INEQUALITIES AND SPECTRAL PROPERTIES OF SOME CLASSES OF OPERATORS IN HILBERT SPACES

By;

Imagiri Kathurima Stanley

A thesis submitted in fulfilment for the requirements of the award for the degree of Doctor of Philosophy in Pure Mathematics of the University of Nairobi.

School of Mathematics, University of Nairobi
Nairobi, Kenya.
October, 2014.

DECLARATION

This thesis is my original work and has not been presented for a degree award in any other university.

S. K. Imagiri

This thesis has been submitted for examination with our approval as university supervisors.

Signature——Date——-   Signature——Date———-

Prof. J. M.Khalagai   *Prof. G. P. Pokhariyal*

AKNOWLEGDEMENTS

Above all, I thank the God Almighty for His faithfulness. It is the Lords doing and it is marvellous in my eyes.

My sincere gratitudes are due to Prof. J.M. Khalagai and Prof. G.P. Pokhariyal. They not only supervised me at M.Sc level, but also taught me all through my undergraduate stage. I am extremely grateful to these two for their patience, perseverance and understanding which, I am sure, have been tested extensively by my sometimes inertial approach to work. In addition, their approachability and sure grasp of a diverse spectrum of functional analysis and operator theory have made them ideal supervisors.

I am also indebted to the University of Nairobi for offering me the Tutorial Fellowship without which, i could not have gone this far.

CONTENTS

CHAPTER ONE: INTRODUCTION

CHAPTER TWO: NORMALITY OF THE PRODUCTS OF NON-NORMAL OPERATORS

CHAPTER THREE: INEQUALITIES AND SEQUENCES OF w-HYPONORMAL OPERATORS

CHAPTER FOUR: EXTENSIONS OF THE PUTNAM-FUGLEDE THEOREM, THE PUTNAM'S AND THE BERGER-SHAW INEQUALITIES

CHAPTER FIVE: CONCLUSIONS AND SUMMARY

ABSTRACT

Every normal operator is diagonalizable and every reducible operator can be expressed as a direct sum decomposition of a normal and a pure operator. Furthermore, any two similar operators have the same spectrum. The Putnam-Fuglede theorem implies similarity and both the Putnam's inequality and the Berger-Shaw inequality are measures used to conjecture how far a given linear operator is from being normal. Boundedness of the self commutator of an operator, also follows from these two inequalities. If the self commutator is bounded, the operator in question becomes reducible, hence decomposable and eventually diagonalizable.

In this thesis, through normality, diagonalizability of n-Power normal, n-Power quasinormal and that of w-hyponormal operators was investigated. In addition, three different operator inequalities, that is, the Putnam-Fuglede theorem, the Putnam's inequality and the Berger-Shaw inequality, were studied for n-Power normal, n-Power quasinormal and w-hyponormal operators.

The main tools used are such as the Lowner-Heinz inequality, the Furuta's inequality, polar decompositions, Aluthge decompositions, direct sum decompositions, matrix decompositions, the kernel condition, similality and quasi-similality.

DEDICATIONS

This work is dedicated to my son, Victor.

Chapter one

INTRODUCTION

In this chapter, a brief history of Hilbert spaces and developments of operator theory is provided. Linear operators are then introduced and their representation by a matrix is demonstrated. Then, the reasons why every normal operator is diagonalizable are explained. Various decompositions of completely non normal operators, especially those which lead to characterizing sufficient conditions under which non normal operators become normal are showcased. After thorough discussions on n-Power quasinormal, w-hyponormal, Putnam-Fuglede theorem, Putnam's and Berger-Shaw inequalities, the problem solved by this thesis is presented. Further, definitions and notations used in this thesis are given and analysis of different classes of operators together with their series of inclusions are outlined. To wind up this chapter, well known results about inequalities satisfied by matrices and consequently, by all operators in general, all of which were important in proving results in different chapters, are introduced.

1.1 Developments of operator theory in Hilbert spaces

Mathematics is useful because it is related to the world in which we live. Elementary mathematics deals with numbers. Anything which has magnitude, or size, can be expressed mathematically in form of numbers. For example the strength of wind is a magnitude and is described by a number, but the wind also has a direction, which is just as important. A vector is a combination of a magnitude and a direction. Examples of vectors are such as displacement, velocity and force. The main thing about vectors is that they can be added together, or even multiplied by a number, giving another vector in each case. Roughly, any set of objects with this property is called a vector space. If we let V to be a vector space over the field of complex numbers C, an inner product on V is a function

$$<,>: V \times V \to C$$

that is both sesquilinear and conjugate-symmetric. In other words, $<,>: V \times V \to C$, is an inner product over a vector space V if given any three objects x, y, z in V and a complex number c, $<,>$ satisfies the following four conditions;

(i) $< x, x > \geq 0$;

(ii) $< x + y, z > = < x, z > + < y, z >$ and $< x, y + z > = < x, y > + < x, z >$;

(iii) $< cx, y > = c < x, y >$ and $< x, cy > = \bar{c} < x, y >$;

(iv) $< x, y > = \overline{< y, x >}$.

A vector space with such an inner product, is known as an inner product space and every complete inner product space is called a Banach space. If the inner product in V induces another function in V, called the norm and denoted by $\|\|$, where $\|x\|^2 = < x, x >$, for each vector x in V, and if in addition, V happens to be complete with respect to this norm, then V is called a Hilbert space. In other words, a Hilbert space is a special kind of a Banach space where the norm is infact an inner product. Thus all vector spaces are not Hilbert spaces. In particular, not every Banach space is a Hilbert space.

The inner product is said to be positive semidefinite, or simply positive, if $\|x\|^2 \geq 0$ always(here, we note that (by i), $\|x\|^2 = 0$ if $x = 0$), and definite if the converse holds. In general, an inner product is said to be positive definite, if it is both positive and definite. It follows that, every complete normed space which is positive definite is a Hilbert space. Therefore, a Hilbert space is simply a vector space equipped with a complete positive definite inner product.

If an inner product is positive, then we can take the principal square root of $\|x\|^2 = < x, x >$ to get the real number $\|x\|$, that is, the norm of x. This norm satisfies all of the requirements of a Banach space. It additionally satisfies the parallelogram law. That is,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

which not all Banach spaces need satisfy. (The name of this law comes from its geometric interpretation: the norms in the left-hand side are the lengths of the diagonals of a parallelogram, while the norms in the right-hand side are the lengths of the sides.)

Furthermore, any Banach space satsifying the parallelogram law has a unique inner product that reproduces the norm, defined by

$$< x, y > = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 - \mathrm{i}\|x + \mathrm{i}y\|^2 + \mathrm{i}\|x - \mathrm{i}y\|^2),$$

or $\frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$ in the real case. Therefore, it is possible to define a Hilbert space as a Banach space that satisfies the parallelogram law.

Hilbert spaces are useful in operator theory, since for instance, if we let H to be a Hilbert space, then for a fixed $y \in H$, the expression $< x, y >$, assigns to each $x \in H$ a number. An assignment F of a number to each element $x \in H$ is called a functional. It follows that $|F(x)| \leq m\|x\|$, for some positive number m and $F(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F(x_1) + \alpha_2 F(x_2)$, for any pair of scalars α_1, α_2 and any two objects x_1, x_2 in H. In other words, for a fixed y, $F(x) =< x, y >$, is a bounded linear functional on the Hilbert space H. It follows that for every bounded linear functional F on H, there exists a unique element y in H, such that $F(x) =< x, y >$, for each x in H. Thus, all bounded linear functionals on a given Hilbert space are just scalar products. This qualification fails in general in Banach spaces. More over, if the dimension of the Hilbert space H under consideration is finite, then every linear operator T on H, has a unique adjoint, T^*. It follows that, $(T^*)^* = T$, $(AB)^* = B^*A^*$, for any other linear operators A, B on H.

In mathematics, operator theory is the branch of functional analysis that focuses on bounded as well as unbounded(but closed), linear operators. This theory also includes the study of algebras of operators. A notable sub branch of operator theory is the single operator theory, where one deals with the properties and classifications of single operators. For example, the classification of normal operators in terms of their spectra falls into this category.
The original model for operator theory is the study of matrices. Although the word 'matrix' was only coined by [Silvester, 1850, [42, Pg 3]], matrix methods have been around for over 2000 years, as attested by the use of what we refer today as the Gaussian elimination method which was first used by [Chinese during the Han Dynasty, 200 B.C. [69, Pg 14]]. In the process of finding normal forms for quadratic functions, [Cauchy, 1826, [90, Pg 29]], discovered eigenvalues and generalizations of square matrices. Cauchy, 1826, also proved the spectral theorem for self adjoint matrices. That is, every real symmetric matrix is diagonalizable. This spectral theorem for Hermitian matrices was later generalized into spectral theorem for normal operators, that is, every normal operator is diagonalizable, [Neumann, 1942]. This simple observation by Neumann, happens to be the most important result in operator theory.
Today, many branches of analysis are insperable from operator theory, notably; differential equations, founded by [Bernoulli family, 1729, [101]]; variational calculus, which was first created by [Euler, 1750, [101]] and the transform theory, which was introduced by [Lagrange, 1782,[101]]. Since all these theories predate operator theory as such by a century or two, it is no suprise that the earliest anticident of operator theory are to be found in them. Unfortunately, the early creators of variational calculus did not avail themselves of operators as abstractly concieved. For instance, one must realize that the technique of calculating the first variation of a functional is a kind of differentiation in a space of functions and that the deriva-

tive in this context, is a linear operator.

The subject of operator theory and that of its subset, spectral theory, came into focus rapidly after 1900. Amajor event was the appearance of Fredholm's theory of integral equations. In a preliminery result based on his dissertation, [Fredholm, 1903, [101]], gave a complete analysis of an important class of integral equations which today are known as Fredholm equation's or Fredholm operator's. An year earlier, [Lebesque, 1902, [101]], had introduced the most important spaces of functions, denoted in his honor as L^p. At about this time, [Hilbert, 1902,[101]], founded the modern spectral theory in a series of articles inspired by Fredholm's work. Like Fredholm, Hilbert begun with the specific idea of integral equations, and noticed that he could obtain more precise results when the space of functions considered was the space of the square integrable functions(L^2), especially when the integral operator was symmetric. This was the discovery of Hilbert spaces and founding of self adjoint operators. Henceforth, operator theory has been the study of bounded or unbounded linear operators on Hilbert spaces. But one might wonder why pay much attention to linear operators.

Linear maps are important since they map straight lines to straight lines while non linear maps, map straight lines into curves. Let us for example consider elementary calculus. Here one deals with all kinds of functions, not just linear ones. One way of analysing a function is to differentiate it. One does so inorder to approximate the graph of a function by the tangent, where the approximation is useful near the line of tangency. Thus the graph is replaced by a straight line, which means the function is approximated by a linear function. A specific application of linear maps is for geometric transformations, such as those performed in computer graphics, where the translation, rotation and scaling of $2D$ or $3D$ objects is performed by the use of a transformation matrix. Linear mappings also are used as a mechanism for describing change. For example, and as we saw earlier, in calculus, correspond to derivatives, or in relativity theory, linear operators are used as devices to keep track of the local transformations of reference frames.

1.2 Linear operators and their representations by Matrices

A linear transformation is a mapping from one vector space to another that preserves vector addition and scalar multiplication. A linear transformation from a vector space to itself, is called a linear operator. As a result, a linear operator is a transformation which maps linear subspaces to linear subspaces, like straight lines to straight lines or straight lines to a single point. In other words, a linear operator is a mapping from a vector space to itself that is compatible with the linear structure of the space. [Pearson, 1898, [32]], gave the first modern definition of a linear operator. The concept of a linear operator, which together with the concept of a vector space, plays a role in very diverse brances of mathematics and physics. In the theory of linear operators, the major problem is that of approximating various classes of linear operators by operators of comparatively simple structures such as self-adjoint and normal operators. The main analytic apparatus for linear operators is the matrix notation since it is known that, [Toeplitz, 1909, [32]], every linear operator can be represented by a matrix. If V and W are two vector spaces of finite dimension, and one has chosen bases in those spaces, then every linear map from V to W can be represented as a matrix.

To demosntrate that, infact it is possible to represent every linear operator by a matrix, consider a linear map f from a vector space V to a vector space W, and choose $\{v_1, v_2, --, v_n\}$, as the bases for V, and $\{w_1, w_2, --, w_m\}$, as the bases for W. Then, for every v in V, there exists real numbers $\{\alpha_1, \alpha_2, -----, \alpha_n\}$ such that, v can be expressed as, $v = \alpha_1 v_1 + \alpha_2 v_2 + ------- + \alpha_n v_n$. Thus, $f(v) = f(\alpha_1 v_1 + \alpha_2 v_2 + ------- + \alpha_n v_n) = f(\alpha_1 v_1) + f(\alpha_2 v_2) + ------- + f(\alpha_n v_n) = \alpha_1 f(v_1) + \alpha_2 f(v_2) + ------- + \alpha_n f(v_n)$, which implies that, the linear map f is entirely determined by the values $f(v_1), f(v_2), ----, f(v_n)$. Therefore, for each $f(v_j)$, there exists real numbers, $\{\beta_1, \beta_2, ----, \beta_m\}$, and $f(v_j)$ has the representation, $f(v_j) = \beta_{1j} w_1 + \beta_{2j} w_2 + ----- + \beta_{mj} w_m$, so that f is entirely determined by the values β_{ij}. Thus, putting these values into an $m \quad by \quad n$ matrix, we can use it to compute the value of f for any vector in V.

It is crucial to note that, since a linear operator maps elements from a given space into itself, then $V = W$, implying, $n = m$, and thus every linear operator can be represented by a square matrix, say T. This T is useful since it allows concrete calculations. Conversely, matrices yield examples of linear maps. A linear operator is said to be bounded if its domain is the whole vector space and in addition if it does not lengthen any vector in this space to infinity. That is, if a linear operator acting on a Hilbert space H, has a matrix representation T, then such an operator is bounded if $D(T) = H$, and there exists a positive real number m, such that, $\|Tx\| \leq m\|x\|$, for every vector x in H, where $D(T)$ denotes the domain of T.

If T is the matrix representation for a given linear operator on H, and the action of this T to a vector say, x in H, is equivalent to multiplying such an x by a number say, λ. That is, $Tx = \lambda x$, then x is called an eigenvector of T corresponding to an eigenvalue λ. The eigenvalues of a linear operator are the roots of the charactestic polynomial of its matrix T, while the spectrum of a linear operator on a finite dimensional Hilbert space, is the set of all of its eigenvalues. We note that, since every linear operator can be represented by a square matrix, say T, and

every square matrix is an example of a certain linear operator, without loss of generality, we henceforth stick to the convention, 'let T be a linear operator', rather than the former version, 'let T be the matrix representation for a linear operator'. We also note that, the eigenvalues of a bounded linear operator T, on a finite dimensional Hilbert space, yields the spectrum of T. Thus, eigenvalues are the major ingredients used in the study of linear operators in operator theory. One might as well note that, if T is a diagonal matrix, then its eigenvalues and eigenvectors follow trivially, One can raise T to any power, by simply raising the diagonal entries to that same power. Geometrically, a diagonal matrix scales the space by a different in each direction, determined by the scale factor on each axis(diagonal entries). Therefore, it is a dream for many researchers, that T is a diagonal matrix, or if not, then atleast T is diagonalizable.
We note that, a matrix T is diagonalizable if it is similar to a diagonal matrix. That is, if there exists an invertible matrix P, such that, $P^{-1}TP$ is a diagonal matrix. That is, $P^{-1}TP = [\Lambda]_i$, where, $[\Lambda]_i$ is a diagonal matrix with non zero entries $\lambda_i's$ on its main diagonal and zeros everywhere. If we let, $P^{-1}TP = [\Lambda]_i$, then we find that, $TP = P[\Lambda]_i$. And, if we allow P to be the square matrix with the column vectors, $(p_1, p_2, ----, p_n)$, then it follows that, $Ap_i = \lambda_i p_i$. So that, the column vectors of P are the eigenvectors of T, and the corresponding diagonal entry is the corresponding eigenvalue, while P^{-1} is the matrix obtained by taking the transpose of the matrix with the eigenvectors of P^* as its columns. It is known that, self-adjoint and normal operators are diagonalizable. In particular, every normal operator is diagonalizable.
A bounded normal operator is a bounded linear operator on a Hilbert space H, which commutes with its adjoint, [Embry, 1966]. That is, a linear operator T on H, is said to be normal if $T^*T = TT^*$. Or T is normal if and only if $\|Tx\| = \|T^*x\|$, for each x in H. It is a remarkable fact that, this simple algebraic condition is strong enough to ensure that a normal operator is, when the ambient Hilbert Space is transformed by an isometric isomorphism, similar to the multiplication by a function on an $L^2 space$. One admirable characteristic of normal operators is that, each one of them has a simple spectrum. Thus, when considered as a single operator, a normal operator has the best spectral theory one might expect.
Putting this in another way, it is entertaining and somehow easier to deal with normal operators, since they 'behave normaly'. Of interest, is the well known fact that, [Halmos, 1967], if an operator say, $T \in B(H)$ is normal, then its operator norm, $\|T\|$, is equal to its largest eigenvalue. In addition, it is known that, [Halmos, 1967], this largest eigenvalue happens to be the same as either its spectral radius, $r(T)$, or its numerical radius, $w(T)$. It also follows that, if T is a normal operator, then, T and its adjoint T^* have the same range and the same kernel. That is, $R(T) = R(T^*)$ and $ker(T) = ker(T^*)$. Consequently, $R(T)$ is dense in H if

and only if T is injective. In other words, the kernel of a normal operator is the orthogonal complement of its range. Thus, every generalized eigenvalue of a normal operator is genuine. That is, a number λ, is an eigenvalue of a normal operator T if and only if its complex conjugate $\bar{\lambda}$ is an eigenvalue of T^*. Also, eigenvectors corresponding to different eigenvalues of a normal operator are orthogonal. More importantly, and one of the qualifications which make normal operators the target of any analyist, and a satisfaction which fails almost every other operator, is the fact that, T and T^* have the same eigenvectors. This property of normal operators, together with the characteristic that, $\|Tx\| = \|T^*x\|$, for each x in H, guarantees the existence of another operator U, such that, $T = UAU^*$, where U is the matrix whose diagonal entries are the eigenvectors of T, A is a diagonal matrix consisting on its main diagonal, the eigenvalues of T, and U^* is the transpose of U. That is, U^* is the matrix consisting of the eigenvectors of T as its rows. This matrix U, is unitary. Thus, $U^* = U^{-1}$. Hence, U is invertible.

In conclusion, if T is a normal operator, then there exists another invertible operator U such that, $T = UAU^{-1}$. And therefore, every normal operator is diagonalizable.

Naturally, problems begins immediately after one gets out of the class of normal operators since it is known that, [Halmos, 1982], completely non-normal operators are not diagolizable in general. This drawback, has led into various screenings and thus groupings of non-normal operators into different classes such as, the hyponormal, p-hyponormal, n-Power quasi normal, w-hyponormal operators, etc, but all of which contain normal operators. Conditions under which these non-normal operators become normal, hence diagonalizable, have attracted numerous attentions in operator theory. However, odds for one to come across a non-normal operator which happens to be not user friendly, are always high. Besides, these generalizations of normal operators into larger and larger classes, is not going to stop soon. It is therefore necessary to develop several alternative techniques of expressing non-normal operators into normal-operators.

1.3 Methods Leading To Normality

In the previous section, we noted that large classes of operators, especially the n-Power quasi normal and the w-hyponormal operators are not diagonalizable in general. These classes are non-normal, but atleast they include all normal operators. The problem of characterizing conditions under which these classes become diagonalizable reduces to that of investigating those which restricts them to the class of normal operators. Some of the familiar approaches of 'loosening' such non-normal operators has been that of, first, obtaining their polar decomposition.

Before we demonstrate this type of decomposition, it is good not to over look one of the earliest methods of breaking any bounded linear operator. That is, the cartesian form of an operator. In other words, if T is a bounded linear operator on an Hilbert space H, then there exists self-adjoint operators A and B on H, such that, $T = A + iB$. It follows that, $||T||^2 \leq ||A||^2 + ||B||^2$ and $|det\ T| \geq detA$.

Also, for operator T, the operator T^*T is always positive and its unique positive square root is denoted by $|T|$. The eigenvalues of $|T|$, counted with multiplicities are called the singular values of T. It is known that, [Gantmaher et al, 1930], for every normal operator $T \in B(H)$, there exists unitary operators W and Q such that $W^*TQ = S$, where S is the diagonal matrix whose diagonal entries are the singular values of T. This type of 'breaking' T, is called the singular value decomposition. Equivalently, if T is a normal operator, then $T = WSQ^*$. This characterization means, in other words that, every normal operator is diagonolizable. It happens to be the major advantage of dealing with normal operators, for it was shown that, [Anderson, 1973], if f is an operator monotone function, then $f(T) = W(f(S))Q^*$. More over, if T is a normal operator, then the largest singular value of T happens to be equal to the norm of T. But this qualification is violated by non-normal operators in general.

The singular value decomposition leads to another form of decomposition, popularly known as the polar decomposition, That is, [Gantmaher, 1930], every operator T can be written as $T = UP$, where U is a partial isometry and P is positive. In this type of decomposition, the positive part P is unique, U is unique if T is invertible, the kernels of both T, U and P are the same and U is unitary if T is normal. It is well known that, [Neumann, 1942], T is normal if and only if U and P commute. This commutativity qualification by normal operators is not satisfied by non-normal operators in general. However, one can easily note that, if a non-normal operator T, (on a finite dimensional Hilbert space), with the polar decomposition $T = UP$, is such that, $UP = PU$, then such an operator happens to be normal. This type of decomposition can also be used to characterize more sufficient conditions for a non-normal operator to be normal. For example, [Moslehian, et al. 2011], noted that for a log-hyponormal or a p-hyponormal operator T, with the polar decomposition $T = UP$, then T is normal if there exists a positive integer n, such that $U^n = U^*$. But w-hyponormal operators contains both p- and log-hyponormal operators, while n-Power quasi-normal operators does not share much with both p- and log-hyponormal operators. It is therefore sensible to ask whether such characterization also guarantees normality for members from these two classes.

In the quest to come up with an easier method of conquering complicated operators, some analysts have, instead of 'decomposing', transformed them into other forms with common major properties, but easier to handle. Aluthge, 1990, generalized the polar decomposition by transforming an operator T into another operator $\tilde{T}$ called the Aluthge transfom of T. That is, an operator $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ is called the Aluthge transformation of T whose polar decomposition is $T = U|T|$, where $|T| = (T^*T)^{1/2}$. More precisely, $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ is called the first Aluthge transform and $\tilde{\tilde{T}} = \tilde{T}_2 = |\tilde{T}|^{1/2}U|\tilde{T}|^{1/2}$, is called the 2^{nd} Aluthge transform of T. In general, $\tilde{T}_n = |\tilde{T}_{n-1}|^{1/2}U|\tilde{T}_{n-1}|^{1/2}$, for every natural number n, is called the n^{th}-Aluthge transform of T, where $\tilde{T}_n = \tilde{U}_n|\tilde{T}_n|$ is the polar decomposition of $\tilde{T}_n$. Infact, $T = |\tilde{T}_0|^{1/2} + |\tilde{T}_0|^{1/2}$, was called the null-Aluthge transform of T. It follows that, [Aluthge, et al. 1999], $\|\tilde{T}_{n-1}\| \geq \|\tilde{T}_n\|$, in general. More importantly, the spectrum of T is invariant under the first two Aluthge transformations. That is, T, $\tilde{T}$ and $\tilde{T}_2$, have the same set of eigenvalues. Aluthge transforms of different classes of operators have been studied extensively. For instance, [Patel, 1996], proved that if T is p-hyponormal and $\tilde{T}$ is normal, then $T = \tilde{T}$. In the same paper, he noted that, Aluthge transformations of an operator 'improves' p-hyponormality of that operator for $p \leq 1$, by proving that, if T is p-hyponormal for $1/2 \leq p < 1$, then $\tilde{T}$ is hyponormal and, if T is p-hyponormal for $0 < p < 1/2$, then $\tilde{T}$ is $p + 1/2$-hyponormal. [Aluthge, et al. 2000], generalized Patel's result by showing it holds true for w-hyponormals. That is, if T is w-hyponormal and $\tilde{T}$ is normal, then $T = \tilde{T}$. They also proved that, if T is a normal operator, then its first Aluthge transform is also normal and that, T is invertible if and only if its first Aluthge transform is invertible. It was shown that, [Aluthge, et al. 2000], if an operator T is w-hyponormal hyponormal, then $\tilde{T}$ is semi-hyponormal, $\tilde{\tilde{T}}$ is hyponormal. In light of this, it is natural for one to ask whether these observations holds true for generalized Aluthge transforms of w-hyponormal operators.

Another common feature of relaxing complicated operators, is that of performing a 'spectral surgery'. That is, investigating the structure of the spectrum of an operator in question, since it is known that, [Kaplansky, 1953], the spectral space of any class of operators includes properly, the spectral space of all operators from its sublasses. It was noted that, [Dunford, 1954], the spectrum of a self-adjoint operator lies along the real line, that of a unitary lies on the unit circle, that of a projection consists of the points 0 and 1, and that of a normal operator can be any compact set in the complex plane. However, not much is known about the locations of the spectra of large classes of operators such as n-power quasi-normal and w-hyponormal operators. Thus, given any class of completely non-normal operators, one would like to investigate the location of its spectra.

The structure of the range and that of the kernel of a linear operator also play a big role in analysing linear operators in Hilbert spaces. It was shown that, [Fuglede, 1950], the range of any linear operator is always orthogonal to the kernel of its self adjoint. Thus, it trivially follows that, if T is a bounded linear operator on an Hilbert space H, and the kernel condition holds for such T, that is $ker(T) \subset ker(T^*)$, then the range of T happens to be orthogonal to the kernel of T. Consequently, [Putnam, 1951], the kernel of T reduces T. It is also known that, [Colojoafa, et al. 1969], every Hilbert space H can be decomposed as; $H = \bar{R(T)} \oplus \bar{R(T)}^{\perp}$ or; $H = \bar{ker(T)} \oplus \bar{ker(T)}^{\perp}$. In this book, he proved that if T is a normal operator, then $ker(T) = ker(T^*)$. Huruya, 1997, proved that if T is p-hyponormal, then $ker(T) \subset ker(T^*)$. Tanahashi, 1999, noted that if T is log-hyponormal, then $ker(T) = ker(T^*)$. It is known that, [Aluthge, et al. 2000], the kernel condition does not hold in general for w-hyponormal operators. However, by combining the kernel condition and the spectral surgery for w-hyponormals, they proved that any w-hyponormal operator which satisfies this condition, happens to be normal if the planar lebesque measure of its spectrum is zero. Therefore, it is important to note that, the study of the shapes of these two structures for members from any large class of completely non-normal operators, can not be belittled.

Another approach of investigating the normality of a given non-normal operator has been that of checking whether such an operator is a non-normal part of a normal operator. One, first multiplies such an operator by another operator from the same class, and then characterizes conditions which imply normality of the product. In other words, given any two normal or non-normal operators, say A and B on H, one might be interested in obtaining sufficient conditions for the operator AB to be normal. The question of characterizing those pairs of normal operators for which the products become normal has been solved for finite dimensional spaces by [Gantmaher et al, 1930] and for compact normal operators by [Wiegmann, 1949]. Actually, in the afore mentioned cases, the normality of AB is equivalent to that of BA. A more general result by [Kittaneh, 1987], implies that, it is sufficient that AB be normal and compact to obtain that BA is also normal. We note that, the product AB becomes the square of A if $A = B$. It is good to note that, if T is a normal operator, then T^n is not normal for every positive integer n. Some of the generalizations of normal operators have inherited this behaviour while some others have not. For example, [Halmos(pbm 164), 1967], gave an example of a hyponormal operator T for which T^2 is not normal. Yanagida, et al. 1999, noted that this behaviour of normal and hyponormal operators is retained by p-hyponormal operators by proving that, if if T is a p-hyponormal operator, for some $p \geq 0$, then T^n is not p-hyponormal for every positive integer n, but p/n-hyponormal. However, this situation is different for both log- and w-hyponormal operators since, [Yanagida, 1999], proved that, if T is log-hyponormal , then T^2 is log-hyponormal and [Yanagida, 2002], showed that, if T is a w-hyponormal operator, then T^n is also w-hyponormal for every positive integer n. By use of generalized powers, several sufficient conditions for a non-normal operator to be normal, have been given by a number of authors. For instance, [Ando, 1972], proved that, if T is a paranormal operator and there exists atleast one positive integer n such that T^n is normal, then T is normal. Aluthge, et al. 2000, showed that Ando's result holds true for w-hyponormal operators. Since both n-Power quasinormal and generalized Aluthge transforms of w-hyponormal operators extends normal operators, it is fundamental to inquire whether similar results holds in these classes.

In addition to the above mentioned methods, the classification of the adjoint, of a non-normal operator, might as well be used to tell whether such an operator is diagonolizable or not. We note that, if T is an operator on H, then its adjoint, that is T^*, trivially satifies; If T is bounded and linear, then T^* is also bounded and linear, $||T|| = ||T^*||$,
, $(T_1 + T_2)^* = T_1^* + T_2^*$, $(T^*)^* = T$, and $(T_1T_2)^* = T_2^*T_1^*$. Majority of the classes which includes normal operators such as the p-hyponormal, n-Power quasinormal and w-hyponormal operators, share a common property of sometimes rejecting T^* after accepting T. It follows that, whenever a non-normal class accepts both T and T^*, it might face the consequences of being restricted back to the class of normal operators. A good example was a result by [Aluthge, 1990], which showed that, if both T and T^* are p-hyponormal for some $p \geq 0$, then T is normal. Aluthge, et al. 2000, later extended this observation and noted that, if both T and T^* are w-hyponormal and either $ker(T) \subset ker(T^*)$ or $ker(T^*) \subset ker(T)$, then T is normal. Recently, [Ahmed, 2011], proved that if both T and T^* are n-Power quasinormal for the same positive integer n, then T^n is a normal operator. Thus, it is crucial for one to keep an eye on the behaviour of T^* while investigating the properties of T.

1.4 Preliminaries

Not every non normal operator is diagonalizable. And since, normal operators are diagonalizable, it is therefore important to come up with several alternatives which imply normality for such operators. Examples of non normal and hence non diagonalizable operators are such as n-Power normal, n-power quasinormal and w-hyponormal operators. Similar operators share the same spectrum, and unitary equivalence preserves reducibility, while quasi-similality, atleast preserves invertibility. The putnam-Fuglede theorem is a tool used to study similality or even quasi-similality between a given pair of linear operators. On the other edge, direct sum decompositions of linear operators breaks-up a given operator into simpler structured parts. In simpler terms, the Putnam's and Berger-Shaw inequalities imply existence of non-trivial invariant subspaces for the linear operator under consideration.

One of the main target of this thesis was to come up with more and more sufficient conditions which imply normality and reducibility for n-Power normal, n-power quasinormal and w-hyponormal operators. In this section, we first discuss these classes and then present the Putnam-Fuglede theorem, the Putnam's inequality and the Berger-Shaw inequality.

1.4.1 n-Power normal and n-power quasinormal operators

The most known immediate extension of normal operators is the class of quasinormal operators. These operators, properly contains all normal operators but are themselves non normal, hence not diagonalizable in general. Since their introduction, the major interest has been that of characterizing sufficient conditions under which these operators become normal. Every quasinormal operator is hyponormal. However, quasinormal operators, are not the only immediate generalizations of normal operators. The classes of n-Power normal and n-Power quasinormal operators also includes normal operators, but are indepedent from hyponormal operators.

Quasinormal operators were introduced by [Stampfli, 1936], n-Power normal by [Jibril, 2007], and n-Power quasinormal by [Sid Ahmed, 2011]. These classes have also been studied by a number of authors. Some of these authors include; [Bala, 1977], [Jeon et al, 2002], [Mecheri, 2005], [Jibril, 2008], [Dragmor et al, 2008], [Moslehian et al, 2011], and recently, by [Panayan, 2012]. [Bala, 1977], was amongst the earliest researchers to study quasinormal operators. Bala proved that, every normal operator is a quasinormal operator and gave some examples of quasinormal operators which are not normal. [Jeon, 2002], and [Mecheri, 2005], also, but indepedently, studied quasinormal operators and characterized several conditions under which these operators become normal. After introducing n-Power normal operators, [Jibril, 2007], proved that this class of operators is not an extension of the quasinormal operators, neither is the class of quasinormal operators an extension of this class. However, [Sid Ahmed, 2011], after introducing n-Power quasinormal operators, studied the relationships between these operators and either, quasinormal or n-Power normal operators. He proved that, if an operator is n-Power normal, for some positive integer n, then the nth power of this operator is normal and conversely. In general, he proved that every n-Power normal operator is a n-Power quasinormal operator. Thus, the class of n-Power quasinormal operators is very large. Even before the introductions of both n-Power normal and n-Power quasinormal operators, conditions under which powers of quasinormal operators become relaxed to powers of normal operators, had been studied by [Mecheri, 2005]. In this paper, the author proved that, the kernel condition does not hold in general in this class, but every n-Power quasinormal operator is n-Power normal, for the same integer n, whenever this condition is satisfied. Ahmed, 2011, generalized results by Mecheri and Jibril. Amongst other beautiful observations, Ahmed proved that; if any operator and its adjoint are both n-Power quasinormal, then the nth power of such an operator is normal; if any operator and its square are both n-Power quasinormal and if in addition, such an operator is in the class of 3-Power quasinormal, then its square happens to be a quasinormal operator and he also in the same paper gave an example of a 2-Power quasinormal operator which is not 3-Power quasinormal. As far as the spectra of these operators is concerned, [Ahmed, 2011], also proved that if an operator is 2-Power quasinormal, then such an operator is also normal provided zero is an isolated point in its spectrum.

If T is a n-Power normal operator, we write $T \in nN$, and $T \in QN$ or $T \in nQN$, if T is a quasinormal or n-Power quasinormal operator respectively. Consequently, $T \in nN$, means T^n commutes with T^* and $T \in nQN$, means T^n commutes with (T^*T). It therefore follows that, if $T \in 3N$, then, T^3 commutes with T^* and, if $T \in 3QN$, then T^3 commutes with (T^*T). Generally, the following inclusion series holds and are known to be proper;

$$(i) \qquad (nN) \subset (nQN)$$

$$(ii) \qquad (QN) \subset (nQN).$$

However, it is crucial to note that, the inclusion series above, hold in general, only for the same positive integer n. In other words, it does not follow in general that, $T \in nQN$, whenever, $T \in (n-1)N$, for any positive integer n. That is, $(nN) \subset (nQN)$, does not imply that, $((n-1)N) \subset (nQN)$, for every positive integer n. For instance, one should not always expect T^4 to commute with (T^*T), even when $T \in 3N$. In addition, it is also good to note that the inclusion series (ii) above is strictly one sided. In other words, existence of a positive integer n such that, T^n commute with (T^*T) does not imply that, T commutes with (T^*T) in general. It was also observed that;

(i) existence of some n satisfying $[T^n, T^*] = 0$, does not generally imply that, $[T^{n-1}, T^*] = 0$, and conversely, [Jibril, 2007], and that;

(ii) existence of some n satisfying $[T^n, (T^*T)] = 0$, does not generally imply that, $[T^{n-1}, (T^*T)] = 0$, and conversely, [Sid Ahmed, 2011]. However, Jibril proved that (i) holds if both T and T^* are n-Power normal operators and [Sid Ahmed, 2011] extended Jibril's result by observing that

(ii) also holds if both T and T^* are n-Power quasinormal operators.

It is at this stage that our deliberations in this thesis, concerning normality of non normal operators start. Recall that, every normal operator is a n-Power normal or a n-Power quasinormal operator, but the reverse inclusion does not hold in general. In this thesis, results about when these operators become normal are proved. For instance, it is proved that, if a n-Power quasi normal operator T is such that, its adjoint commutes with its square, then T is a normal operator. And also, after picking either two n-Power normal or two n-Power quasinormal operators, conditions under which their products become normal are discussed. For example, it is shown that, if any two commutative n-Power quasinormal operators, say A and B are such that, both have the kernel condition, then all powers of their products are normal operators.

1.4.2 w-hyponormal operators

In addition to the classes studied in the subsection above, the research on some operator classes which includes normal operators on a complex Hilbert space H, has been developed by many authors. Especially, the classes of subnormal, hyponormal, log-hyponormal, p-hyponormal and paranormal operators are very famous. It is well known that, every normal operator has the spectral decomposition, and then the structure of normal operators is well-known. The structure of quasinormal operators is also known as a direct sum of a normal operator and an operator valued weighted shift, [Brown, 1978]. It is also well known that, every subnormal operator has a nontrivial invariant subspace, [Brown, 1978]. On the other hand, there are a lot of problems about hyponormal operators and their generalizations. For example, it is not known whether any hyponormal or p-hyponormal operator has a nontrivial invariant subspace or not.

Aluthge transform of an operator T, that is, $\tilde{T}$, and which we defined as, $\tilde{T} = |T|^{1/2}U|T|^{1/2}$, where T has the polar decomposition, $T = U|T|$, and $|T| = (T^*T)^{1/2}$, has been used in the research on semi-hyponormal, log-hyponormal and p-hyponormal operators. In these studies, alot of interest has been paid not only to the first, but to all Aluthge transforms of a given linear operator, which has been simply refered to as, the Aluthge sequences of such an operator. That is, operator sequences of iterated Aluthge transforms. More precisely, $\tilde{T}_n = |\tilde{T}_{n-1}|^{1/2}U|\tilde{T}_{n-1}|^{1/2}$, for every natural number n, is called the n^{th}-Aluthge transform of T, or the Aluthge sequence of T, where $\tilde{T}_n = \tilde{U}_n|\tilde{T}_n|$, is the polar decomposition of the operator $\tilde{T}_n$. Aluthge transform is a good tool for studying some operator classes since, for instance, even if the norm and the numerical radius of $\tilde{T}$, are less than those of T respectively, atleast T and $\tilde{T}$, have the same spectra. It follows that, for a p-hyponormal operator T, where $p \in (0,1]$, $\tilde{T}$ is $p+1/2$-hyponormal, if $p \in (0,1/2]$, and $\tilde{T}$ is hyponormal, if $p \in (1/2,1]$, so that, incase T is a semi-hyponormal operator, then $\tilde{T}$ is hyponormal, and thus, $\tilde{T}$ has better properties than T itself. Aluthge transform may also contribute to non-trivial invariant subspace since it is known that, if T is a bounded linear operator on H and, if there exists atleast one positive integer n, such that $\tilde{T}_n$ has a non-trivial invariant subspace, so does T, [Jung et al. 2000].

Aluthge et al, 2000, used Aluthge transforms to generalize hyponormal operators, by introducing w-hyponormal operators. These operators contains all p- and log- hyponormal operators. For both p-and log hyponormal operators, the kernel condition holds. That is, $ker(T^*) \subset ker(T)$, if T is either a p- or a log-hyponormal operator. Unfortunately, this condition is violated in general by w-hyponormal operators. However, many spectral properties satisfied by p- and log hyponormal operators, are inherited by w-hyponormal operators. Aluthge et al, 2000, studied the spectral properties of w-hyponormal operators. They gave a characterization for an operator T to be w-hyponormal. In addition, they proved that, if a w-hyponormal operator T satisfies the inequality, $|\tilde{T}| \geq |T|$, then T is paranormal. In other words, they showed that, every w-hyponormal operator is paranormal. Using Furuta's inequality, they also observed the following;

(i)If a w-hyponormal operator T satisfies the kernel condition and in addition, if its first Aluthge transform is normal, then this T is also normal.

(ii)Every square of an invertible w-hypnormal operator is also w-hyponormal.

(iii)For w-hyponormal operators, the kernel condition does not hold in general.

(iv)The non-zero points of the approximate and point spectra of a w-hyponormal operator are identical.

Cho et al, 2001, generalized these results and proved that (i) holds without the kernel condition and (ii) holds without the invertibility, if $ker(T) = 0$ and by using examples, they confirmed the existence of a w-hyponormal operator T, where, $ker(T^*) \not\subset ker(T)$, and $ker(T) \not\subset ker(T^*)$. [Yamazaki, 2002], generalized condition (ii), without requiring T to be invertible and proved that, every power of a w-hypnormal operator, is also w-hyponormal.

In the light of these remarks, we also address ourselves in this thesis to the following tasks; Firstly, we improve the results of Aluthge et al, by proving that, repeated Aluthge transforms of any operator, yields the same spectrum. Continuing in the same fashion of Cho and Yamazaki, we exhibit a number of observations on w-hyponormal operators. For instance, we prove that, every Aluthge transform of members of this class, is normaloid and hence spectraloid. On the problem of classifications of the products of the powers of generalized Aluthge transforms, and that of generalized Aluthge transforms of the powers of the products of any given pair of w-hyponormal operators, we give sufficient conditions on when these two become the same operator. Also, by imposing some requirements on the spectra of w-hyponormal operators, we deduce conditions under which, such transformations get restricted to self-adjoint operators. In addition, some sufficient conditions under which a w-hyponormal operator(and products of any two w-hyponormals), become n-Power normal or n-Power quasinormal, and conversely, are deduced.

1.4.3 Putnam-Fuglede theorem

In studying linear operators, the commutativity of any given pair, especially when dealing with the product, can not be disregarded. For instance, it is well known that, if A and B are normal operators such that, $AB = BA$, then both (AB) and $(A + B)$ are also normal. Unfortunately, given any two linear operators, say A and B, it rarely follows that $AB = BA$. Even in cases when $AB \neq BA$, there might exist another operator, say C, such that, C commutes with A and C commutes with B. The behaviour of such an operator has some implications to that of (AB). To study the commutators of a bounded linear operator, say T, it is natural for one first to look at the subspaces which are invariant under T. Recall that, if M is a subspace of a Hilbert space H and $T \in B(H)$, then M is invariant under T, if $T(M) \subset M$, and M reduces T, if both M and $M^{\perp}$ are invariant under T. It follows that, if M is invariant under T, then $M^{\perp}$ is invariant under T^*, and $TP = PTP$, for any projection P onto M. In addition, if M reduces T, then M reduces T^*, $M^{\perp}$ reduces T, M is invariant under both T and T^* and $TP = TP$, for any projection P onto M. It is good to note that, every Hilbert space H can be decomposed as $H = M \oplus M^{\perp}$. Thus, for any $T \in B(H)$, it follows that, $R(\bar{T})$ and $N(T)$ are orthogonal subspaces of H, and therefore, H decomposes as, $H = R(\bar{T}) \oplus N(T)$. This decomposition of H is crucial in studying members of $B(H)$, since as we noted earlier, every $T \in B(H)$, satisfies the characterization $T = U|T|$ and this decomposition of T is called polar if $N(U) = N(|T|)$. It follows that, if T has the polar decomposition, $T = U|T|$, U behaves like T. For instance, if $T = U|T|$, then $T^* = |T|U^*$, $\|Ux\| = \|x\|$, for any x in $R(\bar{T})$ and $\|Ux\| = 0$, for any x in $N(T)$. In this polar decomposition of T, that is, $T = U|T|$, it follows that, if A is any other operator on H, with the polar decomposition $A = V|A|$, and A commutes with both T and T^*, then U and $|T|$ commute with V, V^* and $|A|$. In particular, if T with the polar decomposition $T = U|T|$, is a normal operator, then U and $|T|$ commute with any other operator in $B(H)$, which commutes with both T and T^*.

Putnam-Fuglede theorem is an extension of Fuglede's theorem, [Fuglede, 1950], by Putnam, 1951. If we assume that an operator N on a finite-dimensional Hilbert space H is normal, the spectral theorem says that N is of the form, $N = \Sigma_i \lambda i Pi$, where Pi are pairwise orthogonal projections. One aspects that, if T is any other operator on H, then $TN = NT$, if and only if $TPi = PiT$. Indeed, it can be proved to be true by elementary arguments. For instance, it can be shown that, all Pi are representable as polynomials of N and for this reason, if T commutes with N, it has to commute with each Pi. Therefore, T must also commute with N^*. In general, when the Hilbert space H is not finite-dimensional, the normal operator N gives rise to a projection-valued measure P on its spectrum, $\sigma(N)$, which assigns a projection P_μ to each Borel subset of $\sigma(N)$, [101]. N can be expressed as $N = \int_{\sigma(N)} \lambda dP(\lambda)$.

Differently from the finite dimensional case, it is by no means obvious that, $TN = NT$ implies that, $TP_\mu = P_\mu T$. Thus, it is not so obvious that, T also commutes with any simple function of the form $F = \Sigma_i \lambda i P_{\mu i}$ Indeed, following the construction of the spectral decomposition for a bounded, normal, not self-adjoint, operator T, one sees that to verify that T commutes with P_μ, the most straightforward way is to assume that T commutes with both N and N^*, giving rise to a vicious circle! In particular, Fuglede theorem asserts that, any bounded linear operator on H, which commutes with a given normal operator, also commutes with the adjoint of that operator. In other words, this theorem states;

LEMMA 1.4.3.1, (Fuglede, 1950,[33, lemma 1.1]); Let T and N be bounded operators on a complex Hilbert space H with N being normal. If $TN = NT$, then $TN^* = N^*T$, where N^* denotes the adjoint of N.

This result is useful in studying linear operators. In particular, by this theorem, it follows easily that, the product of any two commutative normal operators is also normal. In other words, if we let M and N to be two commutative normal operators, that is, $MN = NM$, then, $(MN)(MN)^* = (MN)(NM)^* = MNM^*N^*$, and by lemma 1.4.3.1 above, this becomes, $= MM^*NN^*$. But, M and N are normal, so $= M^*N^*MN = (MN)^*(MN)$. Therefore, $(MN)$is also normal.

Putnam, 1951, extended the Fuglede's theorem into what is refered to as the Putnam-Fuglede theorem. This theorem is as follows;

LEMMA 1.4.3.2, (Putnam, 1951,[88, Thm 2]); If A, B are normal operators in $B(H)$, and if $X \in (B(H)$ such that, $AX = XB$, then $A^*X = XB^*$.

From lemma 1.4.3.2 above, it follows that, if $A, B \in B(H)$ are normal operators such that, $AX = XB$, for another operator $X \in B(H)$, then, $A^*X = XB^*$ and that, if $A(AX = XB) = (AX - XB)B$, then $AX = XB$. In general, if $AX = XB \Rightarrow A^*X = XB^*$, it will simply be written, the pair (A, B) satisfies the PF theorem. The PF theorem, is an important tool in operator theory. For example, it is easily implied by this theorem that, any two similar normal operators say, M and N, are unitarily equivalent, since letting $MS = SN$, for some invertible $S \in B(H)$, then by PF it follows that, $M^*S = SN^*$, which implies that, $S^{-1}M^*S = N^*$. Taking adjoints, one gets, $S^*M(S^{-1}) = N$. So that, $S^*M(S^{-1}) = S^{-1}MS \rightarrow SS^*M(SS^*)^{-1} = M$. Therefore, on the $Ran(M)$, SS^* is the identity operator. It follows that, SS^* can be extended to $Ran(M)^{\perp} = Ker(M)$. Thus, by the normality of M, $SS^* = I$. Similarly, $S^*S = I$. Hence, S is unitary. Moreover, $Ran(X)$ reduces A, $(ker \;\; X)^{\perp}$ reduces B and $A/_{ran \;\; X}$, $B/_{(ker \;\; X)^{\perp}}$ are unitarily equivalent normal operators.

One should notice the trivial interplay between the PF theorem and similarity or even quasi similarity of operators. Recall that, two operators, say, $A, B \in B(H)$, are said to be similar if there exists an invertible operator X on H, such that $A = XBX^{-1}$, and that if X is a unitary, then $A = XBX^*$, where in the former case, A and B are said to be unitarily equivalent. In operator theory, similarity is of great importance since it preserves compactness, algebraicity, cyclicity and the spectral picture, (that is, the spectrum, essential spectrum and the index function). In addition, similar operators have isomorphic lattices of invariant and hyper invariant subspaces, [Harmos 1982].

PF theorem simply says that, any operator which is similar to a normal operator, also happens to be similar to the adjoint of that normal operator. This result is one of the fundamental tools in studying decompositions of linear operators, since, for instance, it is known that, if $T_1T_2 = T_2T_3$ implies that, ${T_1}^*T_2 = T_2{T_3}^*$, for $T_1, T_2, T_3 \in B(H)$, with the polar decompositions, $T_k = U_kP_k$, for $k = 1, 2, 3$, then it follows that, $P_3P_2 = P_2P_3$, $U_3P_2 = P_2U_3$, $P_1U_2 = U_2P_3$, $U_1U_2 = U_2U_3$ and ${U_1}^*U_2 = U_2{U_3}^*$. Consequently, $R(\bar{T_2})$ reduces U_1, P_1 and T_1, which implies that, $ker(T_2)$ reduces U_3, P_3 and T_3. In addition, $U_1/_{R(\bar{T_2})}$ (respectively, $P_1/_{R(\bar{T_2})}$, $T_1/_{R(\bar{T_2})}$), is unitarily equivalent to $U_3/_{ker(T_2)^{\perp}}$ (respectively, $P_3/_{ker(T_2)^{\perp}}$, $T_3/_{ker(T_2)^{\perp}}$).

These restrictions of linear operators to reducing subspaces are useful in decomposing completely non normal operators into user friendly parts. Recall that, every Hilbert space H, has the orthogonal decomposition, $H = M \oplus M^{\perp}$, where M is a closed subspace of H, and $M^{\perp}$, is the orthogonal compliment of M. With respect to this orthogonal decomposition, it is known that, every operator $A \in B(H)$, has a direct sum decomposition, $A = A_1 \oplus A_2$, where A_1 and A_2, are normal and pure(completely non normal), parts respectively. Thus, it follows that, if M is a reducing subspace for A, then, $A_1 = A/_M$, and $A_2 = A/_{M^{\perp}}$, on $H = M \oplus M^{\perp}$. That is, the restriction of the operator A to the subspace M, is normal and the restriction of A, to the orthogonal compliment of M, is pure.

This theorem has been studied by several researchers. For instance, Berberian, 1978, relaxed the hypothesis on A and B in this theorem, as the cost of requiring X to be a Hilbert schmidt class. He showed that, this result is true even when A and B are hyponormal operators. Chah, 1994, showed that, the hyponormality in the result of Berbarian can be replaced by the quasi-hyponormality of A and B^*, under some additional conditions. Mecheri, et al, 2005, showed that, the quasihyponormality in the result of Chah, can be replaced by class A operators, A and B^*. In the same paper, they also proved that this equality also holds true for dominant operators. Jeon et al, 2006, showed that, the quasihyponormality in the results of both Chah and Mecheri, can be replaced by (p,k) quasihyponormality of A and B^*, with the additional condition that, $\||A|^{1-p}\|\||B^{-1}|^{1-p}\| \leq 1$. Ouma, 2007, observed that, any pair of quasisimilar w-hyponormal operators satisfy the Putnam-Fuglede theorem. Moslehian, et al, 2011, studied the Putnam-Fuglede theorem and showed that, if any two operators A and B satisfy this theorem, then their Aluthge transforms, $\tilde{A}$ and $\tilde{B}$ respectively, also satisfy this result. Recently, [Bachir, et al, 2012], extended the Putnam-Fuglede theorem to the class of w-hyponormal operators and proved that, any pair of w-hyponormal operators from a subclass of w-hyponormals, whose members satisfy the kernel condition, also satisfy this theorem.

In this thesis, through Putnam-Fuglede theorem, normality of n-Power normal, n-Power quasinormal, and that of w-hyponormal operators is investigated. Then, continuing in the manner of Moslehian, results obtained by Bachir, are extended. For instance, conditions implying similarlity between n-Power normal and w-hyponormal operators, or those which imply similarlity between n-Power quasinormal and w-hyponormal operators are sought via this theorem.

1.4.4 Putnam's and Berger-Shaw inequalities

Before concluding about, how strong or how powerful an operator is, on any space, one can not over look the size of its norm. This operator norm -a scalar- is nothing but, how much an operator stretches or extends a vector of unit length to the fulliest. In most linear operators, this norm is easy to calculate. If otherwise, it can be estimated. In estimations, it can be compared with other scalars of the same operator, such as the spectral radius, the numerical radius, the area of the spectrum or even, the area of the numerical range. In cases when the structure of a given linear operator is complicated, leading to difficulties in estimating its norm, one might investigate whether the operator is reducible, which in turn implies that such an operator can be expressed as a direct sum decomposition of normal and completely non normal operators. A known method of investigating reducibility of an operator say, T, is through seeking whether the self-commutator of T, that is, $T^*T - TT^*$ is bounded. This follows from the fact that, if the self-commutator is bounded, then the operator itself is bounded, hence compact and thus reducible, since every compact operator is reducible. In simpler terms, the Putnam's inequality and the Berger-Shaw inequality are tools used to fore tell whether $T^*T - TT^*$ is bounded, for any non normal operator T on an Hilbert space H.

As we noted earlier, every Hermitian matrix, say A, can be represented as,

$$A = U\Lambda U^* - - - - - - - - - - -(1).$$

Where, Λ is a diagonal matrix consisting of the eigenvalues of A and $U = (u_1, u_2, - - - -, u_n)$, is a unitary matrix, whose u_j^{th} entry, is the normalized eigenvector which corresponds to the eigenvalue λ_j of the matrix A. If we let Λ_j to be the matrix with 1 at the place of λ_j in the diagonal matrix Λ, and substitute every other entry with zero, then, $\Lambda = \lambda_1\Lambda_1 + \lambda_2\Lambda_2 + - - - - - - - - + \lambda_n\Lambda_n$. Thus, equation (1) becomes,

$$A = \lambda_1 U\Lambda_1 U^* + \lambda_2 U\Lambda_2 U^* + - - - - - - - - + \lambda_n U\Lambda_n U^* - - - - - - - - -(2).$$

Putting, $P_j = \lambda_j U\Lambda_j U^*$, for each j in (2), it follows that every P_j is a projection and thus equation (2) becomes,

$$A = \lambda_1 P_1 + \lambda_2 P_2 + - - - - - - - - + \lambda_n P_n - - - - - - - - -(3).$$

Letting $P_j = (E_j - E_{j-1})$, where $E_j's$ are projection for each j, in equation (3) above, then (1) becomes,

$$A = \lambda_1 E_1 + \lambda_2(E_2 - E_1) + \lambda_3(E_3 - E_2) - - - - - - - - + \lambda_n(E_n - E_{n-1}) - - - - - - - - -(4).$$

In general we have that,

$$A = \sum_{j=1}^{n} \lambda_j \Delta E_j - - - - - - - - - - - - - - - - - -(5).$$

Where, each $\Delta E_j = (E_j - E_{j-1})$ and $E_0 = 0$ in equation (5).

Since every self-adjoint operator on a Hilbert space H is an extension of a Hermitian matrix, incase A is a self-adjoint operator, (5) becomes,

$$A = \int \lambda dE_\lambda - - - - - - - - - - - - - - - - - -(6).$$

Where, $\{E_\lambda; \lambda \in R\}$, is a family of projections, such that, $E_{\lambda+0} = E_\lambda$, $E_{-\infty} = 0$ and $E_{+\infty} = 1$. It follows that, if f is a continous function on t, then by equation (6) we have,

$$f(A) = \int f(\lambda dE_\lambda) - - - - - - - - - - - - - - - - - -(7),$$

so that, if $t \in [m, M]$, then (7) becomes,

$$f(A) = \int_m^M f(\lambda dE_\lambda) - - - - - - - - - - - - - - - - - - (8).$$

In particular, if the spectrum of a self-adjoint operator A is bounded by m from below, and by M from above, and in addition, if f is analytic on $\sigma(A)$, then from equation (8) above, we have,

$$f(A) = \int_m^M f(\lambda dE_\lambda) - - - - - - - - - - - - - - - - - - (9),$$

where, $m = inf(\sigma(A))$ and $M = sup(\sigma(A))$.

The norm of A, that is, $\|A\|$, is a continous function on $\sigma(A)$. If A is self-adjoint, or even normal, then it follows that, $\|A\| = r(A)$. Thus, letting $\|A\| = f(A)$, equation (9) above simply becomes, $\|A\| = M$. It follows that, $\|A\| = M$, even when A is a normal operator. Unfortunately, this property by normal operators, is violated by non normal operators in general. But, atleast, if T is an operator on H, whether normal or non normal, the self-commutator norm of T, that is, $\|T^*T - TT^*\|$, is also a continous function on $\sigma(T)$. Thus, ideas about the size of this self-commutator norm, might also be sought from the spectrum of T. It is good to note that, if T is a normal operator, then $[T^*, T] = 0$, so that, $\sigma(T^*T - TT^*) = 0$, since $\|T^*T - TT^*\| = 0$. However, if T is not normal, then $\|T^*T - TT^*\| \geq 0$. Putnam, 1970, compared the self-commutator norm of a hyponormal operator with the planer Lebesque measure of its spectrum. The so called Putnam's inequality. That is;

LEMMA 1.4.4.1, (Putnam, 1970[87, Thm 3]); for a hyponormal operator T;

$$\|T^\star T - TT^\star\| \leq \frac{1}{\pi} Area(\sigma(T)).$$

A number of authors have investigated this inequality and extended it to larger classes of operators, all of which contains the hyponormal operators. For instance, [Cho, et al, 1995], studied the Putnam's inequality and generalized it up to the case of p-hyponormal operators. Uchiyama, 2000, also worked on the Putnam's inequality and extended it up to the case of invertible quasihyponormal operators. Uchiyama's results were generalized to the class of (p,k)-quasi hyponormal operators by Bakir, 2002, and later, [Kim, 2004], also looked at the Putnam's inequality for invertible (p,k) quasi hyponormal operators.

Berger et al, 1973, studied the estimator of the trace norm commutator of a hyponormal operator, the so called Berger-Shaw inequality. That is,

LEMMA 1.4.4.2, (Berger et al,1973[18, Cor 2]); For an n-multicyclic hyponormal operator T; $(T^*T - TT^*)$ is trace class and,

$$tr[T^\star, T] \leq \frac{n}{\pi} Area(\sigma(T)).$$

Unlike with the Putnam's inequality which has been studied by a number of authors, only [Uchiyama, 1999, 2000], and [Kim, 2004], seem to have researched on the Berger-Shaw inequality. Uchiyama, 1999, studied this inequality and laid the upper bound for the trace of the self commutator of an invertible quasihyponormal operator, and a year later, he investigated its behaviour for the case of invertible p-quasihyponormal operators. Uchiyama's results were later extended to the case of invertible (p,k)-quasihyponormal operators by Kim, 2004.

In this thesis, we explore both the Putnam's inequality and the Berger-Shaw inequality for the cases of n-Power normal, n-Power quasinormal and w-hyponormal operators. To extend results due to Uchiyama and those due to Kim even further, a number of corollaries are deduced after combining some of the observations from different earlier sections of this thesis.

1.5 Definitions and notations

Consider a Hilbert space H in which an operator is a bounded linear transformation of H into itself. Let $B(H)$ be the Banach algebra of all bounded linear operators on H. If $T \in B(H)$, then the **norm of** T, denoted by $\|T\|$, is defined as, $||T|| = inf\{c > 0 : ||Tx|| \leq c||x||, \quad \forall x \in H\} = Sup\{||Tx|| : ||x|| = 1\} = Sup\{< Tx, y >: ||x|| = ||y|| = 1\}$.

The adjoint of T, denoted by T^*, is another bounded linear operator on H, such that, if $\forall x \in H$ and a particular $y \in H$, then $< Tx, y >=< x, T^*y >$.

The **spectrum, spectral radius, numerical range** and the **numerical radius** of an operator T are denoted by; $\sigma(T)$, $r(T)$, $W(T)$ and $w(T)$ respectively, and are defined as follows;

$\sigma(T) = \{\lambda : (T - \lambda I) \quad is \ \ not \ \ invertible\}$, where λ is a complex number.

$W(T) = \{< Tx, x >: \|x\| = 1\}$ for x in H.

$r(T) = sup\{\lambda; \lambda \in \sigma(T)\}$.

$w(T) = sup\{\lambda; \lambda \in W(T)\}$

A complex number λ is said to be in the **point spectrum**, of T, denoted by $\sigma_p(T)$, if there is a non zero vector x for which $(T - \lambda)x = 0$.

If in addition, $(T^\star - \overline{\lambda})x = 0$, then λ is said to be in the **normal point spectrum** of T, denoted by $\sigma_{np}(T)$. Thus, λ is said to be in the normal point spectrum of T, if there is an eigenvector $x \in H$, corresponding to λ, which is a normal eigenvector.

A complex number λ is said to be in the **approximate point spectrum** of T, denoted by $\sigma_a(T)$, if there is a sequence $\{x_n\}$ of unit vectors for which $(T-\lambda)x_n \to 0$.

If in addition, $(T^\star - \overline{\lambda})x_n \to 0$, then λ is said to be in the **normal approximate point spectrum** of T, denoted by $\sigma_{na}(T)$.

The operator, $\tilde{T} =\mid T \mid^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}}$, is called Aluthge transformation of an operator T whose polar decomposition is $T = U \mid T \mid$, where $\mid T \mid = (T^\star T)^{\frac{1}{2}}$.

For an operator $T \in B(H)$, T is said to be;

Invertible, if there exists another operator $T^{-1} \in B(H)$, such that, $TT^{-1} = T^{-1}T = I$ and T^{-1}, is called the inverse of T.

Positive, if $< Tx, x > \geq 0$ for all $x \in H$.

A Contraction, if $||T|| \leq 1$.

A Projection, if $T^2 = T$ and $T^* = T$.

Self-adjoint, if $T^* = T$.

Unitary, if $T^*T = TT^* = I$.

Isometry, if $T^*T = I$.

Normal, if $T^*T = TT^*$.

Quasinormal, if $T(T^*T) = (T^*T)T$.

Hyponormal, if $T^*T \geq TT^*$. Where $A \geq B$ means $A - B \geq 0$, for self adjoint operators A and B.

p-hyponormal, if $(T^\star T)^p - (TT^\star)^p \geq 0$, for $0 < p \leq 1$.

q-hyponomal, if $(T^\star T)^q - (TT^\star)^q \geq 0$, for $0 < q \leq p$.

k-hyponormal, if $(T^\star T)^k - (TT^\star)^k \geq 0$, for $k \geq 1$.

∞-hyponormal, if $(T^\star T)^k - (TT^\star)^k \geq 0$, for every positive number k.

Quasi hyponormal, if $(T^\star T) - (TT^\star)T \geq 0$.

p-quasihyponormal, if $T^\star[(T^\star T)^p - (TT^\star)^p]T \geq 0$, for $0 < p \leq 1$.

q-quasi hyponormal, if $T^\star[(T^\star T)^q - (TT^\star)^q]T \geq 0$, for $0 < q \leq p$.

k-quasi hyponormal, if $T^{\star k}[(T^\star T) - (TT^\star)]T^k \geq 0$, for $k \geq 1$.

Log-hyponormal, if T is invertible and $logT^\star T \geq logTT^\star$.

Absolute-(s,t)-paranormal, if $\| \mid T \mid^s \mid T^\star \mid^t x\|^t \geq \| \mid T^\star \mid^t x\|^{s+t}$, for every unit vector x in H where $s \geq 0$ and $t \geq 0$.

Quasinormal iff $[T^*, T]T$ is a positive operator.

Posinormal if there exists another positive operator $P \in B(H)$ such that $T^*PT = TT^*$.

$(\alpha, \beta)-$ **normal,** for $0 \leq \alpha \leq 1 \leq \beta$, if $\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T$.

2-Power normal, if $T^2T* = T^*T^2$.

n-Power normal, for a positive integer n, if $T^nT* = T^*T^n$.

n-Power quasinormal, if $T^nT^*T = T^*T^{n+1}$, $\forall n \in J^+$.

n Power class (Q), if $T^{*^2}T^{2n} = (T^*T^n)$.

Class (Y), if there exists $\alpha \geq 1$ and $k_\alpha > 0$, such that $|T^*T - TT^*|^\alpha \geq k^2{}_\alpha(T-\lambda)^*(T-\lambda), \forall \lambda \in R$.

w-hyponormal, if $|\tilde{T^*}| \leq |T| \leq |\tilde{T}|$.

Class A, if $(| T^2 | \geq | T |^2$.

Class A(k), for $k \geq 0$, if $(| T^\star || T |^{2k} | T^\star |)^{\frac{1}{k+1}} \geq (| T^\star |)^2$.

Class wA(s,t), if for $s \geq 0$ and $t \geq 0$

$$(| T^\star |^t | T |^{2s} | T^\star |^t)^{\frac{t}{s+t}} \geq (| T^\star |)^{2t} \quad and \quad | T |)^{2s} \geq (| T |^s | T^\star |^{2t} | T |^s)^{\frac{s}{s+t}}$$

.

Class A(s,t), if

$$(| T^\star |^t | T |^{2s} | T^\star |^t)^{\frac{t}{s+t}} \geq (| T^\star |)^{2t}$$

.

Normaloid, if $\|T\| = r(T)$.

Spectraloid, if $w(T) = r(T)$.

Convexoid, if $\overline{W(T)} = conv\delta(T)$, where $conv\delta(T)$ means the convex hull of the spectrum of T.

Transaloid, if $T - \mu$ is normaloid for any $\mu \in C$.

Satisfies the condition (G_1) , if $\|(T-\lambda)^{-1}\| \leq \frac{1}{dist(\lambda,\sigma(T))}$.

A (closed) set S in the plane is called a **spectral set** for T, if $\sigma(T) \subset S$ and $\|f(T)\| \leq \|f\|_s$, for any rational function f with poles off S, where $\|f\|_s = sup\{|f(\lambda)|; \lambda \in S\}$.

Numeroid, if $W(T)$ is a spectral set for T.

Spectroid, if $\sigma(T)$ is a spectral set for T.

Hen-spectroid, if the complement of the unbounded component of the complement of $\sigma(T)$ is a spectral set for T.

(p,k) quasihyponormal, for a positive number $0 < p \leq 1$ and positive integer k, if $T^{\star k}[(T^\star T)^p - (TT^\star)^p]T^k \geq 0$.

n-multicyclic, if there are n vectors $x_1, x_2, ..., x_n \in H$, such that, $\bigvee[g(T)x_i : i = 1, 2, ...n, g \in R(\sigma(T))] = H$. Where, $R(\sigma(T))$, denotes the set of all rational funtions analytic on $\sigma(T)$.

Hereditarily normaloid, or HN, if every part of T, (that is, every restriction of T to an invariant subspace) is normaloid.

Totally hereditarily normaloid, or THN, if every invertible part of T is hereditarily normaloid.

Compeletely totally hereditarily normaloid, or CTHN, if either T is totally hereditarily normaloid, or $T - \lambda I$ is hereditarily normaloid for every complex number λ.

Two operators A and B are said to **commute**, denoted by $[A, B] = 0$, if $AB - BA = 0$.

Two operators A and B are said to be **similar**, if there exists an invertible operator $N \in B(H)$, such that, $NA = BN$, or equivalently $A = N^{-1}BN$.

Two operators A and B are said to be **almost similar(a.s)**, if there exists an invertible operator N such that the following two conditions hold:

$A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$.

An operator $N \in B(H)$ is **quasi-invertible or a quasi-affinity**, if it is an injective operator with dense range. That is, an operator $N \in B(H)$ is quasi-invertible or a quasi-affinity, if $Ker(N) = \{0\}$ and $\bar{Ran}(N) = H$, (equivalently, $Ker(N) = \{0\}$ and $Ker(N^*) = \{0\}$). Thus, $N \in B(H)$ is quasi-invertible if and only if, $N^* \in B(H)$ is quasi-invertible.

An operator $A \in B(H)$ is a **quasi-affine transform** of $B \in B(H)$, if there exists a quasi-invertible operator $N \in B(H)$ such that, $NA = BN$. In this case, we say that, N intertwines A and B. Thus, A is a quasi-affine tranform of B, if there exists a quasi-invertible operator intertwining A and B.

Two operators $A \in B(H)$ and $B \in B(H)$, are **quasi-similar**, if they are quasi-affine transforms of each other. Equivalently, two operators $A \in B(H)$ and $B \in B(H)$, are quasi-similar, if there exists quasi-invertible operators $N, M \in B(H)$, such that, $AN = NB$ and $MB = AM$.

Two operators $A \in B(H)$ and $B \in B(H)$, are **unitarily equivalent**, if there exists a unitary operator $U \in B(H)$, such that, $UA = BU$. Equivalently, A and B are said to be unitarily equivalent if $A = U^*BU$, for some unitary operator $U \in B(H)$.

Two operators $A \in B(H)$ and $B \in B(H)$, are said to be **metrically equivalent**, if $\|Ax\| = \|Bx\|$, (equivalently, $| < Ax, Ax > |^{1/2} = | < Bx, Bx > |^{1/2}$), for every vector $x \in H$. That is, operators $A \in B(H)$ and $B \in B(H)$, are said to be metrically equivalent, if $A^*A = B^*B$.

Any other notation or terminology, will be defined at the first instance of its occurrence.

1.6 Analysis of classes of operators in Hilbert spaces

Since this thesis is dedicated to study the diagonalizability and reducibility of mainly n-Power normal, n-Power quasinormal and w-hyponormal operators, in this section, we discuss the relationships between these classes and other higher classes, especially those which contain all normal operators, but are themselves non-normal. Thus, in this section, some of the generalizations of normal operators, together with the interplay between these generalizations is analyzed. One of the earliest extensions of normal operators was by [Stampfli, 1962], when he introduced hyponormal operators. Hyponormal operators generalizes normal operators. Equivalently, every normal operator is hyponormal. This is easly seen since, relaxing the inequality to equality in the definition of a hyponormal operator, we get a normal operator. Later, [Campbell et al, 1978], extended the class of hyponormal operators when he introduced k-quashyponormal operators. By letting $k = 1$, he showed that every hyponormal operator is k-quasihyponormal. $Aluthge$, 1990, also generalized hyponormal operators by introducing the p-hyponormal operators. The p-hyponormal operators generalizes hyponormal operators, since, putting $p = 1$, in the definition of a p-hyponormal operator, one gets a hyponormal operator.

$Arora$, 1993, generalized the hyponormality into p-quasihyponormality. Clearly, p-quasihyponormal operators contains all quasihyponormal operators. In fact, by letting $p = 1$, in the definition of a p-quasihyponormal operator, one obtains a quasihyponormal operator. However, unlike in the case of p- and q-hyponormals, where every p-hyponormal is q-hyponormal whenever $q \leq p$, it is not true in general that every p-quasihyponormal operator is q-quasihyponormal, even when $q \leq p$. This observation followed after [Arora, et al. 1993], gave an example of a quasihyponormal operator which is not 1/2-quasihyponormal.

The class of p-hyponormal operators was also shown to be contained in a larger class when [Furuta, et al. 1997], introduced class A operators.

Another class which contains all invertible hyponormal operators, is that of log-hyponormal operators. This class was introduced by [Tanahashi, 1999]. He went on and proved that every invertible p-hyponormal operator is log-hyponormal and that, every log-hyponormal operator is a class A operator.

Class A operators were later generalized into class A(s,t), by [Fujii, et al. 2000], and in the same year, the latter was generalized into even a larger class, that is the class of absolute-(s,t) paranormal by [Yanagida, 2000]. Fujii, et al. 2000, introduced another class, class $AI(s,t)$, which contains all invertible operators of class $A(s,t)$.

Using Aluthge transformations, [Aluthge, et al. 2000], generalized both log- and p-hyponormal operators into w-hyponormal operators. This class of operators properly contains all p-hyponormal and log-hyponormal operators. It is clear from the definition of w-hyponormality, that every semi-hyponormal operator is w-hypopnormal. But semi-hypnormal operators contains all p-hyponormal operators for every $p \geq \frac{1}{2}$. This shows that w-hyponormal operators contains all p-hyponormal operators.

As a generalisation of w-hyponormality, [Ito, 2001], introduced class $wA(s,t)$. Observe that, letting $s = \frac{1}{2}$ and $t = \frac{1}{2}$ in the definition of $wA(s,t)$ operators, one gets a w-hyponormal operator.

Also, class $wA(1,1)$ is simply called class wA. In otherwords, T is a member of class wA, if and only if $| T^2 | \geq | T |^2$ and $| T^\star |^2 \geq | T^{2\star} |$.

Since from the definition of class A, that is, T belong to class A if $| T^2 | \geq | T |^2$, then it is clear that class A contains class wA, but from the analysis of $wA(s,t)$ operators, it follows that, $wA(s_1,t_1)$ is always contained in $wA(s_2,t_2)$ for $s_2 \geq s_1$ and $t_2 \geq t_1$. This implies that, $wA(\frac{1}{2},\frac{1}{2}) \subseteq wA(1,1)$. But, $wA(\frac{1}{2},\frac{1}{2})$ corresponds to w-hyponormal operators, while $wA(1,1)$ corresponds to class A. Thus, class A is a generalisation of w-hyponormal operators.

As a generalisation of class A, [Yanagida, 2003], also introduced class $A(k)$. Every class A operator is class $A(k)$, since Putting $k = 1$, in the definition of class $A(k)$, we get $| T^2 | \geq | T |^2$. Class $A(k)$ is a subclass in class $A(s,t)$, since from the definition of class$A(s,t)$ operators, it is clear that class $A(k,1)$ equals class $A(k)$.

$Hyoun$, 2003, introduced (p,k)-quasihyponormal operators. The (p,k)-quasihyponormal operators generalises p-quasihyponormal, q-quasihyponormal and q-quasihyponormal operators. Infact, in the definition of a (p,k)-quasihyponormal operator, putting $p = 1$ and $k = 1$, we get a k-quasihyponormal and a p-quasihyponormal operators respectively. Since $0 < q \leq p$, and the (p,k)-quasihyponormal operators contains p-hyponormal operators, then it follows that every q-quasihyponormal operator is a (p,k)-quasihyponormal operator.

It is good also to recall that, [Jibril, 2007], extended normal operators by introducing the class of 2-Power normal operators. He later, [Jibril, 2008], generalized the class of 2-Power normal into n-Power normal, where n can be any positive integer. The results by Jibril were generalized three years later by [Ahmed, 2011], into the class of n-Power quasinormal operators and proved that every n-Power normal operator is a n-Power quasinormal. Recently, [Panayan, 2012], extended all normal operators into another class, which he called the n-Power Class (Q) operators.

1.6.1 Series of inclusions of hilbert space operators

Clearly, among the classes of operators discussed above, the following inclusions holds and are known to be proper;

$(i)\quad self-adjoint \quad \subset normal \quad \subset hyponormal \quad \subset \quad p-hyponormal \quad \subset \quad p-$

$quasihyponormal \subset (p,k)-quasihyponormal$.

(ii) $hyponormal \subset quasihyponormal \subset k-quasihyponormal \subset (p,k)-quasihyponormal$.

(iii) $p-hyponormal \subset semi-hyponormal \subset w-hyponormal \subset wA \subset classA \subset classA(k) \subset classA(s,t)$.

(iv) $p-hyponormal \subset semi-hyponormal \subset w-hyponormal \subset wA \subset classwA(s,t) \subset classA(s,t)$.

(v) $loghyponormal \subset w-hyponormal \subset classAI(s,t) \subset classwA(s,t) \subset classA(s,t)$.

(vi) $\infty-hyponormal \subset k-hyponormal \subset p-hyponormal \subset w-hyponormal \subset classAI(s,t) \subset classwA(s,t) \subset classA(s,t)$.

(vii) $subnormal \subset hyponormal \subset quasihyponormal \subset Class(A) \subset paranormal$.

(viii) $hyponormal \subset transaloid \subset convexoid$.
(ix) $\infty-hyponormal \subset normal \subset quasinormal \subset n-Power-quasinormal$.
(x) $spectroid \subset hen-spectroid \subset numeroid \subset transaloid \subset normaloid \subset spectraloid.$
(xi) $spectroid \subset hen-spectroid \subset numeroid \subset transaloid \subset convexoid.$
(xii) $normal \subset hyponormal \subset p-hyponormal \subset normaloid \subset HN$
(xiii) $CTHN \subset THN \subset HN$
(xiv) $\infty-hyponormal \subset normal \subset n-powernormal \subset n-Power-quasinormal$.
(xv) $\infty-hyponormal \subset normal \subset quasinormal \subset quasihyponormal \subset p-quasihyponormal \subset (p,k)-quasihyponormal.$

1.7 Some inequalities satisfied by square matrices and all Operators in general

As was noted earlier, every linear transformation on an Hilbert space can be represented by a square matrix. Therefore, square matrices are often-if not always- the major tools used to study linear operators on these spaces. In this section, well known inequalities involving square matrices, and which are important in this thesis are presented. Then, some crucial inequalities satisfied by all bounded linear operators, like the Young's inequality, the Lowner-Heinz inequality and the Furuta's inequality, all of which played big roles in this thesis are discussed. The following well known result is satisfied by all matrices in general;

LEMMA 1.7.1.[Berberian, 1976]; Let A, B be any two positive matrices. Then;
(i) $\|A^sB^s\| \leq \|AB\|^s, \forall s \in [0,1]$
(ii) If $\|AB\| \leq 1$, then $\|AB\|^s \leq 1 \forall s \in [0,1]$
(iii)If $\lambda_1(AB) \leq 1$, then $\lambda_1(A^sB^s) \leq 1$ where $\lambda_1(A)$ denotes the largest eigenvalue of A.
(iv) $\|AB\|^t \leq \|A^tB^t\|$ for $t \geq 1$
(v) $det(I + A + B) \leq det(I + A) + det(I + B)$
(vi) $tr(A(logA - logB)) \geq tr(A - B)$
(vii) $(det(A + B))^{1/n} \geq (det(A))^{1/n} + (det(B))^{1/n}$
(viii) If $\|A\| \leq 1$, then $(I - A)$ is invertible.
(ix) $A = T^*T$ for some upper triangular matrix T.
(x) ${\lambda^n}_1(AB) = {\lambda^n}_1(BA)$, where ${\lambda^n}_i(A)$ denotes the ith eigenvalue of the matrix A.

REMARK 1.7.2; Properties $i - ix$ in Lemma 1.7.1 above holds even when A and B are positive linear operators on a Hilbert space H. However, property (x), also holds true but with an additional requirement that, $\sigma(AB) - \{0\} = \sigma(BA) - \{0\}$, since the spectrum of an unbounded operator might contain other points rather than the eigenvalues. In other words, Property (x) implies that AB and BA share the same set of non-zero eigenvalues. That is, AB and (BA) have the same point spectra. Therefore, the invertibility of AB, implies that of BA. Before discussing some operator inequalities which are related to those in Lemma 1.6.1, that is, inequalities satisfied by only positive linear operators, and which were useful in proving some of the results in this thesis, we first look at the generalizations of the Schwarz inequality. It is good to note that, the Schwarz inequality, that is, $|(x, y)| \leq \|x\|\|y\|$, does not involve transformations on H, but vectors in H. Modifications of this inequality, in particular those which involve linear operators have been proved and used extensively, especially when locating the numerical range of a given non normal operator. Recall that, if λ is an element in the spectrum of an operator T, then λ is an element in the numerical range of T. In otherwords, the spectrum of an operator is a subset in the numerical range. A well known fact is that, the convex hull of the spectrum of any operator is properly contained in the numerical range of the same operator and that, $r(T) \leq w(T) \leq \|T\|$, for any linear operator T where this inequality is restricted to an equality in case T is a normal operator. Thus, given some operators say, A and B on H, it is important to investigate their numerical ranges together with the numerical ranges of the powers of their products.

To extend the Schwarz inequality to linear operators on H, [Bombieri, 1976], proved the following result, which is comonly known as the weighted mixed Schwarz inequality, ($WMSI$);

LEMMA 1.7.3, [Bombieri,1976]; For any operator A on a Hilbert space H,

$$|(Ax, y)|^2 \leq (|A|^{2\lambda}x, x)(|A^*|^{2(1-\lambda)}y, y)$$

holds for any $x, y \in H$ and any $\lambda \in [0, 1]$.

Putting $\lambda = 1/2$ in the $WMSI$ above, [Furuta,2001], came up with the following qualification, otherwise known as the mixed Schwarz inequality, (MSI);

LEMMA 1.7.4, [Furuta,2001]; For any operator A on a Hilbert space H,

$$|(Ax, y)|^2 \leq (|A|x, x)(|A^*|y, y)$$

holds for any $x, y \in H$.

The mixed Schwarz inequality above implies the original Schwarz inequality since, letting $\lambda = 1$ in MSI, [Kubo,et al, 1983] came up with the following inequality which is popularly known as the generalized Schwarz inequality, (GSI);

LEMMA 1.7.5, [Kubo, et al.1983]; For any operator A on a Hilbert space H,

$$|(Ax, y)|^2 \leq (|A|x, x)(|A|y, y)$$

holds for any $x, y \in H$.

REMARK 1.7.6; Recall that, if an operator say, A on H is positive, then by Property (ix) of Lemma 1.7.1 above, there exists another operator T, such that, $A = (T^*T)$ and A has a unique positive square root, $(T^*T)^{1/2}$. That is, $A^{1/2} = (T^*T)^{1/2}$, which guarantees one with a quick alternative method of decomposing A since, $A = A^{1/2}A^{1/2}$ and the operator $A^{1/2}$, is more user friendly than the operator A. Generally, $f(A) = \left(f(A^{1/2})\right)^2$, for any analytic function f. However, $A^{1/2}$ is different from $|A| = (A^*A)^{1/2}$. In otherwords, the square root of A is different from the square root of the positive semidefinite operator of A. That is, $A^{1/2} \neq |A|$. It is good to note that $|A|$ exists, regardless of whether A is invertible or not and $|A|$ is useful since for instance, it is known that, for any operator A, $\|A\| = s_1$, where s_1 is the largest eigenvalue of $|A|$.

The next well known results are satisfied by all positive linear operators on a Hilbert space H. For any pair of positive invertible operators A and B on H,[Young,1952], proved the following inequality, which is commonly known as the Young's inequality;

LEMMA 1.7.7, [Young,1952]; Let A, B be positive invertible operators on $B(H)$. Then the following inequality holds for $0 \leq \lambda \leq 1$.

$$(1-\lambda)A + \lambda B \geq A^{1/2}(A^{-1/2}BA^{-1/2^{\lambda}})A^{1/2} \geq [(1-\lambda)A^{-1} + \lambda B^{-1}]^{-1}$$

For any positive linear operators, A and B on H,[Holder,et al. 1958], extended the Young's inequality to the following result, popurlary known as the Holder-McCarthy inequality;

LEMMA 1.7.8, [Holder,et al.1958]; Let A, B be positive linear operators on $B(H)$. Then the following three properties hold;
(i) If $1 \geq \lambda \geq 0$, then $(1-\lambda)A + \lambda B \geq A^{1/2}(A^{-1/2}BA^{-1/2^{\lambda}})A^{1/2}$;
(ii) If $\lambda \geq 1$, then $(1-\lambda)A + \lambda B \leq A^{1/2}(A^{-1/2}BA^{-1/2^{\lambda}})A^{1/2}$;
(iii) If $\lambda \leq 0$, then $(1-\lambda)A + \lambda B \geq A^{1/2}(A^{-1/2}BA^{-1/2^{\lambda}})A^{1/2}$.
Extending the inequality in Lemma 1.6.7, [Wang,1976], shown that inversion is a convex function on the set of positive invertible operators by proving the following result;
LEMMA 1.7.9, [Wang,1976]; Let A, B be positive invertible operators. Then, $[(1-\lambda)A + \lambda B]^{-1} \leq (1-\lambda)A^{-1} + \lambda B^{-1}$, where $1 \geq \lambda \geq 0$.
For a positive linear operator A and $\lambda \in [0, 1]$, [Furuta, 2001] gave an elementary proof for the following result which basically implies that Young inequality and Holder-McCarthy inequality are equivalent.

LEMMA 1.7.10, [Furuta, 2001]; If A is a positive operator, then the following are equivalent;
(i) $(Ax, x)^{\lambda} \geq (A^{\lambda}x, x)$, for all unit vectors $x \in H$;
(ii) $\lambda A = 1 - \lambda \geq A^{\lambda}$.

REMARK 1.7.11; The famous Lowner Heinz inequality, [Heinz, 1934], has been so useful in the study of operators to an extent that, [Arora, et al, 1993], used it in proving that every p-hyponormal operator is q-hyponormal for some p and q such that, $0 \leq p \leq 1$. Some other authors like [Wang, 2003] and [Yanagida, 2002], used this inequality to prove that every invertible p-hyponormal operator is log hyponormal and every log hyponormal operator is w-hyponormal respectively. Clearly, the applications of this result in operator theory are enomous. However, this inequality is restricted in the interval $[0, 1]$. To remove the upper bound in this interval, [Furuta, 1989], extended the Lowner Heinz inequality to what is commonly refered to as the Furuta's inequality. This inequality has also been used by some authors, especially in analysing large classes of operators. For instance, [Fujii, et al, 1990], applied the Furuta's inequality to estimate the value of the relative operator enthropy, that is, $S(A/B)$, for positive invertible operators A and B. In this thesis, these two results, viz, Lowner Heinz and the Furuta inequality, played some roles, especially when studying variations of results from one operator class to another, notably, results where the value of a particular operator norm with its spectral radius or with its numerical radius were compared. The highly cerebrated Lowner-Heinz(LH), inequality relaxes the positivity of an operator and thus, it is satisfied by all operators in general. This inequality states;

LEMMA 1.7.12 [Heinz,1934]; For any two operators A and B in $B(H)$, $A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$, for any $\alpha \in [0, 1]$.

REMARK 1.7.13; We note that, one only talks of $A \geq B$ only when A and B are hermitian operators such that, $A - B$ is positive. That is, $A - B \geq 0$. Lemma 1.7.12 above is useful in the study of operators since, for insntace if $A \geq B \geq 0$, and if it follows that, $A + r \geq B + r \geq r$, for any positive number r, then $A + r$ and $B + r$ are both invertibe operators. However, this inequality does not hold in general for all positive numbers α, as the following result shows;

LEMMA 1.7.14, [Hansen,1980]; $A^{\alpha} \geq B^{\alpha}$ does not hold in general for any $\alpha \geq 1$ even if $A \geq B \geq 0$

REMARK 1.7.15; LH inequality is useful in the study of operators but this result is too restrictive since, $\alpha \in [0, 1]$. However, LH inequality has been extended several times to explain what happens to the inequality $A \geq B \geq$, whenever $\alpha \geq 1$. A good example of such extensions is the following inequality which is popularly known as Furuta's inequality.

LEMMA 1.7.16, [Furuta, 1989]; If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(i) \quad [B^{r/2}A^pB^{r/2}]^{1/q} \geq [B^{r/2}B^pB^{r/2}]^{1/q}.$$

and

$$(ii) \quad [A^{r/2}A^pA^{r/2}]^{1/q} \geq [A^{r/2}B^pA^{r/2}]^{1/q}$$

hold for $p \geq 0$ and $r \geq 0$ with $(1+r)q \geq p+r$.

It can easily be shown that inequalities (i) and (ii) are equivalent. Furuta, 1995, again extended LH inequality into the following result;

LEMMA 1.7.17, [Furuta,1995]; If $A \geq B \geq 0$, then the following inequalities hold,

$$(i) \quad [B^{r/2}A^pB^{r/2}]^{1+r/p+r} \geq [B^{r/2}B^pB^{1+r}$$

and

$$(ii) \quad [A^{1+r} \geq [A^{r/2}B^pA^{r/2}]^{1+r/p+r}$$

for $p \geq 1$ and $r \geq 0$.

REMARK 1.7.18; The proof follows readily by letting $q = p + r/1 + r \geq 1$ and $r \geq 0$ in the Furuta's inequality. The inequality in the lemma above is the essential part of Furuta's inequality since the former is trivial by LH inequality incase, $p \in [0,1]$. This inequality can be applied in coming up with other useful inequalities in the study of completely non normal linear operators. The following inequality is implied by this result;

LEMMA 1.7.19, [Furuta,1995]; If $A \geq C \geq B \geq 0$, then for each $r \geq 0$,

$$(*) \quad [C^{r/2}A^pC^{r/2}]^{1/q} \geq [C^{r/2}C^pC^{r/2}]^{1/q} \geq [C^{r/2}B^pC^{r/2}]^{1/q}$$

holds for $p \geq 0$ and $r \geq 0$ with $(1+r)q \geq p+r$

REMARK 1.7.20; First, we note that Furuta' inequality implies inequality in Lemma 1.7.19 since, the first inequality of $*$ follows from (i) and the second inequality follows from (ii) of Furuta's inequality respectively. On the other edge, this inequality implies Furuta's inequality since putting $B = C$ in $(*)$, one obtains (i) of Furuta's inequality. Also, putting $A = C$ in $*$, one gets (ii) of Furuta's inequality. Thus, these two inequalities are equivalent.

Lemma 1.7.19, implies the following equivalence relation;

LEMMA 1.7.21, [Furuta,1995]; $A \geq C \geq B \geq 0$, holds if and only if

$$(**) \quad [C^{r/2}A^pC^{r/2}]^{1/q} \geq [C^{r/2}C^pC^{r/2}]^{1/q} \geq [C^{r/2}B^pC^{r/2}]^{1/q}$$

holds for $r \geq 0$, $p \in [0,1]$. and $q \geq 1$ with $(1+r)q \geq p+r$

REMARK 1.7.22; A proof of 'only if' part follows by $*$ of Lemma 1.7.19, and also a proof of 'if' part follows by putting $r = 0$ and $p = q = 1$ in ii of lemma 1.7.17. We remark that Lemma 1.7.19 is a characterization of C satisfying the relation $A \geq C \geq B \geq 0$, by using the operator inequality $(**)$.

The following result is an extension of Lemma 1.7.19, and therefore, is also another generalization of Furuta's inequality;

LEMMA 1.7.23, [Furuta,2001]; If $A \geq B \geq 0$, with $A \geq 0$, then for $t \in [0, 1]$ and $p \geq 1$,

$$(G1) \quad [A]^{1-t+r} \geq (A^{r/2}[A^{-t/2}B^p A^{-t/2}]^s A^{r/2})^{1-t+r/(p-t)s+r}$$

for $s \geq 1$ and $r \geq t$

Applications of these inequalities are enomous in operator theory. For example, the following result is a possible consequence of Furuta's inequality;

LEMMA 1.7.24, [Fujii,et al. 2000]; Let $p \geq 0$ $q \geq 0$ and $r \geq 0$. If $(1 + r)q < p + r$ or $0 < q < 1$, then there exists positive invertible operators A and B with $A \geq B > 0$ which does not satisfy the inequality

$$[A]^{p+r/q} \geq [A^{r/2}B^p A^{r/2}]^{1/q}$$

REMARK 1.7.25; Furuta's inequality is also applied extensively in the study of log-hyponormal operators. Recall that the chaotic order is useful in studying log-hyponormal operators since, the log function is not seperable, (that is, the log function is not a function f, for which $f(st) = f(s)f(t)$). In particular, [Fujii, et al. 2000], applied Furuta's inequality to come up with some sufficient conditions under when $log(A) \geq log(B)$ follows from $A \geq B$. There result was as follows;

LEMMA 1.7.26, [Fujii,et al. 2000]; If A and B are positive invertible operators then;

(1) $logA \geq logB$ iff $A^p \geq (A^{p/2}B^p A^{p/2})^{1/2} \forall p \in R^+$, and

(2) $logA \geq logB$ iff $\forall \delta \in (0, 1)$, there exists two positive real numbers α and ϵ, such that, $(e^\delta A)^\alpha \geq B^\alpha + \epsilon I$.

1.8 Locations of spectra of higher classes of operators

Suppose we want to solve an equation of the form, $f(T)x = y$, where x and y are any two elements in an Hilbert space H, T is a bounded linear operator on H and f is a polynomial. If 0 is not in the spectrum of $f(T)$, then $f(T)$ has an inverse -which is also a bounded linear operator on H- and hence our problem has a unique solution for every y in H. To be guaranteed with such unique solutions, one would first like to check the nature of the spectrum of $f(T)$. In particular, one would like to know the location of the eigenvalues of $f(T)$. Generally, it is not always easy to calculate the eigenvalues of all operators. However, in many scientific problems, it is enough to know that the eigenvalues lie in some specified intervals. Such information is provided by inequalities which compare the spectrum of a given operator with its other corresponding scalars such as the norm, the numerical radius, the inner product, or even the commutator norm.

Spectral properties of different classes of operators have been studied by many authors. For instance, when studying the nature of spectra, [Noble, 1977], obtained the following results about the location of spectra;

LEMMA 1.8.1, (Noble,1997[83,Preposition 3]); Let $T \in B(H)$. If;

(i) T is unitary then $\sigma(T)$ is in the unit sphere.

(ii) T is self adjoint then $\sigma(T)$ is on the real line.

(i) T is normal then $\sigma(T)$ is on half plane.

The class of p-hyponormal contains all hyponormal operators, while both includes all normal and unitary operators. To generalize Lemma 1.8.1 above, [Derming, et al, 2003] came up with the following result;

LEMMA 1.8.2, [Derming,et al. 2003]; If T is hyponormal or a p-hyponormal operator then;

(i)T is self adjoint if $\sigma(T) \subset \Re$

(ii) T is positive if $\sigma(T) \subset [0, \infty)$

(iii) T is unitary if $\sigma(T)$ is a unit circle.

Class (A) operators includes all p-hyponormal operators. Thus, the following theorem, is a generalisation of Lemma 1.8.2 above;

LEMMA 1.8.3, [Derming,et al. 2003]; If T is a class (A) operator, then;

(i)T is self adjoint if $\sigma(T) \subset \Re$

(ii) T is positive if $\sigma(T) \subset [0, \infty)$

(iii) T is unitary if $\sigma(T)$ is a unit circle.

REMARK 1.8.4; It is clear that, if T satisfies $(\mid T^2 \mid \geq \mid T \mid^2$ and $\sigma(T)$ lies on the real line then, $T = T^\star$. Also, if $\sigma(T)$ is a circle of radius 1, then $(T^\star T) - (TT^\star) = 0$.

For any operator T, the following holds in general;

$$(a) \quad \sigma_{np}(T) \subseteq \sigma_p(T) \quad and \quad (b) \quad \sigma_{na}(T) \subseteq \sigma_a(T)$$

. [Derming, et al, 2003], also studied some the location of spectra of p-hyponormal, log hyponormal and class A operators. They concluded that, for both semi hyponormal and p-hyponormal operators;

(i)every eigenvector corresponding to a non zero eigenvalue is a normal eigenvector.

(ii)every approximate eigenvector corresponding to a non zero approximate eigenvalue is a normal approximate eigenvector. Inparticular, they proved the following two results;

LEMMA 1.8.5, [Derming,et al. 2003]; If T is hyponormal or semihyponormal operator then,

$$(a) \quad \sigma_{np}(T) = \sigma_p(T) \quad and \quad (b) \quad \sigma_{na}(T) \; = \sigma_a(T)$$

LEMMA 1.8.6, [Derming,et al. 2003]; If T is p-hyponormal operator for $(0 < p \leq \frac{1}{2})$ then,

$$(a) \quad \sigma_{np}(T) = \sigma_p(T) \quad and \quad (b) \quad \sigma_{na}(T) \; = \sigma_a(T)$$

REMARK 1.8.7; If T is hyponormal, then $T - \lambda$ is also hyponormal, for any scalar λ and $\|(T^\star - \lambda^-) \leq (T - \lambda)\|$. In addition, if T is p-hyponormal and $(T - \lambda)x = 0$, then $(T^\star - \overline{\lambda})x = 0$. From Lemma 1.8.5 above, one can conclude that every eigenvector of hyponomal or a p-hyponormal operator is a normal eigenvector. As a consequense of this result, it follows that, if every eigenvalue of T is contained in the real line, then $T = T^\star$.

The following observation is a generalisation of Lemma 1.8.6;

LEMMA 1.8.8, [Derming,et al. 2003]; If T is a w-hyponormal operator, then

$$(a) \quad \sigma_{np}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\} \quad and \quad (b) \quad \sigma_{na}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$$

Every w-hyponormal operator is a class (A) operator. Therefore, the following result, is a generalisation of Lemma 1.8.8;

LEMMA 1.8.9, [Derming,et al. 2003]; If T is a class (A) operator, then

$$(a) \;\; \sigma_{np}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\} \;\; and \;\; (b) \;\; \sigma_{na}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$$

REMARK 1.8.10; For w-hyponormal operators, or in general, for class A operators, every eigenvector corresponding to a non zero eigenvalue is a normal eigenvector and every approximate eigenvector corresponding to a non zero approximate eigenvalue is a normal approximate eigenvector.

Chapter two

NORMALITY OF THE PRODUCTS OF NON-NORMAL OPERATORS

In the current chapter, a brief history on generalizations of normal operators is visited. Sufficient conditions implying normality for both n-Power normal and n-Power quasinormal operators are investigated. In addition, conditions to be satisfied by a pair of operators A and B, each pair picked from any of the above mentioned classes, which guarantee the normality of the product AB and that of the powers of the product, that is, normality of $(AB)^n$, are given. It is also in this chapter where, after imposing more requirements on the numerical ranges of A and B, characterizations under which the normality of A^pB^q, implies that of AB, for any pair of positive integers p and q, are discussed. Lastly, ∞-Power normal and ∞-Power quasinormal operators are introduced and more results outlining conditions under which, either n-Power normal or n-Power quasinormal operators become normal are deduced.

2.1; Introduction

In operator theory, the major attention has been that of characterizing sufficient conditions, which imply normality of completely non-normal operators. Jibril, 1996, introduced 2-Power normal operators. That is, an operator $T \in B(H)$, is said to be 2-Power normal if, $T^2T^* = T^*T^2$. In other words, T is said to be 2-Power normal if, T^2 commutes with T^*. After introducing 2-Power normal operators, he noted that, every 2-Power normal operator is not normal, but every normal operator is 2-Power normal. Jibril, 2008, relaxed the integer 2 in the '2-Power normal' by introducing the n-Power normal operators. That is, an operator $T \in B(H)$, is said to be n-Power normal, for some positive integer n, if $T^nT^* = T^*T^n$, ([62, Dfn 1.2]). In otherwords, T is said to be n-Power normal, if T^n commutes with T^*. The author studied some properties of such class for different values of parameter n. In particular, he proved that, a n-Power normal operator need not be normal, and that, if T is a n-Power normal operator, then T^* is also a n-Power normal operator, for the same n. By giving an example of a 2-Power normal operator which is not 3-Power normal, and that of a 3-Power normal operator which is not 2-Power normal, Jibril confirmed that, a n-Power normal operator need not be either a $(n-1)$-Power normal or a $(n+1)$-Power normal in general. Sid Amhed, 2011, generalized the n-Power normality of an operator T, by intoducing n-Power quasinormal operators. That is, an operator $T \in B(H)$, is said to be n-Power quasinormal, for some positive integer n if, $T^n(T^*T) = (T^*T)T^n$, ([91, Dfn 2.1]). In otherwords, T is said to be n-Power quasinormal if T^n commutes with (T^*T). Continuing in the manner of Jibril, Sid Ahmed confirmed that, a n-Power quasinormal operator need not be either a $(n-1)$-Power quasinormal or a $(n+1)$-Power quasinormal in general. In addition, he made several observations about members from this class. For instance, he proved that, if T is n-Power quasinormal, then T is $2n$-Power quasinormal, and that, if T is invertible, then T^{-1} is also n-Power quasinormal, but unfortunately, T^* is not generally, n-Power quasinormal for the same n. It is known that, n-Power normal and n-Power quasinormal operators are neither diagonalizable nor do they have translation invariant property. Thus, it is natural to come up with more sufficient conditions under which normality of these classes follows. This chapter is purely dedicated to extend the results by Jibril and those due to Sid-Ahmed.

To achieve this, we first discuss some of the well known observations, especially those which form the pillars unto which this thesis stands on.

2.2 Preliminary lemmas

Trivially, every normal operator is a quasinormal operator and if T is normal, then T^n is also normal, $\forall n \in J^+$. However, the non normality of an operator T does not rule out the possibility of finding some positive integer n where, T^n becomes normal. Consequently, normality of T^n does not guarantee normality of T. But at least, the following result by [Jibril, 2008], throws some light about the normality of n-power normal operators

LEMMA 2.2.1, [62, Preposition 2.1]; $T \in nN$ iff $T^n \in N$ for some $n \in J^+$.

The class of n-Power normal operators is indepedent from that of quasinormal operators. However, every n-Power normal operator is n-Power quasinormal as the following result by [Sid Ahmed, 2011] shows;

LEMMA 2.2.2, [91, Preposition 1.3]; If $T \in nN$, then $T \in nQN$, $\forall n \in J^+$

REMARK 2.2.3; We note that the class of n-Power quasinormal operators properly includes that of n-Power normal operators. Thus, there are several n-Power quasinormal operators which are not n-Power normal. More over, for any operator T, if there exists a positive integer n, such that, T is n-Power quasinormal, then, T^{-1} is not n-Power quasinormal in general. That is, $[T^n, T^*T] = 0 \nRightarrow [T^{-1^n}, T^{-1^*}T^{-1}] = 0$. In addititon, the product and sum of any two n-Power quasinormals is not always a n-Power quasinormal. However, the following result by Sid Ahmed, 2011, lays down the conditions under which this class is closed with respect to multiplication and addition.

LEMMA 2.2.4, [91, Thm 2.2]; If $T \in nQN$, then;

(i) $T \in 2nQN$.

(ii) If $ran(T) = H$, then $T \in nN$. In particular, if T is invertable, then $T^{-1} \in nQN$.

(iii) If $A, B \in nQN$, such that, $[A, B] = [A^*, B] = 0$, then $AB \in nQN$.

(iv) If $A, B \in nQN$, such that, $[A, B] = [A^*, B] = 0$, then $(A + B) \in nQN$.

REMARK 2.2.5; We recall that, every partial isometry is not always a normal operator and that, n-Power quasinormality of any operator does not guarantee (n+1)-Power quasinormality of the same operator, for avery positive integer n. Sid Ahmed, 2011, proved the following two conditions under which, T becomes a (n+1)-Power quasinormal, whenever T is n-Power quasinormal, $\forall n \in J^+$.

LEMMA 2.2.6, [91, Preposition 2.3]; If $T \in nQN$, such that, T is a partial isometry, then $T \in (n + 1)QN$.

LEMMA 2.2.7, [91, Preposition 2.5]; If $T \in 2QN \cap 3QN$, then $T \in nQN$, $\forall n \in J^+$, and $n \geq 4$.

REMARK 2.2.8; It is not known whether the kernel condition holds in the class of all n-Power quasinormal operators. That is, if T is a n-Power quasinormal operator, for some $n \in J^+$, it is not known whether, $N(T) \subset N(T^*)$, or $N(T^*) \subset N(T)$. However, if the kernel condition holds for an n-Power quasinormal operator T, then T^n happens to be a normal operator as the following result shows;

LEMMA 2.2.9, [91, Preposition 2.7]; If $T \in nQN$, such that, $N(T^*) \subset N(T)$, then $T \in nN$.

Sid Ahmed, 2011, also proved the following result which relaxes a n-Power quasinormal operator to a normal operator.

LEMMA 2.2.10, [91, Thm 2.9]; If $T \in 2QN \cap 3Qn$ and $T - I \in nQN$, then T is a normal operator.

REMARK 2.2.11; It is important to note that, if T is a normal operator, then T^* is also normal. This is not true in general for n-Power quasinormal operators. But, when both T and T^* happens to be n-Power quasinormal operators for the same $n \in J^+$, then such T ends up being restricted to the class of n-Power normal operators. In other words, T^n becomes a normal operator, as the following result shows;

LEMMA 2.2.12, [91, Thm 2.10]; If $T \in nQN$ and $T^* \in nQN$, then $T \in nN$

REMARK 2.2.13; Recall that, if T is a n-Power quasinormal operator, then it is not known whether, $N(T) \subset N(T^*)$. Fortunately, this condition fails only when $n = 1$, but holds true for any other n, as the following observation shows;

LEMMA 2.2.14, [91, Lemma 2.11]; If $T \in nQN$, then $N(T^{*n}) \subset N(T^n)$, for each $n \geq 2$.

REMARK 2.2.15; If T is a n-Power normal operator, then T^n is normal. This qualification is not inherited by n-Power quasinormal operators. That is, if T is a n-Power quasinormal operator, then T^n might not always be a quasinormal operator for each positive integer n. The following two observations by [Sid Ahmed, 2011], gives conditions under which T^2 becomes a quasinormal operator, whenever T is a 2-Power quasinormal operator;

LEMMA 2.2.16, [91, Thm 2.12]; If $T \in nQN$, $T^2 \in nQN$ and $T \in 3QN$, then $T^2 \in QN$.

LEMMA 2.2.17, [91, Thm 2.13]; If $T, T^2 \in 2QN$ and $N(T) \subset N(T^*)$, then $T^2 \in QN$.

REMARK 2.2.18; Trivially, not every 2-Power quasinormal operator is a normal operator. But, every 2-Power quasinormal operator is associated with a normal operator through its polar decomposition, as the following conclusion by [Sid Ahmed, 2011] tells;

LEMMA 2.2.19, [91, Thm 2.14]; Let $T = U|T|$. $If T \in 2QN$ then an operator $S \in B(H)$ with the polar decomposition $S = U^2|T|$ is a normal operator.

Sid Ahmed, 2011, used the numerical range of a 2-Power quasinormal operator T, to give another condition under which such an operator becomes normal.

LEMMA 2.2.20, [91, Corollary 2.15]; If $T \in 2QN$ and $0 \notin W(T)$, then T is normal.

REMARK 2.2.21; We have already seen that, the 2-Power quasinormality of T does not always guarantee the quasinormality of T^2. However, this becomes true if T commutes with it adjoint, as the following conclusion implies;

LEMMA 2.2.22, [91, Thm 2.16]; If $T \in 2QN$, such that, $[T^*T, TT^*] = 0$, then $T^2 \in QN$.

2.3 Normality of n-Power quasinormal operators

The following result follows from Lemmas 2.2.9 and 2.2.14;

THEOREM 2.3.1, [Imagiri et al, 2013,[51]]; If T is a quasinormal operator, the T^n is normal for each positive integer $n \geq 2$. Consequently, T is not normal.

Proof

Let T be a quasinormal operator. Then, $T \in QN$. Inparticular, $T \in 1QN$. By Lemma 2.2.14, $N(T^{*n}) \subset N(T^n)$, for each $n \geq 2$. By Lemma 2.2.9, T^n is normal for each $n \geq 2$.. But T^1 is not necessarily normal since $N(T^{*1}) \not\subset N(T^1)$. Thus, T is not normal in general.

REMARK 2.3.2; It is good to note that, Theorem 2.3.1 above guarantees the existence of n-Power quasinormal operators which are not n-Power normal for the same integer n. One might as well note that, if T is a quasinormal operator, then T^n commutes with (T^*T), for every positive integer n, and T^n commutes with T^* for every positive integer $n > 1$. It also follows that, if $T \in 1QN$, and if in addition, there exists any positive integer $m \neq 1$, such that, T is m-Power quasinormal, then $N(T^{*n}) = N(T^n)$, for each $n \geq 2$.. To relax the kernel condition to be true for every positive integer n, that is, $N(T^{*n}) = N(T^n)$, for each $n \in J^+$, we applied Lemma 2.2.12 to Theorem 2.3.1 and proved the following result;

THEOREM 2.3.3, [Imagiri et al, 2013,[51]]; If T and T^* are quasinormal operators, then T is a normal operator. Moreover, $N(T^{*n}) = N(T^n)$, for each $n \in J^+$.

Proof

Let $T \in 1QN$ and $T^* \in 1QN$. Then by Theorem 2.3.1, $N(T^{*n}) \subset N(T^n)$ and $N(T^n) \subset N(T^{*n})$, for each $n \geq 2$. Thus, $N(T^{*n}) = N(T^n)$, for each $n \geq 2$. But by Lemma 2.2.12, T^1 is a normal operator since $T, T^* \in 1QN$. In particular, T is a normal operator. Thus, $N(T) = N(T^*) \Rightarrow N(T^{*1}) = N(T^1)$. Hence, $N(T^{*n}) = N(T^n)$, for each $n \in J^+$.

REMARK 2.3.4; We recall that, if T is a normal operator, then both T^* and T^n are normal operators, where n is any positive integer. From Lemma 2.2.1, it follows easily that, the normality of T^n, implies T^n does not only commute with T^{n*}, but commutes with T^* as well. In addition, from Lemma 2.2.2, we realize that if T^n commutes with T^*, then T^n commutes with (T^*T). But when does commutativity of T^n with (T^*T), imply the commutativity of T^n with T^*? In response to this question, we made the following observations;

THEOREM 2.3.5, [Imagiri et al, 2013,[51]]; Let $T \in nN$. If T is normal, then $T \in (n-1)QN$, for every $n \in J^+$.

Proof

$T \in nN \Rightarrow T^nT^* = T^*T^n \Rightarrow T^{n-1}TT^* = T^*T = T^*TT^{n-1}$. Thus, T is a $n-1$ power quasinormal operator.

THEOREM 2.3.6, [Imagiri et al, 2013,[51]]; If T^n is a normal operator, then $T \in nQN$, for some $n \in J^+$.

Proof

Let T^n be a normal operator. Then, by Lemma 2.2.1, $T \in nN$. The proof follows easily from Lemma 2.2.2.

THEOREM 2.3.7, [Imagiri et al, 2013,[51]]; If $T \in nQN$, such that, $[T^*, T] = 0$, then T^n is normal, for each positive integer n.

Proof

$T \in nQN \Rightarrow T^nT^*T = T^*T^{n+1}$ multipying by T^* to the right, we have $T^nT^*TT^* = T^*TT^nT^*$ but $[T, T^n] = 0$ thus, $T^nT^*TT^* = T^*T^nTT^* \Rightarrow T^nT^* = T^*T^n \Rightarrow T^n \in nN$, Hence T^n is normal by Lemma 2.2.1

THEOREM 2.3.8, [Imagiri et al, 2013,[51]]; If $T \in nQN$ and $T \in (n-1)QN$, such that, $[T^*, T^2] = 0$, then $T \in QN$

Proof

If $T \in nQN$, then $T^nT^*T = T^*TT^n$ multiplying to the left by T, we have $TT^nT^*T = TT^*TT^n$ $\Rightarrow T^2T^{n-1}T^*T = TT^*TT^n$, but $T \in (n-1)QN$ that is $T^{n-1}T^*T = T^*T^n$, thus $T^2T^*T^n = TT^*TT^n$, $\Rightarrow T^2T^* = TT^*T$, but $[T^2, T^*] = 0$, thus, $T^*T^2 = TT^*T$, $\Rightarrow T \in QN$.

In particular $T \in 1QN$. Hence $T \in 2QN$ by Lemma 2.2.3.

The following quick results follows readily from Theorem 2.3.8;

COROLLARY 2.3.9, [Imagiri et al, 2013,[51]]; If $T \in 2QN$ such that T is a partial isometry, and $[T^*, T^2] = 0$, then $T \in QN$.

Proof

Since $T \in 2QN$ and T is a partial isometry, then by Lemma 2.2.4, $T \in 3QN$. By Theorem 2.3.8, $T \in QN$.

COROLLARY 2.3.10, [Imagiri et al, 2013,[51]]; If $T \in 2QN$ and $(T - I) \in nQN$, such that, T is a partial isometry, then T is a normal operator.

Proof

Since T is a partial isometry and $T \in 2QN$, then by Corollary 2.3.9, $T \in 3QN$. That is, $T \in (2QN \cap 3QN)$, but $(T - I) \in nQN$, for some $n \in J^+$. Thus by Lemma 2.2.7, T is normal.

2.4 Normality of the products of quasinormal operators

In this section, after picking two operators which are not necessarily normal, some sufficient conditions under which their product become normal are investigated. As was noted earlier, quasinormal operators are not normal operators generally and if an operator happens to be a n-Power quasinormal, for some positive integer n, then it does not follow that such an operator is quasinormal or n-Power normal. The following result gives some conditions under which the product of any two n-power quasinormal operators become normal.

THEOREM 2.4.1, [Imagiri et al, 2013,[51]]; If A and B are invertible operators in nQN, such that, $[A, B] = [A, B^*] = 0$, then $(AB)^n$ is normal.

Proof

From the hypothesis and Lemma 2.2.3, $(AB) \in nQN$. But A^{-1} and B^{-1} exists. Thus,

$(AB)^{-1} \in nQN$, since $(AB)^{-1} = B^{-1}A^{-1}$. That is, $(AB) \in nQN$ and $(AB)^{-1} \in nQN$ $\Rightarrow (AB) \in nN$. Thus, by Lemma 2.2.1, $(AB)^n$ is normal.

REMARK 2.4.2; The normality of AB, does not imply in general that BA is normal, unless when AB is a compact operator. But Theorem 2.4.1, gives another condition on which $(BA)^n$ happens to be normal, whenever $(AB)^n$ is normal, since $[A, B] = 0$, implies $[A^n, B^n] = 0$, for any positive integer n.

We also note that, when $n = 1$, then the following theorem, follows immedietely from Theorem 2.4.1;

COROLLARY 2.4.3, [Imagiri et al, 2013,[51]]; If A and B are invertible quasinormal operators such that, $[A, B] = [A, B^*] = 0$, then AB and BA are normal.

REMARK 2.4.4; It is good to note that, if $A, B \in nQN$, $A^*, B^* \in nQN$, such that, $[A, B] = 0$, then both $(AB)^n$ and $(BA)^n$ are normal operators. This fact follows trivially from Lemma 2.2.12, since, if we let $A, A^* \in nQN$ and $B, B^* \in nQN$, A and B are normal operators. $[A, B] = 0 \Rightarrow [A^n, B^n] = 0 \; \forall n \in J^+$.

That is, A^nB^n is a normal operator, that is, $(AB)^n$ is normal. It easily follows that $(BA)^n$ is also normal. In particular, by imposing an extra condition to any two quasinormal operators A and B, normality of their product AB, can as well follow after dropping the invertibility of A, B, and even without requiring A to commute with B^*, as the following result shows;

COROLLARY 2.4.5, [Imagiri et al, 2013,[51]]; If A, A^* and B, B^* are quasinormal operators, such that, $[A, B] = 0$, then AB and BA are normal.

Proof

Let A, A^*, B, B^*, be quasinormal operators. Then, by Theorem 2.3.3, A and B are normal. But, from the hypothesis, $AB = BA$. Then, it follows that both AB and BA are also normal operators.

REMARK 2.4.6; Recall that, if T is n-Power normal, then T^*, is also n-Power normal for the same n. This qualification is violated by n-Power quasinormal operators. That is, n-Power quasinormality of T does not imply n-power quasinormality of T^*. However, if T happens to be a unitary operator, then n-Power quasinormality of T implies n-power normality of both T and T^*, whenever $T^{n+1} = 0$, as the following observation shows;

THEOREM 2.4.7, [Imagiri et al, 2013,[51]]; If $T \in nQN$, such that, T is unitary and $[T, T^n] = 0$, then $[T^*, T^n] = 0$.

Proof

$T \in nQN \Rightarrow T^nT^*T = T^*TT^n$. Multiplying by T^* to the right we get, $T^nT^*TT^* = T^*TT^nT^* = T^*T^nTT^*$. That is, $T^nT^*TT^* = T^*T^nTT^*$. But $TT^* = T^*T = I \Rightarrow T^nT^* = T^*T^n \; \forall n \in N$. Thus, $[T^*, T^n] = 0$. That is, T^n commutes with T^*. Therefore, T and T^* are n-Power normal operators.

REMARK 2.4.8; It is important to note that, by letting $A = T$ and $B = T^n$, in Theorem 2.4.6, above, then $A, B \in nQN$. If in addition, A commutes with B, then we have that $T^{n+1} = 0$. Thus, this theorem is an application of products of non normal operators. The following results are other consequences from the normality of these products;

THEOREM 2.4.9, [Imagiri et al, 2013,[51]]; If $T \in nQN$ and $T \in (n-1)QN$, for some $n \in J^+$, such that, $[T, T^n] = 0$, then T is normal.

Proof

$T^nT^*T = T^*TT^n$. Multipying through by T^* to the right we get, $T^nT^*TT^* = T^*TT^nT^*$ $\Rightarrow T(T^{n-1})T^*TT^* = T^*T^{n+1}T^{n-1}T^*$. But, $T^{n-1}T^*T = T^*T^n \Rightarrow TT^*T^nT^* = T^*TT^nT^*$ $\Rightarrow TT^* = T^*T$.

THEOREM 2.4.10, [Imagiri et al, 2013,[51]]; If $T \in nQN$ and $T \in (n-1)QN$, such that, $[T^*, T^2] = 0$, then T is a quasinormal operator.

Proof

Let $T \in nQN \Rightarrow T^nT^*T = T^*TT^n$ multipying through by T to the left. $\Rightarrow TT^nT^*T = TT^*TT^n$ $\Rightarrow T^2T^{n-1}T^*T = TT^*TT^n$ if $T^{n-1}T^*T = T^*T^n$ then $T^2T^*T^n = TT^*TT^n$ $T^2T^*T^n = TT^*T^{n+1}$ $\Rightarrow T^2T^* = TT^*T \Rightarrow TTT^* = TT^*T$. If $[T^*, T^2] = 0$, then $TT^*T = T^*T^2$ is quasinormal.

THEOREM 2.4.11, [Imagiri et al, 2013,[51]]; If $T \in nQN$, such that, $[T^n, TT^*] = 0$, then T is a normal operator. In particular, T is the identity operator.

proof

If $T \in nQN$, then $T^nT^*T = T^*TT^n$. If $[T^*, TT^*] = 0$, then $T^nTT^* = TT^*T^n$. That is, $T^nT^*T = T^*T^{n+1}$ and $T^{n+1}T^* = TT^*T^n \Rightarrow T^nT^*TT^{n+1}T^* = T^*T^{n+1}TT^*T^n \Rightarrow T^nT^*T^{n+2}T^* = T^*T^{n+2}T^*T^n \Rightarrow T^*T^{n+1}T^{n+1}T^* = T^*T^{n+2}T^* \Rightarrow T^*T^{n+2} = T^*T^{n+2}T^* \Rightarrow T^* = 1 \Rightarrow T = 1$.

REMARK 2.4.12; The kernel condition holds for normal operators. That is, if A is a normal operator, then $N(A^*) = N(A)$. One might as well note that, if A is n-Power normal, for some positive integer n, then A^n is normal and thus, $N(A^{n*}) = N(A^n)$. But as was noted earlier, it is not known whether this condition holds true in the class of n-Power quasinormal operators. However, any two n-Power quasinormal operators with the kernel condition, yields a normal product provided that they are commutative as the following conclusion says;

THEOREM 2.4.13, [Imagiri et al, 2013,[51]]; If $A, B \in nQN$, such that, $N(A^*) \subset N(A)$, $N(B^*) \subset N(B)$ and $[A, B] = 0$, then $(AB)^n$ and $(BA)^n$ are both normal operators.

proof

Let A, B be any two n-Power quasinormal operators such that, $N(A^*) \subset N(A)$ and $N(B^*) \subset N(B)$. Then, by lemma 2.2.9, A^n and B^n are normal operators. But, the commutativity of A and $B \Rightarrow A^nB^n = B^nA^n = (AB)^n = (BA)^n$. That is, $[A^n, B^n] = 0$ and $A, B \in nN \Rightarrow (AB)^n$ and $(BA)^n$ are normal operators.

REMARK 2.4.14; We again note that, if $n = 1$ in the Theorem 2.4.13 above, then AB and BA are both normal operators, despite the non normality of both A and B, A and B are quasinormal operators). Generally, normality of (AB) does not follow only by imposing more conditions on A and B. In some cases, it might follow that, the product (AB) is not normalbut atleast there exists some positive integer n such that $(AB)^n$ is normal. To come up with sufficient conditions which imply normality of (AB), whenever $(AB)^n$ is normal, one might look at the spectrum or even the numerical range of $(AB)^n$. We also note that, [Embry, 1966], proved that if $T \in B(H)$ is such that T^2 is normal $0 \notin W(T)$, then T is normal. To extend Embry's result, [Duggal, 1977], proved that, if $T^{*n+1}T^{n+1}$ commutes with $T^{*n}T^n$, with zero not in the interior numerical range of T, then T^n is normal, that is, T is a n-Power normal operator.By requiring the numerical range of $(AB)^n$ to satisfy some conditions, more qualifications on when normality of $(AB)^n$ implies that of AB, are deduced and proved.

If $A \in B(H)$ is a normal operator, then A^2 is also normal. But the normality of A^2 does not always imply A is normal. However, for any operator A, if A^2 is normal, that is A is a 2-Power normal operator, then A is also normal if, $0 \notin W(A)$. In this section, the results obtained above were extended. The following lemmas are useful in proving some observations;

LEMMA 2.4.15, [Embry, 1966,[31]]; Let A be any operator. If A^2 is normal, such that,

$0 \notin W(A)$, then A is also normal.

LEMMA 2.4.16, [Embry, 1966,[31]]; Let A be a normal operator. If $0 \notin W(A)$, then any other operator which commutes with A also commutes with A^2.

We first prove the following theorem, and then use it to extend the above Lemmas.

THEOREM 2.4.17, [Imagiri et al, 2013,[51]]; If $0 \notin W(A)$, then $0 \notin W(A^2)$.

proof

Assume $0 \notin W(A)$. Then, $(Ax, x) \neq 0 \quad \forall x \neq 0$. That is, $Ax \neq 0 \quad \forall x \neq 0$. That is, $AAx \neq 0 \;\; \forall x \neq 0$. That is, $A^2x \neq 0 \;\; \forall x \neq 0$. that is, $(A^2x, x) \neq 0 \;\; \forall x \neq 0$. Thus, $0 \notin W(A^2)$.

THEOREM 2.4.18, [Imagiri et al, 2013,[51]]; If $0 \notin W(A)$, then $0 \notin W(A^{2^n}) \;\; \forall n \in J^+$.

proof

From Theorem 2.4.17 above, if $0 \notin W(A)$, then $0 \notin W(A^2)$. It follows that, $0 \notin W(T)$, whenever, $0 \notin W(T_1)$, if $T, T_1 \in B(H)$ such that, ${T_1}^2 = T$. Thus, if $0 \notin W(A)$, then 0 will be an isolated point in the following numerical ranges; $W(A), \;\; W(A^2), \;\; W(A^4), \;\; W(A^8), ..., W(A^{2^n}) \;\; \forall n \in J^+$.

THEOREM 2.4.19, [Imagiri et al, 2013,[51]]; Let $A \in B(H)$ be any operator such that, A^{2^n} is a normal operator for some $n \in J^+$. Then, A^{2^m} is also a normal operator $\forall m \in J^+ \;\; where \;\; m \leq n$, if $0 \notin W(A)$.

proof

We first note that, if $0 \notin W(A)$, then $0 \notin W(A^{2^n}) \quad \forall n \in J^+$. Thus, $0 \notin W(A)$, $\Rightarrow$ $0 \notin W(A^{2^{n-1}}) \;\; \forall n \in J^+$. But $(A^{2^{n-1}})^2 = A^{2^n}$. It follows that, $A^{2^{n-1}}$ is normal whenever A^{2^n} is normal for any positive integer n. Also the following are normal operators; $A^{2^{n-1}}, \;\; A^{2^{n-2}}, \;\; - - - A^2, \;\; A$. Hence A^{2^m} is also a normal operator $\forall m \in J^+ \;\; where \;\; m \leq n.$

THEOREM 2.4.20, [Imagiri et al, 2013,[51]]; Let A, B be any two commuting operators such that, A^{2^m} and B^{2^n}, are both normal for some $m, n \in J^+$. Then, $(A^{2^p}B^{2^q})$ is also a normal operator $\forall p, q \in J^+ \;\; where \;\; p \leq m \;\; and \;\; q \leq n$, if $0 \notin W(A)$ and $0 \notin W(B)$.

proof

If A^{2^m} is normal then by Theorem 2.4.18 above, A^{2^p} is also normal $\forall p \leq m$. Likewise, B^{2^q} is also normal $\forall q \leq n$. From lemma 2.31, A^{2^p} and B^{2^q} commute. That is, A^{2^p} and B^{2^q} are commutative normal operators. Thus, $(A^{2^p}B^{2^q})$ is also a normal operator.

REMARK 2.4.21; [Duggal, 1986], on the operator equation, $AH = KA$, involved the spectrum of A and proved the following;

LEMMA 2.4.22, [Duggal, 1986,[28]]; Let A, B be any operators such that, $[B, A^2] = 0$. Then, $[B, A] = 0$, if $\sigma(A) \cap \sigma(-A) = \phi$.

The following theorem follows from the Lemma 2.4.21 and 2.4.22 above;

THEOREM 2.4.23, [Imagiri et al, 2013,[51]]; If $T \in nQN$, such that, $\sigma(T^{n/2}) \cap \sigma(-T^{n/2}) = \phi$, then $T \in n/2QN$.

proof

Let $T \in nQN$, then $[T^n, T^*T] = 0$. $\Rightarrow [T^*T, T^n] = 0$. The conclusion follows from Lemma 2.4.22, by letting $B = T^*T$ and $A = T^n$. Thus, $[T^*T, T^{n/2} = 0]$. So that, T is a $n/2$-Power quasinormal operator.

REMARK 2.4.24; It is good to note that, if $T \in 2QN$, such that, $\sigma(T) \cap \sigma(-T) = \phi$, then T is a quasinormal operator. We also note that, if $\sigma(T) \cap \sigma(-T) = \phi$, then $0 \notin W(T)$. Theorem 2.4.23 above thus becomes,

COROLLARY 2.4.25, [Imagiri et al, 2013,[51]]; Let $T \in nQN$, such that, $\sigma(T) \cap \sigma(-T) = \phi$, then $T \in n/2N$.

proof

Firstly, note that, it suffices to show that, $\sigma(T) \cap \sigma(-T) = \phi$, $\Rightarrow 0 \notin W(T)$ and the rest of the proof follows from Theorems 2.4.17 and 2.4.23. Now assume to the contrary that, $\sigma(T) \cap \sigma(-T) \neq \phi$. Then, $\sigma(T) \cap \sigma(-T)$ contains atleast one number say λ. Thus $\lambda \in \sigma(T)$. But $\sigma(T)$ is a convex set. It follows that, $0 in \sigma(T)$ and thus, $0 in W(T)$.

THEOREM 2.4.26, [Imagiri et al, 2013,[51]]; Let A, B be any two commuting operators such that, $(AB) \in 2^nQN$, for some $n \in J^+$, and $\sigma(AB)^{2^n} \cap \sigma(-AB)^{2^n} = \phi$, then $(AB)^{2^m}$ is normal for each positive integer $m < n$.

proof

$\sigma(AB)^{2^n} \cap \sigma(-AB)^{2^n} = \phi$, $\Rightarrow 0 \notin W(AB)^{2^n}$, $\Rightarrow 0 \notin W(AB)^{2^m}$, for each positive integer $m < n$. Thus, $(AB)^{2^4}$ is normal and $0 \notin W(AB)^2$. It follows that, $0 \notin W(AB)$ and hence AB is a normal operator.

2.5 ∞-Power normal and ∞-Power quasinormal operators

In this section, ∞-Power normal and ∞-Power quasinormal operators are introduced and some results following from these two classes are discussed.

DEFINITION 2.5.(a), [Imagiri, 2013,[58,Dfn 1.1]]; An operator $T \in B(H)$, is said to be ∞-**Power normal**, if $T^nT^* = T^*T^n$, for every positive integer n.
In other words, T is said to be ∞-Power normal, if T^n commute with T^*, for every positive integer n.

DEFINITION 2.5.(b), [Imagiri, 2013,[58,Dfn 1.2]]; An operator $T \in B(H)$, is said to be ∞-**Power quasinormal**, if $T^n(T^*T) = (T^*T)T^n$, for every positive integer n.

In other words, T is said to be ∞-Power quasinormal, if T^n commute with T^*T, for every positive integer n.
Recall that, the classes of normal, n-Power normal, quasinormal and n-Power quasinormal were denoted as, N, nN, QN, nQN respectively. In a similar fashion, ∞N, and ∞QN, denotes the classes of ∞-Power normal and ∞-Power quasinormal respectively. The following inclusion series follows and are known to be proper;
(a) $N \subset 2N \subset 2QN$.
(b)$N \subset nN \subset nQN$.
(c) $N \subset QN \subset nQN$.

The major target of this section, was to merge results by Jibril[62] and Sid Ahmed[91]. In particular, sufficient conditions under which $(T-\lambda)$ becomes normal for every complex number λ, where T is such that, $T \in nN$ or $T \in nQN$ were investigated. It is good to note that, an operator $T \in B(H)$, is said to be ∞-Power normal if, $[T^n, T^*] = 0$, for each $n \in J^+$, and ∞-Power quasinormal if, $[T^n, (T^*T)] = 0$, for each $n \in J^+$. The following observations follows;

THEOREM 2.5.1, [Imagiri, 2013,[58]]; Let $T \in \infty N$. Then;
(i) $T^* \in \infty N$.
(ii) If T is invertible, then $T^{-1} \in \infty N$.
Proof
(i) Assume that, $T \in \infty N$. Then, $T \in nN$, $\forall n \in J^+$. By Lemma 2.2.1, we have that, T^n is

normal for each n, $\Rightarrow (T^n)^*$ is also normal for every positive integer n. But, $(T^n)^* = (T^*)^n$. Thus, $(T^*)^n$, is normal. It follows that, $T^* \in nN$, for each $n \in J^+$. And thus, $T^* \in \infty N$.

(ii) Since, $T \in nN$, then T^n is normal. But, $(T^n)^{-1} = (T^{-1})^n$, and T^n being normal implies that, $(T^n)^{-1}$ is also normal for each n. Thus, $(T^{-1})^n$, is normal for every $n \in J^+$. $\Rightarrow$ $T^{-1} \in nN$, for each $n \in J^+$. Consequently, $T^{-1} \in \infty N$.

THEOREM 2.5.2, [Imagiri, 2013,[58]]; Let $A \in B(H)$, be such that, $A \in \infty N$. Then;
(i) If $B \in B(H)$, is unitary equivalent to A, then $B \in \infty N$.
(ii) If M is a closed subspace of H, such that M reduces A, then $A/M \in \infty N$.

Proof

(i) $A \in \infty N \Rightarrow A \in nN$, $\forall n \in J^+$. That is, A^n is normal for every n. If B is unitary equivalent to A, then there exists a unitary U satisfying, $B = UAU^* \Rightarrow B^n = UA^nU^*$. Since, A^n is normal for each n, then B^n is also normal for each $n \in J^+$. Thus, $B \in nN$, $\forall n \in J^+$. Hence, B is ∞-Power normal.

(ii) Now let us assume that, $A \in \infty N$. Then, $A \in nN$, $\forall n \in J^+$. Thus, $A^n/M \in N$, for each n. That is, the restriction of A^n to the subspace M is a normal operator for each n. But, $A^n/M = (A/M)^n$, for any positive integer n. It follows that, $(A/M)^n \in N$, $\forall n \in J^+$. Therefore, A/M is is ∞-Power normal.

THEOREM 2.5.3, [Imagiri, 2013,[58]]; If $T \in \infty N$, then $T \in N$.

Proof

Let $T \in \infty N$. Then, $[T^n, T^*] = 0$, for each $n \in J^+$. Letting $n = 1$, we have that, $[T, T^*] = 0$. That is $T^*T = TT^*$. Thus T is normal.

REMARK 2.5.4; Since we noted that, if T is normal, then T^n is also normal for each n, it trivially follows that, every normal operator is ∞-Power normal. Generally, we have;

COROLLARY 2.5.5, [Imagiri, 2013,[58]]; $T \in \infty N$ iff $T \in N$.

Proof

The proof follows easily from Theorem 2.5.3 and Remark 2.5.4 above.

THEOREM 2.5.6, [Imagiri, 2013,[58]]; If $T \in \infty N$, then λT and T^m, are also ∞-Power normal operators, for every complex number λ and every positive integer m.

Proof

Let $T \in \infty N$. Then, by Theorem 2.5.3, T is a normal operator. Thus, λT and T^m, are also normal operators. By Corollary 2.5.5, it follows that, λT and T^m, are also ∞-Power normal operators, for every complex number λ and every positive integer m.

THEOREM 2.5.7, [Imagiri, 2013,[58]]; If $T \in \infty QN$, such that, $N(T) \subset N(T^*)$, then T is normal.

Proof

Let $T \in \infty QN$. Then, $[T^n, (T^*T)] = 0$, for each $n \in J^+$. Now, if , $N(T) \subset N(T^*)$, it follows by Lemma 2.2.6 that, $T \in nN$, for each n. Thus, T is an ∞-Power normal operator. So that, normality of T follows by corollary 2.5.5.

REMARK 2.5.8; It is good to note that, applying Corollary 2.5.5 to Lemmas 2.2.7 and 2.2.9, one can easily conclude that; if $T \ \ and \ \ (T - I) \in 2QN$, then $T \in \infty N$. And that, if $T \in (2QN \cap 3QN)$, such that, $(T - I) \in nQN$, then $T \in \infty N$. Similar to Corollary 2.5.5, we have the following result;

THEOREM 2.5.9, [Imagiri, 2013,[58]]; $T \in \infty QN$ iff $T \in QN$.

Proof

Let $T \in \infty QN$. Then, $[T^n, (T^*T)] = 0$, for each $n \in J^+$. Letting, $n = 1$, we have that, $[T, (T^*T)] = 0$. That is, $T^*T^2 = T(T^*T)$. Thus T is quasinormal. Conversely, let $T \in QN$. Then, $T \in nQN$, for every positive integer n. Thus, $T \in \infty QN$.

COROLLARY 2.5.10, [Imagiri, 2013,[58]]; If $T \ \ and \ \ T^* \in QN$, then T is normal.

Proof

By Theorem 2.5.9, we have that, $T \ \ and \ \ T^* \in QN \Rightarrow T \ \ and \ \ T^* \in nQN$, for every $n \in J^+$. By Lemma 2.2.6, it follows that, $T \in nN$, for each n. Thus, $T \in \infty N$. So that by Theorem 2.5.3, T is a normal operator.

THEOREM 2.5.11, [Imagiri, 2013,[58]]; If $T \in \infty N$, then $(T - \lambda) \in \infty N$, for each complex number λ.

Proof

The proof follows easily by Corollary 2.5.10, since every ∞-Power normal operator is normal, and normal operators have the translation invariant property. That is, if T is a normal operator, then $(T-\lambda)$, is normal for each complex number λ.

REMARK 2.5.12; We note that, [Jibril, 2008], gave an example to show that the classes of $2N$ and $3N$ are not similar and also proved that if T is an operator in both classes, then $T \in nN$ for all positive integers, $n \geq 2$. Thus, T being 2-Power normal and 3-Power normal at the same time, does not imply T is ∞-Power normal in general since, T might fail to be normal. For a T such that, $T \in 2N \cap 3N$ to be ∞-Power normal in general, one might demand more qualifications from T. The following observations, follows after imposing more requirements on T;

THEOREM 2.5.13, [Imagiri, 2013,[58]]; Let $T \in (2N \cap 3N)$. If either T or T^* is injective, then T is an ∞-Power normal operator.

Proof

$T \in 2N \Rightarrow [T^2, T^*] = 0 \Rightarrow T^2T^* = T^*T^2$. $T \in 3N \Rightarrow [T^3, T^*] = 0 \Rightarrow T^3T^* = T^*T^3$. But, $T \in 2N$, thus, $T^3T^* = T^2T^*T \Rightarrow T^2(TT^* - T^*T) = 0$. It therefore follows that, $TT^* - T^*T = 0$, since, either T or T^* is injective. Hence, $T \in N \Rightarrow T \in \infty N$.

THEOREM 2.5.14, [Imagiri, 2013,[58]]; Let $T \in (2N \cap 3N)$. If T is invertible, then $(T-\lambda) \in \infty QN$, for every complex number λ.

Proof

Firstly, we note that, it suffices to prove that T is normal, since, if T is normal, then both T and $(T-\lambda)$ are ∞-Power normal, hence ∞-Power quasinormal. By letting T to be invertible, then T is a bijective, so that T is injective, and thus, from Theorem 2.5.13 above, T is normal. Finally, we prove that this class is not closed with respect to operator addition and operator multiplication as follows;

THEOREM 2.5.15, [Imagiri, 2013,[58]]; Let $A, B \in \infty N$. Then, $A+B$ *and* $AB \notin \infty N$.

Proof

Assume to the contrary that, $A, B \in \infty N \Rightarrow (A+B), (AB) \in \infty N$. Then, it would as well follow that, $A, B \in N \Rightarrow (A+B), (AB) \in N$, which is not true in general.

Chapter three

INEQUALITIES AND SEQUENCES OF ALUTHGE TRANSFORMS OF w-HYPONORMAL OPERATORS

In this chapter, it is shown that, both the size and the structure of the spectrum of every bounded linear operator on Hilbert spaces, are invariant under sequences of Aluthge transforms. That is, the spectrum of any operator is preserved no matter how large is n in the n^{TH}-Aluthge transform. Results leading to the conclusion that every nth-Aluthge transform of a w-hyponormal operator is spectraloid and that, each nth-Aluthge transform of the kth-power of any invertible w-hyponormal operator are not only spectraloid but also totally hereditarily normaloid, are proved. In addition, observations showing how the spectrum of any nth-Aluthge transform of every kth-power of an invertible w-hyponormal operator can be used to relax such an operator to either a self adjoint or a unitary operator, are discussed.

3.1; Introduction

Aluthge transformation is very useful and many authors have obtained results by using it. Mainly, these results are of non normal operators. More over, for each non negative integer n, [Jung et al, 2000], introduced the nth-Aluthge transformation $\tilde{T_n}$. Following this definition, [Yamazaki, 2001], showed some properties of the nth-Aluthge transformations on operator norms as parallel results to those of powers of operators. Unfortunately, after introducing the Aluthge transformation of an operator T of the polar decomposition, $T = U|T|$, [Aluthge, 1990], did not give a complete solution of the polar decomposition for $\tilde{T}$ itself. However, [Masatoshi, et al, 2004], managed to obtain this solution where they showed that $\tilde{T} = VU|\tilde{T}|$, to be the polar decomposition of $\tilde{T}$. Secondly, they showed that, $\tilde{T} = U|\tilde{T}|$, if and only if, T is a binormal. As far as the nth-Aluthge transformation is concerned, they showed that, $\tilde{T_n}$ is binormal for all non negative integers n, if and only if T is centered. That is, $\tilde{T_n}$ is binormal if and only if,

$[T^n(T^n)^*, (T^m)^*T^m] = 0$ for any pair of natural numbers m and n. We noted in chapter one that, nth-Aluthge transformations of an operator T, that is, $\tilde{T_n}$, is the first Aluthge transformation of the $(n-1)$th-Aluthge transformation of T. That is, $\tilde{T_n} = \tilde{T_{n-1}}$, for each natural number n. So that, $\tilde{T_n}$ is the n^{th} term of a sequence of Aluthge transformations of T. Therefore, talking of nth-Aluthge transformations of T is the same talking of sequences of Aluthge transformations of T. One might as well recall that, in studying linear operators, especially those which are not user friendly, $\tilde{T}$ yields results faster than T, since, for instance, $\tilde{T}$ does not depend on the partial isometry part of the polar decomposition of T. It is also of importance to recall that, even if $\|\tilde{T}\| \leq \|T\|$, atleast $\sigma(\tilde{T}) = \sigma(T)$ and $N(\tilde{T}) = N(T)$, thus $r(\tilde{T}) = r(T)$ in general. It also follows that $\tilde{T}$ is invertible if and only if, T is invertible.

The second section of this chapter, was dedicated to the generalizations of these results for each natural number n. In particular, it was proved in this section that, $\sigma(\tilde{T_n}) = \sigma(T)$ and $N(\tilde{T_n}) = N(T)$, thus $r(\tilde{T_n}) = r(T)$ in general. In addition, it was proved that, $\tilde{T_n}$ is invertible if and only if, $\tilde{T_{n-1}}$ is invertible for every natural number n, so that invertibility of T implies that of $\tilde{T_n}$, and conversely.

Clearly, w-hyponormal operators owe their definition to Aluthge transformations. These operators are completely non normal but includes all p- and log-hyponormal operators. As was shown in chapter one, many authors have researched on w-hyponormal operators. It is natural to note that the major target of these investigations has been that of finding sufficient conditions which imply normality of members form this class. For instance, immediately after introducing w-hyponormal operators, [Aluthge et al, 2000], observed that, if T is a w-hyponormal operator such that, $\tilde{T}$ is normal, then T is normal. They also found out that, if T is a w-hyponormal, then so is T^{-1}, whenever T is invertible, but T^* is not always a w-hyponormal and that, if T is invertible, then T^2 is also a w-hyponormal operator. One of the major results they obtained was that, the first and the second Aluthge transformations of a w-hyponormal operator are semi-hyponormal and hyponormal respectively. Thus, one mught easily realize that all sequences of Aluthge transformations of a w-hyponormal are w-hyponormal since semi-hyponormals and hyponormals are p-hyponormal, hence w-hyponormal. Later, while studying w-hyponormal operators, [Jung et al, 2002], came up with the unfortunate observation that, the kernel condition does not hold in general in this class. That is, if T is a w-hyponormal operator, then neither $N(T) \subset N(T^*)$, nor $N(T^*) \subset N(T)$. But at least, [Yamazaki, 2002], managed to relax the invertibility condition imposed on a w-hyponormal operator T by [Aluthge et al, 2000], in order for T^2 to be w-hyponormal, by proving that, T^n is a w-hyponormal for each positive integer n, regardless of whether T is invertible or not.

In the remaining sections of this chapter, more results, similar to those of Aluthge, Yamamazaki and Yanagida, were stated and proved. For instance, in Section 3.3, it was proved that, sequences of Aluthge transformations are not only spectraloid but also normaloid. In Section 3.4, normality of a w-hyponormal operator T, resulting from requiring nth-Aluthge transformations of T to satisfy some conditions, are studied and in Section 3.5, we looked at the normality of the products of two w-hyponormal operators. Sections 3.6 and 3.7, were dedicated to studying relationships between powers of sequences of Aluthge transformations and sequences of powers of Aluthge transformations of w-hyponormal operators.

3.2; Equality of the spectra of generalized aluthge transforms

Recall that, any bounded linear operator T on a Hilbert space H, satisfies the power inequality, $w(T^n) \leq w(T)^n$. However, it is important to note that, if such T happens to be a spectraloid, that is, $r(T) = w(T)$, then, the power inequality becomes an equality. To compute $w(T)$, one need first to locate the numerical range of T, which is not always a trivial task. Whenever, the numerical range of T is not easily found, one can atleast compute the spectrum, thence, the numerical radius of T, and then check whether this operator is normaloid, since $r(T) = \|T\| \Rightarrow r(T) = w(T)$. In other words, normaloidness of T, implies spectraloidness of T. Therefore, given any operator, it is natural to determine either, the location of its spectrum, or that of its numerical range. In this section, we prove that all Aluthge transforms of any operator have equal spectra. We begin by stating the following known result, which guarantees the invertibility of any operator, if its first Aluthge transform happens to be invertible.

LEMMA 3.2.1, [Aluthge et al, 1990,[2]]; If $\tilde{T}$ is invertible, then $|T|$ is invertible.
The following observation, is an extension of Lemma 3.2.1 above. That is, the converse in the Lemma above, also happens to hold.

LEMMA 3.2.2, [Aluthge et al, 2000,[4]]; The operator T is invertible, if and only if, the operator $\tilde{T}$ is invertible.
The first and the second Aluthge transforms of any operator have the same spectrum as that of the operator, hence the same spectral radius as the following result shows;

LEMMA 3.2.3, [Aluthge et al, 2000,[5]]; The spectra of T, $\tilde{T}$ and $\tilde{T}_2$ are identical, that is, $\sigma(T) = \sigma(\tilde{T}) = \sigma(\tilde{T}_2)$.

To extend Lemma 3.2.3, to any positive integer n, we first proved the following result;

THEOREM 3.2.4, [Imagiri et al, 2011,[50]]; If $\tilde{T_n}$ is invertible, then $|\tilde{T_{n-1}}|$ is invertible, for any natural number n.

proof

Suppose to the contrary that $|\tilde{T_{n-1}}|$ is not ivertible. Then, $|\tilde{T_{n-1}}|^{\frac{1}{2}}$, is not invertible and either the range, $ran(|\tilde{T_{n-1}}|^{\frac{1}{2}})$ of $(|\tilde{T_{n-1}}|^{\frac{1}{2}})$ is not dense, or $|\tilde{T_{n-1}}|^{\frac{1}{2}}$ is not bounded below. Since,

$$(\tilde{T_n})x = |\tilde{T_{n-1}}|^{\frac{1}{2}}U(|\tilde{T_{n-1}}|^{\frac{1}{2}}x),$$

for each vector x in H, $ran(\tilde{T_n})$ is contained in $ran(|\tilde{T_{n-1}}|^{\frac{1}{2}})$. If $ran(|\tilde{T_{n-1}}|^{\frac{1}{2}})$ is not dense, then $ran(\tilde{T_n})$ is not dense and hence, $\tilde{T_n}$ is not invertible.

On the other hand, if $|\tilde{T_{n-1}}|^{\frac{1}{2}}$ is not bounded below, then there is a sequence x_n of unit vectors such that, $\||T|^{\frac{1}{2}}x_n\|$ tends to zero as n tends to infinity. Since, $\tilde{T_n} = |\tilde{T_{n-1}}|^{\frac{1}{2}}U|\tilde{T_{n-1}}|^{\frac{1}{2}}$,

$$\|(\tilde{T_n})x_n\| < \||\tilde{T_{n-1}}|^{\frac{1}{2}}U\|\||\tilde{T_{n-1}}|^{\frac{1}{2}}x_n\|,$$

$\|(\tilde{T_n})x_n\|$ tends to zero as n tends to infinity. Thus, $\tilde{T_n}$ is not bounded below and is therefore not invertible.

THEOREM 3.2.5, [Imagiri et al, 2011,[50]]; The operator $\tilde{T_{n-1}}$ is invertible if and only if the operator $\tilde{T_n}$ is invertible, for any positive integer n.

proof

Suppose $|\tilde{T_{n-1}}|$ is invertible. Then $|\tilde{T_{n-1}}|^{\frac{1}{2}}$ is invertible and thus $\tilde{T_n}$ is invertible since,

$$\tilde{T_n} = |\tilde{T_{n-1}}|^{\frac{1}{2}}U|\tilde{T_{n-1}}|^{\frac{1}{2}},$$

Conversely, if $\tilde{T_n}$ is invertible, we have by $Theorem$ 1.4, that, $|\tilde{T_{n-1}}|$ is invertible.
Since $\tilde{T_{n-1}} = |\tilde{T_{n-1}}|^{-\frac{1}{2}}\tilde{T_n}|\tilde{T_{n-1}}|^{\frac{1}{2}}$, then $\tilde{T_{n-1}}$ is invertible.

THEOREM 3.2.6, [Imagiri et al, 2011,[50]]; The spectra of

$$T, \tilde{T}, \tilde{T_2}, -----, \tilde{T_{n-1}}, \tilde{T_n}$$

are identical, that is,

$$\sigma(T) = \sigma(\tilde{T}) = \sigma(\tilde{T_2}) = ---- = \sigma(\tilde{T_{n-1}}) = \sigma(\tilde{T_n}),$$

for every natural number n.

proof

Firstly, we note that, the non zero points of $\sigma(\tilde{T_{n-1}}) = \sigma(U\tilde{T_{n-1}})$, and $\sigma(\tilde{T_n}) = \sigma(|\tilde{T_{n-1}}|^{\frac{1}{2}}U(|\tilde{T_{n-1}}|^{\frac{1}{2}})$, are identical and since, $0 \in \sigma(\tilde{T_{n-1}})$ if and only if, $0 \in \sigma(\tilde{T_n})$, by $Theorem$ 3.2.5, we have that, $\sigma(\tilde{T_{n-1}}) = \sigma(\tilde{T_n})$, for any natural number n.

3.3; Classifications of Aluthge transforms of w-hyponormal operators

For any operator T, it is known that, $\|\tilde{T}\| \leq \|T\|$. Similarly, $\|(\tilde{T_{n-1}})\| \leq \|(\tilde{T_n})\|$. However, for w-hyponormal operators, the inequality is restricted to an equality.

In this section, we generalize this result to any natural number n. But first, we cite the following known results, and then apply them in proving that every $n^{TH} - Aluthge$ transform of any w-hyponormal operator is spectraloid.

LEMMA 3.3.1, [Ando, 1987,[8]]; For any operator T,

$$r(T) \leq w(T) \leq \|T\|$$

REMARK 3.3.2; We recall that, majority of classes of operators such as normal, subnormal, hyponormal and log-hyponormal are normaloid and thus, the inequality in Lemma 3.3.1 above, is relaxed to an equality. All these classes are properly contained in the class of w-hyponormal operators. This large class shares several properties with its subclasses. The following result shows that every w-hyponormal operator is also normaloid.

LEMMA 3.3.3, [Aluthge et al, 2000,[4]]; If T is a w-hyponormal operator, then;

$$\|\tilde{T}\| = \|T\| = r(T).$$

REMARK 3.3.4; It is good to note that, repeated Aluthge transforms of the same operator yields different operators. That is, $\tilde{T_{n-1}} \neq \tilde{T_n}$, in general. However, if the norm is invariant of repeated Aluthge transforms, then such an operator happens to be normaloid and conversely, as the following result shows;

LEMMA 3.3.5, [Aluthge et al, 2000,[4]]; $\|\widetilde{T_{n-1}}\| = \|\tilde{T_n}\|$, if and only if, T is normaloid.

REMARK 3.3.6; We have already seen that, for any operator T, $\|T\| \geq \|\tilde{T}\|$, and $\|T^k\| \leq \|T\|^k$, for each natural number k. However, equality of any of these inequalities, implies that of the other as the following result shows;

LEMMA 3.3.7, [Aluthge et al, 2000,[4]]; For any operator T, the following are equivalent;

$$(i) \quad \|T^k\| = \|T\|^k.$$

$$(ii) \quad \|T\| = \|\tilde{T}\|.$$

The following result, which shows that every Aluthge transform of any w-hyponormal operator is normaloid, follows from the Lemmas 3.3.5 and 3.3.7 above;

THEOREM 3.3.8, [Imagiri et al, 2011,[50]]; If T is a w-hyponormal operator, $\|(\widetilde{T_{n-1}})\| = \|(\tilde{T_n})\|$, for every natural number n, and $\|T^k\| = \|T\|^k$, for every natural number k.

proof

Trivially, from *lemma* 3.3.3, $\|T\| = r(T)$.

So that, every w-hyponormal operator is normaloid. And thus, by lemma 3.3.5, $\|\widetilde{T_{n-1}}\| = \|\tilde{T_n}\|$.

Letting $n = 1$, we have that, $\|T\| = \|\tilde{T}\|$. So that, $\|T^k\| = \|T\|^k$, follows from *lemma* 3.3.7.

REMARK 3.3.9; We recall that, every normaloid operator is spectraloid, and thus spectraloidness of w-hyponormal operators follows from Theorem 3.3.8 above. That is, every w-hyponormal operator is normaloid and thus, spectraloid. The following result also confirms this observation, even without requiring normaloidness of T.

COROLLARY 3.3.10, [Imagiri et al, 2011,[50]]; Every w-hyponormal operator is spectraloid.

proof

For w-hyponormal operators, we have from *Lemma* 3.3.3 that, $\|\tilde{T}\| = \|T\| = r(T)$. But, we have in general from *Lemma* 3.3.1 that, $r(T) \leq w(T) \leq \|T\|$. This implies that, $r(T) = w(T)$, for w-hyponormal operators. Thus, every w-hyponormal operator is spectraloid.

To generalise Corollary 3.3.10, for every natural number n, we first state the following well known result;

LEMMA 3.3.11, [Derming et al, 2003,[25]]; For any operator T; $w(\tilde{T}) \leq w(T)$.

The following result shows that the class of spectraloid operators is closed with respect to repeated Aluthge transforms of w-hyponormal operators.

THEOREM 3.3.12, [Imagiri et al, 2011,[50]]; If T is a w-hyponormal operator, $r(\tilde{T_n}) = w(\tilde{T_n})$.

proof

Note first, we have from *Lemma* 3.3.11 that, $w(\tilde{T_n}) \leq w(T) - - - (i)$, for any natural number n.

Remember that, $\sigma(\tilde{T_{n-1}}) = \sigma(\tilde{T_n})$, for any natural number n, implies that, $r(\tilde{T_n}) = r(\tilde{T_{n-1}})$. That is, for w-hyponormal operators, $r(\tilde{T_n}) = w(T) - - - (ii)$. But, we have in general that, $r(T) \leq w(T)$, thus $r(\tilde{T_n}) \leq w(\tilde{T_n}) - - - (iii)$. From (i),(ii) and (iii), it follows that, $r(\tilde{T_n}) = w(\tilde{T_n})$. That is, every n^{TH}-Aluthge transform of a w-hyponormal operator is spectraloid.

3.4; Powers of generalized Aluthge transforms of w-hyponormal operators

In this section, results obtained in Sections 3.2 and 3.3 above are extended. To avoid confusions while talking about the nth-power and the nth-Aluthge transforms, another positive integer k will be used for the power. We would also like to assert that, there is a difference between the nth-Aluthge transform of the kth power and the kth power of the nth-Aluthge transform of an operator, say T. In other words, $\tilde{T^k}_n \neq \tilde{T_n}^k$ generally, for every pair of positive integers n, k. Powers of operators from any class, including the hyponormal operators, are not in general members of the same class. For instance, if T is a class (A) operator, then T^2 is not neccessarily a class (A) operator. But, if T is an invertible class (A) operator, all of its powers happen to be class (A) operators. In addition, it is known that, if an operator T is invertible, then all Aluthge transforms of T are similar. Unlike in Class (A) operators, every power of a w-hyponormal operator is a w-hyponormal operator, as the following result shows;

LEMMA 3.4.1, [Mashahiro, 2003,[73]]; If T is a w-hyponormal operator, then T^k is a w-hyponormal operator, for every positive integer k.

The following theorem is a generalization of the Lemma 3.4.1 above;

THEOREM 3.4.2, [Imagiri et al, 2013,[52]]; If T is an invertible w-hyponormal operator, then both $\tilde{T^k}_n$ and $\tilde{T_n}^k$ are also invertible w-hyponormal operators, $\forall n, k \in J^+$. Consequently, $\tilde{T^k}_n = \tilde{T_n}^k$.

proof

We first note that, the invertibility of T implies that, T^k is also an invertible w-hyponormal operator for any positive integer k. If T^k is invertible, then by Theorem 3.2.5 above, we have that, all Aluthge transforms of T^k are invertible. That is, $\tilde{T^k}_n$ is invertible, $\forall n, k \in J^+$.

Now assume T is an invertible w-hyponormal operator. Then, $\tilde{T_n}$ is also an invertible w-hyponormal operator, $\forall n \in J^+$. Thus, all of its powers are invertible. Hence, $\tilde{T_n}^k$ is also an invertible operator, $\forall k \in J^+$. The equality follows quickly since, if an operator T is invertible, then all Aluthge transforms of T are the same operator.

REMARK 3.4.3; The equality in Theorem 3.4.2 holds since T is invertible. In general, if T is a w-hyponormal operator, then $\tilde{T^k}_n \neq \tilde{T_n}^k$, for every pair of positive integers n and k. However, the two operators share the same spectra as the following result shows;

THEOREM 3.4.4, [Imagiri et al, 2013,[52]]; If T is a w-hyponormal operator, then, $\sigma(\tilde{T^k}_n) = \sigma(\tilde{T_n}^k)$, $\forall n, k \in J^+$.

REMARK 3.4.5; In order to prove Theorem 3.4.4 above, we first prove Corollary 3.4.11 below, which follows immediately from the following well known results;

LEMMA 3.4.6, [Lummer et al, 1954,[72]]; If $\lambda \in \sigma(T)$, then $|\lambda| \leq r(T)$.

REMARK 3.4.7; We note that, Lemma 3.4.6 above, confirms the fact that, the spectrum of any operator is bounded from above by its radius. That is, any number larger than the spectral radius of any operator, will be found in the resolvent of that operator. One might as well note that, if $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$, since we have seen that, $\|T\| \geq r(T)$ for any bounded linear operator. As an extension of this result, we state the following well known observation;

LEMMA 3.4.8, [Nelson, 1959,[82]]; If $\lambda \in \sigma(T)$, then for each positive integer k, we have, $\lambda^k \in \sigma(T^k)$.

REMARK 3.4.9; By Lemma 3.4.8 above, it is easy to conclude that, the spectrum of any operator is always contained in the spectrum resulting from raising the said operator to any power. That is, $\sigma(T) \subset \sigma(T^k)$. In general, the following result is well known and follows as a consequence of the spectral mapping theorem;

LEMMA 3.4.10, [Crabble, 1970,[24]]; If $\lambda \in \sigma(T)$, then $f(\lambda) \in \sigma(f(T))$, for any polynomial f.

The following Corollary follows from the Lemmas 3.4.6, 3.4.8 and 3.4.10 above;

COROLLARY 3.4.11, [Imagiri et al, 2013,[52]]; If $r(T) = r$ then, $r(T^k) = r^k$.

Proof

Assume that, $\lambda \in \sigma(T)$ is the scaler with the largest positive value $|\lambda|$. Then, $r(T) = |\lambda|$ and $\lambda^k \in \sigma(T^k)$. It follows that, $|\lambda^k|$ is the largest positive value in $\sigma(T^k)$ since if not, then, there exists another number say, $n \geq k$, such that, $|\lambda^n|$ is the largest positive value in $\sigma(T^k)$, which contradicts the selection of $\lambda \in \sigma(T)$. Consequently, $|\lambda^k| = r(T^k)$. Thus, from Lemma 3.4.8, $r(T^k) = r^k$.

Proof of Theorem 3.4.4

For any bounded linear operator on H, all of its Aluthge transforms have the same spectra, (Theorem 3.2.6). Thus, $\sigma(\tilde{T^k}_n) = \sigma(T^k)$.

On the other edge, $\sigma(\tilde{T_n}^k)$ includes $\sigma(\tilde{T_n})$. But, $\sigma(\tilde{T_n}) = \sigma(T)$ and $\sigma(T) \subset \sigma(T^k)$. Now letting $r(T) = r$ then, $r(T^k) = r^k$, $r(\tilde{T_n}) = r$ and thus, $r(\tilde{T_n}^k) = r^k$. Hence, $\sigma(\tilde{T^k}_n) = \sigma(\tilde{T_n}^k)$, $\forall n, k \in J^+$.

THEOREM 3.4.12, [Imagiri et al, 2013,[52]]; If T is a w-hyponormal operator, then both $\tilde{T^k}_n$ and $\tilde{T_n}^k$ are spectraloid $\forall n, k \in J^+$.

Proof

We only need to prove that, $\tilde{T^k}_n$ is spectraloid, since from Theorem 3.4.4 above, $\tilde{T^k}_n$ and $\tilde{T_n}^k$ have the same spectra $\forall n, k \in J^+$. We first note that, T^k is also a w-hyponormal operator, by Lemma 3.4.1, and from Theorem 3.3.12, $\tilde{T^k}_n$ is spectraloid for any pair of positive integers n, k.

REMARK 3.4.13; The class of w-hyponormal operators, contains all self-adjoint and all unitary operators. In the following last observation, locations of specra of nth-Aluthge transforms of the kth power of a w-hyponormal operator, are used to deduce conditions under which such operators end up being restricted to either, self-adjoint or unitary operators.

THEOREM 3.4.14, [Imagiri et al, 2013,[52]]; If T is a w-hyponormal operator, such that, $0 \notin W(T)$, then;
(i) $\tilde{T^k}_n$, is self adjoint if, $\sigma(\tilde{T^k}_n) \subset R$.
(ii) $\tilde{T^k}_n$, is positive if, $\sigma(\tilde{T^k}_n) \subset [0, \infty)$.
(iii) $\tilde{T^k}_n$, is unitary if, $\sigma(\tilde{T^k}_n)$, is a unit circle.

Proof

Let T be a w-hyponormal operator, such that, zero is an isolated point in the numerical range of T. It follows that, T is invertible. Thus, every power of such an operator, is also a w-hyponormal operator. That is, T^k is w-hyponormal for every positive integer k. But, every Aluthge transform of w-hyponormal operator is also a w-hyponormal operator since, Aluthge transformation reduces w-hyponormality to hyponormality. Thus, $\tilde{T^k}_n$ is a w-hyponormal for any pair of positive integers n and k. The rest of the proof follows from Lemma 1.6.2.3, since every w-hyponormal operator, is a class (A) operator.

REMARK 3.4.15; It is important to note that, Theorem 3.4.14 above, does not hold only for the nth-Aluthge transform of the kth power, that is, $\tilde{T^k}_n$, of w-hyponormal operators, but also holds for the kth power of the nth-Aluthge transform, that is, $\tilde{T_n}^k$. This conclusion follows from the fact that, zero is an isolated point in the numerical range of T, which instead ensures T is invertible, and thus, $\tilde{T^k}_n$ and $\tilde{T_n}^k$ are the same operator. However, this observation also holds, even without requiring T to be an invertible operator as shown by the following corollary;

COROLLARY 3.4.16, [Imagiri et al, 2013,[52]]; If T is a w-hyponormal operator, then;
(i) $\tilde{T_n}^k$, is self adjoint if, $\sigma(\tilde{T_n}^k) \subset R$.
(ii) $\tilde{T_n}^k$, is positive if, $\sigma(\tilde{T_n}^k) \subset [0, \infty)$.
(iii) $\tilde{T_n}^k$, is unitary if, $\sigma(\tilde{T_n}^k)$, is a unit circle.

Proof

Firstly, we have from Theorem 3.4.4 above, that, $\sigma(\tilde{T^k}_n) = \sigma(\tilde{T_n}^k)$, $\forall n, k \in J^+$. The rest of the proof follows from Theorem 3.4.14 above, since, even if $\tilde{T^k}_n \neq \tilde{T_n}^k$ generally, atleast $\tilde{T_n}^k$ and $\tilde{T^k}_n$, are both w-hyponormal operators for any chosen pair of positive integers n, k.

3.5; Normality and power quasinormality of w-hyponormal operators

In this section, by using results obtained in chapter two, we investigate the normality of w-hyponormal operators. The major tool used to come up with conditions under which w-hyponormal operators end up being restricted to the class of normal operators, is the Aluthge transform. In particular, it is proved that, if the nth-Aluthge transform of a w-hyponormal operator T is normal, then T is 2^n-Power normal, for any positive integer n. To begin with, we recall that, the class of w-hyponormal operators contains all p-hyponormal and log hyponormal operators. It is also known that, every invertible p-hyponormal operator is a log-hyponormal operator. One should also note that, classes of operators are not in general, invariant under generalized Aluthge transformations. That is, repeated Aluthge transforms of operators from a given class, do not always yield members of the class in question. However, the following result, shows that the first two Aluthge transforms of a w-hyponormal operator, are w-hyponormal operators;

LEMMA 3.5.1, [Aluthge et al, 2000,[4, Thm 2.4]]; If T is w hyponormal, then $\tilde{T}$ is semi hyponormal and $\tilde{\tilde{T}}$ is hyponormal.

REMARK 3.5.2; From Lemma 3.5.1 above, it easily follows that the class of w-hyponormal operators, is invariant under these transformations since both semi-hyponormal and hyponormal operators are w-hyponormals, and thus, the semi-hyponormality of $\tilde{T}$, and the hyponormality of $\tilde{\tilde{T}}$, implies that the 3^{rd} *and* 4^{th}-Aluthge transforms of a w-hyponormal T, are also w-hyponormal. It is also well known that, if an operator T is invertible, then all Aluthge transformations of T yields similar operators and that, if T is a normal operator, then all Aluthge transformations of T are also normal and conversely, as the following result shows;

LEMMA 3.5.3, [Aluthge et al, 2000,[4, Thm 2.1]]; If an operator T is invertible, then all Aluthge transforms of T are similar operators and, T is normal if and only if, every Aluthge transform of T is normal.

REMARK 3.5.4; Trivially, every normal operator is a semi-hyponormal operator. By Lemma 3.5.1, it follows that, the first Aluthge transform of a w-hyponormal operator is a semi-hyponormal operator. However, if this transform happens to be normal, then the original w-hyponormal operator also happens to be a normal operator, as the following result shows;

LEMMA 3.5.5, [Aluthge et al, 2000,[4]]; Let $T = U|T|$ be the polar decomposition of a w-hyponormal operator. If $\tilde{T}$ is normal, then $T = \tilde{T}$. That is, T is normal.

We noted that, if an operator T on a finite dimensional Hilbert space H is normal and $T = U|T|$ is its polar decomposition, then $[T, |T|] = 0$. This qualification is not generally inherited by larger classes of operators apart from the class of quasinormal operators, as the following result by [Bala, 1977] shows;

LEMMA 3.5.6, [16, Thm 2.4]; Let $T = U|T|$ be the polar decomposition of an operator T. Then T is quasinormal if and only if $U|T| = |T|U$.

REMARK 3.5.7; It is good to note that, the behaviour of the partial isometry operator U in the polar decomposition, $T = U|T|$, of any operator T, is in most cases, like that of T, as the following three lemmas shows;

LEMMA 3.5.8, [Kato,1965,[68]]; Let $T = U|T|$ be the polar decomposition of an operator T. Then $T^2 = 0$ if and only if $U^2 = 0$.
LEMMA 3.5.9, [Halmos,1967,[42]]; Let $T = U|T|$ be the polar decomposition of an operator T. Then;
(i) if T is binormal then so is U.
(ii) if T is quasinormal, then so is U.
(iii) if T is normal then so is U.
(iv) if T is self adjoint, then so is U.
(v) if T is positive, then so is U, and U is a projection.

LEMMA 3.5.10, [Kaplansky,1953,[67]]; Every normal operator can be written in the form UP, where P is positive and U can be taken to be unitary such that, $UP = PU$ and U commutes with all operators that commute with T and T^*.
We extended the above lemmas as follows;

THEOREM 3.5.11, [Imagiri et al, 2013,[52]]; Let T be a w-hyponormal operator. If $\tilde{T}$ is normal, then TT^*T is normal.
Proof
Let T be a w-hyponormal operator such that $\tilde{T}$ is normal. Then T is normal. But T being a bounded linear operator implies T^*T is self adjoint. From the normality of T, it follows that T

is quasinormal and thus, $[T, T^*T] = 0$. Notice that, both T^2 and T^* are also normal, and that, $TT^*T = T^2T^*$. This implies that, T is 2-Power normal. That is, $[T^2, T^*] = 0$. We have that, $(T^2T^*)^*(T^2T^*) = TT^{*2}T^2T^* = TT^*T^*TTT^* = TT^*TT^*TT^* = TTT^*TT^*T^* = T^2T^*TT^{*2} = (T^2T^*)(TT^{*2}) = (T^2T^*)(T^2T^{**})$. Thus, T^2T^* is normal. Hence, TT^*T is a normal.

THEOREM 3.5.12, [Imagiri et al, 2013,[52]]; Let T be a w-hyponormal operator. If $\tilde{T_2}$ is normal, then T is normal.

Proof

Assume that T is a w-hyponormal operator such that $\tilde{T_2}$ is normal. From Lemmas 3.5.3 and 3.5.5 above, $\tilde{T_1} = \tilde{T}$, is a semi-hyponormal operator, thus a w-hyponormal operator. But, $\tilde{T_2}$ is the first Aluthge transform of $\tilde{T}$. Thus, it follows that, $\tilde{T}$ is normal. Hence, T is also normal.

THEOREM 3.5.13, [Imagiri et al, 2013,[52]]; Let T be a w-hyponormal operator. If $\tilde{T_n}$ is normal for some $n \in J^+$, then T is normal.

Proof

Trivially, by induction, letting $n = 3$, then it follows that $\tilde{T_2}$ is normal since $\tilde{T_3}$ is the first Aluthge transform of $\tilde{T_2}$. In general, we have that $\tilde{T_{n-1}}$ is normal whenever $\tilde{T_n}$ is normal for any positive integer n. Thus, T is a normal operator if $\tilde{T_n}$ is normal.

THEOREM 3.5.14, [Imagiri et al, 2013,[52]]; Let T be a w-hyponormal operator. If $\tilde{T_n}$ is normal, then T is a 2^n-Power normal operator for any positive integer n.

Proof

We note that, the normality of $\tilde{T_n}$ implies that, T is also normal. But, T being normal on the other hand implies that, T^{2^n} is a normal operator $\forall n \in J^+$. Thus, from Lemma 2.2.1, it follows that T is a 2^n-Power normal operator. Hence, T is a 2^n-Power normal if $\tilde{T_n}$ is normal for any positive integer n.

THEOREM 3.5.15, [Imagiri et al, 2013,[52]]; Let T be a w-hyponormal operator such that $\tilde{T_n}$ is normal for some positive integer $n \geq 2$. Then, $ker(T^m) \subset ker(T^{m*}) \forall m \in J^+$.

Proof

Assume that $\tilde{T_n}$ is normal for some $n \geq 2$. Then, T is normal. It follows that, T^{2m} is also normal for any $m \in J^+$. That is, $T \in 2mQN$, $\forall m \in J^+$. Hence by Lemma 2.2.14, $ker(T^m) \subset ker(T^{m*})$, for any positive integer m.

COROLLARY 3.5.16, [Imagiri et al, 2013,[52]]; Let T be a w-hyponormal operator. If $\tilde{T_n}$ is normal for some $n \in J^+$, then $ker(T) = ker(T^*)$.

Proof

$\tilde{T_n}$ being normal for some $n \in J^+ \Rightarrow T$ is a normal operator. Thus, $ker(T) = ker(T^*)$.

3.6 Normality of the product of two w-hyponormal operators

In this part, some of the conditions under which the product of any two w-hyponormal operators become normal are proved.

THEOREM 3.6.1, [Imagiri et al, 2013,[53]]; Let A, B be any two w-hyponormal operators. If $\tilde{A}$ and $\tilde{B}$ are normal and $[A, B] = 0$, then the following holds;

(i) $[\tilde{A}, \tilde{B}] = 0$.

(ii) $[\tilde{A^*}, \tilde{B^*}] = 0$.

iii) $\tilde{AB}$ and $\tilde{BA}$ are normal operators.

Proof of (i)

We first note that, A, B, (AB) and (BA) are normal operators. Assume that, A and B have the polar decompositions $A = U|A|$ and $B = V|B|$ respectively. Thus, $\tilde{A} = |A|^{1/2}U|A|^{1/2}$ and $\tilde{B} = |B|^{1/2}V|B|^{1/2}$. Therefore, $(\tilde{A}\tilde{B}) = |A|^{1/2}U|A|^{1/2}.|B|^{1/2}V|B|^{1/2} = |A|UV|B|$, since the normality of A, B and Lemma 3.5.6 and 3.5.8 above, imply that, $[U, |A|] = 0$, $[|A|, |B|] = 0$ $[V, |A|] = 0$ and $[U, |B|] = 0$. Hence, $|A|UV|B| = V|B|U|A| = |B|^{1/2}V|B|^{1/2}|A|^{1/2}U|A|^{1/2} = \tilde{AB}$. That is, $[\tilde{A}, \tilde{B}] = 0$.

Proof of (ii)

$\tilde{A}$ and $\tilde{B}$ being normal implies that, $\tilde{A^*}$ and $\tilde{B^*}$ are normal. Since A, B are normal, then A^*, B^* are also normal, hence w-hyponormal. Trivially, $[A, B] = 0 \Rightarrow [A^*, B^*] = 0$. The rest of the proof follows from part (i) above.

Proof of (iii)

Since $\tilde{A}$, $\tilde{B}$ are normal and $[\tilde{A}, \tilde{B}] = 0$, then $(\tilde{A}\tilde{B})$ and $(\tilde{B}\tilde{A})$ are also normal. The normalities of A, B, (AB) ,(BA), $(\tilde{A}\tilde{B})$ and $(\tilde{B}\tilde{A})$ implies that, $(\tilde{A}\tilde{B}) = \tilde{AB}$ and $(\tilde{B}\tilde{A}) = \tilde{BA}$. Hence, $\tilde{AB}$ and $\tilde{BA}$ are normal operators.

COROLLARY 3.6.2, [Imagiri et al, 2013,[53]]; Let A, B be any two w-hyponormal operators. If $\tilde{A_n}$ and $\tilde{B_n}$ are normal, for some $n \in J^+$ and $[A, B] = 0$, then the following holds;
(1) $[\tilde{A_n}, \tilde{B_n}] = 0$
(2) $[\tilde{A^*}_n, \tilde{B^*}_n] = 0$
(3) $(\tilde{AB})_n$ and $(\tilde{BA})_n$ are normal operators.

Proof

The normalities of $\tilde{A_n}$ and $\tilde{B_n} \Rightarrow \tilde{A_{n-1}}$ and $\tilde{B_{n-1}}$ are also normal, for some $n \in J^+$. We have only to show that, $[\tilde{A_n}, \tilde{B_n}] = 0$ and the rest of the proof, follows from Theorem 3.6.1 above. Recall that, $[A, B] = 0 \Rightarrow [\tilde{A}, \tilde{B}] = 0$. Likewise, $[\tilde{A}, \tilde{B}] = 0 \Rightarrow [\tilde{A_2}, \tilde{B_2}] = 0$. In general, $[\tilde{A_{n-1}}, \tilde{B_{n-1}}] = 0 \Rightarrow [\tilde{A_n}, \tilde{B_n}] = 0$, for any positive integer n. Thus, $[\tilde{A_n}, \tilde{B_n}] = 0$, whenever $[A, B] = 0$.

3.7; Normality of the powers of sequences of Aluthge transforms of w-hyponormal operators

Recall that, Aluthge transforms of an operator 'improves' p-hyponormality of that operator for $p \leq 1$, since if, T is p-hyponormal for $1/2 \leq p < 1$, then $\tilde{T}$ is hyponormal, and if, T is p-hyponormal for $0 < p < 1/2$, then $\tilde{T}$ is $p + 1/2$-hyponormal. In addition, we have already seen that, if an operator T is w-hyponormal, then $\tilde{T}$ is semi-hyponormal, $\tilde{\tilde{T}}$ is hyponormal. It follows easily that, repeated Aluthge transformations of a w-hyponormal operator, results into a p-hyponormal operator and thus, the class of all w-hyponormal operators is closed with respect to generalized Aluthge transformations, since every p-hyponormal operator is w-hyponormal. We also noted that, if $\tilde{T}$ happens to be a normal operator, then its counter part, that is, T is normal, and conversely. Thus, repeated Aluthge transformations of a normal operator, begets another normal operator, no matter the number of transformations. However, if an operator T is normal, then T^n is not normal in general. It is also known that, an operator T being not normal, does not imply that, one can not find some positive integer n, that allows T^n to be normal. In other words, normality of T^n, for some positive integer n, does not always follow from that of T. The following theorem is inline with these observations;

THEOREM 3.7.1, [Imagiri et al, 2013,[53]]; Let T be a w-hyponormal operator. If there exists a pair of positive integers k *and* n, such that, $\tilde{T^k}_n$ is a normal operator, then T is a k-Power normal operator, but T is not normal in general.

Proof

It suffices only to show that, T is a k-Power normal operator since, there are a number of k-Power normal operators which are not normal. In fact, from Lemma 2.2.1, we only need to show that, T^k is a normal operator. Now assume to the contrary that, there does not exist a positive integer k, such that, T^k is a normal operator. By Lemma 3.4.1, T^k is a w-hyponormal operator. Thus, there exists a positive integer n, such that, $\tilde{T^k}_n$ is a normal operator. It follows that, T^k is also normal. Hence, T is a k-Power normal operator.

REMARK 3.7.2; When does the normality of $\tilde{T^k}_n$, imply the normality of T? To answer this question, we prove the following result;

THEOREM 3.7.3, [Imagiri et al, 2013,[53]]; Let T be a w-hyponormal operator, such that, $0 \notin W(T)$. If there exists a pair of positive integers k, n, such that, $\tilde{T^{2^k}}_n$ is a normal operator, then T is a normal operator.

Proof

From the proof of Theorem 3.7.1 above, T^{2^k} is a normal operator. By Theorem 2.4.19, T^{2^m} is also a normal operator for every positive integer $m \leq k$. Thus, if $m = 1$, we have that, T^2 is also normal. Consequently, the normality of T follows from the fact that, $0 \notin W(T)$.

THEOREM 3.7.4, [Imagiri et al, 2013,[53]]; Let T be a w-hyponormal operator. If there exists a pair of positive integers k and n, such that, $\tilde{T^k}_n$ is a normal operator, then $\tilde{T^{k^2}}_{n_1}$ is a normal operator, for every positive integer $n_1 \leq n$.

Proof

By Theorem 3.7.1, T^k is a w-hyponormal operator. And from Theorem 3.5.13, it folllows that, if there exists a positive integer n, such that, the nth-Aluthge transform of T^k is normal, then T^k is also a normal operator. Thus, from a property of normal operators, T^{k^2} is a normal operator. Hence, every Aluthge transform of T^{k^2} is normal.

THEOREM 3.7.4, [Imagiri et al, 2013]; Let T be a w-hyponormal operator. If there exists a pair of positive integers k and n, such that, $(\tilde{T_n})^{2^k}$ is a normal operator, then the following are normal operators;

(i) $(\tilde{T_{n_1}})^{2^k}$, for every positive integer n_1 less than n.

(ii) $(\tilde{T_n})^{2^{k_1}}$, for each positive integer k_1 less than k, if $0 \notin W(T)$.

(iii) $(\tilde{T_{n_1}})^{2^{k_1}}$, for each pair of positive integers $k_1 \leq k$ and $n_1 \leq n$, if $0 \notin W(T)$.

Proof of (i)

From theorem 3.7.3, we have that, $\tilde{T_{n_1}}$ is normal for each positive integer n_1 less than n. Thus, $(\tilde{T_{n_1}})^2$ is also normal. Generally, $(\tilde{T_{n_1}})^{2^k}$, is also a normal operator for each positive integer k.

Proof of (ii)

If $0 \notin W(T)$, then $0 \notin W(\tilde{T})$. Moreover, $0 \notin W(\tilde{T_n})$. From Theorem 3.33, it follows that, $0 \notin W(\tilde{T_n}^{2^k})$. From Theorem 3.7.3, we have that, $(\tilde{T_n})^{2^{k_1}}$, is normal for each positive integer k_1 less than k, since $(\tilde{T_n})^{2^k}$ is a normal operator for any positive integer n.

Proof of (iii)

From the proof of part (i) above, it follows that, $\tilde{T_{n_1}}$ is normal for each positive integer n_1 less than n. Thus, $(\tilde{T_{n_1}})^{2^k}$ is also normal for any positive integer k. And from the proof of part (ii), we have that, $(\tilde{T_{n_1}})^{2^{k_1}}$, is normal for each pair of positive integers $k_1 \leq k$ and $n_1 \leq n$.

Chapter four

EXTENSIONS OF THE PUTNAM-FUGLEDE THEOREM, THE PUTNAM'S INEQUALITY AND THE BERGER-SHAW INEQUALITY

In this chapter, it is first shown that, the class of w-hyponormal operators is not the same as that of n-Power normal operators, and thus w-hyponormal operators are different from n-Power quasinormal operators. Then, sufficient conditions implying self-adjointedness of either any two similar n-Power normal, or any two similar n-Power quasinormal operators are presented. In addition, the Putnam-Fuglede theorem, the Putnam's inequality and the Berger-Shaw's inequality are studied for n-Power normal, n-Power quasinormal and w-hyponormal operators.

4.1; Difference between w-hyponormal and n-Power quasinormal operators

Since it is known that, if T is a normal operator, then there exists a unitary operator U such that, $T = UP = PU$, where, $P = (T^*T)^{1/2}$, and both U and P commute with V^*, V and $|A|$, of the polar decomposition $A = V|A|$, of any A commuting with both T and T^*, then one might conclude that, if there exists a positive integer n, such that, B is n-Power normal, or equivalently, B^n is normal, for some non normal operator $B \in B(H)$, then B^n decomposes as $B^n = U'P' = P'U'$, where U' is a unitary and P' is a projection such that both U' and P' commute with V', V'^* and $|C|$ of the polar decomposition $C = V'|C|$ of any operator $C \in B(H)$ which commute with both B^n and B^{n*}. This behaviour of a non normal operator B comes in from the fact that $[B^n, B^*] = 0$, if and only if, B^n is normal, for the same positive integer n. However, such interplay between normality and n-Power normality of linear operators, is not exhibited between quasinormality and n-Power quasinormality. In other words, it is unfortunate that, existence of a positive integer n, which allows n-Power quasinormality of a linear operator T, does not guarantee generally the quasinormality of T^n. This limitation makes the study of n-Power quasinormal operators non user friendly since they are not easily diagonalizable. To worsen their study, if T is n-Power quasinormal, for some positive integer n, then T^n commutes with (T^*T), but the polar decomposition of T^n, that is, $T^n = U|T^n|$, does not readily send any light to that of T, since so far, the classification of U, rather than being a partial isometry,(a general case for all bounded linear operators), is not known. Unlike n-Power quasinormal operators, one should note that if T is n-Power normal, for some positive integer n, then T^n is normal, and thus, the polar decomposition of T^n, that is, $T^n = U|T^n|$, sends some useful informations about the decompositions of T, since this time round, the 'usual' partial isometry U, is reduced to a unitary. Nevertheless, every n-Power normal operator is n-Power quasinormal. And atleast, just like n-Power normals, the class of n-Power quasinormal operators, also enjoys one of the fundamental feature in decompositions, of being invariant under restrictions. That is, the restriction of n-Power quasinormal operator to any invariant subspace, is also n-Power quasinormal. One might also be aware of the fact that, a non normal operator T, might fail to be either n-Power normal or n-Power quasi normal, till when n becomes a very large positive integer. This worsens the study of these classes, since given any non normal matrix A for example, it is not easy to fore tell whether, all powers of A, do not commute with either A^T or with (A^TA). That is, it is not easy for one to confidently claim that, there does not exist a positive integer n, such that, $[A^n, A^T] = 0$ or $[A^n, (A^TA)] = 0$. Like as if these obstructions are not enough, [Jibril, 2008], observed that n-Power normal operators are independent from p-hyponormal operators. Trivially, as it was confirmed by [Ahmed, 2011], n-Power quasinormal operators are also different from p-hyponormal operators. Recall that, w-hyponormal operators includes all p-hyponormal operators. Consequently, n-Power quasinormal and w-hyponormal operators, are both non normal but atleast, their intersection includes all normal and all quasinormal operators. It is therefore natural to ask whether the observations by Jibril and Ahmed, hold true when p-hyponormality is replaced by w-hyponormality.

We start off this chapter by proving that, the class of n-Power normal, and hence that of n-Power quasinormal, are independent from w-hyponormal operators. It is good to recall that, every w-hyponormal operator is paranormal. That is , if $T \in B(H)$ satisfies the inequality, $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$, then T also satisfies the inequality, $\|T^2x\| \geq \|Tx\|^2$, for every unit vector x in H. However, if T is such that, $[T^n, T^*] = 0$, or $[T^n, T^*T] = 0$, then T does not in general satisfy the inequality, $\|T^2x\| \geq \|Tx\|^2$. By using paranormality of T, it is shown in this section that, n-Power normal operators are different from w-hyponormal operators and that, every operator

$T \in B(H)$, which is both n-Power normal and w-hyponormal at the same time, happens to be normal. To achieve this, the following well known results will be needed;

LEMMA 4.1.1, [Embry, 1966,[31]]; Let T be a paranormal operator. If T^n is normal for some positive integer n, then T is normal.

LEMMA 4.1.2, [Aluthge, 2000,[4]]; Every w-hyponormal operator is paranormal.

REMARK 4.1.3; It can easily be seen from the two Lemmas above that, if T is w-hyponormal and there exists atleast one positive integer n such that T^n is normal, then T is normal. Using this qualification of paranormal and w-hyponormal operators, we prove the following two results which leads to the conclusion that these classes are indepedent from that of n-Power normal operators.

THEOREM 4.1.4, [Imagiri, 2014,[54]]; Let $T \in nN$, for some positive integer n. Then T is not paranormal.

Proof

Let $T \in nN$. Assume to the contrary that T is paranormal. From Lemma 2.2.1, it follows that T^n is normal. By Lemma 4.1.1, T is normal. Thus, $T \in nN$, for some positive integer n, implies that T is normal. Which is false.

REMARK 4.1.5; Notice that Theorem 4.1.4 above holds true by substituting n-Power normal with n-Power quasinormal since every n-Power normal operator is n-Power quasinormal. The paranormality of T in this theorem can be restricted to w-hyponormality as shown by the following observation.

COROLLARY 4.1.6, [Imagiri, 2014,[54]]; Every n-Power normal operator is not w-hyponormal and vice-versa.

Proof

The proof of if T is such that $[T^n, T^*] = 0 \nRightarrow |\tilde{T}| \geq |T| \geq |\tilde{T^*}|$, for any positive integer n, follows trivially from Theorem 4.1.4, since every w-hyponormal operator is paranormal. To prove the reverse implication, we assume to the contrary that, every w-hyponormal operator T is such that, there must exist some positive integer n, for which T is n-Power normal. Then, T is normal since T is w-hyponormal and T^n is normal. Thus, every w-hyponormal is normal. A contradiction.

REMARK 4.1.7; Recall that n-Power normal operators have the characteristic of taking-in the adjoint of T whenever they take-in T itself, but for the same positive integer n. That is, if $T \in B(H)$ is such that, $T \in nN$, for some $n \in J^+$, then $T^* \in nN$, for the same n. This behaviour of n-power normal operators can also be used to disqualify them from the class of w-hyponormal operators, since if it follows to the contrary that every n-Power normal operator is w-hyponormal, then it would also follow that, if T is w-hyponormal, then so is T^*, which is not true in general. But what happens when both T and T^* are w-hyponormal? To answer this question, we prove the following result.

THEOREM 4.1.8, [Imagiri, 2014,[54]]; Let $T \in B(H)$. If T and T^* are w-hyponormal operators, then T is ∞-Power normal.

Proof

Assume that both T and T^* are w-hyponormal operators. Then, $|\tilde{T}| \geq |T| \geq |\tilde{T^*}|$, and $|\tilde{T^*}| \geq |T| \geq |\tilde{T}|$. Thus, $|\tilde{T}| = |T| = |\tilde{T^*}|$. In particular, $|\tilde{T}| = |\tilde{T^*}|$. But, $|\tilde{T}| = (\tilde{T^*}\tilde{T})^{1/2}$ and $|\tilde{T^*}| = (\tilde{T}\tilde{T^*})^{1/2}$. So that, $|\tilde{T}| = (\tilde{T^*}\tilde{T})^{1/2} = |\tilde{T^*}| = (\tilde{T}\tilde{T^*})^{1/2}$. And thus, $(\tilde{T^*}\tilde{T})^{1/2} = (\tilde{T}\tilde{T^*})^{1/2}$ $\Rightarrow (\tilde{T^*}\tilde{T}) = (\tilde{T}\tilde{T^*})$. Therefore, $\tilde{T}$ is normal. From Lemma 3.5.3, we have that T is normal, and from Theorem 2.5.9, it follows that T is ∞-Power normal.

REMARK 4.1.9; It can easily be noticed from Theorem 4.1.8 above that, if both T and T^* are w-hyponormal operators, then T is normal. Therefore, it is sensible to come up with more conditions which allows T^* to be w-hyponormal, whenever T is. In the light of this, we have the following result.

THEOREM 4.1.10, [Imagiri, 2014,[54]]; Let T be a w-hyponormal operator. If there exists atleast one positive integer n, such that, T^* is a n-Power normal operator, then T is normal.

Proof

It suffices to prove that, $[T^*, T] = 0$, whenever T satisfies both $|\tilde{T}| \geq |T| \geq |\tilde{T^*}|$ and $[T^n, T^*] = 0$, for some $n \in J^+$, since, the normality of T^{n*}, implies trivially that of T^n. In other words, notice that, n-Power normality of T^*, implies that T^{n*} is normal, (by Lemma 2.2.1), and thus, T^n is normal. Hence, T is w-hyponormal and T^n is normal. Therefore, from Lemma 4.1.1, we have that, T is normal.

REMARK 4.1.11; To this end, it is important to note that, in addition to Theorem 4.1.4, Corollary 4.1.6 and Theorem 4.1.8 also hold true by substituting power normality with power quasinormality. However, Theorem 4.1.10 holds true for n-Power quasinormal operators, only after imposing more requirements on T^*. One good example of such a requirement is by demanding that, T should also be n-Power quasinormal, and in addition that, $ker(T) \subset ker(T^*)$. All in all, one ends up concluding that, $nQN \quad \not\subset \quad w-hyponormal$ and $w-hyponormal \quad \not\subset \quad nQN$. Recall that, the class of w-hyponormal operators is also independent from that of (p,k)-quasi hyponormal operators, but both classes contains all p-hyponormal operators. Despite the indepedence, it is known that these classes share many ineteresting properties with hyponormal operators. It is also known that, if $T \in B(H)$, the existence of another operator $S \in B(H)$, satisfying the conditions, $T = ST^*S^{-1}$ and $0 \notin \bar{W(S)}$, ends up sometimes restricting T to the class of normal operators, especially in cases when T happens to be non normal. For instance, [Stampfli, 1962], observed that if T is hyponormal, and there exists some S on H, satisfying the above given conditions, then T is self-adjoint. [Xia, 1983], proved that existence of such $S \in B(H)$, implies that, all eigenvalues of T are real numbers, for every $T \in B(H)$. Later, [In Hyoun Kim, 2004], used the results by Williams, and concluded that, T remains self adjoint even when the hyponormality in the results due to Sheth is replaced by p-hyponormality. We wind up this section by extending the results of Sheth and In Hyoun Kim, to the class of n-Power quasinormal operators. Firstly, we state the results of Stampfli, Xia and In Hyoun Kim.

LEMMA 4.1.12, [93, Preposition 4]; If T is a hyponormal operator and $T = ST^*S^{-1}$, for any operator S, where $0 \notin \bar{W(S)}$, then T is self-adjoint.

REMARK 4.1.13; Observe that, if T is self-adjoint, then, the spectrum of T lies on the real line. But, the locations of spectra of operators from higher classes is not trivial. Atleast, the following result sheds more light about the location of spectrum of any operator, but after imposing some extra requirements.

LEMMA 4.1.14, [97, Thm 2.6]; If T is any operator and $T = ST^*S^{-1}$, for any operator S, where $0 \notin \bar{W(S)}$, then $\sigma(T) \subset |R$.

LEMMA 4.1.15, [60, Lemma 2.4]; If T or T^* is a p-hyponormal operator, and $T = ST^*S^{-1}$, for any operator S, where $0 \notin \bar{W(S)}$, then T is self-adjoint.

REMARK 4.1.16; For any operator $A \in B(H)$, let $A = U|A|$ be the polar decomposition of A. It is known that, if A is injective, then $|A|$ is injective, and in general, $|A|^n$ is injective for each positive integer n. As a consequence, $(A^*A)^n$ is injective for every n. Conversely, the injectiveness of A is obvious from the injectiveness of A^*A. Thus, an operator A is injective if and only if, an operator $(A^*A)^k$ is injective for each positive integer k. Due to these observations, we substitute the hyponormality in the result by Sheth, with n-power normality or even with n-Power quasinormality, and prove the following results.

THEOREM 4.1.17, [Imagiri, 2014,[54]]; If T is a n-Power normal operator, and $T = ST^*S^{-1}$, for any operator S, where $0 \notin \bar{W(S)}$, then T is self-adjoint.

Proof

Firstly, note that, $T = ST^*S^{-1}$, implies that, $TS = ST^*$, and thus T is similar to T^*. Thus, $T^k = ST^{k^*}S^{-1}$, for every positive integer k. In particular, $T^n = ST^{n*}S^{-1}$, for some positive integer n. Since T is n-Power normal, then by Lemma 2.2.1, T^n is also normal. It follows that, T^n is hyponormal. So that from Lemma 4.1.12 above, T^n is self-adjoint. $\Rightarrow T^n = T^{n*} = T^{*n}$. If $T^n = U|T^n|$, is the polar decomposition for T^n, then U is unitary since T^n is normal and thus, $T^{n*} = |T^{n*}|U^*$, is the polar decomposition for T^{n*}. Therefore, $|T^{n*}|U^* = U|T^n| \Rightarrow |T^{n*}| = |T^n| \Rightarrow |T^*|^n = |T|^n \Rightarrow |T^*| = |T| \Rightarrow (TT^*)^{1/2} = (T^*T)^{1/2} \Rightarrow (TT^*) = (T^*T)$, and thus T is normal.

COROLLARY 4.1.18, [Imagiri, 2014,[54]]; If T and T^* are n-Power quasinormal operators, and $T = ST^*S^{-1}$, for any operator S, where $0 \notin \bar{W(S)}$, then T is self-adjoint.

Proof

Let both T and T^* be n-Power quasinormal operators, for the same positive integer n. Then by Lemma 2.2.12, T is n-Power normal. The rest of the proof follows from Theorem 4.1.17 above.

REMARK 4.1.19; It is good to note that, by applying Lemma 2.1.9, the additional requirement that T^* should also be n-Power quasinormal, in Corollary 4.1.18 above, can be dropped and the rest of the result holds true, but after demanding the kernel condition to hold for T. That is, if T is a n-Power quasinormal operator such that $ker(T^*) \subset ker(T)$, and $T = ST^*S^{-1}$, for any operator S, where $0 \notin \bar{W(S)}$, then T is self-adjoint.

4.2; Putnam-Fuglede theorem for n-Power normal, n-Power quasi-normal and w-hyponormal operators

In studying linear operators which are not normal, the major step has been that of finding methods of decomposing such operators into various parts which are easier to handle. Naturally, one first identifies subspaces which are invariant under such operators. Recall that, by a subspace of a Hilbert space H, we mean a closed linear manifold of H, which is also a Hilbert space. Recall that, a subspace M of a Hilbert space H, is invariant under T, if $T(M) \subseteq M$. That is, M is invariant under T, if for every vector $x \in M$, $Tx \in M$. And that, a subspace M of H is said to reduce T, if both M and $M^{\perp}$, are invariant under T. In addition, an operator T is said to be completely non-unitary, (respectively, completely non-normal or pure), if the restriction of T to any nonzero reducing subpace, is not unitary, (respectively, not normal). This identification of invariant subspaces is good since, it is known that, if M is an invariant subspace for T, then H can be decomposed as, $H = M \bigoplus M^{\perp}$, and relative to this decomposition, T has the matrix decomposition,

$$T = \begin{bmatrix} T|_M & X \\ 0 & Y \end{bmatrix}$$

for operators $X : M^{\perp} \to M$ and $Y : M^{\perp} \to M^{\perp}$, where $T|_M$ denotes the restriction of T to M. Conversely, if an operator T can be written as the triangulation,

$$T = \begin{bmatrix} Z & X \\ 0 & Y \end{bmatrix}$$

in terms of the decomposition $H = M \bigoplus M^{\perp}$, then $Z = T|_M : M \to M$, X and Y are parts of T. It is well known that, $X = 0$, if and only if, M reduces T. In such a case, the operator T, is decomposed into the orthogonal direct sum of the operators $Z = T|_M$ and $Y = T|_{M^{\perp}}$: Thus, $T = Z \bigoplus Y$. And therefore, if M reduces T, then T has the matrix decomposition,

$T = \begin{bmatrix} Z & 0 \\ 0 & Y \end{bmatrix}$. Hence, studying the properties of T, is reduced into studying the properties of its direct summands, Z and Y, which are known to be less complicated than T.

A subspace M of H, is said to be T - hyperinvariant, if M is invariant for every operator that commutes with T, that is, M is hyperinvariant for T if $S(M) \subset M$, for every operator $S \in B(H)$, such that, $ST = TS$. Clearly, if M is a reducing hyperinvariant subspace for T, then M is invariant under T. Generally, one might observe that; reducing subspaces $\subseteq$ invariant subspaces, and thus, hyperinvariant subspaces $\subseteq$ invariant subspaces.

It follows that, direct summands of an operator T, are therefore the restrictions of T to reducing subspaces. It is known that, every $T \in B(H)$, has the trivial reducing subspaces H and

$\{0\}$, and that, there are some operators on H, whose reducing subspaces are only the trivial ones. In this connection, an operator $T \in B(H)$, is said to be reducible, if T has atleast one nontrivial reducing subspace (equivalently, $T \in B(H)$, is said to be reducible, if T has a proper nonzero direct summand). Otherwise, T is said to be irreducible. It is also good to recall that, a lattice is a set in which every pair of any two elements, have a supremum (least upper bound), and an infimum(greatest lower bound). Trivially, the set of all invariant subspaces for $T \in B(H)$, is a lattice. In this chapter, this set is denoted by $Lat(T)$, and is defined as, $Lat(T) = \{M \subseteq H : T(M) \subseteq M\}$.

Putnam-Fuglede theorem provides more light on the study of operators which commute with two normal operators A and B, even in cases where $AB \neq BA$. Given any two normal operators, say A and B on H, Putnam-Fuglede theorem, [Putnam, 1951], claims that, if $AX = XB$ for some $X \in B(H)$, then, $A^*X = XB^*$. In other words, this theorem says that, if we define $\delta_{A,B} : B(H) \to B(H)$ by $\delta_{A,B}(X) = AX - XB$, for any $X \in B(H)$, and if A and B are normal operators and $X \in ker(\delta_{A,B})$, then $X \in ker(\delta_{A^*,B^*})$. Moreover, $[ran \;\; X]$ reduces A, $(ker \;\; X)^{\perp}$ reduces B and $A/_{ran \;\; X}$, $B/_{(ker \;\; X)^{\perp}}$ are normal operators. It is important to note that, one quick implication which follows from this theorem is that, if A, B, T are normal operators such that, $TA = AT$ and $TB = BT$, then $TA^* = A^*T$, $TB^* = B^*T$, $T^*A = AT^*$ and $T^*B = BT^*$. In general, if any pair of operators, say $A, B \in B(H)$, satisfies the PF theorem, then the similarity of A, B, implies that of A^*, B^*, provided that the intertwining operator X is a quasi affinity. Naturally, by imposing more requirements on the pair A, B, these two becomes quasi similar, and hence more light is thrown on their direct sum decompositions. Thus, the interplay between the PF theorem and similarity, (or even quasi simirality), is of great importance in studying linear operators since these two, almost imply one another. For example, [Duggal, 2000,([29])], after coming up with sufficient conditions for any two p-hyponormal operators to satisfy the PF theorem, concluded that, the normal parts of quasi similar p-hyponormal operators, are unitarily equivalent. In the same paper, the author showed that, a p-hyponormal operator which is quasi-similar to an isometry, is a unitary. [Duggal, et al, 2004,([30])], proved that the results in [29], hold true when p-hyponormality is substituted by log-hyponormality. [Ouma, 2007,([84])], extended the results in [30], and proved that these results hold true for w-hyponormal operators as well. As was seen in Section 1.4.3 above, more generalizations of the PF Theorem have appeared over the first four decades. In this section, attention is only paid to assymetric extensions of the this theorem. That is, we study the PF theorem in cases where the normality of the pair is replaced by a weaker requirement such as n-Power quasinormality or w-hyponormality. In other words, we set out in this section, to find sufficient conditions which

imply similarity or quasi similarity of the pair A, B, where both are either n-Power normal or, both are n-Power quasinormal. Then, we investigate conditions under which the pair A, B, satisfies the PF theorem, for a w-hyponormal operator A, and n-Power normal, (or, an n-Power quasinormal) operator B.
Before presenting our results, we first discuss the following well known observations, which form the background of this study.

Recall that the product and the sum of two normal operators, are also normal if the two operators under consideration commute. That is, if $A, B \in B(H)$, are both normal such that, $[A, B] = 0$, then both AB and $A + B$, are also normal. In relation to PF Theorem, [Putnam, 1970], obtained the following result, that is called the second degree PF Theorem, and is denoted by (SPF) theorem, henceforth.

LEMMA 4.2.1, [87, Preposition 3]; If N, M are normal operators in $B(H)$, and if $X \in (B(H)$, such that, $N(NX - XM) = (NX - XM)M$, then $NX = XM$.
REMARK 4.2.2; It is good to recall that, any two normal operators satisfy the PF Theorem, and thus any two similar normal operators, are unitarily equivalent. It also follows by the PF Theorem that, their adjoints are also unitarily equivalent. It follows from Lemma 4.2.1 above that, if there exists an operator $T \in B(H)$, such that, $T = NX - XM$, for some normal operators M and N, and any other operator X, then these two normal operators are unitarily equivalent provided, $NT = TM$.
Recall that, every invertible p-hyponormal operator is log-hyponormal and all p-hyponormal operators are properly contained in the class of (p,k)-quasi hyponormal. [Bakir, 2000], extended the PF Theorem to the class of log hyponormal and (p,k) quasihyponormal operators and proved the following two results;

LEMMA 4.2.3, [15, Thm 2.2]; Let $A \in B(H)$ and A^* be a (p,k) quasihyponormal operator such that, $ker \;\; A \subset ker \;\; A^*$ and B^* be a dominant operator $\in B(H)$. if $AC = CB$, for some $C \in B(H)$, then $CA^* = B^*C$. Moreover, $[ran \;\; C]$ reduces A, $(ker \;\; C)^{\perp}$ reduces B and $A/_{ran \;\; C}$, $B/_{(ker \;\; C)^{\perp}}$ are unitarily equivalent normal operators.

REMARK 4.2.4; If T is a dominant operator, then $ker(T) = ker(T^*)$. However, if T is a (p,k)-quasi hyponormal, this kernel condition does not hold in general. One might note that, even if dominant operators and (p,k)-quasi hyponormal operators are not normal operators in general, it follows by Lemma 4.2.3 above that, these classes are restricted back to normal operators, if the intertwinning operator C is a quasiaffinity, provided the kernel condition holds true for the (p,k)-quasi hyponormal class. This is trivial since, if C is a quasiaffinity, then $ran(C)$ is dense in H. Thus, $ran(C) \cap Ker(C) = \{0\}$, and thus, $A/_{ran \ \ C} = H$, $B/_{(ker \ \ C)^{\perp}} = H$. Therefore, A, B are not only normal, but are also unitarily equivalent. The following result, also by [Bakir, 2000], implies that dominance and (p,k)-quasi hyponormality in Lemma 4.2.3, can be substituted by log-hyponormality and class Y operators respectively, and yet the result follows even after dropping the kernel condition on class Y operators.

LEMMA 4.2.5, [15, Thm 2.4]; Let $A \in B(H)$ be a log hyponormal operator and $B^* \in B(H)$ be a class Y operator. If $AC = CB$, for some $C \in B(H)$, then $A^*C = CB^*$. Moreover, $[ran \ \ C]$ reduces A, $(ker \ \ C)^{\perp}$ reduces B and $A/_{ran \ \ C}$, $B/_{(ker \ \ C)^{\perp}}$, are unitarily equivalent normal operators.

REMARK 4.2.6; From elementary mathematics, recall that, if M is any set, then a relation R (or equivalently, a-non empty subset of the cartesian product $M \times M$), from M to M, is said to be equivalent, if for any three elements $x, y \ \ and \ \ z \in M$, R satisfies the following conditions; (i) $(x, x) \in R$, (ii) If $(x, y) \in R, \ \ then \ \ (y, x) \in R$ and (iii) If $(x, y) \ \ and \ \ (y, z) \in R, \ \ then \ \ (x, z) \in R$. Equivalence relations play a big role in $B(H)$, since for instance, these relations imply transitivity. That is, if two operators say, (A, B) are paired by a given relation and C is any other operator, then (A, C) are also paired by this relation provided that, (B, C) are paired. From the Fuglede theorem and the PF Theorem, it easily follows that, this PF is both reflexive and symmetric respectively. If we let $A = (N_1, N_2)$ and $B = (M_1, M_2)$, denote tuples of commuting normal operators in $B(H)$, and define the elementary operators $\Delta_{A,B}$ and $\Delta_{A^*,B^*} \in B(H)$, by $\Delta_{A,B}(X) = N_1XN_2 - M_1XM_2$ and $\Delta_{A^*,B^*}(X) = {N_1}^*X{N_2}^* - {M_1}^*X{M_2}^*$, then the following result by [Yin, 2004], shows that the PF Theorem is transitive as well, and thus an equivalence relation on $B(H)$.

LEMMA 4.2.7, [100, Preposition 2.1]; If the operators $N_i, M_i \in B(H)$, $i = 1, 2$ are normal operators, then $\Delta_{M_i,N_i}(X) = 0$, for some $X \in B(H)$, implies $\Delta_{{M^*}_i,{N^*}_i}(X) = 0$.

REMARK 4.2.8; Recall that, an operator $T \in B(H)$, is said to be a quasinilpotent if there exists some positive integer n, such that, $T^n = 0$. Thus, if T is a nilpotent, then $TT^{n-1} = 0$. And from the fact that, if $AB = 0$, for any two operators A and B on H, $N(A)$ is a non trivial invariant subspace for B and $R(\bar{B})$ is a non trivial invariant subspace for A, it therefore follows that, every nilpotent operator has a non trivial invariant subspace. The following result, still by [Yin, 2004], is an extension of PF Theorem under perturbation by quasinilpotents;

LEMMA 4.2.9, [100, Preposition 2.3]; Let A, B be normal operators and C, D be quasinilpotents such that, $AC = CA$, $BD = DB$. If $(A + C)X = X(B + D)$, for some $X \in B(H)$, then $AX = XB$.

REMARK 4.2.10; We note that as was seen earlier, one of the famous problems in operator theory, has been that of coming up with a non trivial invariant subspace for any linear transformation on H. It is not known whether every operator on H, especially when the dimension of H is not finite, has a non trivial invariant subspace. Fortunately, if any pair of operators satisfy the PF Theorem, then both operators involved in the pair are reducible, as the following result by [Kotaro, 2004], shows;

LEMMA 4.2.11, [71, Corollary 2.5]; Let $A, B \in B(H)$, then the following assertions are equivalent;
(1) A, B satisfy PF Theorem.
(2) If $AC = CB$, for some $C \in B(H)$, then $ran \quad C$ reduces A, $(ker \quad C)^{\perp}$ reduces B, and $A/_{ran \quad C}$, $B/_{(ker \quad C)^{\perp}}$ are unitarily equivalent normal operators.

REMARK 4.2.12; Class Y operators are indepedent of p-hyponormal operators. To come up with conditions under which a class Y operator happens to be unitarily equivalent to a p-hyponormal operator, [Mecheri et al. 2006], extended the PF result as follows;
LEMMA 4.2.13, [75, Thm 3]; Let $A \in B(H)$ be an injective p-hyponormal operator, and $B^* \in B(H)$, be a class Y operator. If $AC = CB$, for some $C \in B(H)$, then $A^*C = CB^*$. Moreover, $[ran \quad C]$ reduces A, $(ker \quad C)^{\perp}$ reduces B and $A/_{ran \quad C}$, $B/_{(ker \quad C)^{\perp}}$ are unitarily equivalent normal operators.

REMARK 4.2.14; We racall that, every invertible p-hyponormal operator is log-hyponormal and the class of w-hyponormal operators contains all p-hyponormal and log-hyponormal operators. It is known that, if an operator T is p-hyponormal, then $ker(T) \subset ker(T^*)$, and if T is a log-hyponormal operator, then $ker(T) = ker(T^*)$. Unfortunately, if T is a w-hyponormal operator, the kernel condition does not always hold. However, if this kernel condition holds true for w-hyponormal operators, then any pair of operators from this class, also satisfy the PF Theorem as the following result by [Ouma, 2007], shows below;

LEMMA 4.2.15, [84, Thm 1]; Let $A, B^* \in B(H)$ be w-hyponormal operators with $ker \ \ A \subset ker \ \ A^*$ and $ker \ \ B^* \subset ker \ \ B$. If $AC = CB$ for some $C \in B(H)$, then $A^*C = CB^*$. Moreover, $[ran \ \ C]$ reduces A, $(ker \ \ C)^\perp$ reduces B and $A/_{ran \ \ C}$, $B/_{(ker \ \ C)^\perp}$ are unitarily equivalent normal operators.

REMARK 4.2.16; We noted earlier that, if any two operators are unitarily equivalent, and one of them, has a non trivial reducing subspace, so is the other. To characterize sufficient conditions for when a w-hyponormal operator ends up being unitarily equivalent to a dominant operator, [Bachir et al, 2012], observed as follows;

LEMMA 4.2.17, [14, Thm 3.3]; Let $A \in B(H)$ be a dominant operator, and $B^* \in B(H)$ be a w-hyponormal operator such that, $ker(B^*) \subset ker(B)$. Then, the pair (A, B) satisfies the PF Theorem.

REMARK 4.2.18; We need to assert that, a necessary condition for the pair (T, T^*) to satisfy the PF Theorem is that, $ker(T) \subset ker(T^*)$. As we have already mentioned, for w-hyponormal operators, this is not generally true. To confirm this, assume P is the orthogonal projection onto $ker(A)$, where $A \in B(H)$ is a w-hyponormal operator. Then, $AP = PA^*$, but $A^*P \neq PA$. Thus, $ker(A) \not\subset ker(A^*)$, and also $ker(A^*) \not\subset ker(A)$, for every w-hyponormal operator A. Hence, these operators do not always satisfy the PF Theorem. However, if A, B^* are w-hyponormal operators such that, $ker(A)$ reduces B^*, then the pair (A, B) satisfies the PF Theorem as the following two observations, again by [Bachir et al, 2012], shows;

LEMMA 4.2.19, [14, Thm 3.5]; Let $A \in B(H)$ be a w-hyponormal operator, and $B^* \in B(H)$, be an injective w-hyponormal operator. Then the pair (A, B) satisfies the PF Theorem.

LEMMA 4.2.20, [14, Thm 3.6]; Let $A \in B(H)$ be a w-hyponormal operator, such that, $ker(A) \subset ker(A^*)$, and $B^* \in B(H)$, such that $ker(B^*) \subset ker(B)$, then the pair (A, B) satisfies the PF Theorem.

REMARK 4.2.21; To generalize the PF Theorem, for cases when the operators A, B are n-Power normal, we trivially have;

COROLLARY 4.2.22, [Imagiri, 2014,[59]]; Let A, B be n-Power normal operators, for some positive integer n. If there exists another operator X, such that, $A^nX = XB^n$, then $A^{n*}X = XB^{n*}$.

Proof

From Lemma 2.2.1 above, it follows that, A^n is normal since $A \in nN$. Also, if $B \in nN$, then B^n is normal. Inparticular, A^n *and* B^n, are normal operators. Thus, the pair, (A^n, B^n), do satisfy the PF Theorem.

REMARK 4.2.23; Corollary 4.2.22 above follows easily from the normality of A^n *and* B^n. Recall that, all powers of a normal operator are normal. In other words, if T is a normal operator, then T^k is normal for every positive integer k, and this in turn yields, $T \in nN$, for every n. However, the converse, in this observation holds true only in cases when T is n-Power normal for every positive integer n. But, existence of a positive integer n, which sends T^n to the class of normal operators, does not generally, guarantee the normality of T. This is the reason why, in Corollary 4.2.22 above, the existence of some X satisfying, $A^nX = XB^n$, does not imply $A^*X = XB^*$. To guarantee this implication, we first prove the following result, then apply it;

THEOREM 4.2.24, [Imagiri, 2014,[59]]; Let A be n-Power normal operator, for some positive integer n, and B be a normal operator. If there exists another operator X, such that, $A^nX = XB$, then $A^*X = XB^*$.

Proof

Without loss of generallity, let X be a quasiaffinity. Then, if $A^nX = XB$, where A^n and B are normal operators, then the pair (A^n, B), satisfies the PF Theorem. Thus, $A^{n*}X = XB^*$. Since, $A^nX = XB$, then A^n and B are similar normal operators, and thus unitarily equivalent. Likewise, from the fact that, $A^{n*}X = XB^*$, and since normality of T^n implies that of T^{n*}, A^{n*} and B^* are similar normal operators, and thus, unitarily equivalent. Inparticular, B^* is unitarily equivalent to A^{n*}, thus every subspace of H, which reduces B^*, also reduces A^{n*}, (unitary equivalence preseves reducing subspaces). Note that, if A and B have the direct sum decompositions, $A = A_1 \oplus A_2$, and $B = B_1 \oplus B_2$, respectively, where A_1, B_1 are normal parts, and A_2, B_2, are pure parts, then, $A^* = {A_1}^* \oplus {A_2}^*$, and $B^* = {B_1}^* \oplus {B_2}^*$, respectively, where

$A_1{}^*, B_1{}^*$ are normal parts, and $A_2{}^*, B_2{}^*$, are pure parts. X being a quasiaffinity, implies XB is also a quasiaffinity, thus XB is injective, so is A^n and thus,A^{n*} is also injective. Therefore, A^{n*} has the direct sum decomposition, $A^{n*} = A_1{}^{n*} \oplus A_2{}^{n*}$. B^* being normal, implies $B_2{}^* = 0$. It follows that, $A_2{}^{n*} = 0$, (direct summands of unitarily equivalent operators are pairwise unitarily equivalent). And thus, $A_2{}^* = 0$. Hence, $A_1{}^*$ is normal, which implies A is also normal since, $A = A_1 \oplus 0$. Eventually, we have that, $AX = XB$, which begets the requirement, $A^*X = XB^*$.

THEOREM 4.2.25, [Imagiri, 2014,[59]]; Let A, B be injective n-Power normal operators, for some positive integer n. If there exists another operator X, such that, $A^nX = XB^n$, then $A^*X = XB^*$.

Proof

Firstly, we note that, A^n and B^n are injective normal operators. Thus, A and B are also injective, but not necessarily normal. Now assume that the intertwinning operator X is a quasiaffinity. Then, from Corollary 4.2.22 above, $A^{n*}X = XB^{n*}$, which implies that, A^{n*} and B^{n*}, are symetric normal operators, hence unitarily equivalent. We prove that, both A *and* B are normal operators and the rest of the proof will follow from Theorem 4.2.24 above. We therefore, only need to prove that, either B^* or B is normal, and a similar process can be used to show that A is normal. The operator X, being a quasiaffinity on H, implies that, $ker(X) = \{0\}$, and X has the matrix decomposition,

$X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$. Since B^n is injective, XB^n is also a quasiaffinity on H, and H has the orthogonal decomposition, $H = \bar{ran}(X) \bigoplus ran(X)^{\perp}$. With respect to this decomposition, let B^n have the direct sum decomposition, $B^n = B_1{}^n \oplus B_2{}^n$. Then, B has the decomposition, $B = B_1 \oplus B_2$, and therefore, $XB^n = X_1B_1{}^n \oplus X_2B_2{}^n = X_1B_1{}^n$, since $X_2 = 0$. It follows that, $XB = X_1B_1$. We note that, $B_1{}^n$ being normal, implies that, $X_1B_1{}^n$ is also normal, since X_1 is another quasiaffinity. This in turn implies that, B_1 is a normal operator. It therefore follows that, B is normal since, $B = B_1$.

REMARK 4.2.26; Every normal operator is n-Power normal for some positive integer n, and every n-Power normal operator is n-Power quasinormal, for the same positive integer n. In other words, n-Power quasinormal operators contains properly, all n-Power normal operators, which in turn includes properly, all normal operators. We in what follows, extend Theorems 4.2.24 and 4.2.25 above, to the class of n-power quasinormal operators. That is, we prove the following theorem;

THEOREM 4.2.27, [Imagiri, 2014,[55]]; Let A, B be invertible n-Power quasinormal operators, for some positive integer n. If there exists another operator X, such that, $A^nX = XB^n$, then;
$(i)A^{n*}X = XB^{n*}$.
$(ii)A^*X = XB^*$.

REMARK 4.2.28; Before proving Theorem 4.2.27(i) above, we need the following result by [Anderson and Foias, 1975], popularly refered to as the Restriction Invariant Property, and due to which one concludes that, if an operator T belongs to a certain class and M is an invariant subspace for T, then the restriction of T to M, (that is $T/_M$), happens also to be of the same class as that of T.

LEMMA 4.2.29, [6, Thm 11]; Let an operator $T \in B(H)$ satisify a property μ and let M be an invariant subspace for T. Then $T/_M$, also satisies μ.

REMARK 4.2.30; Lemma 4.2.29 above, is important in studying linear operators, since it follows trivially from this lemma that, if T is n-Power quasinormal and a subspace M of an Hilbert space H is T-invariant, then $T/_M$, is also n-Power quasinormal, for the same positive integer n. In other words, if T is n-Power quasinormal, and has the direct sum decomposition, $T = T_1 \oplus T_2$, where T_1 is normal, then T_2 is a pure n-Power quasinormal operator. By applying this lemma, we can now prove the first part of Theorem 4.2.27 as follows;

Proof of theorem 4.2.27(i)

Without loss of generality, assume that X is a quasiaffinity. Since A and B are invertible, then they are injective. Trivially, for any positive integer n, A^n and B^n are also invertible and thus injective. And since, $A^nX = XB^n$, then either, $\bar{ran}(X)$ is invariant by A^n,(implying, $ker(X)^\perp$, is invariant by B^n), or $\bar{ran}(X)$ is invariant by B^n, (implying, $ker(X)^\perp$, is invariant by A^n). Whichever the case, H has the orthogonal decomposition, $H = \bar{ran}(X) \bigoplus ran(X)^\perp$, which yields;
$A^n = \begin{bmatrix} A^n{}_1 & A^n{}_2 \\ 0 & A^n{}_3 \end{bmatrix}$ $B^n = \begin{bmatrix} B^n{}_1 & B^n{}_2 \\ 0 & B^n{}_3 \end{bmatrix}$, and $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$. With respect to this representation, and from the fact that, $A^nX = XB^n$, it therefore follows that, ${A_1}^nX_1 = X_1{B_1}^n$. We need to prove that, ${A_1}^{*n}X_1 = X_1{B_1}^{*n}$. It suffices to prove that, ${A_1}^n$ and ${B_1}^n$ are both quasinor-

mal operators, and the rest of the proof will follow from Lemma 4.2.19, since every quasinormal operator is hyponormal, and every hyponormal operator is w-hyponormal. Firstly, note that, from Remark 4.2.30, it follows that, ${A_1}^n$ and ${B_1}^n$ are invertible n-Power quasinormal operators, for the same positive integer n. Thus, to prove that, ${A_1}^n$ and ${B_1}^n$ are both quasinormal operators, we only need to prove that ${A_1}^n$ is quasinormal and the same procedure can be applied to obtain the quasinormality of ${B_1}^n$. The n-Power quasinormality of ${A_1}^n$, implies that ${A_1}^n$ commutes with $({A_1}^*A_1)$. Recall that, ${A_1}^n$ is invertible. Thus, both A_1 and ${A_1}^*$ are invertible. It follows that, $({A_1}^*A_1)$ is also invertible. But $({A_1}^*A_1)$ is naturally self-adjoint. Thus, $({A_1}^*A_1)$ is an invertible self-adjoint operator, which implies that, $({A_1}^*A_1)$ is unitary. From the n-Power quasinormality of ${A_1}^n$, we have that, $[{A_1}^n, ({A_1}^*A_1)] = 0$. Implying, $[{A_1}^n, ({A_1}^*A_1)^k] = 0$, for every positive integer k, since $({A_1}^*A_1)$ is unitary. Thus, $[{A_1}^n, ({A_1}^*A_1)^n] = 0$, for some positive integer n. It also follows that, $({A_1}^*A_1)^n = {A_1}^{*n}{A_1}^n$, since both A_1 and ${A_1}^*$ are invertible. And thus, $[{A_1}^n, ({A_1}^{*n}{A_1}^n)] = 0$, so that ${A_1}^n$ is a quasinormal operator.

REMARK 4.2.31(a); Recall that, ([39, Pg 23]), if a, b, c are any three real numbers, such that, $ab = ac$, then $b = c$. In other words, the cancellation law holds in the set of real numbers. Unfortunately, if A, B, C are any three matrices of the same size such that, $AB = AC$, then it does not follow in general that, $B = C$. That is, the cancellation law does not hold in square matrices. Consequently, for any three operators $A, B, C \in B(H)$, $B \neq C$, even incases when $AB = AC$. However if A is invertible, and $AB = AC$, then $B = C$. By applying this observation to Theorem 4.2.25 above, the proof of the remaining part of Theorem 4.2.27 is as follows;

Proof of theorem 4.2.27(ii)

Again, asssume that X is a quasiaffinity and $A^nX = XB^n$, where A and B are invertible n-Power quasinormal operators. Then, $A^n(A^*A) = (A^*A)A^n$ and $B^n(B^*B) = (B^*B)B^n$. To prove that, $A^*X = XB^*$, we only need to prove that both A and B are n-Power normal operators, and the rest of the proof will follow from Theorem 4.2.25. To do this, we only require to show that A is n-Power normal, since the same procedure can be applied to prove that B is also n-Power normal. Now, $A^n(A^*A) = (A^*A)A^n$, implies that, $A^nA^*A = A^*A^{n+1} = A^*A^nA$. Thus, $A^nA^*A = A^*A^nA$. Multiplying both sides, to the right, by A^{-1}, we have that, $A^nA^* = A^*A^n$. Which implies that, A^n is normal, and thus, A is n-Power normal.

REMARK 4.2.31(b); In the first section of this chapter, that is in Section 4.1 above, it is evident that w-hyponormal operators are different from both n-Power normal and n-Power

quasi normal operators. In what follows, the PF Theorem for a pair of operators (A, B), where A is w-hyponormal and B is either an n-Power normal or an n-Power quasinormal operator is studied. As usual, $T \in N$, $T \in nN$, $T \in QN$ and $T \in nQN$, denotes, T belongs to the class of normal, n-power normal, quasinormal and n-power quasinormal operators respectively. We need also to emphasize that, if $A \in B(H)$, has the direct sum decomposition, $A = A_1 \oplus A_2$, where A_1 and A_2 are normal and pure parts respectively, it follows that, A_2 being pure, is injective since it is completely non normal. That is, there does not exist a non trivial invariant subspace of H under which, the restriction of A_2 to it, is normal. In this direct sum decomposition, it also follows that, $|A_2|^{1/2}$ is a quasiaffinity. In addition, $\tilde{A} = \tilde{A}_1 \oplus \tilde{A}_2 = A_1 \oplus \tilde{A}_2$, (since all Aluthge transforms of a normal operator are similar, and thus unitarily equivalent). And thus, $\tilde{\tilde{A}} = \tilde{\tilde{A}}_1 \oplus \tilde{\tilde{A}}_2 = A_1 \oplus \tilde{\tilde{A}}_2$. In general, $\tilde{A}_n = A_1 \oplus (\tilde{A}_2)_n$, for every positive integer n, where, $(\tilde{A})_n$, denotes, the n^{TH}-Aluthge transform of A.
Due to these remarks, the asymmetric version of the PF Theorem is as follows;

THEOREM 4.2.32, [Imagiri, 2014,[58]]; Let $B \in nN$ and A be a w-hyponormal operator such that, $ker(A) \subset ker(A^*)$. If there exists a quasiaffinity $X \in B(H)$, such that, $AX = XB^n$, then $A^*X = XB^{n*}$.

THEOREM 4.2.33, [Imagiri, 2014,[55]]; Let A be an invertible nQN operator, and B^* be an injective w-hyponormal operator. If there exists some $X \in B(H)$, such that, $A^nX = XB$, then $A^{n*}X = XB^*$.

REMARK 4.2.34; It is important to recall that, if an operator A on H has the kernel condition, $ker(A) \subset ker(A^*)$, then $ker(A)$ reduces $A$and thus, on the decomposition $H = ker(A)^{\perp} \oplus ker(A)$, A can be written as the direct sum decomposition, $A = A_1 \oplus A_2$. With respect to such a decomposition, it has already been noted in Remark 4.2.31(a) above that, A_2 is completely non normal, which implies that, if there exists a reducing subspace M for A_2, such that, the restriction of A_2 to M is normal, then $M = \{0\}$. In other words, in such direct decompositions of A, if it happens that A_2 is normal, then A is normal. From Lemma 4.2.29 above, one might as well observe that, if T is an n-Power normal operator, or an n-Power quasinormal operator, then the restriction of T to any T-invariant subspace, is also n-Power normal or n-Power quasinormal respectively. That is, if $T \in nN$, or $T \in nQN$, such that, $T = T_1 \oplus T_2$, then, $T_1, T_2 \in nN$, or $T_1, T_2 \in nQN$ respectively.
In addition, recall that, [Nelson, 1954,([])] an operator A on an Hilbert space H, is said to have

a 'single valued extension property', if there exists no non-zero analytic H-valued function $f(z)$, such that, $(A-z)f(z)=0$. Also recall that, if a subspace M of H is invariant under A, then $H=M\bigoplus M^{\perp}$, and with respect to this decomposition, $A=A_1\oplus A_2$, where $A_1=A/_M$ and $A_2=A/_{M^{\perp}}$. It also follows that, $x=x_1\oplus x_2$, for any $x\in H$, where $x_1\in M$ and $x_2\in M^{\perp}$. It is known that, if C is a unit disc around the origin, $f:C\rightarrow H$ is a bounded function such that, $(A-z)f(z)=x$, for some $x\in H$, then f is analytic. Consequently, if A has the single valued property, then $(A-z)f(z)=(A_1-z)f(z)\oplus(A_2-z)f(z)=x_1\oplus x_2=x$. And that, if A is similar to another operator, say B, such that A has this property, then so is B, [Stampfli, 1962,([93])]. Unfortunately, [Colojoara et al, 1968,([22])], showed that not every operator in $B(H)$ which has this single valued extension property. However, [Stampfli et al, 1977,([92])], proved that every normal and every hyponormal operator has this property. Observations in [92] were extended and proved to hold true in the case of dominant operators by [Radjabalipour, 1987,([89])].
Also, recall that, if T is a normal operator, then $(T-\lambda)$ is normal for every complex number λ, and that, if T is a w-hyponormal operator, then $\tilde{T}$ is semi-hyponormal and $\tilde{\tilde{T}}=(\tilde{T})_1$, is hyponormal. We state, without proof, the following three results which will be required to prove Theorems 4.2.32 and 4.2.33 above.

LEMMA 4.2.35, [Putnam, 1970,[87]]; If B is a normal operator on H, then $\bigcap(B-\lambda)H=\{0\}$, for every complex number λ

LEMMA 4.2.36, Rudin et al, 1973,[90, Pg 112]]; Let $A,B\in B(H)$, be such that, $0\leq B\leq t(A-\lambda)(A-\lambda)^{*}$, for every complex number λ , where t is a positive real number. Then, for every $X\in B(H)$, there exists a bounded function $f:C\rightarrow H$, such that, $(A-\lambda)f(\lambda)=X$.

LEMMA 4.2.37, [Duggal, 1986,[28]]; If T is p-hyponormal, then there exists a quasiaffinity X and a hyponormal operator A, such that, $AX=XT$.

Proof of Theorem 4.2.32

Let $A\in B(H)$, have the direct sum decomposition, $A=A_1\oplus A_2$, where A_1 and A_2 are normal and pure parts respectively. Since A is w-hyponormal, then by Remark 4.2.34 above, $\tilde{\tilde{A}}=\tilde{\tilde{A}}_1\oplus\tilde{\tilde{A}}_2=A_1\oplus\tilde{\tilde{A}}_2$, is hyponormal. Thus by Lemma 4.2.37 above, there exists a quasiaffinity T such that, $\tilde{\tilde{A}}T=TA$. Hence, $\tilde{\tilde{A}}TX=TAX=TXB^{n}$, for some $X\in B(H)$, such that, $\bar{ran}(X)=H$. Also, if $W\in B(H)$, is a quasiaffinity, then $\tilde{\tilde{A}}W=WB^{n}$, so that $W^{*}\tilde{\tilde{A}}^{*}=B^{n*}W^{*}$, where W^{*}, is also such that, $\bar{ran}(W^{*})=H$. Letting $X\in|\tilde{\tilde{A}}|^{1/2}H$, then $X\in(\tilde{\tilde{A}}^{*}\tilde{\tilde{A}}-\tilde{\tilde{A}}\tilde{\tilde{A}}^{*})^{1/2}H$, and by Lemma 4.2.36 above, there exists a bounded function

$f : C \to H$, such that, $(\tilde{\tilde{A}}^* - \lambda)f(\lambda) = X$. Hence, $W^*X = W^*(\tilde{\tilde{A}}^* - \lambda)f(\lambda)W^*$. So that by Lemma 4.2.35 above, $W^*X \in ran(B^{*n} - \lambda)$, for every complex number λ, and thus, $W^*X = 0 \Rightarrow X = 0$. Therefore, by Lemma 4.2.15 above, it follows that, $A^*X = XB^{n*}$, since B^n is normal, hence w-hyponormal.

Proof of theorem 4.2.33

Since B^* is injective, and $A^nX = XB$, then $\bar{ran}(X)$ is invariant by A^n and $(ker(X))^{\perp}$, is invariant by B^*. Thus, H has the orthogonal decomposition, $H = \bar{ran}(X) \bigoplus (ran(X))^{\perp}$, (equivalently, $H = ker(X) \bigoplus (ker(X))^{\perp}$), which yields;
$A^n = \begin{bmatrix} {A^n}_1 & {A^n}_2 \\ 0 & {A^n}_3 \end{bmatrix}$ $B = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}$, and $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$. From $A^nX = XB$, we get ${A^n}_1X_1 = X_1B_1$. By Remarks 4.2.31(a and b) above, ${A^n}_1 \in nQN$. That is, ${A^n}_1$ is also n-Power quasinormal, and thus, ${A^n}_1$ commutes with $({A_1}^*A_1)$. Implying, $[{A_1}^n, ({A_1}^*A_1)] = 0$. In addition, the invertibility of A, implies that of A^n, which eventually implies that of, ${A^n}_1$. In-particular, ${A^n}_1$ is an invertible operator. Thus, $ker{A^n}_1 = \{0\}$, which implies, $kerA_1 = \{0\}$, and thus, $ker({A_1}^*A_1) = \{0\}$. It therefore follows that, A_1 and $({A_1}^*A_1)$ are also invertible linear operators. But, in addition to being invertible, $({A_1}^*A_1)$ is a positive, hence self-adjoint, which inturn implies that, $({A_1}^*A_1)$ is unitary. From the n-Power quasinormality of ${A_1}^n$, we have that, $[{A_1}^n, ({A_1}^*A_1)] = 0$. Implying, $[{A_1}^n, ({A_1}^*A_1)^k] = 0$, for every positive integer k, since $({A_1}^*A_1)$ is unitary. Thus, $[{A_1}^n, ({A_1}^*A_1)^n] = 0$, for some fixed positive integer n. It also follows that, $({A_1}^*A_1)^n = {A_1}^{*n}{A_1}^n$, since both A_1 and ${A_1}^*$ are invertible. And thus, $[{A_1}^n, ({A_1}^{*n}{A_1}^n)] = 0$, so that ${A_1}^n$ is a quasinormal operator. Recall that, every quasinormal operator is hyponormal and every hyponormal operator is w-hyponormal, thus ${A_1}^n$ is an invertible, (hence, injective), w-hyponormal operator. From the injectivity of a w-hyponormal B^*, and from the matrix decomposition of B, it follows that, B^* has the matrix decomposition; $B^* = \begin{bmatrix} {B_1}^* & 0 \\ {B_2}^* & 0 \end{bmatrix}$, and ${B_1}^*$ is also an injective w-hyponormal operator. In particular, we have that, ${A_1}^n$ and ${B_1}^*$ are injective w-hyponormal operators, and that, ${A^n}_1X_1 = X_1B_1$. Applying Lemma 4.2.19 above, we have that, ${{A^n}_1}^*X_1 = X_1{B_1}^*$, which eventually yields the required equality, $A^{n*}X = XB^*$.

REMARK 4.2.38; Notice that, in Theorems 4.2.32 above, it follows that, $A^*X = XB^{n*}$, but we are not sure whether, $A^*X = XB^*$. It is therefore natural for one to ask, under what conditions does $AX = XB^n$, for a w-hyponormal A and a n-Power normal B, imply the equality $A^*X = XB^*$? To answer this question, we need first revisit some observations.

Recall that, if two operators are almost similar and one of them happens to be self-adjoint, so is the other. That is, almost similarity preserves self-adjointedness. Also recall that, any two normal operators which are similar, also happens to be unitarily equivalent. It is known that, any two operators, which are unitarily equivalent are also similar, (and almost similar), but the converse is not true in general. It therefore follows that, unitarily equivalence is a sharper tool for studying linear operators. Thus unitarily equivalence preserves all properties preserved by either similarlity or almost similarlity, but similarlity does not preserve all properties which remain invariant under unitarily equivalence. For instance, unitarily equivalence preserves quasi-triangularlity in square matrices, but similarlity and almost similarity do not. However, similarity preserves several properties in $B(H)$, and sometimes implies normality for non normal operators. Good examples of when non normal operators get restricted to normality under similarity, are results by [Duggal, 2000] and [Ito, 2001]. Indepedently, the authors showed that, any p-hyponormal or log-hyponormal operator which is similar to a normal operator is normal. [Duggal et al, 2005], extended these observations and proved that, any invertible p-quasi hyponormal operator which is similar to a normal is normal as well. These results were also proved to hold true when p-hyponormality is replaced with w-hyponormality, again by [Duggal et al, 2005]. By using these results, we intend to state conditions under which the equalities obtained in Theorems 4.2.32 and 4.2.33 above, can be obtained without the powers involved. Afterwards, an extension of these observations will be proved. First, we state these results;

LEMMA 4.2.39, [29, Preposition 4]; Let T_1 be a p-hyponormal operator and T_2 be normal. If there exists an operator X with dense range such that, $T_1X = XT_2$, then T_1 is normal.

LEMMA 4.2.40, [61, Thm 1.5]; Let T_1 be a log-hyponormal operator and T_2 be normal. If there exists an operator X with dense range such that, $T_1X = XT_2$, then T_1 is normal.

LEMMA 4.2.41, [30, Lemma 2.4]; Let T_1 be an invertible p-quasihyponormal operator and T_2 be normal. If there exists an operator X with dense range such that, $T_1X = XT_2$, then T_1 is normal.

LEMMA 4.2.42, [30, Lemma 2.5]; Let T_1 be a w-hyponormal operator and T_2 be normal. If there exists an operator X with dense range such that, $T_1X = XT_2$, then T_1 is normal.

By using these results, we answer the question posed in Remark 4.2.34 above, as follows;

COROLLARY 4.2.43, [Imagiri, 2014,[58]]; Let $B \in nN$ and A be a w-hyponormal operator such that, $ker(A) \subset ker(A^*)$. If there exists a quasiaffinity $X \in B(H)$, such that, $AX = XB^n$, then $A^*X = XB^*$.

Proof

Firstly, note X being a quasiaffinity, implies that X has a dense range. From Lemma 2.2.1 above, we have that, B^n is normal since, $B \in nN$. Applying Lemma 4.2.42 to the equetion $AX = XB^n$, we find that A is normal. Finally, $A^*X = XB^*$, by applying Theorem 4.2.24 above, since A and B^n are similar normal operators, hence unitarily equivalent.

COROLLARY 4.2.44, [Imagiri, 2014,[55]]; Let A be an invertible nQN operator, and B^* be an injective w-hyponormal operator. If there exists some $X \in B(H)$, such that, $A^nX = XB^*$, then $A^*X = XB$.

Proof

If B^* is an injective w-hyponormal operator, then B is also injective. Thus, $ker(B^*) = ker(B) = \{0\}$, and thus, $ker(B^*) \subset ker(B)$. On the other edge, invertibilty of n-Power quasinormal operator A, implies that, $ran(A) = H$. From Lemma 2.2.4(ii) above, we have that, $A \in nN$ and thus, by Lemma 2.2.1 above, A^n is normal. Now, we have that, B^* is a w-hyponormal operator with the kernel condition, A^n is normal, and in addition, $A^nX = XB^*$, for some quasiaffinity X. From Corollary 4.2.43 above, it follows that, $A^*X = XB$.

REMARK 4.2.45; It is important to recall that every quasinormal operator is n-Power quasinormal for every positive integer n and that, every quasinormal operator is hyponormal. It easily follows from Lemma 4.2.39 above, that every quasinormal operator which happens to be similar to a normal operator, ends up becoming restricted to a normal operator, since every hyponormal operator is p-hyponormal. Does this restriction to normality of non-normal operators affect the class of n-Power normal, or that of n-Power quasinormal operators? To answer this question, we extend the immediate four lemmas above, by proving the following two results;

THEOREM 4.2.46, [Imagiri, 2014,[58]]; Let T_1 be an n-Power normal operator, for some positive integer n, and T_2 be normal. If there exists an operator X with dense range such that, $(T_1)^nX = XT_2$, then T_1 is normal.

Proof

Let $T_1 = A$ and $T_2 = B$. Then, $A^n X = XB$, where $A \in nN$ and B is normal. From Theorem 4.2.24 above, we have that, $A^*X = XB^*$. In this case, X is a quasiaffinity and B^* is normal, which implies XB^* is also a quasiaffinity hence normal. Notice that, A^*X is also a quasiaffinity, hence A^* is also normal. It therefore follows that, A is normal, and hence, T_1 is a normal operator.

THEOREM 4.2.47, [Imagiri, 2014,[55]]; Let T_1 be an invertible n-Power quasinormal operator, for some positive integer n, and T_2 be normal. If there exists an operator X with dense range such that, $T_1X = XT_2$, then T_1 is normal.

Proof

Likewise, let $T_1 = A$ and $T_2 = B$. Then, $A^n X = XB$, where $A \in nQN$ and B is normal. We are only required to prove that, $A \in nN$, and the rest of the proof will follow from Theorem 4.2.46 above. By the invertibilty of n-Power quasinormal operator A, we have that, A is injective, and thus, $ran(A) = H$. From Lemma 2.2.4(ii) above, we get that, $A \in nN$ and therefore, by Lemma 2.2.1 above, A^n is normal.

REMARK 4.2.48; Recall that, in chapters two and three above, several results which imply normality for n-Power normal, n-Power quasinormal and w-hyponormal operators were proved. By using some of these results, we now deduce a number of corollaries which give more sufficient conditions for a given pair of operators to satisfy the PF Theorem. We first give an alternative proof for Theorem 4.2.27, by applying Lemma 2.2.4 to Theorem 4.2.25.

COROLLARY 4.2.49, [Imagiri, 2014,]; Let A, B be invertible n-Power quasinormal operators, for some positive integer n. If there exists another operator X, such that, $A^n X = XB^n$, then; $A^*X = XB^*$.

Proof

Since $A, B \in nQN$, and both are invertible, then both are injective. Thus, both A and B are n-Power quasinormal operators, with dense ranges. That is, $ran(A) = ran(B) = H$. It follows trivially, from Lemma 2.2.4 above, that $A, B \in nN$. In particular, A and B are invertible and hence, injective n-Power normal operators. And thus, by Theorem 4.2.25 above, $A^*X = XB^*$.

COROLLARY 4.2.50, [Imagiri, 2014]; If $A, B \in nQN$ such that, $N(A^*) \subset N(A)$, $N(B^*) \subset N(B)$, and $A^n X = XB^n$, then $(A^n)^*X = X(B^n)^*$, for another operator $X \in B(H)$ and some $n \in J^+$.

Proof

By Lemma 2.2.9 above, $A \in nN$ and $B \in nN$. That is, A^n and B^n are normal operators. Thus, by Theorem 4.2.22 above, if $A^nX = XB^n$, then $(A^n)^*X = X(B^n)^*$.

COROLLARY 4.2.51, [Imagiri, 2014]; If $A, B \in 2QN \cap 3Qn$, and $(A-I), (B-I) \in nQN$, then, the pair (A, B) satisfies the PF Theorem.

Proof

Let $A, B \in 2QN \cap 3Qn$, and $(A-I), (B-I) \in nQN$. From Lemma 2.2.10 above, it follows trivially that, A and B are normal operators. Thus, $AX = XB \Rightarrow A^*X = XB^*$, for another operator $X \in B(H)$.

COROLLARY 4.2.52, [Imagiri, 2014]; If $A, B \in nQN$ and $A^*, B^* \in nQN$, for some $n \in J^+$, then, the pair (A^n, B^n) satisfies the PF Theorem.

Proof

Let $A, B \in nQN$ and $A^*, B^* \in nQN$, for some $n \in J^+$. Then, by Lemma 2.2.12 above, $A \in nN$ and $B \in nN$. $\Rightarrow A^n$ and B^n are normal operators. It therefore follows that, if X is any operator in $(B(H))$ such that, $A^nX = XB^n$, then $(A^n)^*X = X(B^n)^*$.

COROLLARY 4.2.53, [Imagiri, 2014]; If $A, B \in 2QN$, such that, $0 \notin W(A)$ and $0 \notin W(B)$, then, the pair (A, B) satisfies the PF Theorem.

Proof

By lemma 2.2.20 above, and the hypothesis, it follows that, A and B are normal operators. Trivially, $AX = XB \Rightarrow A^*X = XB^*$, where $X \in B(H)$.

COROLLARY 4.2.54, [Imagiri, 2014]; If $A, B \in nQN$ and $A, B \in (n-1)QN$, for some $n \in J^+$, such that, $[A^*, A^2] = [B^*, B^2] = 0$, then, the pair (A, B) satisfies the PF Theorem.

Proof

By Theorem 2.3.8 above and the hypothesis, it follows that, A and B are quasinormal operators. But every quasinormal operator is hyponormal. Thus, A and B are hyponormal operators. Trivially, $AX = XB \Rightarrow A^*X = XB^*$, where $X \in B(H)$, since hyponormal operators satisfy the PF Theorem.

REMARK 4.2.55(a); It is good to recall the major importance of the direct sum decomposition, $T = T_1 \oplus T_2$, of an operator T acting on an Hilbert space H, follows from the well known fact, that, properties satisfied by the direct summands, that is, T_1 and T_2, are always the same properties satisfied by the direct sum, that is, T. Thus, studying the behaviour of T gets relaxed to studying the behaviour of the parts T_1 and T_2, since these parts are known to have a simpler structure than 'their mother operator', T. Unfortunately, for any T to be guaranteed of such a direct sum decomposition, there must exists atleast one non-trivial reducing subspace in H, for T. From the invariant problem for linear operators, (that is, the question of coming up with a non-trivial invariant subspace for every operator on H), one expects to encounter some operators on H, (especially, when the dimension of H is neither 1 nor finite), which cant be expressed as direct sum decompositions. Notice that, any subspace M of H, which reduces the product (AB), of any two operators, A and B, also reduces both A and B. By combining some results from chapters above, the following observations are deduced;

COROLLARY 4.2.56, [Imagiri, 2014]; Let $A, B \in nQN$ be commuting operators such that, $N(A^*) \subset N(A)$, $N(B^*) \subset N(B)$, and $(AB)^n X = X(BA)^n$, for another operator X on H. Then, $[ran \ \ X]$ reduces $(AB)^n$, and $(ker \ \ X)^{\perp}$ reduces $(BA)^n$.

Proof

Notice that, we need only to prove that the pair $((AB)^n, (BA)^n)$ satisfies the PF Theorem and then the conclusion will follow by applying Lemma 4.2.11 above. Now assume that, $N(A^*) \subset N(A)$, $N(B^*) \subset N(B)$, and $(AB)^n X = X(BA)^n$, for another operator X on H. Then, from Lemma 2.2.9 above, we have that, both A and B are n-Power normal operators. From Lemma 2.2.1 above, it follows that A^n and B^n are normal operators. The commutativity of A and B, implies that of, A^n and B^n. That is, $[A, B] = 0 \Rightarrow [A^n, B^n] = 0$. Thus, A^n and B^n are commuting normal operators. It follows that, their products are also normal. Therefore, both $(A^n B^n)$ and $(B^n A^n)$ are normal operators. But, $(A^n B^n) = (AB)^n$ and $(B^n A^n) = (BA)^n$, implying $(AB)^n$ and $(BA)^n)$ are normal. And since, $(AB)^n X = X(BA)^n$, then by the Fuglede theorem, $(AB)^{n*} X = X(BA)^{n*}$. Thus, the pair $((AB)^n, (BA)^n))$ satisfies the PF Theorem. Consequently, $[ran \ \ X]$ reduces $(AB)^n$, $(ker \ \ X)^{\perp}$ reduces $(BA)^n$ and $(AB)^n/_{ran \ \ X}$, $(BA)^n/_{(ker \ \ X)^{\perp}}$ are unitarily equivalent normal operators.

COROLLARY 4.2.57, [Imagiri, 2014]; If $A, B \in nQN$ and $A^*, B^* \in nQN$, for some $n \in J^+$, such that, $[A, B] = 0$, and $(AB)^n X = X(BA)^n$, for another operator X on H. Then, $[ran \ \ X]$ reduces $(AB)^n$, and $(ker \ \ X)^{\perp}$ reduces $(BA)^n$.

Proof

Again note that, it suffices to prove that both $(AB)^n$ and $(BA)^n$ are normal operators and the rest of the proof will be the same as that of Corollary 4.2.56 above. Now since, $A, B \in nQN$ and $A^*, B^* \in nQN$, for the same $n \in J^+$, it follows easily from Lemma 2.2.12 above that, A^n and B^n are normal operators. In addition, $[A, B] = 0 \Rightarrow [A^n, B^n] = 0$. Thus, A^n and B^n are commuting normal operators. It follows that, both (A^nB^n) and (B^nA^n) are normal operators. But, $(A^nB^n) = (AB)^n$ and $(B^nA^n) = (BA)^n$, implying $(AB)^n$ and $(BA)^n$) are normal.

REMARK 4.2.55(b); Recall that, w-hyponormal operators do not satisfy the PF Theorem in general, since the kernel condition is not always satisfied by all members from this class. In the following part, by using Aluthge transforms of w-hyponormal operators, more conditions under which operators from this class satisfy the PF Theorem are given. In particular, via generalized Aluthge transformations of w-hyponormal operators, we extend results by [Bachir et al, 2012]. That is, by applying Theorem 3.5.13 above, we extend Lemma 4.2,20 above, as follows;

COROLLARY 4.2.58, [Imagiri, 2014]; If A and B are w-hyponormal operators, such that, $\tilde{A_{n_1}}$ and $\tilde{B_{n_2}}$, are normal operators, for any pair of positive integers n_1, n_2, (not necessarily equal), then, the pair (A, B) satisfies the PF Theorem.

Proof

We assume that, there exists some operator X on H satisfying, $AX = XB$, where A and B are w-hyponormal operators. Since there exist some positive integers n_1, n_2, such that, $\tilde{A_{n_1}}$ and $\tilde{B_{n_2}}$, are normal operators, it follows from Theorem 3.5.13, both A and B are normal operators. Trivially, $AX = XB \Rightarrow A^*X = XB^*$.

THEOREM 4.2.24, [Imagiri, 2014]; Let A and B^* be w-hyponormal operators such that, $ker(A) \subset ker(A^*)$ and $ker(B^*) \subset ker(B)$. If there exists a quasiaffinity $X \in B(H)$, such that, $X \in ker(\delta_{A,B})$, then A and B^* are unitarily equivalent normal operators.

4.3; Putnam's inequality for n-Power normal, n-Power quasinormal and w-hyponormal Operators

For any operator T on a Hilbert space H, there are some natural self-adjoint operators associated with it. These are such as; the real and imaginery parts of T, (equivalently, the operators A, B, in the cartesian decomposition $T = A + iB$), the absolute value of T, (equivalently, the operator $|T|^2 = (T^*T)$), and the self-commutator of T. If one claims to understand an operator or class of operators, then one ought to understand these naturally associated operators. For any two operators, say A and B on a Hilbert space H, recall that, $[A, B] = AB - BA$, is called the commutator of A and B, and also, $[A^*, A] = A^*A - AA^*$, is called the self-commutator of A. Letting the operator, $D_A = [A^*, A]$, then D_A is the self-commutator of A. The operator D_A is self-adjoint, thus normal and therefore, its structure is simple. In other words, D_A is a diagonalizable operator. It is good to note that, for every $T \in B(H)$, the correspodence, $f : T \to D_T$, is one-one, and thus, D_T provides yet another alternative method of studying the orignal operator T, no matter how complicated is the structure of T. For instance, it easily follows that, if $D_T = 0$, then T is normal, and if $D_T \geq 0$, then T is hyponormal. It is also known that, if the spectrum of D_T has zero radius, then $D_T = 0$, and thus T is normal, and if $ker(D_T)$ is not a trivial subspace of H, that is, $ker(D_T) \neq \{0\} \neq H$, then $ker(D_T)$ reduces T, thus T becomes reducible, and thus, T ends up having the desired direct sum decomposition, $T = T_1 \oplus T_2$. Unfortunately as we noticed earlier, not every operator acting on H, boosts this qualification. In particular, it is still unknown whether every contraction or even every hyponormal operator has a non trivial invariant subspace. Results that come close to affirmative answers are such as, the one due to [Bercovici, 1990], which says that, every contraction whose spectrum contains the unit circle has a non trivial invariant subspace, and a result due to, [Brown, 1987], which says that, any hyponormal operator whose spectrum has non empty interior has a non trivial invariant subspace. Recall that, unlike hyponormal operators and contractions, normal operators are known to be reducible. Putnam's inequality for hyponormal operators, that is, Lemma 1.6.4.3 above, is also a tool which fore tells when a given hyponormal operator is reducible. Note that, this result says that, if T is hyponormal, then $\|D_T\| \leq \pi^{-1} Area(\sigma(T))$, and thus ensures that, a hyponormal operator which has zero area, is normal hence reducible. As was seen in chapter one above, a number of authors, by using the spectral theorem as a tool, have investigated the form of this inequality, for other large classes of operators, especially those which includes all hyponormal operators, such as p-hyponormal and (p,k)-hyponormal operators. In this section, we first discuss the spectral theorem, then present, (without proofs), some of the known extensions of Putnam' inequality, and later investigate the form of this inequality for both n-Power normal, n-Power quasinormal and w-hyponormal operators.

Before presenting the spectral theorem, we first need the following definition of a spectral measure;

DEFINITION 4.3.1; Let Ω be a subset in the set of complex numbers and let Σ_Ω, be

the δ-algebra of all Borel sets in Ω. A (complex) spectral measure in a Hilbert space H is a function $P : \Sigma_\Omega \to B(H)$, such that;
(a) $P(\Lambda)$ is a projection for every $\Lambda \in \Sigma_\Omega$.
(b) $P(theta) = 0$ and $P(\Omega) = I$.
(c) $P(\Lambda_1 \cap \Lambda_2) = P(\Lambda_1)P(\Lambda_2)$, for every $\Lambda_1, \Lambda_2 \in \Sigma_\Omega$.
(d) If $\{\Lambda_k\}$ is a countable collection of pairwise disjoint sets from Σ_Ω, then $P(\sqcup)_k\Lambda_k = \sum_k P(\Lambda_k)$.

If $\{\Lambda_k : k \geq 0\}$, is a countably infinite collection, then the above identity is to be understood as convergence in the strong topology, that is, $P(\sqcup)_k\Lambda_k \to \sum_k P(\Lambda_k)$, strongly. Note that this actually generalizes the concept of a resolution of the identity on H. For instance, if Ω is a countably infinite set, say $\Omega = \{\Lambda_k \in C : k \geq 0\}$, then $\{P(\{\Lambda_k\}) \in B(H) : k \geq 0\}$, is a sequence of projections, orthogonal to each other, and $\sum_{k=0}^{\infty} P(\{\lambda_k\}) \to I$, strongly. For every $x, y \in H$, the map, $P_{x,y} : \Sigma_\Omega \to C$, defined by $P_{x,y}(\Lambda) =< P(\Lambda)x, y >$, for all $\Lambda \in \Sigma_\Omega$, is an ordinary complex valued countably additive measure on Σ_Ω, (called the spectral measure associated with P). For any bounded Σ_Ω-measurable function $\mu : \Omega \to C$, the integral of μ with respect to this measure, that is, $\int \mu(\lambda)dP(\lambda)$, will be denoted by $\int \mu(\lambda)d(< P(\lambda)x, y >)$, for all $x, y \in H$. The notation, $F = \int \mu(\lambda)dP(\lambda)$, is a mere abbreviation for the above relation. It follows that, if $F = \int \mu(\lambda)dP(\lambda)$, then $F^* = \int \mu(\bar{\lambda})dP(\lambda)$. With respect to this definition, the spectral theorem for normal operators, (a result which has been a corner stone in operator theory, since amongst other goodies, it expresses a way of partitioning every normal operator into 'user friendly' parts), is born. This result states as follows;

LEMMA 4.3.2, [Halmos, 1967]; If $N \in B(H)$ is a normal operator, then there exists a unique spectral measure on $\Sigma_{\sigma(N)}$, such that, $N = \int \lambda dP(\lambda)$.
Moreover, if Λ is a non empty open subset of $\sigma(N)$, then $P(\Lambda) \neq 0$. Further more, an operator $T \in B(H)$, commutes with N and with N^*, if and only if, T commutes with $P(\Lambda)$, for every $\Lambda \in \Sigma_{\sigma(N)}$.

REMARK 4.3.3; Lemma 4.3.2 above, roughly asserts that, any normal operator N can be obtained by integrating the co-ordinate function on the compact subset, $\sigma(N)$, of the complex plane with respect to a 'nice' projection-valued measure. Moreover, one can make sense out of $F(N)$, for a class of functions F which include, in particular, continuous functions. In a simpler language, the spectral theorem for normal linear transformations, says that every normal matrix can be expressed as a direct sum decomposition of orthogonal pro-

jections. Recall that, if a subset M is a subspace of H and an operator $P \in B(H)$, is a projection of H onto M, then M reduces P. Also recall that, direct sum decompositions of operators, succeed as tools of great importance in studying linear operators, since they enjoy the property of transfering invariant subspaces from the direct summands to the direct sums. In other words, consider a vector space R^n, whith the property that, R^n can be partitioned by m mutually orthogonal subspaces $M_1, ------, M_m$. Then, R^n has the orthogonal decomposition, $R^n = M_1 \bigoplus M_2 \bigoplus M_3 \bigoplus ----- \bigoplus M_m$. With respect to this decomposition, assume that, a square matrix T acting on R^n, achieves the orthogonal decomposition, $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus ----- \oplus T_m$, where T_i is the restriction of T to M_i, for each $i \in [1, m]$. That is, $T_i = T/_{M_i}$, and are such that, every T_i is orthogonal to any T_j, whenever $i \neq j$. It is known that, every subspace M_i of R^n is invariant under T_i,(a direct summand), and in this case, M_i gets invariant under T, (the direct sum), as well. One might firstly recall that, every linear transformation can be represented by a square matrix, and every square matrix represents a certain linear transformation. And therefore, Lemma 4.3.2 above, guarantees the existence of a non trivial reducing subspace for any normal operator N acting on H, since, N can be expressed as a direct summand of orthogonal projections $P(\Lambda)s$, where the range of each $P(\Lambda)$, that is, $ran(P(\Lambda))$, reduces $P(\Lambda)$, hence $ran(P(\Lambda))$, reduces N, and from the fact that, $P(\Lambda) \neq 0$ and Λ is a non empty open subset of $\sigma(N)$, it follows that, $ran(P(\Lambda)) \neq \{0\} \neq H$. And thus, $ran(P(\Lambda))$, ends up becoming a non trivial reducing subspace for N. .
We have already observed that, every hyponormal operator is p-hyponormal, but there are p-hyponormal operators, which are not hyponormal. While studying the Putnam's inequality for p-hyponormal operators, [Muneo Cho, et al, 1995], stated and proved the following result;

LEMMA 4.3.4, [80, Thm 4]; For a p-hyponormal operator T, $[T^\star, T] \leq \varphi(\frac{1}{p})\|T\|^{2(1-p)}\ min\{\frac{p}{\pi}\int_{\sigma(T)} r^{2p}$

REMARK 4.3.5; Recall that, every p-hyponormal operator is quasihyponormal, regardless of whether $p \leq 1$ or $p \geq 1$. Putnam's inequality was extended to quasihyponormal operators by [Atsushi, 2000], as follows;

LEMMA 4.3.6, [13, Thm 1]; If T is quasihyponormal then,

$$[T^\star, T] \leq 2\|T\|[\frac{1}{\pi}Area(\sigma(T))]^{\frac{1}{2}}.$$

REMARK 4.3.7; Quasihyponormality was generalized to p-quasihyponormality by [Tanahashi, 1993]. [Atsushi, 2000], extended Lemma 4.3.6 above, to p-quasihyponormal operators as follows;

LEMMA 4.3.8, [13, Preposition 5]; If T is a p-quasihyponormal operator then,

$$\|PT^\star T - TT^\star P\| \leq \quad min\{\frac{p}{\pi}\int_{\sigma(T)} r^{2p-1}drd\theta, \frac{1}{\pi^p}(\int_{\sigma(T)} rdrd\theta)^p\}$$

where P is the projection onto $[TH] = H - kernel \ \ of \ \ T^\star$.

REMARK 4.3.9; [In Hyoun Kim, 2004], introduced the (p,k)-quasihyponormal operators which contains all p-quasihyponormal operators. [In Hyoun kim, 2004], proved that the Putnam's inequality also holds in (p,k)-quasihyponormal operators. His result, which is a generalization of Lemma 4.3.8 above, was as follows;

LEMMA 4.3.10, [60, Thm 1]; If T is a (p,k)-quasihyponormal operator then,

$$\|P\{(T^\star T)^p - (TT^\star)^p\}P\| \leq \quad min\{\frac{p}{\pi}\int_{\sigma(T)} r^{2p-1}drd\theta, \frac{1}{\pi^p}(\int_{\sigma(T)} rdrd\theta)^p\},$$

where P is the projection onto $[TH] = H - kernel \ \ of \ \ T^{\star k}$.

REMARK 4.3.11; Letting $T \in B(H)$, be an invertible p-hyponormal operator with, $0 < p \leq 1$, it follows trivially from Lowner-Heinz inequality that, T is q-hyponormal, for every q such that, $0 < q < p$. Hence, $\|\frac{1}{q}(T^*T)^q - (TT^*)^q\| \leq \frac{1}{\pi}\int\int_{\sigma(T)} r^{2q-1}drd\theta$, and also, letting $q \to 0^+$, [In Hyoun Kim, 2004], proved the following result which is an extension of Putnam's inequality to the case of log-hyponormal operators;

LEMMA 4.3.12, [60, Lemma 3]; let $T \in B(H)$ be a log-hyponormal operator. Then,

$$\|log(T^*T) - log(TT^*)\| \leq \frac{1}{\pi}\int\int_{\sigma(T)} r^{-1}drd\theta$$

.

REMARK 4.3.13; The classes of p-hyponormal and (p,k)-quasihyponormal are generalizations of hyponormal operators. The class of w-hyponormal operators, also generalizes hyponormal operators but is indepedent from the class of (p,k)-quasihyponormal operators. Recall also that, even if every n-Power normal operator is n-Power quasinormal, for the same positive integer n, n-Power quasinormal operators are neither w-hyponormal nor (p,k)-quasihyponormal. In fact, n-Power normal operators are not hyponormal and hyponormal operators are not n-Power normal. However, it is good to note that, both hyponormal and n-Power quasinormal operators, atleast, contains all quasinormal operators. Also recall that, an operator $T \in B(H)$, is said to be subnormal if T has a normal extension. That is, T is subnormal if their exist another Hilbert space say, K containing H and a normal operator N on K, such that, the restriction of N on H is T. Trivially, every normal operator T on H, is subnormal. Importantly, notice that if T is normal, then the self-commutator norm of T is zero. That is, $\|T^*T - TT^*\| = 0$, for every normal T acting on a Hilbert space H. This implies that, the Putnam's inequality has nothing much to tell incase of a normal T. Similarly, if T is n-Power normal, then T^n is normal, and thus, $T^{n*}T^n - T^nT^{n*} = 0$, (equivalently, $T^*T^n - T^nT^* = 0$), which begets the equality, $\|T^{n*}T^n - T^nT^{n*}\| = 0$, (equivalently, $\|T^*T^n - T^nT^*\| = 0$). However, it is unfortunate that this trivial size of the self-commutator norm of any n-Power normal T, does not always guarantee the equality, $\|T^*T - TT^*\| = 0$, since every n-power normal operator is not normal. Our first task in this section, is to investigate the size of the self-commutator norm of T, (equivalently, the size of $\|D_T\|$), for every $T \in B(H)$, satisfying the equality, $\|T^{n*}T^n - T^nT^{n*}\| = 0$, where n is a positive integer. In other words, we extend the Putnam's inequality to the case of n-power normal operators, as follows;

THEOREM 4.3.14, [Imagiri, 2014,[56]]; Let T be n-Power normal, for some positive integer n. If there exists a T^n-invariant subspace M of H, such that, $T^n/_M = T$, then;

$$\|T^\star T - TT^\star\| \leq \frac{1}{\pi} Area(\sigma(T)).$$

Proof

Since M is a subspace of a Hilbert space H, M is a Hilbert space by itself, with an extension H. Since T^n is a normal operator on H and $T^n/_M = T$, where $M \subset H$, then T is subnormal on M with an extension T^n. With respect to the decomposition, $H = M \bigoplus M^\perp$, T^n can be written as;

$T^n = \begin{bmatrix} T & A \\ 0 & B \end{bmatrix}$. So that, $T^{n*} = \begin{bmatrix} T^* & 0 \\ A^* & B^* \end{bmatrix}$. From the n-Power normality of T, it follows that T^n is normal, and thus, T^n commutes with T^{n*}. That is, $T^{n*}T^n - T^nT^{n*} = 0 - - - - - - -(i)$. Substituting, T^n and T^{n*} with their matrix representations above, and 0 with the correspoding zero matrix, in equation (i) above, and then simplifying, we get,

$$\begin{bmatrix} T^*T & T^*A \\ A^*T & A^*A+B^*B \end{bmatrix} - \begin{bmatrix} TT^*+AA^* & AB^* \\ BA^* & BB^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This implies that, $T^*T - TT^* = AA^*$. Since $A \in B(H)$, both (A^*A) and (AA^*) are positive operators. Inparticular, $(AA^*) \geq 0$. Thus, $T^*T - TT^* \geq 0$. Therefore, T is hyponormal. And thus, from Lemma 1.4.4.1 above, it follows that, $\|T^\star T - TT^\star\| \leq \frac{1}{\pi} Area(\sigma(T))$.

REMARK 4.3.15; Notice that, hyponormality of T follows after demanding that M must be invariant under T^n. Is this hyponormality affected when M goes on and reduces T^n? In response to this question, we state and prove the following two results, which in fact happen to be sharper than Theorem 4.3.14 above.

THEOREM 4.3.16, [Imagiri, 2014,[56]]; Let T be n-Power normal, for some positive integer n. If there exists a T^n-reducing subspace M of H, such that, $T^n/_M = T$, then T is normal. **Proof**

Let $T^n = \begin{bmatrix} T & A \\ 0 & B \end{bmatrix}$, $H = M \bigoplus M^\perp$. Since M reduces T^n, $A = 0$, and thus, $T^n = \begin{bmatrix} T & 0 \\ 0 & B \end{bmatrix}$ and $T^{n*} = \begin{bmatrix} T^* & 0 \\ 0 & B^* \end{bmatrix}$. Since T^n is normal, $T^{n*}T^n - T^nT^{n*} = 0$. Which implies,

$\begin{bmatrix} T^*T & 0 \\ 0 & B^*B \end{bmatrix} - \begin{bmatrix} TT^* & 0 \\ 0 & BB^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and thus, $T^*T - TT^* = 0$. Hence, T is normal.

THEOREM 4.3.17, [Imagiri, 2014,[56]]; Let T be n-Power normal, for some positive integer n. Then, T^n does not have a non-trivial reducing subspace M of H, such that, $T^n/_M = T$.

Proof

Assume to the contary that, M is a non-trivial reducing subspace of T^n such that, $T^n/_M = T$. Then, T^n has the direct sum decomposition $T^n = T \oplus B$ with respect to the decomposition $H = M \bigoplus M^\perp$, where T is normal and B is pure, (that is, there does not exist a non-trivial reducing subspace for T^n of H in which B is normal). But, $T^n = \begin{bmatrix} T & 0 \\ 0 & B \end{bmatrix}$ and

$T^{n*} = \begin{bmatrix} T^* & 0 \\ 0 & B^* \end{bmatrix}$. Since T^n is normal, $T^{n*}T^n - T^nT^{n*} = 0$. Which implies,

$\begin{bmatrix} T^*T & 0 \\ 0 & B^*B \end{bmatrix} - \begin{bmatrix} TT^* & 0 \\ 0 & BB^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and thus, $B^*B - BB^* = 0$. Hence, B is normal.

Note that, $T^n/_{M^\perp} = B$, and if M is a non-trivial subspace of H, that is $M \neq \{0\} \neq H$, then

$H \neq \{0\} \neq M^{\perp}$. This implies that, $M^{\perp}$ is also a non-trivial subspace of H which reduces T^n, and the restriction of T^n to $M^{\perp}$, that is B, is normal. This contradicts the purety of B. Therefore, M is a trivial subspace of H.

REMARK 4.3.18; Recall that, n-Power quasinormal operators generalizes all n-Power normal operators but are indepedent from hyponormal operators. The following result is an extension of the Putnam's inequality to the class of n-Power quasinormal operators;

THEOREM 4.3.19, [Imagiri, 2014,[56]]; Let T be an invertible n-Power quasinormal operator, for some positive integer n. If there exists a T^n-invariant subspace M of H, such that, $T^n/_M = T$, then;

$$\|T^{\star}T - TT^{\star}\| \leq \frac{1}{\pi}Area(\sigma(T)).$$

Proof

From the invertibility of T^n, it follows that T is also invertible. Since T is n-Power quasinormal, we have that, $T^n(T^*T) = (T^*T)T^n \Rightarrow T^nT^*T = T^*T^nT$. Post multiplying by T^{-1}, we obtain $T^nT^* = T^*T^n$. Thus, T^n commutes with T^*, and thus T^n is normal. Since $T^n/_M = T$, decomposing T^n as, $T^n = \begin{bmatrix} T & A \\ 0 & B \end{bmatrix}$, it follows from the proof of Theorem 4.3.14 above, that $T^*T - TT^* \geq 0$. Thus, T is hyponormal and therefore, $\|T^{\star}T - TT^{\star}\| \leq \frac{1}{\pi}Area(\sigma(T))$.

REMARK 4.3.20; Recall that, w-hyponormal operators generalizes all p-hyponormal and log-hyponormal operators but w-hyponormal operators are indepedent from both (p,k)-quasihyponormal and n-Power quasinormal operators. It is important to note that, if an operator T is normal, then its adjoint, T^* is also normal. Unlike normal operators, hyponormal, p-hyponormal, w-hyponormal and n-Power quasinormal operators have an habit of sometimes rejecting T^* after taking in T. Also recall that, an operator T is said to be co-hyponormal, if T^* is hyponormal. That is, T is co-hyponormal if T^* satisfies the inequality, $TT^* - T^*T \geq 0$. It follows trivially that, every operator T, which is both hyponormal and co-hyponormal ends up becoming normal. [Aluthge et al, 2000], proved that if T and T^* are both w-hyponormal, such that $ker(T^*) \subset ker(T)$, then T is normal. Without imposing the kernel condition on T, we extend the Putnam's inequality to the class of w-hyponormal operators as follows;

THEOREM 4.3.21, [Imagiri, 2014,[56]]; Let T be a w-hyponormal operator. If T^* is also w-hyponormal, then;

$$\|T^{\star}T - TT^{\star}\| \leq \frac{1}{\pi}Area(\sigma(T)).$$

Proof

If T and T^{*} are w-hyponormal operators, then from Lemma 3.5.1, it follows that their first Aluthge transforms are semi-hyponormal, and their second Aluthge transforms are hyponormal. That is, $\tilde{T}$ and $\tilde{T^{*}}$ are semi-hyponormal operators, and $\tilde{\tilde{T}}$ and $\tilde{\tilde{T^{*}}}$ are hyponormal operators. In particular, $\tilde{\tilde{T}}$ is hyponormal and thus,

$$\|\tilde{\tilde{T}}^{\star}\tilde{\tilde{T}} - \tilde{\tilde{T}}\tilde{\tilde{T}}^{\star}\| \leq \frac{1}{\pi}Area(\sigma(\tilde{\tilde{T}})) - - - (i)$$

. But by Theorem 3.2.6 we had that, all Aluthge transforms of any w-hyponormal operator have equal spectra. Thus, (i) above becomes,

$$\|\tilde{\tilde{T}}^{\star}\tilde{\tilde{T}} - \tilde{\tilde{T}}\tilde{\tilde{T}}^{\star}\| \leq \frac{1}{\pi}Area(\sigma(T)) - - - (ii)$$

. From Theorem 3.3.8, we have that, $\|\tilde{\tilde{T}}\| = \|\tilde{T}\| = \|T\|$ and $\|\tilde{\tilde{T^{*}}}\| = \|\tilde{T^{*}}\| = \|T^{*}\|$. Recall that, for any operator T, $\|T\| = \|T^{*}\|$. Thus,

$$\|\tilde{\tilde{T}}\| = \|\tilde{T}\| = \|T\| = \|\tilde{\tilde{T^{*}}}\| = \|\tilde{T^{*}}\| = \|T^{*}\| - - - (iii)$$

. $\tilde{\tilde{T}}$ and $\tilde{\tilde{T^{*}}}$ are w-hyponormal operators since they are hyponormal operators. Therefore applying Theorem 3.3.8 to equetion (iii) above, we have that, $\|\tilde{\tilde{T}}\tilde{\tilde{T^{*}}}\| = \|\tilde{T}\tilde{T^{*}}\| = \|TT^{*}\|$ and $\|\tilde{\tilde{T^{*}}}\tilde{\tilde{T}}\| = \|\tilde{T^{*}}\tilde{T}\| = \|T^{*}T\|$. So that, $\|\tilde{\tilde{T}}^{\star}\tilde{\tilde{T}} - \tilde{\tilde{T}}\tilde{\tilde{T}}^{\star}\| = \|T^{*}T - TT^{*}\|$. Hence, substituting in equetion (ii) above, we eventually obtain the required inequality, $\|T^{\star}T - TT^{\star}\| \leq \frac{1}{\pi}Area(\sigma(T))$.

REMARK 4.3.22; By applying some of the results obtained in chapter two and chapter three above, we now check the size of the self-commutator norm for the product of either two n-Power quasinormal operators, or two w-hyponormal operators. The following corollaries follows immediatery;

COROLLARY 4.3.23, [Imagiri, 2014,[56]]; Let A and B be invertible operators in nQN such that, $[A, B] = [A, B^{*}] = 0$. Then;

$$\|((AB)^{n})^{\star}(AB)^{n} - (AB)^{n}((AB)^{n})^{\star}\| \leq \frac{1}{\pi}Area(\sigma((AB)^{n})).$$

Proof

From Theorem 2.4.1 above, $(AB)^{n}$ is normal. Thus, $[(AB)^{n}, ((AB)^{n})^{*}] = 0 \Rightarrow \|((AB)^{n})^{\star}(AB)^{n} -$

$(AB)^n((AB)^n)^\star\| = 0$. Hence, $\|((AB)^n)^\star(AB)^n - (AB)^n((AB)^n)^\star\| \le \frac{1}{\pi}Area(\sigma((AB)^n))$.

COROLLARY 4.3.24, [Imagiri, 2014,[56]]; Let $A, B \in nQN$, and $A^*, B^* \in nQN$, such that, $[A, B] = 0$. Then;

$$\|((AB)^n)^\star(AB)^n - (AB)^n((AB)^n)^\star\| \le \frac{1}{\pi}Area(\sigma((AB)^n)).$$

Proof

From Lemma 2.2.9 above, we have that, A^n and B^n are normal operators. Since $[A, B] = 0$, then A^n and B^n commute. That is, A^n and B^n are commutative normal operators, thus their product, (A^nB^n) is normal. But, $(A^nB^n) = (AB)^n$, for any positive integer n. It follows that, $(AB)^n$ is normal. Thus, $[(AB)^n, ((AB)^n)^*] = 0 \Rightarrow \|((AB)^n)^\star(AB)^n - (AB)^n((AB)^n)^\star\| = 0$. Hence, $\|((AB)^n)^\star(AB)^n - (AB)^n((AB)^n)^\star\| \le \frac{1}{\pi}Area(\sigma((AB)^n))$.

COROLLARY 4.3.25, [Imagiri, 2014,[56]]; If T is a w-hyponormal operator such that, $\tilde{T_n}$ is normal for some $n \in J^+$, then;

$$\|T^\star T - TT^\star\| \le \frac{1}{\pi}Area(\sigma(T)).$$

Proof

If $\tilde{T^n}$ is normal for any positive integer n, then T is also normal. Thus, $[T, T^*] = 0$. That is, $T^*T - TT^* = 0 \Rightarrow \|T^*T - TT^*\| = 0$. Hence, $\|T^\star T - TT^\star\| \le \frac{1}{\pi}Area(\sigma(T))$.

COROLLARY 4.3.26, [Imagiri 2014,[56]]; If T is a w-hyponormal operator such that, $\tilde{T_n}$ is normal for some $n \in J^+$, then;

$$\|\tilde{T_m^\star}\tilde{T_m} - \tilde{T_m}\tilde{T_m^\star}\| \le \frac{1}{\pi}Area(\sigma(\tilde{T_m})).$$

Where, $m \in J^+$ such that, $m \le n$.

Proof

We know that, if $\tilde{T_n}$ is normal, then $\tilde{T_{n-1}}$ is also normal for any $n \in J^+$. In particular, $\sigma\tilde{T_n} = \sigma\tilde{T_{n-1}}\ \forall n \in J^+$. Thus, $\tilde{T_m}$ is normal for all $m \in J^+$ such that, $m \le n$. Trivially, $\|\tilde{T_m^\star}\tilde{T_m} - \tilde{T_m}\tilde{T_m^\star}\| \le \frac{1}{\pi}Area(\sigma(\tilde{T_m}))$.

REMARK 4.3.27; The powers of any w-hyponormal operator are again w-hyponormal, but the product of any two w-hyponormal operators is not a w-hyponormal operator in general. Incase of two commuting w-hyponormal operators, the following observation can quickly be deduced;

COROLLARY 4.3.28, Imagiri 2014,[56]; Let A, B be any two w-hyponormal operators. If $\tilde{A_n}$ and $\tilde{B_n}$ are normal for some $n \in J^+$ and $[A, B] = 0$, then;

$$\|((AB)^{2m})^{\star}(AB)^{2m} - (AB)^{2m}((AB)^{2m})^{\star}\| \leq \frac{1}{\pi}Area(\sigma((AB)^{2m})).$$

Proof

If $\tilde{A_n}$ and $\tilde{B_n}$ are normal, then by Theorem 3.5.13 above, A, B are normal. Since $[A, B] = 0$, then both AB and BA are normal operators. Since, every power of a normal operator is again a normal operator, it follows that, $(AB)^m$, is normal for each positive integer m. And thus, $(AB)^{2m}$ is also normal $\forall m \in J^+$. This implies that, $[(AB)^{2m^*}, (AB)^{2m}] = 0$. Therefore, $\|((AB)^{2m})^{\star}(AB)^{2m} - (AB)^{2m}((AB)^{2m})^{\star}\| \leq \frac{1}{\pi}Area(\sigma((AB)^{2m}))$.

4.4; Berger-Shaw's inequality for n-Power normal, n-Power quasinormal and w-hyponormal Operators

The strengh of an operator is expressed by the size of its norm, and fortunately, there are several other types of norms, such as the Hilbert-Schmidt norm or even the trace of a given operator. Unlike the operator norm, which we denoted by $\|T\|$, and defined as, $\|T\| = (< Tx, Tx >)^{1/2} \quad \forall x \in H$, and which is used in the Putnam's inequality, the Berger-Shaw's inequality estimates the strentgh of the self commutator of Hilbert Schmidt operators by their trace. The following definitions therefore features in this section;

DEFINITION 4.4.1

Let $T \in B(H)$ and $\{e_n\}$ be an orthonormal basis for H. Then, the **Hilbert-Schmidt norm** of T, denoted by $\|T\|_2$, is defined as; $\|T\|_2 = (\sum_{n=1}^{\infty} \|Te_n\|^2)^{1/2}$. If $\|T\|_2 \leq \infty$, T is said to be a **Hilbert-Schmidt operator**. Any operator on H, that can be expressed as a product of two Hilbert-Schmidt operators, is called a **trace class operator**. If T is a trace class operator, then the **trace of** T, denoted by $tr(T)$, is a linear functional which associates T with a positive complex number and is defined as, $tr(T) = \sum_{n=1}^{\infty} < Te_n, e_n >$. For every Hilbert-Schmidt operator T on H, it follows that $\|T\|_2 = \|T^*\|_2$ and the above mentioned two norms are related by the inequality, $\|T\| \leq \|T\|_2$, [Conway, 1981]. In addition, a result by [Murphy, 1990], shows that, if $T \in B(H)$ and A is a Hilbert-Schmidt operator, then $\|TA\|_2 \leq \|T\|\|A\|_2$ and $\|AT\|_2 \leq \|A\|_2\|T\|$. One should easily note that these two inequalities implies that, if

$T \in B(H)$ and A is a Hilbert-Schmidt operator, then both (TA) and (AT) are Hilbert-Schmidt operators.

DEFINITION 4.4.2

Let $T \in B(H)$ and x be a vector in H such that, $\bigvee\{T^n x\}_{n\geq 0} = H$. Then, x is said to be a **cyclic vector** for T, where $\bigvee\{T^n x\}_{n\geq 0} = H = span(\{\bar{T^n x}\}_{n\geq 0})$, is a subspace of H. If $T \in B(H)$ has a cyclic vector, then T is called a **cyclic operator**. A subspace M of H is **totally cyclic** for $T \in B(H)$, if every non zero vector in M is cyclic. Observe that, T has no non trivial invariant subspace, if and only if every non zero vector in H is a cyclic vector for T. For if $M \subset H$, is T-invariant, then $T^n(M) \subset M$: That is if and only if, $\bigvee\{T^n x\}_{n\geq 0} = H$, for every $x \neq 0$ in H. This inturn implies that, H is itself totally cyclic for T. Generally, if $\Re(\sigma(T))$ denotes the set of all rational functions analytic on $\sigma(T)$, then for any positive integer k, T is said to be k **multicyclic**, if there are k cyclic vectors $x_1, x_2, ---, x_k$ in H, such that, $\bigvee\{g(T)x_i, i = 1, 2, --, k, g \in \Re(\sigma(T))\} = H$.

REMARK 4.4.3;Recall that, an operator T is compact if its range is a compact subset of H. It is known that, every compact operator is reducible and that, if H is of finite dimension, then every operator on H is compact. Thus every bounded operator is compact. Therefore, from the fact that, every operator is bounded, if its norm is finite, and that $\|A\| \leq tr(A)$ for any square matrix A, it follows readily that, any linear operator is bounded if its trace is finite. Problems begin when the dimension of H is not finite, since the reducibilty of any T(not neccessarily bounded), on such H cant be concluded via the compactness of T. Luckily, for any T, whether bounded or not bounded, $[T^*, T]$ exists naturally, and it is known that, if this self commutator of T is compact, then T may atleast be written as a direct sum of irreducible operators. Therefore, an operator T is reducible if $[T^*, T]$ is compact. Equivalently, an operator $T$is compact if the $tr[T^*, T]$ is finite. Berger-Shaw inequality estimates the least upper bound of $[T^*, T]$, hence implying that $[T^*, T]$ is compact. In other words, this inequality observes that, for any cyclic hyponormal operator T, the self commutator norm of T, is trace class, and the trace of this self commutator is atmost $\pi^{-1/2}Area(\sigma(T))$. In this section, we first present without proofs, some of the well known extensions of the Berger-Shaw's inequality, then investigate this inequality for n-power normal, n-Power quasinormal and w-hyponormal operators. For simplicity, B.S.I, denotes the Berger-Shaw's inequality. The following result by [Atsushi, 1999], is an extension of the B.S.I, to p-hyponormal operators, (Area means the Planar Lebesque measure);

LEMMA 4.4.4, [12, Thm 1]; If T is n-multicyclic p-hyponormal operator, then; (T^*T-TT^*) is trace class and,

$$tr(| T |^{1-p} (T^{\star}T)^p - (TT^{\star})^p | T |^{1-p}) \leq \frac{n}{\pi} Area(\sigma(T))$$

REMARK 4.4.5; Every invertible p-hyponormal operator is log-hyponormal. However, p-hyponormal operators are not invertible in general. Berger-Shaw inequality is as follows for the case of invertible p-hyponormal operators;

LEMMA 4.4.6, [13, Preposition 4]; If T is an invertible n-multicyclic p-hyponormal operator, then; $(T^*T - TT^*)$ is trace class and,

$$tr((T^{\star}T)^p - (TT^{\star})^p) \leq \|T^{-1}\|^{2(1-p)} \frac{n}{\pi} Area(\sigma(T)$$

REMARK 4.4.7; Every hyponormal operator is quasihyponormal. But quasihyponormal operators are different from p-hyponormal operators. The following observation is an extension of Lemma 4.4.6 above, to the class of quasihyponormal operators;

LEMMA 4.4.8, [13, Preposition 5]; For a n-multicyclic quasihyponormal operator T, $tr[T^{\star}, T] \leq 4\|T\| \frac{n}{\pi} Area(\sigma(T))$

REMARK 4.4.9; Even if quasihyponormal and p-hyponormal operators are different classes, they are both contained properly in the class of p-quasihyponormal operators. The following result is a generalization of the B.S.I, to the class of p-quasihyponormal operators;

LEMMA 4.4.10, [13, Thm 3]; If T is a n-multicyclic p-quasihyponormal operator, then; $(T^*T - TT^*)$ is trace class and,

$$tr((P(T^{\star}T)^p P - P(TT^{\star})^p P)^{\frac{1}{p}}) \leq \frac{n}{\pi} Area(\sigma(T))$$

where P is the projection onto $[TH]^{\neg} = H - ker(T^{\star})$.

REMARK 4.4.11; The class of p-quasihyponormal is invariant under invertibilty. That is, if T is an invertible member of this class, then so is T^{-1}. Generally, every p-quasihyponormal operator is not invertible. The Berger-Shaw's inequality is as follows for the case of invertible p-quasihyponormal operators;

LEMMA 4.4.12, [13, Thm 4]; If $A = T\backslash_{[TH]^{\neg}}$ (restriction of T onto $[TH]^{\neg}$), is an invertible n-multicyclic p-quasihyponormal operator, then,

$$\|(T^{\star}T - TT^{\star})^2\|_1 \leq 6\|T\|^{2(2-q)}\|A^{-1}\|^{2(1-q)}\varphi(\frac{1}{q})\frac{n}{\pi}Area(\sigma(T))$$

. Where, $\varphi(\frac{1}{q}) = \sum_{2+\frac{1}{q}, \quad otherwise}^{\frac{1}{q}, \quad if \frac{1}{q} \in (natural numbers)}$.

REMARK 4.4.13; Notice that, the restriction A of a n-multicyclic p-quasihyponormal operator T to an invariant subspace of T, is also n-multicyclic. This follows from the fact that $\sigma(A) \subset \sigma(T)$.

The class of (p,k)-quasihyponormal includes all p-quasihyponormal operators. [In Hyoun Kim, 2004], came up with the following result which is an extension the B.S.I, to the case of (p,k)-quasihyponormal operators;

LEMMA 4.4.14, [60, Thm 3]; If T is a n-multicyclic (p,k)-quasihyponormal operator, then; $(T^*T - TT^*)$ is trace class and,

$$tr((P(T^{\star}T)^pP - P(TT^{\star})^pP)^{\frac{1}{p}}) \leq \frac{n}{\pi}Area(\sigma(T))$$

. Where, $(P(T^{\star}T)^pP - P(TT^{\star})^pP)^{\frac{1}{p}}$ belongs to the schatten $\frac{1}{p}$ class, and P is the projection onto $[TH]^{\neg} = H - ker(T^{\star p})$.

REMARK 4.4.15; Every quasinormal operator is n-Power quasinormal and the class of quasihyponormal operators properly contains the class of quasinormal operators. However, if $T \in B(H)$ is a n-Power normal operator, it is not known whether T is quasihyponormal for every positive integer $n \geq 1$. Ingeneral, the class of all n-Power normal operators neither contains, (nor is it contained), in the class of all quasihyponormal operators. The following results, give sufficient conditions for when the trace of the self commutator of a k-multicyclic n-Power normal, k-multicyclic n-Power quasinormal, or k-multicyclic w-hyponormal operator becomes a finite number;

THEOREM 4.4.16, [Imagiri, 2014,[57]]; Let T be a k-multicyclic n-Power normal, for positive integers n, k. If there exists a T^n-invariant subspace M of H, such that, $T^n/_M = T$, then; $(T^*T - TT^*)$ is trace class and,

$$tr[T^\star, T] \leq \frac{k}{\pi} Area(\sigma(T)).$$

THEOREM 4.4.17, [Imagiri, 2014,[57]]; Let T be an invertible k-multicyclic n-Power quasinormal operator, for some positive integers n, k. If there exists a T^n-invariant subspace M of H, such that, $T^n/_M = T$, then; $(T^*T - TT^*)$ is trace class and,

$$tr[T^\star, T] \leq \frac{k}{\pi} Area(\sigma(T)).$$

THEOREM 4.4.18, [Imagiri, 2014,[57]]; Let T be a k-multicyclic w-hyponormal operator. If T^* is also a k-multicyclic w-hyponormal, then; $(\tilde{\tilde{T}}^\star \tilde{\tilde{T}} - \tilde{\tilde{T}}\tilde{\tilde{T}}^\star)$ is trace class and,

$$tr(\tilde{\tilde{T}}^\star \tilde{\tilde{T}} - \tilde{\tilde{T}}\tilde{\tilde{T}}^\star) \leq \frac{k}{\pi} Area(\sigma(\tilde{\tilde{T}}))$$

.

REMARK 4.4.19; Observe that, apart from requiring T to be k-multicyclic, all other conditions demanded by Theorems 4.4.16 and 4.4.17 above, are the same as those demanded by Theorems 4.3.14 and 4.3.19 above respectively. Thus, it suffices to prove that the restriction of a k-multicyclic n-Power normal, (or that of a k-multicyclic n-Power quasinormal), operator to a invariant subspace, is again k-multicyclic, then henceforth, apply the same procedure used in the proofs of Theorems 4.3.14 and 4.3.19 above. Since every n-power normal operator is n-power quasinormal for the same positive integer n, we only need to show that the restriction of a k-multicyclic n-Power quasinormal operator to a invariant subspace, is again k-multicyclic. That is, we first prove the following result;

THEOREM 4.4.20, [Imagiri, 2014,[57]]; If T is a k-multicyclic n-Power quasinormal operator, then the restriction A of T to a T-invariant subspace M of H, is k-multicyclic.

Proof

Let T be a k-multicyclic n-Power quasinormal operator, $M \subset H$, be invariant for T, and A

be the restriction of T to M. Then, A is a part of T, which implies $\sigma(A) \subset \sigma(T)$. Thus, $\Re(\sigma(T)) \subset \Re(\sigma(A))$. Since, T is k multicyclic, there exist vectors, $x_1, - - -, x_k$ in H, such that, $\bigvee\{g(T)x_i, i = 1, 2, --, k, g \in \Re(\sigma(T))\} = H$. Putting $y_i = Tx_i, i = 1, --, k$, we have that, $\bigvee\{g(A)y_i, i = 1, 2, --, k, g \in \Re(\sigma(A))\} \supset \bigvee\{g(A)y_i, i = 1, 2, --, k, g \in \Re(\sigma(T))\} = \bigvee\{g(A)Tx_i, i = 1, 2, --, k, g \in \Re(\sigma(T))\} = \bigvee\{g(T)Tx_i, i = 1, 2, --, k, g \in \Re(\sigma(T))\} = \bigvee\{Tg(T)x_i, i = 1, 2, --, k, g \in \Re(\sigma(T))\} = M$. That is, there exist cyclic vectors $y_1, --, y_k$ of A in M, such that, $\bigvee\{g(A)y_i, i = 1, 2, --, k, g \in \Re(\sigma(A))\} = M$. And thus, A is also k multicyclic.

Proof of Theorem 4.4.16

Firstly, notice that M being a subspace of a Hilbert space H, M is a Hilbert space by itself, with an extension H. Since T^n is a normal operator on H and $T^n/_M = T$, where $M \subset H$, then T is subnormal on M with an extension T^n. With respect to the decomposition, $H = M \bigoplus M^{\perp}$, T^n can be written as;

$T^n = \begin{bmatrix} T & A \\ 0 & B \end{bmatrix}$. So that, $T^{n*} = \begin{bmatrix} T^* & 0 \\ A^* & B^* \end{bmatrix}$. From the n-Power normality of T, it follows that T^n is normal, and thus, T^n commutes with T^{n*}. That is, $T^{n*}T^n - T^nT^{n*} = 0 - - - - - - -(i)$. Substituting, T^n and T^{n*} with their matrix representations above, and 0 with the correspoding zero matrix, in equation (i) above, and then simplifying, we get,

$$\begin{bmatrix} T^*T & T^*A \\ A^*T & A^*A + B^*B \end{bmatrix} - \begin{bmatrix} TT^* + AA^* & AB^* \\ BA^* & BB^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This implies that, $T^*T - TT^* = AA^*$. Since $A \in B(H)$, both (A^*A) and (AA^*) are positive operators. Inparticular, $(AA^*) \geq 0$. Thus, $T^*T - TT^* \geq 0$. Therefore, T is hyponormal. Applying Theorem 4.4.20 above, we have that T is a k multicyclic hyponormal operator. It finally follows from Lemma 1.4.4.2 above that, $(T^*T - TT^*)$ is trace class and, $tr[T^\star, T] \leq \frac{k}{\pi} Area(\sigma(T))$.

Proof of Theorem 4.4.17

From the invertibility of T^n, it follows that T is also invertible. Since T is n-Power quasinormal, we have that, $T^n(T^*T) = (T^*T)T^n \Rightarrow T^nT^*T = T^*T^nT$. Post multiplying by T^{-1}, we obtain $T^nT^* = T^*T^n$. Thus, T^n commutes with T^*, and thus T^n is normal. Since $T^n/_M = T$, decomposing T^n as, $T^n = \begin{bmatrix} T & A \\ 0 & B \end{bmatrix}$, it follows from the proof of Theorem 4.3.14 above, that $T^*T - TT^* \geq 0$. Therefore, T is hyponormal. Applying Theorem 4.4.20 above, we have that T is a k multicyclic hyponormal operator. It finally follows from Lemma 1.4.4.2 above that,

$(T^*T - TT^*)$ is trace class and, $tr[T^\star, T] \leq \frac{k}{\pi} Area(\sigma(T))$.

REMARK 4.4.21; Before proving Theorem 4.4.17 above, we first need to prove the following result which guarantees k-multicyclicity of all Aluthge transforms of a k-multicyclic w-hyponormal operator.

THEOREM 4.4.22, [Imagiri, 2014,[57]]; If T is a k-multicyclic w-hyponormal operator, then every Aluthge transform of T is also k-multicyclic.

Proof

Let T is a k-multicyclic w-hyponormal operator and $A = \tilde{T_n}$, for any positive integer n. From Theorem 3.2.6 above, $\sigma(T) = \sigma(A)$. Thus, $\Re(\sigma(T)) = \Re(\sigma(A))$ and $ker(T) = ker(A)$. From Theorem 3.2.5 above, that is T is invertible if and only if, A is invertible, it follows that T is one-one and onto $ran(T)$, if and only if, A is one-one and onto $ran(A)$. This implies that, if there exist a pair of vectors $(x, b) \in H$, such that, $Tx = b$, then there exists a unique vector $y \in H$, such that $Ay = b$. Since T is k multicyclic, there exist vectors, $x_1, - - -, x_k$ in H, such that, $\bigvee\{g(T)x_i, i = 1, 2, --, k, g \in \Re(\sigma(T))\} = H$. And thus, $H = \bigvee\{g(T)x_i, i = 1, 2, --, k, g \in \Re(\sigma(T))\} \bigvee\{g(T)y_i, i = 1, 2, --, k, g \in \Re(\sigma(A))\} = \bigvee\{g(A)y_i, i = 1, 2, --, k, g \in \Re(\sigma(A))\}$. Therefore, for every positive integer n, there exist vectors $y_1, - - -, y_k$ in H, such that, $\bigvee\{g(\tilde{T_n})y_i, i = 1, 2, --, k, g \in \Re(\sigma(\tilde{T_n}))\} = H$. And hence, every Aluthge transform of T is k-multicyclic.

Proof of Theorem 4.4.18

Since T is w-hyponormal, then $\tilde{\tilde{T}}$ is hyponormal. Applying Theorem 4.2.21 above, we have that, $\tilde{\tilde{T}}$ is a k-multicyclic hyponormal operator. By Lemma 1.4.4.1 above, it follows that, $(\tilde{\tilde{T}}^\star \tilde{\tilde{T}} - \tilde{\tilde{T}}\tilde{\tilde{T}}^\star)$ is trace class and, $tr(\tilde{\tilde{T}}^\star \tilde{\tilde{T}} - \tilde{\tilde{T}}\tilde{\tilde{T}}^\star) \leq \frac{k}{\pi} Area(\sigma(\tilde{\tilde{T}}))$.

Chapter five

CONCLUSION AND SUMMARY

5.1 Conclusion

Diagonal or equivalently, diagonalizable operators are easy to study since they have simple structures. Linear operators which are not diagonalizable, might atleast be expressed as direct sum decompositions of probably diagonalizable operators. Unfortunately, linear operators acting on Hilbert spaces are neither diagonalizable, nor reducible in general. However, every normal operator is either diagonalizable or similar to a known diagonalizable operator. On the other edge, every reducible operator can be expressed as a direct sum decomposition of a normal and a completely non-normal operator. Generally, n-Power normal, n-power quasinormal and w-hyponormal operators are not only non-normal, but also irreducible.

For any linear transformation T, no matter how complicated is its structure, the positive square root of T, (that is, $T^{1/2}$), and the positive square root of (T^*T), (that is, $|T|$), both behave like T and have simpler structures than that of T. To compute $T^{1/2}$ or $|T|$, one need to compute the eigenvalues of T or those of (T^*T) respectively. In particular, the eigenvalues of T are the fundamental tools used in computing the positive square root of T. For any operator T, the spectrum of T includes properly all the eigenvalues of T, (equivalently, the point spectrum of T). If T is bounded, then both the residue and the continous spectra of T are empty sets. This implies that, the spectrum of a bounded linear operator T on a Hilbert space H, coincides with the point spectrum of T. And thus, locating the spectrum of T, means simply locating the set of all eigenvalues of T. Unfortunately, locations of spectra for higher classes of operators such as n-Power normal, (for a very large positive integer n), n-Power quasinormal and that of w-hyponormal operators are not easily understood. Thus, the shape of the spectrum for the nth-Aluthge transform of an operator picked from any of these classes is without doubt, not trivial.

Results in this thesis show sufficient conditions under when n-Power normal operators, n-Power quasinormal operators and w-hyponormal operators become normal, hence diagonolizable. In addition, conditions implying the normality of the product of any two n-Power normal or that of any two n-Power quasinormal, or even that of any two w-hyponormal operators have been discussed. On the question of reducibility, results leading to existence of non trivial invariant subspaces have been presented. Normality of both n-power normal and n-Power quasinormal operators, has been dealt with in chapter two, while that of w-hyponormal operators, together with an attempt to locate the spectra of these operators, have been given in chapter three. To aid existence of non-trivial reducing subspaces, Putnam-Fuglede theorem, the Putnam's inequality and the Berger-Shaw's inequality for n-Power normal, n-power quasinormal and w-hyponormal operators have been discussed in chapter four.

5.2 Summary of main results

In this thesis, by extending known results to higher classes of operators, we have made several key contributions especially, regarding decompositions and spectral properties for these classes. In Chapter two, we have considered possible conditions under which an n-Power normal or an n-Power quasinormal operator gets restricted to a normal operator and also looked at conditions under which the product of any two operators -each pair picked from the above mentioned classes- become normal.

To this end, we have proved indepedent results and deduced some valuable consequences. Firstly, recall for instance that, if an operator T acting on a Hilbert space H is normal, then T^n is normal for any positive integer n but the converse is not always true. That is, for any T on H, existence of a positive integer n for which T^n becomes normal does not generally imply T is normal. Regarding conditions under when normality of T^n implies that of T or to some extent, when quasinormality of T^n implies quasinormality of T, or even when quasinormality of T implies normality of T, we have managed to prove a number of results. For example, in Theorem 2.3.1, we have shown that if T is a quasinormal operator, then T^n is normal for every positive integer $n \geq 2$, and in Theorem 2.3.3, we have proved that, if T and T^* are quasinormal, then T is normal.

The product of two normal operators or that of two non-normal operators, is not a normal operator in general. However, in Theorem 2.4.1, we have succeeded to lay down sufficient conditions, under which the product of any two n-Power quasinormal operators become normal.

The kernel condition does not hold in general in the class of n-Power quasinormal operators. By assuming this condition to be true in this class, in Theorem 2.4.13, we have managed to prove that the nth-power of the product of any two n-Power quasinormal operators is normal, provided that these two operators are commutative.
Even though it is known in general that the numerical range of an operator, say T, is properly contained in the numerical range of T^n for any positive integer n larger than one, which instead implies that, the numerical range of T^n includes some points which are not points in the numerical range of T, in Theorem 2.4.18, we have succeeded to prove that, if zero is an isolated point in the numerical range of T, then zero is an isolated point in the numerical range of the operator T^{2^n}. By applying this observation, in Theorem 2.4.19, we have managed to generalize a result due to embry, (Lemma 2.4.15) by proving that, if there exists some positive integers m and n, such that the operator T^{2^n} is normal, then the operator T^{2^m} is also normal provided m is less than n and zero is an isolated point in the numerical range of T. After introducing ∞-Power normal and ∞-Power quasinormal operators, (Definitions 2.5(a) and 2.5(b)), we have managed to generalize several results due to Sid Ahmed. For example, in Corollary 2.5.10, we have given an alternative proof for Theorem 2.3.3. One major difference between n-Power normal and normal operators is that, unlike normal operators, n-Power normal operators do not have the translation invariant property. That is, if an operator T is n-Power normal, then $(T - \lambda)$ is not n-Power normal for every complex number λ. However, as we have shown in Theorem 2.5.11, this is not true for ∞-Power normal operators. In other words, in thisTheorem we have succeeded in proving that ∞-Power normal operators have translation invariant property.

In Chapter three, we have first adressed ourselves to the problem of identifying the location of spectra of any nth-Aluthge transform of all operators in general and the classifications of any nth-Aluthge transform of w-hyponormal operators. To this end, in Theorems 3.2.4, 3.2.5 and 3.2.6, we have succeeded in extending results by Derming and Aluthge, (see Lemmas 3.2.1, 3.2.2 and 3.2.3) by proving that, if an operator T is invertible, then every nth-Aluthge transform of T is also invertible and conversely. And that, all Aluthge transforms of T have equal spectra. The inequalities, $\|\tilde{T}\| \leq \|T\|$ and $r(T) \leq w(T) \leq \|T\|$, hold for any operator T on H. In Theorem 3.3.8 and Corollary 3.3.10, we have succeeded in proving that, incase of w-hyponormal operators, these two inequalities are relaxed to equalities. Consequently, these two observations have led to Theorem 3.3.12, which is a generalization of another result by Derming and others, (Lemma 3.3.11), where we have proved that, the spectral radius of all Aluthge transforms of a

w-hyponormal operator, is always equal to the corresponding numerical radius. This in turn has made us to arrive to the conclusion that, every nth-Aluthge transform of a w-hyponormal operator is not only spectraloid, but also normaloid. Even if generally, the nth-Aluthge transform of the kth power of any operator, is different from the kth power of the nth-Aluthge transform of the same operator, (that is, $(\tilde{T^k}_n) \neq (\tilde{T_n}^k)$, where m, n are any pair of positive integers), in Theorem 3.4.2, we have managed to prove that, incase of w-hyponormal operators, these two transformations, are equal. That is, we have shown that, $(\tilde{T^k}_n) = (\tilde{T_n}^k)$, for a w-hyponormal operator T. This observation has led to several other good results. For example, in Theorem 3.4.12, we have attained the classification of $(\tilde{T^k}_n)$ and $(\tilde{T_n}^k)$, where it has been proved that, both transformations are spectraloid whenever T is w-hyponormal.
By proving Theorem 3.4.14, we have managed to extend results due to Nobble,(Lemma 1.8.1), and those due to Wang et al, (Lemma 1.8.8), to the effect of stating conditions under which, the kth power of the nth-Aluthge transform of any invertible w-hyponormal operator ends up being sharpened, either into a self adjoint or a unitary operator, by merely looking at the corresponding spectra.

Our major task in Chapter four, was to extend the three operator inequalities, that is; the Putnam-FugledeTheorem, the Putnam's inequality and the Berger-Shaw's inequality to the higher classes of n-Power normal, n-Power quasinormal and w-hyponormal operators. Before attempting these generalizations, especially those which involved the assymetric extensions of Putnam-Fuglede theorem, we had first to make sure that, n-Power normal operators are indepedent from w-hyponormal operators. For instance, in Theorem 4.1.4, we have shown that n-power normal operators are not always paranormal, and thus, by using paranormality as a tool, we have managed to deduce Corollary 4.1.6, in which it has been concluded that, n-Power normal operators are different from w-hyponormal operators.
In the second section of Chapter four, we have succeeded in proving several results concerning the Putnam-Fuglede theorem. For example, while studying thisTheorem for n-Power normal operators, we have observed in Theorem 4.2.24 that, if A is n-Power normal, B is normal and there exists another operator X on H, such that, X intertwines A^n to B, then X intertwines A^* to B^*. In other words, by Theorem 4.2.24, we have managed to prove that, every n-Power normal operator which is similar to a normal operator, is normal. And in Theorem 4.2.25, by proving that, if A, B are both injective n-Power normal operators, and there exists another operator X on H, such that, X intertwines A^n to B^n, then X intertwines A^* to B^*, we have managed to prove that, any two injective similar n-Power normal operators are normal. Ev-

ery n-Power normal operator is n-Power quasinormal. To extend our results to the case of a bigger class which includes all n-Power normal operators, in Theorem 4.2.27, we have laid down sufficient conditions under which any pair of n-Power quasinormal operators satisfy the Putnam-Fuglede theorem, by proving that, if A, B are both invertible n-Power quasinormal operators, and there exists another operator X on H, such that, X intertwines A^n to B^n, then X intertwines A^* to B^*. We have also gone further than this, and used the Putnam-Fuglede theorem, to investigate conditions under which a n-Power normal or a n-Power quasinormal operator becomes similar or quasi similar to a w-hyponormal operator. For example, inTheorem 4.2.32 and then later in Corollary 4.2.43, we have managed to claim that any n-Power normal operator which is similar to a w-hyponormal operator, ends up becoming normal under an addition requirement that the said w-hyponormal operator has the kernel condition. And also in Theorems 4.2.33 and then by Corollary 4.2.44, we have found out that any invertible n-Power quasinormal operator, which happens to be similar to a w-hyponormal operator, ends up being restricted back to normality, provided the kernel condition holds true for the w-hyponormal operator in question.

While studying the size of the self-commutator norm, (equivalently, the Putnam's inequality), we have proved several results. Good examples are such as Theorems 4.3.14 and 4.3.19 for n-Power normal and n-Power quasinormal operators respectively. In these results, we have managed to extend the Putnam's inequality to n-Power normal operators and n-Power quasinormal operators, only after imposing some tricky conditions on T. In both results, we have required the existence of a subspace M of H such that, M is invariant under T^n, and in addition we have demanded that the restriction of T^n to M, to be T. And worse still, in Theorem 4.3.19, we have forced invertibility of T. Incase of the Putnam's inequality for w-hyponormal operators, we have proved in Theorem 4.3.21 that, the self-commutator norm of a w-hyponormal operator T is finite, if T^* is also a w-hyponormal operator.
Concerning the size of the trace of the self-commutator, (equivalently, the Berger-Shaw's inequality), for both n-power normal and n-Power quasinormal operators, we have managed to prove Theorems 4.4.16 and 4.4.17. Apart from requiring that T should be k-nulticyclic in bothTheorems, all other conditions demanded inTheorems 4.4.16 and 4.4.17 are similar to the conditions demanded in Theorems 4.3.14 and 4.3.19 respectively. But to make sure that the restriction T to M is also k-multicyclic, we have first succeeded to prove Theorem 4.4.19, and then after applying this observation to Theorems 4.3.14 and 4.3.19, we have at the end managed to prove Theorems 4.4.16 and 4.4.17. Another main achievement in this last part of chapter

four is Theorem 4.4.22 in which we have proved that every Aluthge transform of a k-multicyclic w-hyponormal operator, is again k-multicyclic. This observation has eventually led into proving Theorem 4.4.18 and thus, extending the Berger-Shaw's inequality to w-hyponormal operators.

5.3 Future research

Results in this thesis have clearly shown that it is of great importance to investigate diagonalizability of other higher classes of operators through normality. It is clear that direct summands and factors of a linear operator, reveal information about the operator. When reducibility and normality cannot be established, more analysis might be carried out in order to determine structures and properties of these operators. For instance, by using Aluthge transformation as a tool, we may be able to discern the location of the spectrum of an operator. This study has produced many new results on normality and on direct sum decompositions and factorization of some classes of operators. The treatment of the topic is however far from complete. Next, we give a list of some possibilities for future research.

(a) It is well known from this thesis that, all Aluthge transformations of a given linear operator acting on a Hilbert space H have equal spectra. The nature of the spectrum of a n-Power normal or a n-Power quasinormal operator T on H is not known. Knowing the location of the spectrum of $(\tilde{T})_n$, atleast for one positive integer n, means knowing the location of the spectrum of T. Therefore, given an n-Power normal or an n-Power quasinormal operator T, what is the form of the first and the second Aluthge transforms of T?

(b) Existence of higher classes of operators which have an habbit of rejecting T^* after accepting T, (and which ends up suffering some consequences if they incase accept both), is also a clear observation realized in this thesis. for example, if T is hyponormal then T^* is not hyponormal in general and if T is n-Power quasinormal, then T^* might or might not be n-Power quasinormal. We noticed that if both T and T^* are hyponormal then T is normal, and that if both T and T^* are n-Power quasinormal, then T^n is normal. Knowing the Aluthge transformations of T^* is also another thing of crucial importance in fore telling about the behaviour of T. In this regard then, under what conditions does the adjoint of the nth-Alluthge transform of an operator T, ends up being equal to the nth-Aluthge transform of the adjoint of T?

(c) In this thesis it has been proved that the classes of w-hyponormal and n-Power quasinormal operators are indepedent from one another. It has also been shown that, these two classes includes properly all the quasinormal operators and that, every w-hyponormal operator is a $class\ \ A$ operator. If T is a w-hyponormal operator, then it has already been realized that, the first and the second Aluthge transforms of T are semi-hyponormal and hyponormal operators respectively. However if T is a $class\ \ A$ operator, then both the first and the second Aluthge transforms of T are not known. Therefore then;

(i) What are the first and the second Aluthge transforms of a $class\ \ A$ operator T?
(ii) Does the intersection of w-hyponormal operators with n-Power quasinormal operators contain more materials rather than quasihyponormal operators?
(iii) What is the form of the smallest class which includes both n-Power quasinormal and w-hyponormal operators?

(d) Generalizations of classes of operators into larger classes gives us a unified way of studying such operators. For instance, normal operators have been generalized in different ways leading into different series of inclusions of classes of operators. Some of the well known immediate generalizations of normal operators are such as the classes of quasinormal and that of n-power normal operators. Hyponormal operators extends quasinormal operators. Hyponormal operators have also several immediate but indepedent generalizations. Examples of which are such as the classes of p-hyponormal, log-hyponormal, or even that of all p-quasihyponormal operators. Recall that, an operator T is normal if $T^*T = TT^*$ and hyponormal if $T^*T \geq TT^*$. Since T is said to be n-Power normal, for some integer n, if $T^nT^* = T^*T^n$, then one might as well introduce a new class of n-Power hyponormal operators, by defining an operator T, an n-Power hyponormal operator, if T satisfies the inequality $T^nT^* \geq T^*T^n$. It is easy to observe that, this new class includes all normal, all n-Power normal and all hyponormal operators. For such an n-power hyponormal operator T, we pose the following questions;

(i) Is T^* an n-Power hyponormal operator?
(ii) If T is invertible, is T^{-1} n-Power hyponormal?
(iii) What is the spectrum of T?
(iv) What is the form of both the first and second Aluthge transforms of T?
(v) Is the restriction of T to an invariant subspace an n-Power hyponormal operator?
(vi) Assuming T is not bounded, is T reducible?
(vii) It as also been observed that, as a consequence of the Putnam-FugledeTheorem, any w-

hyponormal or any n-Power quasinormal operator which happens to be similar to a normal operator, ends up being normal. Can we conclude the same after substituting w-hyponormality or n-Power quasinormality with n-Power hyponormality?

(viii) We have also noticed that, Putnam's inequality and the Berger-Shaw inequality tells about the boundedness of T through the boundedness of the self-commutator of T, which in turn implies compactness of T, hence reducibility of T. What are some of the sufficient conditions required by an n-Power hyponormal operator T, in order for such T to satisfy these inequalities?

REFERENCES

[1] Agler and Stankus, m-Isometric transformations of Hilbert space I, Integral Equations and Operator Theory, 21 (1995), 383-429.

[2] Aluthge, On p-hyponormal operators for 0 ¡ p ¡ 1, Integral Equations Operator Theory 13 (1990), 307315.

[3] Aluthge Aliyadasa and Derming Wang, Powers of p-hyponormal operators, J. Inequal. Appl. 3 (1999), 279-84.

[4] Aluthge Aliyadasa and Derming Wang, w-hyponormal operators I, Integral Equations Operator Theory 36 (2000), 1-10.

[5] Aluthge Aliyadasa and Derming Wang, w-hyponormal operators II, Integral Equations Operator Theory 37(2000), 324-331.

[6] Anderson and Foias, properties which normal operators share with normal derivations and related operators, Pacific J. Math., 61 (1975), 313325.

[7] Andrew Browder, States, numerical range, etc., Proceedings of the Brown informal Analysis Seminar, Summer, 1969.

[8] Ando Tandao, On some operator inequalities, Math. Ann. 279 (1987), 157-159.

[9] Ando and Gohberg, Operator Theory and Complex Analysis. Workshop on Operator Theory and Complex Analysis. Sapporo (Japon), June (1991)

[10] Arora Sanjeev and Arora P, On p-quasihyponormal operators for $0 \geq p \geq 1$,Yokohama Math.J.41(1993), 25-29.

[11] Atsushi Uchiyama, Kotaro Tanahashi and Jun Ik Lee, Spectrum of class $A(s,t)$ operators.Actasci.math 70(2004)279-287.

[12] Atsushi Uchiyama, Berger ShawTheorem for p-hyponormal operators.-integr. eqn. ope.theory 33(1999) 229-230.

[13] Atsushi Uchiyama, On inequalities of Putnam and Berger-Shaw for p-quasihyponormal operators.-integr. eqn. ope.theory(2000).

[14] Bachir.A and Lombarkia.F, Fuglede-Putnam's for w-hyponormal operators. Mathematical Inq. and Appln. Vol 12, no 4 (2012), 777-786.

[15] Bakir A.N, Putnam FugledeTheorem on (p,k) quasihyponormal and class Y operators. Chlef university, Chlef, 02000, Algeria.

[16] Bala, A Note on Quasinormal Operators. Indian J.pue appl.Math. 8, (1977), 463-465

[17] Berbarian S.K, Introduction to Hilbert spaces. New York. Chelsea Publishing Company. (1976)

[18] Berger Carrie and Shaw Ben, On self commutators of multiciclic hyponormal operators are always trace class, Bull.Amer.Math.Soc.79(1973), 1193-1199.

[19] Campbell and Gupta, On k-quasihyponormal operators, mathematica japonica 23(1978/1979), no. 1, 25-29.

[20] Cho Timothy Chou, On spectra of AB and BA, Proc. KOTAC 3 (2000), 15-19.

[21] Cho Timothy Chou, Huruya Tadasi and Kim Yo, A note on w-hyponormal operators, J. Inequal.Appl. 23(2000),Pg 102-111.

[22] Colojoafa and Foias, Theory of Generalized Spectral Operators, Gordon and Breach, New York, 1968.

[23] Conway J.B, A course in Functional analysis, second edition. Springer (1990).

[24] Crabbe M.J, Some results on the numerical range of an operator, J. London Math. Soc, (2) 2 (1970), 741-745.

[25] Derming Wang and Jun Ik Lee. spectral properties of class A operators.info. centre for mathematical sciences, volume 6, number 2 (Dec 2003) 93-98.

[26] Dragmier and Moslehian. Some inequalities for (α, β)-normal operators in Hilbert Spaces. Facta Universitatis (NIS) Ser. Math. Inform. Vol. 23 (2008), 39-47.

[27] Duggal B.P, Quasi-similar p-hyponormal operators, Integral Equation and Operator Theory 26 (1996), 338345.

[28] Duggal, On dominant operators, Arch. Math. (Basel) 46(1986), 353-359.

[29] Duggal, A remark on generalised Putnam-FugledeTheorems, Proc. Amer. Math. Soc. 129(2000), 83-87.

[30] Duggal and Jeon H.I, Remarks on spectral properties of p- hyponormal and log-hyponormal operators, Bull. Korean Math. Soc. 42(2005), 543-554.

[31] Embry M.R, Conditions implying Normality in Hilbert Space. Pacific J. Math. 18 (1966), 457-460.

[32] Fujii Mitsuru, Jung.D, Lee.S.H, Lee Mike.Y and Nakamoto.R, Some classes of operators related to paranormal and log-hyponormal operators, Math. Japon. 51 (2000), 395-402.

[33] Fuglede B, A commutativityTheorem for normal operators, Proc. Nat. Acad. Sci. U. S. A. 36 (1950), 3540.

[34] Furuta Takayuki, An elementary proof of an order preserving inequality, Proc. Japan Acad. Ser. A Math. Sci. 65 (1989), 126.

[35] Furuta Takayuki, Ito Masuo and Yamazaki Tetsuo, A subclass of paranormal operators including class of log hyponormal and several related classes, Sci. Math. 1 (1998), 389-403.

[36] Furuta Takayuki and Yanagida Mashahiro,, On powers of p-hyponormal operators,

Sci.Math. 2 (1999), 279-284.

[37] Furuta Takayuki and Yanagida Mashahiro,, On powers of p-hyponormal and log-hyponormal operators, J. Inequal. Appl. 5 (2000), 367-380.

[38] Furuta Takayuki. Invitation to Linear operators. Tylor and Francis Group. 11 New Fetter Lane, London EC4P 4EE. (2001)

[39] Gantmaher. Theory of Matrices, volume 1. Chelsea, New York, 1959.

[40] Gantmaher and Krein, O normalnyh operatorah v ermitovom prostranstve [Russian], Izv. fiz-mat ob-vapri Kazanskom Univ., tom IV, vyp. 1, ser. 3, (1929-1930), 7184.

[41] Gohberg I.C, Krein M.G, Introduction to the Theory of Nonselfadjoint Operators, Transl. Math. Mon. vol. 18, Amer. Math. Soc, Providence RI 1969.

[42] Halmos P.R, A Hilbert Space Problem Book, Van Nostrand, Princeton, New Jersey, 1967.

[43] Halmos P.R, A Hilbert Space Problem Book, Second Edition, Springer-Verlag, New York, 1974.

[44] Halmos Peter Richard, A Hilbert Space Problem Book 2nd ed., Springer Verlag, New York, 1982.

[45] Hansen F, An inequality, Math. Ann. 246 (1980), 249250.

[46] Heinz E, Beitrage zur St Orungstheorie der Spektralzerlegung, Math. Ann. 123 (1951), 415438.

[47] Hirzallah O, Kittaneh F and Moslehian M.S, Schatten p-norm inequalities related to a charac- terization of inner product spaces, Math. Inequal. Appl. 13 (2010), no. 2, 235-241.

[48] Ho Jeon, Se Hee Kim. Eungil Ko. and Je Eun Park. On positive normal operators. Bull Korean Math. Soc(39), 2002, No. 1, pp 33-41.

[49] Huruya Tadasi, A note on p-hyponormal operators, Proc. Amer. Math. Soc. 125 (1997), 3617-3624.

[50] Imagiri S.K, Khalagai J.M and Pokhariyal.G.P, nth-Aluthge transforms of w-hyponormal operators. Far East Jnr. of Appld. Maths. Vol. 56, Number 1, 2011, pp 65-74.

[51] Imagiri S.K, Khalagai J.M and Pokhariyal.G.P, On The Normality Of The Products Of n-Power quasinormal Operators. Far East Jnr of Appld. Maths.Vol. 82, Number 1, 2013, pp 41-53.

[52] Imagiri S.K, Khalagai J.M and Pokhariyal.G.P, On the Normality of the Products of Generalized Aluthge Transforms of w-Hyponormal Operators, Far East Jnr of Appld. Maths. Vol 84, No.2, 2013, pp 95-111.

[53] Imagiri S.K, Khalagai J.M and Pokhariyal.G.P, Restrictions On The Powers of nth - Aluthge Transforms of w-Hyponormal Operators, Far East Jnr. of Appld. Maths.Vol. 85,

Number 1, 2013, pp 25-45.

[54] Imagiri Kathurima, *An Interplay Between n-Power quasinormal and w-Hyponormal operators,* Far East Jnr of Appld. Maths. Accepted for publication, 2014.

[55] Imagiri Kathurima, *Putnam-FugledeTheorem for n-Power quasinormal and w-hyponormal operators,* Far East Jnr of Appld. Maths. Accepted for publication, 2014.

[56] Imagiri Kathurima, *Putnam's inequality for n-Power quasinormal and w-hyponormal operators,* Pioneer jnl of mathematics and mathematical sciences. Accepted for publication, 2014.

[57] Imagiri Kathurima, *Berger-Shaw inequality for n-Power quasinormal and w-hyponormal operators,* Far East Jnr of Appld. Maths. Accepted for publication, 2014.

[58] Imagiri Kathurima, *∞-Power normal and ∞-Power quasinormal operators,* Pioneer jnl of mathematics and mathematical sciences, Accepted for publication, 2014.

[59] Imagiri Kathurima, *Putnam-FugledeTheorem for n-Power normal and w-hyponormal operators,* Pioneer jnl of mathematics and mathematical sciences, Accepted for publication, 2014.

[60] In Hyoun Kim, on (p,k) quasihyponormal operators. Mathematics of inequalities and appln, volume 7, number 4(2004)629-638.

[61] Ito Masuo, Generalizations of the results on powers of p-hyponormal operators, J. Inequal. Appl. 6 (2001), 1-15.

[62] Jibril A.A.S, On n-Power Normal operators. The jnr for Sc and Eng. Vol. 33, number 2A. (2008), 247-251.

[63] Jibril A.A.S, On 2-Normal operators. Dirasat Vol. 23, No.2.

[64] Joel Anderson, On normal derivations, Proc. Amer. Math. Soc, 38 (1973), 135-140.

[65] Jung I, Ko and Pearcy, Aluthge transform of operators, Integral Equations Operator Theory 37 (2000) 437448.

[66] Kadison R.V and Ringrose, Fundamentals of the Theory of Operator Algebras, Vol. 1, Elementary Theory, Vol. 2, Advanced Theory, Amer. Math. Soc., Providence R.I. 1997. 150 Aurelian Gheondea

[67] Kaplansky, Products of normal operators, Duke Math. J. 20(1953), 257260.

[68] Kato T, Some mappingTheorems for the numerical range, Proc. Japan. Acad. 41 (1965), 652-655.

[69] Kato T, Perturbation theory for linear operators, 2nd ed., Springer-Verlag, New York, 1980.

[70] Kittaneh F, Inequalities for the Schatten p-norm II, Glasgow Math. J. 29 (1987), 99104.

[71] Kotaro Tanahashi, On isolated points of spectrum of (p,k)-quasihyponormal operators volume 7 (2004)

[72] Lumer and Rosenblum, Linear operator equations, Proc. Amer. Math. Soc, 10 (1959), 32-41.

[73] Mashahiro Yanagida, On operator inequalities and their applications to non-normal operators, Tokyo University (2003).

[74] Mashahiro Yanagida, Powers 0f Class wA(s,t) Operators Associated with Generalized Aluthge Transformation, J. Inequal. Appl. (2002), Vol.7(2), pp 143-168.

[75] Mecheri Sarah, On the normality of operators. Revista Colombiana de Mathematicas. Vol. 39, 87-95 (2005).

[76] Mecheri S, Tanahashi K and Uchiyama A, FugledePutnamTheorem for p-hyponormal or class Y operators, Bull. Korean Math. Soc. 43 (2006), no. 4, 747753.

[77] Moore R, An asymptotic FugledeTheorem, Proc. Amer. Math. Soc. 50 (1975), 138142.

[78] Moslehian M.S and Nabavi Sales, Some conditions implying normality of operators, C. R. Math. Acad. Sci. Paris, Ser. I 349 (2011), no. 5-6, 251254.

[79] Moslehian M.S, and Nabavi Sales, Fuglede-Putnam typeTheorems Via the Aluthge Transforms, Math-FA, (Dec 2011).

[80] Muneo Cho and Masuo Itoh.-Putnam inequality for p-hyponormal operators, proceedings for the Ameican mathematical society. volume 123,number 8 (1995) 2435-2441.

[81] Nelson Dunford and Jacob T. Schwartz, Linear Operators, Part III, Spectral Operators, Wiley Interscience, Wiley Classics, 1988.

[82] Nelson Dunford, Spectral operators, Pacific J. Math., 4 (1954), 321-354.

[83] Noble Barnes, Applied linear algebra, 2^{nd} ed. prentice hall, Englewood cliffs, NJ., (1977).

[84] Ouma O.M, on quasi-similarity and w-hyponormal operators Opscula Mathematica.Vol.27.No.1 2007

[85] Panayan, On n-Power Class (Q) operators,Int.Journal of Maths.Analysis. Vol.6, 2012, No. 31, 1513-1518.

[86] Patel S.M, On intertwining p-hyponormal operators, Indian J. Math, 38 (1996), 287-290.

[87] Putnam.C.R, An inequality for the area of hyponormal spectra, Math.Z.116(1970), 323-330.

[88] Putnam C.R, On normal operators in Hilbert space, Amer. J. Math. 73(1951), 357362.

[89] Radjabalipour M: An extension of Putnam-FugledeTheorem for hyponormal operators. Math. Z. 194 (1987), no. 1, 117120. MR 88c:47065

[90] Rudin and Walter. Functional Analysis. Mc Graw-Hill (1973)

[91] Sid Ahmed. On the class of n-Power quasinormal operators on Hilbert space. Bulletin of Mathematical Analysis and Applications. Vol. 3, Issue 2 pp213-228 (2011)

[92] Stampfli and Wadhwa on dominant operators, Monatshefte. F ur. Math. 84 (1977), 3336.

[93] Stampfli, hyponormal operators, Pacific J. Math 12 (1962), 14531458.

[94] Tanahashi K and Uchiyama A, On the Riesz idempotent of class A Operators, Math. In.Appl. 5 (2002), 291298.

[95] Tanahashi Kotaro, On log-hyponormal operators, Integral Equations Operator Theory 34 (1999), 364-372.

[96] Von Neumann, Approximative properties of matrices of high finite order, Portugaliae Math. 3(1942). 162.

[97] Xia Du, spectral theory for hyponormal operators, Birkhauser Verlag, Basel., 1983.

[98] Yamazaki Tetsuo, Extensions of the results on p-hyponormal and log-hyponormal operators by Aluthge and Wang, SUT J. Math. 35 (1999), 139-148.

[99] Yamazaki Tetsuo, On powers of class A(k) operators including p-hyponormal and log-hyponormal operators, Math. Inequal. Appl. 3 (2000), 97-104.

[100] Yin Chen, On the Putnam FugledeTheorem, IJMMS 2004:53 2821-2834

[101] http:/ / www. pnas. org/ content/ 36/ 1/ 35. full. pdf+ html?ck=nck