_i) by independence is

$$\prod_{i=1}^{N} e^{\mu v i}\, \mu^{d_i} = e^{-\mu(V_1+\cdots+V_n)}\mu^{d_{1\ldots}d_N}$$

$= e^{\mu v}\mu^d Where\ d = \sum_{i=1}^{N} d_i \ \ and\ v = \sum_{i=1}^{N} v_i$ (3.9)

In other words , define random variables $D\ and\ V$ to be the total number of deaths and the total waiting time , respectively , and the joint probability function of all the (D_i, V_i) can be simply expressed in terms of $D\ and\ V$(Lecture Notes UoN, 2012).

For known transition intensity, we can calculate the likelihood of nay combination of deaths and waiting time. However, in practice the value of the transition intensity is unknown. We use statistical inference to calculate the value of the transition intensity that is most plausible given the observed data i.e. the maximum likelihood estimates of .

The probability function immediately furnishes the likelihood for $L(\mu; d, v) = e^{-\mu v}\mu^d$ which yields the maximum likelihood estimates (MLE) for

$\bar{\mu} = \frac{d}{v}$Now we need to prove that the likelihood is maximized by $\bar{\mu} = \frac{d}{v}$. The solution is that the likelihoods is

$L(\mu; d, v) = e^{-\mu v}\mu^d$ So that $\log L = -\mu v + d\log\mu$.

Differentiating

$\frac{d}{du}\log L = -v + \frac{d}{\mu}$ (3.10)

And setting $-v + \frac{d}{\mu} = 0 \rightarrow \check{\mu} = \frac{d}{v}$

Which is the maximum since$\frac{d^2}{du^2}\log L = -\frac{d}{\mu^2} < 0$

It is reassuring that the mathematical approach produces a result that is intuitive , i.e. that the maximum likelihood estimates of the hazard rate is the number of observed deaths divided by the total time for which lives were exposed to the hazard.

The measurements of the total time for which lives are exposed to the hazard is one of the fundamental techniques covered by this course. It enables accurate assessment of risks, from the probability of a policyholder dying to the probability of a claim under a motor insurance policy(Lecture Notes UoN, 2012).

3.7 Properties of The Maximum Likelihood Estimator

The estimate$\bar{\mu}$, being a function of the sample of the samples values d and v, can itself be regarded as a sample value drawn from the distribution of the corresponding estimator: $\bar{\mu} = \frac{d}{v}$

So, the estimator $\bar{\mu}$ is a random variable and the maximum likelihood estimate$\bar{\mu}$ is the observed value of that random variable(Lecture Notes UoN, 2012).

It is important in application to be able to estimate the moments of the estimator $\bar{\mu}$, for example to compare the experience with that of a standard table. At least, we need to estimate$E[\bar{\mu}]$ and $var\ [\bar{\mu}]$.

In order to derive the properties of the estimator $\bar{\mu}$ we will use two results that link the random variables$D$ and V.

The following exact results are obtained

$E[D_i - \mu V_i] = 0$ and $var[D_i - \mu V_i] = E[D_i]$ (3.11)

Note that the first of these can also be written as $E[D_i] = \mu . E[V_i]$

In the case that the $\{a_i\}$ and $\{b_i\}$ are known constants, this follows from integrating /summing the probability function of (D_i, V_i) over all possible events to obtain:

$$\int_0^{b_i - a_i} e^{-\mu v_i} \mu . dv_i + e^{-\mu(b_i - a_i)} = 1$$

And then differentiating with respect to $\bar{\mu}$, once to obtain the mean and twice to obtain the variance

We will show how to use this to prove result (1) above in moment, but first we need to derive two other results. (The derivation of result (2) is covered in the question & answer bank).

We can how that

$E[D_i] = \int_0^{b_i - a_i} e^{-\mu t}. \mu. dt$ (3.12)

As a solution, **s**ince $D_i = 0$ if the life is not observed to die, we only need to consider the probability of death occurring

$$D_i = 1, i.e\ E[D_i] = 1 \times \int_0^{b_i - a_i} tP_{x+a_i} \mu_{x+a_i+t} dt$$

We are assuming constant transition intensity(Lecture Notes UoN, 2012).

$\mu\ so\ tP_{x+a_i} = e^{-\mu t}$Hence $E[D_i] = \int_0^{b_i - a_i} e^{-\mu t}\ \mu dt$

In order to show that

$$E[V_i] = \int_0^{b_i - a_i} t. e^{-\mu t}. \mu. dt + (b_i - a_i) e^{-\mu(b_i - a_i)}$$

The solution will be as follows:

$$E[V_i] = \int_0^{b_i - a_i} P[life\ dies\ at\ time\ t] \times t. dt + (b_i - a_i) \times P[life\ survives]$$

Hence

$E[V_i] = \int_0^{b_i - a_i} t. e^{-\mu t}. \mu. dt + (b_i - a_i) e^{-\mu(b_i - a_i)}$ (3.13)

Proof of (1)

Differentiating (*) with respect to $\bar{\mu}$ gives;

$\int_0^{b_i - a_i} e^{-\mu v_i} dv_i - \mu \int_0^{b_i - a_i} v_i\, e^{\mu v_i} dv_i - (b_i - a_i) e^{-\mu(b_i - a_i)} = 0$

(Because the limits of the integrals don't depend on μ, this just involves differentiating the expressions inside the integral with respect to μ)$(\text{Scott W.F.}, 1991)$.

Multiplying throughout by μ then gives:

$$\int_0^{b_i-a_i} e^{-\mu v_i} dv_i - \mu\left\{\int_0^{b_i-a_i} v_i\, e^{\mu v_i}.\mu.dv_i + (b_i - a_i)e^{-\mu(b_i-a_i)}\right\} = 0 \qquad (3.14)$$

We can see that the first term is $E[D_i]$ and the expression is curly brackets is $E[V_i]$

So $E[D_i] - \mu.E(V_i) = 0$ As required (Lecture Notes UoN, 2012).

3.8 The Distribution of μ

To find the asymptotic distribution of μ consider: $\frac{1}{N}(D - \mu V) = \frac{1}{N}\sum_{i=1}^{N}(D_i - \mu V_i)$ (Lecture Notes UoN, 1999).

(3.15)

$$We\ know\ that\ E[(D_i - \mu V_i)] = 0\ and\ that\ var\ [D_i - \mu V_i] = E[D_i]$$

So, by the central limit theorem:

$\frac{1}{N}(D - \mu V) \sim Normal\left(0, \frac{1}{N^2}\sum_{i=1}^{N} E[D_i]\right)$ and $\frac{1}{N}(D - \mu V) \sim Normal\ \left(0, \frac{E[D]}{N^2}\right)$

Then note that (not rigorously): $\lim_{N\to\infty}(\tilde{\mu} - \mu) = \lim_{N\to\infty} \frac{N}{V}\left(\frac{D}{N} - \frac{\mu V}{N}\right)$

By the law of large numbers, $V/N \to E[V_i]$ (technically, this refers to "convergence in probability ") and $(\tilde{\mu} - \mu) \sim \frac{1}{E[V_i]} Normal\ \left(0, \frac{E[D]}{N^2}\right)$ (3.16)

$$\sim Normal\ \left(0, \frac{E[D]}{N.E(V_i)^2}\right) \sim Normal\ \left(0, \frac{E[D]}{(E[V_1+V_2+\cdots+V_N])^2}\right) \sim Normal\ \left(0, \frac{E[D]}{E[V]^2}\right)$$

But we know that $E[D - \mu V] = 0 \to E[D] = \mu.E[V]$

So, asymptotically: $\mu \sim Normal\ \left(\mu, \frac{\mu}{E[V]}\right)$ (3.17)

3.9 Alternative Derivation of μ

In this section we describe another way of deriving the asymptotic distribution ofμ, the maximum likelihood estimator ofμ.

We start from the likelihood function:

$$L = \mu^d e^{-\mu v}$$

The log likelihood is then

$\log L = d \log \mu - \mu v$ and differentiating this with respect to μ gives:

$\frac{d \log L}{d\,\mu} = \frac{d}{\mu} - v$ setting this equal to 0 and solving for μ yields the required maximum likelihood estimate: $\tilde{\mu} = \frac{d}{v}$

We can check that this does maximize the likelihood, by examining the sign o the second derivative of the log likelihood.

$$\frac{d^2 \log L}{d\mu^2} = -\frac{d}{\mu^2} < 0 \rightarrow max \qquad (3.18)$$

The corresponding maximum likelihood estimator is: $\bar{\mu} = \frac{D}{V}$

Where D and V are random variables denoting the number of deaths and the total waiting time, respectively.

3.10 Exact Calculation of The Central Exposed to Risk E_X^C

Central exposed to risk (or waiting times) are very natural quantities, intrinsically observable even if observation may be incomplete in practice, that is, just record the time spent under observation by each life. Note that this is so even if lives are observed for only part of the year of age [x, **x** +1] whatever reason(Lecture Notes UoN, 2012).

(a)Working with complete data

The procedure for the exact calculation of E_X^C is obvious: record all dates of birth, record all dates of entry into observation, record all dates of exit from observation and then compute. If we add to the data above the cause of the cessation of observation, we have dx as well, and we have finished. The central exposed to risk $E_X^C$for a life with age label x is the time from Date A to Date B where: Date A is the latest of: the date of reaching age label x of the start of the investigation and the date of entry. Date B is the earliest of; the date of reaching age label x +1, the end of the investigation and the date of exit (for whatever reason).

(b) Census approximation of E_x^c from available data

We will consider how to calculate E_X^C approximately when the exact dates of entry to and exit from observation have not been recorded. Suppose that we have death data of the form: d_x =Total number of deaths age x last birthday during calendar years $K, K+1, ... K+N$2(Scott W.F., 1991).

That is, we have observations over ***N***+1 calendar years of all deaths between ages *x* and *x* +1.

However, instead of the times of entry to and exit from observation of each fife being known, we have instead only the following census data:

$\boldsymbol{P_{X,t}} =$ = Number of lives under observation aged x last birthday at time *t* where *t 1* January in calendar years K, K + 1,..., K + N,K + N + 1

Now define $\boldsymbol{P}_{x,t}$ to be the number of lives under observation, aged x last birthday, at **any**time ***t***. Note that:

$$E_x^c = \int_K^{K+N+1} P_{x,t} dt$$

During any short time interval $(t, t+dt)$there will be$P_{x,t}$, lives each contributing a fraction of a year dt to the exposure. So, integrating $P_{x,j} \times dt$ over the observation period gives the total central exposed to risk for this age. In other words.$E_x^c$is the area under the $P_{x,t}$"curve" between $t = K \; and \; t = K + N + 1$(Lecture Notes UoN, 2012).

The problem is that we do not know the value of $P_{x,t}$for all $\boldsymbol{t}$. So we cannot work out the exact value of the integral. We have the values of $P_{x,t}$only if t is a 1 January (a census date), so we must estimate $E_x^c$from the given census data. The problem reduces to estimating an integral, given the integrand at just a few points (in this case, integer spaced calendar times). This is a routine problem in numerical analysis.

The simplest approximation, and the one most often used, is that $P_{x,t}$is linear between census dates, leading to the trapezium approximation(Scott W.F., 1991).

The area of a trapezium is:

$$\boldsymbol{base \times \frac{1}{2}(length\ of\ side\ A + length\ of\ side\ B)}$$

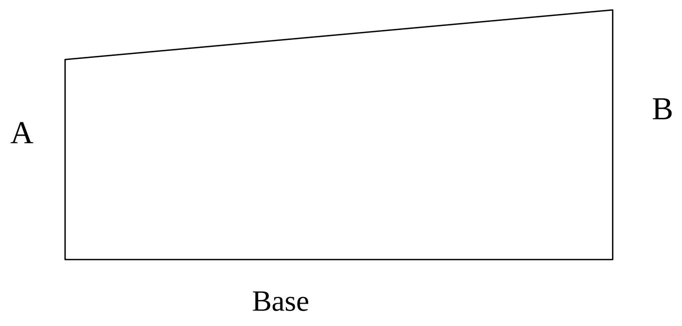


Figure 3.5: A trapezium of census data

In this case the base of the trapezium is equal to 1, ie the period between census dates, the length offside A is $P_{x,t}$the number of policies in force at the start of the year (at time) and the length of side B is

$\boldsymbol{P_{x,t+1}}$the number of polices in force at the end of the year at time $\boldsymbol{t+1}$.

Now, using the trapezium approximation

$$\boldsymbol{E_x^c = \int_K^{K+N+1} P_{X,t}\, dt \cong \sum_{t=K}^{K+N} \frac{1}{2}\left(P_{x,t} + P_{x,t+1}\right)} \quad (3.19)$$

This is the method used by the CMIB. It is easily adapted to census data available at more or less frequent intervals, or at irregular intervals(Lecture Notes UoN, 2012).

Example: Estimate based on the following data:

Calendar year	Population aged 55 last birthday on 1 January
2001	46,233
2002	42,399
2003	42,618
2004	42,020

Solution: Using the census approximation, the central exposed to risk for the 5 year period 1 January 2001 to 1 January 2004 is:

$$E_{55}^{c} \approx \frac{1}{2}[P_{55}(1.1.01) + P_{55}(1.1.02)] + \frac{1}{2}[P_{55}(1.1.02) + P_{55}(1.1.03)]$$
$$+ 1/2\ [P_{55}(1.1.03) + P_{55}(1.1.04)]$$

$$= 1/2P_{55}(1.1.01) + p_{55}(1.1.02) + P_{55}(1.1.03) + \frac{1}{2}P_{55}(1.1.04)$$

$$= \frac{1}{2}(46{,}233) + 42{,}399 + 42{,}618 + \frac{1}{2}(42{,}020)$$

$$= 129{,}143.5$$

3.11 Estimating transition intensities

DefineD^{ij}as the number of transitions from state i to state j, and let V^{i}denote the "waiting time" in state i. These random variables have realizationsd^{ij}/v^{i} depending on whether we are discussing estimators or estimates) (Scott W.F.1991). We state the following important results: that the vector μ is the maximum likelihood estimator of the true value, $\mu$and that the asymptotic distribution of the vector $\mu$of maximum likelihood estimators is multivariate Normal with mean equal to the vector of true transition intensities, μ the variance matrix has diagonal elements(Lecture Notes UoN, 2012).

$\frac{\mu^{ij}}{E(V^{i})} = \frac{(\mu^{ij})^2}{E(D^{ij})}$ and off-diagonal elements zero. Hence the asymptotic distribution of the component

$\mu^{ij}\ is\ N\left(\mu^{ij}, \mu^{ij}/E(V^{i})\right) = N(\mu^{ij}, \mu^{ij})^2/E(D^{ij})$.

Now the asymptotic posterior distribution of $\mu^{ij} is$ approximately
$N(\tilde{\mu}^{ij}, (\tilde{\mu}^{ij})^2/d^{ij})$(Scott W.F., 1991).

We now label the N lives from $k = 1$ to $k = N$, since the symbol i may be used to index the states. We can use an example to illustrate the methodology: In a certain investigation, 40 "active" lives become "ill" and there were 8,176 years of exposure among the "active" lives. (Some "ill" lives recovered, and some of the "active" lives and "ill" lives died). Estimate the force of transition from the "active" to the "ill" state (this being assumed to be the same at all ages) and give an approximate 95% confidence interval(Scott W.F.1991).

The solution will be as follows:

$$\tilde{\mu}^{ai} = \frac{d^{ai}}{v^a} = \frac{40}{8{,}176} = 0.004892$$

is the M.L.E. of μ^{ai}. The estimated s.d. of $\tilde{\mu}^{ai}$ is $\tilde{\mu}^{ai}/\sqrt{d^{ai}} = 0.004892/\sqrt{40} = 0.0007735$so an approximate 95% confidence interval forμ^{ai} is $0.004892 \mp 1.96 \times 0.0007735 = (0.003376, 0.006408)$ (Scott W.F., 1991).

3.12 Graduation by Parametric Formula

There are usually three methods of carrying out a graduation which are graduation by parametric formula, graduation by reference to a standard table and graphical graduation (Lecture notes, UoN, 2012). The most appropriate method of graduation to use will depend on the quality of the data available and the purpose for which the graduated rates will be used.

The general methodology of graduation is essentially the same under each method. Once we have decided on the appropriate method, we will choose a model of represent the underlying force (or rates) of mortality or transition, fit the model to the crude observed rates and test the graduation for adherence to data and (if necessary) smoothness. Each method can produce many possible graduations. The graduation chosen will be the one whose adherence and smoothness best meet the requirements for which the rates are intended(Lecture Notes UoN, 2012).

Graduation is a compromise between adherence to data (goodness of fit) and smoothness. The balance that we want between these two conflicting objectives is a subjective choice and will depend on how the graduated rates will be used. For example if we are constructing a standard table of national population mortality, we will be interested in maximizing the accuracy. We will put more emphasis on adherence and less emphasis on smoothness. If the rates are to be used to calculate premiums and reserves for a life insurance company, we will want to ensure that the rates (and hence the premiums and reserves) progress smoothly from age to age to avoid sudden changes and inconsistencies. We will put more emphasis on smoothness and less emphasis on adherence. The mortality rates at ages around the accident hump will be less important in this situation are few policy holders are likely to be in the age range 18-22 years.

In this study however, we will focus on graduation by formula because of its pragmatic nature.

The method of graduation most often used for reasonably large experiences is to fit a parametric formula to the crude estimates(London, D. 1985).

The underlying assumption is μ_x (or q_x) can be modeled using an appropriate mathematical formula with unknown parameters. The parameters are typically calculated automatically by a computer using numerical methods.

If the formula used does not include enough parameters, it will not be flexible enough to follow the crude rates closely, which may result in over graduation. If too many parameters are included, sudden bends may appear in the graduated curve, which may result in under graduation.

For different values of the parameters, we can assess the smoothness and adherence to data of the fitted model. (In practice we will not need to check smoothness if the number of parameters is sufficiently small).

We will choose the values of the parameters that provide the most appropriate model, according to some pre-defined criterion in respect of goodness of fit. Two simple (but useful) formulae are:

Gompertz (1825) $\mu_x = Bc^x$ (3.20)

Makeham (1860) μ_x = A + $Bc^{2(66)}$

(Scott W.F., 1999)

In practice, it is usually found that μ_x follows an exponential curve quite closely over middle and older ages (in human populations) so most successful formulae include a Gompertz term. Makeham's formula is interpreted as the addition of accidental deaths, independent of age, to a Gompertz term representing senescent deaths.

The most recent standard tables produced for us by UK life insurance companies used formulae of the form.

μ_x= pollynomial_1 + exp (polyinomial_2)

Which includes Gompertz and Makeham as special cases(London, D. 1985). The formulae were of the form:

$\dot{u_x}$ =f $(a_{1,} a_{2,} \dots\dots, a_{r+1,} a_{1+2,\dots.} a_{r+s,} x)$ Where

Polynomial (1) =a_1 + a_2x + a_3x2 + …. + $a_r x^{r-1}$

Polynomial (2)=$a_{r+1} + a_{r+2^{x}}, + a_{r+3x^{2}} + \ldots . a_{r+sx^{s-1}}$

In other words, a formula with (r +s) parameters was fitted for each table of the form μ_x = polynomial$_1$ + exp (polynomial$_2$).

The researcher chose to use a Perks formula specification, a(x), to graduate transition intensities to sickness states. Moreover, an additional parameter, H, was included for the purposes of a more suitable fit.

$$a(x) = \frac{A + Bc^{x}}{1 + Dc^{x} + Kc^{-x}} + H \quad (3.21)$$

Graduations using the Gompertz-Makeham and Logit Gompertz-Makeham formula of type (r,s) have been investigated previously using health and disability related data (CMIR 6 (1983) and CMIR 17 (1991)). Generally, the LogitGompertz-Makeham formula is expressed as:

$$LGM_{\beta}^{r,5}(x) = \frac{GM_{\beta}^{r,5}(x)}{1 + GM_{\beta}^{r,5}(x)} \text{ Where } GM_{\beta}^{r,5}(x) = \sum_{i=1}^{r} \beta_i x^{i-1} + \exp\left\{ \sum_{i=r+1}^{r+5} \beta_i x^{i-r-1} \right\}$$

(5.21)

is the Gompertz-Makeham formula of type (r,s) (Forfar et al 1985).

A LogitGompertz-Makeham formula, LGM (1,2) , was found to fit sufficiently well here for recovery intensities, that is:

$$\mu_x^{0ji} = \frac{\beta_1 + \exp(\beta_2 + \beta_3 x)}{1 + \beta_1 + \exp(\beta_2 + \beta_3 x)} \quad (3.22)$$

3.13 Calculation of Premiums using the equivalence principle

So far we have shown that multiple state models are a natural way of modeling cash flows for insurance policies and we have also shown how to evaluate probabilities for such models given only the transition intensities between pairs of states. The next stage in our study of multiple state models is to calculate premiums and policy values for a policy represented by such a model and to show how we can evaluate them(Lecture Notes UoN, 1999).

To do this we can generalize our definitions of insurance and annuity functions to a multiple state framework. We implicitly use the indicator function approach, which leads directly to intuitive formulae for the expected present values, but does not give higher moments.

Suppose we have a life aged x currently in state i of a multiple state model. We wish to value an annuity of 1 per year payable continuously while the life is in some state j (which may be equal to i)(London, D. 1985).

The EPV of the annuity, at force of interest δ per year, is

$$\bar{a}_x^{ij} = E\left[\int_0^\infty e^{-\delta t}\, i(Y(t) = j|Y(0) = i)dt\right]$$

$$= \int_0^\infty e^{-\delta t}\, E[I(Y(t) = j|Y(0) = i)]dt$$

$= \int_0^\infty e^{-\delta t}\, {}_tp_x^{ij}\, dt$ Where I is the indicator function

Similarly, if the annuity is payable at the start of each year, from the current time, conditional on the life being in state j, given that the life is currently in state i, the expected present value is

$$\bar{a}_x^{ij} = \sum_{j\neq k} e^{-\delta t}\, {}_tp_x^{ij}\, \mu_{x+t}^{jk} dt \qquad (3.23)$$

Annuity benefits payable more frequently can be valued similarly. For insurance benefits, the payment is usually conditional on making a transition. A death benefit is payable on transition into the dead state; a critical illness insurance policy might pay a sum insured on death or earlier diagnosis of one of a specified group of illnesses(Scott W.F., 1999).

Suppose a unit benefit is payable immediately on each future transfer into state k, given that the life is currently in state j (which may be equal to$k$). Then the expected present value of the benefit is

$$\bar{A}_x^{ik} = \int_0^\infty \sum_{j\neq k} e^{-\delta t}\, {}_tp_x^{ij}\, \mu_{x+t}^{jk} dt. \qquad (3.24)$$

To derive this, we consider payment in the interval $t, t + dt$ so that the amount of the payment is 1, the discount factor (for sufficiently small dt)is, $e^{-\delta t}$ and the probability that the benefit is paid is the probability that the life transfers into state k in $(t, t + dt)$ given that the life is in state i at time 0. In order to transfer into state k in $(t, t + dt)$ the life must be in some state j that is not k immediately before (the probability of two transitions in infinitesimal time being negligible), with probability ${}_tp_x^{ij}$ then transfer from j to k during the interval $(t, t + dt)$ with probability $\mu_{x+t}^{jk} dt$. Summing (that is, integrating) over all the possible time intervals gives equation (3.24).

Other benefits and annuity designs are feasible; for example, a lump sum benefit might be paid on the first transition from healthy to sick, or premiums may be paid only during the first sojourn in state 0. Most practical cases can be managed from first principles using the indicator function approach(Lecture Notes UoN, 2012).

In general, premiums are calculated using the equivalence principle and we assume that lives are in state 0 at the policy inception date(London, D. 1985).

Example 3.1: Suppose an insurer issues a 10-year disability income insurance policy to a healthy life aged 60. Calculate the premiums for the following two policy designs using the model and parameters given from Example: Assume an interest rate of *5%* per year effective, and that there are no expenses.

(a) Premiums are payable continuously while in the healthy state. A benefit of $20000 per year is payable continuously while in the disabled state. A death benefit of $50000 is payable immediately on death.

(b) Premiums are payable monthly in advance conditional on the life being in the healthy state at the premium date. The sickness benefit of $20000 per year is payable monthly in arrear, if the life is in the sick state at the payment date. A death benefit of $50000 is payable immediately on death(Dickson et al, 2009)..

Solution :(a) We equate the EPV of the premiums with the EFV of the benefits.

The computation of the EPV of the benefits requires numerical integration. All values below have been calculated using the repeated Simpson's rule, with h = 1/12. Using

Table 8.1, Let P denote the annual rate of premium. Then the EPV of the premium income is

$$pa_{60:10}^{-00} = P\int_{0}^{10} e^{-\delta t}{}_tp_{60}^{00}dt$$

(Scoot W.F., 1999)

And numerical integration gives $\boldsymbol{a_{60:10}^{-00} = 6.57}$ Next, the EPV of the sickness benefit is $20000\,a_{60:10}^{-01} = 20000 \int e^{-\delta t}{}_tp_{60}^{01}\,dt$ and numerical integration gives

$a_{60:10}^{-01} = 0.66359$. Last, the EPV of the death benefit is

$$50000\,A_{60:10}^{-02} = 50000 \int e^{-\delta t}\left({}_tp_{60}^{00}\mu_{60+t}^{02} + {}_tp_{60}^{01}\mu_{60+t}^{12}\right)dt$$

Using numerical integration, we find $A_{60:10}^{-01} = 0.16231$.

Hence, the annual premium rate is

$$P = \frac{20000\,a_{60:10}^{-01} + 50000\,A_{60:10}^{-01}}{a_{60:10}^{-00}} = 3254.65$$

We now need to find the EPV of annuities payable monthly, and we can calculate these from Table 8.1. First, to find the EPV of premium income we calculate

$$a_{60:10}^{(12)00} = \frac{1}{12}\left(1 + \frac{1}{12}p_{60}^{00}v^{1/12} + \frac{2}{12}p_{60}^{01}v^{2/12} + \frac{3}{12}p_{60}^{00}v^{3/12} + \cdots + 10p_{60}^{01}v^{10}\right)$$

$$= 0.66877$$

And to find the EPV of the sickness benefit we require

Note that the premiums are payable in advance, so that the final premium payment date is at time 9 However, the disability benefit is payable in arrear so that a payment will be made at time 10if the policyholder is disabled at that lime.

The death benefit is unchanged from part (a), so the premium is \$3257.20per year, or \$271.43per month(Dickson et al, 2009).

3.14 Policy values or Reserves and Thiele's differential equation

The definition of the time t policy value for a policy modeled using a multiple state model is the expected value at that time of the future loss random variable - with one additional requirement. For a policy described by a multiple state model, the future loss random variable, and hence the policy value, at duration t years depends on which state of the model the policyholder is in at that time. We can express this formally as follows: a policy value is the expected value at that time of the future loss random variable conditional on the policy being in a given state at that time(Dickson et al, 2009). We use the following notation for policy values(Scott W.F.1991).

Notation ${}_tV^{(i)}$ denotes the policy value at duration *t* for a policy which is in state *i* at that time. A policy value depends numerically on the basis used in its calculation, that is the transition intensities between pairs of states as functions of the individual's age, the force of interest, the assumed expenses, and the assumed bonus rates in the case of participating policies. The key to calculating policy values is Thiele's differential equation, which can be solved numerically using Euler's method(Dickson et al, 2009).

Notation, ${}_tV^{(i)}$ denotes the policy value at duration t for a policy which is in state i at that time. The key to calculating policy values is Thiele's differential equation, which can be solved numerically using Euler's, or some more sophisticated, method. To establish some ideas we start by considering the disability income insurance model.

Suppose we consider a policy with a term of n years issued to a life aged x. Premiums are payable continuously throughout the term at rate P per year while the life is healthy, an annuity benefit is payable continuously at rate B per year while the life is sick, and a lump sum, 5, is payable immediately on death within the term. Recovery from sick to healthy is possible and the disability income insurance model is appropriate(Scoot W.F., 1999). We are interested in calculating policy values for this policy and also in calculating the premium using the equivalence principle. For simplicity we ignore expenses in this section, but these could be included as extra 'benefits' or negative 'premiums' provided only that they are payable continuously at a constant rate while the life is in a given state and/or are payable as lump sums immediately on transition between pairs of states. Also for simplicity, we assume that the premium, the benefits and the force of interest per year, are constants rather than functions of time(Dickson et al, 2009)..

Example 3.2: Show that, for $0 \leq t < n$,

$$t^{V^{(0)}} = Ba^{-01}_{x+tn-t} + SA^{-02}_{x+En-t} - Pa^{-00}_{x+Bn-t}$$

(a)

And derive expressions for t^{v} (3.25)

(b) Show that, for $0 \leq t <$

(c) $\frac{d}{dt} t^{V^{(0)}} = \delta t^{V^{(0)}} + P - \mu^{01}_{x+t}\left(t^{V^{(0)}} - t^{V^{(0)}}\right) - \mu^{02}_{x+t}\left(S - t^{V^{(0)}}\right)$

And $\frac{d}{dt} t^{V^{(1)}} = \delta t^{V^{(1)}} - B - \mu^{10}_{x+t}\left(t^{V^{(0)}} - t^{V^{(1)}}\right) - \mu^{12}_{x+t}\left(S - t^{V^{(1)}}\right)$ (3.26)

(d) Suppose that

$x = 40, n = 20, \delta = 0.04, B = \$\ 100000, S = \$\ 500000$

And $\mu^{01}_{x} = a_1 + b_1\, exp\{c_1 x\}$, $\mu^{10}_{x} = 0.1\, \mu^{01}_{x}$, $\mu^{02}_{x} = a_2 + b_2\, exp\{c_2 x\}$ and $\mu^{12}_{x} = \mu^{02}_{x}$

where $\mu^{01}_{x} = 0.0279, \mu^{02}_{x} = 0.0229, \mu^{12}_{x} = \mu^{12}_{x} = \mu^{02}_{x}$

Calculate $10^{V^{(0)}}, 10^{V^{(1)}}$ $and\ 0^{V^{(0)}}$ $for\ n = 20$ using Euler's method with a step size of 1/12 given that

$1. P = \$5500\ and$

$2. P = \$\ 6000$

Calculate P using the equivalence principle

Solutions: (a) The policy value $t^{V^{(0)}}$ equals

EPV of future benefits — EPV of future premiums

Conditional on being in state 0 at time t

= EPV of future disability income benefit +EPV of future death benefit — EPV of future premiums conditional on being in state 0 at lime /

This leads directly to formula (3.45).

The policy value for a life in state I is similar, but conditioning on being in state 1 at time t so that

$$t^{V^{(1)}} = Ba^{-11}_{x+tn-t} + SA^{-12}_{x+En-t} - Pa^{-10}_{x+Bn-t} \qquad (3.27)$$

where the annuity and insurance functions are defined(Dickson et al, 2009)..

(b)We could derive formula (3.27) directly. To do this it is helpful to think of ,t^{v} as the amount of cash the insurer is holding at time ***t.*** given that the policyholder is in state 0 and that, in terms of expected values, this amount is exactly sufficient to provide for future losses(Scoot W.F., 1999)

.

Let be such that $t < t + h <$and let be small. Consider what happens between times and $t +$Premiums received and interest earned will increase the insurer's cash to $t^{v} e^{on} + P_{S_h}$

Recall that $e^{on} = 1 + \delta h + 0(h)$ and $s_h = (e^{on} - 1)/\delta = h + 0(h)$

So that $e^{\delta h} + p_{S_h} = t^{V^{(0)}}(1 + \delta h) + Ph + 0(h)$

This amount must be sufficient to provide the amount the insurer expects to need at

time this amount is a policy value of and possible extra amounts of

$S - t + h^{V^{(o)}}$ if the policyholder dies: the probability of which *is* $\mu^{02}_{x+t} + 0(h)$ and

$t + h^{V^{(1)}} - t + h^{V^{(0)}}$ if the policyholder falls sick: the probability of which is $h\mu^{01}_{x+t} + 0(h)$

Hence

$) + ph = t + h^{v} + h\{\mu^{02}_{x+t}(S - t + h^{v}) + \mu^{01}_{x+t}(t + h^{v} - t + h^{v})\} + 0(h)$

Rearranging, dividing by **h** and letting $h \to 0$ gives formula (3.28) (c)(i) Euler's method for the numerical evaluation of t^{v} and, t^{v} is based on replacing the differentials on the left-hand sides of formulae (3.27) and (3.28) by discrete time approximations based on a step size h. which are correct up to 0(h). We could write, for example,

$$\frac{d}{dt}t^{V^0} = \left(t + h^{V^0} - t^{V^0}\right)/h + 0(h)/h$$

Putting this into formula (3.47) would give a formula for $t + h^{V^{(0)}}$ in terms of t^{V^0} and t^{V^1}.This is not ideal since the starting values for using Euler's method are

$$\boldsymbol{n^{V^0} = 0 = n^{V^1}}$$

and so we will be working backwards, calculating successively policy values at durations

$h, \dots h, 0$

for this reason, it is more convenient to have formulae for and in terms of, and, We can achieve this by writing

$$\frac{d}{dt}t^{V^{(0)}} = \left(V_t^{(0)} - V_{T-H}^{(01)}\right)/H + 0(h)/h$$

And $\frac{d}{dt}t^{V^{(1)}} = \left(V_t^{(1)} - t - h^{V^{(1)}}\right)/h + 0(h)/h$

Putting these expressions into formulae (3.26)and(3.27), multiplying through by h, rearranging and ignoring terms which are o(h). Gives the following two (approximate) equations

$$V_{t-h}^{(0)} = t^{V^{(0)}}(1 - \delta h) - Ph + h\mu_{x+t}^{01}\left(t^{V^{(1)}} - t^{V^{(0)}}\right) + h\mu_{x+t}^{02}\left(S - t^{V^{(0)}}\right) \quad (3.29)$$

And

$$V_{t-h}^{(1)} = t^{V^{(1)}}(1 - \delta h) - Bh + h\mu_{x+t}^{10}\left(t^{V^{(0)}} - t^{V^{(1)}}\right) + h\mu_{x+t}^{12}\left(S - t^{V^{(1)}}\right) \quad (3.30)$$

These equations, together with the starting values at time n and given values of the step size, h, and premium rate P, can be used to calculate successively

$$V_{n-h}^{(0)}, V_{n-h}^{(1)}, V_{n-2h}^{(0)}, \dots . V_{10}^{(0)}, V_{10}^{(1)}, \dots . V_0^{(0)}$$

1) for $n = 20, h = \frac{1}{2}$ and $p = \$5500$ we get

2) $V_{10}^{(0)} = \$18084$, $V_{10}^{(1)} = \$829731$, $V_0^{(0)} = -\$3815$

For

$n = 20, h = \frac{1}{2}$ and $p = \$6000$ we get

3)

4) $V_{10}^{(0)} = \$14226$, $V_{10}^{(1)} = \$829721$, $V_0^{(0)} = -\$2617$

(c)(ii) Let P*be the premium calculated using the equivalence principle. Then for this premium we have by definition $V_n^{(0)} = \mathbf{0}$. Using the results in **0**part (i) and assuming $V_0^{(0)}$ is (approximately) a linear function of p we have

$$\frac{p^* - 5500}{6000 - 5500} \approx \frac{0 - 3815}{-2617 - 3815}$$

So that

$p^* = \$5797$

Using Solver or Goal Seek in Excel, setting 0^{V^0} to be equal to zero, by varying P. The equivalence principle premium is \$5796.59. Using the techniques of Example 3.2 gives

$$a_{40:20}^{-00} = 12.8535, a_{40:20}^{-01} = 0.31593, A_{40:20}^{-02} = 0.08521$$

and hence an equivalence principle premium of \$5772.56. The difference arises because we are using two different approximation methods(Dickson et al, 2009)..

The above example illustrates why, for a multiple state model, the policy value at duration / depends on the state the individual is in at that time. If in this example, the individual is in state 0 at time 10. Then it is quite likely that no benefits will ever be paid and so only a modest policy value is required. On the other hand, if the individual is in state 1 , it is very likely that benefits at the rate of \$100000 per year will be paid for the next 10 years and no future premiums will be received. In this case, a substantial policy value is required the difference between the values of $V_{10}^{(0)}$ and $V_{10}^{(1)}$ in part (c), and the fact that the Latter are not much affected by the value of the premium, demonstrate this point(Scoot W.F., 1999)

.

3.15 Thiele's differential equation - the general case

Consider an insurance policy issued at age x and with term years described by a multiple state model with n +states, labeled 0,1 , 2,..., n. Let us denote the transition intensity between stales i and/ age y, 6, denote the force of interest per year at time t and denote the rate of payment of benefit while the policyholder is in state , and $s_t^{(ij)}$Denote the lump sum benefit payable instantaneously at time t on transition from state i to state j.

We assume that $\delta_1, B_t^i, and\ S_t^{ij}$and are continuous functions of t. Note that premiums are included within this model as negative benefits and expenses can be included as additions to the benefits(Dickson et al, 2009)..

For this very general model, Thiele's differential equation is as follows.

$for\ i = 0.1\ , \ldots n\ and\ 0 \le t \le n$

$$\frac{d}{dt} t^{V^{(i)}} = \delta_i\, t^{V^{(i)}} - B_t^{(i)} - \sum_{j=0, j\neq i}^{n} \mu_{x+t}^{ij} \left(S_t^{(ij)} + t^{V^{(j)}} - t^{V^{(i)}} \right) \tag{3.31}$$

At time t the policy value tor a policy in state $i.\, t^{V^{(i)}}$ is changing as a result of interest being earned at rate $\delta_t\, t^{V^{(i)}}$ and Benefits being paid at rate $B_t^{(i)}$. The policy value will also change if the policyholder jumps from state i to any other state j at this time.

Formula (3.50) can be derived more formally by writing down an integral equation for and differentiating it(Dickson et al, 2009). We choose a small step size hand replace the left-hand side by

$$\left(t^{V^{(i)}} - t - h^{V^{(i)}} + 0(h) \right) / h$$

Multiplying through by h rearranging and ignoring terms which are 0(h),we have a formula for. in terms of the policy values at duration t. We can then use Euler's method, starting with $V_n^{(i)} = 0$ to calculate the policy values at durations $- h, n - 2h, \ldots h, 0$.

CHAPTER FOUR: APPLICATIONS, DATA PRESENTATION AND ANALYSIS

4.1 Introduction

In Kenya, the lack of a scientifically modeled national health financing insurance can be attributed to the lack of complete, comprehensive and reliable data. This is particularly problematic in the case of LTC insurance given the unusually broad range of services and modes of disability associated with LTC (Eagles 1992). Clearly choice of data will influence the modeling methodology. In this chapter, therefore, a case study of National Hospital Insurance Fund (N.H.I.F.) is undertaken for the purposes of both introducing the current and available data sets in Kenya and, more importantly, in justifying the choice of data used in this project. The choice of N.H.I.F. was particularly informed by the fact it is the only LTC insurer with a scheme of more than 200,000 principal members. Other private insurers have schemes in tens of thousands which the researcher found to be insufficient for the purpose of multi state modeling.

4.2 Data Requirements

Ideal data for LTC insurance pricing is a longitudinal data set that tracks both levels of disability and LTC utilization patterns of a large representative population. As discussed by Meiners (1989), the benefit of longitudinal data for LTC pricing is primarily to enable an understanding of LTC utilization changes as the cohort ages.

Many nations, including Kenya, lack a systematic LTC data-reporting program enabling comprehensive information to be collected across service sectors, care programs and jurisdictions (Reif 1985). Given that Kenya currently has no private insurance coverage for LTC, there is clearly a need to gather data on virtually all aspects of LTC insurance covers including costs, risk management, marketing and underwriting. From a pure actuarial pricing and reserving perspective, utilization/demand data for LTC segregated by age and sex in conjunction with changes to utilization(ie functional changes) as a function of age are essential. The following sections discuss and evaluate the various options for obtaining this information.

4.3 Kenyan Long Tern Care Data

In order to access LTC insurance, the researcher approached N.H.I.F. with a request to provide time-segregated claims for the year 2012 on a monthly basis, according to various levels of disability with healthy/able being free of any claim records of the civil servants scheme with over 200,000 principal scheme members. Outpatient claims had low medications requirements, possibly normal check-ups and claims which involved some pharmaceutical purchases and minor surgery such as tooth removals. In patient claims were characterized by in patient hospitalization for more than 3 months sometimes with critical illnesses such as cancer, diabetes and high blood pressure being involved. Recoveries from both out patient and in patient claims were subtracted from the last expenses, meaning death cases since LTC covers were primarily concerned with sickness benefits. Death claims were assumed to be covered by other insurance classes such as life and endowments and term assurances.

4.4 Approximation from 1-step Transition Probabilities

The researcher used Ms Excel 2007 and Matlab 7.0.4 to model the data in terms of Chapman-Kolmogorov equations and calculate the 1-step transition probabilities at 10-yearly age intervals and the results are reported in Tables 4.1 and 4.2 females and males (from the healthy/able state) respectively. The calculation was done using the maximum likelihood estimates of the $t-$year probability ${}_t p_x^{ab}$ of a life aged (x) making a transition from state *a* to state *b* using the formulae in the four state markov model in chapter four.

The calculation was done using the maximum likelihood estimates of the year probability of a life aged (x) making a transition form state a to state b using an equation of the form :

$$\mu = \frac{nx_{x,x+t}^{ab}}{E_X^C} \tag{4.1}$$

Where

$$E_x^c = \int_K^{K+N+1} P_{X,t}\,dt \cong \sum_{t=K}^{K+N} \frac{1}{2}\left(P_{x,t} + P_{x,t+1}\right)$$

Several observations can be made with transition intensities or forces of transition and probabilities for both males and females generally behaving as expected with transition probabilities to sickness states increasing with age. Transition through disability levels is reasonably progressive. That is, given that a transition out of the sickness state occurs, there is a higher probability of moving to a lower sickness like out patient than directly to a more severe sickness level such as the in patient sickness level. At higher ages, however, transition to the in patient sickness exceeding other out patient sickness levels. This seems reasonable owing to the effects of ageing and chronic frailty.

Table 4.1: Male one-step transition probabilities

	Healthy	Outpatient	Inpatient	Dead
Healthy				
20	0.990045	0.005229	0.001793	0.001199
30	0.988251	0.006233	0.002137	0.001313
40	0.983648	0.008726	0.002992	0.001742
50	0.971556	0.01486	0.005095	0.003562
60	0.940283	0.029544	0.01013	0.010245
70	0.897377	0.04365	0.015034	0.029305
80	0.702715	0.119224	0.046477	0.077222
Outpatient 20	0.15	0.844587	0.002142	0.001199
30	0.15	0.843664	0.002554	0.001313
40	0.15	0.841226	0.003575	0.001742
50	0.15	0.834462	0.006088	0.003562
60	0.15	0.815941	0.012106	0.010245
70	0.15	0.785242	0.017965	0.029305
80	0.15	0.652276	0.05554	0.077222
In patient 20	0	0	0.1	0.002825
30	0	0	0.1	0.005195
40	0	0	0.1	0.01009
50	0	0	0.1	0.018562
60	0	0	0.1	0.031897
70	0	0	0.1	0.055423
80	0	0	0.1	0.105596
Dead				
20	0	0	0	0
30	0	0	0	0
40	0	0	0	0
50	0	0	0	0
60	0	0	0	0
70	0	0	0	0
80	0	0	0	0

Secondly, transition probabilities out of the sickness state appear higher for males than females. Transition out of the healthy/able state to outpatient and inpatient sickness states appears higher for females than males. Again, mortality in the inpatient sickness state is higher for males than females.

These last two points above are particularly interesting as they form the basis of an a priori expectation that the likelihood of LTC utilization by females will be higher than by males in the Kenyan population, therefore resulting in more expensive premiums for females.

Table 4.2: Female 1-step transition probabilities

	Healthy	**Outpatient**	**Inpatient**	**Dead**
	0.991502	0.004906	0.001299	0.000417
20	0.990264	0.005612	0.001486	0.000492
30	0.986776	0.007486	0.001983	0.000893
40	0.97731	0.012414	0.00329	0.002231
50	0.95288	0.024985	0.006659	0.005772
60	0.920111	0.037427	0.010481	0.015529
70	0.745522	0.088349	0.033458	0.048903
80	0.15	0.845614	0.001624	0.000417
Out patient 20	0.15	0.844967	0.001858	0.000492
30	0.15	0.84305	0.002479	0.000893
40	0.15	0.837712	0.004113	0.002231
50	0.15	0.823774	0.008324	0.005772
60	0.15	0.800804	0.013102	0.015529
70	0.15	0.654558	0.041825	0.048903
80	0	0.15	0.846652	0.000417
In patient	0	0.15	0.846155	0.000492
30	0	0.15	0.844634	0.000893
40	0	0.15	0.840339	0.002231
50	0	0.15	0.829065	0.005772
60	0	0.15	0.808763	0.015529
70	0	0.15	0.670198	0.048903
80				
Dead **20**	0	0	0	0
30	0	0	0	0
40	0	0	0	0
50	0	0	0	0
60	0	0	0	0
70	0	0	0	0
80	0	0	0	0

4.5 Calculating Transition Intensities from Transition Probabilities

The researcher imposed a Markov assumption to describe the process in the model. That is, we consider a stochastic process $\{S(t), 0 < t < \infty\}$ with state space { 1, 2,...,6} where S(t) represents the state of the process at time t. $\{S(t), 0 < t < \infty\}$ is a continuous time Markov chain if for states $g, h \in \{1, 2,...6\}$ and $x, t, \geq,$

$$\Pr\{S(x+t=)\ h\ |S(x)=g, S(r)\ \text{for}\ 0 \leq rx\} = \Pr\{S(x+t)=h|S(x)=g\}.$$

In other words, the future development of $S(t)$ can be determined only from its present state and without regard to the process history. We denote ${}_t p_x^{gh} = \Pr\{S(x+t) = h \mid S(x) = g\}$,

${}_t p_x^{\overline{gg}} = \Pr\{S(x+u) = g\ \forall u \in [x, x+t] | S(x) = g\}$ and assume a closed system whereby

$\sum_{h=1}^{6} {}_t p_x^{gh} = 1$ for all $x \leq 0\ r$ and $t \geq 0$. The transition probabilities also obey the Chapman - Kolmogorov equations:

$$t{+}_u p_x^{gh} = \sum_{1=1}^{6} {}_t p_x^{gl} \cdot {}_u p_{x+t}^{lh} \tag{4.2}$$

The existence of transition intensity functions is also assumed such that $\mu_x^{gh} = \lim_{t \to 0+} \frac{{}_t p_x^{gh}}{t}$ or, alternatively, that ${}_t p_x^{gh} = \mu_x^{gh} + o(t)$ Transition and occupancy probabilities are related to transition intensities via the relations:

$$\frac{d}{dt} {}_t p_x^{gh} = \sum_{l+h} ({}_t p_x^{gl} \mu_{x+1}^{lh} - {}_t p_x^{gh} \mu_{x+l}^{hl}) \text{ and } {}_t p_x^{\overline{gg}} = \exp\left(\int_0^t \sum_{l+g} \mu_{x+r}^{gl} dr \right)$$

Where these equations are better known as the Kolmogorov forward equations (Cox and Miller 1965). The assumption that there is constant force of transition in the transition intensities for each age in the data is required. Consequently, if P(t) is defined to be the matrix of transition probabilities over t years and Q to be the matrix of constant transition intensities per annum, then it can be shown directly from the Chapman-Kolmogorov equations (Jones 1992b) that

P(t) = exp (**Q**t). Thus, calculating transition intensities requires finding the infinitesimal generator Q for the transition probability matrix **P(t).**

The researcher chose to use a Schur-Parlett method purely because of its straightforward implementation through software such as MS-EXCEL and MATLAB, which was used by the researcher.

The method initially requires the computation of a Schur decomposition **P =UTU`** **,**where **U** is a unitary matrix (ie its entries are complex and its inverse is the conjugate-transpose), **U*** is the conjugate transpose of U, and T is an upper triangular matrix. We can then determine functions of matrices (including natural logarithms) using the formula:

$$f(\mathbf{P}) = \mathrm{U}f(\mathbf{T})^{\mathbf{U^*}} \qquad \mathbf{(4.3)}$$

Parlett (1974) proposes a recursive relationship for determining the matrix F, defined as f(T), which is derived from equating elements (i,j) where *i<j,(* ie strictly upper triangular) in the commutivity relation FT=TF. The elements (i,j) in the commutivity result satisfy

$$\sum_{k=i}^{j} f_{tk} t_{kj} = \sum_{k=i}^{j} t_{ik} f_{kj}$$

and as long as *ti, ~ t jj*(ie the eigen values are distinct), then:

$$f_{ij} = t_{ij} \frac{f_{jj} - f_{ii}}{t_{jj} - t_{ii}} + \frac{\sum_{k=i+l}^{j-i} [t_{ik} f_{kj} - f_{ik} t_{kj}]}{t_{jj} - t_{ii}} \qquad (4.4)$$

Thus, starting with $f_{ii} = f(t_{ii})$, all other elements of F can be calculated one super diagonal at a time. Tables 4.3 and 4.4 report the calculated annual transition intensities calculated from 1-step transition probabilities at 10-yearly age intervals for males and females respectively. As anticipated, there were a number of calculated transition intensities which are negative and thus have no physical interpretation. They remain useful, however, as starting values for our constraining algorithm. We will refer to these `unconstrained estimates' as *jx* for the transition intensity at age x from state i to state j constituting matrix generator Q_x . The researcher found the Schur-Parlett approach to give satisfactory results over the majority of the age range. We note, however, that the computational procedure was unstable at the extremely high ages.

Table 4.3 :Male unconstrained transition intensities calculated from 1-step transition probabilities in 10 yearly age intervals.

	Healthy	**Outpatient**	**Inpatient**	**Dead**
Healthy				
20		0.005552	0.001896	0.001198
30		0.006628	0.002262	0.001308
40		0.009315	0.003172	0.001725
50		0.016034	0.005429	0.003511
60		0.032798	0.010957	0.01013
70		0.050588	0.016731	0.029426
80		0.172025	0.058298	0.078928
Out patient 20	0.163936		0.0023	0.001197
30	0.16419		0.002744	0.001306
40	-0.164851		0.00385	0.001721
50	0.166644		0.006599	0.003498
60	-0.171567		0.013363	0.010091
70	0.179329		0.020487	0.029352
80	0.225628		0.07359	0.078519
In patient 20	0.00399	0.17751		0.001197
30	0.00485	0.177686		0.001305
40	-0.00469	0.178158		0.001716
50	0.00397	0.179494		0.003484
60	-0.00311	0.183309		0.010051
70	-0.00494	0.190219		0.029274
80	0.00469	0.227448		0.07813
Dead20	0	0	0	
30	0	0	0	
40	0	0	0	
50	0	0	0	
60	0	0	0	
70	0	0	0	
80	0	0	0	

Table 4.4: Female unconstrained transition intensities calculated from 1-step transition probabilities in 10 yearly age intervals.

	Healthy	**Outpatient**	**Inpatient**	**Dead**
Healthy				
20		0.005238		0.000415
30		0.005998	0.001357	0.000486
40		0.008025	0.001553	0.000875
50		0.01342	0.002075	0.002177
60		0.027604	0.003457	0.00562
70		0.042664	0.007069	0.015295
80		0.123111	0.01133	0.047905
Out patient 20	-0.163707		0.041157	0.000414
30	0.163882		0.001728	0.000484
40	0.164386		0.001978	0.000869
50	-0.16578		0.002643	0.002161
60	0.16951		0.004409	0.005571
70	-0.175197		0.009048	0.015189
80	0.217371		0.014566	0.047106
In patient20	0 0 0.006102 0.004074 0.000562 0.00515 0.00775 0.00619 0.00742 0.00604 0.00935	-0.177359	0.054786	0.000413
30	0	0.006047		
40	0	-0.177491		0.000482
50	0	0.177885		0.000863
60	0	-0.178994		0.002143
70	0	0.181963		0.005517
80	0	0.18715		0.015072
Dead		0.230334	0	0.046253
30	0	0	0	
40	0	0	0	
50	0	0	0	
60	0	0	0	
70	0	0	0	
80	0	0	0	

Transition intensities which are positive are required. Determining an appropriate method to deal with this requires care as adjusting negative 'transition intensities' to non-negative values will inevitably force other transition intensities, particularly those complementary to transition intensities that are negative, to compensate accordingly.

The researcher implemented this approach by estimating $\hat{Q}$ using Israel et al's (2001) algorithm instead of our original Schur-Parlett method. Tables 4.5 and 4.6 show the eigen values $\theta_1, \theta_2, \ldots\ldots\ldots\theta_n$ for the transition probability matrices estimated from data at 10-yearly age intervals for both males and females respectively.

Table 4.5 Eigen values for transition probability matrices estimated from the N.H.I.F. survey data in 10-yearly age intervals - Males.

Age	θ_1	θ_2	θ_3	θ_4
10	0.999821	0.838121	0.848222	1
20	0.998724	0.836277	0.846855	1
30	0.998483	0.834203	0.84593	1
40	0.997715	0.828986	0.843431	1
50	0.995003	0.81566	0.836371	1
60	0.986077	0.782183	0.816708	1
70	0.96448	0.737629	0.784764	1
80	0.534709	0.646476	0.721884	1

Table 4.6: Eigen values for transition probability matrices estimated from N.H.I.F. data in 10-yearly age intervals - Females.

Age	θ_1	θ_2	θ_3	θ_4
10	0.999838	0.839261	0.847249	1
20	0.999496	0.838461	0.846717	1
30	0.999287	0.83703	0.845997	1
40	0.998546	0.833052	0.843896	1
50	0.996327	0.82245	0.838032	1
60	0.990413	0.795729	0.821848	1
70	0.977043	0.760464	0.795421	1
80	0.567567	0.624489	0.706487	1

The researcher chose to implement a simple constraining algorithm to constrain the transition intensities to lie in the non-negative region. That is, we estimate Q using

$$\hat{Q} = \min \| \mathbf{P} - \exp(Q) \| \qquad (4.5)$$

such that the elements ($i \sim j$) of Q are non – negative.

The procedure incorporates a least squares routine, using the unconstrained transition intensities as starting values and $\exp(Q)$ evaluated using a Taylor series expansion used by Moler and Van Loan (1978) for series computations of matrix exponentials, thus:

$$\text{Exp(Q)} = \sum_{n=0}^{\infty} \frac{1}{n!} Q^n = 1 + Q + \frac{1}{2!} Q^2 + ... + \frac{1}{n!} Q^n + ...$$

Tables 4.7 and 4.8 show the annual constrained transition intensities calculated using the above algorithm at 10-yearly age intervals for both males and females respectively. Interestingly, the constraining procedure results in Q having the non-negative off diagonal entries estimated as zero and recovery transitions only occurring progressively by one state - a likely feature of estimating transition intensities from transition probabilities estimated using Rickayzen and Walsh's (2002) framework.

Figures 4.4 and 4.5 illustrate the constrained transition intensities for both males and females from the healthy/able state estimated using the simple constraining algorithm described above.

Several important observations may be made here among them the fact that Qs no longer contains any negative `transition intensities'. Secondly, the constraining procedure does not impact on the unconstrained negative intensities in isolation. All elements of the transition intensity matrix will be affected. However, a comparison of the unconstrained transition intensities against the resulting constrained transition intensities for both males and females reveals only marginal differences to other elements as a result of the constraining procedure.

Table 4.7: Female constrained transition intensities in 10 yearly age intervals.

		Healthy	Outpatient	Inpatient	Dead
Healthy	**20**		0.005219	0.001353	0.000416
	30		0.005978	0.001551	0.000487
	40		0.008004	0.002078	0.000875
	50		0.013394	0.003471	0.002177
	60		0.027578	0.007069	0.005631
	70		0.042624	0.011333	0.015317
	80		0.122889	0.041199	0.048004
Outpatient	**20**	0.16252		0.002092	0.000444
	30	0.162691		0.002321	0.000522
	40	0.163172		0.002919	0.000939
	50	0.16452		0.00461	0.002345
	60	0.168141		0.009264	0.005796
	70	0.173658		0.014801	0.015443
	80	0.214095		0.055272	0.047636
In patient	**20**	0	0.170766		0.0001
	30	0	0.17114		0.0001
	40	0	0.172247		0.0001
	50	0	0.174135		0.0001
	60	0	0.17685		0.003165
	70	0	0.181535		0.012649
	80	0	0.219895		0.043193
Dead	**20**	0	0	0	
	30	0	0	0	
	40	0	0	0	
	50	0	0	0	
	60	0	0	0	
	70	0	0	0	
	80	0	0	0	

Table 4.8: Male constrained transition intensities in 5 yearly age intervals.

	Healthy	Outpatient	In patient	Dead
Healthy				
20		**0.005535**	0.001894	0.001198
30		0.006611	0.002262	0.001307
40		0.009299	0.003174	0.001725
50		0.016017	0.005431	0.003514
60		0.032777	0.010964	0.010139
70		0.050562	0.016749	0.029442
80		0.171942	0.058377	0.07903
Outpatient				
20	0.16274		0.002654	0.001293
30	0.162984		0.003072	0.001412
40	0.163619		0.004112	0.001864
50	0.165353		0.006806	0.00371
60	0.170124		0.013593	0.010323
70	0.177647		0.020748	0.029616
80	0.221915		0.074159	0.079062
In patient				0.0001
20	0	0.171054		
30	0	0.171502		0.0001
40	0	0.172716		0.0001
50	0	0.174649		0.001143
60	0	0.178058		0.007668
70	0	0.184316		0.026776
80	0	0.216654		0.075003
Dead20		0	0	0
30		0	0	0
40		0	0	0
50		0	0	0
60		0	0	0
70		0	0	0
80		0	0	0

The study also noted that several transition intensities have the tendency to change direction abruptly at the extremely high ages. Recovery intensities appear to be increasing as a function of age for both males and females. This initially seems counter intuitive. However, if one considers the conditional probability that a recovery transition occurs given a departure from the life's current state, it is easily verifiable that this quantity is indeed decreasing as a function of age - which is consistent with the underlying recovery process. A further reason lies with the Rickayzen and Walsh's (2002) feature of recovery transition probabilities which are constant for each age.

Finally, the researcher noted that although the constraining procedure produces a matrix Q that has row-sums 0 and non-negative off diagonal entries, it no longer satisfies P(1) = exp(Q) exactly. Overall, the method used here to constrain the transition intensities is not critical as these intensities must ultimately be graduated in order to apply Thiele's differential equation approach.

4.6 Graduating Transition Intensities to Sickness States

The graduation by mathematical formulae is pursued purely because of the need for functional forms for the constrained transition intensities for use in the Thiele's differential equations pricing and reserving framework. Graduation by mathematical formulae is discussed in detail in Benjamin and Pollard (1980), London (1985) and Forfar et al (1988). The graduation of transition intensities are in three parts. The start is to graduate transition intensities to the levels of sickness states, then graduate recovery transition intensities and finally graduate mortality transition intensities. Furthermore, smoothness and goodness of fit criteria are discussed here in relation to the absence of exposed to risk information.

The transition intensities considered are $\mu_x^{12}, \mu_x^{13}, \mu_x^{14}, \mu_x^{23}, \mu_x^{24}, and \mu_x^{34}$ for both males and females. The choice of formulae was directly influenced by the functional forms used to estimate the original 1-step transition probabilities in discrete time from which these intensities were derived. The researcher estimated transition probabilities to sickness levels states according to a logistic type function motivated by Perks (1932). The researcher chose to use a Perks formula specification, a(x), to graduate transition intensities to sickness states. Moreover, an additional parameter, H, was included for the purposes of a more suitable fit.

$$a(x) = \frac{A + Bc^x}{1 + Dc^x + Kc^{-x}} + H \qquad (4.6)$$

Table4.9. Parameter estimates for graduating transition intensities to sickness states for males using Perks specification.

Parameters	$\mu_x^{1,2}$	$\mu_x^{1,3}$	$\mu_x^{1,4}$	$\mu_x^{2,3}$	$\mu_x^{2,4}$	$\mu_x^{3,4}$
A	0.001716	0.001192	-0.00174	0.001762	0.002586	-0.00208
B	0.000112	0.0000422	0.0000316	0.000053	0.0000265	0.0000447
C	1.097952	1.093898	1.090271	1.093061	1.097779	1.098795
D	0.000127	-0.000048	-0.00019	-0.0001	0.001027	0.001346
K	110	110	110	110	110	110
H	0.006186	0.002114	0.001342	0.002894	0.001114	0.000185

Table 4.10; Parameter estimates for graduating transition intensities to sickness states for females using Perks specification

Parameters	$\mu_x^{1,2}$	$\mu_x^{1,3}$	$\mu_x^{1,4}$	$\mu_x^{2,3}$	$\mu_x^{2,4}$	$\mu_x^{3,4}$
A	0.005823	0.001667	0.001234	0.002054	-0.01028	-0.00027
B	0.000125	0.0000313	0.0000219	0.000832	-0.01985 90	-0.00035
C	1.097723	1.097345	1.097263	1.092977	1.091327	1.089583
D	0.001315	0.001196	0.001006	0.000041	0.0000803	0.0000693
K	110	110	110	110	110	110
H	0.004864	0.001246	0.000901	0.000066	0.003899	0.001207

4.7 Graduating Recovery Transitions

The transition intensities considered here are those concerning recovery p32, psi and 1u. Graduations using the Gompertz- Makeham and Logit Gompertz-Makeham formula of type (r,s) have been investigated previously using health and disability related data (CMIR 6 (1983) and CMIR 17 (1991)). Generally, the Logit Gompertz-Makeham formula is expressed as:

$$LGM_{\beta}^{r,5}(x) = \frac{GM_{\beta}^{r,5}(x)}{1+GM_{\beta}^{r,5}(x)} \text{ Where } GM_{\beta}^{r,5}(x) = \sum_{i=1}^{r} \beta_i x^{i-1} + \exp\left\{ \sum_{i=r+1}^{r+5} \beta_i x^{i-r-1} \right\} \qquad (4.7)$$

is the Gompertz- Makeham formula of type (r,s) (Forfar et al 1985).

A LogitGompertz-Makeham formula, LGM (1,2) , was found to fit sufficiently well here for recovery intensities, that is:

$$\mu_x^{0ji} = \frac{\beta_1 + \exp(\beta_2 + \beta_3 x)}{1 + \beta_1 + \exp(\beta_2 + \beta_3 x)} \qquad (4.8)$$

Female recovery transition intensities had the tendency to change direction abruptly at extremely high ages. Note, however, that male recovery transition intensities did not have this problem. The researcher chose to extrapolate over the higher ages for the female graduations. This was chosen purely to remain consistent with formulae used to graduate male recovery intensities. In any case, for our ultimate purpose of pricing calculations, it is anticipated that the impact of this assumption will be minimal.

The parameters $\{\beta_1, \beta_2, \beta_3\}$ were estimated using unweighted least squares. The parameter estimates for graduating recovery transition intensities for both males and females by mathematical formula as specified in equation (4.8) are presented in Tables 4.11 and 4.12 respectively.

Table 4.11; Parameter estimates for graduating recovery transition intensities for males using a LGM $(1,2)$ specification.

Parameter	$\mu_x^{2,1}$	$\mu_x^{3,2}$
β_1	0.207171	0.215534
β_2	-30.05004	-26.27263
β_3	0.326204	0.282975

Table 4.12: Parameter estimates for graduating recovery transition intensities for females using a LGM $(1,2)$ specification.

Parameter	$\mu_x^{2,1}$	$\mu_x^{3,2}$
β_1	0.19639	0.22901
β_2	19.6237	67.1497
β_3	0.211458	0.744763

4.8 Graduating Mortality Transition Intensities

The final set of transition intensities to be considered are those concerning mortality - is:

$$\mu_x^{0\ i6} = \gamma_1 + \gamma_2 + \exp(\gamma_3 + \gamma_4 x) \qquad (4.9)$$

Again, the parameters $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ were estimated using un weighted least squares. The parameter estimates for graduating mortality transition intensities for both males and females by mathematical formula as specified in equation 4.9 are presented in Tables 4.13 and 4.14 respectively.

Table 4.13. Parameter estimates for graduating mortality transition intensities for males using a GM $(2,2)$ specification.

		Transition	Intensity
Parameter	$\mu_x^{1,4}$	$\mu_x^{2,4}$	$\mu_x^{3,4}$
Yl	0.0066	0.007678	0.007857
72	-0.000378	-0.000451	-0.000467
73	-7.189564	-6.869178	-6.750137
74	0.062122	0.058743	0.056746

Table 4.14 Parameter estimates for graduating mortality transition intensities for females using a GM (2.2) specification.

		Transition	Intensity
Parameter	$\mu_x^{1,4}$	$\mu_x^{2,4}$	$\mu_x^{3,4}$
Yl	0.004367	0.004367	0.000805
72	0.00024	-0.00024	-0.00011 -
73	-10.1279	-7.66023	7.06937
74	0.089905	0.061344	0.059315

4.9 Smoothness and Goodness of Fit Criteria

One of the main advantages of graduating by mathematical formulae is that the resulting graduations are smooth. There is therefore no issue concerning smoothness here except in the case where two curves have been blended for graduating transitions to sickness states. As already discussed, the researcher endeavoured to ensure a smooth transition across both curves by forcing endpoints of both curves to meet and first derivatives at end points to be equal.

The researcher chose to use the Theil Inequality Coefficient (TIC) (Theil 1958) which is a scale invariant statistic typically used to assess econometric forecast samples. It is expressed as:

$$TIC = \frac{\sqrt{\sum_{x=0}^{n}\left(\overset{0}{\mu}_x - \hat{\mu}_x\right)}}{\sqrt{\frac{\sum_{x=1}^{n}\overset{0}{\mu}{}_x^{2}}{n}} + \sqrt{\frac{\sum_{x=1}^{n}\hat{\mu}^{2}}{n}}} \qquad (4.10)$$

and lies between 0 and 1 with 0 being a perfect fit. Graduated curves with a coefficient of 10% or less were accepted. Table 4.15 reports the TIC for the graduated transition intensities for both males and females.

Overall, the reported TIC are generally low suggesting that the graduated curves provide a good fit to the observed transition intensities. Furthermore, it is interesting to note that the inequality coefficients for males appear to be better than the female counterparts despite there being no intuitive reason as to why this should occur. Three reported inequality coefficients for females ($\mu_x^{2,4}$ and $\mu_x^{1,4}$) are slightly greater than 10% suggesting that the formula specification for these transition intensities was sub-optimal. The researcher chose not to change the formula specification for these three transition intensities and to retain consistency with the other intensities as the reported coefficients were only marginally greater than 10%.

Table 4.15: Theil Inequality Coefficient (TIC) for graduated transition intensities

Transition intensity	**Male TIC**	**Female TIC**
μ_x^{14}	0.02317	0.034338
$\mu_x^{2,4}$	0.00861	0.031603
$\mu_x^{3,4}$	0.01514	0.02387
$\mu_x^{2,3}$	0.01406	0.02537
Recovery intensity		
$\mu_x^{2,1}$	0.06229	0.07065
$\mu_x^{3,2}$	0.04052	0.11509
Mortality intensity		
$\mu_x^{1,4}$	0.04469	0.01787
$\mu_x^{2,4}$	0.05063	0.04332
$\mu_x^{3,4}$	0.05801	0.11178

4.10 The Premium Pricing Calculations

In this section, the focus is on pricing of LTC products. The LTC benefit types and benefit triggers and the application of Thiele's differential equations as a framework for pricing and reserving LTC policies in Kenya is analyzed.

The pricing methodology adopted in this project is essentially an application of Thiele's differential equations to derive formulae concerning the expected development of the mathematical reserve for a closed LTC insurance portfolio.

If we let $V_i(r,u)$ denote the expected present value (EPV) of LTC benefits in the time interval *(r, u)*, given that the policyholder is in state i at time r with a prevailing force of interest of δ over the period (r,u).

In general, for a multiple state model with n states, let $B_{jk}(t)$ denote the benefit payable at time t upon transition from state j to state k, and let $b\ j\ (t)$ denote the rate of benefit payment at time t if the policyholder is in state j.

Then, $V_i(r,u)$ may be expressed as:

$$V_i(r,u) = \int_r^u e^{-\delta(t-r)}\ {}_{t-r}\, p_{x+r}^{\bar{ii}} b_i(t)dt + \int_r^u e^{-\delta(t-r)}\, p_{x+r}^{\bar{ii}} \sum_{j \neq i} \mu_{x+1}^{ij} \left(B_{ij}(t) + V_j(t,u)\right) \qquad (4.11)$$

which leads to the generalisations of Thiele's differential equations:

$$\frac{d}{dr}V_i(r,u) = \delta V_i(r,u) - b_i(r) - \sum_{j \neq i} \mu_{x+r}^{ij} (B_{ij}(r) + V_j(t,u)) \qquad (4.12)$$

For i = 1,2,...,n (see Hoem (1969)).

Now turning to the issue of pricing some illustrative LTC products, lets consider first a whole life stand-alone LTC policy where premiums are payable continuously at rate P per annum while the life is healthy/able (ie no sickness) and an annuity is payable to the policyholder at rate A per annum while enduring out or inpatient sickness state. That is, A per annum is paid to the policyholder when in need of LTC. Note that no death benefit is payable. For the purposes of premium calculation, we require the expected present value at time t=0 of a unit payment while the individual is in each of the able and LTC claiming states.

Therefore, consider first, the case where:

b, (t) = 1, b2 (t) = b3 (t) = b4 (t) = b5 (t) = b6 (t) = 0, and B_{ij} (t) = 0

for all i and j which allows us to calculate the present value of a unit payment, payable as long as the life is healthy/able - which ultimately translates to the calculation of premiums.

Thus we have the following equations:

$$\frac{d}{dr}V_1(r,u) = \delta v_1(r,u) - 1 - \begin{bmatrix} \mu^{12}_{x+r}(V_2(r,u) - V_1(r,u) - V_1(r,u)) + \\ \mu^{13}_{x+r}(V_3(r,u) - V_1(r,u)) + \mu^{14}_{x+r}(V_4(r,u) - V_1(r,u) + \mu^{15}_{x+r}(V_5(r,u) - V_1(r,u) \end{bmatrix} +$$
$$\mu^{16}_{x+r}V_1(r,u)$$

$$\frac{d}{dr}V_2(r,u) = \delta V_2(r,u) - \left[\mu^{21}_{x+r}(V_1(r,u) - V_2(r,u) + \mu^{23}_{x+r}(V_3(r,u) - V_2(r,u))\right] +$$
$$\mu^{24}_{x+4}(V_4(r,u) - V_2(r,u)) + \mu^{25}_{x+r}(V_5(r,u) - V_2(r,u)\ \ ] + \mu^{26}_{x+r}V_2(r,u)$$

$$\frac{d}{dr}V_3(r,u) = \delta V_3(r,u) - \left[\mu^{23}_{x+r}(V_2(r,u) - V_3(r,u) - V_3(r,u)) + \mu^{34}_{x+r}(V_4(r,u) - V_3(r,u))\right] +$$
$$\mu^{35}_{x+r}(V_5(r,u) - V_3(r,u)\ \ ] + \mu^{36}_{x+r}V_3(r,u)$$

$$\frac{d}{dr}V_4(r,u) = \delta V_4(r,u) - \left[\mu^{43}_{x+r}(V_3(r,u) - V_4(r,u)) + \mu^{45}_{x+r}(V_5(r,u) - V_4(r,u)\right] + \mu^{46}_{x+r}V_4(r,u)$$

$$\frac{d}{dr}V_5(r,u) = \delta V_5(r,u) - \left[\mu^{54}_{x+r}(V_4(r,u) - V_5(r,u))\right] + \mu^{56}_{x+r}V_5(r,u) \qquad (4.13)$$

Solving for VI *(0, u)* gives the expected present value of a unit payment to the individual while in the able/healthy state, say EPV_I .

We also do the same for: $b_4(t) = b_1(t) = b_2(t) = 0,\ and\ B_{ij}(t) = 0$ for all *i* and *j* which allows us to calculate the EPV of a unit payment while the individual is in the out patient state, say EPV2. And $b_3(t) = 1, b_1(t) = b2(t) = b_4(t) = 0,\ and\ B_{ij} = 0$ for all *i* and *j* which allows us to calculate the EPV of a unit payment while the individual is in the in patient state, say EPV_3.

Using the principle of equivalence, the net annual premium, P, may be calculated as:

P x EPV_I = A x (EPV4 + EPV3) (1.04)

Note that the system of Thiele's differential equations may not be solved analytically. We therefore solve numerically. Note also that *u* is required to be sufficiently large to mimic a whole of life assurance.

A nominal rate of interest of 8% per annum is assumed where premiums are increasing at an assumed inflation rate of 4% per annum and benefits are similarly increased by 4% per annum whether the insured is claiming or not. Thus a 4% effective net interest rate per annum is appropriate for comparative purposes. The benefit level used was the one given by N.H.I.F. as illustrated in the Annex I. Table 4.16 reports the net annual premium for a whole of life stand-alone LTC policy calculated at 5 yearly age intervals.

Table 4.16: Net annual premiums (Kshs) for a whole of life stand-alone LTC policy calculated using Thiele's differential equations for both males and females compared to other studies.

Age	**Male**	**Female**
20	6800	10080
25	7250	11430
30	9370	12200
35	10840	14560
40	12830	18730
45	15550	21960
50	19510	28890
55	24590	37070
60	32420	48520

This pricing framework may easily be extended to other LTC product types. For instance, consider a LTC rider benefit policy where premiums are payable continuously at rate P per annum while able (ie no sickness) and an annuity is payable to the policyholder at rate A per annum while enduring out patient or in patient sickness. In addition, a sum assured, S, is payable immediately on death from any live state.

That is, we need to consider the case where $b_1; (t) = 0$ for $i = 1,2,...,5$ and $B_{j4}(t) = 1$ for *j*=1, 2,...,5 which allows us to calculate the EPV of a unit payment when the individual transits to the dead state, say EPV4. Again, using the principle of equivalence, the calculation of the net annual premium for this rider benefit policy may be calculated as:

$$P \times EPV, = A \times (EPV_2 + EPV3) + S \times EPV4 \qquad (4.14)$$

Table 18 presents the net annual premium for a LTC rider benefit policy, calculated at 5 yearly age intervals using the same basis as the stand-alone policy with a sum assured, S, of Kshs 2,000, 000.

Table 4.17 Net annual premiums(Kshs) for a LTC rider benefit policy calculated using Thiele's differential equations for both males and females.

Age	***Male***	***Female***
20	8290	11200
25	9570	12160
30	12225	14340
35	13904	17330
40	16058	21340
45	22646	27440
50	29140	34420
55	36850	45060
60	48780	60300
65	67650	83480

The premium rates for the LTC rider benefit policy are clearly heavier than the stand alone LTC policy reflecting the addition of the death benefit. Moreover, they are proportionally higher at the older ages as expected.

In contrast to net annual premiums, the single premium for a LTC rider benefit policy where premiums are payable continuously at rate P per annum while healthy/able and an annuity payable to the policyholder at rate A per annum while enduring out patient or in patient sickness state with sum assured S, payable immediately on death from any live state, may be calculated directly by including all benefit payments and sums assured concurrently.

Table 4.18 presents the single premium for both a LTC stand alone policy and LTC rider benefit policy, calculated at 5 yearly age intervals using the same bases as per calculations for net annual premiums(Kshs).

Age	**Male**	**Female**	**Male**	**Female**
20	135570	182170	175530	204300
25	154920	201460	191360	234540
30	166460	224380	211030	286880
35	179300	256550	232580	293110
40	193460	282390	258030	332890
45	208770	317140	286160	378430
50	224680	357540	312330	427560
55	241390	392300	346350	482370
60	255680	476660	376090	536530
65	277220	479880	466610	589860

4.11 Reserving for illustrative LTC products

Having solved for the net annual premium, P, we may calculate the development of the reserve for each state -$V_1(r,u), V_2(r,u), V_3(r,u), V_4(r,u)\ and\ V_5(r,u),$ All we need to specify are the boundary conditions given as:

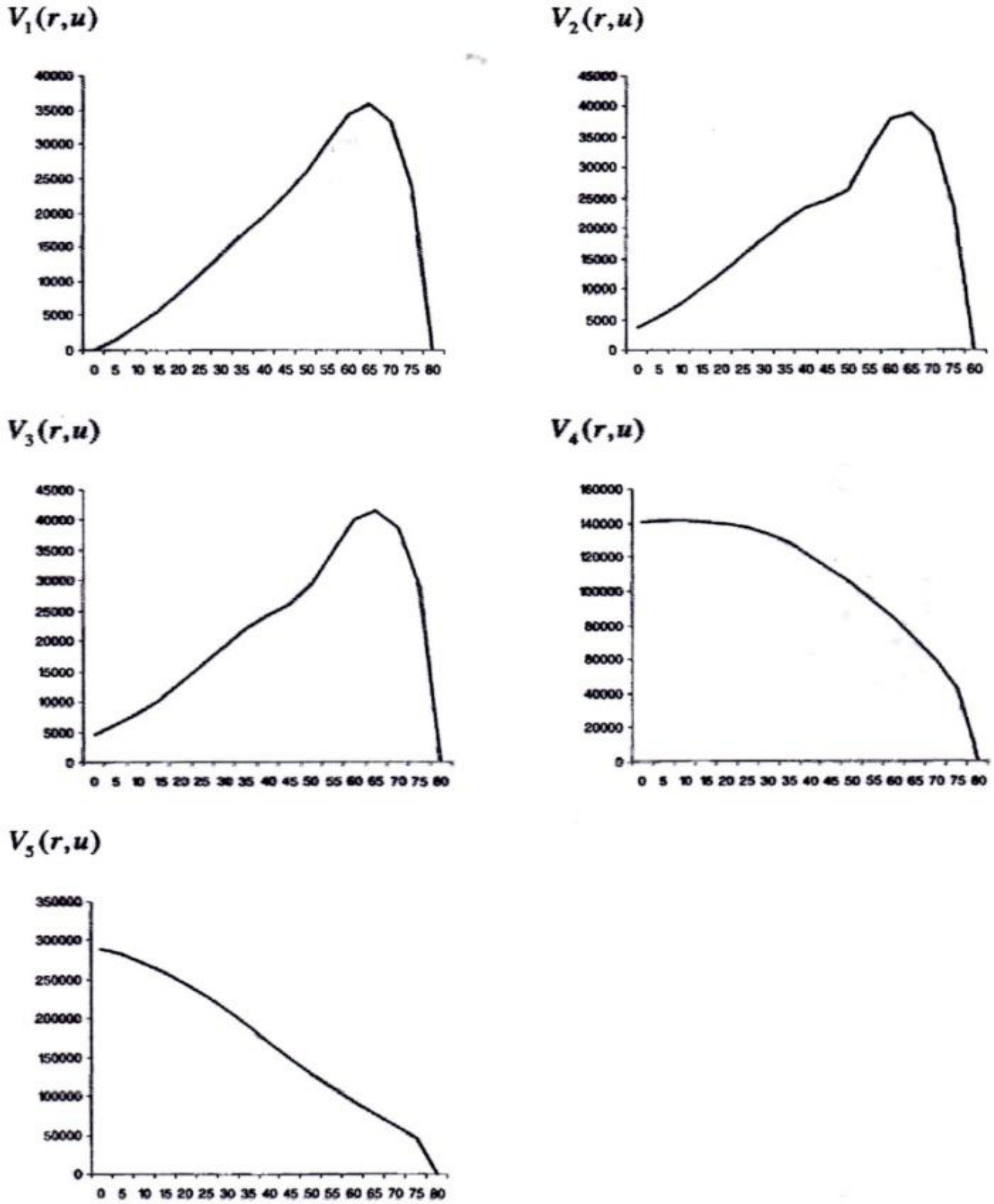
$$V_1(u,u) = V_2(u,u) = V_3(u,u) = V_4(u,u) = V_5(u,u) = 0 \qquad (4.15)$$

For illustrative purposes, we present results for the reserve profile for an LTC stand-alone policy at different states in figure 4.1 .

Overall, the results in Figure 4.1 show that the behavior of $V_1(r,u)$ is largely as expected. Reserves for non-LTC claiming states $(V_1(r,u), and\ V_4(r,u))$ begin at zero and gradually build before falling and ultimately releasing the entire reserve at the end of the policy term. Reserves in LTC claiming states, however, begin at very high levels and gradually fall to zero at the end of the policy term.

Insurers are likely to be most concerned with $V_1(r,u)$as the vast majority of LTC policies would ordinarily be affected while the individual is in the healthy state . In each of the reserve profiles calculated here, $V_1(r,u)$ has a zero reserve at both contract issue and termination which is directly attributable to the equivalence principle. An interesting point to note is that the reserve levels for both the outpatient and inpatient states $V_2(r,u)$ and $V_3(r,u)$ begin at a positive non-zero level.

Figure 4.1: Reserve (Kshs-'000s) profile for a LTC stand alone policy at different states



CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

This chapter summarizes the study findings according to the research objectives. Conclusions of major study results are then outlined. Finally, recommendations based on the study findings are made to the N.H.I.F. Board, the Insurance Regulatory Authority (I.R.A), The Ministry of Health and actuarial scientists and practitioners on the pricing of Long Term Care Products offered by the national LTC insurer.

5.2 Summary of the Study Findings

The study used the four state multi state modeling framework to calculate transition intensities and probabilities of N.H.I.F. members transitioning from the healthy through to out and in patient sickness states up to death.

The forces of transition, recovery and mortality were estimated using the maximum likelihood approach with the crude estimates of the same being graduated using parametric methodologies of Compertz-Mekahem with their logarithmic modifications using logistic variations and Perks formula.

The membership data sample was categorized according to gender with all the calculations and estimates of transition intensities, probabilities, graduations, and pricing being based on each gender.

The study analysis was able to price the net and single premiums for whole life stand alone and LTC rider benefit policies.

In doing this, a nominal rate of interest of 8% per annum was assumed where premiums were increasing at an assumed inflation rate of 4% per annum and benefits similarly increased by 4% per annum whether the insured was claiming or not. Thus a 4% effective net interest rate per annum was appropriate for comparative purposes.

The study found that transition intensities or forces of transition and probabilities for both males and females generally behaved as expected with transition probabilities to sickness states increasing with age. Transition through sickness/disability levels was reasonably progressive. That is, given that a transition out of the sickness/disabled state occurs, there was a higher probability of moving to a lower sickness/disability level than directly to a more severe sickness/disability level. At higher ages, however, transition to the in patient sickness exceeding outpatient sickness/disability levels. This seems reasonable owing to the effects of ageing and chronic frailty.

Secondly, transition probabilities out of the sick/disabled state appear higher for males than females. Transition out of the healthy/able state to outpatient and inpatient sickness states appears higher for females than males. Again, mortality in the inpatient sickness state is higher for males than females.

These last two points above are particularly interesting as they form the basis of an a priori expectation that the likelihood of LTC utilization by females will be higher than by males in the Kenyan population, therefore resulting in more expensive premiums for females.

On the reserving calculations, the study findings revealed that the reserve behavior is largely as expected. Reserves for non-LTC claiming states begin at zero and gradually build before falling and ultimately releasing the entire reserve at the end of the policy term.

The study found that for LTC claiming states of outpatient and inpatient the reserves begin at a positive non zero level since the premiums are not required to be paid by the insured during these states.

5.3 Conclusions

Based on the study findings, the research concludes that the male premium rates are uniformly less than female premium rates. Again, the transition intensities or forces of transition and probabilities for both males and females generally behaved as expected with transition probabilities to sickness states increasing with age.

Transition through sickness/disability levels was reasonably progressive because of treatment so that there was a higher probability of moving to a lower sickness/disability level than directly to in patient sickness/disability level. At higher ages, however, transition to the inpatient sickness exceeded the out patient sickness/disability levels. This seems reasonable owing to the effects of ageing and chronic frailty.

Secondly, transition probabilities out of the sick/disabled state appear higher for males than females. Transition out of the healthy/able state to outpatient and inpatient sickness states appears higher for females than males. Again, mortality in the in-patient sickness state is higher for males than females.

The two observations made form the basis of an a priori expectation that the likelihood of LTC utilization by females will be higher than by males in the Kenyan population, therefore resulting in more expensive premiums for females. Although the study did not investigate the nature of sicknesses for both in and out patient hospital attendances, the conclusion that female utilization of the medical insurance facility could be attributed to other visits such as prenatal, maternity and postnatal medical care.

Overall, the study concluded that the net annual premium rates for both males and females calculated using Thiele's differential equations within a multiple state model framework appear both reasonable and consistent with past LTC studies with higher rates expected at higher ages.

The study concluded that the reserves for the non LTC claiming state begin at zero and gradually build up and ultimately releases the entire reserve at the end of the policy term.

The study concludes that for LTC claiming states of outpatient and inpatient, the reserves begin at a positive non zero level since the premiums are not required to be paid by the insured during these states.

5.4 Recommendations

The non-existence of adequate data in Kenya has been, and will be, a significant obstacle in the introduction of private LTC insurance in Kenya. It is possible, however, to develop a model for pricing LTC insurance using the currently available data in Kenya. This can be done via the application of Thiele's differential equations for a multiple state model. This model, despite its complexity, offers a significant degree of modeling flexibility and robustness which makes it preferable to traditional annuity inception approaches.

This study, to the researcher's knowledge, represents the first stochastic model developed for the purposes of pricing reserving LTC in Kenya.

The researcher therefore recommends this pricing and reserving model for the development and design of the LTC products in the Kenyan insurance industry.

There are, however, a number of limitations here, largely a result of inadequate data. In particular, the researcher acknowledges the inconsistency of a continuous time Markov chain with the discrete state model framework. The focus to a great extent in this project has been the development of a model which, when adequate data becomes available, will produce increasingly accurate results.

The study recommends that a comparative study using both the multi state and multiple decrement approaches be done in order to arrive at more accurate results in the pricing and reserving of LTC products in the Kenyan insurance industry.

The study recommended that reserves for non LTC claiming states begin at zero and gradually build up and ultimately release the entire reserve at the end of the policy term.

For LTC claiming states of outpatient and inpatient, the study recommends that reserves begin at a positive non zero level since the premiums are not required to be paid by the insured during these states.

5.5 Room for Further Research

Once appropriate LTC specific data become available for Kenya, the following extensions to this study could be undertaken: One is to allow for all possible modes of recovery in the multiple state model; secondly, other scholars and actuarial scientists could allow for lapses in the multiple state model.

It is possible to allow for duration in the multiple state models by implementing a semi-Markov assumption as opposed to a Markov assumption. That is, allow transition intensities to depend on both age and the duration of stay in the current state.

For comparative purposes, the technical actuarial bases used to price several illustrative LTC products come from a survey of earlier relevant literature. The model bases here, however, may be easily modified at the insurer's discretion in another scholarly research.

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N.H.I.F.'s MEMBERS' SERVICES LTC INSURANCE COVER

Characteristics of the cover

1. Employees actively in civil service and disciplined services are eligible for cover
2. Only the principle member, spouse and three (3) children are covered. (Additional children can be covered at an extra premium as stipulated in annex II)
3. Dependant children are eligible for cover upto 18 years of age. However with proof that the child is enrolled in full time learning institution, they are covered upto 25 years of age. Children with disabilities are also covered past 18 years of age.
4. Medical cover commences immediately upon registration and payment of full premium for new employees. No waiting period is imposed.
5. Annual medical checkup for the member is required for the principal member.
6. Out-patient visits will attract a co-payment of KES 100 and KES 200 in government and other facilities per visit .
7. Out-patient dental and optical cover available only on referral basis.
8. Out-patient facilities shall be based on choice made by member with an option of changing after three months except in cases of emergency, travel on duty or transfer. In order to change outpatient facility, one must fill in a consent form which will be available online or at the NHIF offices.

How to Register As an NHIF Member

1. Fill in an NHIF 2 Application form (available at any NHIF office & on our website). Online registration is also acceptable (please see instructions below).
2. Attach colored passport size photos of contributor and all declared dependant(s)
3. Attach photocopies of identification cards for contributor and spouse (If married)
4. Attach copies of birth certificate of child(ren) or birth notification for child(ren) below six (6) months,adoption certificate where applicable. Also include details of additional children you may wish to cover in the enhanced scheme and you are willing to pay additional premiums.
5. Submit duly filled application form to the nearest NHIF office

BENEFITS PACKAGE

1. **In-patient cover**

Under this cover, members will access:

- Hospital accommodation charges
- Nursing care
- Diagnostic, laboratory or other medically necessary facilities and services
- Physician's, surgeon's, anesthetist's, or physiotherapist's fees
- Operating theater charges
- Specialist consultations or visits
- Drugs, dressing and medications prescribed by treating physician for in-hospital use
- Day care services such as surgery and other medical services deemed fit by the physician
- Pre-hospitalization procedures such as laboratory, x-ray or other medical diagnostic procedures which results in a member being admitted on the same day the tests are done.
- Ambulance services for transportation and transfer of a sick member/dependant to another facility

2. **Out-patient cover**

The out-patient cover shall include:

- Consultation
- Laboratory investigations and X-ray services
- Drugs administration and dispensing
- Dental health care services
- Radiological examinations
- Nursing and midwifery services
- Minor surgical services
- Physiotherapy services
- Optical care
- Occupational therapy services

- Referral for specialized services

3. **Maternity cover and reproductive health**
    - Consultation and treatment for both mother and child
    - Normal and caesarian section
    - Family planning services excluding fertility treatment
4. **Dental cover**
    - Dental consultation
    - Cost of filling
    - X-rays
    - Extractions including surgical extractions together with anaesthetics fees, hospital and operating theatre cost.

The cover does not include:

- The cost of replacement or repairs of old dentures, bridges and plates unless damage arises directly from accidental, external and visible means
- Orthodontic treatment of cosmetic nature

5. **Optical Cover**
    - Cost of eyeglasses (limited to a pair per family unless proven medically necessary)
    - Frames
    - Eye testing fees

The cover does not include:

- The cost of frames replacement unless directly caused by an injury to an eye
- Replacement of lenses unless necessitated in course of further treatment

6. **Ex-gratia payments**

   NHIF shall not be liable for Ex-gratia payments upon the limits being exhausted.

7. **Group life cover**

NHIF shall pay death benefit upon the demise of a member

8. **Last expense cover**

NHIF shall, upon written notification of death of a member while this cover is in force, pay to the client or such other person or persons as the client may in writing direct, the amount specified in the Annex I to cater for the funeral expenses.

EXCLUSIONS

1. Expenses incurred by a member as a result of:

- Investigations, treatment, surgery for obesity or its sequel, cosmetic or beauty treatment and /or surgery
- Massage unless necessary for treatment i.e following an accident
- Stays at sanatoria, old age homes, places of rest etc
- Treatment by chiropractors, acupuncturitst and herbalists, stays and /or maintenance or treatment received in nature cure clinics or similar establishments or private beds registered within a nursing home, convalescent and /or rest homes
- Claim for expenses incurred whilst the member was outside territorial limits of kenya unless a kenyan who is temporarily abroad and needs emergency treatment for illness or injury that occurs during the period of travel provided six weeks is not exceeded.
- Fertility treatment related to infertility and impotence
- Vaccines other than those of KEPH

2. Charges recoverable under any Work Injury Benefits Act or any other medical plan

ANNEX I: N.H.I.F.'s LIMITS OF LIABILITY

CIVIL SERVICE JOB GROUPS	DISCIPLINED FORCES GRADES	STATE LAW OFFICE GRADES	IN-PATIENT LIMIT (KES)	OUT-PATIENT LIMIT(KES)
A-G	PG,1,2,3	-	No Limit	No Limit
H	PG 4	-	No Limit	No Limit
J	PG 5	-	No Limit	No Limit
K	PG 6	SL 1	No Limit	No Limit
L	PG 7	SL 2	No Limit	No Limit
M	PG 8 & 9	SL 3	No Limit	No Limit
N	PG 10	SL 4	1,130,000	80,000
P	PG 11	SL 5	1,250,000	80,000
Q	PG 12	SL 6	1,500,000	100,000
R,S,T	PG 13,14	SL 7,8,9	2,000,000	150,000

DENTAL & OPTICAL COVER

CIVIL SERVICE JOB GROUPS	DISCIPLINED FORCES GRADES	STATE LAW OFFICE GRADES	DENTAL COVER	OPTICAL COVER
A-M	PG1-9	SL 1-3	10,000	10,000
N-Q	PG 10-12	SL 4-6	20,000	20,000
R-T	PG 13-14	SL 7-9	30,000	30,000

GROUP LIFE & LAST EXPENSE

CIVIL SERVICE JOB GROUPS	DISCIPLINED FORCES GRADES	STATE LAW OFFICE GRADES	GROUP LIFE (KES)	LAST EXPENSE (KES)
A-G	PG,1,2,3	-	200,000	40,000
H	PG 4	-	200,000	40,000
J	PG 5	-	200,000	50,000
K	PG 6	SL 1	250,000	50,000
L	PG 7	SL 2	250,000	50,000
M	PG 8 & 9	SL 3	300,000	60,000
N	PG 10	SL 4	300,000	70,000
P	PG 11	SL 5	350,000	80,000
Q	PG 12	SL 6	400,000	80,000
R,S,T	PG 13,14	SL 7,8,9	500,000	100,000

ANNEX II: N.H.I.F.'s ADDITIONAL DEPENDANT(S)

Additional dependant(s') premium for in-patient and out-patient limits within the unit per annum

JOB GROUP	PREMIUM OF EACH ADDITIONAL DEPENDANT (ONE) (KES) OPTION 1	PREMIUM OF UNLIMITED ADDITIONAL DEPENDANT(S) (KES) OPTION 2
A-G,PG,1,2,3	3,500	6,000
H,PG 4	3,500	6,000
J, PG 5	3,500	6,000
K, PG 6, SL 1	3,500	6,000
L, PG 7,SL 2	3,500	6,000
M, PG 8,9,SL 3	3,500	6,000
N, PG 10,SL 4	6,000	12,000
P, PG 11,SL 5	6,000	12,000
Q, PG 12,SL 6	6,000	12,000
R,S,T,PG 13,14 & SL 7,8,9	9,000	20,000