UNIVERSITY OF NAIROBI

COLLEGE OF BIOLOGICAL AND PYSICAL SCIENCES

SCHOOL OF MATHEMATICS

TRANSITION PROBABILITIES AS APPLIED IN ACTUARIAL CALCULATIONS OF LONG TERM CARE INSURANCE PRODUCTS

BY

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DECLARATION

This project is my original work and has never been submitted for a degree in any other university.

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This project has been submitted for examination with our approval as university of Nairobi supervisors

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DEDICATION

I dedicate this work to my family, whose support carried me through when things got tough and whose encouragement has seen me achieve completion of this project. Special thanks to my father Prof. Manene who accorded me guidance and walked with me step by step throughout the duration of my research.

I thank all my friends for their understanding and support throughout the process.

Most importantly I thank God for providing me good health throughout the period of my research. Truly, this work could not have been done if not for His grace and mercy.

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Abstract

Long term care refers to services provided to persons who, for one reason or another are unable to carry out activities of daily living. In developing models for pricing LTC insurance products, actuaries have applied, among other approaches, multiple state models.

The current project seeks to construct a five-state multiple state model (allowing for recoveries) that depicts LTC needs at different ADL failure levels and can be used in Actuarial calculations of premiums and reserves.

We begin by introducing the study of multiple state models using Markov approach.Calculation of transition intensities for the five-state model is then done. We use a matrices approach to calculate transition probabilities from the calculated transition intensities and conclude by illustrating how the calculated transition probabilities can be applied in actuarial calculations of premiums and reserves.

Contents

Chapter 1

GENERAL INTRODUCTION

1.1 Background Information

Long-term care (LTC) refers to services provided to persons who are unable to carry out "Activities of Daily Living" (ADLs) without some form of assistance. These persons could be dependent due to some form of mental or physical impairment.They could also be persons across all ages, but in this research project we concentrate on LTC for the elderly.

LTC is a combination of medical, nursing, custodial, social and community services. It may be provided at home or in an institution and is classified as formal or informal care. Formal care is provided by government organizations, local, national or international NGOs, or by profit organizations. The personnel who provide formal care include recognized professionals such as doctors and social workers and/or para-professionals such as personal care workers. On the other hand, informal care is provided by family members (nuclear and/or extended), neighbors, friends, independent volunteers as well as organized volunteer work through organizations such as religious groups.

As countries in the world continue to advance in medical technology and treatments, longevity increases, i.e. people tend to live longer. A need to design programmes providing LTC for the elderly then arises.

Solutions that have been developed to cater for the increasing LTC needs can be classified as Insurance based and Non-insurance based. In insurance-based programs, a policyholder who fulfils the set eligibility criteria must be granted benefits, regardless of available budgets. On the other hand, in non-insurance (budget constrained) programs, service provision is dependent on limited funds. Once the funds run out, no services are provided. Finance for insurance based systems is through contributory payment such as premiums and employer-employee contributions whereas finance for the non-insurance based systems is through general taxation.

One example of a country that uses an insurance based system is Germany. In 1994 the German Parliament passed into law measures establishing a social insurance scheme for LTC. Costs for LTC under this scheme were to be met by mandatory contributions from employers and their employees. The cover was to extend to children and non-employed married partners at no extra costs. Eligibility for benefits was to be determined on the basis of an assessment of need, with no account of family or financial circumstances being taken into consideration. The more in need an individual was deemed to be, the more the benefit they were entitled to. Due to the increasing costs of offering LTC services, the scheme has at one point run down reserves and changes have had to be made.

Sweden LTC system is an example of a budget constrained system. The system is tax financed therefore enabling service providers to focus more carefully and narrowly on those older people whose needs are greatest. Eligibility for publicly-provided LTC services depends on the presence of need and on the inability to meet these needs by other means.

Such developed countries and others which have adopted different strategies in dealing with the ever increasing LTC needs provide useful lessons for the middle-income and low-income countries in their development of LTC systems that fit their specific demographics. It is worth noting, however, that developing countries are facing increasing needs for LTC services at levels of income far lower than those in developed countries.

In most developing countries the LTC needs of the elderly have been met mostly by the family units, more so the women. However, as opportunities for ladies to further their education and join the professional world increase, the family is no longer able to cater to the needs of their old without external assistance. Also, the family unit is quickly disintergrating due to urbanization and dependence on it for LTC needs will soon need to be replaced with paid care whether at home or in an institution.

In coming up with a solution to the rising need for LTC, actuaries have developed LTC insurance products. Generally, in pricing disability benefits, actuaries have used approaches such as inception annuity approach, Manchester-Unity approach and multiple state modelling approach among others. Of interest to us is the multiple state modelling approach.

A multiple state model is a probability model that describes the random movements of a subject among various states. These random movements are also referred to as transitions and form the basis of multiple state models.

Multiple state models consist of:

- A finite number of states
- Arrows indicating possible movements between some, but not necessarily all, pairs of states
- Defined states representing the status of a subject(s) without loss of generality.

As long as disbursement of benefits is contigent upon a transition from one state to another, then multiple state models are applicable in the insurance product development process. Also, the appropriate model, in terms of number of states and possible transitions, is determined by the kind of benefits provided by the insurance product.

In this research project we bank on the fact that a LTC insurance product is one in which an individual is in one of a possible number of states at each point in time and that transition from one state to another has some financial impact on the insurer.

1.1.1 Background of LTC insurance products

In this section we provide a brief description of a selection of available LTC insurance products.

Long term care insurance is offered in a variety of forms and the products offered can be classified as:

- Pre-funded products
- Unit-linked products
- Immediate care annuities
- Pension linked products

among others.

1. **Pre-funded products**

Pre-funded products are aimed at persons concerned about their future and are in reasonably good health. These products offer protection against costs arising from future deterioration in health. Payment of claims is based on either the inability to perform a certain number of ADLs or a significant cognitive impairment. These products can be categorized as:

- Stand-alone policies
- Riders to other policies
- Extensions to permanent health insurance (PHI)

Stand-alone policies provide an income payable in the event of a valid claim. Premiums are either regular or single. Regular premiums are aimed at young policyholders while single premiums are aimed at the older policyholders. There is a choice of cover based on various levels of disability and care.These policies are not intended as investment vehicles. As such a surrender value is not provided.

LTC insurance can be offered as a rider to a whole life plan or as a natural addition to critical illness cover where LTC is considered as an additional illness. If the claims criteria is satisfied, an accelerated death benefits is payable by monthly installments.

LTC insurance can also be offered as an extension to permanent health insurance which pays out if the policyholder is unable to work through sickness or accident, or if they need care as specified by ADL criteria or cognitive failure.

2. **Unit-linked products**

Unit-linked products allow policyholders to invest capital in a number of unit-linked investment funds. As the value of the underlying assets grows, the value of the invested capital also grows. LTC charges and other expenses incurred by the insurer are then deducted. These products provide surrender values which is a major shortfall of the pre-funded products.

3. **Immediate care annuities**

Immediate care annuities are aimed at people who are just about to go into care or are already receiving care but need to provide for future costs. Such annuities attract individuals whose health has already deteriorated and are in need of immediate guaranteed income. These products provide guaranteed monthly payments to cover the specified care costs (all or partial) in exchange for a single premium. The payments will generally continue for as long as care is needed.The minimum age for these products is usually 60 with a maximum of between ages 90 and 100.

4. **Pension linked products**

These products are also referred to as enhanced pension LTC products. They are sold at retirement and are a combination of a standard pension annuity paid while the policyholder is in good health and a higher income paid while they are claiming LTC benefits.

1.2 Problem Statement

LTC costs have a significant likelihood of depleting the elderly populations resources in terms of pensions and retirement investments. As such, continuous exploration of models for pricing and reserving LTC insurance products becomes a central issue in ensuring this does not happen. These models include,but are not confined to, multiple-state models and inception annuity models. In this project we concentrate on the application of multiple-state models in pricing and reserving LTC insurance products.

In applying multiple state models to pricing long-term care insurance products, most authors have not allowed for recoveries in their models. In reality, however, multiple-state models for pricing and reserving LTC insurance products should be able to allow for recoveries unless the disabilities considered in the model are permanent disabilities.

1.3 Objectives

General Objective

In this project we seek to:

- Construct a five-state multiple state model (allowing for recoveries) that depicts LTC needs at different ADL failure levels and can be used in Actuarial calculations of premiums and reserves

Specific Objectives

- Introduce the Markov framework to Multiple state models
- Calculate transition intensities for the five-state model
- Use a matrices approach to calculate transition probabilities from the calculated transition intensities
- Illustrate how the calculated transition probabilities can be applied in actuarial calculations of premiums and reserves

1.4 Scope of the Study

In this project we apply a Markov framework to the study of multiple state models. We proceed to introduce matrices in the calculations of the transition probabilities and show how we can apply these probabilities to the calculation of discrete-time actuarial values.

We then use an illustrative approach to show how the transition probabilities are applied in actuarial calculations.

1.5 Significance of the study

Due to the catastrophic nature of LTC costs, savings and pensions of the elderly have the potential of being swept out once they become in need of LTC. Studying models that aim at alleviating this financial burden is therefore among the central issues in dealing with LTC costs of the elderly.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

In this chapter we review literature on the application of multiple state models to pricing LTC insurance products.

2.2 Multiple state models in designing LTC insurance products

Waters et. al (1984) distinguishes between a multiple state model and a multiple decrement model. The paper puts forward an alternative approach to multiple state models, the TI (transition intensities) approach as opposed to the flow, orientation and integration equations (FOI) technique proposed by Haberman (1983) . The approach uses the forces of transition, or transition intensities between states as the fundamental quantities of the model.

In the paper, four aspects of the study of multiple-state models where the TI-approach is considered better than the FOI-approach are discussed. These are:

- The specification of the model
- Natural assumptions that aid computations
- Comparison of different models
- Estimation of probabilities or intensities and the statistical properties of the estimators

A simple example of a multiple state model is then used to illustrate the TI-approach.

In this project, we consider a transition intensities approach with a discussion on the first three aspects.

(Jones,1994) presents a method for finding probabilities needed for actuarial calculations in applications that can be modelled as multi-state processes. The paper begins with a review of the properties of the Markov process. A key result that exploits the tractability of Markov processes with constant forces of transition is then presented. The paper explains also how the results can be used in the case of piecewise constant forces.

A decomposition of the force of transition matrix is used which leads to a convenient representation of the transition probability matrix. The situation in which the Markov assumption is inappropriate is then addressed.

Jones suggests that duration dependence be reflected by increasing the number of states in the model instead of using the Semi-Markov model. This is justified by a limiting result that illustrates the convergence of the approximating Markov process to the semi-Markov process. Finally an example is demonstrated that applies the approach to select and ultimate mortality.

In this paper, we review the Markov process, perform a decomposition of the transition intensities matrix with piece-wise constant intensities and finally present our transition probabilities matrices.

(Leung, 2004) develops a model for pricing LTC insurance contracts in Australia using the disability prevalence contained in the 1998 Australian Bureau of Statistics (ABS) Survey of Disability, Ageing and Carers. Premium and Reserve calculations are performed by applying generalizations of Thiele's differential equation for a multiple state model within a Markov framework. The sets of results presented capture a varying range of possible scenarios and demonstrate the flexibility of the model.

Leung's primary objective is to develop and test a multiple state model for pricing and reserving LTC insurance products. In Leung (2004), a discrete time multiple state model is developed for projecting the needs and costs of LTC in Australia. However, in the current paper, the assumption of discrete time is relaxed and the underlying process is modelled in a continuous time Markov framework. This enables calculation of transition intensities for application in Thiele's differential equation.

The aforementioned data which is deemed relevant in Australia for pricing and reserving LTC insurance is surveyed. This is followed by a brief review of existing LTC pricing and reserving literature emerging from Australia and abroad. Multiple state models are then developed and a discussion on the probabilistic structure used to calculate premiums and reserves for a set of illustrative hypothetical insurance products is put forward. Finally, an analysis of the sensitivities of the model is done and avenues for further research are presented.

In this project, we employ a probabilistic structure in calculations of premiums and reserves for a hypothetical benefits structure.

Haberman et. al. (1997) illustrates how mathematics of Markov stochastic processes can be used through the framework of multiple state models in the actuarial modelling of different types of Long-Term Care Insurance. The paper begins by describing multiple state models and actuarial values of streams of payments. A review of long term care insurance benefits is then put forward and actuarial calculations for time-continuous and time-dicrete cashflows are discussed.

In this project, we discuss actuarial calculations for a stand-alone LTC insurance product using time-dicrete cashflows.

2.3 Summary of the Literature Review

From the literature reviewed, we have been able to come up with a clear guideline for our research. The guideline is as follows:

- We consider a transition intensities approach where the transition intensities are the fundamental quantities of the multiple state model
- We review the Markov process, perform a decomposition of the transition intensities matrix with piece-wise constant intensities and finally present our transition probabilities matrices
- We employ a probabilistic structure in calculations of premiums and reserves for a hypothetical benefits structure of a stand-alone LTC insurance product with dicrete-time cashflows

In the study of Multiple state models using a Markovian approach, most authors have not allowed for recoveries in their models. In our model, we relax the "no recovery"assumption.

Chapter 3

METHODOLOGY

3.1 Introduction

Actuaries use various methods in pricing disability related insurance products. The main reason for using different approaches is the availability or lack thereof of data. Actuaries face the challenge of scanty statistical data and more often than not, calculation procedures used for pricing and reserving have to be simplified. In this section we embark on the mathematics of applying a Markov framework to multiple state models for LTC insurance products.

We also introduce actuarial calculations based on discrete time cashflows.

3.2 Multiple state modelling approach

A multiple state model is a model in which the subject of interest is in one of a number of states at each point in time. In modelling insurance products, actuaries are interested in modelling the payments made while the policyholder is in a particular state. These payments are premiums,benefits and reserves.

The study of multiple state models can be approached in a Markov or semi-Markov framework. These approaches can be defined in a time-continuous and/or a time-discrete context. The current project focusses on multiple state modelling of LTC insurance products in a time-discrete Markov framework.

In the following sections, we provide a description of the general setup of Markov multi-state models.Transition probabilities and transition intensities are then defined and their Chapman-Kolmogorov equations derived.The relationship between the transition probabilities and the transition intensities is given by the Kolmogorov Forward Differential equation. Since we are interested with multiple state models in which transitions are dependent on age, we introduce notations that take into account the age of the subject. Thereafter multiple state models are introduced beginning with the simplest case of the alive-dead model and working our way to a five-state model. For each of the state models we introduce, we show how transition probability matrices are derived by using Chapman-Kolmogorov equations. These expressions show the relationship between transition probabilities and transition intensities.

Ideally, transition intensities for a disability related model should progress smoothly with age. We deal with this as an assumption but explain the methods that can be used in graduating these intensities.

3.2.1 Setup of Markov multiple state models

Suppose there are n states. Denote the state space by S where S is a countably finite set such that:

$$S = \{1, 2, ..., n\}$$

Denote the set of direct transitions by Λ :

$$\Lambda \subseteq (i,j)\,|i \neq j; \qquad\qquad i,j \in S$$

The pair (S,Λ) is called a multiple state model. Define $X(t)$ as the state occupied by the subject under consideration at time t, where $t \geq 0$ and the time unit is 1-year. $\{X(t);t \geq 0\}$ is said to be a time-continuous Markov process if, for each finite set of times $0 \leq t_0 \leq t_1 \leq ... \leq t_n$ and corresponding set of states $i_0, i_1, ..., i_n, j \in S$ with

$$pr\left[X(t_n) = i_n, X(t_{n-1}) = i_{n-1}, ..., X(t_0) = i_0\right] > 0$$

the process satisfies the Markov property:

$$\begin{aligned} & pr\left[X(t_n) = i_n | X(t_{n-1}) = i_{n-1}, X(t_{n-2}) = i_{n-2}, ..., X(t_0) = i_0\right] \\ = \;& pr\left[X(t_n) = i_n | X(t_{n-1}) = i_{n-1}\right] \end{aligned}$$

The Markov property shows that this probability does not depend on the history of the event but depends only on the immediate past. We say that $\{X(t);t \geq 0\}$ is a time-continuous Markov process since we are dealing with continuous time.

To deal with a time-discrete Markov model we define $X(t)$ as above but with $t = 0,1,2,3,....$ We can again say that $\{X(t);t = 0,1,2,3,...\}$ is a time-dicrete Markov process if it satisfies the Markov property.

Transition probabilities of the Markov process are denoted by $p_{ij}(s,t)$ and defined by

$$
\begin{aligned}
p_{ij}(s,t) &= \text{conditional probability that an individual is in state } j \text{ at time } t \\
&\quad \text{given that they were in state } i \text{ at time } s \\
&= pr\left[X(t)=j|X(s)=i\right] \qquad (3.1)\\
&= \frac{pr\left[X(t)=j, X(s)=i\right]}{pr\left[X(s)=i\right]}; \quad t \geq s \geq 0, \quad i,j \in N
\end{aligned}
$$

if $pr\left[X(s)=i\right] > 0$, otherwise $p_{ij}(s,t) = 0$.

We also have:

$$p_{ij}(s,s) = \delta_{ij}, \qquad s \geq 0$$

δ_{ij} is referred to as the Kronecker delta and is equal to 0 for $i \neq j$ and equal to 1 for $i = j$.

The transition probabilities satisfy the following properties:

$$0 \leq p_{ij}(s,t) \leq 1; \quad i,j \in N\ ; \qquad (3.2)$$

$$\sum_{j \in N} p_{ij}(s,t) = 1, \quad 0 \leq s \leq t \qquad (3.3)$$

We assume that transition probabilities for each fixed period of time, vary in time. As such, we need to specify the beginning and the end of the time interval $[s,t]$, instead of just its length $t-s$. Hence, the time-continuous or time-discrete Markov process is assumed to be time inhomogeneous. Equation (3.4) is the Chapman-Kolmogorov equation for a time inhomogenous Markov chain. The equation states that a process that starts in state i at time s and is in state j at time t occurs via some state $k \in N$ at an arbitrary intermediate time τ.

$$p_{ij}(s,t) = \sum_{k=1}^{n} p_{ik}(s,\tau)\, p_{kj}(\tau,t) \tag{3.4}$$

where

$$0 \le s \le \tau \le t$$

Proof

$$\begin{aligned}
p_{ij}(s,t) &= \sum_{k=1}^{n} pr\left[X(s,t)=j, X(s,\tau)=k | X(s)=i\right] \\
&= \sum_{k=1}^{n} \frac{pr\left[X(s,t)=j, X(s,\tau)=k, X(s)=i\right]}{pr\left[X(s)=i\right]} \\
&= \sum_{k=1}^{n} \frac{\left\{ \begin{array}{c} pr\left[X(s,t)=j | X(s,\tau)=k, X(s)=i\right] \\ *pr\left[X(s,\tau)=k, X(s)=i\right] \end{array} \right\}}{pr\left[X(s)=i\right]} \\
&= \sum_{k=1}^{n} pr\left[X(s,t)=j | X(s,\tau)=k, X(s)=i\right] * pr\left[X(s,\tau)=k | X(s)=i\right]
\end{aligned}$$

using the Markov property we have:

$$\begin{aligned}
p_{ij}(s,t) &= \sum_{k=1}^{n} pr\left[X(s,t)=j | X(s,\tau)=k\right] * pr\left[X(s,\tau)=k | X(s)=i\right] \\
&= \sum_{k=1}^{n} p_{kj}(\tau,t)\, p_{ik}(s,\tau) \\
&= \sum_{k=1}^{n} p_{ik}(s,\tau)\, p_{kj}(\tau,t)
\end{aligned}$$

Kolmogorov Forward Equation

Using the Chapman-Kolmogorov formula given in equation (3.4), we deduce that:

$$\begin{aligned}
p_{ij}(s,t+h) &= \sum_{k\neq j} p_{ik}(s,t)\, p_{kj}(t,t+h) \qquad (3.5)\\
&= \sum_{k\neq j} p_{ik}(s,t)\, p_{kj}(t,t+h) + p_{ij}(s,t)\, p_{jj}(t,t+h)\\
p_{ij}(s,t+h) - p_{ij}(s,t) &= \sum_{k\neq j} p_{ik}(s,t)\, p_{kj}(t,t+h) + p_{ij}(s,t)\, p_{jj}(t,t+h) - p_{ij}(s,t)\\
&= \sum_{k\neq j} p_{ik}(s,t)\, p_{kj}(t,t+h) - [1 - p_{jj}(t,t+h)]\, p_{ij}(s,t)
\end{aligned}$$

$$\begin{aligned}
\lim_{h\to 0} \frac{p_{ij}(s,t+h) - p_{ij}(s,t)}{h} &= \lim_{h\to 0} \frac{\sum_{k\neq j} p_{ik}(s,t)\, p_{kj}(t,t+h) - [1 - p_{jj}(t,t+h)]\, p_{ij}(s,t)}{h}\\
\frac{\partial}{\partial t} p_{ij}(s,t) &= \sum_{k\neq j} p_{ik}(s,t) \lim_{h\to 0} \frac{p_{kj}(t,t+h)}{h} - p_{ij}(s,t) \lim_{h\to 0} \frac{[1 - p_{jj}(t,t+h)]}{h}\\
&= \sum_{k\neq j} p_{ik}(s,t)\, \mu_{kj} - p_{ij}(s,t)\, \mu_j\\
\text{where } \lim_{h\to 0} \frac{p_{kj}(t,t+h)}{h} &= \mu_{kj} \text{ and } \lim_{h\to 0} \frac{[1 - p_{jj}(t,t+h)]}{h} = \mu_j,\, k \neq j
\end{aligned}$$

The Kolmogorov Forward equation is thus:

$$\frac{\partial}{\partial t} p_{ij}(s,t) = \sum_{k\neq j} p_{ik}(s,t)\, \mu_{kj} - p_{ij}(s,t)\, \mu_j$$

$\mu_{ij}(t)$ is defined as the transition intensity between two states i and j. To be precise, $\mu_{ij}(t)$ is the rate of change of the probability p_{ij} in a very small time interval, h:

$$\mu_{ij}(t) = \lim_{h\to 0} \frac{p_{ij}(t,t+h)}{h}, \qquad i \neq j$$

for any given time $\{t : 0 < t < T\}$ and interval length $h > 0$.

For the interest of our study, we introduce a slightly different definition of transition probabilities and transition intensities:

$$
\begin{aligned}
p_{ij}(x,t) &= \text{The probability that a life now aged } x+t \text{ and in state} \\
&\quad\ j \text{ was in state } i \text{ at age } x \\
\mu_{ij}(x+t) &= \text{The transition intensity/rate from state } i \text{ to state } j \text{ at} \\
&\quad\ \text{age } x+t
\end{aligned}
$$

In both these cases $i,j = 1,2,3,...,n; \quad x = 0,1,2,3...,$ and $0 \leq t \leq 1$.

Assuming piecewise constant forces of transition would imply that:

$$\mu_{ij}(x+t) = \mu_{ij}(x) \text{ for } x = 0,1,2,3... \text{ and } 0 \leq t \leq 1$$

Equation (3.5) which we use in deriving expressions of transition probabilities now becomes:

$$p_{ij}(x,t+h) = \sum_{k=1}^{n} p_{ik}(x,t)\, p_{kj}(t,t+h) \tag{3.6}$$

Also the Kronecker delta which gives us our initial conditions is now given as:

$$p_{ij}(x,x) = \delta_{ij}, \qquad\qquad x \geq 0$$

and is equal to 0 for $i \neq j$ and equal to 1 for $i = j$. x is taken to mean exact age.

3.3 A two-state model

In understanding multiple state models, it is only practical to start from the simplest model and build our way to more complex models. As such we begin by reviewing the simple alive-dead model which has transition intensity in one direction and can be considered as a single decrement model. We extend this school of thought to the healthy-ill model which brings in the idea of recovery. For each of the models, we derive the Kolmogorov Forward Differential Equation and explicit expressions of the transition probabilities matrix.

3.3.1 Alive-Dead model

Figure 1 represents our first two-state model. It is considered to be the simplest multiple state model in actuarial literature and as such forms an essential building block in reviewing multiple state models. This model can also be considered as a single-decrement model.

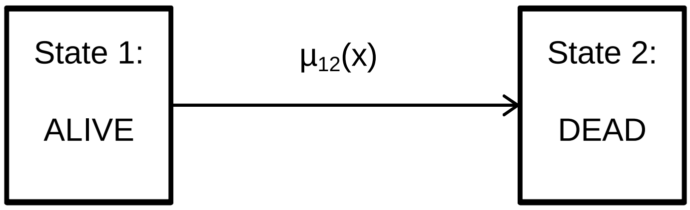

FIGURE 1: Alive-Dead Model

We begin by deriving the Kolmogorov Forward Differential Equation for this model. Recall equation (3.6):

$$p_{ij}(x, t+h) = \sum_{k=1}^{n} p_{ik}(x,t)\, p_{kj}(t, t+h)$$

In the case of the alive-dead model above, $k = 1, 2$. We proceed as follows:

$$\begin{aligned} p_{11}(x,t+h) &= \sum_{k=1}^{2} p_{1k}(x,t)\, p_{k1}(t,t+h) \qquad (3.7) \\ &= p_{11}(x,t)\, p_{11}(t,t+h) + p_{12}(x,t)\, p_{21}(t,t+h) \\ &= p_{11}(x,t)\, p_{11}(t,t+h) + p_{12}(x,t) \times 0 \end{aligned}$$

To proceed, we need to define $o(h)$ (order h). A function f(h) is said to be of $o(h)$ if:

$$\lim_{h\to 0} \frac{f(h)}{h} = 0$$

and from calculus we have:

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Therefore from equation (3.7) we have:

$$p_{11}(x,t+h) - p_{11}(x,t) = p_{11}(x,t)\left[-\left(\mu_{12}(x)\, h + o(h)\right)\right]$$

Dividing through by h we get

$$\begin{aligned} \frac{d}{dt} p_{11}(x,t) &= \lim_{h\to 0} p_{11}(x,t)\, \frac{\left[-\left(\mu_{12}(x)\, h + o(h)\right)\right]}{h} \\ \frac{d}{dt} p_{11}(x,t) &= p_{11}(x,t)\left(-\mu_{12}(x)\right) \qquad (3.8) \end{aligned}$$

Next:

$$\begin{aligned} p_{12}(x,t+h) &= \sum_{k=1}^{2} p_{1k}(x,t)\, p_{k2}(t,t+h) \\ &= p_{11}(x,t)\, p_{12}(t,t+h) + p_{12}(x,t)\, p_{22}(t,t+h) \\ &= p_{11}(x,t)\, p_{12}(t,t+h) + p_{12}(x,t) \times 1 \end{aligned}$$

$$
\begin{aligned}
p_{12}(x,t+h) - p_{12}(x,t) &= p_{11}(x,t)\left[\mu_{12}(x)\,h + o(h)\right] \\
\frac{d}{dt}p_{12}(x,t) &= \lim_{h\to 0}\frac{p_{11}(x,t)\left[\mu_{12}(x)\,h + o(h)\right]}{h} \\
&= p_{11}(x,t)\left[\mu_{12}(x)\right] \qquad (3.9)
\end{aligned}
$$

In matrix form, the Kolmogorov Forward Differential Equation is given by:

$$
\begin{bmatrix} p\prime_{11}(x,t) & p\prime_{12}(x,t) \end{bmatrix} = \begin{bmatrix} p_{11}(x,t) & p_{12}(x,t) \end{bmatrix} \begin{bmatrix} -\mu_{12}(x) & \mu_{12}(x) \\ 0 & 0 \end{bmatrix} \qquad (3.10)
$$

A review of some matrix algebra

Here we introduce some matrix notation that aid in simplifying calculations.

Define:

$$
\begin{aligned}
P(x,t) &= \{p_{ij}(x,t)\}_{i,j=1}^{n} = \text{ The transition probability matrix} \\
P\prime(x,t) &= \left\{\frac{d}{dt}p_{ij}(x,t)\right\}_{i,j=1}^{n} \\
Q(x) &= \left\{\mu_{ij}(x)\right\}_{i,j=1}^{n} = \text{The matrix of piecewise constant transition intensities}
\end{aligned}
$$

The Kolmogorov Forward differential equation may now be written as:

$$
P\prime(x,t) = P(x,t) \times Q(x) \qquad (3.11)
$$

with $x = 0, 1, 2, ...$ and $0 \le t \le 1$ defined as before and with boundary condition $P(0,t) = I$ (where I is the identity matrix).

The solution to (3.11) can be found as follows:

$$\begin{aligned}
\frac{P\prime(x,t)}{P(x,t)} &= Q(x) \\
\frac{d}{dt}\ln P(x,t) &= Q(x) \\
\int d\ln P(x,t) &= \int Q(x)\,dt \\
\ln P(x,t) &= tQ(x) \\
P(x,t) &= \exp(tQ(x)) \\
&= I + \frac{tQ(x)}{1!} + \frac{(tQ(x))^2}{2!} + \frac{(tQ(x))^3}{2!} + ... \\
&= I + \sum_{k=1}^{\infty}\frac{(tQ(x))^k}{k!} \qquad (3.12)
\end{aligned}$$

If the matrix $Q(x)$ has distinct eigenvalues then it can be expressed in the form:

$$Q(x) = A(x)\,D(x)\,C(x) \qquad (3.13)$$

where:

$A(x)$ = the matrix of right eigenvectors

$D(x)$ = the diagonal matrix whose elements are eigenvalues of $Q(x)$

$C(x) = A(x)^{-1}$ exists

Thus $Q(x) = A(x)\,D(x)\,A(x)^{-1}$

Further:

$$\begin{aligned}
Q(x)^k &= \left(A(x)\,D(x)\,A(x)^{-1}\right)^k \\
&= \left(A(x)\,D(x)\,A(x)^{-1}\right)\left(A(x)\,D(x)\,A(x)^{-1}\right)\ldots\left(A(x)\,D(x)\,A(x)^{-1}\right) \\
&= A(x)\,D(x)\,A(x)^{-1}\,A(x)\,D(x)\,A(x)^{-1}\ldots A(x)\,D(x)\,A(x)^{-1} \\
&= A(x)\,D(x)\,ID(x)\,ID(x)\,ID(x)\ldots D(x)\,ID(x)\,ID(x)\,A(x)^{-1} \\
&= A(x)\,D(x)\,D(x)\ldots D(x)\,D(x)\,A(x)^{-1} \\
&= A(x)\,D(x)^k\,A(x)^{-1} \qquad (3.14)
\end{aligned}$$

Substituting (3.14) in (3.12) we get:

$$\begin{aligned}
P(x,t) &= I + \sum_{k=1}^{\infty} \frac{(Q(x)\,t)^k}{k!} \\
&= I + \sum_{k=1}^{\infty} \frac{t^k}{k!} Q(x)^k \\
&= I + \sum_{k=1}^{\infty} \frac{t^k}{k!} A(x)\,D(x)^k\,A(x)^{-1} \\
&= I + A(x)\left(\sum_{k=1}^{\infty} \frac{t^k}{k!} D(x)^k\right) A(x)^{-1} \qquad (3.15)
\end{aligned}$$

For this problem we wish to determine $D(x)$ and $A(x)$, i.e. to determine the eigenvalues and eigenvectors of:

$$Q(x) = \begin{bmatrix} -\mu_{12}(x) & \mu_{12}(x) \\ 0 & 0 \end{bmatrix}$$

To find the eigenvalues we solve the equation

$$
\begin{aligned}
|Q(x) - \lambda I| &= 0 \\
\text{i.e.} \begin{vmatrix} -\mu_{12}(x) - \lambda & \mu_{12}(x) \\ 0 & -\lambda \end{vmatrix} &= 0 \\
\lambda\left(\mu_{12}(x) + \lambda\right) &= 0 \\
\lambda &= 0 \text{ or } -\mu_{12}(x) \\
\lambda_1 &= -\mu_{12}(x) \text{ and } \lambda_2 = 0 \text{ are the eigenvalues}
\end{aligned}
$$

The corresponding eigenvectors are:

$$
\begin{aligned}
i) \begin{bmatrix} -\mu_{12}(x) & \mu_{12}(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \lambda_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
-\mu_{12}(x) x_1 + \mu_{12}(x) x_2 &= \lambda_1 x_1 \text{ since } \lambda_1 = -\mu_{12}(x) \text{ we have} \\
-\mu_{12}(x) x_1 + \mu_{12}(x) x_2 &= -\mu_{12}(x) x_1 \\
\text{therefore } \mu_{12}(x) x_2 &= 0 \\
x_2 &= 0 = 0 * x_1 \\
\text{eigen vector} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 * x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

we take the eigenvector for $\lambda_1 = -\mu_{12}(x)$ as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$
\begin{aligned}
ii) \begin{bmatrix} -\mu_{12}(x) & \mu_{12}(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \lambda_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ since } \lambda_2 = 0 \\
-\mu_{12}(x) x_1 + \mu_{12}(x) x_2 &= 0 \\
x_1 &= x_2
\end{aligned}
$$

Therefore the eigenvector $= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We take the eigenvector for $\lambda_2 = 0$ as $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since the eigenvalues are distinct, we can write:

$$Q(x) = A(x) D(x) A(x)^{-1}$$

where:

$$A(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ which implies that } A(x)^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

We deduce that:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(tD(x))^k}{k!} &= \sum_{k=1}^{\infty} \frac{t^k}{k!} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^k \\ &= \sum_{k=1}^{\infty} \frac{t^k}{k!} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=1}^{\infty} \frac{(\lambda_1 t)^k}{k!} & 0 \\ 0 & \sum_{k=1}^{\infty} \frac{(\lambda_2 t)^k}{k!} \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda_1 t} - 1 & 0 \\ 0 & e^{\lambda_2 t} - 1 \end{bmatrix} \end{aligned}$$

Now:

$$
\begin{aligned}
P(x,t) &= I + A(x)\left(\sum_{k=1}^{\infty}\frac{t^k}{k!}D(x)^k\right)A(x)^{-1} \\
&= I + \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}e^{\lambda_1 t}-1 & 0\\ 0 & e^{\lambda_2 t}-1\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} \\
&= I + \begin{bmatrix}e^{\lambda_1 t}-1 & e^{\lambda_2 t}-1\\ 0 & e^{\lambda_2 t}-1\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} \\
&= I + \begin{bmatrix}e^{\lambda_1 t}-1 & -e^{\lambda_1 t}+1+e^{\lambda_2 t}-1\\ 0 & e^{\lambda_2 t}-1\end{bmatrix} \\
&= I + \begin{bmatrix}e^{\lambda_1 t}-1 & -e^{\lambda_1 t}+e^{\lambda_2 t}\\ 0 & e^{\lambda_2 t}-1\end{bmatrix}
\end{aligned}
$$

Setting $\lambda_1 = -\mu_{12}(x)$ and $\lambda_2 = 0$, we have:

$$
\begin{aligned}
P(x,t) &= I + \begin{bmatrix}e^{-\mu_{12}(x)t}-1 & -e^{-\mu_{12}(x)t}+e^{0}\\ 0 & e^{0}-1\end{bmatrix} \\
&= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}e^{-\mu_{12}(x)t}-1 & -e^{-\mu_{12}(x)t}+1\\ 0 & 0\end{bmatrix}
\end{aligned}
$$

Thus, transition probabilities for the alive-dead model in terms of the piecewise constant transition intensities can be expressed as:

$$
\begin{bmatrix}p_{11}(x,t) & p_{12}(x,t)\\ p_{21}(x,t) & p_{22}(x,t)\end{bmatrix} = \begin{bmatrix}e^{-\mu_{12}(x)t} & 1-e^{-\mu_{12}(x)t}\\ 0 & 1\end{bmatrix}
$$

3.3.2 Healthy-ill model

The healthy-ill model is a two-state model with forces of transition in both directions.This model introduces the idea of recovery.

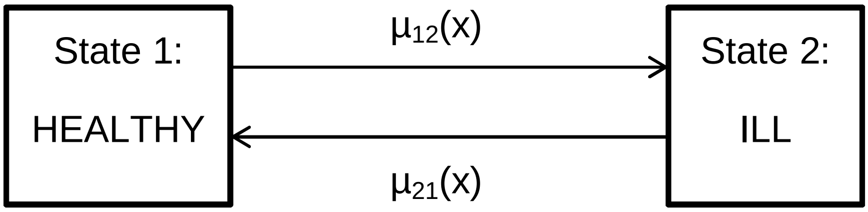

FIGURE 2: Healthy-Ill model

Recovery is the ability of a person to go back to a state that they had visited before.

Using equation (3.6) we derive the Kolmogorov Forward Differential Equations as follows:

$$\begin{aligned} p_{11}(x,t+h) &= \sum_{k=1}^{2} p_{1k}(x,t)\, p_{k1}(t,t+h) \\ &= p_{11}(x,t)\, p_{11}(t,t+h) + p_{12}(x,t)\, p_{21}(t,t+h) \\ &= p_{11}(x,t)\left[1 - \left(\mu_{12}(x)\, h + o(h)\right)\right] + p_{12}(x,t)\left[\mu_{21}(x)\, h + o(h)\right] \\ \frac{d}{dt} p_{11}(x,t) &= p_{11}(x,t)\left[-\mu_{12}(x)\right] + p_{12}(x,t) * \mu_{21}(x) \end{aligned}$$

$$\begin{aligned} p_{12}(x,t+h) &= \sum_{k=1}^{2} p_{1k}(x,t)\, p_{k2}(t,t+h) \\ &= p_{11}(x,t)\, p_{12}(t,t+h) + p_{12}(x,t)\, p_{22}(t,t+h) \\ &= p_{11}(x,t)\left[\mu_{12}(x)\, h + o(h)\right] + p_{12}(x,t)\left[1 - \left(\mu_{21}(x)\, h + o(h)\right)\right] \\ \frac{d}{dt} p_{12}(x,t) &= p_{11}(x,t) * \mu_{12}(x) + p_{12}(x,t) * \left[-\mu_{21}(x)\right] \end{aligned}$$

$$
\begin{aligned}
p_{21}(x,t+h) &= \sum_{k=1}^{2} p_{2k}(x,t)\, p_{k1}(t,t+h) \\
&= p_{21}(x,t)\, p_{11}(t,t+h) + p_{22}(x,t)\, p_{21}(t,t+h) \\
&= p_{21}(x,t)\left[1-\left(\mu_{12}(x)\, h + o(h)\right)\right] + p_{12}(x,t)\left[\mu_{21}(x)\, h + o(h)\right] \\
\frac{d}{dt} p_{21}(x,t) &= p_{21}(x,t) * \left[-\mu_{12}(x)\right] + p_{12}(x,t) * \mu_{21}(x)
\end{aligned}
$$

$$
\begin{aligned}
p_{22}(x,t+h) &= \sum_{k=1}^{2} p_{2k}(x,t)\, p_{k2}(t,t+h) \\
&= p_{21}(x,t)\, p_{12}(t,t+h) + p_{22}(x,t)\, p_{22}(t,t+h) \\
&= p_{21}(x,t)\left[\mu_{12}(x)\, h + o(h)\right] + p_{22}(x,t)\left[1-\left(\mu_{21}(x)\, h + o(h)\right)\right] \\
\frac{d}{dt} p_{22}(x,t) &= p_{21}(x,t) * \mu_{12}(x) + p_{22}(x,t) * \left[-\mu_{21}(x)\right]
\end{aligned}
$$

In matrix form, we have

$$
\begin{bmatrix} p'_{11}(x,t) & p'_{12}(x,t) \\ p'_{21}(x,t) & p'_{22}(x,t) \end{bmatrix} = \begin{bmatrix} p_{11}(x,t) & p_{12}(x,t) \\ p_{21}(x,t) & p_{22}(x,t) \end{bmatrix} \begin{bmatrix} -\mu_{12}(x) & \mu_{12}(x) \\ \mu_{21}(x) & -\mu_{21}(x) \end{bmatrix}
$$

In compact form, we have

$P\prime(x,t) = P(x,t) * Q(x)$

where $Q(x) = \begin{bmatrix} -\mu_{12}(x) & \mu_{12}(x) \\ \mu_{21}(x) & -\mu_{21}(x) \end{bmatrix}$

We find the eigenvalues and eigenvectors for $Q(x)$:

$$\begin{aligned}
|Q(x) - \lambda I| &= 0 \\
\begin{vmatrix} -\mu_{12}(x) - \lambda & \mu_{12}(x) \\ \mu_{21}(x) & -\mu_{21}(x) - \lambda \end{vmatrix} &= 0 \\
\left(-\mu_{12}(x) - \lambda\right)\left(-\mu_{21}(x) - \lambda\right) - \mu_{12}(x) * \mu_{21}(x) &= 0 \\
\left(\mu_{12}(x) + \lambda\right)\left(\mu_{21}(x) + \lambda\right) - \mu_{12}(x) * \mu_{21}(x) &= 0 \\
\mu_{12}(x) * \mu_{21}(x) + \mu_{12}(x)\lambda + \mu_{21}(x)\lambda + \lambda^2 - \mu_{12}(x) * \mu_{21}(x) &= 0 \\
\lambda^2 + \lambda\left(\mu_{12}(x) + \mu_{21}(x)\right) &= 0
\end{aligned}$$

Using the quadratic formula:

$$\begin{aligned}
\lambda &= \frac{-\left(\mu_{12}(x) + \mu_{21}(x)\right) \pm \sqrt{\left(\mu_{12}(x) + \mu_{21}(x)\right)^2}}{2} \\
&= \frac{-\left(\mu_{12}(x) + \mu_{21}(x)\right) \pm \left(\mu_{12}(x) + \mu_{21}(x)\right)}{2} \\
\text{therefore } \lambda &= 0 \text{ or } -\left(\mu_{12}(x) + \mu_{21}(x)\right) \\
\lambda_1 &= -\left(\mu_{12}(x) + \mu_{21}(x)\right) \text{ and } \lambda_2 = 0 \text{ are the eigenvalues}
\end{aligned}$$

The corresponding eigenvectors are:

$$
\begin{aligned}
i)\ \begin{bmatrix} -\mu_{12}(x) & \mu_{12}(x) \\ \mu_{21}(x) & -\mu_{21}(x) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \lambda_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
-\mu_{12}(x)\, x_1 + \mu_{12}(x)\, x_2 &= \lambda_1 x_1 \text{ since } \lambda_1 = -\left(\mu_{12}(x) + \mu_{21}(x)\right) \\
\text{we have } -\mu_{12}(x)\, x_1 + \mu_{12}(x)\, x_2 &= -\left(\mu_{12}(x) + \mu_{21}(x)\right) x_1 \\
&= -\mu_{12}(x)\, x_1 - \mu_{21}(x)\, x_1 \\
&\Longrightarrow \mu_{12}(x)\, x_2 = -\mu_{21}(x)\, x_1 \\
&\Longrightarrow x_2 = \frac{-\mu_{21}(x)}{\mu_{12}(x)} x_1 \\
\text{also } \mu_{21}(x)\, x_1 - \mu_{21}(x)\, x_2 &= -\left(\mu_{12}(x) + \mu_{21}(x)\right) x_2 \\
&= -\mu_{12}(x)\, x_2 - \mu_{21}(x)\, x_2 \\
&\Longrightarrow \mu_{21}(x)\, x_1 = -\mu_{12}(x)\, x_2 \\
&\Longrightarrow x_2 = \frac{-\mu_{21}(x)}{\mu_{12}(x)} x_1 \\
\text{therefore the eigenvector is } &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{-\mu_{21}(x)}{\mu_{12}(x)} x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ \frac{-\mu_{21}(x)}{\mu_{12}(x)} \end{bmatrix}
\end{aligned}
$$

we take the eigenvector for $\lambda_1 = -\left(\mu_{12}(x) + \mu_{21}(x)\right)$ as $\begin{bmatrix} 1 \\ \frac{-\mu_{21}(x)}{\mu_{12}(x)} \end{bmatrix}$

$$
\begin{aligned}
ii)\ \begin{bmatrix} -\mu_{12}(x) & \mu_{12}(x) \\ \mu_{21}(x) & -\mu_{21}(x) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \lambda_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ since } \lambda_2 = 0 \\
-\mu_{12}(x)\, x_1 + \mu_{12}(x)\, x_2 &= \lambda_2 x_1 = 0 \Longrightarrow -\mu_{12}(x)\, x_1 = -\mu_{12}(x)\, x_2 \\
&\Longrightarrow x_1 = \frac{\mu_{12}(x)}{\mu_{12}(x)} x_2 \Longrightarrow x_1 = x_2 \\
\mu_{21}(x)\, x_1 - \mu_{21}(x)\, x_2 &= \lambda_2 x_2 = 0 \Longrightarrow \mu_{21}(x)\, x_1 = \mu_{21}(x)\, x_2 \\
&\Longrightarrow x_1 = \frac{\mu_{21}(x)}{\mu_{21}(x)} x_2 \Longrightarrow x_1 = x_2
\end{aligned}
$$

we take the eigenvector for $\lambda_2 = 0$ as $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

From equation 3.14 :

$Q(x) = A(x)\, D(x)\, A(x)^{-1}$

where $A(x) = \begin{bmatrix} 1 & 1 \\ \frac{-\mu_{21}(x)}{\mu_{12}(x)} & 1 \end{bmatrix}$ and $A(x)^{-1} = \frac{1}{1+\frac{\mu_{21}(x)}{\mu_{12}(x)}} \begin{bmatrix} 1 & -1 \\ \frac{\mu_{21}(x)}{\mu_{12}(x)} & 1 \end{bmatrix}$

We also had:

$$\sum_{k=1}^{\infty} \frac{(tD(x))^k}{k!} = \begin{bmatrix} e^{\lambda_1 t} - 1 & 0 \\ 0 & e^{\lambda_2 t} - 1 \end{bmatrix}^k$$

But in this case $\lambda_2 = 0$ and the equation becomes:

$$\sum_{k=1}^{\infty} \frac{(tD(x))^k}{k!} = \begin{bmatrix} e^{\lambda_1 t} - 1 & 0 \\ 0 & 0 \end{bmatrix}^k \tag{3.16}$$

Replacing $A(x)$, $A(x)^{-1}$ and equation (3.16) in equation (3.15) we get:

$$\begin{aligned} P(x,t) &= I + \begin{bmatrix} 1 & 1 \\ \frac{-\mu_{21}(x)}{\mu_{12}(x)} & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} - 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \frac{\mu_{21}(x)}{\mu_{12}(x)} & 1 \end{bmatrix} \frac{\mu_{12}(x)}{\mu_{12}(x) + \mu_{21}(x)} \\ &= I + \begin{bmatrix} e^{\lambda_1 t} - 1 & 0 \\ \frac{-\mu_{21}(x)}{\mu_{12}(x)} \left(e^{\lambda_1 t} - 1\right) & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \frac{\mu_{21}(x)}{\mu_{12}(x)} & 1 \end{bmatrix} \frac{\mu_{12}(x)}{\mu_{12}(x) + \mu_{21}(x)} \\ &= I + \begin{bmatrix} e^{\lambda_1 t} - 1 & 0 \\ \frac{-\mu_{21}(x)}{\mu_{12}(x)} \left(e^{\lambda_1 t} - 1\right) & 0 \end{bmatrix} \begin{bmatrix} \frac{\mu_{12}(x)}{\mu_{12}(x)+\mu_{21}(x)} & -\frac{\mu_{12}(x)}{\mu_{12}(x)+\mu_{21}(x)} \\ \frac{\mu_{21}(x)}{\mu_{12}(x)+\mu_{21}(x)} & \frac{\mu_{12}(x)}{\mu_{12}(x)+\mu_{21}(x)} \end{bmatrix} \\ &= I + \begin{bmatrix} \frac{\mu_{12}(x)}{\mu_{12}(x)+\mu_{21}(x)} \left(e^{\lambda_1 t} - 1\right) & -\frac{\mu_{12}(x)}{\mu_{12}(x)+\mu_{21}(x)} \left(e^{\lambda_1 t} - 1\right) \\ \frac{-\mu_{21}(x)}{\mu_{12}(x)+\mu_{21}(x)} \left(e^{\lambda_1 t} - 1\right) & \frac{\mu_{21}(x)}{\mu_{12}(x)+\mu_{21}(x)} \left(e^{\lambda_1 t} - 1\right) \end{bmatrix} \\ \text{but } \lambda_1 &= -\left(\mu_{12}(x) + \mu_{21}(x)\right) \end{aligned}$$

We define $a = \frac{\mu_{12}(x)}{\mu_{12}(x)+\mu_{21}(x)} \left(e^{-\left(\mu_{12}(x)+\mu_{21}(x)\right)t} - 1\right)$

and $b = \frac{\mu_{21}(x)}{\mu_{12}(x)+\mu_{21}(x)}\left(e^{-\left(\mu_{12}(x)+\mu_{21}(x)\right)t} - 1\right)$

We now have: $P(x,t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$

Thus the transition probabilities for the healthy-ill model in terms of the piecewise constant transition intensities can be given as:

$$\begin{bmatrix} p_{11}(x,t) & p_{12}(x,t) \\ p_{21}(x,t)\,0 & p_{22}(x,t) \end{bmatrix} = \begin{bmatrix} 1+a & -a \\ -b & 1+b \end{bmatrix}$$

3.4 A Three-State Model

Figure 3 represents a three- state model in which there are no recoveries. It can also be referred to as a two-decrement model.

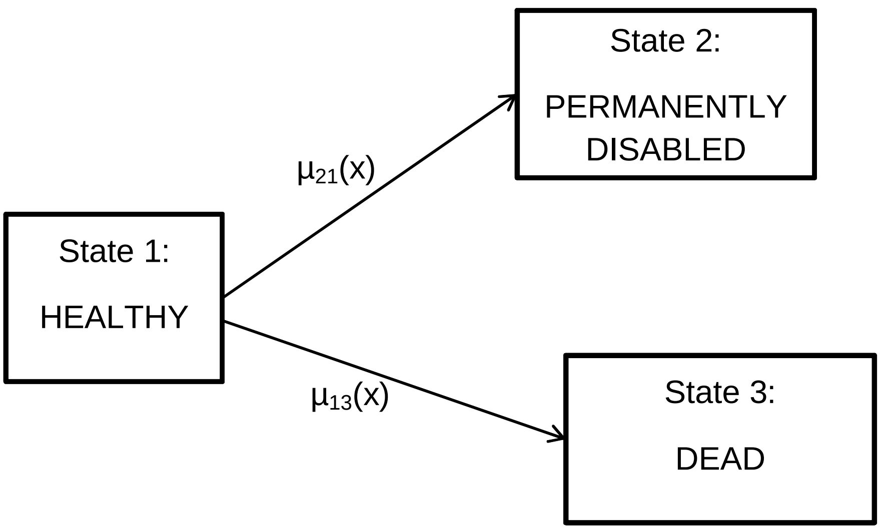

FIGURE 3: A three-state model

By the Chapman-Kolmogorov equation:

$$p_{ij}(x,t+h) = \sum_k p_{ik}(x,t)\, p_{kj}(t,t+h)$$

We proceed to derive Kolmogorov Forward Differential Equations. In this section we derive equations for transition probabilities using solutions to ordinary differential equations.

In this case $k = 1, 2, 3$.

$$\begin{aligned}
p_{11}(x,t+h) &= p_{11}(x,t)\, p_{11}(t,t+h) + p_{12}(x,t)\, p_{21}(t,t+h) + p_{13}(x,t)\, p_{31}(t,t+h) \\
&= p_{11}(x,t)\left[1 - \left(\mu_{12}(x) + \mu_{13}(x)\right) h + o(h)\right] + p_{12}(x,t) * 0 \\
&\quad + p_{13}(x,t) * 0 \\
\frac{d}{dt} p_{11}(x,t) &= p_{11}(x,t) * \left[- \left(\mu_{12}(x) + \mu_{13}(x)\right) h + o(h)\right] \\
\frac{d}{dt} p_{11}(x,t) &= -\left(\mu_{12}(x) + \mu_{13}(x)\right)
\end{aligned}$$

We assume that observation is done over the interval $(0, t)$ to get:

$$\begin{aligned}
\int_0^t dp_{11}(x,t) &= \int_0^t -\left(\mu_{12}(x) + \mu_{13}(x)\right) ds \\
p_{11}(x,t) &= -\left[\left(\mu_{12}(x) + \mu_{13}(x)\right) s\right]_0^t \\
p_{11}(x,t) &= -\left(\mu_{12}(x) + \mu_{13}(x)\right) t
\end{aligned}$$

Next:

$$\begin{aligned}
p_{12}(x,t+h) &= p_{11}(x,t)\, p_{12}(t,t+h) + p_{12}(x,t)\, p_{22}(t,t+h) + p_{13}(x,t)\, p_{32}(t,t+h) \\
&= p_{11}(x,t)\left[\mu_{12}(x)\, h + o(h)\right] + p_{12}(x,t) * 1 + p_{13}(x,t) * 0
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}p_{12}(x,t) &= p_{11}(x,t) * \mu_{12}(x) \\
\frac{d}{dt}p_{12}(x,t) &= -\left(\mu_{12}(x) + \mu_{13}(x)\right) t * \mu_{12}(x) \\
\int_0^t dp_{12}(x,t) &= -\mu_{12}(x) \int_0^t \left(\mu_{12}(x) + \mu_{13}(x)\right) sds \\
p_{12}(x,t) &= -\mu_{12}(x) \left[\left(\mu_{12}(x) + \mu_{13}(x)\right) \frac{s^2}{2}\right]_0^t
\end{aligned}$$

Therefore:

$$p_{12}(x,t) = -\mu_{12}(x) \left(\mu_{12}(x) + \mu_{13}(x)\right) \frac{t^2}{2}$$

Next we have:

$$\begin{aligned}
p_{13}(x,t+h) &= p_{11}(x,t)\, p_{13}(t,t+h) + p_{12}(x,t)\, p_{23}(t,t+h) + p_{13}(x,t)\, p_{33}(t,t+h) \\
&= p_{11}(x,t) \left[\mu_{13}(x)\, h + o(h)\right] + p_{12}(x,t) * 0 + p_{13}(x,t) * 1 \\
\frac{d}{dt}p_{13}(x,t) &= p_{11}(x,t) * \mu_{13}(x) \\
\int_0^t dp_{13}(x,t) &= -\mu_{13}(x) \int_0^t \left(\mu_{12}(x) + \mu_{13}(x)\right) sds \\
&= -\mu_{13}(x) \left[\left(\mu_{12}(x) + \mu_{13}(x)\right) \frac{s^3}{3}\right]_0^t \\
p_{13}(x,t) &= -\mu_{13}(x) \left[\left(\mu_{12}(x) + \mu_{13}(x)\right) \frac{t^3}{3}\right]
\end{aligned}$$

We proceed with expressions of the transition probabilities:

$p_{21}(x,t) = 0, p_{22}(x,t) = 1, p_{23}(x,t) = 0$

$p_{31}(x,t) = 0, p_{32}(x,t) = 0, p_{33}(x,t) = 1$

The matrix form of expressions for the transition probabilities in terms of the piecewise constant transition intensities is therefore given by:

$$P(x,t) = \begin{bmatrix} -yt & -\mu_{12}(x)\frac{t^2}{2}y & -\mu_{13}(x)\frac{t^3}{3}y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $y = \mu_{12}(x) + \mu_{13}(x)$

When Kolmogorov equations can be solved using ordinary differential equations, the matrix $P(x,t)$ can be calculated directly using transition intensities between the various states in the model.

3.5 A Four-State Model

Figure 4 represents a 4-state multiple state model which can be considered as a three-decrement model.

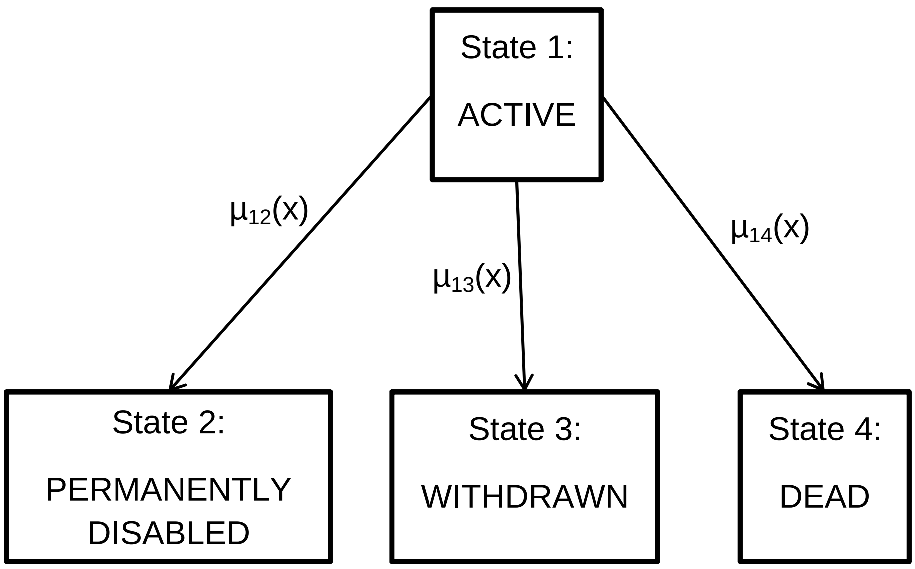

FIGURE 4: Four-state model

By Chapman-Kolmogorov Equation:

$$p_{ij}(x,t) = \sum_{k=1}^{j} p_{ik}(x,t)\, p_{kj}(t,t+h)$$

We proceed to derive the Kolmogorov Forward Differential Equations.

In this case $k = 1,2,3,4$. Solution to the Kolmogorov Equations is obtained by solving ordinary differential equations as follows:

$$\begin{aligned}
p_{11}(x,t+h) &= \sum_{k=1}^{4} p_{1k}(x,t)\, p_{k1}(t,t+h) \\
&= p_{11}(x,t)\, p_{11}(t,t+h) + p_{12}(x,t)\, p_{21}(t,t+h) + p_{13}(x,t)\, p_{31}(t,t+h) \\
&\quad + p_{14}(x,t)\, p_{41}(t,t+h) \\
&= p_{11}(x,t)\left[1 - \left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x)\right) h + o(h)\right] + p_{12}(x,t) * 0 \\
&\quad + p_{13}(x,t) * 0 + p_{14}(x,t) * 0 \\
&= p_{11}(x,t)\left[1 - \left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x)\right) h + o(h)\right] \\
\frac{d}{dt} p_{11}(x,t) &= -\left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x)\right) p_{11}(x,t) \\
\frac{1}{p_{11}(x,t)} \frac{d}{dt} p_{11}(x,t) &= -\left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x)\right) \\
\frac{d}{dt} \ln p_{11}(x,t) &= -\left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}\right) \\
\int_0^t d \ln p_{11}(x,t) &= \int_0^t -\left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x)\right) ds \\
\ln p_{11}(x,t) &= -\left[\left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x)\right) s\right]_0^t \\
\ln p_{11}(x,t) &= -\left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x)\right) t \\
\text{Therefore } p_{11}(x,t) &= e^{-\left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x)\right) t}
\end{aligned}$$

Next:

$$
\begin{aligned}
p_{12}(x,t+h) &= \sum_{k=1}^{4} p_{1k}(x,t)\,p_{k2}(t,t+h) \\
&= p_{11}(x,t)\,p_{12}(t,t+h)+p_{12}(x,t)\,p_{22}(t,t+h)+p_{13}(x,t)\,p_{32}(t,t+h) \\
&\quad +p_{14}(x,t)\,p_{42}(t,t+h) \\
&= p_{11}(x,t)\big(\mu_{12}(x)\,h+o(h)\big)+p_{12}(x,t)*1+p_{13}(x,t)*0 \\
&\quad +p_{14}(x,t)*0 \\
&= p_{11}(x,t)\big(\mu_{12}(x)\,h+o(h)\big)+p_{12}(x,t) \\
\frac{d}{dt}p_{12}(x,t) &= \mu_{12}(x)\,p_{11}(x,t) \\
&= \mu_{12}(x)\,e^{-\left(\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)\right)t} \\
\int_0^t dp_{12}(x,t) &= \mu_{12}(x)\int_0^t e^{-\left(\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)\right)s}ds \\
p_{12}(x,t) &= \left[\frac{-\mu_{12}(x)\,e^{-\left(\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)\right)s}}{\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)}\right]_0^t \\
\text{Therefore } p_{12}(x,t) &= \frac{-\mu_{12}(x)\,e^{-\left(\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)\right)t}}{\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)}
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
p_{13}(x,t+h) &= \sum_{k=1}^{4} p_{1k}(x,t)\,p_{k3}(t,t+h) \\
&= p_{11}(x,t)\,p_{13}(t,t+h)+p_{12}(x,t)\,p_{23}(t,t+h)+p_{13}(x,t)\,p_{33}(t,t+h) \\
&\quad +p_{14}(x,t)\,p_{43}(t,t+h) \\
&= p_{11}(x,t)\big[\mu_{13}(x)\,h+o(h)\big]+p_{12}(x,t)*0+p_{13}(x,t)*1 \\
&\quad +p_{14}(x,t)*0 \\
&= p_{11}(x,t)\big[\mu_{13}(x)\,h+o(h)\big]+p_{13}(x,t)*1
\end{aligned}
$$

$$\begin{aligned}
\frac{d}{dt}p_{13}(x,t) &= \mu_{13}(x)\,p_{11}(x,t) \\
&= \mu_{13}(x)\,e^{-\left(\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)\right)t} \\
\int_0^t dp_{13}(x,t) &= \int_0^t \mu_{13}(x)\,e^{-\left(\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)\right)s}dt \\
p_{13}(x,t) &= \left[\frac{-\mu_{13}(x)\,e^{-\left(\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)\right)s}}{\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)}\right]_0^t \\
\text{Thus } p_{13}(x,t) &= \frac{-\mu_{13}(x)\,e^{-\left(\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)\right)t}}{\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)}
\end{aligned}$$

Using a similar argument, it can be shown that:

$$p_{14}(x,t) = \frac{-\mu_{14}(x)\,e^{-\left(\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)\right)t}}{\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)}$$

For this model we have:

$p_{21}(x,t) = 0, p_{22}(x,t) = 1, p_{23}(x,t) = 0, p_{24}(x,t) = 0, p_{31}(x,t) = 0,$

$p_{32}(x,t) = 0, p_{33}(x,t) = 1, p_{34}(x,t) = 0, p_{41}(x,t) = 0, p_{42}(x,t) = 0,$

$p_{43}(x,t) = 0, p_{44}(x,t) = 1$

We define:

$c = e^{-\left(\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)\right)t}$

and the matrix representation of our transition probabilities is given as:

$$P(x,t) = \begin{bmatrix} c & \frac{-\mu_{12}(x)c}{\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)} & \frac{-\mu_{13}(x)c}{\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)} & \frac{-\mu_{14}(x)c}{\mu_{12}(x)+\mu_{13}(x)+\mu_{14}(x)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.6 A five-state long-term care model

We now introduce a five-state model in which recovery is allowed in multiple directions. In reality some of the recoveries in this model are impossible because an individual would be permanently disabled.

We define the notations used in the Figure 5 as follows:

- 0 ADLs means that no activities of daily living have been failed
- 1 ADL means that only one activity of daily living has been failed
- 2ADLs means that two activity of daily living have been failed
- 3ADLs means that more than three activities of daily living have been failed

The activities of daily living (ADLs) considered in order of severity from least severe to most severe are:

- Eating
- Bathing
- Dressing
- Transference
- Mobility

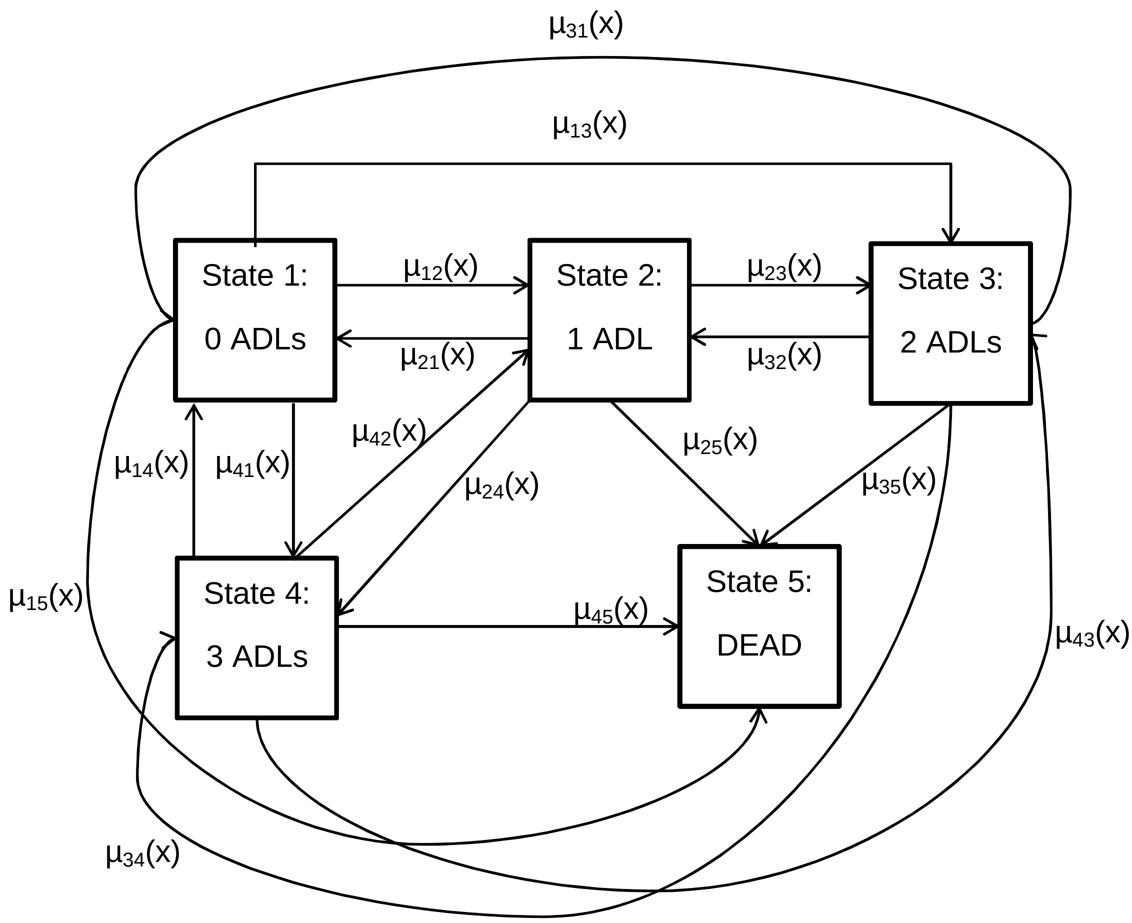

FIGURE 5: Five-state model

To derive equations for transition probabilities, recall equation 3.6 :

$$p_{ij}\left(x,t+h\right)=\sum_{k=1}^{n}p_{ik}\left(x,t\right)p_{kj}\left(t,t+h\right)$$

We use this equation to derive expressions for our transition probabilities. In this case $k=1,2,3,4,5$.

$$
\begin{aligned}
p_{11}(x,t+h) &= \sum_{k=1}^{5} p_{1k}(x,t)\, p_{k1}(t,t+h) \\
&= p_{11}(x,t)\, p_{11}(t,t+h) + p_{12}(x,t)\, p_{21}(t,t+h) \\
&\quad + p_{13}(x,t)\, p_{31}(t,t+h) + p_{14}(x,t)\, p_{41}(t,t+h) \\
&\quad + p_{15}(x,t)\, p_{51}(t,t+h) \\
&= p_{11}(x,t)\left[1 - \left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x) + \mu_{15}(x)\right) h + o(h)\right] \\
&\quad + p_{12}(x,t) * \mu_{21}(x)\, h + p_{13}(x,t) * \mu_{31}(x)\, h \\
&\quad + p_{14}(x,t)\, \mu_{41}(x)\, h \\
\lim_{h\to 0} \frac{p_{11}(x,t+h) - p_{11}(x,t)}{h} &= p_{11}(x,t)\left[-\left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x) + \mu_{15}(x)\right)\right] \\
&\quad + p_{12}(x,t) * \mu_{21}(x) + p_{13}(x,t) * \mu_{31}(x) + p_{14}(x,t)\, \mu_{41}(x) \\
\text{Therefore } \frac{d}{dt} p_{11}(x,t) &= p_{11}(x,t)\left[-\left(\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x) + \mu_{15}(x)\right)\right] \\
&\quad + p_{12}(x,t) * \mu_{21}(x) + p_{13}(x,t) * \mu_{31}(x) + p_{14}(x,t)\, \mu_{41}(x)
\end{aligned}
$$

Next we have:

$$
\begin{aligned}
\lim_{h\to 0} \frac{p_{12}(x,t+h) - p_{12}(x,t)}{h} &= p_{11}(x,t) * \mu_{12}(x) \\
&\quad + p_{12}(x,t)\left[-\left(\mu_{21}(x) + \mu_{23}(x) + \mu_{24}(x) + \mu_{25}(x)\right)\right] \\
&\quad + p_{13}(x,t) * \mu_{32}(x) + p_{14}(x,t)\, \mu_{42}(x)
\end{aligned}
$$

$$
\begin{aligned}
\text{Therefore } \frac{d}{dt} p_{12}(x,t) &= p_{11}(x,t) * \mu_{12}(x) \\
&\quad + p_{12}(x,t)\left[-\left(\mu_{21}(x) + \mu_{23}(x) + \mu_{24}(x) + \mu_{25}(x)\right)\right] \\
&\quad + p_{13}(x,t) * \mu_{32}(x) + p_{14}(x,t)\, \mu_{42}(x)
\end{aligned}
$$

$$
\begin{aligned}
p_{13}(x,t+h) &= \sum_{k=1}^{5} p_{1k}(x,t)\, p_{k3}(t,t+h) \\
&= p_{11}(x,t)\, p_{13}(t,t+h) + p_{12}(x,t)\, p_{23}(t,t+h) \\
&\quad + p_{13}(x,t)\, p_{33}(t,t+h) + p_{14}(x,t)\, p_{43}(t,t+h) \\
&\quad + p_{15}(x,t)\, p_{53}(t,t+h) \\
&= p_{11}(x,t) * \mu_{13}(x)\, h + p_{12}(x,t)\, \mu_{23}(x)\, h \\
&\quad + p_{13}(x,t)\left[1 - \left(\mu_{31}(x) + \mu_{32}(x) + \mu_{34}(x) + \mu_{35}(x)\right) h + o(h)\right] \\
&\quad + p_{14}(x,t)\, \mu_{43}(x)\, h
\end{aligned}
$$

$$
\begin{aligned}
p_{14}(x,t+h) &= \sum_{k=1}^{5} p_{1k}(x,t)\, p_{k4}(t,t+h) \\
&= p_{11}(x,t)\, p_{14}(t,t+h) + p_{12}(x,t)\, p_{24}(t,t+h) \\
&\quad + p_{13}(x,t)\, p_{34}(t,t+h) + p_{14}(x,t)\, p_{44}(t,t+h) \\
&\quad + p_{15}(x,t)\, p_{54}(t,t+h) \\
&= p_{11}(x,t) * \mu_{14}(x)\, h + p_{12}(x,t)\, \mu_{24}(x)\, h \\
&\quad + p_{13}(x,t)\, \mu_{34}(x)\, h \\
&\quad + p_{14}(x,t)\left[1 - \left(\mu_{41}(x) + \mu_{42}(x) + \mu_{43}(x) + \mu_{45}(x)\right) h + o(h)\right]
\end{aligned}
$$

$$
\begin{aligned}
\lim_{h \to 0} \frac{p_{14}(x,t+h) - p_{14}(x,t)}{h} &= p_{11}(x,t) * \mu_{14}(x) + p_{12}(x,t)\, \mu_{24}(x) + p_{13}(x,t)\, \mu_{34}(x) \\
&\quad + p_{14}(x,t)\left[-\left(\mu_{41}(x) + \mu_{42}(x) + \mu_{43}(x) + \mu_{45}(x)\right)\right] \\
\text{Therefore } \frac{d}{dt} p_{14}(x,t) &= p_{11}(x,t) * \mu_{14}(x) + p_{12}(x,t)\, \mu_{24}(x) + p_{13}(x,t)\, \mu_{34}(x) \\
&\quad + p_{14}(x,t)\left[-\left(\mu_{41}(x) + \mu_{42}(x) + \mu_{43}(x) + \mu_{45}(x)\right)\right]
\end{aligned}
$$

Finally:

$$\begin{aligned}
p_{15}(x,t+h) &= \sum_{k=1}^{5} p_{1k}(x,t)\, p_{k5}(t,t+h) \\
&= p_{11}(x,t)\, p_{15}(t,t+h) + p_{12}(x,t)\, p_{25}(t,t+h) \\
&\quad + p_{13}(x,t)\, p_{35}(t,t+h) + p_{14}(x,t)\, p_{45}(t,t+h) \\
&\quad + p_{15}(x,t)\, p_{55}(t,t+h) \\
&= p_{11}(x,t) * \mu_{15}(x)\, h + p_{12}(x,t)\, \mu_{25}(x)\, h + p_{13}(x,t)\, \mu_{35}(x)\, h \\
&\quad + p_{14}(x,t) * \mu_{45}(x)\, h + p_{15}(x,t) * 1 \\
\lim_{h \to 0} \frac{p_{15}(x,t+h) - p_{15}(x,t)}{h} &= p_{11}(x,t) * \mu_{15}(x) + p_{12}(x,t)\, \mu_{25}(x) + p_{13}(x,t)\, \mu_{35}(x) \\
&\quad + p_{14}(x,t) * \mu_{45}(x) \\
\text{Therefore } \frac{d}{dt} p_{15}(x,t) &= p_{11}(x,t) * \mu_{15}(x) + p_{12}(x,t)\, \mu_{25}(x) + p_{13}(x,t)\, \mu_{35}(x) \\
&\quad + p_{14}(x,t) * \mu_{45}(x)
\end{aligned}$$

Using the compact form of Kolmogorov forward differential equations:

$$P\prime(x,t) = P(x,t) * Q(x)$$

where:

$$P\prime(x,t) = \begin{bmatrix}
p_{11}\prime(x,t) & p_{12}\prime(x,t) & p_{13}\prime(x,t) & p_{14}\prime(x,t) & p_{15}\prime(x,t) \\
p_{21}\prime(x,t) & p_{22}\prime(x,t) & p_{23}\prime(x,t) & p_{24}\prime(x,t) & p_{25}\prime(x,t) \\
p_{31}\prime(x,t) & p_{32}\prime(x,t) & p_{33}\prime(x,t) & p_{34}\prime(x,t) & p_{35}\prime(x,t) \\
p_{41}\prime(x,t) & p_{42}\prime(x,t) & p_{43}\prime(x,t) & p_{44}\prime(x,t) & p_{45}\prime(x,t) \\
p_{51}\prime(x,t) & p_{52}\prime(x,t) & p_{53}\prime(x,t) & p_{54}\prime(x,t) & p_{55}\prime(x,t)
\end{bmatrix}$$

$$P(x,t) = \begin{bmatrix}
p_{11}(x,t) & p_{12}(x,t) & p_{13}(x,t) & p_{14}(x,t) & p_{15}(x,t) \\
p_{21}(x,t) & p_{22}(x,t) & p_{23}(x,t) & p_{24}(x,t) & p_{25}(x,t) \\
p_{31}(x,t) & p_{32}(x,t) & p_{33}(x,t) & p_{34}(x,t) & p_{35}(x,t) \\
p_{41}(x,t) & p_{42}(x,t) & p_{43}(x,t) & p_{44}(x,t) & p_{45}(x,t) \\
p_{51}(x,t) & p_{52}(x,t) & p_{53}(x,t) & p_{54}(x,t) & p_{55}(x,t)
\end{bmatrix}$$

$$\text{and } Q(x) = \begin{bmatrix} -d & \mu_{12}(x) & \mu_{13}(x) & \mu_{14}(x) & \mu_{15}(x) \\ \mu_{21}(x) & -e & \mu_{23}(x) & \mu_{24}(x) & \mu_{25}(x) \\ \mu_{31}(x) & \mu_{32}(x) & -f & \mu_{34}(x) & \mu_{35}(x) \\ \mu_{41}(x) & \mu_{42}(x) & \mu_{43}(x) & -g & \mu_{45}(x) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where:

$$d = (\mu_{12}(x) + \mu_{13}(x) + \mu_{14}(x) + \mu_{15}(x))$$

$$e = (\mu_{21}(x) + \mu_{23}(x) + \mu_{24}(x) + \mu_{25}(x))$$

$$f = (\mu_{31}(x) + \mu_{32}(x) + \mu_{34}(x) + \mu_{35}(x))$$

$$g = (\mu_{41}(x) + \mu_{42}(x) + \mu_{43}(x) + \mu_{45}(x))$$

The next step is to find the eigenvalues and eigen vectors for $Q(x)$:

$$\begin{aligned} |Q(x) - \lambda I| &= 0 \\ \left| \begin{bmatrix} -d-\lambda & \mu_{12}(x) & \mu_{13}(x) & \mu_{14}(x) & \mu_{15}(x) \\ \mu_{21}(x) & -e-\lambda & \mu_{23}(x) & \mu_{24}(x) & \mu_{25}(x) \\ \mu_{31}(x) & \mu_{32}(x) & -f-\lambda & \mu_{34}(x) & \mu_{35}(x) \\ \mu_{41}(x) & \mu_{42}(x) & \mu_{43}(x) & -g-\lambda & \mu_{45}(x) \\ 0 & 0 & 0 & 0 & -\lambda \end{bmatrix} \right| &= 0 \\ -\lambda \left| \begin{bmatrix} -d-\lambda & \mu_{12}(x) & \mu_{13}(x) & \mu_{14}(x) \\ \mu_{21}(x) & -e-\lambda & \mu_{23}(x) & \mu_{24}(x) \\ \mu_{31}(x) & \mu_{32}(x) & -f-\lambda & \mu_{34}(x) \\ \mu_{41}(x) & \mu_{42}(x) & \mu_{43}(x) & -g-\lambda \end{bmatrix} \right| &= 0 \end{aligned}$$

Clearly, solving for explicit expressions of the eigenvalues and eigenvectors of the expression above is no easy task. The number of unknowns is great and it would take a great amount of time and concentration to come up with explicit expressions for the eigenvalues and their corresponding eigenvectors.

In Chapter 4, where we perform our data analysis, we will employ a software in performing these calculations.

3.6.1 Graduation of Transition Intensities

We are working with the aim of coming up with expressions for transition probabilities in terms of transition intensities. An important assumption imposed on the transition intensities is therefore that:

- The rates are representative of the graduated transition rates

Although graduation of transition intensities is beyond the scope of this project, we go ahead to describe its importance and the different methods of performing it.

(Haberman and Pitacco, 1999) describe graduation as the methods by which a set of observed probabilities are fitted and smoothed to provide a suitable basis for inferences (such as calculation of premiums) to be made. There are three main methods of graduating probabilities:

- Graduation by using a mathematical formulae (parametric graduation)
- Graphical graduation (non-parametric graduation)
- Graduation by reference to a standard table (non-parametric graduation)

Parametric graduation is the preferred method for large data sets. The underlying assumption is that the transition intensities can be modelled using a mathematical formula.

Graphical graduation involves drawing by hand a curve which is thought by the researcher as the best fit of the transition intensities. As such, this method falls short on accuracy.

Graduation by reference to a standard table is applied mostly to a small dataset, if the lives under consideration in the dataset are believed to be similar to those of a larger number of lives that form the basis of a standard table. The basic features of the standard table can then be imported to the new graduated rates.

3.7 Actuarial Calculations

In this research project, we wish to consider discrete-time Actuarial calculations. As such our Markov chain is a time-discrete inhomogeneous Markov chain. We consider premiums and reserves for stand alone annuities.We will use the five-state model in Figure 5 as our reference point.

3.7.1 Premiums and reserves for stand-alone annuities

Assume that benefits are only payable when an individual is in states 2 through to 4, with no benefit being provided to an individual in state 1. We do not consider death benefits.

Let b^2, b^3 and b^4 denote the annual payments related to LTC states $2, 3$ and 4 respectively. It is logical to assume that $b^2 \leq b^3 \leq b^4$. We also assume that benefits are paid at policy annivesaries while the insured is claiming LTC benefits.

Denote the actuarial value at time 0, with $X(0) = i$ as $\beta_i^{LTC}(0, \infty)$.

Now:

$$\beta_i^{LTC}(0, \infty) = \sum_{h=1}^{+\infty} \left(b^2 p_{i2}(x, h) + b^3 p_{i3}(x, h) + b^4 p_{i4}(x, h) \right) v^h$$

$v = \frac{1}{1+i}$ denotes the annual discount factor where i is the effective interest rate per annum.

Assume that annual level premiums are paid for m years while the insured is in state 1 i.e. 0 ADLs. Let P denote the annual premium. The equivalence principle implies that:

$$P\ddot{a}^{11}_{x:\overline{m}|} = \beta^{LTC}_1(0,\infty)$$

where:

$$\ddot{a}^{11}_{x:\overline{m}|} = \sum_{k=0}^{m-1} v^k p_{11}(x,k)$$

Now let us consider prospective reserves at integer times. First we have reserves relating to state 1 where an individual is not claiming LTC benefits. This reserve is given by:

$$\begin{aligned} V^1_t &= \beta^{LTC}_1(t,\infty) - P\ddot{a}^{11}_{x+t:\overline{m-t}|} && \text{if } t < m \\ V^1_t &= \beta^{LTC}_1(t,\infty) && \text{if } t \geq m \end{aligned}$$

and is conditional on the life being in state 1 at age $x+t$.

Next we have the reserves corresponding to the three LTC states:

$$\begin{aligned} V^2_t &= \beta^{LTC}_2(t,\infty) \\ V^3_t &= \beta^{LTC}_3(t,\infty) \\ V^4_t &= \beta^{LTC}_4(t,\infty) \end{aligned}$$

conditional on a life being in state $2,3,4$ at age $x+t$ respectively.

Chapter 4

NUMERICAL ANALYSIS

4.1 Introduction

We begin by defining our data and describing how we intend to use it.

Data Description

We have a longitudinal dataset derived from Transactions of Society of Actuaries 1995 Vol.47. It is consists of:

- Three age groups: $65-74, 75-84$ and $85+$ by gender
- Information on ADL status in 1982 and 1984 for each of the age groups by gender

We assume that transition intensities are not significantly different for males and females, especially for the LTC claimable states. As such, we aggregate their numbers to come up with single tables on the number of individuals in particular ADL statuses in the given age-groups.

We use this data to calculate the rate at which individuals transit from one ADL status to another as follows:

$$\mu_{ij}(x,t) = \frac{\begin{array}{c}\text{The number of individuals in state } i \text{ in 1982 and in state } j \\ \text{in 1984 from a particular age group}\end{array}}{\begin{array}{c}\text{The total number of individuals in state } i \text{ from} \\ \text{this age group in 1982}\end{array}} \tag{4.1}$$

In this case x represents an age group and not a specific age as we had stated earlier. (Rickayzen, 2002) proposes that we divide these rates by 2 so as to get the annual rates. We employ this school of thought.

Also the diagonal entries of our $Q(x)$ matrix are given by:

$$entry\ \ (i,j) = -\left(\text{sum of all other entries in row } i\right) \tag{4.2}$$

Once the $Q(x)$ matrix is written, we employ an online matrix calculator (bluebit) to perform the decomposition of this matrix into the form:

$$Q(x) = A(x)\, D(x)^k\, A(x)^{-1}$$

by assuming that $Q(x)$ has distinct eigenvalues and corresponding eigenvectors. This assumption is proved true by the online calculator. With this decomposition we then proceed to calculate the $P(x,t)$ matrix using the following equation:

$$P(x,t) = I + A(x)\left(\sum_{k=1}^{\infty} \frac{t^k}{k!} D(x)^k\right) A(x)^{-1}$$

We note that since the transition intensities have been converted to annual rates, we assume that the transition probabilities are also for a one year period. As such our $P(x,t)$ matrix can be written as $P(x,1)$.

Time-inhomogeinity would now mean that we have $P(x,1), P(x,2), P(x,3), ...$ As such we would require a dataset that has recorded observations of initial and final states of individuals at different time intervals with each observation period being short enough for us to be able to claim that the transition intensities are piece-wise constant.

From our data, we will be able to come up with $P(x,1)$ for each of the age-groups. We will then give illustrative expressions of time-inhomogeneous Actuarial calculations for stand alone annuities.

4.2 Calculation of transition intensities

For these calculations we assume that the totals given for each row (in the original data set) represent the individuals starting from that particular state in 1982. For each age group we present:

1. The table of total number of individuals in different states in 1982 and 1984
2. The table of calculated $Q(x)$ matrix using equations 4.1 and 4.2
3. The calculated eigenvalues and eigenvectors for the $Q(x)$ matrix
4. The matrices $A(x), A(x)^{-1}$ and $\sum_{k=1}^{\infty} \frac{(tD(x))^k}{k!}$
5. The computed $P(x,t)$ matrix which we expect to be bounded between 0 and 1 i.e. $p_{ij}(x,t) \in (0,1)$

4.2.1 65-74 age group

The table below consists of the total number of individuals starting from various states in 1982 and the corresponding number of people in those states in 1984 for the ages between 65 and 74.

1982 Status	Age 65-74 (1984 status)					
	Initial Number	0 ADLs	1 ADL	2 ADLs	3+ ADLs	Dead
0 ADLS	18667	17019	214	84	142	1208
1 ADL	285	105	65	28	24	63
2 ADLs	115	24	21	16	23	31
3+ ADLs	145	15	7	12	53	58

TABLE 1: 1982 and 1984 status of persons aged 65-74

The meaning of a transition in our case is that subjects are in a particular state at one point in time and then they are in another state at another point in time. As such when an individual is in the same state at the start and at the end of the observation period, we represent this as NIL in our transition intensities table as no transition has taken place.

The table below represents calculated transition intensities using equation 4.1

1982 Status	Annual $\mu_{ij}(x)$ Age 65-74				
	0 ADLs	1 ADL	2 ADLs	3+ ADLs	Dead
0 ADLs	NIL	0.0057	0.0022	0.0038	0.0324
1 ADL	0.1842	NIL	0.0491	0.0421	0.1105
2 ADLs	0.1043	0.0913	NIL	0.1	0.1348
3+ ADLs	0.0517	0.0241	0.0414	NIL	0.2

TABLE 2: Transition Intensities for ages 65-74

Next, using the rates in TABLE 2 and equation 4.2, we can now come up with our $Q(x)$ matrix

$$Q(x) = \begin{bmatrix} -0.0441 & 0.0057 & 0.0022 & 0.0038 & 0.0324 \\ 0.1842 & -0.386 & 0.0491 & 0.0421 & 0.1105 \\ 0.1043 & 0.0913 & -0.4304 & 0.1 & 0.1348 \\ 0.0517 & 0.0241 & 0.0414 & -0.3172 & 0.2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where x represents the age-group $64 - 75$

We now employ online software (bluebit) to calculate eigenvalues and eigenvectors for $Q(x)$ yielding the following results:

- Eigenvalues are:

$$\begin{aligned} \lambda_1 &= -0.0382 \\ \lambda_2 &= -0.2647 \\ \lambda_3 &= -0.3883 \\ \lambda_4 &= -0.4866 \\ \lambda_5 &= 0 \end{aligned}$$

- Corresponding eigenvectors are given by:

$$\lambda_1 = \begin{bmatrix} -0.7561 \\ -0.482 \\ -0.3738 \\ -0.2373 \\ 0 \end{bmatrix}, \lambda_2 = \begin{bmatrix} -0.0289 \\ 0.4337 \\ 0.6167 \\ 0.6563 \\ 0 \end{bmatrix}, \lambda_3 = \begin{bmatrix} 0.0099 \\ -0.7542 \\ -0.4278 \\ 0.4981 \\ 0 \end{bmatrix}, \lambda_4 = \begin{bmatrix} -0.0017 \\ 0.3768 \\ -0.9107 \\ 0.1693 \\ 0 \end{bmatrix},$$

$$\lambda_5 = \begin{bmatrix} 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \end{bmatrix}$$

The eigenvalues are distinct, therefore as before we have:

$$Q(x) = A(x) D(x) A(x)^{-1}$$

where:

$$A(x) = \begin{bmatrix} -0.7561 & -0.0289 & 0.0099 & -0.0017 & 0.4472 \\ -0.482 & 0.4337 & -0.7542 & 0.3768 & 0.4472 \\ -0.3738 & 0.6167 & -0.4278 & -0.9107 & 0.4472 \\ -0.2373 & 0.6563 & 0.4981 & 0.1693 & 0.4472 \\ 0 & 0 & 0 & 0 & 0.4472 \end{bmatrix}$$

We use online software again to calculate $A(x)^{-1}$, yielding:

$$A(x)^{-1} = \begin{bmatrix} -1.2906 & -0.0267 & -0.0135 & -0.0265 & 1.3573 \\ -0.7082 & 0.4013 & 0.3367 & 0.9109 & -0.9408 \\ 0.3585 & -0.7588 & -0.1859 & 0.692 & -0.1058 \\ -0.1182 & 0.6392 & -0.7771 & 0.3026 & -0.0464 \\ 0 & 0 & 0 & 0 & 2.2361 \end{bmatrix}$$

and

$$\sum_{k=1}^{\infty} \frac{t^k}{k!} D(x)^k = \begin{bmatrix} e^{\lambda_1 t} - 1 & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} - 1 & 0 & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} - 1 & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} - 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_5 t} - 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0375 & 0 & 0 & 0 & 0 \\ 0 & -0.2326 & 0 & 0 & 0 \\ 0 & 0 & -0.3218 & 0 & 0 \\ 0 & 0 & 0 & -0.3853 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we have:

$P(x,t) = I + A(x) \left(\sum_{k=1}^{\infty} \frac{t^k}{k!} D(x)^k \right) A(x)^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -0.0425 & 0.0048 & 0.002 & 0.0034 & 0.0324 \\ 0.1523 & -0.3179 & 0.0335 & 0.0317 & 0.1005 \\ 0.0914 & 0.0619 & -0.3468 & 0.0704 & 0.1231 \\ 0.0469 & 0.0184 & 0.029 & -0.2699 & 0.1756 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9575 & 0.0048 & 0.002 & 0.0034 & 0.0324 \\ 0.1523 & 0.6821 & 0.0335 & 0.0317 & 0.1005 \\ 0.0914 & 0.0619 & 0.6532 & 0.0704 & 0.1231 \\ 0.0469 & 0.0184 & 0.029 & 0.7301 & 0.1756 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Due to rounding off the rows may not add up to 1 but using the original values they add up to 1.

4.2.2 75-84 age group

The table below consists of the total number of individuals starting from various states in 1982 and the corresponding number of people in those states in 1984 for the ages between 75 and 84.

1982 Status	Age 75-84 (1984 status)					
	Initial Number	0 ADLs	1 ADL	2 ADLs	3+ ADLs	Dead
0 ADLs	9572	7715	352	111	175	1219
1 ADL	345	98	63	26	48	110
2 ADLs	126	19	12	20	28	47
3+ ADLs	157	13	7	15	60	62

Table 3: 1982 and 1984 status of persons aged 75-84

Next, we have the table of calculated transition intensities using equation 4.1

1982 Status	Annual $\mu_{ij}(x,t)$ Age 75-84				
	0 ADLs	1 ADL	2 ADL	3 ADL	Dead
0 ADLs	NIL	0.0184	0.0058	0.0091	0.0637
1 ADL	0.142	NIL	0.0377	0.0696	0.1594
2 ADLs	0.0754	0.0476	NIL	0.1111	0.1865
3+ ADLs	0.0414	0.0223	0.0478	NIL	0.1975

TABLE 4: Transition intensities for ages 75-84

Now, we use TABLE 4 and equation 4.2 to come up with the $Q(x)$ matrix for this age-group.

$$Q(x) = \begin{bmatrix} -0.097 & 0.0184 & 0.0058 & 0.0091 & 0.0637 \\ 0.1420 & -0.4087 & 0.0377 & 0.0696 & 0.1594 \\ 0.0754 & 0.0476 & -0.4206 & 0.1111 & 0.1865 \\ 0.0414 & 0.0223 & 0.0478 & -0.3089 & 0.1975 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where x represents the age-group $75-84$

Our eigenvalues are calculated to be:

$$\begin{aligned}
\lambda_1 &= -0.0816 \\
\lambda_2 &= -0.2640 \\
\lambda_3 &= -0.4237 \\
\lambda_4 &= -0.4658 \\
\lambda_5 &= 0
\end{aligned}$$

Corresponding eigenvectors are given by:

$$\lambda_1 = \begin{bmatrix} 0.7985 \\ 0.4385 \\ 0.3232 \\ 0.2564 \\ 0 \end{bmatrix}, \lambda_2 = \begin{bmatrix} 0.1020 \\ -0.3915 \\ -0.5745 \\ -0.7114 \\ 0 \end{bmatrix}, \lambda_3 = \begin{bmatrix} 0.0450 \\ -0.8217 \\ -0.4592 \\ 0.3343 \\ 0 \end{bmatrix}, \lambda_4 = \begin{bmatrix} -0.0083 \\ 0.3399 \\ -0.9115 \\ 0.2313 \\ 0 \end{bmatrix},$$

$$\lambda_5 = \begin{bmatrix} 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \end{bmatrix}$$

The eigenvalues are distinct, therefore as before we have:

$Q(x) = A(x) D(x) A(x)^{-1}$

where:

$$A(x) = \begin{bmatrix} 0.7985 & 0.1020 & 0.0450 & -0.0083 & 0.4472 \\ 0.4385 & -0.3915 & -0.8217 & 0.3399 & 0.4472 \\ 0.3232 & -0.5745 & -0.4592 & -0.9115 & 0.4472 \\ 0.2564 & -0.7114 & 0.3343 & 0.2313 & 0.4472 \\ 0 & 0 & 0 & 0 & 0.4472 \end{bmatrix}$$

calculation of $A(x)^{-1}$, yields:

$$A(x)^{-1} = \begin{bmatrix} 1.1639 & 0.0776 & 0.0413 & 0.0908 & -1.3736 \\ 0.5419 & -0.1854 & -0.3197 & -0.9677 & 0.9309 \\ 0.3246 & -0.8503 & -0.1572 & 0.6415 & 0.0414 \\ -0.0925 & 0.5728 & -0.8017 & 0.319 & 0.0024 \\ 0 & 0 & 0 & 0 & 2.2361 \end{bmatrix}$$

and $\sum_{k=1}^{\infty} \frac{(tD(x))^k}{k!} = \begin{bmatrix} -0.0784 & 0 & 0 & 0 & 0 \\ 0 & -0.2320 & 0 & 0 & 0 \\ 0 & 0 & -0.3454 & 0 & 0 \\ 0 & 0 & 0 & -0.3724 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Therefore:

$P(x,t) = I + A(x) \left(\sum_{k=1}^{\infty} \frac{t^k}{k!} D(x)^k \right) A(x)^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -0.0910 & 0.0145 & 0.0049 & 0.0082 & 0.0633 \\ 0.1131 & -0.3333 & 0.0264 & 0.0507 & 0.1432 \\ 0.0629 & 0.0329 & -0.3408 & 0.0787 & 0.1663 \\ 0.0365 & 0.0167 & 0.0336 & -0.2631 & 0.1763 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.909 & 0.0145 & 0.0049 & 0.0082 & 0.0633 \\ 0.1131 & 0.6667 & 0.0264 & 0.0507 & 0.1432 \\ 0.0629 & 0.0329 & 0.6592 & 0.0787 & 0.1663 \\ 0.0365 & 0.0167 & 0.0336 & 0.7369 & 0.1763 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4.2.3 85+ age group

The table below consists of the total number of individuals starting from various states in 1982 and the corresponding number of people in those states in 1984 for ages 85 and above.

	Age 85+ (1984 status)					
1982 Status	Initial Number	0 ADLs	1 ADL	2 ADLs	3+ ADLs	Dead
0 ADLs	2134	1341	173	66	94	460
1 ADL	206	39	49	13	35	70
2 ADLs	84	6	11	11	16	40
3+ ADLs	104	8	5	10	28	53

TABLE 5: 1982 and 1984 status of persons aged 85+

Again, we have the table of calculated transition intensities using equation 4.1

	Annual $\mu_{ij}(x,t)$ Age 85+				
1982 Status	0 ADLs	1 ADL	2 ADL	3 ADL	Dead
0 ADLs	NIL	0.0405	0.0155	0.022	0.1078
1 ADL	0.0947	NIL	0.0316	0.085	0.1699
2 ADLs	0.0357	0.0655	NIL	0.0952	0.2381
3+ ADLs	0.0385	0.024	0.0481	NIL	0.2548

TABLE 6: Transition intensities for ages 85+

Now, using TABLE 6 and equation 4.1, we calculate $Q(x)$ for this age-group.

$$Q(x) = \begin{bmatrix} -0.1858 & 0.0405 & 0.0155 & 0.0220 & 0.1078 \\ 0.0947 & -0.3811 & 0.0316 & 0.085 & 0.1699 \\ 0.0357 & 0.0655 & -0.4345 & 0.0952 & 0.2381 \\ 0.0385 & 0.0240 & 0.0481 & -0.3654 & 0.2548 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where x represents the age-group 85+

Calculation of the eigenvalues yields:

$$\lambda_1 = -0.1496$$
$$\lambda_2 = -0.3131$$
$$\lambda_3 = -0.4321$$
$$\lambda_4 = -0.472$$
$$\lambda_5 = 0$$

The corresponding eigenvectors are given by:

$$\lambda_1 = \begin{bmatrix} 0.7972 \\ 0.4607 \\ 0.2922 \\ 0.2585 \\ 0 \end{bmatrix}, \lambda_2 = \begin{bmatrix} 0.3245 \\ -0.4987 \\ -0.5963 \\ -0.539 \\ 0 \end{bmatrix}, \lambda_3 = \begin{bmatrix} 0.1085 \\ -0.7820 \\ -0.3727 \\ 0.4876 \\ 0 \end{bmatrix}, \lambda_4 = \begin{bmatrix} 0.0324 \\ -0.1138 \\ -0.8998 \\ 0.4199 \\ 0 \end{bmatrix},$$

$$\lambda_5 = \begin{bmatrix} 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \end{bmatrix}$$

The eigenvalues are distinct and we therefore have:

$$Q(x) = A(x) D(x) A(x)^{-1}$$

where:

$$A(x) = \begin{bmatrix} 0.7972 & 0.3245 & 0.1085 & 0.0324 & 0.4472 \\ 0.4607 & -0.4987 & -0.7820 & -0.1138 & 0.4472 \\ 0.2922 & -0.5963 & -0.3727 & -0.8998 & 0.4472 \\ 0.2585 & -0.539 & 0.4876 & 0.4199 & 0.4472 \\ 0 & 0 & 0 & 0 & 0.4472 \end{bmatrix}$$

We use online software again to calculate $A(x)^{-1}$, yielding:

$$A(x)^{-1} = \begin{bmatrix} 0.9943 & 0.2342 & 0.1216 & 0.2473 & -1.5973 \\ 0.5736 & -0.2927 & -0.3433 & -0.8595 & 0.9219 \\ 0.2429 & -1.0571 & 0.4399 & 0.6376 & -0.2633 \\ -0.1579 & 0.7078 & -1.0265 & 0.3857 & 0.0909 \\ 0 & 0 & 0 & 0 & 2.2361 \end{bmatrix}$$

and we have:

$$\sum_{k=1}^{\infty} \frac{(tD(x))^k}{k!} = \begin{bmatrix} -0.1389 & 0 & 0 & 0 & 0 \\ 0 & -0.2688 & 0 & 0 & 0 \\ 0 & 0 & -0.3509 & 0 & 0 \\ 0 & 0 & 0 & -0.3762 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the expression:

$$P(x,t) = I + A(x) \left(\sum_{k=1}^{\infty} \frac{t^k}{k!} D(x)^k \right) A(x)^{-1}$$

We have:

$$P(x,t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -0.1675 & 0.0312 & 0.0123 & 0.0186 & 0.1054 \\ 0.0732 & -0.314 & 0.023 & 0.0604 & 0.1575 \\ 0.0299 & 0.045 & -0.35 & 0.0662 & 0.2089 \\ 0.0308 & 0.0182 & 0.0328 & -0.3034 & 0.2216 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8325 & 0.0312 & 0.0123 & 0.0186 & 0.1054 \\ 0.0732 & 0.6860 & 0.023 & 0.0604 & 0.1575 \\ 0.0299 & 0.045 & 0.65 & 0.0662 & 0.2089 \\ 0.0308 & 0.0182 & 0.0328 & 0.6966 & 0.2216 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4.3 Transition Probabilities

We would like to collect transition probabilities for the different age-groups into tables according to the initial states so that we can see if our expectations are met.

The table below represents the transition probabilities for the different age-groups with initial state 1 (0 ADLs).

State 1			
	65-74	75-84	85+
p11	0.9575	0.909	0.8325
p12	0.0048	0.0145	0.0312
p13	0.002	0.0049	0.0123
p14	0.0034	0.0082	0.0186
p15	0.0324	0.0633	0.1054

TABLE 7: Initial state 1

Remarks:

- The probability of remaining in state 1 reduces as age increases
- The probability of moving to states 2,3,4 and 5 increases with age

The following table represents the transition probabilities for the different age-groups with initial state 2 (1 ADL).

State 2			
	65-74	75-84	85+
p21	0.1523	0.1131	0.0732
p22	0.6821	0.6667	0.686
p23	0.0335	0.0264	0.023
p24	0.0317	0.0507	0.0604
p25	0.1005	0.1432	0.1575

TABLE 8: Initial state 2

-

Remarks:

- The probability of remaining in state 2 is high for the age-group 65-74, it then reduces for the age-group 75-84 and increases for the 85+ age-group
- The probability of recovery reduces with age
- The probability of moving to state 3 reduces with age
- The probability of moving to states 4 and 5 increases with age

Next, we have the table of transition probabilities for the different age-groups with initial state 3 (2 ADLs).

State 3			
	65-74	75-84	85+
p31	0.0914	0.0629	0.0299
p32	0.0619	0.0329	0.045
p33	0.6532	0.6592	0.65
p34	0.0704	0.0787	0.0662
p35	0.1231	0.1663	0.2089

TABLE 9: Initial state 3

Remarks:

- The probabilities of remaining in state 3 or moving to state 4 increases from the first age-group to the next and then declines for the 85+ age-group
- The probability of complete recovery (i.e. moving to state 1) reduces with age
- The probability of partial recovery (i.e. moving to state 2) is highest for age-group 65-74 with a decline for ages 75-84 and an increase for ages 85+
- The probability of moving to state 5 increases with age

Finally, we have the table of transition probabilities for the different age-groups with initial state 4 (3 ADLs).

State 4			
	65-74	75-84	85+
p41	0.0469	0.0365	0.0308
p42	0.0184	0.0167	0.0182
p43	0.029	0.0336	0.0328
p44	0.7301	0.7369	0.6966
p45	0.1756	0.1763	0.2216

TABLE 10: Initial state 4

Remarks:

- The probabilities of remaining in state 4 or moving to state 3 increases from the first age-group to the next and then declines for the 85+ age-group
- The probability of complete recovery (i.e. moving to state 1) reduces with age
- The probability of moving to state 2 is highest for age-group 65-74 with a decline for ages 75-84 and an increase for ages 85+
- The probability of moving to state 5 increases with age

4.4 Actuarial Calculations

In this section we wish to use the probabilities in the previous section to form part of the equations of time-inhomogeneous Actuarial calculations for stand alone-annuities.

We assume that the transition probability, $p_{ij}(x, t)$, for a particular age-group is the same for all the ages within the age-group where $i, j = 1, 2, 3, 4, 5$.

Illustration 1

Consider an individual in the 65-74 age-group. Let $b^2 = 1000, b^3 = 1700$ and $b^4 = 2500$ and let the starting state be state 1. Assume further that the interest rate is 5% effective annually.

Now the actuarial value is given by:

$$\begin{aligned}
\beta_1^{LTC}(0,\infty) &= \sum_{h=1}^{+\infty}\left[b^2 p_{12}(x,h) + b^3 p_{13}(x,h) + b^4 p_{14}(x,h)\right](1.05)^{-h} \\
&= \sum_{h=1}^{+\infty}\left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right](1.05)^{-h} \\
&= \left[1000 p_{12}(x,1) + 1700 p_{13}(x,1) + 2500 p_{14}(x,1)\right](1.05)^{-1} + \\
&\quad \sum_{h=2}^{+\infty}\left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right](1.05)^{-h}
\end{aligned}$$

where x refers to the age-group 65-74.

Since we have the values for the transition probabilities when $h = 1$, we replace them in the equation to get:

$$\begin{aligned}
\beta_1^{LTC}(0,\infty) &= \left[1000 * 0.0048 + 1700 * 0.002 + 2500 * 0.0034\right](1.05)^{-1} + \\
&\quad \sum_{h=2}^{+\infty}\left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right](1.05)^{-h} \\
&= 15.9048 + \sum_{h=2}^{+\infty}\left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right](1.05)^{-h}
\end{aligned}$$

We can then express the premium as:

$$\begin{aligned}
P\ddot{a}^{11}_{x:m\rceil} &= \beta_1^{LTC}(0,\infty) \\
&= 15.9048 + \sum_{h=2}^{+\infty}\left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right](1.05)^{-h}
\end{aligned}$$

and

$$
\begin{aligned}
\ddot{a}^{11}_{x:m\rceil} &= \sum_{k=0}^{m-1} v^k p_{11}(x,k) \\
&= v^0 p_{11}(x,0) + v^1 p_{11}(x,1) + \sum_{k=2}^{m-1} v^k p_{11}(x,k) \\
&= 1 + (1.05)^{-1} 0.9575 + \sum_{k=2}^{m-1} v^k p_{11}(x,k) \\
&= 1.9119 + \sum_{k=2}^{m-1} v^k p_{11}(x,k)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
P\ddot{a}^{11}_{x:m\rceil} &= \beta_1^{LTC}(0,\infty) \\
P\left(1.9119 + \sum_{k=2}^{m-1} v^k p_{11}(x,k)\right) &= 15.9048 \\
&\quad + \sum_{h=2}^{+\infty} \left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right] \\
&\quad * (1.05)^{-h}
\end{aligned}
$$

Therefore:

$$
P = \frac{15.9048 + \sum_{h=2}^{+\infty} \left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right]}{\left(1.9119 + \sum_{k=2}^{m-1} v^k p_{11}(x,k)\right)} (1.05)^{-h}
$$

Now,

$$V_t^1 = \beta_1^{LTC}(t,\infty) - P\ddot{a}^{11}_{x+t:\overline{m-t|}} \qquad \text{if } t < m$$
$$V_t^1 = \beta_1^{LTC}(t,\infty) \qquad \text{if } t \geq m$$

We assume $t = 0$ to get:

$$V_0^1 = 15.9048 + \sum_{h=2}^{+\infty} \left[1000p_{12}(x,h) + 1700p_{13}(x,h) + 2500p_{14}(x,h)\right](1.05)^{-h}$$
$$-P\left(1.9119 + \sum_{k=2}^{m-1} v^k p_{11}(x,k)\right)$$

V_t^2, V_t^3 and V_t^4 can be calculated using a similar argument and assuming $t = 0$.

Illustration 2

Consider an individual in the 75-84 age-group. Let $b^2 = 1000, b^3 = 1700$ and $b^4 = 2500$ and let the starting state be state 1. Assume further that the interest rate is 5% effective annually.

Now the actuarial value is given by:

$$\begin{aligned}
\beta_1^{LTC}(0,\infty) &= \sum_{h=1}^{+\infty} \left[b^2 p_{12}(x,h) + b^3 p_{13}(x,h) + b^4 p_{14}(x,h)\right](1.05)^{-h} \\
&= \sum_{h=1}^{+\infty} \left[1000p_{12}(x,h) + 1700p_{13}(x,h) + 2500p_{14}(x,h)\right](1.05)^{-h} \\
&= \left[1000p_{12}(x,1) + 1700p_{13}(x,1) + 2500p_{14}(x,1)\right](1.05)^{-1} + \\
&\quad \sum_{h=2}^{+\infty} \left[1000p_{12}(x,h) + 1700p_{13}(x,h) + 2500p_{14}(x,h)\right](1.05)^{-h}
\end{aligned}$$

where x refers to the age-group 75-84.

Since we have the values for the transition probabilities when $h = 1$, we replace them in the equation to get:

$$\begin{aligned}\beta_1^{LTC}(0,\infty) &= [1000 * 0.0145 + 1700 * 0.0049 + 2500 * 0.0082](1.05)^{-1} + \\ &\quad \sum_{h=2}^{+\infty}[1000p_{12}(x,h) + 1700p_{13}(x,h) + 2500p_{14}(x,h)](1.05)^{-h} \\ &= 41.2667 + \sum_{h=2}^{+\infty}[1000p_{12}(x,h) + 1700p_{13}(x,h) + 2500p_{14}(x,h)](1.05)^{-h}\end{aligned}$$

We can then express the premium as:

$$\begin{aligned}P\ddot{a}_{x:m\rceil}^{11} &= \beta_1^{LTC}(0,\infty) \\ &= 41.2667 + \sum_{h=2}^{+\infty}[1000p_{12}(x,h) + 1700p_{13}(x,h) + 2500p_{14}(x,h)](1.05)^{-h}\end{aligned}$$

and

$$\begin{aligned}\ddot{a}_{x:m\rceil}^{11} &= \sum_{k=0}^{m-1} v^k p_{11}(x,k) \\ &= v^0 p_{11}(x,0) + v^1 p_{11}(x,1) + \sum_{k=2}^{m-1} v^k p_{11}(x,k) \\ &= 1 + (1.05)^{-1} 0.909 + \sum_{k=2}^{m-1} v^k p_{11}(x,k) \\ &= 1.8657 + \sum_{k=2}^{m-1} v^k p_{11}(x,k)\end{aligned}$$

Therefore:

$$
\begin{aligned}
P\ddot{a}^{11}_{x:m\rceil} &= \beta_1^{LTC}(0,\infty) \\
P\left(1.8657+\sum_{k=2}^{m-1} v^k p_{11}(x,k)\right) &= 41.2667 \\
&+\sum_{h=2}^{+\infty}\left[1000p_{12}(x,h)+1700p_{13}(x,h)+2500p_{14}(x,h)\right] \\
&*(1.05)^{-h}
\end{aligned}
$$

Therefore:

$$
P=\frac{41.2667+\sum_{h=2}^{+\infty}\left[1000p_{12}(x,h)+1700p_{13}(x,h)+2500p_{14}(x,h)\right]}{1.8657+\sum_{k=2}^{m-1} v^k p_{11}(x,k)}(1.05)^{-h}
$$

We have,

$$
\begin{aligned}
V_t^1 &= \beta_1^{LTC}(t,\infty)-P\ddot{a}^{11}_{x+t:m-t\rceil} && \text{if } t<m \\
V_t^1 &= \beta_1^{LTC}(t,\infty) && \text{if } t\geq m
\end{aligned}
$$

We assume $t=0$ to get:

$$
\begin{aligned}
V_0^1 &= 41.2667+\sum_{h=2}^{+\infty}\left[1000p_{12}(x,h)+1700p_{13}(x,h)+2500p_{14}(x,h)\right](1.05)^{-h} \\
&\quad -P\left(1.8657+\sum_{k=2}^{m-1} v^k p_{11}(x,k)\right)
\end{aligned}
$$

V_t^2, V_t^3 and V_t^4 can be calculated using a similar argument and assuming $t=0$.

Illustration 3

Consider an individual in the 85+ age-group. Let $b^2 = 1000, b^3 = 1700$ and $b^4 = 2500$ and let the starting state be state 1. Assume further that the interest rate is 5% effective annually.

Now the actuarial value is given by:

$$\begin{aligned}
\beta_1^{LTC}(0,\infty) &= \sum_{h=1}^{+\infty}\left[b^2 p_{12}(x,h) + b^3 p_{13}(x,h) + b^4 p_{14}(x,h)\right](1.05)^{-h} \\
&= \sum_{h=1}^{+\infty}\left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right](1.05)^{-h} \\
&= \left[1000 p_{12}(x,1) + 1700 p_{13}(x,1) + 2500 p_{14}(x,1)\right](1.05)^{-1} + \\
&\quad \sum_{h=2}^{+\infty}\left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right](1.05)^{-h}
\end{aligned}$$

where x refers to the age-group 85+.

Since we have the values for the transition probabilities when $h = 1$, we replace them in the equation to get:

$$\begin{aligned}
\beta_1^{LTC}(0,\infty) &= \left[1000 * 0.0312 + 1700 * 0.0123 + 2500 * 0.0186\right](1.05)^{-1} + \\
&\quad \sum_{h=2}^{+\infty}\left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right](1.05)^{-h} \\
&= 93.9143 + \sum_{h=2}^{+\infty}\left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right](1.05)^{-h}
\end{aligned}$$

We can then express the premium as:

$$\begin{aligned} P\ddot{a}^{11}_{x:m\rceil} &= \beta_1^{LTC}(0,\infty) \\ &= 93.9143 + \sum_{h=2}^{+\infty}\left[1000p_{12}(x,h) + 1700p_{13}(x,h) + 2500p_{14}(x,h)\right](1.05)^{-h} \end{aligned}$$

and

$$\begin{aligned} \ddot{a}^{11}_{x:m\rceil} &= \sum_{k=0}^{m-1} v^k p_{11}(x,k) \\ &= v^0 p_{11}(x,0) + v^1 p_{11}(x,1) + \sum_{k=2}^{m-1} v^k p_{11}(x,k) \\ &= 1 + (1.05)^{-1}\, 0.8325 + \sum_{k=2}^{m-1} v^k p_{11}(x,k) \\ &= 1.7929 + \sum_{k=2}^{m-1} v^k p_{11}(x,k) \end{aligned}$$

Therefore:

$$\begin{aligned} P\ddot{a}^{11}_{x:m\rceil} &= \beta_1^{LTC}(0,\infty) \\ P\left(1.7929 + \sum_{k=2}^{m-1} v^k p_{11}(x,k)\right) &= 93.9143 \\ &\quad + \sum_{h=2}^{+\infty}\left[1000p_{12}(x,h) + 1700p_{13}(x,h) + 2500p_{14}(x,h)\right] \\ &\quad * (1.05)^{-h} \end{aligned}$$

Therefore:

$$P = \frac{93.9143 + \sum_{h=2}^{+\infty} \left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right]}{1.7929 + \sum_{k=2}^{m-1} v^k p_{11}(x,k)} (1.05)^{-h}$$

We have,

$$\begin{aligned} V_t^1 &= \beta_1^{LTC}(t,\infty) - P\ddot{a}^{11}_{x+t:m-t\rceil} && \text{if } t < m \\ V_t^1 &= \beta_1^{LTC}(t,\infty) && \text{if } t \geq m \end{aligned}$$

We assume $t = 0$ to get:

$$\begin{aligned} V_0^1 &= 93.9143 + \sum_{h=2}^{+\infty} \left[1000 p_{12}(x,h) + 1700 p_{13}(x,h) + 2500 p_{14}(x,h)\right] (1.05)^{-h} \\ &\quad - P\left(1.7929 + \sum_{k=2}^{m-1} v^k p_{11}(x,k)\right) \end{aligned}$$

V_t^2, V_t^3 and V_t^4 can be calculated using a similar argument and assuming $t = 0$.

We can perform similar calculations from different starting states.

Chapter 5

CONCLUSIONS AND RECCOMMENDATIONS

5.1 Conclusions

Multiple state models are effective in pricing LTC products because:

- Given a suitably specified model, they provide an accurate representation of the true process of insurance. In particular these models can account correctly for the effect of lives that recover and return to the population exposed to risk.
- Developing and testing these models can give a high level of understanding of the product, and particularly the possible outcomes that can occur over time and in particular scenarios.

On the other hand, these models have the following shortcomings:

- They can be complex to construct, difficult to maintain and take a considerable amount of computing power.

- They require a large number of assumptions and many of these assumptions are either unknown or poorly specified at the present time.
- The most commonly used multiple state models - Markov processes - do not deal well with durational variation in transition rates, which are common in long-term care.To do so either requires a great many individual states or a semi-Markov model.

5.2 Recommendations

We propose the following improvements to our work:

- For completeness, we propose that a longitudinal dataset be analyzed which has records of observations of initial and final states of individuals at different time intervals with each observation period being short enough for us to be able to assume that the transition intensities are piece-wise constant.
- A study on Semi-Markov modelling of multiple state models be done so as to allow for the durational variation in transition intensities.

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