UNIVERSITY OF NAIROBI
School of Mathematics
College of Biological and Physical sciences

NEGATIVE BINOMIAL DISTRIBUTIONS FOR FIXED AND RANDOM PARAMETERS

BY

NELSON ODHIAMBO OKECH

I56/70043/2011

A PROJECT PRESENTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICAL STATISTICS.

DECEMBER 2014

DECLARATION

I declare that this is my original work and has not been presented for an award of a degree to any other university

Signature……………………………………… **Date……………………………………….**

Mr. Nelson Odhiambo Okech

This thesis is submitted with my approval as the University supervisor

Signature……………………………………………. **Date………………………………………….**

Prof. J.A.M. Ottieno

DEDICATIONS

This project is dedicated to my mother Mrs. Lorna Okech, my late father Mr. Joseph Okech and my only sister Hada Achieng Okech for having initiated this project, inspired me and sponsored the fulfillment of this dream.

ACKNOWLEDGMENT

I would like to acknowledge with gratitude the effort and guidance from my supervisor Prof. JAM Ottieno. The fulfillment of the project was as a result of his selfless devotion to help.

I'm deeply indebted to the entire School of Mathematics (University of Nairobi) for the support accorded to me at whatever level. The entire academic journey in the mentioned institution was made possible by the relentless support from the dedicated lectures and support staff. Your contribution was immense and may God shower you with his blessings

It cannot be gainsaid that my friends, relatives and well – wishers played a pivotal role in the accomplishment of this project. They provided moral support, encouraged me whenever the journey seemed rocky and gave me the necessary peace of mind to enable me channel all my energy to this remarkable venture: Duncan Okello, Beryl Anyango, Mary Okech, Evelyn Okech, Shadrack Okech, Danson Ngiela and George Oluoch

Last but not least, it is indeed my great pleasure to appreciate the support from my teammates and fellow students during the entire period.

Above all, I thank the Almighty God for the good health and continued protection for the whole period.

ABSTRACT

The first objective of this project is to construct Negative Binomial Distributions when the two parameters *p* and *r* are fixed using various methods based on: Binomial expansion; Poisson – Gamma mixture; Convolution of *iid* Geometric random variables; Compound Poisson distribution with the *iid* random variables being Logarithmic series distributions; Katz recursive relation in probability; Experiments where the random variable is the number of failures before the rth success and the total number of trials required to achieve the rth success.

Properties considered are the mean, variance, factorial moments, Kurtosis, Skewness and Probability Generating Function.

The second objective is to consider *p* as a random variable within the range 0 and 1. The distributions used are:

i. The classical Beta (Beta I) distribution and its special cases (Uniform, Power, Arcsine and Truncated beta distribution).
ii. Beyond Beta distributions: Kumaraswamy, Gamma, Minus Log, Ogive and two – sided Power distributions.
iii. Confluent and Gauss Hypergeometric distributions.

The third objective is to consider *r* as a discrete random variable. The Logarithmic series and Binomial distributions have been considered. As a continuous random variable, an Exponential distribution is considered for *r.*

The Negative Binomial mixtures obtained have been expressed in at least one of the following forms.

a. Explicit form
b. Recursive form
c. Method of moments form.

Comparing explicit forms and the method of moments, some identities have been derived.

For further work, other discrete and continuous mixing distributions should be considered. Compound power series distributions with the *iid* random variables being Geometric or shifted Geometric distributions are Negative Binomial mixtures which need to be studied.

Properties, estimations and applications of Negative Binomial mixtures are areas for further research.

Table of Contents

CHAPTER 1

GENERAL INTRODUCTION

1.1 Background information

Negative Binomial Distribution

Pierre de Montmort first mentioned Negative Binomial distribution in 1713. He considered a series of Binomial trials and came up with a finding of the probability of the number of failures ‘’x’’, before the rth success in the series.

In 1838 Poisson Simeon (1781 - 1840) developed the Poisson regression. He first introduced the new distribution as the limiting case of the Binomial. He later discovered that we can derive Poisson from the Binomial distribution and he demonstrated how the two distributions actually relate. Poisson regression was developed to handle count data, and later became the standard method used to model counts.

It is important to note that Poisson assume equality in its mean and variance. This is a very rare characteristic in real data. Data that has greater variance than the mean is termed as “*Poisson over dispersed”*, yet more commonly known as “*overdispersed”*. It is recommended that we apply Negative Binomial distribution instead of Poisson distribution when dealing with count data that is “*overdispersed”*.

Some of the most prominent and well known discrete distributions are the Binomial, the Poisson and the Negative Binomial distribution. The theoretical connection between these distributions is too close that it is hardly convenient to discuss any one of them without referring to the other.

For instance, the Negative Binomial distribution is based on the other two distributions (Poisson and Binomial) in relation to its construction as you will see later in this project.

Negative Binomial distribution can be expressed in two different ways depending on the definition of the parameter r as follows.

a. 1st Form of Negative Binomial Distribution

Consider a sequence of independent Bernoulli (p) trials. In each trial the probability of success is *p.* Let the random variable X denote the trial, at which the rth success occurs, where r is a fixed integer, then,

$$Pr(X = x/r, p) = \binom{x-1}{x-r} p^r (1-p)^{x-r} \qquad x = r, r+1, r+2, \ldots \qquad (1.01)$$

And we say that X has a Negative Binomial distribution with parameters r and p expressed as $X \sim NB(r, p)$

b. 2nd Form of Negative Binomial Distribution

Negative Binomial distribution can as well be expressed as follows.

$x = the\ Number\ of\ failures\ before\ the\ rth\ success\ in\ an\ infinite\ series\ of$

$indipendent\ trials\ with\ a\ constant\ probability\ of\ success\ p.$

$x + r - 1 = the\ number\ of\ trials\ excluding\ the\ rth\ success$

$\binom{x+r-1}{x} = the\ number\ of\ ways\ of\ obtaining\ x\ failures\ and\ r-1\ success$

Thus the alternative form of the Negative Binomial distribution is expressed as follows

$$Prob(X = x/r, p) = \binom{r+x-1}{x} p^r (1-p)^x \qquad (1.02)$$

$$for\ x = 0,1,2, \ldots and\ r > 0$$

Distribution Mixtures

A **mixture distribution** is the probability distribution of random variable whose values can be interpreted as being derived from an underlying set of other random variables: specifically, the final outcome value is selected at random from among the underlying values, with a certain probability of selection being associated with each. Here the underlying random variables may be random vectors each having the same dimension, in which case the mixture distribution is a multivariate distribution.

In cases where each of the underlying random variable is continuous, the outcome variable will also be continuous and its probability density function is sometimes referred to as a **mixture density**.

A mixture distribution constitutes a number of components which are often restricted to being FINITE, although in some cases the components may be COUNTABLE. More general cases (i.e. an UNCOUNTABLE set of component distributions), as well as the countable case, are referred to as COMPOUND DISTRIBUTIONS

Finite or countable mixtures

$$F(x) = \sum_{j=1}^{n} w_i P_i(x)$$

$$f(x) = \sum_{j=1}^{n} w_i p_i(x)$$

The sum is finite and the mixture is called a **finite mixture,** and in applications, an unqualified reference to a "mixture density" usually means a finite mixture. The case of a countable set of components is covered formally by allowing$n = \infty$.

Uncountable mixtures

Consider a probability density function *p*(*x*;*a*) for a variable *x*, parameterized by *a*. That is, for each value of *a* in some set *A*, *p*(*x*;*a*) is a probability density function with respect to *x*. Given a probability density function *w* (meaning that *w* is nonNegative and integrates to 1), the function

$$f(x) = \int_A \; w(a) p(x; a) \mathrm{da}$$

is again a probability density function for *x*. A similar integral can be written for the cumulative distribution function.

Mixtures of parametric families

Parameters in a mixture distribution can be grouped together into a parametric family and they don’t follow any arbitrary probability distributions. In such cases, assuming that it exists, the density can be written in the form of a sum as:

$$f(x; a_1, \dots \dots \dots, a_n) = \sum_{j=1}^{n} w_i p_i(x; a_i)$$

for one parameter, or

$$f(x; a_1, \dots \dots \dots, a_n, b_1, \dots \dots \dots, b_n) = \sum_{j=1}^{n} w_i p_i(x; a_i b_i)$$

for two parameters, and so forth.

Negative Binomial mixtures

The Negative Binomial distribution is constituted of two parameters $r \; and \; p$, and either of the parameters can be randomized to achieve the Negative Binomial mixture.

This project entails the scenarios where the parameter p has a continuous mixing distribution with the probability *g(p)* such that

$$f(x) = \int_0^1 \binom{r+x-1}{x} p^r (1-p)^x g(p) dp$$

where *f(x)* is the Negative Binomial mixture

1.2 Problem statement

The project considers the Negative Binomial of the following format

$$Pr(X = k/p) = \binom{r+k-1}{k} p^r (1-p)^k$$

The problem statement is to find the Negative Binomial mixtures in the following distribution

i. $f(x) = \int_0^1 \binom{r+k-1}{k} p^r (1-p)^k \, g(p) dp$

p is within the [0,1] domain

ii. $f(x) = \int_0^\infty \binom{r+k-1}{k} p^r (1-p)^k \, g(r) dr$

We need to develop these new distributions to help in solving the problem associated with over dispersed data.

Evaluating data that has several factors that affect the final outcome of the analysis need to be fitted using a heterogeneous distribution that will capture a majority of the aspects. This will reduce the risk that is associated with data loss or generality

1.3 Objectives

Main objective

To construct Negative Binomial mixtures when the mixing distributions come from the probability of success and the number of success as random variable.

Specific objectives

a. To express the Negative Binomial mixed distributions in explicit forms, recursive forms and expectation forms when the mixing distributions are:
   i. Classical beta distribution and its special cases.
   ii. The beyond beta distributions
b. To construct the Negative binomial mixtures when the number of successes takes
   i. Logarithmic distribution
   ii. Binomial distribution
   iii. Negative Binomial distribution
c. To construct generalized Negative Binomial mixture when the number of successes (r) is Exponential distribution
d. To construct Geometric Mixtures as special cases of Negative Binomial distribution

1.4 Methodology

The methods applied to construct the Negative Binomial mixtures include:

i. Explicit or direct method
ii. Moment Generating Function method
iii. Recursive relation method

1.5 Literature review

Here, we will analyze and access some of the works that has been done in relation to Negative Binomial distribution and its mixtures

Wang, Z. (2010) has done research on a three parameter distribution which is called the Beta Negative binomial (BNB) distribution. He derived the closed form and the factorial moment of the BNB distribution. A recursion on the pdf of the BNB stopped-sum distribution and a stochastic comparison between BNB and NB distributions are derived as well. He observed that BNB provides a better fit with a heavier tail compared to the Poisson and the NB for count data especially in the insurance company claim data

Li Xiaohu et al (2011) have studied the Kumaraswamy Binomial Distribution. They considered two models of the Kumaraswamy Distribution, derived their pdfs and other basic properties. The stochastic orders and dependence properties are also worked on by the group. Applications based on the incident of international terrorism and drinking days in two weeks were highlighted as well.

Nadarajah, S. et al (2012) proposed a new three – parameter distribution for modeling lifetime date. This is the Exponential – Negative Binomial distribution. It is advocated as most reasonable among the many exponential mixture type distributions proposed in the recent years.

Kotz S. et al (2004) came up with other Continuous families of Distributions that are Beyond Beta with Bounded Support and applications. Properties studied included moments, CDF, Quartiles, maximum likelihood method of estimating parameters amongst others. The distributions highlighted include Triangular distribution, Standard Two sided Power series

Sarabia J. et al (2008) have done some work on construction of multivariate distributions. The paper reviews the following set of methods: (a) Construction of multivariate distributions based on order statistics, (b) Methods based on mixtures, (c) Conditionally specified distributions, (d) Multivariate skew distributions, (e) Distributions based on the method of the variables in common and (f) Other methods, which include multivariate weighted distributions, vines and multivariate Zipf distributions.

Furman E.(2007) has done some work on the generation of the Negative Binomial Distribution from the sum of random variables. He also talks about the reasons why the negative Binomial distribution has been frequently proposed as a reasonable model for the number of insurance claims.

Ghitany et al (2001) has also shown that Hypergeometric generalized negative binomial distribution has moments of all positive orders, is overdispersed, skewed to the right and leptokurtic.

1.6 Significance of the study and its applications

This is an important project both in substance and timing. The objectives of this project as well as the analysis, as scheduled for discussion are important in identifying new distributions, their properties, identities as well as relevant applications.

Statistical distributions are at the core of statistical science and are a leading requisite tool for its applications. Negative Binomial and especially its mixtures are used widely in the insurance industry in the measure of total claims. This can be done by calculating the total claims distribution (according to the different methods known) by spending a reasonable computing time and without incurring in underflow and overflow (this problem could be defined as a “compatibility” problem of the parameters).

A finite mixture of Negative Binomial (NB) regression models has been proposed to address the unobserved heterogeneity problem in vehicle crash data. This approach can provide useful information about features of the population under study. For a standard finite mixture of regression models, previous studies have used a fixed weight parameter that is applied to the entire dataset. However, various studies suggest modeling the weight parameter as a function of the explanatory variables in the data.

Chapter 2

CONSTRUCTIONS AND PROPERTIES OF NEGATIVE BINOMIAL DISTRIBUTION

2.1. Introduction

Negative Binomial distribution can be constructed from a variety of methods. Some of the techniques are based on:

1. Binomial expansion
2. Mixtures
3. Compound Poisson distribution
4. Katz Recursive relation in probability
5. Logarithmic series
6. Sums of a fixed number of Geometric random variables
7. From experiment

Below are the brief discussions of these methods.

2.2. NBD based on Binomial expansion

Expanding $(a+b)^{-r} where\ r > 0, we\ get$

$$(a+b)^{-r} = \binom{-r}{0} a^{-r} + \binom{-r}{1} a^{-r-1} b^1 + \binom{-r}{2} a^{-r-2} b^2 + \cdots$$

$$= \textstyle\sum_{x=0}^{\infty} \binom{-r}{x} a^{-r-x} b^x \qquad (2.01)$$

$Let\ a = 1\ and\ b = -\theta$

Then

$$(1-\theta)^{-r} = \textstyle\sum_{x=0}^{\infty} \binom{-r}{x} (-\theta)^x \qquad (2.02)$$

Dividing both sides by $(1-\theta)^{-r}$

$$1 = \sum_{x=0}^{\infty} \frac{\binom{-r}{x}(-\theta)^x}{(1-\theta)^{-r}}$$

$$= \sum_{x=0}^{\infty} \binom{-r}{x} (-1)^x (\theta)^x (1-\theta)^r \qquad (2.03)$$

$$\therefore p_x = \binom{-r}{x}(-1)^x(\theta)^x(1-\theta)^r \; for \; r > 0; \; x = 0,1,2,.. \qquad (2.04)$$

Is a pmf

But

$$\binom{-r}{x} = \frac{-r(-r-1)(-r-2)(-r-3)\ldots\ldots..\big(-r-(x-2)\big)\big(-r-(x-1)\big)}{1\cdot 2\cdot 3\ldots\ldots.(x-1)x}$$

$$\binom{-r}{x} = \frac{(-1)^x r(r+1)(r+2)(r+3)\ldots\ldots..\big(r+(x-2)\big)\big(r+(x-1)\big)}{x!}$$

$$\binom{-r}{x} = (-1)^x\binom{r+x-1}{x}$$

Thus

$$(-1)^x\binom{-r}{x} = \binom{r+x-1}{x}$$

Replacing this in equation above

$$p_x = \binom{-r}{x}(-1)^x(\theta)^x(1-\theta)^r \; for \; x = 0,1,2,\ldots$$

$$p_x = \binom{r+x-1}{x}\theta^x(1-\theta)^r \; for \; x = 0,1,2,\ldots.; 0 < \theta < 1 \qquad (2.05)$$

This is the Negative Binomial distribution with parameters $r \; and \; \theta = 1-p$

2.3. NBD based on mixtures

Negative Binomial distribution can be developed from mixing Poisson distribution with Gamma distribution. Gamma distribution is considered as the prior distribution while the Poisson distribution a posterior distribution

Poisson distribution

This is expressed as the probability of a given number of events occurring in a fixed interval of time and/or space if these events occur with a known average rate and independent of time since the last event.

A discrete random variable *X* is said to have a Poisson distribution with parameter $\lambda > 0$, if for *x=0, 1, 2, …*, the probability mass function of *X* is expressed as:

$$Pr(X = x/\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}; \quad x = 0,1,2, \ldots; \;\; \lambda > 0 \tag{2.06}$$

Gamma distribution

$$g(\lambda; \alpha, \beta) = \frac{\lambda^{\alpha-1}e^{-\frac{\lambda}{\beta}}}{\Gamma\alpha\beta^{\alpha}} \; ; for \;\; \lambda \geq 0 \; and \alpha, \beta > 0 \tag{2.07}$$

The mixture

$$f(x) = \int_0^\infty Pr(X = x/\lambda) \, g(\lambda; \alpha, \beta) d\lambda$$

Inserting the equations

$$f(x) = \frac{1}{x! \, \Gamma\alpha\beta^{\alpha}} \int_0^\infty e^{-\lambda}\lambda^{x} \, \lambda^{\alpha-1}e^{-\lambda/\beta} d\lambda$$

$$f(x) = \frac{1}{x! \, \Gamma\alpha\beta^{\alpha}} \int_0^\infty e^{-\lambda-\lambda/\beta} \lambda^{x+\alpha-1} \, d\lambda$$

$$f(x) = \frac{1}{x! \, \Gamma\alpha\beta^{\alpha}} \int_0^\infty e^{-\lambda\left(\frac{1+\beta}{\beta}\right)} \lambda^{x+\alpha-1} \, d\lambda$$

but

$$\int_0^\infty e^{-\lambda\left(\frac{1+\beta}{\beta}\right)} \lambda^{x+\alpha-1} \, d\lambda = \Gamma(\alpha + x) \left\{\frac{\beta}{1+\beta}\right\}^{\alpha+x}$$

$$f(x) = \frac{1}{x! \, \Gamma\alpha\beta^{\alpha}} \Gamma(\alpha + x) \left\{\frac{\beta}{1+\beta}\right\}^{\alpha+x}$$

$$f(x) = \frac{(\alpha + x - 1)!}{x! \, (\alpha - 1)! \, \beta^{\alpha}} \left\{\frac{\beta}{1+\beta}\right\}^{\alpha+x}$$

$$f(x) = \binom{\alpha + x - 1}{x} \left\{\frac{1}{1+\beta}\right\}^{\propto} \left\{1 - \frac{1}{1+\beta}\right\}^{x} \tag{2.08}$$

$$for \; x = 0,1,2, \ldots.; \beta, \propto > 0$$

The marginal distribution of X is a Negative Binomial distribution with $r = \alpha \; and \; p = \frac{1}{1+\beta}$

2.4. Construction from a fixed number of Geometric random variables

$Let\ s_r = X_1 + X_2 + X_3 + \cdots + X_r\ denote\ the\ sum\ of\ iid\ random\ variables\ X_i and\ r\ is\ fixed$

The PGF of $s_N$is given by

$$H(s) = E(s^{s_r}) \tag{2.09}$$

$$= E(s^{X_1+X_2+\cdots+X_r})$$

$$= E(s^{X_1}s^{X_2}s^{X_3} \dots \dots . s^{X_r})$$

$$= E(s^{X_1})E(s^{X_2})E(s^{X_3}) \dots \dots . E(s^{X_r}) \tag{2.10}$$

(since X_i are independent and identical)

$$H(S) = [E(s^X)]^r \qquad (because\ X_i^{'}s\ are\ identical)$$

$$= [G(s)]^r$$

where $G(s) is\ the\ pgf\ of X_i$

$$Let\ X_i{\sim}Geometric(p)$$

Case 1

The pmf of X is

$$p_x = pq^x \qquad for\ x = 0,1,2, \dots; q = 1 - p; 0 < p < 1$$

and the pgf of $X$is

$$G(s) = E(s^X) = \frac{p}{1 - qs}; |s| < (1 - p)^{-1}$$

Therefore

$$H(s) = \{G(s)\}^r$$

$$H(s) = \left[\frac{p}{1-qs}\right]^r \tag{2.11}$$

$\left[\frac{p}{1-qs}\right]^r$ is the pgf of a Negative Binomial distribution with $0 < p < 1, q + p = 1 and\ r > 0$

Proof

$$H(s) = \left[\frac{p}{1-qs}\right]^r$$

$$= p^r(1-qs)^{-r}$$

$$= p^r \sum_{x=0}^{\infty} (-1)^x \binom{-r}{x} (qs)^x$$

$$= p^r \sum_{x=0}^{\infty} \binom{r+x-1}{x} (qs)^x$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r q^x s^x$$

$$= \sum_{x=0}^{\infty} p_x s^x$$

$$= E(s^x)$$

where p_x is the pdf of the negative Binomial distribution and hence the above statement is validated

Case 2

The pmf of x is

$$p_x = pq^{x-1} \qquad for\ x = 1,2,3, \dots;\ q = 1-p;\ 0 < p < 1$$

The pgf of X in this case is given by

$$G(s) = \frac{ps}{1-qs} \qquad if\ |s| < \frac{1}{q}$$

Therefore

$$H(s) = \{G(s)\}^r$$

$$H(s) = \left[\frac{ps}{1-qs}\right]^r$$

$\left[\frac{ps}{1-qs}\right]^r$ is the pgf of a Negative Binomial distribution with $0 \le p \le 1, q + p = 1 and\ r > 0$

Proof

$$H(s) = \left[\frac{ps}{1-qs}\right]^r$$

$$= (ps)^r (1-qs)^{-r}$$

$$= (ps)^r \sum_{x=0}^{\infty} (-1)^x \binom{-r}{x} (qs)^x$$

$$= (ps)^r \sum_{x=0}^{\infty} \binom{r+x-1}{x} (qs)^x$$

$$= (ps)^r \left[\binom{r-1}{0}(qs)^0 + \binom{r}{1}(qs)^1 \binom{r+1}{2}(qs)^2 + \cdots + \binom{r+x-1}{x}(qs)^x + \cdots\right]$$

$$= (ps)^r \sum_{x=r}^{\infty} \binom{x-1}{x-r} (qs)^{x-r}$$

$$= (ps)^r \sum_{x=r}^{\infty} \binom{x-1}{x-r} (qs)^{x-r}$$

$$= \sum_{x=r}^{\infty} \binom{x-1}{x-r} p^r q^{x-r} s^x$$

$$= \sum_{x=r}^{\infty} p_x \, s^x$$

$$= E(s^x)$$

where p_x is the pdf of the negative Binomial distribution and hence the above statement is validated

$$p_x = \binom{x-1}{x-r} p^r q^{x-r} \;\; for \; 0 \le p \le 1, p+q=1, x=r, r+1, r+2, \ldots$$

2.5. Construction from logarithmic series

Construction of Logarithmic distribution.

$$\frac{1}{1-p} = 1 + p + p^2 + p^3 + p^4 + p^5 + \cdots$$

Integrating both sides with respect to *p*we get

$$\int \frac{dp}{1-p} = \int (1 + p + p^2 + p^3 + p^4 + p^5 + \cdots)\text{dp}$$

$$-\log(1-p) = p + \frac{p^2}{2} + \frac{p^3}{3} + \frac{p^4}{4} + \frac{p^5}{5} + \cdots + \frac{p^x}{x} + \cdots \qquad (2.12)$$

$$= \sum_{x=1}^{\infty} \frac{p^x}{x}$$

$$1 = \sum_{x=1}^{\infty} \frac{p^x}{-x\log(1-p)}$$

And hence the logarithmic distribution takes the form below

$$p_x = \frac{p^x}{-x\log(1-p)} \qquad for\ x = 1,2,3,\ldots.; 0 < p < 1 \qquad (2.14)$$

If we consider the associated power equation 2.12 and find its derivative, we get

$$1 + p + p^2 + p^3 + \cdots + p^x + \cdots = \frac{1}{1-p} \qquad (2.15)$$

multiplying both sides of (2.15) by $1 - p$

$$(1-p) + (1-p)p + (1-p)p^2 + (1-p)p^3 + \cdots + (1-p)p^x + \cdots = 1$$

Note that the generating term of this series $(1-p)p^x$ is the pmf of the Geometric distribution where $x = 0,1,2,\ldots$ represents the number of successes before the first failure in a sequence of independent Bernoulli trials with parameter p

Derivative of both sides of equation (2.15)

$$1 + 2p + 3p^2 + 4p^3 + \cdots + xp^{x-1} + (x+1)p^x + \cdots = \frac{1}{(1-p)^2} \qquad (2.16)$$

Multiply across by$(1-p)^2$ and consider the associated generating distribution

$$(x+1)p^x\,(1-p)^2 = (x+1)(1-p)^2p^x; \quad x = 0,1,2,\ldots$$

$$= \binom{x+1}{1}(1-p)^2p^x$$

$$= \binom{2+x-1}{2-1}(1-p)^2p^x$$

which is the pmf of the negative Binomial distribution with parameters 2 and p for $x = 0,1,2, \ldots$

Again take the derivative of (2.16)

$$2 + 3\text{x}2p + 4\text{x}3p^2 + 5\text{x}4p^3 + \cdots + (x+1)xp^{x-1} + (x+2)(x+1)p^x + \cdots = \frac{2}{(1-p)^3}$$

Multiplying across by the reciprocal of $\frac{2}{(1-p)^3}$and taking the associated pmf

$$\frac{(x+2)(x+1)}{2}(1-p)^3p^x = \binom{x+2}{2}(1-p)^3p^x$$

$$= \binom{3+x-1}{3-1}(1-p)^3p^x$$

$$for\ x = 0,1,2, \ldots$$

which is the pmf of the negative Binomial distribution with parameter 3 and p

Taking the rth derivative of each side the power series, we find a series from which the NBD with parameters r and p can be obtained

$$(r-1)! + \frac{r!}{1!}p + \frac{(r+1)}{2!}p^2 + \frac{(k+2)!}{3!}p^3 + \cdots + \frac{(r+x-1)!}{x!}p^x + \cdots = \frac{(r-1)!}{(1-p)^r} \quad (2.17)$$

which is associated with NBD with parameters r and p, and hence

$$\frac{(r+x-1)!}{(r-1)!\,x!}(1-p)^rp^x = \binom{r+x-1}{x}(1-p)^rp^x \quad (2.18)$$

$$for\ x = 0,1,2, \ldots; r > 0; 0 \le p \le 1$$

This is a negative Binomial distribution with parameters r and p

2.6. Representation as compound Poisson distribution

Let $Y_n\ n = 1,2,3, \ldots$ denote a sequence of identical and independent distributed random variables each having a logarithmic distribution $\log(p)\ \ with\ a\ pmf$

$$p_y = \frac{p^y}{-y\log(1-p)} \qquad for\ y = 1,2,3, \ldots.; 0 < p < 1 \quad (2.19)$$

Let N be a random variable independent of the sequence and suppose

$$N \sim Poisson\big(\lambda = rln(1-p)\big)$$

$$let\ s_N = X_1 + X_2 + X_3 + \cdots \ldots \ldots + X_N\ be\ sum\ of\ the\ independent\ random\ variables$$

To calculate the pgf $H(s)\ of\ X\ which\ is\ the\ composition\ of\ the$Probability Generating functions $G_N(s)\ and\ G_y(s)$

$$G_N(s) = e^{\lambda(s-1)} \tag{2.20}$$

and

$$G_y(s) = \frac{\ln(1-ps)}{\ln(1-p)}\,|s| < \frac{1}{p} \tag{2.21}$$

We get

$$H(s) = G_N\left(G_y(s)\right)$$

$$= e^{\lambda\left(G_y(s)-1\right)}$$

$$= \exp\lambda\left(\frac{\ln(1-ps)}{\ln(1-p)} - 1\right)$$

$$= \exp\big(-r(\ln(1-ps) - \ln(1-p))\big)$$

$$H(s) = \left[\frac{1-p}{1-ps}\right]^r \tag{2.22}$$

which is the probability generating function for the negative Binomial distribution

Proof

$$H(s) = \left[\frac{q}{1-ps}\right]^r$$

$$= q^r(1-ps)^{-r}$$

$$= q^r \sum_{x=0}^{\infty} (-1)^x \binom{-r}{x} (ps)^x$$

$$= q^r \sum_{x=0}^{\infty} \binom{r+x-1}{x} (ps)^x$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} q^r p^x s^x$$

$$= \sum_{x=0}^{\infty} p_x \, s^x$$

$$= E(s^x)$$

where p_x is the pdf of the negative Binomial distribution and hence the above statement is validated

2.7. Construction using Katz recursive relation in probability

Consider the following recursive relation in probabilities

$$\frac{f(x+1)}{f(x)} = \frac{P(x)}{Q(x)} \qquad (2.23)$$

Where $P(x) and\ Q(x) are\ polynomials\ of\ x$

$f(.)$ is the probability mass function in particular

Let

$$\frac{P(x)}{Q(x)} = \frac{\alpha+\beta x}{1+x} \qquad (2\ .24)$$

which is the Katz relationship

This implies

$$\frac{f(x+1)}{f(x)} = \frac{\alpha+\beta x}{1+x} \quad ; x = 0,1,2 \ldots$$

Let $\alpha \neq 0\ and\ \beta \neq 0$

Then

$$f(x+1) = \frac{\alpha+\beta x}{1+x} f(x) \quad ; x = 0,1,2 \ldots \qquad (2.25)$$

We shall use two approaches to obtain the negative Binomial distribution.

2.7.1. Iteration technique

When $x = 0$ $\quad f(1) = \alpha f(0)$

When $x = 1$ $\quad f(2) = \frac{\alpha+\beta}{2} f(1) = \left[\frac{\alpha+\beta}{2}\right] \alpha f(0)$

When $x = 2$ $\quad f(3) = \frac{\alpha+2\beta}{3} f(2) = \left[\frac{\alpha+2\beta}{3}\right]\left[\frac{\alpha+\beta}{2}\right] \alpha f(0)$

When $x = 3$ $\quad f(4) = \frac{\alpha+3\beta}{4} f(3) = \left[\frac{\alpha+3\beta}{4}\right]\left[\frac{\alpha+2\beta}{3}\right]\left[\frac{\alpha+\beta}{2}\right] \alpha f(0)$

When $x = k - 1$ $\quad f(k) = \frac{\alpha+(k-1)\beta}{k} f(k-1)$

$$f(k) = \left[\frac{\alpha + (k-1)\beta}{k}\right]\left[\frac{\alpha + (k-2)\beta}{k-1}\right] \dots \dots \left[\frac{\alpha + 3\beta}{4}\right]\left[\frac{\alpha + 2\beta}{3}\right]\left[\frac{\alpha + \beta}{2}\right] \alpha f(0) \qquad (2.26)$$

$$f(k) = \frac{\left[\beta\left(\frac{\alpha}{\beta} + k - 1\right)\right]\left[\beta\left(\frac{\alpha}{\beta} + k - 2\right)\right] \dots \dots .. \left[\beta\left(\frac{\alpha}{\beta} + 2\right)\right]\left[\beta\left(\frac{\alpha}{\beta} + 1\right)\right]\left[\beta\left(\frac{\alpha}{\beta}\right)\right] f(0)}{k!}$$

$$f(k) = \beta^k \frac{\left(\frac{\alpha}{\beta} + k - 1\right)!}{\left(\frac{\alpha}{\beta} - 1\right)!\, k!} f(0)$$

Therefore

$$f(k) = \beta^k \binom{\frac{\alpha}{\beta} + k - 1}{k} f(0); \qquad k = 0,1,2,3 \dots \qquad (2.27)$$

Since equation 2.27 is a pmf, then;

$$\sum_{k=0}^{\infty} f(k) = 1$$

$$f(0) + \sum_{k=1}^{\infty} f(k) = 1$$

$$f(0) + \sum_{k=1}^{\infty} \beta^k \binom{\frac{\alpha}{\beta} + k - 1}{k} f(0) = 1$$

But

$$f(0)\left(1+\sum_{k=1}^{\infty}\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}\right)=1$$

$$f(0)=\frac{1}{1+\sum_{k=1}^{\infty}\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}}$$

But

$$f(k)=\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}f(0);\quad k=0,1,2,3\ldots$$

This implies that $f(k)$ will be

$$f(k)=\frac{\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}}{\sum_{k=0}^{\infty}\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}} \tag{2.28}$$

$$k=0,1,2,\ldots$$

If $r=\frac{\alpha}{\beta}$ is a positive integer

$$\binom{\frac{\alpha}{\beta}+k-1}{k}=\binom{r+k-1}{k}=(-1)^k\binom{-r}{k}$$

$$\sum_{k=0}^{\infty}\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}=\sum_{k=0}^{\infty}(-\beta)^k\binom{-r}{k}$$

$$=(1-\beta)^{-r}$$

$$0<\beta<1$$

Conclusion

$$f(x+1)=\left(\frac{\alpha+\beta x}{1+x}\right)f(x);\quad x=0,1,2\ldots \tag{2.29}$$

For

1. $r = \frac{\alpha}{\beta}$ is a positive integer
2. $0 < \beta < 1$
3. $\alpha > 0$

$$f(k) = \frac{\beta^k \binom{\frac{\alpha}{\beta} + k - 1}{k}}{(1-\beta)^{-r}} = \binom{\frac{\alpha}{\beta} + k - 1}{k} \beta^k (1-\beta)^r$$

$$f(k) = \binom{r + k - 1}{k} \beta^k (1-\beta)^r \qquad (2.30)$$

$$0 < \beta < 1;\ k = 0,1,2,3 \ldots.$$

This is a Negative Binomial distribution with parameters $k\ and\ p = 1 - \beta$

2.7.2. Using the Probability Generating Function (PGF) technique

Remember

$$f(x+1) = \frac{\alpha + \beta x}{1 + x} f(x) \qquad ; x = 0,1,2, \ldots ; \alpha > 0; 0 < \beta < 1$$

Let $\alpha \neq 0\ and\ \beta \neq 0$

Then

$$\sum_{x=0}^{\infty} [1 + x]\, f(x+1) s^x = \sum_{x=0}^{\infty} [\alpha + \beta x]\, f(x) s^x \qquad (2.31)$$

Define

$$G(s) = \sum_{x=0}^{\infty} f(x)\, s^x \qquad (2.32)$$

$$G^{'}(s) = \sum_{x=0}^{\infty} x f(x)\, s^{x-1}$$

$$G^{'}(s) = \sum_{x=0}^{\infty}(x+1)f(x+1)\,s^{x}$$

$$G^{'}(s) = \alpha\sum_{x=0}^{\infty}f(x)\,s^{x} + \beta\sum_{x=0}^{\infty}xf(x)\,s^{x}$$

$$G^{'}(s) = \alpha G(s) + \beta s\sum_{x=0}^{\infty}xf(x)\,s^{x-1} \tag{2.33}$$

$$G^{'}(s) = \alpha G(s) + \beta s G^{'}(s)$$

$$(1-\beta s)G^{'}(s) = \alpha G(x)$$

$$(1-\beta s)\frac{\partial G(s)}{\partial s} = \alpha G(s)$$

$$\int\frac{\partial G(s)}{G(s)} = \int\frac{\alpha}{(1-\beta s)}\partial s \tag{2.34}$$

$$\ln G(s) = \frac{\alpha}{-\beta}\ln(1-\beta s) + \ln C$$

$$\ln G(s) = \ln(1-\beta s)^{\frac{\alpha}{-\beta}} + \ln C$$

$$\ln G(s) = \ln C(1-\beta s)^{\frac{\alpha}{-\beta}}$$

$$G(s) = C(1-\beta s)^{\frac{\alpha}{-\beta}} \tag{2.35}$$

Let $s = 1$

$$G(1) = C(1-\beta)^{\frac{\alpha}{-\beta}}$$

$$1 = C(1-\beta)^{\frac{-\alpha}{\beta}}$$

$$C = \frac{1}{(1-\beta)^{\frac{-\alpha}{\beta}}} = (1-\beta)^{\frac{\alpha}{\beta}} \tag{2.36}$$

$$\therefore G(s) = \left(\frac{1-\beta s}{1-\beta}\right)^{\frac{-\alpha}{\beta}}$$

Let $\frac{\alpha}{\beta} = r$ be a positive integer

$$G(s) = \left(\frac{1-\beta s}{1-\beta}\right)^{-r}$$

$$G(s) = \left(\frac{1-\beta}{1-\beta s}\right)^{r} \tag{2.37}$$

Suppose $p = 1 - \beta \ and \ p + q = 1, then$

$$G(s) = \left(\frac{p}{1-qs}\right)^{r} \tag{2.38}$$

This is a PGF of a NBD with parameters $r > 0 \ and \ p$

Note: proof is carried out in section 2.4 above under case 1

2.7. From experiment

Let X be the number of failures preceding the rth success in an infinite series of independent trials with a constant probability of success p

$$r + x - 1 = the\ toatal\ number\ of\ trials\ excluding\ the\ rth\ success$$

$$\binom{r+x-1}{x} = The\ number\ of\ ways\ of\ obtaining\ x\ failures\ and\ r-1\ success$$

$$\therefore Prob(X = x) = \binom{r+x-1}{x} p^{r-1}(1-p)^x.p$$

$$= \binom{r+x-1}{x} p^{r}(1-p)^x \tag{2.39}$$

$$for\ \ r > 0; 0 \le p \le 1;\ x = 0,1,2,3 \ldots.$$

Alternatively, denote the probability of X=x by $p_x(r), k = 0,1,2, \ldots$

We can formulate a difference equation for $p_x(r)$as follows.

$p_x(r) = The\ probability\ of\ the\ first\ trial\ being\ a\ success\ followed\ by\ a\ prob\ of\ a\ failure\ with\ \ r-1\ successes$

Or

$p_x(r) = The\ probability\ of\ the\ first\ trial\ being\ a\ failure\ followed\ by\ x-1 failures\ with\ \ r\ successes$

$$p_x(r) = pp_x(r-1) + qp_{x-1}(r); \quad x = 1,2,3, \ldots and\ r \ge 1 \tag{2.40}$$

In terms of pgf,

$$\sum_{x=1}^{\infty} p_x(r)\, s^x = p\sum_{x=1}^{\infty} p_x(r-1)\, s^x + q\sum_{x=1}^{\infty} p_{x-1}(r) s^x \qquad (2.41)$$

But $G(s,r) = \sum_{x=0}^{\infty} p_x(r)\, s^x$

$$\therefore G(s,r) - p_0(r) = p[G(s,r-1) - p_0(r-1)] + qsG(s,r) \qquad (2.42)$$

But $p_0(r) = p^r\ and\ p_0(r-1) = p^{r-1}$

$$\therefore G(s,r) - p^r = pG(s,r-1) - p^r + qsG(s,r)$$

$$(1-qs)G(s,r) = pG(s,r-1)$$

$$G(s,r) = \frac{p}{1-qs}G(s,r-1)$$

Put $r = 1 \Rightarrow G(s,1) = \frac{p}{1-qs}G(s,0)$

But

$$G(s,0) = \sum_{x=0}^{\infty} p_x(0)\, s^x = p_0(0) + \sum_{x=1}^{\infty} p_x(0)\, s^x$$

$$p_x(0) = The\ prob\ of\ x\ failures\ before\ zero\ success$$

$$= 0$$

$$p_x(0) = 0\ and\ \ p_0(0) = 1\ for\ x \neq 0$$

$$G(s,1) = \frac{p}{1-qs}G(s,0) = \frac{p}{1-qs}$$

Put $r = 2$ $\qquad G(s,2) = \frac{p}{1-qs}G(s,1) = \left(\frac{p}{1-qs}\right)^2$

In general

$$G(s,r) = \left(\frac{p}{1-qs}\right)^r \qquad (2.43a)$$

This is the pgf of a NBD where the random variable X is the number of failures before the rth success.

The other case is to consider X to be the total number of trials required to achieve r successes. Let us denote X=x with Probability $p_x(r)$. The corresponding difference equation is:

$$p_x(r) = pp_{x-1}(r-1) + qp_{x-1}(r); \quad x = 1,2,3, \dots and\ r \geq 1 \qquad (2.44)$$

In terms of pgf,

$$\sum_{x=1}^{\infty} p_x(r)\, s^x = p \sum_{x=1}^{\infty} p_{x-1}(r-1)\, s^x + q \sum_{x=1}^{\infty} p_{x-1}(r) s^x \qquad (2.45)$$

But $G(s,r) = \sum_{x=0}^{\infty} p_x(r)\, s^x$

$$\therefore G(s,r) - p_0(r) = p[sG(s,r-1)] + qsG(s,r)$$

But $p_0(r) = p^r\ and\ p_0(r-1) = p^{r-1}$

$$\therefore G(s,r) - p^r = psG(s,r-1) + qsG(s,r)$$

$$(1-qs)G(s,r) - 0 = psG(s,r-1)$$

$$G(s,r) = \frac{ps}{1-qs} G(s,r-1)$$

Put $r = 1 \; => G(s,1) = \frac{ps}{1-qs} G(s,0)$

But

$$G(s,0) = \sum_{x=0}^{\infty} p_x(0)\, s^x$$

$$= p_0(0) + \sum_{x=1}^{\infty} p_x(0)\, s^x$$

$$p_x(0) = The\ prob\ of\ x\ failures\ before\ zero\ success$$

$$= 0$$

$$p_x(0) = 0\ and\ \ p_0(0) = 1\ for\ x \neq 0$$

$$\therefore\ G(s,0) = 1$$

$$\therefore G(s,1) = \frac{ps}{1-qs}.1 = \frac{ps}{1-qs} \qquad (2.46)$$

Put $r = 2$ $\quad G(s,2) = \frac{ps}{1-qs} G(s,1) = \left(\frac{ps}{1-qs}\right)^2$

In general

$$G(s,r) = \left(\frac{ps}{1-qs}\right)^2 \qquad (2.43b)$$

This is the pgf of a NBD where the random variable X is the number of trials required to achieve r>1 successes.

Properties

From the construction of Negative Binomial distribution, we have established that Negative Binomial distribution can be expressed in the following formats

1. $p_x = \binom{r+x-1}{x} p^r (1-p)^x \qquad for\ \ r > 0; 0 < p < 1;\ x = 0,1,2,3 \ldots.$

This is a Negative Binomial distribution with parameters $r\ and\ p$, x represents the total number of failures before the rth success

2. $p_x = \binom{x-1}{r-1} p^r (1-p)^{x-r} \qquad x = r, r+1, r+2, \ldots$

If p is a sequence of independent Bernoulli trials and random variable x is taken to denote the trial, at which the rth success occurs, where r is a fixed integer

Probability generating function

The probability generating function of a Negative Binomial distribution is given by the following equation

$$G(s) = E(s^x) = \sum_{x=0}^{\infty} p_x\, s^x$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r q^x s^x \qquad (2.47)$$

$$= p^r \sum_{x=0}^{\infty} \binom{r+x-1}{x} (qs)^x$$

$$= p^r \sum_{x=0}^{\infty} (-1)^x \binom{-r}{x} (qs)^x = p^r (1 - qs)^{-r}$$

$$G(s) = \left[\frac{p}{1 - qs}\right]^r \qquad (2.48)$$

This is the probability generating function for the negative Binomial distribution with

$$|qs| < 1, 0 \leq p \leq 1, q + p = 1 \; and \; r > 0$$

Consider

$$p_x = \binom{r + x - 1}{x} p^r (1 - p)^x \qquad for \;\; r > 0; 0 < p < 1; \; x = 0,1,2,3 \ldots.$$

The first derivative of the pgf of this distribution is

$$G^{'}(s) = \frac{rqp^r}{(1 - qs)^{r+1}} \qquad (2.49)$$

$let \; s = 1$

$$G^{'}(1) = \frac{rqp^r}{(1 - q)^{r+1}}$$

$$G^{'}(1) = \frac{rq}{p} \qquad (2.50)$$

The second derivative of the pgf of the negative Binomial distribution is

$$G^{''}(s) = \frac{r(r + 1)q^2 p^r}{(1 - qs)^{r+2}} \qquad (2.51)$$

$let \; s = 1$

$$G^{''}(1) = \frac{r(r + 1)q^2}{p^2} \qquad (2.52)$$

Mean

$$E(X) = G^{'}(1)$$

$$E(X) = \frac{r(1 - p)}{p} \qquad (2.53)$$

Variance

$$var(X) = G^{''}(1) + G^{'}(1) - \left(G^{'}(1)\right)^2$$

$$var(X) = \frac{rq}{p^2} \qquad (2.54)$$

Factorial moments of Negative Binomial distribution

For a natural number *r,* the *rth* factorial moment of a probability distribution on the real or complex numbers, or in other words, a random variable X with that probability distribution is

$$E((X)_k) = \mu_k(X) = E[X(X-1)(X-2)\dots(X-k+1)] \qquad (2.55)$$

Where E refers to the expectation and $(x)_k = x(x-1)(x-2)\dots(x-k+1)$is the falling factorial.

Thus for the negative binomial distribution

$$\mu_k(X) = \frac{\Gamma(r+k)}{\Gamma r}\frac{(1-p)^k}{p^k} \qquad for\ k = 1,2,3\dots \qquad (2.56)$$

$$\mu_1(X) = E(X) = \frac{r(r-1)!}{(r-1)!}\frac{q}{p} = \frac{rq}{p} \qquad (2.57)$$

$$\mu_2(X) = E(X^2 - X) = \frac{\Gamma(r+2)}{\Gamma r}\frac{(1-p)^2}{p^2} \qquad (2.58)$$

$$\mu_2(X) = r(r+1)\frac{q^2}{p^2} \qquad (2.59)$$

So

$$E(X^2 - X) = r(r+1)\frac{q^2}{p^2} \qquad (2.60)$$

$$E(X^2) = r(r+1)\frac{q^2}{p^2} + \frac{rq}{p}$$

$$E(X^2) = \frac{rq}{p^2}[(r+1)q + p] \qquad (2.61)$$

$$\mu_3(X) = E(X^3 - 3X^2 + 2X) \qquad (2.62)$$

$$= (r+2)(r+1)r\frac{q^3}{p^3} \qquad (2.63)$$

So

$$E(X^3 - 3X^2 + 2X) = (r+2)(r+1)r\frac{q^3}{p^3}$$

$$E(X^3) = (r+2)(r+1)r\frac{q^3}{p^3} + 3E(X^2) + 2E(X)$$

$$E(X^3) = (r+2)(r+1)r\frac{q^3}{p^3} + 3\frac{rq}{p}\left[(r+1)\frac{q}{p} + 1\right] - 2\frac{rq}{p} \tag{2.64}$$

$$= \frac{rq}{p^3}\left\{(r+2)(r+1)q^2 + 3p^2\left[(r+1)\frac{q}{p} + 1\right] - 2p^2\right\}$$

$$= \frac{rq}{p^3}\left\{(r+2)(r+1)q^2 + p^2\left[3(r+1)\frac{q}{p} + 1\right]\right\}$$

$$E(X^3) = \frac{rq}{p^3}\{(r+2)(r+1)q^2 + p[3(r+1)q + p]\} \tag{2.65}$$

$$\mu_4(X) = E[X(X-1)(X-2)(X-3)] \tag{2.66}$$

$$= E[(X^4 - 6X^3 + 11X^2 - 6X)]$$

$$= E(X^4) - 6E(X^3) + 11E(X^2) - 6E(X)$$

$$\mu_4(X) = (r+3)(r+2)(r+1)r\frac{q^4}{p^4} \tag{2.67}$$

Hence

$$E(X^4) = (r+3)(r+2)(r+1)r\frac{q^4}{p^4} + 6E(X^3) - 11E(X^2) + 6E(X) \tag{2.68}$$

$$= (r+3)(r+2)(r+1)r\frac{q^4}{p^4} + 6\frac{rq}{p^3}\{(r+2)(r+1)q^2 + p[3(r+1)q + p]\} - 11\frac{rq}{p^2}\big((r+1)q + p\big) + 6\frac{rq}{p}$$

$$E(X^4) = \frac{rq}{p^4}\{(r+3)(r+2)(r+1)q^3 + 6p\{(r+2)(r+1)q^2 + p[3(r+1)q + p]\} - 11p^2\big((r+1)q + p\big) + 6p^3\} \tag{2.69}$$

Skewness and kurtosis

Skewness is a measure of symmetry or the lack of symmetry. A distribution, or data set, is symmetric if it looks the same to the left and right of the center point.

Kurtosis is a measure of whether the data are peaked or flat relative to a normal distribution. That is, data sets with high kurtosis tend to have a distinct peak near the mean, decline rather rapidly, and have heavy tails. Data sets with low kurtosis tend to have a flat top near the mean rather than a sharp peak. A uniform distribution would be the extreme case.

According to Pearson's moment coefficient of skewness, the skewness of a random variable X is the third Standard moment denoted by γ_1

Suppose $X_i, i = 1,2,3, \dots . N$ are univariate data that follows a Negative Binomial distribution then

$$\gamma_1 = E\left[\left(\frac{x-\mu}{s}\right)^3\right] \tag{2.70}$$

$$= E[x^3 - 3x^2\mu + 3x\mu^2 + \mu^3]\left(\frac{1}{s}\right)^3$$

$$= [E(x^3) - 3\mu E(x^2) + 3\mu^2 E(x) + \mu^3]\left(\frac{1}{s}\right)^3$$

$$= [E(x^3) - 3\mu\{E(x^2) + 3\mu^2\} + \mu^3]\left(\frac{1}{s}\right)^3$$

$$= [E(x^3) - 3\mu s^2 + \mu^3]\left(\frac{1}{s}\right)^3$$

$\therefore$

$$\gamma_1 = \left[\frac{rq}{p^3}\{(r+2)(r+1)q^2 + p[3(r+1)q + p]\} - 3\left(\frac{rq}{p}\right)\left(\frac{rq}{p^2}\right) + \left(\frac{rq}{p}\right)^3\right]\left(\frac{p^2}{rq}\right)^{\frac{3}{2}} \tag{2.71}$$

Negative values for the skewness indicate data that are skewed left and positive values for the skewness indicate data that are skewed right

$kurtosis$

$$Kurtosis = \frac{\sum_{i=1}^{N}(X_i - \mu)^4}{(N-1)s^4}$$

$$= E\left[\left(\frac{x-\mu}{s}\right)^3\right] \tag{2.72}$$

$$= [E(X)^4 - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4]\left(\frac{1}{s}\right)^4$$

$$= \left[\frac{rq}{p^4}\left\{(r+3)(r+2)(r+1)q^3 + 6p\{(r+2)(r+1)q^2 + p[3(r+1)q+p]\}\right.\right.$$
$$\left. - 11p^2\big((r+1)q+p\big) + 6p^3\right\} - 4\mu\frac{rq}{p^3}\{(r+2)(r+1)q^2 + p[3(r+1)q+p]\}$$
$$\left. + 6\mu^2\frac{rq}{p^2}[(r+1)q+p] - 3\left(\frac{rq}{p}\right)^4\right]\left(\frac{p^2}{rq}\right)^2$$

CHAPTER 3

BETA - NEGATIVE BINOMIAL MIXTURES

3.1. Introduction

From chapter 2, we have identified the following forms of Negative Binomial distribution.

1. $p_k = \binom{r+k-1}{k} p^r (1-p)^k \; for \; k = 0,1,2,\dots.; 0 \le p \le 1$ (3.01)

with parameters $r \; and \; p$. k represents the total number of failures before the rth success and;

2. $p_k = \binom{k-1}{r-1} p^r (1-p)^{k-r} \qquad k = r, r+1, r+2, \dots; 0 \le p \le 1$ (3.02)

If p is a sequence of independent Bernoulli trials and random variable k is taken to denote the number of trials required to produce r successes, where r is a fixed integer

In this chapter we are going to consider the Negative Binomial distribution as given in (3.01) when r is fixed and p is varying between 0 and 1.

The distribution of p is the classical beta distribution is given by

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \qquad 0 < x < 1; \; \alpha, \beta > 0 \qquad (3.03)$$

This will act as acts as the mixing distribution.

We shall also use the special cases of the classical beta distribution. These are

1. Uniform distribution
2. Power function distribution
3. Truncated beta distribution
4. Arc – sine distribution

Apart from the beta distribution and its special cases, we shall also consider

5. Confluent Hypergeometric distribution
6. Gauss Hypergeometric distribution

The mixed negative Binomial distributions obtained by various mixing (prior) distributions will be expressed

1. Explicitly (where integration is possible)
2. Recursively
3. Using method of Moments

In section 3.2 we shall have a brief discussion of the forms.

For the other sections we shall briefly introduce the mixing distributions before mixing them with the negative Binomial distribution.

3.2. A brief discussion of the various forms of expressing the mixed distribution

3.2.1. Explicit form

The mixed distribution is expressed as

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp \qquad (3.04)$$

If the integration is possible then we say that f*(x)* is expressed explicitly.

However, in most cases this is not possible so we resort to alternative forms.

3.2.2. Method of moments

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp \qquad (3.05)$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \int_0^1 p^{(r+k)} g(p) dp \qquad (3.06)$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k E\left(p^{(r+k)}\right)$$

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r) x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k E\left(p^{(r+k)}\right) \qquad (3.07)$$

$$for\ 0 \le p \le 1; r > 0; x = 0,1,2, \dots$$

$$E(p^{r+k}) is\ the\ moment\ of\ order\ r+k\ about\ the\ origin\ of\ the\ mixing\ distribution$$

3.2.3. Recursion

One way of obtaining recursions is by considering the ratios of two consecutive probabilities i.e. *f(x)/ f(x-1)*

3.3. Classical beta – Negative Binomial distribution

3.3.1. Construction of Classical Beta Distribution

Classical beta distribution can be constructed in various ways.

Method 1

We can consider a beta function which is expressed in the following format.

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx \qquad (3.08)$$

If we divide both sides by$B(\alpha,\beta)$, we get

$$1 = \int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}\,dx \qquad (3..09)$$

The right hand side of equation 3.06 is a pdf since the integral is equal to 1 and hence the pdf is expressed as follows

$$f(X=x;\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)};\ 0<x<1;\ \alpha,\beta>0 \qquad (3.10)$$

This is the **Beta distribution.**

Method 2

An alternative way of constructing a beta distribution is shown below.

Let $x_1 and\, x_2$ be two stochastically independent random variables that have Gamma distributions and joint pdf

$$f(x_1,x_2) = \frac{1}{\Gamma\alpha}{x_1}^{\alpha-1}e^{-x_1}\frac{1}{\Gamma\beta}{x_2}^{\beta-1}e^{-x_2} \qquad (3.11)$$

$$f(x_1, x_2) = \frac{1}{\Gamma\alpha\Gamma\beta} {x_1}^{\alpha-1} {x_2}^{\beta-1} e^{-x_1} e^{-x_2} \quad 0 < x_1 < \infty, 0 < x_2 < \infty$$

Let $Y = x_1 + x_2 \quad and\ p = \frac{x_1}{x_1+x_2}$

Therefore

$$x_1 = yp, \quad x_2 = y - yp = y(1-p)$$

Then

$$g_1(p, y) = \frac{1}{\Gamma\alpha\Gamma\beta} (py)^{\alpha-1} [y(1-p)]^{\beta-1} e^{-yp} e^{-y(1-p)} |J| \tag{3.12}$$

Where

$$|J| = \begin{vmatrix} \frac{dx_1}{dy} & \frac{dx_1}{dp} \\ \frac{dx_2}{dy} & \frac{dx_2}{dp} \end{vmatrix} = \begin{vmatrix} p & y \\ 1-p & -y \end{vmatrix}$$

$$|J| = -yp - y(1-p) = |-y| = y$$

Therefore equation 3.12 becomes

$$g_1(p, y) = \frac{1}{\Gamma\alpha\Gamma\beta} p^{\alpha-1} [1-p]^{\beta-1} y^{\alpha-1+\beta-1+1} e^{-yp-y(1-p)}$$

$$g_1(p, y) = \frac{1}{\Gamma\alpha\Gamma\beta} p^{\alpha-1} [1-p]^{\beta-1} y^{\alpha+\beta+1} e^{-y} \tag{3.13}$$

$$with\ 0 < y < \infty \quad and\ 0 < p < 1$$

Integrating $g_1(p, y)$ with respect to y, the results to the marginal pdf is given by

$$g_2(p, y) = \int_0^\infty \frac{1}{\Gamma\alpha\Gamma\beta} p^{\alpha-1} [1-p]^{\beta-1} y^{\alpha+\beta-1} e^{-y}\, dy \tag{3.14}$$

Introducing

$$\Gamma(\alpha + \beta)$$

$$g_2(p,y) \ = \ \frac{p^{\alpha-1}[1-p]^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma\alpha\Gamma\beta}\int_0^\infty \frac{y^{\alpha+\beta-1}e^{-y}}{\Gamma(\alpha+\beta)}dy$$

$$g_2(p,y) \ = \ \frac{p^{\alpha-1}[1-p]^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma\alpha\Gamma\beta}.1 \quad = \quad \frac{p^{\alpha-1}[1-p]^{\beta-1}}{B(\alpha,\beta)}$$

$$g_2(p,y) = \frac{p^{\alpha-1}[1-p]^{\beta-1}}{B(\alpha,\beta)}\ ; \quad 0 \le p \le 1, \alpha,\beta > 0 \qquad (3.15)$$

Equation 3.16 is also called the **Classical Beta distribution with parameters $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$**

3.3.2. Properties of the Classical Beta distribution

The jth moment of this pdf (classical beta) about the origin is given by

$$E(P^i) = \int_0^1 \frac{p^{\alpha+i-1}[1-p]^{\beta-1}}{B(\alpha,\beta)}\ dp$$

$$E(P^i) = \frac{B(\alpha+j,\beta)}{B(\alpha,\beta)} \qquad (3.16i)$$

$$E(P^i) = \frac{(\alpha+j-1)!\,(\alpha+\ \beta-1)!}{(\alpha+\beta+j-1)!\,(\alpha-1)!} \qquad (3.16ii)$$

The **Mean** of the classical beta is therefore

$$E(P) = \frac{\alpha}{\alpha+\beta} \qquad (3.17)$$

The 2nd moment about the origin is

$$E(P^2) = \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)} \qquad (3.18)$$

And finally the variance of the mixing distribution becomes

$$Var(P) = \ E(P^2) - (E(P))^2$$

$$Var(P) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \qquad (3.19)$$

3.3.3 The mixture

In explicit form

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)} dp$$

$$= \binom{r+x-1}{x} \frac{1}{B(\alpha,\beta)} \int_0^1 p^{r+\alpha-1}(1-p)^{x+\beta-1} dp \qquad (3.20)$$

$$= \binom{r+x-1}{x} \frac{B(r+\alpha, x+\beta)}{B(\alpha,\beta)} \quad where\ x = 0,1,2,\ldots \qquad (3.21a)$$

$$= \frac{(r+x-1)!}{(r-1)!\,x!} \frac{B(r+\alpha, x+\beta)}{B(\alpha,\beta)} \quad where\ x = 0,1,2,,\ldots\ldots \qquad (3.21b)$$

$$= \frac{\Gamma(r+x)}{\Gamma(r)x!} \frac{B(r+\alpha,x+\beta)}{B(\alpha,\beta)}; \quad x = 0,1,2,\ldots; r,\alpha,\beta > 0 \qquad (3.21c)$$

$$= \frac{\Gamma(r+x)\Gamma(\alpha+\beta)\Gamma(r+\alpha)\Gamma(x+\beta)}{\Gamma(r)x!\,\Gamma\alpha\Gamma\beta\Gamma(r+\alpha+x+\beta)}; \quad x = 0,1,2,\ldots; r,\alpha,\beta > 0 \quad (3.21d)$$

Z. Wang (2010). One mixed negative binomial distribution with application. *Journal of Statistical Planning and Inference*

Using Method of Moments

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k E(p^{(r+k)})$$

$E(p^{r+k})$ *is the moment of order* $r+k$ *about the origin of the mixing distribution*

But

$$E(P^i) = \frac{B(\alpha+j,\beta)}{B(\alpha,\beta)}$$

$$\therefore f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B(\alpha+r+k,\beta)}{B(\alpha,\beta)}; for\ x = 0,1,2,\ldots; r,\alpha,\beta > 0 \ (3.22)$$

Mixing using Recursive relation

There are three ways of applying the recursive relation in this mixture, namely

1. Using ratio of the conditional distribution
2. Using ratio of the mixed distribution
3. Using a dummy function

1. Using ratio of the conditional distribution

$$p_r(x) = \int_0^1 f(x/p)g(p)dp \tag{3.23}$$

$p(x/p)$is a Negative Binomial distribution in this case

$$f(x/p) = \binom{r+x-1}{x} p^r (1-p)^x \tag{3.24}$$

Substituting $x \; with \; x-1$ in equation 3.24 we get

$$f((x-1)/p) = \binom{r+x-2}{x-1} p^r (1-p)^{x-1} \tag{3.25}$$

Dividing equation 3.24 by equation 3.25 we get

$$\frac{f(x/p)}{f((x-1)/p)} = \frac{\binom{r+x-1}{x} p^r (1-p)^x}{\binom{r+x-2}{x-1} p^r (1-p)^{x-1}}$$

$$f(x/p) = \frac{(r+x-1)(r+x-2)!}{x(x-1)!\,(r-1)!} \frac{(x-1)!\,(r-1)!}{(r+x-2)!} (1-p) f((x-1)/p)$$

$$f(x/p) = \frac{r+x-1}{x} (1-p) f((x-1)/p) \tag{3.26}$$

Substituting equation (3.26) into equation (3.23)

$$p_r(x) = \int_0^1 \frac{r+x-1}{x} (1-p) f((x-1)/p) g(p) dp$$

$$p_r(x) = \frac{r+x-1}{x}\left[\int_0^1 f((x-1)/p)g(p)dp - \int_0^1 pf((x-1)/p)g(p)dp\right] \quad (3.27)$$

Consider

$$\int_0^1 f((x-1)/p)g(p)dp \quad (3.28)$$

This can be expressed as

$$\int_0^1 \binom{r+x-2}{x-1} p^r(1-p)^{x-1}g(p)dp = p_r(x-1) \quad (3.29a)$$

Consider

$$\int_0^1 pf((x-1)/p)g(p)dp \quad (3.30)$$

$$= \int_0^1 \binom{r+x-2}{x-1} p^{r+1}(1-p)^{x-1}g(p)dp$$

$$= \frac{\binom{r+x-2}{x-1}}{\binom{r+1+x-2}{x-1}} \int_0^1 \binom{r+1+x-2}{x-1} p^{r+1}(1-p)^{x-1}g(p)dp$$

$$= \frac{\binom{r+x-2}{x-1}}{\binom{r+1+x-2}{x-1}} p_{r+1}(x-1)$$

$$\int_0^1 pf((x-1)/p)g(p)dp = \frac{r}{r+x-1} p_{r+1}(x-1) \quad (3.30a)$$

Substituting equation 3.29a and 3.30a into 3.27 the Beta - Negative Binomial recursive mixture becomes

$$p_r(x) = \frac{r+x-1}{x}\left[p_r(x-1) - \frac{r}{r+x-1} p_{r+1}(x-1)\right] \quad (3.31)$$

$$\boldsymbol{r} > 1, x = 0,1,2 \ldots\ldots;$$

2. Using ratio of the mixed distribution

From equation 3.21b

$$f(x) = \frac{(r + x - 1)!}{(r - 1)!\, x!} \frac{B(r + \alpha, x + \beta)}{B(\alpha, \beta)} \quad where\ x = 0,1,2, \dots$$

Considering the below ratio.

$$\frac{f(x)}{f(x - 1)} = \frac{(r + x - 1)!}{(r - 1)!\, x!} \frac{B(r + \alpha, x + \beta)}{B(\alpha, \beta)} \frac{(r - 1)!\, x!}{(r + x - 1)!} \frac{B(\alpha, \beta)}{B(r + \alpha, x - 1 + \beta)}$$

$$\frac{f(x)}{f(x - 1)} = \frac{r + x - 1}{x} \left[\frac{\beta + x - 1}{r + x + \alpha + \beta - 1}\right]$$

$$f(x) = \frac{(\beta + x - 1)(r + x - 1)}{x(r + x + \alpha + \beta - 1)} f(x - 1) \tag{3.32}$$

$$for\ \boldsymbol{r}, \alpha, \beta > 0, x = 0,1,2 \dots;$$

3. Using a dummy function

Consider the mixture equation below

$$f(x) = \binom{r + x - 1}{x} \frac{1}{B(\alpha, \beta)} \int_0^1 p^{r+\alpha-1} (1 - p)^{x+\beta-1} dp \tag{3.33}$$

Introducing the dummy function

$$I_x(r, \alpha, \beta) = \frac{B(\alpha, \beta) f(x)}{\binom{r+x-1}{x}} = \int_0^1 p^{r+\alpha-1} (1 - p)^{x+\beta-1} dp \tag{3.34}$$

Integrating the integral by parts.

$$\int_0^1 p^{r+\alpha-1} (1 - p)^{x+\beta-1} dp = \int_0^1 u dv$$

$$\int_0^1 u dv = uv - \int_0^1 v du$$

Let

$$u = p^{r+\alpha-1} \qquad and \qquad dv = (1-p)^{x+\beta-1}dp$$

Hence

$$du = (r+\alpha-1)p^{r+\alpha-2}dp \quad and \quad v = -\frac{(1-p)^{x+\beta}}{x+\beta}$$

∴

$$\int_0^1 p^{r+\alpha-1}(1-p)^{x+\beta-1}dp = \left[-\frac{(1-p)^{x+\beta}p^{r+\alpha-1}}{x+\beta}\right]_0^1 + \frac{(r+\alpha-1)}{x+\beta}\int_0^1 (1-p)^{x+\beta}p^{r+\alpha-2}dp$$

$$= \frac{(r+\alpha-1)}{x+\beta}I_{x+1}(r-1,\alpha,\beta) \tag{3.35}$$

$$I_x(r) = \frac{B(\alpha,\beta)f(x)}{\binom{r+x-1}{x}} = \frac{(r+\alpha-1)}{x+\beta}I_{x+1}(r-1)$$

$$\frac{B(\alpha,\beta)f(x)}{\binom{r+x-1}{x}} = \frac{(r+\alpha-1)}{x+\beta}\frac{B(\alpha,\beta)f(x+1)}{\binom{r+x}{x+1}} \tag{3.36}$$

This leads to

$$f(x) = \frac{(r+x-1)(x+\beta-1)}{x(r+x-2)}f(x-1) \tag{3.37}$$

$$for\ r,\beta,\alpha > 0\ and\ x = 1,2,3,\ldots$$

Identity

We can draw an identity based on the result from explicit mixing and that from method of moments as follows.

$$B(r+\alpha, x+\beta) = \sum_{k=0}^{x}\binom{x}{k}(-1)^k B(\alpha+r+k,\beta)\,;for\ x = 0,1,2,\ldots;r,\alpha,\beta > 0 \tag{3.38}$$

3.3.4 Properties of Beta – Negative Binomial Distribution

Mean

Considering the result got from the recursive relation mixture i.e. *equation 3.31*

$$p_r(x) = \frac{r+x-1}{x}\left[p_r(x-1) - \frac{r}{r+x-1}p_{r+1}(x-1)\right]$$

$$E(X) = \sum_{x=0}^{\infty} x\, p_r(x)$$

$$E(X) = \sum_{x=0}^{\infty}(r+x-1)\left[p_r(x-1) - \frac{r}{r+x-1}p_{r+1}(x-1)\right] \qquad (3.39)$$

$$= \sum_{x=0}^{\infty}(r+x-1)\, p_r(x-1) - r\sum_{x=0}^{\infty} p_{r+1}(x-1)$$

$$E(X) = \sum_{x=0}^{\infty}(r+x-1)\binom{r+x-2}{x-1}p^r(1-p)^{x-1} - r\sum_{x=0}^{\infty}\binom{r+x-1}{x-1}p^{r+1}(1-p)^{x-1} \quad (3.40)$$

Consider the first part of the equation

$$\sum_{x=0}^{\infty}(r+x-1)\binom{r+x-2}{x-1}p^r(1-p)^{x-1} \qquad (3.40a)$$

This can be expressed as follows

$$\sum_{x=0}^{\infty}\frac{(r+x-1)!}{(r-1)!\,(x-1)!}p^r(1-p)^{x-1} \qquad (3.40b)$$

But

$$\frac{(r+x-1)!}{(r-1)!\,(x-1)!} = r\binom{r+x-1}{x-1}$$

Hence

$$\sum_{x=0}^{\infty} \frac{(r+x-1)!}{(r-1)!\,(x-1)!} p^r (1-p)^{x-1} = r \sum_{x=0}^{\infty} \binom{r+x-1}{x-1} p^r (1-p)^{x-1} \tag{3.41}$$

$$= \frac{r}{p} \sum_{x=0}^{\infty} \binom{r+x-1}{x-1} p^{r+1} (1-p)^{x-1} = \frac{r}{p}$$

Since

$$\sum_{x=0}^{\infty} \binom{r+x-1}{x-1} p^{r+1} (1-p)^{x-1} = 1$$

Consider the second part of the equation

$$r \sum_{x=0}^{\infty} \binom{r+x-1}{x-1} p^{r+1} (1-p)^{x-1} \tag{3.42}$$

This can be expressed as

$$r \sum_{x=0}^{\infty} \binom{r+x-1}{x-1} p^{r+1} (1-p)^{x-1} = r$$

Since

$$\sum_{x=0}^{\infty} \binom{r+x-1}{x-1} p^{r+1} (1-p)^{x-1} = 1$$

The mean therefore becomes

$$E(X) = \frac{r(1-p)}{p} \tag{3.43}$$

Variance

$$E(X^2) = \sum_{x=0}^{\infty} X^2 \, p_r(x)$$

$$E(X^2) = \sum_{x=0}^{\infty} x(r+x-1) \binom{r+x-2}{x-1} p^r (1-p)^{x-1} - r \sum_{x=0}^{\infty} x \binom{r+x-1}{x-1} p^{r+1} (1-p)^{x-1} \tag{3.44}$$

Consider the first part of the equation above

$$\sum_{x=0}^{\infty} x(r+x-1)\binom{r+x-2}{x-1} p^r (1-p)^{x-1} \qquad (3.44a)$$

It can be expressed or manipulated as follows

$$\frac{1}{1-p}\sum_{x=0}^{\infty} \frac{x(r+x-1)!}{(r-1)!\,(x-1)!} p^r (1-p)^x$$

$$= \frac{1}{1-p}\sum_{x=0}^{\infty} \frac{x^2(r+x-1)!}{(r-1)!\,x!} p^r (1-p)^x$$

$$= \frac{1}{1-p}\sum_{x=0}^{\infty} x^2 \binom{r+x-1}{x} p^r (1-p)^x$$

But considering a Negative Binomial Distribution

$$E(X^2/p) = \sum_{x=0}^{\infty} x^2 \binom{r+x-1}{x} p^r (1-p)^x$$

$$E(X^2/p) = \frac{r(1-p)\big(1+r(1-p)\big)}{p^2}$$

And hence

$$\frac{1}{1-p}\sum_{x=0}^{\infty} \frac{x(r+x-1)!}{(r-1)!\,(x-1)!} p^r (1-p)^x = \frac{r\big(1+r(1-p)\big)}{p^2}$$

Consider the second part of the equation

$$r\sum_{x=0}^{\infty} x\binom{r+x-1}{x-1} p^{r+1}(1-p)^{x-1} \qquad (3.44b)$$

This can be expressed as follows

$$= \frac{rp}{1-p}\sum_{x=0}^{\infty} \frac{x(r+x-1)!}{r(r-1)!\,(x-1)!} p^r (1-p)^x$$

$$= \frac{rp}{1-p}\sum_{x=0}^{\infty}\frac{x^2(r+x-1)!}{r(r-1)!\,x!}p^r(1-p)^x$$

$$= \frac{p}{1-p}\sum_{x=0}^{\infty}x^2\binom{r+x-1}{x}p^r(1-p)^x$$

$$= \frac{p}{(1-p)}X\frac{r(1-p)\big(1+r(1-p)\big)}{p^2}$$

$$= \frac{r\big(1+r(1-p)\big)}{p}$$

The second moment therefore becomes

$$E(X^2) = \frac{r\big(1+r(1-p)\big)}{p^2} - \frac{r\big(1+r(1-p)\big)}{p}$$

$$E(X^2) = \frac{r\big(1+r(1-p)\big)}{p}\left(\frac{1}{p}-1\right) \tag{3.45}$$

Hence

$$var(X) = \frac{r\big(1+r(1-p)\big)}{p}\left(\frac{1}{p}-1\right) - \frac{r^2(1-p)^2}{p^2}$$

$$var(X) = \frac{r(r^2-p)}{p^2} \tag{3.46}$$

3.3.5. Moment Generating Function

Consider the explicit mixture of Beta distribution with the Negative Binomial distribution

$$f(X=x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\frac{B(r+\alpha,x+\beta)}{B(\alpha,\beta)}; \quad x = 0,1,2,\dots; r,\alpha,\beta > 0$$

$$MGF\ g(t) = E(t^n);\ n = 0,1,2,\dots.$$

$$g(t) = H(r,\beta; r+\alpha+\beta, t)pr(Z=0) \tag{3.47}$$

Where

$$H(a,b;c,t) = \sum_{n=0}^{\infty} \frac{\left(\frac{a_{(n)} b_{(n)}}{c_{(n)}}\right) t^n}{n!} \quad is\ the\ Hypergeometric\ function$$

$$x_{(n)} = \frac{\Gamma(x+n)}{\Gamma x}$$

$$a = r;\ \ b = \beta;\ \ c = r + \alpha + \beta; t = t$$

$$\therefore g(t) = \sum_{n=0}^{\infty} \left(\frac{r_{(n)} \beta_{(n)}}{(r+\alpha+\beta)_{(n)}}\right) \frac{t^n}{n!}$$

$$g(t) = \sum_{n=0}^{\infty} \left[\frac{\Gamma(r+n)}{\Gamma r} \frac{\Gamma(\beta+n)}{\Gamma \beta} \frac{\Gamma(r+\alpha+\beta)}{\Gamma(r+\alpha+\beta+n)}\right] \frac{t^n}{n!} \qquad (3.48)$$

Differentiate $g(t)$ with respect to t

$$g'(t) = n \sum_{n=0}^{\infty} \left[\frac{\Gamma(r+n)}{\Gamma r} \frac{\Gamma(\beta+n)}{\Gamma \beta} \frac{\Gamma(r+\alpha+\beta)}{\Gamma(r+\alpha+\beta+n)}\right] \frac{t^{n-1}}{n!}$$

$$g^{'}(t) = \sum_{n=0}^{\infty} \left[\frac{\Gamma(r+n)}{\Gamma r} \frac{\Gamma(\beta+n)}{\Gamma \beta} \frac{\Gamma(r+\alpha+\beta)}{\Gamma(r+\alpha+\beta+n)}\right] \frac{t^{n-1}}{(n-1)!} \qquad (3.49)$$

Hence

$$g^{''(t)} = n(n-1) \sum_{n=0}^{\infty} \left[\frac{\Gamma(r+n)}{\Gamma r} \frac{\Gamma(\beta+n)}{\Gamma \beta} \frac{\Gamma(r+\alpha+\beta)}{\Gamma(r+\alpha+\beta+n)}\right] \frac{t^{n-2}}{n!} \qquad (3.50)$$

$$g^{''}(t) = \sum_{n=0}^{\infty} \left[\frac{\Gamma(r+n)}{\Gamma r} \frac{\Gamma(\beta+n)}{\Gamma \beta} \frac{\Gamma(r+\alpha+\beta)}{\Gamma(r+\alpha+\beta+n)}\right] \frac{t^{n-2}}{(n-2)!} \qquad (3.51)$$

3.4. Special cases of beta – negative Binomial distribution

3.4.1. Uniform – Negative Binomial distribution

3.4.1.1. Uniform distribution

Construction

Consider a beta distribution defined as follows

$$g(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)} \qquad 0 < p < 1; \quad \alpha, \beta > 0$$

If we let $\alpha = \beta = 1$

We get the uniform distribution [0,1] given by

$$g(p) = \begin{cases} 1 & 0 < p < 1 \\ 0 & elsewhere \end{cases} \tag{3.52}$$

Properties

Moment of order j about the origin of the uniform distribution g(p)

$$E(P^j) = \int_0^1 p^j \, dp$$

$$= \left[\frac{p^{j+1}}{j+1}\right]_0^1$$

$$E(p^j) = \frac{1}{j+1} \tag{3.53}$$

Mean

$$E(P) = \frac{1}{2} \tag{3.54}$$

Variance

$$var(P) = \frac{1}{12} \tag{3.55}$$

3.4.1.2. The mixtures

Explicit mixing

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$$

where $g(p)$is the uniform distribution.

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x dp$$

$$f(x) = \binom{r+x-1}{x} B(r+1, x+1) \int_0^1 \frac{p^r (1-p)^x}{B(r+1, x+1)} dp$$

$$f(x) = \binom{r+x-1}{x} B(r+1, x+1)$$

$$f(x) = \frac{(r+x-1)!}{(r-1)!\, x!} \frac{\Gamma(r+1)\Gamma(x+1)}{\Gamma(r+x+2)}$$

$$f(x) = \frac{r}{(r+x+1)(r+x)} \qquad x = 0,1,2, \ldots ; r > 0 \qquad (3.56)$$

Using recursive relation

The mixture between Negative Binomial and uniform distribution can be expressed in a recursive format in the following two ways.

1st Form

Here we consider the mixture from explicit mixing i.e. equation 3.56

$$f(x) = \frac{r}{(r+x+1)(r+x)}$$

Formulation and working the below ratio.

$$\frac{f(x)}{f(x-1)} = \frac{r}{(r+x+1)(r+x)} \frac{(r+x)(r+x-1)}{r}$$

Hence

$$f(x) = \frac{r+x-1}{r+x+1} f(x-1) \tag{3.57}$$

2nd Form

Here we introduce a dummy function$I_x(r)$ that is of a Beta format as follows.

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x \, dp$$

$$\frac{f(x)}{\binom{r+x-1}{x}} = \int_0^1 p^r (1-p)^x \, dp$$

$$I_x(r) = \frac{f(x)}{\binom{r+x-1}{x}} = \int_0^1 p^r (1-p)^x \, dp \tag{3.58}$$

Using integral by parts

Let

$$u = (1-p)^x$$

$$du = -x(1-p)^{x-1} dp$$

$$v = \frac{p^{r+1}}{r+1}$$

$$dv = p^r \, dp$$

$$\int u \; dv = uv - \int v \; du$$

$$I_x(r) = \left[(1-p)^x \frac{p^{r+1}}{r+1} \right]_0^1 + \frac{x}{r+1} \int_0^1 p^{r+1} (1-p)^{x-1} dp$$

$$I_x(r) = \frac{x}{r+1} \int_0^1 p^{r+1} (1-p)^{x-1} dp \tag{3.59}$$

$$I_x(r) = \frac{x}{r+1} I_{x-1}(r+1)$$

$$I_x(r) = \frac{f(x)}{\binom{r+x-1}{x}} = \frac{x}{r+1}\frac{f(x-1)}{\binom{r+x-1}{x-1}}$$

$$f(x) = \frac{x}{r+1}\frac{\binom{r+x-1}{x}}{\binom{r+x-1}{x-1}}f(x-1)$$

Hence

$$f(x) = \frac{r}{r+1}f(x-1) \tag{3.60}$$

Using Method of Moments

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\sum_{k=0}^{x}\binom{x}{k}(-1)^k E(p^{(r+k)})$$

$$E(p^{r+k}) is\ the\ moment\ of\ order\ r+k\ about\ the\ origin\ of\ the\ mixing\ distribution$$

$$E(p^j) = \frac{1}{j+1}$$

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\sum_{k=0}^{x}\binom{x}{k}(-1)^k \frac{1}{r+k+1} \tag{3.61}$$

$$for\ r > 0; x = 0,1,2, \dots$$

Identity

We can draw an identity based on the result from explicit mixing and that from method of moments as follows.

$$\frac{r}{(r+x+1)(r+x)} = \frac{\Gamma(r+x)}{\Gamma(r)x!}\sum_{k=0}^{x}\binom{x}{k}(-1)^k \frac{1}{r+k+1}$$

$$\frac{r!\,x!}{(r+x+1)(r+x)!} = \sum_{k=0}^{x}\binom{x}{k}(-1)^k \frac{1}{r+k+1}$$

$$for\ r > 0; x = 0,1,2, \dots$$

3.4.2. Power function - negative Binomial distribution

3.4.2.1. Power function distribution

Construction

Consider the beta function

$$g(\mathrm{p}) = \begin{cases} \dfrac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)} & 0 < p < 1;\ \alpha,\beta > 0 \\ 0 \quad elsewhere \end{cases}$$

Let β *=1* then

$$g(p) \ = \ \frac{p^{\alpha-1}(1-p)^{0}}{B(\alpha,1)}$$

$$g(p) = \begin{cases} \alpha p^{\alpha-1} & 0 < p < 1;\ \alpha > 0 \\ 0 & elsewhere \end{cases} \quad (3.62)$$

This is the pdf of a power function distribution with parameter α

Properties

The moments of order j about the origin is

$$E(P^j) = \int_0^1 \alpha p^{\alpha-1} p^j \, dp$$

$$= \alpha \int_0^1 p^{\alpha+j-1} dp$$

$$= \ \alpha \ \left[\frac{p^{\alpha+j-1}}{\alpha+j}\right]_0^1$$

$$E(P^j) = \ \frac{\alpha}{\alpha+j} \quad (3.63)$$

Mean

$$E(P) = \frac{\alpha}{\alpha+1} \tag{3.64}$$

Variance

$$var(p) = E(p^2) - \{E(p)\}^2$$

$$var(p) = \frac{\alpha}{\alpha+2} - \frac{a^2}{(\alpha+1)^2}$$

$$var(p) = \frac{\alpha}{(\alpha+1)^2(\alpha+2)} \tag{3.65}$$

3.4.2.2 The mixing

Explicit mixing

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$$

$g(p)$is the power function distribution

$$g(p) = \alpha p^{\alpha-1}$$

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x \alpha p^{\alpha-1} dp$$

$$f(x) = \binom{r+x-1}{x} \alpha \int_0^1 p^{r+\alpha-1} (1-p)^x dp$$

$$f(x) = \binom{r+x-1}{x} \alpha\, B(r+\alpha, x+1) \int_0^1 \frac{p^{r+\alpha-1}(1-p)^x}{B(r+\alpha, x+1)} dp$$

$$f(x) = \binom{r+x-1}{x} \alpha\, B(r+\alpha, x+1) \tag{3.66}$$

$$for\ x = 0{,}12 \ldots. \quad ; a, r > 0$$

This is the density function of the Negative Binomial – Power function expressed explicitly

Using Method of Moments

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\sum_{k=0}^{x}\binom{x}{k}(-1)^k E(p^{(r+k)})$$

$E(p^{r+k})$ *is the moment of order* $r+k$ *about the origin of the mixing distribution*

$$E\left(P^j\right) = \frac{\alpha}{\alpha+j}$$

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\sum_{k=0}^{x}\binom{x}{k}(-1)^k \frac{\alpha}{\alpha+r+k} \qquad (3.67)$$

$$for\ r,\alpha > 0;\ x = 0,1,2,\ldots$$

Using the recursive relation expression

We can achieve this by the use of two different ways.

1st Form

Consider the result from explicit mixture as shown below.

$$f(x) = \binom{r+x-1}{x}\alpha\, B(r+\alpha, x+1)$$

Expressing this a ratio

$$\frac{f(x)}{f(x-1)} = \frac{\binom{r+x-1}{x}B(r+\alpha,x+1)}{\binom{r+x-2}{x-1}B(r+\alpha,x)}$$

Hence

$$f(x) = \frac{r+x-1}{r+\alpha+\ x}f(x-1) \qquad (3.68)$$

2nd Form

Consider the explicit expression below

$$f(x) = \binom{r+x-1}{x} \alpha \int_0^1 p^{r+\alpha-1}(1-p)^x dp$$

Manipulate this to a dummy function $I_x(r,\alpha)$as shown below

$$\frac{f(x)}{\binom{r+x-1}{x}\alpha} = \int_0^1 p^{r+\alpha-1}(1-p)^x dp$$

Let

$$I_x(r,\alpha) \ = \frac{f(x)}{\binom{r+x-1}{x}\alpha} = \int_0^1 p^{r+\alpha-1}(1-p)^x dp \qquad (3.69)$$

Integrating the integral by parts

Let

$$u = (1-p)^x$$

$$du = -x(1-p)^{x-1} dp$$

$$dv = \ p^{r+\alpha-1} dp$$

$$v = \frac{p^{r+\alpha}}{r+\alpha}$$

$$\int \mathrm{u\ dv} = \mathrm{vu} - \int \mathrm{v\ du}$$

$$I_x(r,\alpha) \ = \left[(1-p)^x \frac{p^{r+\alpha}}{r+\alpha}\right]_0^1 + \frac{x}{r+\alpha}\int_0^1 p^{r+\alpha}\ (1-p)^{x-1} dp$$

$$I_x(r,\alpha) \ = \frac{x}{r+\alpha}\int_0^1 p^{r+\alpha}\ (1-p)^{x-1} dp \qquad (3.70)$$

$$I_x(r,\alpha) = \frac{f(x)}{\binom{r+x-1}{x}\alpha} = \frac{x}{r+\alpha} I_{x-1}(r-1,\alpha)$$

$$\frac{f(x)}{\binom{r+x-1}{x}\alpha} = \frac{x}{r+\alpha}\frac{f(x-1)}{\binom{r+x-3}{x-1}\alpha} \qquad (3.71)$$

$$f(x) = \frac{x}{r+\alpha} \frac{\binom{r+x-1}{x}}{\binom{r+x-3}{x-1}} f(x-1)$$

Hence

$$f(x) = \frac{(r+x-1)(r+x-2)}{(r+\alpha)(r-1)} f(x-1) \tag{3.72}$$

Identity

We can draw an identity based on the result from explicit mixing and that from method of moments as follows.

$$\frac{r}{(r+x+1)(r+x)} = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{\alpha}{\alpha+r+k}$$

$$\frac{r!\,x!}{(r+x+1)(r+x)!} = \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{\alpha}{\alpha+r+k} \tag{3.73}$$

$$for \quad x = 0,1,2,\dots; r,\alpha > 0$$

3.4.3. Arcsine – negative Binomial distribution

3.4.3.1 Arcsine distribution

Construction

The standard Arcsine distribution is a special case of the beta distribution with $\alpha = \beta = \frac{1}{2}$

Consider a beta distribution below

$$g(p) = \frac{p^{\alpha-1}[1-p]^{\beta-1}}{B(\alpha,\beta)} \qquad 0 \le p \le 1;\ \alpha,\beta > 0$$

If $\alpha = \beta = \frac{1}{2}$

Then

$$g(p) = \frac{p^{-1/2}[1-p]^{-1/2}}{B(\frac{1}{2},\frac{1}{2})} \qquad 0 \le p \le 1 \; ; \; \alpha, \beta > 0$$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi$$

$$g(p) = \frac{1}{\pi\sqrt{p(1-p)}} \qquad 0 \le p \le 1; a, b > 0 \tag{3.74}$$

This is the pdf of Arcsine distribution

Properties of Arcsine distribution

Jth moment about the origin

$$g(p) = \frac{p^{-\frac{1}{2}}[1-p]^{-\frac{1}{2}}}{\pi} \qquad 0 \le p \le 1 \; ; \; \alpha, \beta > 0$$

$$E\left(p^j\right) = \frac{1}{\pi}\int_0^1 p^{j-\frac{1}{2}}[1-p]^{-\frac{1}{2}} \, dp$$

$$\therefore E\left(p^j\right) = \frac{B\left(j+\frac{1}{2},\frac{1}{2}\right)}{\pi}$$

$$E(P) = \frac{B\left(\frac{3}{2},\frac{1}{2}\right)}{\pi}$$

$$E(P) = \frac{B\left(\frac{3}{2},\frac{1}{2}\right)}{B\left(\frac{1}{2},\frac{1}{2}\right)}$$

$$\therefore E(P) = \frac{1}{2} \tag{3.75}$$

$$Var(P) = E(p^2) - [E(p)]^2$$

$$\therefore Var(P) = \frac{1}{8} \tag{3.76}$$

3.4.3.2 The mixture

a) Explicit mixing

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp \tag{3.77}$$

Where $g_1(p)$is the arcsine distribution

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x \frac{1}{\pi\sqrt{p(1-p)}} dp$$

$$f(x) = \binom{r+x-1}{x} \frac{B\left(r+\frac{1}{2}, x+\frac{1}{2}\right)}{\pi} \tag{3.78}$$

$$for\ r > 0; x = 0,1,2, \dots; and\ \ \pi = B\left(\frac{1}{2}, \frac{1}{2}\right)$$

b) Using recursive relation

We can achieve this by the use of two different ways.

1st Form

Consider the result from explicit mixture as shown below.

$$f(x) = \binom{r+x-1}{x} \frac{B\left(r+\frac{1}{2}, x+\frac{1}{2}\right)}{\pi}$$

Expressing this a ratio

$$\frac{f(x)}{f(x-1)} = \frac{\binom{r+x-1}{x} B\left(r+\frac{1}{2}, x+\frac{1}{2}\right)}{\binom{r+x-2}{x-1} B\left(r+\frac{1}{2}, x-\frac{1}{2}\right)}$$

Hence

$$f(x) = \frac{(r+x-1)(2x-1)}{2x(r+x)} f(x-1) \tag{3.79}$$

$$for\ r > 0; x = 0,1,2, \dots;$$

2nd Form

Consider the mixture equation

$$f(x) = \binom{r+x-1}{x}\frac{1}{\pi}\int_0^1 p^{r-\frac{1}{2}}(1-p)^{x-\frac{1}{2}}g(p)dp$$

Manipulate this to a dummy function $I_x(r)$ as shown below

$$I_x(r,p) = \frac{\pi f(x)}{\binom{r+x-1}{x}} = \int_0^1 p^{r-\frac{1}{2}}(1-p)^{x-\frac{1}{2}}g(p)dp \quad (3.80)$$

Using integration by parts to solve the integral

We let

$$u = (1-p)^{x-\frac{1}{2}}$$

$$dv = p^{r-\frac{1}{2}}dp$$

$$du = \left(\frac{1}{2} - x\right)(1-p)^{x-\frac{3}{2}}dp$$

$$v = \frac{p^{r+\frac{1}{2}}}{r+\frac{1}{2}}$$

$$I_x(r,p) = \left[(1-p)^{x-\frac{1}{2}}\frac{p^{r+\frac{1}{2}}}{r+\frac{1}{2}}\right]_0^1 + \frac{\left(\frac{1}{2} - x\right)}{r+\frac{1}{2}}\int_0^1 p^{r+\frac{1}{2}}(1-p)^{x-\frac{3}{2}}dp \quad (3.81)$$

$$I_x(r,p) = \frac{\left(\frac{1}{2} - x\right)}{r+\frac{1}{2}}\int_0^1 p^{r+\frac{1}{2}}(1-p)^{x-\frac{3}{2}}dp$$

$$= \frac{(1-2x)}{2r+1}\int_0^1 p^{r+\frac{1}{2}}(1-p)^{x-\frac{3}{2}}dp$$

$$I_x(r,p) = \frac{1-2x}{2r+1}\, I_{x-1}(r+1,p) \quad (3.82)$$

$$I_x(r,p) = \frac{\pi f(x)}{\binom{r+x-1}{x}} = \frac{1-\ 2x}{2r+1}\frac{\pi f(x-1)}{\binom{r+x-1}{x-1}}$$

$$f(x) = \frac{1-\ 2x}{2r+1}\frac{\binom{r+x-1}{x}}{\binom{r+x-1}{x-1}}f(x-1)$$

hence

$$f(x) = \frac{r(1-\ 2x)}{x(2r+1)}f(x-1) \tag{3.83}$$

a. Mixture from the method of moment

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\sum_{k=0}^{x}\binom{x}{k}(-1)^k E(p^{(r+k)})$$

$E(p^{r+k})$*is the moment of order* $r+k$ *about the origin of the mixing distribution*

$$E(P^j) = \frac{B\left(j+\frac{1}{2},\frac{1}{2}\right)}{\pi}$$

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\frac{B\left(r+k+\frac{1}{2},\frac{1}{2}\right)}{\pi} \tag{3.84}$$

$$for\ r,\alpha > 0;\ x = 0,1,2,\dots$$

Identity

We can draw an identity based on the result from explicit mixing and that from method of moments as follows.

$$B\left(r+\frac{1}{2},x+\frac{1}{2}\right) = \sum_{k=0}^{x}\binom{x}{k}(-1)^k B\left(r+k+\frac{1}{2},\frac{1}{2}\right) \tag{3.85}$$

$$for\ r,\alpha > 0;\ x = 0,1,2,\dots$$

3.4.4. Negative Binomial – Truncated Beta distribution

3.4.4.1 Truncated Beta distribution

Construction

Consider a two sided truncated function given by

$$\int_{\alpha}^{\beta} p^{a-1}(1-p)^{b-1}\,dp \quad with\; 1 < \alpha < p < \beta < 1; a, b > 0 \qquad (3.86)$$

This function can be expressed in terms of incomplete beta function as

$$\int_{\alpha}^{\beta} p^{a-1}(1-p)^{b-1}\,dp \;= \int_{0}^{\beta} p^{a-1}(1-p)^{b-1}\,dp \;-\; \int_{0}^{\alpha} p^{a-1}(1-p)^{b-1}\,dp$$

$$= B_{\beta}(a,b) - B_{\alpha}(a,b)$$

Thus

$$1 = \int_{\alpha}^{\beta} \frac{p^{a-1}(1-p)^{b-1}}{B_{\beta}(a,b) - B_{\alpha}(a,b)}\,dp$$

This gives a distribution referred to as truncated beta distribution g(p)with parameters α, β, a, b

$$g(p) = \frac{p^{a-1}(1-p)^{b-1}}{B_{\beta}(a,b) - B_{\alpha}(a,b)};\; for\; 1 < \alpha < p < \beta < 1; a, b > 0 \qquad (3.87)$$

Properties

Moments of order *j* about the origin of de distribution can be expressed as

$$E(P^j) = \int_{\alpha}^{\beta} \frac{p^{a-1}(1-p)^{b-1}P^j}{B_{\beta}(a,b) - B_{\alpha}(a,b)}\,dp$$

$$E(P^j) = \int_{\alpha}^{\beta} \frac{p^{a+j-1}(1-p)^{b-1}}{B_{\beta}(a,b) - B_{\alpha}(a,b)}\,dp$$

$$E(P^j) = \frac{B_{\beta}(a+j,b) - B_{\alpha}(a+j,b)}{B_{\beta}(a,b) - B_{\alpha}(a,b)} \qquad (3.88)$$

3.4.4.2 The mixture

a) Explicit mixing

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$$

Where g(p) is the truncated beta distribution

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x \frac{p^{a+j-1}(1-p)^{b-1}}{B_\beta(a,b) - B_\alpha(a,b)} dp \quad (3.89)$$

$$f(x) = \binom{r+x-1}{x} \int_0^1 \frac{p^{a+r+j-1}(1-p)^{x+b-1}}{B_\beta(a,b) - B_\alpha(a,b)} dp$$

$$f(x) = \binom{r+x-1}{x} \frac{B_\beta(a+r, b+x) - B_\alpha(a+r, b+x)}{B_\beta(a,b) - B_\alpha(a,b)}$$

$$f(x) = \binom{r+x-1}{x} \frac{B_\beta(a+r, b+x) - B_\alpha(a+r, b+x)}{B_\beta(a,b) - B_\alpha(a,b)} \quad (3.90)$$

$$for\ a, b > 0; 1 < \alpha < p < \beta < 1; x = 0,1,2 \dots$$

This is the pdf of the Negative Binomial – truncated beta mixture

b) Using method of moments

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k E(p^{(r+k)})$$

$E(p^{r+k})$*is the moment of order* $r+k$ *about the origin of the mixing distribution*

And

$$E\left(P^i\right) = \frac{B_\beta(a+j,b) - B_\alpha(a+j,b)}{B_\beta(a,b) - B_\alpha(a,b)}$$

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \; \frac{B_\beta(a+r+k,b) - B_\alpha(a+r+k,b)}{B_\beta(a,b) - B_\alpha(a,b)} \qquad (3.91)$$

$$for\; r > 0;\; 1 < \alpha < p < \beta < 1; a,b > 0;\; x = 0,1,2, \ldots$$

Identity

We can draw an identity based on the result from explicit mixing and that from method of moments as follows.

$$B_\beta(a+r,b+x) - B_\alpha(a+r,b+x) = \sum_{k=0}^{x} \binom{x}{k} (-1)^k \{\; B_\beta(a+r+k,b) - B_\alpha(a+r+k,b)\} \qquad (3.92)$$

$$for\; r > 0;\; 1 < \alpha < p < \beta < 1; a,b > 0;\; x = 0,1,2, \ldots$$

3.5. Negative Binomial – Confluent Hypergeometric distribution

3.5.1. Confluent Hypergeometric

Construction

Given the confluent Hypergeometric function

$${}_1F_1(a, a+b;\; -\mu) = \int_0^1 \frac{p^{a-1}(1-p)^b}{B(a,b)} e^{-p\mu}\, dp$$

Dividing both sides by ${}_1F_1(a, a+b;\; -\mu)$ we get

$$\int_0^1 \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)\,{}_1F_1(a, a+b;\; -\mu)} e^{-p\mu}\, dp = 1$$

Since the LHS of the above equation is equated to 1, then the LHS equation qualifies to be a pdf. This is the pdf of Confluent Hypergeometric and is expressed as follows.

$$g(p) = \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)_1F_1(a, a+b;\ -\mu)} e^{-p\mu}\ for\ 0 < p < 1; a, b > 0;\ -\infty < \mu < \infty \quad (3.93)$$

$$ref\ Nadarajah\ and\ Kotz\ (2007)$$

Finding the jth moment of the Confluent Hypergeometric distribution

$$E(P^j) = \int_0^1 p^j g(p) dp$$

$$E(P^j) = \int_0^1 \frac{p^j p^{a-1}(1-p)^{b-1}}{B(a,b)_1F_1(a, a+b;\ -\mu)} e^{-p\mu} dp$$

$$E(P^j) = \frac{1}{B(a,b)_1F_1(a, a+b;\ -\mu)} \int_0^1 p^{j+a-1}(1-p)^{b-1} e^{-p\mu} dp$$

$$E(P^j) = \frac{B(j+a,b)}{B(a,b)_1F_1(a, a+b;\ -\mu)} \int_0^1 \frac{p^{j+a-1}(1-p)^{b-1} e^{-px}}{B(j+a,b)} dp$$

But

$$\int_0^1 \frac{p^{j+a-1}(1-p)^{b-1} e^{-p\mu}}{B(j+a,b)} dp = {}_1F_1(a+j, a+j+b;\ -\mu)$$

Thus

$$E(P^j) = \frac{B(j+a,b)_1F_1(a+j, a+j+b;\ -\mu)}{B(a,b)_1F_1(a, a+b;\ -\mu)} \quad (3.94)$$

3.5.2 Confluent Hypergeometric – Negative Binomial mixing

a) Explicit mixing

$$f(x) = \int_0^1 f(x/p)g(p)dp$$

Where

$$f(x/p) = \binom{r+x-1}{x} p^r (1-p)^x \quad 0 < p < 1; r > 0; \;\; x = 0,1,2, \dots.$$

$$g(p) = \frac{p^{a-1}(1-p)^b}{B(a,b)_1F_1(a, a+b; \; -\mu)} e^{-p\mu} \; for \; 0 < p < 1; a, b > 0; \; -\infty < \mu < \infty$$

$$f(x) = \frac{1}{B(a,b)_1F_1(a, a+b; \; -\mu)} \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x p^{a-1} (1-p)^b e^{-p\boldsymbol{\mu}} \, dp$$

$$= \frac{1}{B(a,b)_1F_1(a, a+b; \; -\mu)} \binom{r+x-1}{x} \int_0^1 p^{r+a-1} (1-p)^{x+b} e^{-p\boldsymbol{\mu}} \, dp$$

$$= \frac{B(r+a, x+b)}{B(a,b)_1F_1(a, a+b; \; -\mu)} \binom{r+x-1}{x} \int_0^1 \frac{p^{r+a-1}(1-p)^{x+b-1} e^{-p\boldsymbol{\mu}}}{B(r+a, x+b)} dp$$

$$f(x) = \frac{B(r+a, x+b)_1F_1(a+r, a+b+r+x; \; -\mu)}{B(a,b)_1F_1(a, a+b; \; -\mu)} \binom{r+x-1}{x} \qquad (3.95)$$

For $r, a, b > 0; \;\; -\infty < \mu < \infty; \;\; x = 0,1,2, \dots$

Properties of Confluent Hypergeometric – Negative Binomial Mixture

First 3 moments about the origin

$$E(X) = \frac{a_1F_1(a+1, a+b+1; -\mu)}{(a+b)_1F_1(a, a+b; -\mu)}$$

$$E(X^2) = \frac{a(a+1)}{(a+b)(a+b+1)} \frac{{}_1F_1(a+2, a+b+2; -\mu)}{{}_1F_1(a, a+b; -\mu)}$$

$$E(X^3) = \frac{a(a+1)(a+2)}{(a+b)(a+b+1)(a+b+2)} \frac{{}_1F_1(a+3, a+b+3; -\mu)}{{}_1F_1(a, a+b; -\mu)}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = \frac{a(a+1)_1F_1(a+2, a+b+2; -\mu)}{(a+b)(a+b+1)_1F_1(a, a+b; -\mu)} - \left[\frac{a_1F_1(a+1, a+b+1; -\mu)}{(a+b)_1F_1(a, a+b; -\mu)}\right]^2$$

Ref; A paper by E.Gomez; J.M. Perez – Sanchez; F.J Vazquez – Polo and A. Hernandez – Bastida

b) **Mixing using method of moments**

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k E(p^{(r+k)})$$

$$E(p^{r+k}) \textit{is the moment of order } r+k \textit{ about the origin of the mixing distribution}$$

$$E(P^j) = \frac{B(j+a, b)_1F_1(a+j, a+j+b;\ -\mu)}{B(a, b)_1F_1(a, a+b;\ -\mu)}$$

Therefore

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B(r+k+a, b)_1F_1(a+r+k, a+r+k+b;\ -\mu)}{B(a, b)_1F_1(a, a+b;\ -\mu)} \qquad (3.96)$$

$$for\ x = 0,1,2, \dots; a, b > o\ ; r > 0$$

Identity

We can draw an identity based on the result from explicit mixing and that from method of moments as follows.

$$B(r+a, x+b)_1F_1(a+r, a+b+r+x;\ -\mu) = \sum_{k=0}^{x} \binom{x}{k} (-1)^k B(r+k+a, b)_1F_1(a+r+k, a+r+k+b;\ -\mu)$$

$$for\ x = 0,1,2, \dots; a, b > o\ ; r > 0 \qquad (3.97)$$

Applications of Confluent Hypergeometric – Negative Binomial mixture

Confluent Hypergeometric – Negative Binomial mixture is a common distribution used in insurance to a variety of claims

E.Gomez; J.M. Perez – Sanchez; F.J Vazquez – Polo and A. Hernandez – Bastida found out that when you calculate the expected frequencies for automobile insurance claims and using the Confluent Hypergeometric – Negative Binomial mixture then the outcome came out satisfactorily fit.

The mixture was found to be a better alternative to the standard Negative Binomial distribution and other mixtures.

Suppose the number of claims in a portfolio of policies in a time period is denoted by N

Let $X_i, i = 1,2,3,, \ldots$ be the amount of the ith claim

$S = X_1 + X_2 + X_3 + \cdots \ldots\ldots + X_N$ will be the aggregate or total claims generated by the potfolio.

Note

1. The random variables $X_i, i = 1,2,3,, \ldots N$ are i.i.d with a CDF F(x) and pdf f(x)
2. The random variable N; X_1, X_2, X_3 are mutually independent

Suppose we are using the Confluent Hypergeometric – Negative Binomial mixture model for N, the CDF of the distribution of total claims become

$$F_s(X) = \sum_{k=0}^{\infty} F^{*k}(\mathrm{X})\Pr(N = k)$$

Where

F^{*k} is the kth fold convolution of F and $\Pr(N = k)$ is defined above

3.6. Gauss Hypergeometric – Negative Binomial Distribution

3.6.1. Gauss Hypergeometric distribution

Construction

Given the Gauss Hypergeometric function

$${}_2F_1(a, \varepsilon, a + b; -z) = \int_0^1 \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)(1+zt)^{\varepsilon}}$$

$$for\ 0 < t < 1; a, b > 0;\ -\infty < \varepsilon < \infty$$

Divide both sides by ${}_2F_1(a, \varepsilon, a + b; -z)$ to get

$$\int_0^1 \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)(1+zt)^{\varepsilon}\,{}_2F_1(a,\varepsilon,a+b;-z)} = 1$$

This forms the pdf of gauss Hypergeometric given by

$$g(t) = \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)(1+zt)^{\varepsilon}\,{}_2F_1(a,\varepsilon,a+b;-z)} \quad (3.98)$$

$$for\ 0 < t < 1; a,b > 0;\ -\infty < \varepsilon < \infty$$

(see Armero and Bayarri (1994))

Finding the jth moment about the origin

$$E(T^j) = \frac{1}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)} \int_0^1 \frac{t^{j+a-1}(1-t)^{b-1}}{(1+zt)^{\varepsilon}} dt$$

$$= \frac{B(j+a,b)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)} \int_0^1 \frac{t^{j+a-1}(1-t)^{b-1}}{B(j+a,b)(1+zt)^{\varepsilon}} dt$$

$$E(T^j) = \frac{B(j+a,b)_2F_1(j+a,\varepsilon,a+b+j;-z)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)} \quad (3.99)$$

3.6.2. Gauss Hypergeometric – Negative Binomial mixture

a) Explicit mixing

$$f(x) = \int_0^{\infty} \binom{r+x-1}{x} p^r (1-p)^x g(p) dp$$

Where *g(p)* is the pdf of the Gauss Hypergeometric distribution

$$g(p) = \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)(1+zt)^{\varepsilon}\,{}_2F_1(a,\varepsilon,a+b;-z)}$$

$$f(x) = \binom{r+x-1}{x} \frac{1}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)} \int_0^{\infty} \frac{p^r(1-p)^x p^{a-1}(1-p)^{b-1}}{(1+zt)^{\varepsilon}} dp$$

$$= \binom{r+x-1}{x} \frac{B(a+r,b+x)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)} \int_0^\infty \frac{p^{r+a-1}(1-p)^{x+b-1}}{B(a+r,b+x)(1+zt)^\varepsilon} dp$$

$$f(x) = \binom{r+x-1}{x} \frac{B(a+r,b+x)_2F_1(a+r,\varepsilon;a+b+r+x;-z)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)} \quad (3.100)$$

For $x = 0,1,2 \ldots; r = 0,1,2 \ldots.; a, b > 0; -\infty < \varepsilon < \infty$

b) Using method of moments

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k E(p^{(r+k)})$$

$E(p^{r+k})$ *is the moment of order* $r+k$ *about the origin of the mixing distribution*

And the jth moment for the Gauss Hypergeometric distribution is given by

$$E(P^j) = \frac{B(j+a,b)_2F_1(j+a,\varepsilon,a+b+j;-z)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)}$$

Hence

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B(j+a,b)_2F_1(r+k+a,\varepsilon,a+b+r+k;-z)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)} \quad (3.101)$$

Identity

We can draw an identity based on the result from explicit mixing and that from method of moments as follows.

$$B(a+r,b+x)_2F_1(a+r,\varepsilon;a+b+r+x;-z) = \sum_{k=0}^{x} \binom{x}{k} (-1)^k B(j+a,b)_2F_1(r+k+a,\varepsilon,a+b+r+k;-z)$$

For $x = 0,1,2 \ldots; r = 0,1,2 \ldots.; a, b > 0; -\infty < \varepsilon < \infty$ (3.102)

CHAPTER 4

NEGATIVE BINOMIAL MIXTURES BASED ON DISTRIBUTIONS BEYOND BETA

4.1 Introduction

All the distributions we are going to consider in this category are within the [0,1] domain and are not based on beta distribution. Some of them are listed below;

1. Kumaraswamy (I) and (II) distribution
2. Triangular distribution
3. Truncated Exponential distribution
4. Truncated Gamma distribution
5. Minus Log distribution
6. Two – Sided Ogive distribution
7. Ogive distribution
8. Two – sided power distribution

When carrying out the mixing we will use the following methods

a. Moments method
b. Direct integration and substitution also referred to as explicit mixing
c. Iteration method(where applicable)

4.2 Negative Binomial – Kumaraswamy (I) Distribution

4.2.1 Kumaraswamy (I) Distribution

Construction

Given the Kumaraswamy (II) Distribution with a pdf

$$g(p) = ab(1-p^a)^{b-1}p^{a-1} \quad 0 < p < 1; a, b > 0 \tag{4.01}$$

Let $p = U^{1/a}$ $\quad 0 < u < 1; \quad a > 0$

Then

$$g(u) = ab(1-u)^{b-1}u^{1-\frac{1}{a}}|J|$$

But

$$|J| = \left|\frac{dp}{du}\right| = \frac{u^{\frac{1}{a}-1}}{a}$$

Substituting $|J|$

$$g(u) = ab(1-u)^{b-1}u^{1-\frac{1}{a}}\frac{u^{\frac{1}{a}-1}}{a}$$

Therefore

$$g(u) = \begin{cases} b(1-u)^{b-1} & for\ 0 < u < 1; b > 0 \\ 0 & elsewhere \end{cases} \tag{4.02}$$

This is the pdf of Kumaraswamy (I) distribution with parameter b

It is always denoted by Kw (I) distribution

Properties of Kw (I) distribution

The jth moment about the origin of the Kw (I) distribution is given below

$$E\left(U^j\right) = b\int_0^1 u^j (1-u)^{b-1} dp$$

$$E\left(U^j\right) = bB(j+1, b) \qquad (4.03)$$

$$E(U) = bB(2, b) = \frac{1}{(b+1)(b+2)} \qquad (4.04)$$

$$var\ (U) = E(U^2) - \left(E(U)\right)^2$$

$$= \frac{2}{(b+3)(b+2)(b+1)} - \frac{1}{(1+b)^2(2+b)^2}$$

$$var\ (U) = \frac{2b^2 + 5b + 1}{(1+b)^2(2+b)^2(3+b)} \qquad (4.05)$$

4.2.2 Negative Binomial – Kumaraswamy (I) distribution mixing

4.2.2.1 Explicit mixing

This refers to the mixing on the direct substitution basis

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$$

Such that $g(p)$ is he pdf of Kw (I) distribution

$$g(p) = \begin{cases} b(1-p)^{b-1} & for\ 0 < p < 1; b > 0 \\ 0 & elsewhere \end{cases}$$

Replacing $g(p)$

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x b(1-p)^{b-1} dp$$

$$f(x) = \binom{r+x-1}{x} b \int_0^1 p^r (1-p)^{x+b-1}\, dp$$

$$f(x) = \binom{r+x-1}{x} bB(r+1, x+b)$$

$$f(x) = \binom{r+x-1}{x} bB(r+1, x+b) \quad x = 0,1,2, \dots; \quad b, r > 0 \qquad (4.06)$$

See Li Xiaohuet al (2011) on Binomial mixture

4.2.2.2 Mixing using the moments method

$$Prob(X = x) = \frac{\Gamma(r+x)}{\Gamma(\mathrm{r})x!} \sum_{k=0}^{x} (-1)^k \binom{x}{k} E(p^{(r+k)})$$

$$E(p^{r+k}) \textit{is the moment of order } r+k \textit{ about the origin of the mixing distribution}$$

$$Prob(X = x) = \frac{\Gamma(r+x)}{\Gamma(\mathrm{r})x!} b \sum_{k=0}^{x} (-1)^k \binom{x}{k} B(r+k+1, b) \qquad (4.07)$$

$$for\ r, b > 0; x = 0,1,2, \dots$$

Identity

We can draw an identity based on the result from explicit mixing and that from method of moments as follows.

$$B(r+1, x+b) = \sum_{k=0}^{x} (-1)^k \binom{x}{k} B(r+k+1, b)$$

$$for\ r, b > 0; x = 0,1,2,3, \dots$$

4.3 Negative Binomial – Truncated Exponential Distribution

4.3.1 Truncated Exponential Distribution ($TEX(\lambda, b)$)

Construction

Let Y be a one sided truncated Exxponential randon variable, the pdf of Ycan be evaluated as follows

$$\int_0^b \frac{e^{-y/\lambda}}{\lambda} dy = [1 - e^{-b/\lambda}]$$

Dividing both sides by $[1 - e^{-b/\lambda}]$

$$\int_0^b \frac{\frac{1}{\lambda} e^{-y/\lambda}}{[1 - e^{-b/\lambda}]} dy = 1$$

Let $y = pb$

$dy = bdp$

$$\int_0^1 \frac{\frac{b}{\lambda} e^{-bp/\lambda}}{[1 - e^{-b/\lambda}]} dy = 1$$

This is a pdf which can be expressed as

$$g(p) = \begin{cases} \dfrac{\frac{b}{\lambda} e^{-\frac{bp}{\lambda}}}{\left[1 - e^{-\frac{b}{\lambda}}\right]} & 0 < p < 1;\;\; b, \lambda > 0 \\ 0 & elsewhere \end{cases} \quad (4.08)$$

This the Truncated Exponential distribution with parameters $\lambda \; and \; b$

4.3.2 Negative Binomial – Truncated Exponential distribution mixing

4.3.2.1 Explicit mixing

This is direct substitution and integration expressed as follows

$$f(x) = \binom{r + x - 1}{x} \int_0^1 p^r (1 - p)^x g(p) dp$$

$g(p)$is the pdf of truncated exponential distribution with parameters $\lambda \; and \; b$

$$f(x)=\binom{r+x-1}{x}\int_0^1 p^r(1-p)^x\frac{\frac{b}{\lambda}e^{-\frac{bp}{\lambda}}}{\left[1-e^{-\frac{b}{\lambda}}\right]}dp$$

$$f(x)=\binom{r+x-1}{x}\frac{b}{\lambda\left[1-e^{-\frac{b}{\lambda}}\right]}\int_0^1 p^r(1-p)^x e^{-\frac{bp}{\lambda}}dp \quad (4.09)$$

Consider a Confluent Hypergeometric function

$$_1F_1(a,a+b;-x)B(a,b)=\int_0^1 p^{a-1}(1-p)^{b-1}e^{-px}dp$$

Therefore

$$\int_0^1 p^r(1-p)^x e^{-\frac{bp}{\lambda}}dp={}_1F_1\left(r+1,r+x+2;-\frac{b}{\lambda}\right)B(r+1,x+1) \quad (4.10)$$

Thus equation 4.09 results to

$$f(x)=\binom{r+x-1}{x}\frac{b_1F_1\left(r+1,r+x+2;-\frac{b}{\lambda}\right)B(r+1,x+1)}{\lambda\left[1-e^{-\frac{b}{\lambda}}\right]}$$

$$f(x)=\frac{b_1F_1\left(r+1,r+x+2;-\frac{b}{\lambda}\right)}{\lambda\left[1-e^{-\frac{b}{\lambda}}\right]}\frac{(r+x-1)!}{(r-1)!\,x!}\frac{r(r-1)!\,x!}{(r+x+1)(r+x)(r+x-1)!}$$

$$f(x)=\frac{b_1F_1\left(r+1,r+x+2;-\frac{b}{\lambda}\right)}{\lambda\left[1-e^{-\frac{b}{\lambda}}\right]}\frac{r}{(r+x+1)(r+x)} \quad for\ x=0,1,2,..;b,r,\lambda>0 \quad (4.11)$$

This the mixture distribution of Negative Binomial distribution and Truncated Exponential distribution in its simplest form

4.4 Negative Binomial – Truncated Gamma Distribution

4.4.1 Truncated Gamma Distribution

Construction

Consider an incomplete Gamma function given by

$$\gamma(b,a) = \int_0^a t^{b-1} e^{-t} dt$$

Divide both sides by $\gamma(a,b)$

$$\int_0^a \frac{t^{b-1} e^{-t}}{\gamma(a,b)} dt = 1$$

Then let $t = ap \quad which\ implies\ dt = a\ dp$

Therefore

$$\int_0^1 \frac{a^b p^{b-1} e^{-ap}}{\gamma(a,b)} dp = 1$$

It forms a pdf which can be expressed as

$$g(p) = \begin{cases} \dfrac{a^b p^{b-1} e^{-ap}}{\gamma(a,b)} & 0 < p < 1; a, b > 0 \\ 0 & elsewhere \end{cases} \qquad (4.12)$$

ith moment

$$E(P^j) = \int_0^1 \frac{a^b p^{b+j-1} e^{-ap}}{\gamma(a,b)} dp$$

$$E(P^j) = \frac{a^b \gamma(a,b+j)}{a^{b+j} \gamma(a,b)} \int_0^1 \frac{a^{b+j} p^{b+j-1} e^{-ap}}{\gamma(a,b+j)} dp$$

$$E(P^j) = \frac{a^b \gamma(a,b+j)}{a^{b+j} \gamma(a,b)}$$

$$E(P^j) = \frac{\gamma(a,b+j)}{a^j \gamma(a,b)}$$

4.4.2 Negative Binomial – Truncated Gamma Distribution mixing

4.4.2.1 Explicit mixing

By substituting $g(p)$ and integrating the integral

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$$

Where $g(p)$ is he pdf of truncated gamma distribution

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x \frac{a^b p^{b-1} e^{-ap}}{\gamma(a,b)} dp$$

$$f(x) = \binom{r+x-1}{x} \frac{a^b}{\gamma(a,b)} \int_0^1 p^r (1-p)^x p^{b-1} e^{-ap} dp$$

$$f(x) = \binom{r+x-1}{x} \frac{a^b}{\gamma(a,b)} \int_0^1 p^{r+b-1} (1-p)^x e^{-ap} dp \qquad (4.13)$$

Consider a Confluent Hypergeometric Function

$${}_1F_1(a, a+b; -x) B(a,b) = \int_0^1 p^{a-1} (1-p)^{b-1} e^{-px} dp$$

Therefore

$$\int_0^1 p^{r+b-1} (1-p)^x e^{-ap} dp = {}_1F_1(r+b, r+b+x+1; -a) B(r+b, x+1)$$

Thus

$$f(x) = \binom{r+x-1}{x} \frac{a^b {}_1F_1(r+b, r+b+x+1; -a) B(r+b, x+1)}{\gamma(a,b)} \qquad (4.14)$$

$$for\ x = 0,1,2,\ldots; r,a,b > 0$$

(see Bhattacharya (1968) on Binomial mixture)

4.4.2.2. Using the method of moments

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\sum_{k=0}^{x}(-1)^k\binom{x}{k}E(p^{(r+k)})$$

$E(p^{r+k})$ *is the moment of order* $r+k$ *about the origin of the mixing distribution*

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\sum_{k=0}^{x}(-1)^k\binom{x}{k}\frac{\gamma(a,b+r+k)}{a^{(r+k)}\gamma(a,b)}$$

$$for\ x = 0,1,2,\ldots; r,a,b > 0$$

Identity

We can draw an identity based on the result from explicit mixing and that from method of moments as follows.

$$a^b\,{}_1F_1(r+b,r+b+x+1;-a)B(r+b,x+1) = \sum_{k=0}^{x}(-1)^k\binom{x}{k}\frac{\gamma(a,b+r+k)}{a^{(r+k)}}$$

$$for\ x = 0,1,2,\ldots; r,a,b > 0$$

4.5. Negative Binomial – Minus Log Distribution

4.5.1. Minus Log Distribution

Construction

Let the Random variable X have the uniform pdf $U[0,1]$, Let $[x_1, x_2]$denote a random sample from the distribution.

The joint pdf $x_1\ and x_2$is then

$$h(x_1x_2) = \begin{cases} f(x_1)f(x_2) & 0 < x_1 < 1;\ 0 < x_2 < 1 \\ 0 & elsewhere \end{cases}$$

Consider the two random variables $P = x_1x_2\ and\ Y = x_2$

The joint pdf of $P\ and\ Y$ is

$$g(p, y) = 1|J|$$

Where

$$|J| = \begin{vmatrix} \frac{dx_1}{dp} & \frac{dx_1}{dy} \\ \frac{dx_2}{dp} & \frac{dx_2}{dy} \end{vmatrix}$$

$$|J| = \begin{vmatrix} \frac{1}{y} & -\frac{p}{y^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{y}$$

Thus $\qquad g(p, y) = \frac{1}{y} \qquad 0 < p < y < 1$

The marginal pdf of p is

$$g(p) = \int_p^1 \frac{1}{y} dy$$

$$g(p) = [\log y]_p^1 = 0 - \log p$$

$$g(p) = -\log p \qquad 0 < p < 1 \tag{4.16}$$

This is the pdf of minus log distribution

Properties of Minus log distribution

The jth moment of the minus log distribution is given below

$$E(P^j) = \int_0^1 P^j (-\log p)\, dp$$

Let $a = -\log p \qquad p = e^{-a} \qquad dp = -e^{-a} da$

$$E(P^j) = -\int_0^1 a e^{-a(j+1)} da$$

Using the minus sign outside the integral to swap the limits, we will have.

$$E\left(P^{j}\right) = \int_0^{\infty} a e^{-a(j+1)} da$$

$$E\left(P^{j}\right) = \frac{1}{(j+1)^2} \tag{4.17}$$

$$E(P) = \frac{1}{4} \tag{4.18}$$

$$var\ (P) = \frac{7}{144} \tag{4.19}$$

4.5.2Negative Binomial – Minus Log Distribution mixing

4.5.2.1 Explicit mixing

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$$

$g(p)$ is he pdf of minus log distribution.

Thus

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x (-\log p) dp \tag{4.20}$$

Let $a = -\log p$ $\quad p = e^{-a}$ $\quad dp = -e^{-a}$

We have

$$f(x) = \binom{r+x-1}{x} \left\{ -\int_0^1 a e^{-ra} (1-e^{-a})^x e^{-a} da \right\}$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \left\{ -\int_0^1 a e^{-ak} e^{-ra} e^{-a} da \right\}$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \left\{ -\int_0^1 a e^{-a(r+k+1)} da \right\}$$

Using the minus sign outside the integral to swap the limits, we will have.

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \left\{ \int_0^{\infty} a e^{-a(r+k+1)} da \right\}$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{1}{(r+k+1)^2}$$

The mixture can be expressed as

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{1}{(r+k+1)^2} \quad x = 0,1,2,\ldots r > 0 \qquad (4.21)$$

4.5.2.1 Mixing using method of moments

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)(x)!} \sum_{k=0}^{x} (-1)^k \binom{x}{k} E(p^{(r+k)})$$

$E(p^{r+k})$ *is the moment of order* $r+k$ *about the origin of the mixing distribution*

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)(x)!} \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{1}{(r+k+1)^2} \qquad (4.22)$$

$$for \;\; x = 0,1,2,\ldots r > 0$$

4.6 Negative Binomial – Standard Two – Sided Power Distribution

4.6.1 Standard Two – Sided Power Distribution

Construction

The standard II sided power distribution can be viewed as a particular case of the general two sided continuous family with support [0, 1] given by the density below

$$g(p/\theta, h(./\varphi)) = \begin{cases} h\left(\left[\frac{p}{\theta}\right]/\varphi\right) & 0 < p < \theta \\ h\left(\left[\frac{1-p}{1-\theta}\right]/\varphi\right) & \theta < p < 1 \end{cases} \tag{4.23}$$

Where $h(./\varphi)$ is an appropriately selected continuous pdf on [0, 1] with parameter(s) φ

$h(./\varphi)$ is called a general density such that

$$h(y) = ky^{k-1} \qquad 0 < y < 1\,; k > 0$$

which is a power function distribution

Then

$$g(p) = \begin{cases} k\left(\frac{p}{\theta}\right)^{k-1} & 0 < p < \theta \\ k\left(\frac{1-p}{1-\theta}\right)^{k-1} & \theta < p < 1 \end{cases} \tag{4.24}$$

And this is the pdf of a two – sided power distribution with parameters $k \; and \; \theta$

4.6.2 Negative Binomial – Standard Two – Sided Power Distribution

4.6.2.1 Explicit mixing

This entails direct substitution into the following equation

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$$

Where $g(p)$ is he pdf of Standard Two – Sided Power Distribution

$$f(x) = \binom{r+x-1}{x} k \left\{ \frac{1}{\theta^{k-1}} \int_0^\theta p^{r+k-1} (1-p)^x dp + \frac{1}{(1-\theta)^{k-1}} \int_\theta^1 p^r (1-p)^{x+k-1} dp \right\}$$

$$f(x) = \binom{r+x-1}{x} k \left\{ \frac{B_\theta(r+k, x+1)}{\theta^{k-1}} + \frac{[B(r+1, x+k) - B_\theta(r+1, x+k)]}{(1-\theta)^{k-1}} \right\} \quad (4.25)$$

This is the mixture between Negative Binomial distribution and the standard two – side power distribution.

4.7. Negative Binomial – Ogive Distribution

4.7.1. Ogive Distribution

Construction

The general form of an Ogive distribution is given by

$$g(p) = \frac{2m(m+1)}{3m+1} p^{(m-1)/2} + \frac{1-m^2}{3m+1} p^m \qquad 0 < p > 1\,; m > 0 \qquad (4.26)$$

From ($Dorp\ and\ Kotz$ (2003))

4.7.2. Negative Binomial – Ogive Distribution mixing

4.7.2.1. Explicit mixing

This entails direct substitution into the following equation

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$$

Where $g(p)$ is he pdf of the Ogive distribution

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x \left\{ \frac{2m(m+1)}{3m+1} p^{\frac{m-1}{2}} + \frac{1-m^2}{3m+1} p^m \right\} dp \qquad (4.26)$$

$$f(x) = \binom{r+x-1}{x} \frac{(m+1)}{3m+1} \left\{ 2m \int_0^1 p^{r+\frac{m-1}{2}} (1-p)^x dp + (1-m) \int_0^1 p^{r+m} (1-p)^x \, dp \right\}$$

$$= \binom{r+x-1}{x} \frac{(m+1)}{3m+1} \left\{ 2m \int_0^1 p^{\frac{2r+m-1}{2}} (1-p)^x dp + (1-m) \int_0^1 p^{r+m} (1-p)^x \, dp \right\}$$

$$f(x) = \binom{r+x-1}{x} \frac{(m+1)}{3m+1} \left\{ 2mB\left(\frac{2r+m-1}{2} + 1, x+1 \right) + (1-m)B(r+m+1, x+1) \right\} \quad (4.27)$$

$$for\ x = 0,1,2, \dots.; r > 0; m > 0$$

4.8. Negative Binomial – Standard Two – Sided Ogive Distribution

4.8.1. Standard Two – Sided Ogive Distribution

Construction

The two sided ogive distribution can be viewed as a particular case of the general two sided continuous family with [0, 1] given by the density

$$g(p/\theta, h(./\varphi)) = \begin{cases} h\left(\left[\frac{p}{\theta} \right] / \varphi \right) & 0 < p < \theta \\ h\left(\left[\frac{1-p}{1-\theta} \right] / \varphi \right) & \theta < p < 1 \end{cases} \qquad (4.28)$$

Where $h(./\varphi)$ is an appropriately selected continuous pdf on [0, 1] with parameter(s) φ

$h(./\varphi)$ is called a general density such that

$$z(y) = \frac{2m(m+1)}{3m+1} y^{\frac{m-1}{2}} + \frac{1-m^2}{3m+1} y^m \qquad (4.29)$$

$$0 < y < 1; m > 0$$

This is an Ogive distribution and so

$$g(p) = \begin{cases} \dfrac{2m(m+1)}{3m+1}\left[\dfrac{p}{\theta}\right]^{\frac{m-1}{2}} + \dfrac{1-m^2}{3m+1}\left[\dfrac{p}{\theta}\right]^{m} & 0 < p < \theta \,; m > 0 \\ \dfrac{2m(m+1)}{3m+1}\left[\dfrac{1-p}{1-\theta}\right]^{\frac{m-1}{2}} + \dfrac{1-m^2}{3m+1}\left[\dfrac{1-p}{1-\theta}\right]^{m} & \theta < p < 1 \,; m > 0 \end{cases} \tag{4.30}$$

And this is the pdf of the two sided Ogive Distribution with parameters $m \text{ and } \theta$

When $p = \theta$ then the two – sided Ogive distribution is smooth and this is the reflection point.

Although this contradicts the situation at the reflection point of the two – sided power function

4.8.2. Negative Binomial – Standard Two – Sided Ogive Distribution mixing

This entails direct substitution into the following equation

$$f(x) = \binom{r+x-1}{x}\int_0^1 p^r(1-p)^x g(p)dp$$

$g(p)$ is the pdf of the Standard Two – Sided Ogive Distribution

$$f(x) = \binom{r+x-1}{x}\int_0^1 p^r(1-p)^x \left\{\left[\frac{2m(m+1)}{3m+1}\left[\frac{p}{\theta}\right]^{\frac{m-1}{2}} + \frac{1-m^2}{3m+1}\left[\frac{p}{\theta}\right]^{m}\right]_0^\theta + \left[\frac{2m(m+1)}{3m+1}\left[\frac{1-p}{1-\theta}\right]^{\frac{m-1}{2}} + \frac{1-m^2}{3m+1}\left[\frac{1-p}{1-\theta}\right]^{m}\right]_\theta^1\right\} dp$$

$$f(x) = \binom{r+x-1}{x}\frac{(m+1)}{3m+1}\left\{\frac{2m}{[\theta]^{\frac{m-1}{2}}}\int_0^\theta p^{\frac{2r+m-1}{2}}(1-p)^x dp + \frac{1-m}{\theta^m}\int_0^\theta p^{r-m}(1-p)^x dp + \frac{2m}{(1-\theta)^{\frac{m-1}{2}}}\int_\theta^1 p^r(1-p)^{\frac{2x+m-1}{2}} dp + \frac{1-m}{(1-\theta)^m}\int_\theta^1 p^r(1-p)^{x+m} dp\right\}$$

$$f(x) = \binom{r+x-1}{x}\frac{(m+1)}{3m+1}\left\{\frac{2m}{[\theta]^{\frac{m-1}{2}}}B_\theta\left(\frac{2r+m-1}{2}+1, x+1\right) + \frac{1-m}{\theta^m}B_\theta(r-m+1, x+1) + \frac{2m}{(1-\theta)^{\frac{m-1}{2}}}\left[B\left(r+1, \frac{2x+m-1}{2}+1\right) - B_\theta\left(r+1, \frac{2x+m-1}{2}+1\right)\right]\frac{1-m}{(1-\theta)^m}\left[B(r+1, x+m+1) - B_\theta(r+1, x+m+1)\right]\right\} \quad (4.31)$$

$$for\ x = 0,1,2, \ldots. \quad ; m > 0 \,; r > 0$$

4.9. Negative Binomial – Triangular Distribution

4.9.1. Triangular Distribution

Construction

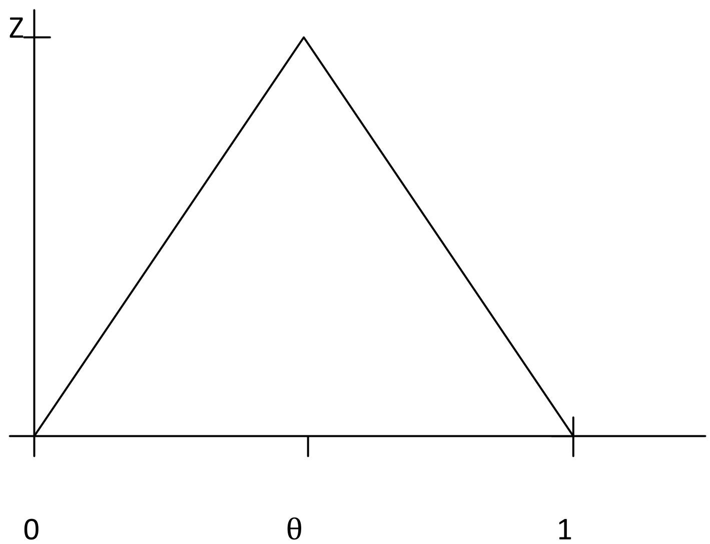


Diagram 1

The Triangular distribution $T(0,1,\theta)$ arise from the conjuction of two lines which share a common vertex.

The density of the triangular distribution is defined by

$$g(p) = \begin{cases} g_1(p) & if\ 0 < p < \theta \\ g_2(p) & if\ \theta < p < 1 \\ 0 & else\ where \end{cases} \quad (4.32)$$

When

$g_1(p)$ is the equation of the line $(0{,}0), (\theta, 2)$ computed as

$$\frac{g_1(p)}{p} = \frac{2}{\theta}$$

Therefore

$$g_1(p) = \frac{2p}{\theta} \quad (4.32i)$$

$g_2(p)$ is the equation of the line $(\theta, 2), (1, 0)$ computed as

$$\frac{g_2(p)}{p - 1} = \frac{-2}{1 - \theta}$$

Therefore

$$g_2(p) = \frac{2(1 - p)}{1 - \theta} \quad (4.32ii)$$

Thus the density of the triangular distribution becomes

$$g(p) = \begin{cases} \dfrac{2p}{\theta} & 0 < p < \theta \\ \dfrac{2(1-p)}{1-\theta} & \theta < p < 1 \\ 0 & elsewhere \end{cases} \quad (4.33)$$

Ref (Kotz S et al (2004) - Beyond Beta – Other Continuous families of Distributions with Bounded Support and applications page 1 – 31)

Properties of Triangular distribution

The moment of order j about the origin of triangular distribution is worked out below

$$E(P^j) = \int_0^\theta \frac{2p}{\theta} P^j \, dp + \int_\theta^1 \frac{2(1-p)}{1-\theta} P^j \, dp$$

$$E(P^j) = \left[\frac{2p^{j+2}}{\theta(j+2)}\right]_0^\theta + \frac{2}{1-\theta}\left[\frac{p^{j+1}}{j+1} - \frac{p^{j+2}}{j+2}\right]_\theta^1$$

$$E(P^j) = \frac{2\theta^{j+2}}{\theta(j+2)} + \frac{2}{1-\theta}\left[\frac{1}{j+1} - \frac{1}{j+2} - \frac{\theta^{j+1}}{j+1} + \frac{\theta^{j+2}}{j+2}\right]$$

$$E(P^j) = \frac{2\theta^{j+2}}{\theta(j+2)} + \frac{2}{1-\theta}\left[\frac{1}{(j+1)(j+2)} + \frac{(j+1)\theta^{j+2} - (j+2)\theta^{j+1}}{(j+1)(j+2)}\right]$$

$$E(P^j) = \frac{2}{1-\theta}\left[\frac{\theta^{j+2}(1-\theta)}{\theta(j+2)} + \frac{1}{(j+1)(j+2)} + \frac{(j+1)\theta^{j+2} - (j+2)\theta^{j+1}}{(j+1)(j+2)}\right]$$

$$E(P^j) = \frac{2}{1-\theta}\left[\frac{\theta^{j+1}(1-\theta)}{(j+2)} + \frac{1}{(j+1)(j+2)} + \frac{(j+1)\theta^{j+2} - (j+2)\theta^{j+1}}{(j+1)(j+2)}\right]$$

$$E(P^j) = \frac{2}{1-\theta}\left[\frac{(j+1)\theta^{j+1}(1-\theta) + 1 + (j+1)\theta^{j+2} - (j+2)\theta^{j+1}}{(j+1)(j+2)}\right]$$

$$= \frac{2}{1-\theta}\left[\frac{(j+1)\theta^{j+1} - (j+1)\theta^{j+2} + 1 + (j+1)\theta^{j+2} - (j+2)\theta^{j+1}}{(j+1)(j+2)}\right]$$

$$E(P^j) = \frac{2}{1-\theta}\left[\frac{1-\theta^{j+1}}{(j+1)(j+2)}\right] \tag{4.34}$$

Mean is therefore

$$E(P) = \frac{1+\theta}{3} \tag{4.35}$$

Variance od p is given by

$$Var(P) = E(P^2) - \left(E(P)\right)^2$$

$$Var(P) = \frac{\theta^2 - \theta + 1}{18} \tag{4.36}$$

4.9.2. Negative Binomial – Triangular distribution mixing

4.9.2.1. Explicit mixing

This entails direct substitution into the following equation

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$$

Where $g(p)$ is he pdf of the Triangular distribution

$$f(x) = \binom{r+x-1}{x} \left\{ \frac{2}{\theta} \int_0^\theta p^{r+1} (1-p)^x \, dp + \frac{2}{1-\theta} \int_\theta^1 p^r (1-p)^{x+1} dp \right\} \tag{4.37}$$

$$f(x) = \binom{r+x-1}{x} \left\{ \frac{2}{\theta} B_\theta(r+2, x+1) + \frac{2}{1-\theta} \left[B(r+1, x+2) - B_\theta(r+1, x+2) \right] \right\}$$

$$f(x) = \frac{2\Gamma(r+x)}{x!\,\Gamma r} \left\{ \frac{B_\theta(r+2, x+1)}{\theta} + \frac{B(r+1, x+2) - B_\theta(r+1, x+2)}{1-\theta} \right\} \tag{4.38}$$

$$for\ x = 0,1,2, \dots \dots ; 0 < \theta < 1, r > 0$$

And this is the mixture of Triangular distribution and Negative Binomial distribution expressed explicitly

4.9.2.2. Mixing using the method of moments

Here we have two ways of carrying out mixing using this method

Case 1

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$$

Where $g(p)$ is he pdf of the triangular distribution

But we know that

$$(1-p)^x = \sum_{k=0}^{x} \binom{x}{k} (-1)^{x-k} p^{x-k}$$

Therefore

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r \sum_{k=0}^{x} \binom{x}{k} (-1)^{x-k} p^{x-k} \, g(p) dp$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^{x-k} \int_0^1 p^{r+x-k} g(p) dp \qquad (4.39)$$

$$f(x) \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^{x-k} \, E(p^{r+x-k}) \qquad (4.40)$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^{x-k} \frac{2(1-\theta^{r+x-k+1})}{(r+x-k+1)(r+x-k+2)(1-\theta)} \qquad (4.41)$$

Case 2

From the formula

$$f(x) = \sum_{k=0}^{x} \frac{(r+x-1)!}{(r-1)!\,(x-k)!\,k!}(-1)^k E(p^{(r+k)})$$

$$E(p^{r+k}) is\ the\ moment\ of\ order\ r+k\ about\ the\ origin\ of\ the\ mixing\ distribution$$

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)!\,(x)!} \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{2}{(1-\theta)} \left[\frac{1-\theta^{r+k+1}}{(r+k+1)(r+k+2)} \right] \qquad (4.42)$$

Properties of Negative Binomial – Triangular distribution mixture

The mean is given by

$$E(X) = \int_0^1 \frac{\alpha(1-p)}{p} g(p) \qquad (4.43)$$

$$E(X) = \alpha E(P^{-1}) - \alpha$$

$$E(X) = \alpha \left(\frac{-2\log\theta}{1-\theta} - 1 \right) \qquad (4.44)$$

Since $0 < \theta \leq 1$

Therefore

$$E(X) > \alpha \quad \vee \theta$$

Variance

Variance of this mixture doesn't exist since the second inverse moment of the triangular distribution doesn't exist. It is important to note that this distribution has a very long tail.

Identity

We can draw an identity based on the result from explicit mixing and that from method of moments as follows.

$$\frac{(1-\theta)}{2}\left\{\frac{4}{(1-\theta)\theta}B_\theta(r+2,x+1)+[B(r+1,x+2)-\ B_\theta(r+1,x+2)]\right\}=\sum_{k=0}^{x}(-1)^k\binom{x}{k}\left[\frac{1-\theta^{r+k+1}}{(r+k+1)(r+k+2)}\right]$$

$$\text{For } x=0,1,2,\ldots\ldots;0<\theta<1,r>0$$

CHAPTER 5

GEOMETRIC DISTRIBUTION MIXTURES WITH BETA GENERATED DISTRIBUTIONS IN THE [0, 1] DOMAIN

5.1. Introduction

Geometric distribution is a special form of the Negative Binomial Distribution when $r = 1$. It is therefore important to note that the results of Negative Binomial mixtures can be used to generate the geometric mixtures by substituting the value of $r = 1$in the Negative Binomial mixtures. We are going to review the Geometric distribution mixtures with Beta generated distributions in the [0,1] domain in this chapter.

5.2. Geometric – Classical Beta Distribution

Classical Beta Distribution

$$f(X = x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}$$

$$for\ \ 0 < x < 1;\ \alpha, \beta > 0$$

5.2.1. Classical Beta – Geometric distribution from explicit mixing

Classical Beta – Negative Binomial distribution from explicit mixing

$$f(x) = \frac{(r+x-1)!}{(r-1)!\,x!}\frac{B(r+\alpha, x+\beta)}{B(\alpha, \beta)}$$

$$for\ \alpha, \beta > 0; x = 0,1,2,, \ldots$$

From the above expression of the BND, when r=1, we attain the following distribution which is the **Beta - geometric mixture**

$$f(x) = \frac{B(1+\alpha, x+\beta)}{B(\alpha, \beta)} \tag{5.01}$$

$$for\ \alpha, \beta > 0; x = 0,1,2,, \ldots$$

5.2.2. Classical Beta – Geometric distribution from method of moments mixing

Classical Beta – Negative Binomial distribution from method of moments mixing

$$f(x) = \frac{\Gamma(r+x)!}{\Gamma(r)!\,(x)!}\sum_{k=0}^{x}(-1)^k \binom{x}{k}\frac{B(\alpha+r+k,\beta)}{B(\alpha,\beta)}$$

$$for\ \alpha,\beta > 0; x = 0,1,2,,\ldots$$

From the above expression of the BND we substitute r with 1 to attain the following distribution which is the Beta - geometric mixtures

$$f(x) = \sum_{k=0}^{x}(-1)^k \binom{x}{k}\frac{B(\alpha+k+1,\beta)}{B(\alpha,\beta)} \quad (5.02)$$

$$for\ \alpha,\beta > 0; x = 0,1,2,,\ldots$$

5.2.3. Classical Beta – Geometric distribution from recursive relation

There are three ways of applying the recursive relation in this mixture, namely

a. Using ratio of the conditional distribution
b. Using ration of the mixed distribution
c. Using a dummy function

a. Using ratio of the conditional distribution

$$Prob(x) = \int_0^1 f(x/p)g(p)dp \quad (5.03)$$

$f(x/p)$is a Geometric distribution in this case

$$f(x/p) = p(1-p)^x \quad (5.04)$$

Substituting $x\ with\ x-1$ in equation 5.04 we get

$$f((x-1)/p) = p(1-p)^{x-1} \quad (5.05)$$

Dividing equation 5.04 by equation 5.05 we get

$$\frac{f(x/p)}{f((x-1)/p)} = \frac{p(1-p)^x}{p(1-p)^{x-1}}$$

$$f(x/p) = (1-p)f((x-1)/p)$$

$$f(x/p) = (1-p)f((x-1)/p) \tag{5.06}$$

Substituting equation (5.06) into equation (5.03)

$$Prob(x) = \int_0^1 (1-p)f((x-1)/p)g(p)dp$$

$$Prob(x) = \int_0^1 f((x-1)/p)g(p)dp - \int_0^1 pf((x-1)/p)g(p)dp \tag{5.07}$$

Consider

$$\int_0^1 f((x-1)/p)g(p)dp \tag{5.08}$$

This can be expressed as

$$\int_0^1 p(1-p)^{x-1}g(p)dp = p_1(x-1) \tag{5.08a}$$

Consider

$$\int_0^1 pf((x-1)/p)g(p)dp \tag{5.09}$$

$$= \int_0^1 p^2(1-p)^{x-1}g(p)dp$$

$$= B(3,x)\int_0^1 \frac{p^2(1-p)^{x-1}}{B(3,x)}g(p)dp = B(3,x)$$

$$\int_0^1 pf((x-1)/p)g(p)dp = B(3,x) \tag{5.09a}$$

Substituting equation 5.08a and 5.09a into 5.07 the Beta - Geometric recursive mixture becomes

$$Prob(x) = [Prob(x-1) - B(3,x)]; \quad for\ x = 0,1,2 \ldots\ldots; \tag{5.10}$$

b. Using ratio of the mixed distribution

From equation 5.01

$$f(x) = \frac{B(1+\alpha, x+\beta)}{B(\alpha,\beta)} \quad where\ x = 0,1,2,\dots$$

Considering the below ratio.

$$\frac{f(x)}{f(x-1)} = \frac{B(1+\alpha, x+\beta)}{B(\alpha,\beta)} \frac{B(\alpha,\beta)}{B(1+\alpha, x+\beta-1)}$$

$$\frac{f(x)}{f(x-1)} = \left[\frac{x+\beta-1}{x+\beta+\alpha}\right]$$

$$f(x) = \left[\frac{x+\beta-1}{x+\beta+\alpha}\right] f(x-1) \qquad (5.11)$$

$$for\ \boldsymbol{r}, \alpha, \beta > 0, x = 0,1,2 \dots\dots;$$

c. Using a dummy function

Consider the mixture equation below

$$f(x) = \frac{1}{B(\alpha,\beta)} \int_0^1 p^{\alpha}(1-p)^{x+\beta-1} dp \qquad (5.11)$$

Introducing the dummy function

$$I_x(\alpha,\beta) = B(\alpha,\beta) f(x) = \int_0^1 p^{\alpha}(1-p)^{x+\beta-1} dp \qquad (5.13)$$

Integrating the integral by parts.

$$\int_0^1 p^{\alpha}(1-p)^{x+\beta-1} dp = \int_0^1 u dv$$

$$\int_0^1 u dv = uv - \int_0^1 v du$$

Let

$$u = p^{\alpha} \qquad and \qquad dv = (1-p)^{x+\beta-1} dp$$

Hence

$$du = \alpha p^{\alpha-1} dp \quad and \quad v = -\frac{(1-p)^{x+\beta}}{x+\beta}$$

∴

$$\int_0^1 p^{\alpha}(1-p)^{x+\beta-1} dp = \left[-\frac{(1-p)^{x+\beta}p^{\alpha}}{x+\beta}\right]_0^1 + \frac{\alpha}{x+\beta}\int_0^1 (1-p)^{x+\beta}p^{\alpha-1}dp \qquad (5.14)$$

$$= \frac{\alpha}{x+\beta} I_{x+1}(\alpha-1,\beta)$$

$$I_{(x)}(\alpha,\beta) = B(\alpha,\beta)f(x) = \frac{\alpha}{x+\beta} I_{x+1}(\alpha-1,\beta) \qquad (5.15)$$

$$B(\alpha,\beta)f(x) = \frac{\alpha}{x+\beta} B(\alpha-1,\beta)f(x+1)$$

∴

$$f(x) = \frac{(\alpha-1)}{\alpha} f(x+1) \qquad (5.16)$$

$$for\ \alpha > 0\ and\ x = 1,2,3,\ldots$$

This is the **beta geometric distribution expressed iteratively**

5.2.4. Properties of beta – Geometric distribution

Moment generating function

Consider the moment generating function from the beta – Negative Binomial distribution as expressed in chapter 2

$$g(t) = \sum_{n=0}^{\infty}\left(\frac{r_{(n)}\beta_{(n)}}{(r+\alpha+\beta)_{(n)}}\right)\frac{t^n}{n!}$$

$$g(t) = \sum_{n=0}^{\infty}\left[\frac{\Gamma(r+n)}{\Gamma r}\frac{\Gamma(\beta+n)}{\Gamma\beta}\frac{\Gamma(r+\alpha+\beta)}{\Gamma(r+\alpha+\beta+n)}\right]\frac{t^n}{n!} \qquad (4.05)$$

When $r = 1$ we will end up with the moment generating function for the beta geometric distribution as stipulated below.

$$g(t) = \sum_{n=0}^{\infty} \left[\frac{\Gamma(1+n)\Gamma(\beta+n)}{\Gamma\beta} \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha+\beta+n)} \right] \frac{t^n}{n!} \tag{5.06}$$

Differentiate $g(t)$ with respect to t

$$g^{'(t)} = n \sum_{n=0}^{\infty} \left[\frac{\Gamma(1+n)\Gamma(\beta+n)}{\Gamma\beta} \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha+\beta+n)} \right] \frac{t^{n-1}}{n!} \tag{5.07}$$

$$g^{'(t)} = \sum_{n=0}^{\infty} \left[\frac{\Gamma(1+n)\Gamma(\beta+n)}{\Gamma\beta} \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha+\beta+n)} \right] \frac{t^{n-1}}{(n-1)!} \tag{5.08}$$

$$g^{''(t)} = n(n-1) \sum_{n=0}^{\infty} \left[\frac{\Gamma(1+n)\Gamma(\beta+n)}{\Gamma\beta} \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha+\beta+n)} \right] \frac{t^{n-2}}{n!} \tag{5.09}$$

$$g^{''(t)} = \sum_{n=0}^{\infty} \left[\frac{\Gamma(1+n)\Gamma(\beta+n)}{\Gamma\beta} \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha+\beta+n)} \right] \frac{t^{n-2}}{(n-2)!} \tag{5.10}$$

5.3. Geometric – Uniform distribution

The uniform distribution [0,1] given by

$$g(p) = \begin{cases} 1 & 0 < p < 1 \\ 0 & elsewhere \end{cases} \tag{5.11}$$

The Negative Binomial – uniform distribution have the following formats from the respective methods used in mixing the two distributions

Negative Binomial – uniform distribution from explicit mixing

$$f(x) = \frac{r}{(r+x+1)(r+x)}; \quad for\, r > 0\; and\; x = 0,1,2,\dots \tag{5.12}$$

Negative Binomial – uniform distribution from recursive relation

1st form

$$f(x) = \frac{r+x-1}{r+x+1} f(x-1) \quad for\ r > 0\ and\ x = 0,1,2,\dots \qquad (5.13a)$$

2nd form

$$f(x) = \frac{r}{(r+1)} f(x-1) \quad for\ r > 0\ and\ x = 0,1,2,\dots \qquad (5.13a)$$

Negative Binomial – uniform distribution from method of Moments

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{1}{r+k+1} \qquad (5.14)$$

In the above Negative Binomial mixtures it is important to note that if r=1 then they become geometric - uniform distribution mixtures from the respective mixing methods as shown below.

5.3.1. Geometric – uniform distribution from explicit mixing

$$f(x) = \frac{1}{(2+x)(1+x)} \quad for\ x = 0,1,2,\dots \qquad (5.15a)$$

$$f(x) = B(2, x+1) \quad for\ x = 0,1,2,\dots \qquad (5.15b)$$

5.3.2. Geometric – uniform distribution from recursive relation

1st form

Here we consider the mixture from explicit mixing method.

$$f(x) = \frac{x}{x+2} f(x-1); \quad for\ x = 0,1,2,\dots \qquad (5.16a)$$

2nd form

$$f(x) = \frac{1}{2} f(x-1); \quad for\ x = 0,1,2,\dots \qquad (5.16b)$$

5.3.3. Geometric – uniform distribution from Method of moments

$$f(x) = \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{1}{k+2} \; ; \;\; for\ x = 0,1,2,\dots \qquad (5.16)$$

5.4. Negative Binomial – power function distribution

5.4.1. Power function distribution

$$g(p) = ap^{a-1} \qquad 0 < p < 1;\ \alpha > 0 \qquad (5.17)$$

This is the pdf of a power function distribution with parameter α

The following are the results of from chapter 3 in relation to the distributions emanating from the Negative Binomial mixture with the power function.

a. **Negative Binomial – power function mixing from explicit format**

$$f(x) = \binom{r+x-1}{x} \alpha\, B(r+\alpha, x+1) \qquad (5.18)$$

$$for\ r, a > 0; x = 0,12 \dots.$$

This is the density function of the Negative Binomial – power function expressed explicitly

b. **Negative Binomial – power function from method of moments**

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)(x)!} \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{\alpha}{\alpha+r+k} \qquad (5.19)$$

$$for\ r, \alpha > 0\ and\ x = 0,1,2,\dots$$

c. **Negative Binomial – power distribution in a recursive expression**

1st form (from explicit mixture result)

$$f(x) = \frac{r+x-1}{x+a+1} f(x-1) \qquad (5.20a)$$

2nd form (using a dummy function)

$$f(x) = \frac{(r+x-1)(r+x-2)}{(r+a)(r-1)} f(x-1) \qquad (5.20b)$$

When r=1 the above mixtures become geometric – power mixtures

5.4.2. Geometric – Power distribution from explicit mixture

$$f(x) = \alpha\, B(1+\alpha, x+1) \tag{5.21}$$

$$for\ a > 0;\ x = 0{,}12 \ldots.$$

5.4.3. Geometric – Power distribution from moments method

$$f(x) = \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{\alpha}{\alpha + k + 1} \tag{5.22}$$

$$for\ \alpha > 0\ and\ x = 0,1,2, \ldots$$

5.4.4. Geometric – Power distribution in a recursive format

1st form (from explicit mixture result)

$$f(x) = \frac{x}{x + a + 1}\, f(x-1) \tag{5.23a}$$

2nd form (using a dummy function)

Replacing r by 1 in the denominator of the below equation will nullify the result.

$$f(x) = \frac{(r + x - 1)(r + x - 2)}{(r + a)(r - 1)}\, f(x-1)$$

We can however calculate this from scratch to achieve the below result

Consider the explicit expression below

$$f(x) = \alpha \int_0^1 p^\alpha (1-p)^x\, dp$$

Manipulate this to a dummy function $I_{(\alpha)}(x)$as shown below

$$I_{(\alpha)}(x) = \frac{f(x)}{\alpha} = \int_0^1 p^{\alpha}(1-p)^x dp \tag{5.24}$$

Integrating the integral by parts

Let

$$u = (1-p)^x$$

$$du = -x(1-p)^{x-1}dp$$

$$dv = \ p^{\alpha} dp$$

$$v = \ \frac{p^{\alpha+1}}{\alpha+1}$$

$$\int \mathrm{u\ dv} = \mathrm{vu} - \int \mathrm{v\ du}$$

$$I_{\alpha}(x) = \left[(1-p)^x \frac{p^{\alpha+1}}{\alpha+1}\right]_0^1 + \frac{x}{\alpha+1}\int_0^1 p^{\alpha+1}\,(1-p)^{x-1}dp \tag{5.25}$$

$$I_{\alpha}(x) \ = \frac{x}{\alpha+1}\int_0^1 p^{\alpha+1}\,(1-p)^{x-1}dp \tag{5.26}$$

$$I_{\alpha}(x) = \frac{f(x)}{\alpha} = \frac{x}{\alpha+1} I_{\alpha+1}(x-1)$$

$$\frac{f(x)}{\alpha} = \frac{x}{(r+\alpha)}\frac{f(x-1)}{(\alpha-1)} \tag{5.27}$$

Hence

$$f(x) = \frac{\alpha x}{r+\alpha} f(x-1) \tag{5.23b}$$

5.5. Geometric – Arcsine distribution

5.5.1. Arcsine distribution

$$g(p) = \frac{1}{\pi\sqrt{p(1-p)}} \qquad 0 < p < 1; a, b > 0 \ and \ \pi = B\left(\frac{1}{2}, \frac{1}{2}\right) \tag{5.28}$$

This is the pdf of arcsine distribution

The following are some of the expressions of the Negative Binomial – arcsine distributions

a. Negative Binomial – Arcsine from Explicit mixing

$$f(x) = \binom{r+x-1}{x} \frac{B\left(r+\frac{1}{2}, x+\frac{1}{2}\right)}{\pi} \tag{5.29}$$

b. Negative Binomial– Arcsine distribution in a recursive format

1st form (from explicit mixture result)

$$f(x) = \frac{(r+x-1)(2x-1)}{2x(r+x)} f(x-1) \tag{5.30a}$$

2nd form (using a dummy function)

$$f(x) = \frac{r(r-2x)}{x(2r-1)} f(x-1) \tag{5.30b}$$

c. Negative Binomial– Arcsine distribution from method of moments

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B\left(r+k+\frac{1}{2}, \frac{1}{2}\right)}{\pi} \tag{5.31}$$

$$for \ r, \alpha > 0; x = 0,1,2, \dots$$

When r=1 the above mixtures become geometric – arcsine distributions mixtures

5.5.2 Geometric – Arcsine from explicit mixing

$$f(x) = \frac{B\left(\frac{3}{2}, x+\frac{1}{2}\right)}{\pi} \tag{5.32}$$

5.5.3. Geometric – Arcsine distribution in a recursive format

1st form (from explicit mixture result)

$$f(x) = \frac{2x-1}{2(1+x)}\, f(x-1) \qquad for\ \alpha > 0; x = 0,1,2, \dots \qquad (5.33a)$$

2nd form (using a dummy function)

$$f(x) = \frac{1-2x}{x}\, f(x-1)\ for\ x = 0,1,2, \dots \qquad (5.33b)$$

5.5.4. Geometric – Arcsine distribution from method of moments

$$f(x) = \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B\left(k+\frac{3}{2},\frac{1}{2}\right)}{\pi} \qquad (5.34)$$

$$for\ \alpha > 0; x = 0,1,2, \dots$$

5.6 Geometric – Truncated beta distribution

5.6.1. Truncated beta

This below distribution is referred to as **truncated beta** distribution g(p)with parameters α, β, a, b

$$g(p) = \frac{p^{a-1}(1-p)^{b-1}}{B_\beta(a,b) - B_\alpha(a,b)} \qquad for\ 1 < \alpha < p < \beta < 1; a, b > 0 \qquad (5.35)$$

The following are some of the expressions of the Negative Binomial – arcsine distributions

a. **Negative Binomial – Truncated beta distribution explicitly mixed**

$$f(x) = \binom{r+x-1}{x} \frac{B_\beta(a+r,b+x) - B_\alpha(a+r,b+x)}{B_\beta(a,b) - B_\alpha(a,b)} \qquad (5.36)$$

$$for\ r, a, b > 0; 1 < \alpha < p < \beta < 1; x = 0,1,2 \dots$$

This is the pdf of the Negative Binomial – truncated beta mixture

b. Negative Binomial – Truncated beta distribution from method of moments

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)!\,x!}\sum_{k=0}^{x}(-1)^k \binom{x}{k} \frac{B_\beta(a+k+r,b) - B_\alpha(a+k+r,b)}{B_\beta(a,b) - B_\alpha(a,b)} \quad (5.37)$$

$$for\ r,a,b > 0; 1 < \alpha < p < \beta < 1; x = 0,1,2 \ldots$$

When r=1 we will have truncated beta – geometric as follows

5.6.2 Geometric – Truncated beta from explicit mixing

$$f(x) = \frac{B_\beta(a+1,b+x) - B_\alpha(a+1,b+x)}{B_\beta(a,b) - B_\alpha(a,b)} \quad (5.38)$$

$$for\ a,b > 0; 1 < \alpha < p < \beta < 1; x = 0,1,2 \ldots$$

5.6.3. Geometric – Truncated beta distribution from method of moments

$$f(x) = \sum_{k=0}^{x}(-1)^k \binom{x}{k} \frac{B_\beta(a+k+1,b) - B_\alpha(a+k+1,b)}{B_\beta(a,b) - B_\alpha(a,b)} \quad (5.39)$$

$$for\ a,b > 0; 1 < \alpha < p < \beta < 1; x = 0,1,2 \ldots$$

5.7 Geometric – Confluent Hypergeometric

5.7.1. Confluent Hypergeometric

The pdf of confluent Hypergeometric expressed as follows

$$g(p) = \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)\,{}_1F_1(a,a+b;\ -\mu)} e^{-p\mu}\ for\ 0 < p < 1; a,b > 0;\ -\infty < \mu < \infty \quad (5.40)$$

$$ref\ Nadarajah\ and\ Kotz\ (2007)$$

The Negative Binomial – confluent Hypergeometric distribution have the following formats from the respective methods used in mixing the two distributions

a. Confluent Hypergeometric – Negative Binomial distribution from explicit mixing

$$f(x) = \frac{B(r+a,x+b)\,{}_1F_1(a+r,a+b+r+x;\ -\mu)}{B(a,b)\,{}_1F_1(a,a+b;\ -\mu)}\binom{r+x-1}{x} \quad (5.41)$$

$$for\ r,a,b > 0;\ -\infty < \mu < \infty;\ x = 0,1,2,\ldots$$

Properties of confluent Hypergeometric – Negative Binomial Mixture

First 3 moments about the origin

$$E(X) = \frac{a}{a+b} \frac{{}_1F_1(a+1, a+b+1; -\mu)}{{}_1F_1(a, a+b; -\mu)} \tag{5.42}$$

$$E(X^2) = \frac{a(a+1)}{(a+b)(a+b+1)} \frac{{}_1F_1(a+2, a+b+2; -\mu)}{{}_1F_1(a, a+b; -\mu)} \tag{5.43}$$

$$E(X^3) = \frac{a(a+1)(a+2)}{(a+b)(a+b+1)(a+b+2)} \frac{{}_1F_1(a+3, a+b+3; -\mu)}{{}_1F_1(a, a+b; -\mu)} \tag{5.44}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = \frac{a(a+1){}_1F_1(a+2, a+b+2; -\boldsymbol{\mu})}{(a+b)(a+b+1){}_1F_1(a, a+b; -\boldsymbol{\mu})} - \left[\frac{a(a+1)}{(a+b)(a+b+1)} \frac{{}_1F_1(a+2, a+b+2; -\boldsymbol{\mu})}{{}_1F_1(a, a+b; -\boldsymbol{\mu})}\right]^2 \tag{5.45}$$

b. Confluent Hypergeometric – Negative Binomial distribution from Method of Moments

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)(x)!} \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{B(r+k+a, b) \quad {}_1F_1(a+r+k, a+r+k+b;\ -\mu)}{B(a,b){}_1F_1(a, a+b;\ -\mu)} \tag{5.46}$$

$$for\ x = 0,1,2, \dots; a, b > o\ ; r > 0$$

In the above negative Binomial mixtures it is important to note that if r=1 then they become geometric Hypergeometric mixtures from the respective mixing methods as shown below.

5.7.2 Geometric – Confluent Hypergeometric distribution from explicit mixing

$$f(x) = \frac{B(1+a, x+b)_1F_1(a+1, a+b+1+x;\ -\mu)}{B(a,b)_1F_1(a, a+b;\ -\mu)} \tag{5.47}$$

$$for\ a,b > 0;\ \ -\infty < \mu < \infty;\ \ x = 0,1,2,\dots$$

Properties of Geometric – Confluent Hypergeometric Mixture

First 3 moments about the origin

$$E(X) = \frac{a}{a+b}\frac{{}_1F_1(a+1, a+b+1; -\mu)}{{}_1F_1(a, a+b; -\mu)} \tag{5.48}$$

$$E(X^2) = \frac{a(a+1)}{(a+b)(a+b+1)}\frac{{}_1F_1(a+2, a+b+2; -\mu)}{{}_1F_1(a, a+b; -\mu)} \tag{5.49}$$

$$E(X^3) = \frac{a(a+1)(a+2)}{(a+b)(a+b+1)(a+b+2)}\frac{{}_1F_1(a+3, a+b+3; -\mu)}{{}_1F_1(a, a+b; -\mu)} \tag{5.50}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = \frac{a(a+1)_1F_1(a+2, a+b+2; -\mu)}{(a+b)(a+b+1)_1F_1(a, a+b; -\mu)} - \left[\frac{a(a+1)}{(a+b)(a+b+1)}\frac{{}_1F_1(a+2, a+b+2; -\mu)}{{}_1F_1(a, a+b; -\mu)}\right]^2 \tag{5.51}$$

5.7.2 Geometric – Hypergeometric distribution from Method of Moments

$$f(x) = \sum_{k=0}^{x}(-1)^k\binom{x}{k}\frac{B(1+k+a, b)\ \ {}_1F_1(a+k+1, a+k+b+1;\ -\mu)}{B(a,b)_1F_1(a, a+b;\ -\mu)} \tag{5.52}$$

$$for\ x = 0,1,2,\dots; a,b > o\ ; r > 0$$

5.8. Geometric – Gauss Hypergeometric distribution

5.8.1. Gauss Hypergeometric distribution

The pdf of gauss Hypergeometric given by

$$g(t) = \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)(1+zt)^{\varepsilon}{}_2F_1(a,\varepsilon,a+b;-z)} \quad (5.53)$$

$$for\ 0 < t < 1; a,b > 0;\ -\infty < \varepsilon < \infty$$

(see Armero and Bayarri (1994))

a. Gauss Hypergeometric – Negative Binomial distribution from Explicit mixing

$$f(x) = \binom{r+x-1}{x}\frac{B(a+r,b+x)_2F_1(a+r,\varepsilon;a+b+r+x;-z)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)} \quad (5.54)$$

$$\text{For } r,a,b,z > 0; -\infty < \varepsilon < \infty; x = 0,1,2 \ldots;$$

b. Gauss Hypergeometric – Negative Binomial distribution from method of moments mixing

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)(x)!}\sum_{k=0}^{x}(-1)^k\binom{x}{k}\frac{B(r+k+a,b)_2F_1(r+k+a,\varepsilon,a+b+r+k;-z)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)} \quad (5.55)$$

$$\text{For } r,a,b,z > 0; -\infty < \varepsilon < \infty; x = 0,1,2 \ldots;$$

5.8.3. Gauss Hypergeometric – Geometric distribution from Explicit mixing

$$f(x) = \frac{B(a+1,b+x)_2F_1(a+1,\varepsilon;a+b+x+1;-z)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)} \quad (5.56)$$

$$\text{For } a,b,z > 0; -\infty < \varepsilon < \infty; x = 0,1,2 \ldots;$$

5.8.4. Gauss Hypergeometric – Geometric distribution from method of moments mixing

$$f(x) = \sum_{k=0}^{x}(-1)^k\binom{x}{k}\frac{B(r+k+a,b)_2F_1(k+a+1,\varepsilon,a+b+k+1;-z)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)} \quad (5.57)$$

CHAPTER 6

GEOMETRIC MIXTURES BASED ON DISTRIBUTIONS BEYOND BETA FROM NEGATIVE BINOMIAL MIXTURES

6.1. Introduction

All the distributions we are going to consider in this category are within the [0,1] domain and are not based on beta distribution. Below distributions will be studied;

1. Kumaraswamy (I) and (II) distribution
2. Triangular distribution
3. Truncated exponential distribution
4. Truncated gamma distribution
5. Minus log distribution
6. Two – sided ogive distribution
7. Ogive distribution
8. Two – sided power distribution

5.2. Geometric – Kumaraswamy (I) Distribution

5.2.1. Kumaraswamy (I) Distribution

$$g(u) = b(1-u)^{b-1} \quad for\ 0 < u < 1; b > 0 \tag{6.01}$$

This is the pdf of Kumaraswamy (I) distribution with parameter b`

The Negative Binomial – Kumaraswamy (I) Distribution have the following formats from the respective methods used in mixing the two distributions

a. Negative Binomial – Kumaraswamy (I) distribution from explicit mixing

$$f(x) = \binom{r+x-1}{x} bB(r+1, x+b) \;\; x = 0,1,2,\dots; \;\; b, r > 0 \tag{6.02}$$

See Li Xiaohuet et al (2011) on Binomial mixture

b. Negative Binomial – Kumaraswamy (I) distribution mixing from the method moments

$$P(X=x) = \frac{\Gamma(r+x)}{\Gamma(r)(x)!}\sum_{k=1}^{x}(-1)^k\binom{x}{k} bB(r+k+1, b) \tag{6.03}$$

$$for\ r, b > 0\ and\ x = 0,1,2,\dots$$

When r=1 we will have the following geometric – Kumaraswamy (I) distribution as obtained from explicit mixing of the geometric distribution and the Kumaraswamy (I) distribution

5.2.2. Geometric – Kumaraswamy (I) distribution from explicit mixing

$$f(x) = bB(2, x+b); \;\; x = 0,1,2,\ldots; \;\; b > 0 \qquad (6.04)$$

5.2.2. Geometric – Kumaraswamy (I) distribution from method of moments mixing

When r=1 we will have geometric – KW (I) mixtures from the above negative Binomial mixtures as follows.

$$P(X=x) = \sum_{k=1}^{x} (-1)^k \binom{x}{k} bB(k+2, b) \qquad (6.05)$$

$$for\; r, b > 0 \; and \; x = 0,1,2,\ldots$$

6.3. Geometric – Truncated Exponential Distribution (*TEX(λ, b)*)

6.3.1. Truncated Exponential Distribution (*TEX(λ, b)*)

This is a pdf which can be expressed as

$$g(p) = \frac{\frac{b}{\lambda} e^{-bp/\lambda}}{[1 - e^{-b/\lambda}]} \qquad 0 < p < 1; \;\; b, \lambda > 0 \qquad (6.06)$$

This the truncated exponential distribution with parameters $\lambda \; and \; b$

Negative Binomial – Truncated Exponential distribution from explicit mixing

$$f(x) = \frac{b_1F_1\left(r+1, r+x+2; -\frac{b}{\lambda}\right)}{\lambda\left[1 - e^{-\frac{b}{\lambda}}\right]} \frac{r}{(r+x+1)(r+x)} \qquad for\; x = 0,1,2,..; b, r, \lambda > 0 \qquad (6.07)$$

6.3.1. Geometric – Truncated Exponential distribution from explicit maxing

When r=1 the above mixture attains the status of a geometrical - truncated exponential distribution from explicit mixing of the geometric distribution with the truncated exponential distribution

$$f(x) = \frac{b_1F_1\left(2, 1+x+2; -\frac{b}{\lambda}\right)}{\lambda\left[1 - e^{-\frac{b}{\lambda}}\right]} \frac{1}{(x+2)(1+x)} \qquad for\; x = 0,1,2,..; b, \lambda > 0 \qquad (6.08)$$

6.4. Geometric – Truncated Gamma Distribution

6.4.1. Truncated Gamma Distribution

Below is the pdf of Truncated Gamma Distribution

$$g(p) = \begin{cases} \dfrac{a^b p^{b-1} e^{-ap}}{\gamma(a,b)} & 0 < p < 1; a,b > 0 \\ 0 & elsewhere \end{cases} \quad (6.09)$$

a. Negative Binomial – Truncated Gamma Distribution from explicit mixing

$$f(x) = \binom{r+x-1}{x} \frac{a^b {}_1F_1(r+b, r+b+x+1; -a) B(r+b, x+1)}{\gamma(a,b)} \quad x = 0,1,2,\dots; r,a,b > 0 \quad (6.10)$$

(see Bhattacharya, S.K. (1968) on Binomial mixture)

b. Negative Binomial – Truncated Gamma Distribution from moments method mixing

$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)(x)!} \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{\gamma(a, b+r+k)}{a^{r+k}\gamma(a,b)}$$

$$for\ a,b,r > 0; x = 0,1,2,\dots$$

6.4.2. Geometric – Truncated Gamma distribution from explicit mixing

When r=1 the above mixture attains the status of a Geometric – Truncated Gamma distribution stated below

$$f(x) = \frac{a^b {}_1F_1(1+b, b+x+2; -a) B(1+b, x+1)}{\gamma(a,b)} \quad x = 0,1,2,\dots; a,b > 0 \quad (6.11)$$

6.4.1. Geometric – Truncated Gamma distribution from moments method mixing

When r=1 the above mixture attains the status of a geometric – truncated gamma distribution stated below

$$f(x) = \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{\gamma(a, b+k+1)}{a^{1+k}\gamma(a,b)} \quad (6.12)$$

$$for\ a,b,r > 0; x = 0,1,2,\dots$$

6.5. Geometric – Minus Log Distribution

6.5.1. Minus Log Distribution

$$g(p) = -\log p \;\; for\; 1 \le p \le 0 \qquad (6.13)$$

This is the pdf of minus log distribution

a. **Negative Binomial – Minus Log Distribution from method of moments mixing**

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{1}{(r+k+1)^2} \quad x = 0,1,2, \dots r > 0 \qquad (6.14)$$

6.5.1Geometric – Minus Log distribution from method of moments mixing

When r=1 the above mixtures attains the status of a geometric – minus log distribution which is expressed in the equation below

$$f(x) = \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{1}{(k+2)^2} \quad x = 0,1,2, \dots \dots. \qquad (6.15)$$

6.6. Geometric – Standard Two – Sided Power Distribution

6.6.1. Standard Two – Sided Power Distribution

$$g(p) = \begin{cases} k\left(\frac{p}{\theta}\right)^{k-1} & 0 < p < \theta \\ k\left(\frac{1-p}{1-\theta}\right)^{k-1} & \theta < p < 1 \end{cases} \qquad (6.16)$$

This is the pdf of a two – sided power distribution with parameters $k\; and\; \theta$

a. **Negative Binomial – Standard Two – Sided Power Distribution from explicit mixing**

$$\boldsymbol{f(x)} = \binom{r+x-1}{x} k \left\{ \frac{1}{\theta^{k-1}} B_\theta(r+k, x+1) + \frac{1}{(1-\theta)^{k-1}} [B(r+1, x+k) - B_\theta(r+1, x+k)] \right\} \; (6.17)$$

This is the distribution obtained from the mixture of Negative Binomial distribution and the standard two – side power distribution.

6.6.1. Geometric – Standard Two – Sided Power distribution from explicit mixing

When r=1, the above mixture reduces to geometric – standard two – sided power distribution which is stated below.

$$\boldsymbol{f}(\boldsymbol{x}) = k\left\{\frac{1}{\theta^{k-1}}B_{\theta}(1+k, x+1) + \frac{1}{(1-\theta)^{k-1}}[B(2, x+k) - B_{\theta}(2, x+k)]\right\} \qquad (6.18)$$

6.7. Geometric – Ogive Distribution

6.7.1. Ogive Distribution

The general form of an Ogive distribution is given by

$$g(p) = \frac{2m(m+1)}{3m+1}p^{(m-1)/2} + \frac{1-m^2}{3m+1}p^m \qquad 0 < p > 1\,; m > 0 \qquad (6.19)$$

From ($Dorp\ and\ Kotz$ (2003))

a. **Negative Binomial – Ogive Distribution from explicit mixing**

$$f(x) = \binom{r+x-1}{x}\frac{(m+1)}{3m+1}\left\{2mB\left(\frac{2r+m-1}{2}+1, x+1\right) + (1-m)B(r+m+1, x+1)\right\} \quad (6.20)$$

$$for\ x = 0,1,2, \ldots.; r > 0; m > 0$$

6.7.2. Geometric – Ogive distribution from explicit mixing

When r=1, the above mixture reduces to geometric – ogive distribution which is stated below.

$$f(x) = \frac{m+1}{3m+1}\left\{2mB\left(\frac{m+1}{2}+1, x+1\right) + (1-m)B(m+2, x+1)\right\} \qquad (6.21)$$

6.8. Geometric – Standard Two – Sided Ogive Distribution

6.8.1. Standard Two – Sided Ogive Distribution

$$g(p) = \begin{cases} \dfrac{2m(m+1)}{3m+1}\left[\dfrac{p}{\theta}\right]^{\frac{m-1}{2}} + \dfrac{1-m^2}{3m+1}\left[\dfrac{p}{\theta}\right]^{m} & 0 < p < \theta\,; m > 0 \\ \dfrac{2m(m+1)}{3m+1}\left[\dfrac{1-p}{1-\theta}\right]^{\frac{m-1}{2}} + \dfrac{1-m^2}{3m+1}\left[\dfrac{1-p}{1-\theta}\right]^{m} & \theta < p < 1\,; m > 0 \end{cases} \quad (6.22)$$

This is the pdf of the two sided Ogive Distribution with parameters $m\ and\ \theta$

When $p = \theta$ then the two – sided Ogive distribution is smooth and this is the reflection point.

Although this contradicts the situation at the reflection point of the two – sided power function

a. **Negative Binomial – Standard Two – Sided Ogive Distribution from explicit mixing**

$$f(x) = \binom{r+x-1}{x}\frac{(m+1)}{3m+1}\left\{\frac{2m}{[\theta]^{\frac{m-1}{2}}}B_\theta\left(\frac{2r+m-1}{2}+1, x+1\right) + \frac{1-m}{\theta^m}B_\theta(r-m+1, x+1) + \frac{2m}{(1-\theta)^{\frac{m-1}{2}}}\left[B\left(r+1, \frac{2x+m-1}{2}+1\right) - B_\theta\left(r+1, \frac{2x+m-1}{2}+1\right)\right]\frac{1-m}{(1-\theta)^m}\left[B(r+1, x+m+1) - B_\theta(r+1, x+m+1)\right]\right\} \quad (6.23)$$

$$for\ x = 0,1,2,\ldots.\ \ ; m > 0\,; r > 0$$

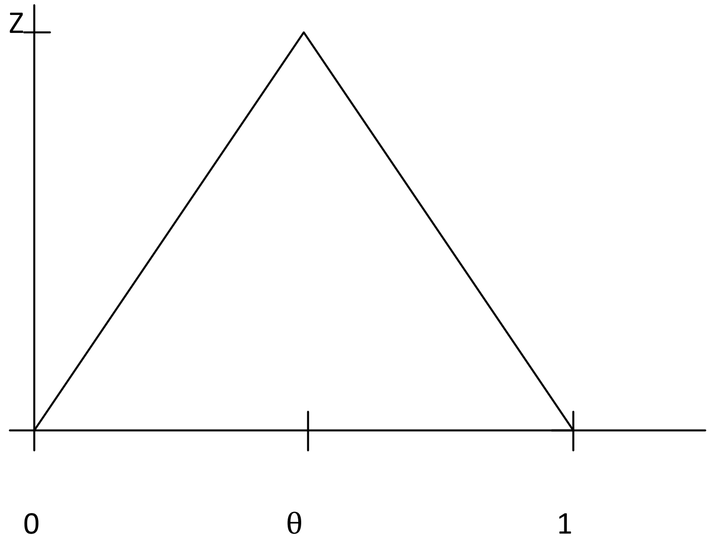
6.8.1. Geometric – Standard Two – Sided Ogive Distribution from explicit mixing

When r=1, the above mixture reduces to geometric – Two Sided Ogive distribution which is stated below.

$$f(x) = \frac{m+1}{3m+1}\left\{\frac{2m}{[\theta]^{\frac{m-1}{2}}} B_\theta\left(\frac{m+3}{2}, x+1\right) + \frac{1-m}{\theta^m} B_\theta(2-m, x+1) + \frac{2m}{(1-\theta)^{\frac{m-1}{2}}}\left[B\left(2, \frac{2x+m+1}{2}\right) - B_\theta\left(r+1, \frac{2x+m+1}{2}\right)\right] \frac{1-m}{(1-\theta)^m}[B(2, x + m + 1) - B_\theta(2, x+m+1)]\right\} \quad (6.24)$$

6.9. Geometric – Triangular Distribution

6.9.1. Triangular Distribution



Thus the density of the triangular distribution is

$$g(p) = \begin{cases} \frac{2p}{\theta} & 0 < p < \theta \\ \frac{2(1-p)}{1-\theta} & \theta < p < 1 \\ 0 & elsewhere \end{cases} \quad (6.25)$$

a. **Negative Binomial – triangular distribution mixing**

$$f(x) = \frac{2\Gamma(r+x)}{x!\,\Gamma r}\left\{\frac{B_\theta(r+2, x+1)}{\theta} + \frac{B(r+1, x+2) - B_\theta(r+1, x+2)}{1-\theta}\right\} \quad (6.26)$$

$$for\ x = 0,1,2, \ldots \ldots; 0 < \theta < 1, r > 0$$

This is the mixture of triangular distribution and Negative Binomial distribution expressed explicitly

b. **Negative Binomial – Triangular distribution from method of moments mixing**

Case 1

$$p(x) = \frac{\Gamma(r+x)!}{\Gamma(r)!\,(x)!}\sum_{k=0}^{x}\binom{x}{k}(-1)^{x-k}\frac{2(1-\theta^{r+x-k+1})}{(r+x-k+1)(r+x-k+2)(1-\theta)} \quad (4.27)$$

$$for\ r > 0;\ 0 < p < \theta < 1\ and\ x = 0,1,2,\ldots$$

Case 2

$$f(x) = \sum_{k=0}^{x}(-1)^k\binom{x}{k}\frac{2}{(1-\theta)}\left[\frac{1-\theta^{r+k+1}}{(r+k+1)(r+k+2)}\right] \quad (6.28)$$

$$for\ r > 0;\ 0 < p < \theta < 1\ and\ x = 0,1,2,\ldots$$

6.9.2. Geometric – Triangular distribution from explicit mixing

When r=1, the above mixture reduces to geometric – Triangular distribution which is stated below.

$$f(x) = 2\left\{\frac{B_\theta(3,x+1)}{\theta} + \frac{B(2,x+2) - B_\theta(2,x+2)}{1-\theta}\right\} \quad (6.29)$$

$$for\ x = 0,1,2,\ldots\ldots;\ 0 < \theta < 1$$

6.9.2. Geometric – Triangular distribution from method of moments mixing

Case 1

When r=1, equation 4.27 above reduces to Geometric – Triangular distribution which is stated below.

$$p(x) = \sum_{k=0}^{x}\binom{x}{k}(-1)^{x-k}\frac{2(1-\theta^{x-k+2})}{(x-k+2)(x-k+3)(1-\theta)} \quad (6.30)$$

Case 2

When r=1, equation 2.28 above reduces to geometric – Triangular distribution which is stated below.

$$f(x) = \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{2}{(1-\theta)} \left[\frac{1-\theta^{k+2}}{(k+2)(k+3)} \right] \qquad (6.31)$$

$$for\; r > 0;\; 0 < p < \theta < 1 \; and \; x = 0,1,2,\ldots$$

Properties of geometric – Triangular distribution mixture

The mean is given by

$$E(X) = \alpha \left(\frac{-2\log\theta}{1-\theta} - 1 \right) \qquad (6.32)$$

Since $0 < \theta \leq 1$

Therefore

$$E(X) > \alpha \quad \vee \theta$$

Variance

Variance of this mixture doesn't exist since the second inverse moment of the triangular distribution doesn't exist. It is important to note that this distribution has a very long tail.

CHAPTER 7

NEGATIVE BINOMIAL MIXTURES WITH P AS A CONSTANT IN THE MIXING DISTRIBUTION

7.1. Introduction

This chapter talks about mixing of the negative Binomial with other distributions having r as a variable and holding p constant.

We are going to consider four mixing distributions, namely

1. Logarithmic distribution
2. Exponential distribution
3. Binomial distribution

To get the mean and variance of the mixture, we use the method of **probability generation function** by getting the moments i.e.

$$G(s) = \sum_{k=0}^{\infty} p_k \, s^k \tag{7.01}$$

7.2. Logarithmic distribution/ logarithmic series distribution/ log series distribution

Construction

Consider a Maclaurin Series expansion

$$-\log(1-p) = p + \frac{p^2}{2} + \frac{p^3}{3} + \cdots$$

$$1 = \frac{-1}{\log(1-p)}\left(p + \frac{p^2}{2} + \frac{p^3}{3} + \cdots\right) \tag{7.02}$$

$$1 = \sum_{i=1}^{\infty} \frac{-1}{\log(1-p)} \frac{p^k}{k} \tag{7.03}$$

This qualifies to be a PMF since it cumulates to 1.

The probability mass function is a log(p) distribution expressed as

$$f(X=k) = \frac{-1}{\log(1-p)}\frac{p^k}{k} \quad ; for\ k = 0,1,2\ldots\ and\ 0 \le p \le 1 \tag{7.04}$$

Its Cumulative Distribution Function is

$$F(k) = \int_0^k \frac{-1}{\log(1-p)}\frac{p^x}{x}dx \tag{7.05}$$

$$F(k) = \frac{-1}{\log(1-p)}\int_0^k \frac{p^x}{x}dx$$

$$F(k) = 1 + \frac{B(p;k+1.0)}{\log(1-p)} \tag{7.06}$$

$$where\ B\ is\ an\ incomplete\ Beta\ function$$

Negative Binomial – logarithmic mixture

$$p_k = \int_0^\infty \binom{r+k-1}{k} p^r q^k\, g(r)dr \tag{7.07}$$

$$g(r) = \frac{-1}{\log(1-p)}\frac{p^r}{r}$$

$$p_k = \int_0^\infty \binom{r+k-1}{k} p^r q^k \frac{-1}{\log(1-p)}\frac{p^r}{r}dr \tag{7.08}$$

The PGF is given by the following expression

$$G(s) = \sum_{k=0}^{\infty} p_k\, s^k$$

$$G(s) = \frac{-1}{\log(1-p)}\int_0^\infty \frac{p^{2r}}{r}\sum_{k=0}^{\infty}\binom{r+k-1}{k}(sq)^k dr$$

$$G(s) = \frac{-1}{\log(1-p)} \int_0^\infty \frac{p^{2r}}{r} \sum_{k=0}^{\infty} \binom{-r}{k} (-sq)^k dr$$

$$G(s) = \frac{-1}{\log(1-p)} \int_0^\infty \frac{p^{2r}(1-sq)^{-r}}{r} dr \qquad (7.09)$$

$$G(s) = \frac{-1}{\log(1-p)} \int_0^\infty \left[\frac{p^2}{(1-sq)}\right]^r \frac{1}{r} dr \qquad (7.10)$$

7.3. Negative Binomial – exponential distribution

7.3.1. Exponential distribution

$$g(r) = \beta e^{-\beta r} \qquad (7.11)$$

7.3.2. Negative Binomial – exponential mixture

$$p_k = \int_0^\infty \binom{r+k-1}{k} p^r q^k \, g(r) dr$$

$$p_k = \beta \int_0^\infty \binom{r+k-1}{k} p^r q^k e^{-\beta r} \, dr \qquad 6.12$$

The PGF is given by the following expression

$$G(s) = \sum_{k=0}^{\infty} p_k \, s^k$$

$$G(s) = \beta \int_0^\infty p^r e^{-\beta r} \sum_{k=0}^{\infty} \binom{r+k-1}{k} (qs)^k dr \qquad (7.13)$$

$$G(s) = \beta \int_0^\infty p^r e^{-\beta r} \sum_{k=0}^{\infty} \binom{-r}{k} (-qs)^k dr$$

$$G(s) = \beta \int_0^\infty p^r e^{-\beta r} \, (1-qs)^{-r} dr$$

$$G(s) = \beta \int_0^\infty e^{-\beta r} \left[\frac{1-qs}{p}\right]^{-r} dr$$

$$G(s) = \beta \int_0^{\infty} e^{-\beta r}\, e^{-r ln\left[\frac{1-qs}{p}\right]} dr$$

$$G(s) = \beta \int_0^{\infty} e^{-r\beta\left(1+\frac{1}{\beta} ln\left[\frac{1-qs}{p}\right]\right)}\, dr \qquad (7.15)$$

Let

$$y = r\beta\left(1 + \frac{1}{\beta} ln\left[\frac{1-qs}{p}\right]\right)$$

$$r = \frac{y}{\beta\left(1 + \frac{1}{\beta} ln\left[\frac{1-qs}{p}\right]\right)}$$

$$dr = \frac{dy}{\beta\left(1 + \frac{1}{\beta} ln\left[\frac{1-qs}{p}\right]\right)}$$

Hence

$$G(s) = \frac{\beta}{\beta\left(1+\frac{1}{\beta} ln\left[\frac{1-qs}{p}\right]\right)} \int_0^{\infty} e^{-y}\, dy \qquad (7.16)$$

$$G(s) = \frac{1}{1 + \frac{1}{\beta} ln\left[\frac{1-qs}{p}\right]} \qquad (7.17)$$

or

$$G(s) = \frac{\beta}{\beta + ln\left[\frac{1-qs}{p}\right]} \qquad (7.18)$$

Mean

Differentiate the PGF with respect to s

$$G(s) = \frac{\beta}{\beta + ln\left[\frac{1-qs}{p}\right]}$$

$$G'(s) = \frac{\left\{\beta + ln\left[\frac{1-qs}{p}\right]\right\}.0 - \frac{\beta p}{1-qs}.\left(-\frac{q}{p}\right)}{\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]^2}$$

$$G^{'}(s) = \frac{\beta q}{(1-qs)\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]^2} \qquad (7.19)$$

$$G'(1) = \frac{\beta q}{(1-q)} \frac{1}{\left[\beta + ln\left[\frac{1-q}{p}\right]\right]^2}$$

$$G^{'}(1) = \frac{q}{\beta(1-q)} \qquad (7.20)$$

$$E(X) = G^{'}(1) = \frac{q}{\beta(1-q)} = \frac{1-p}{\beta p} \qquad (6.21)$$

$$p = 1 - q$$

Variance

Let

$$(\mathbf{1} - \boldsymbol{qs})\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]^2 = V$$

$$\frac{\boldsymbol{dV}}{\boldsymbol{ds}} = 2(\mathbf{1} - \boldsymbol{qs})\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]\left[\frac{\boldsymbol{p}}{\mathbf{1} - \boldsymbol{qs}}\right]\left[-\frac{\boldsymbol{q}}{\boldsymbol{p}}\right] + \left[\beta + ln\left[\frac{1-qs}{p}\right]\right]^2 [-q]$$

$$\frac{\boldsymbol{dV}}{\boldsymbol{ds}} = -2(\mathbf{1} - \boldsymbol{qs})\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]\left[\frac{\boldsymbol{q}}{\mathbf{1} - \boldsymbol{qs}}\right] - \boldsymbol{q}\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]^2$$

Now

$$G^{''}(s) = \frac{(1-qs)\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]^2 .0 - \beta q\left(-2(\mathbf{1} - \boldsymbol{qs})\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]\left[\frac{\boldsymbol{q}}{\mathbf{1-qs}}\right] - \boldsymbol{q}\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]^2\right)}{(1-qs)^2\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]^4}$$

$$G''(s) = \frac{\beta q\left(2(\mathbf{1-qs})\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]\left[\frac{\boldsymbol{q}}{\mathbf{1-qs}}\right] + \boldsymbol{q}\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]^2\right)}{(1-qs)^2\left[\beta + ln\left[\frac{1-qs}{p}\right]\right]^4} \tag{7.22}$$

$$G''(1) = \frac{\beta q\left(2(\mathbf{1-q})\left[\beta + ln\left[\frac{1-q}{p}\right]\right]\left[\frac{\boldsymbol{q}}{\mathbf{1-q}}\right] + \boldsymbol{q}\left[\beta + ln\left[\frac{1-q}{p}\right]\right]^2\right)}{(1-q)^2\left[\beta + ln\left[\frac{1-q}{p}\right]\right]^4}$$

$$G''(1) = \frac{\beta q(2\beta q + q\beta^2)}{(1-q)^2\beta^4} \tag{7.23}$$

$$G''(1) = \frac{q^2(2+\beta)}{(1-q)^2\beta^2}$$

Or

$$G''(1) = \frac{(1-p)^2(2+\beta)}{p^2\beta^2}$$

$$Var(X) = G''(1) - [G'(1)]^2 - G'(1)$$

$$Var(X) = \frac{(1-p)^2(2+\beta)}{p^2\beta^2} - \left[\frac{1-p}{\beta p}\right]^2 - \frac{1-p}{\beta p}$$

$$Var(X) = \frac{(1-p)^2(2+\beta) - (1-p)^2 + \beta p(1-p)}{p^2\beta^2} \tag{7.24}$$

$$Var(X) = \frac{(1-p)^2[2+\beta-1] + \beta p(1-p)}{p^2\beta^2}$$

$$Var(X) = \frac{(1-p)^2[\beta+1] + \beta p(1-p)}{p^2\beta^2}$$

Negative Binomial – Binomial

Binomial distribution

$$g(r) = \begin{cases} \binom{n}{r} p^r (1-p)^{n-r} & 0 < p < 1; r = 0,1.2 \dots n \\ 0 & elsewhere \end{cases}$$

$$p_k = \int_0^\infty \binom{r+k-1}{k} p^r q^k \, g(r) dr$$

$$p_k = \int_0^\infty \binom{r+k-1}{k} p^r q^k \binom{n}{r} p^r (1-p)^{n-r} dr$$

The PGF is given by the following expression

$$G(s) = \sum_{k=0}^{\infty} p_k \, s^k$$

$$G(s) = \int_0^\infty \binom{n}{r} p^{2r} \, (1-p)^{n-r} \sum_{k=0}^{\infty} \binom{r+k-1}{k} (sq)^k dr$$

$$G(s) = \int_0^\infty \binom{n}{r} p^{2r} \, (1-p)^{n-r} \sum_{k=0}^{\infty} \binom{-r}{k} (-sq)^k dr$$

$$G(s) = \int_0^\infty \binom{n}{r} p^{2r} \, q^{n-r} (1-sq)^{-r} dr$$

$$G(s) = q^n \int_0^\infty \binom{n}{r} p^{2r} \, [q(1-sq)]^{-r} dr \qquad (7.24)$$

CHAPTER 8

SUMMARY AND CONCLUSION

New distributions were generated as anticipated in the project statement and objectives.

Chapter 1 briefly explained the concept around the Negative Binomial distribution. It goes to the extent of explaining the evolution and the reason why negative Binomial distribution is more efficient or more preferred as compared to other discrete distributions like Binomial and Poisson. The importance of probability mixing was highlighted as well. Applications areas were also briefly studied in this chapter

In the second chapter, we have shown various methods constructing Negative Binomial distribution. The main aim of this section was to not only identify various ways of developing Negative Binomial distribution, but to also verify the various forms of the Negative Binomial distribution. Some of the properties of the Negative Binomial distribution were identified in this chapter. Eg. Moments, mean, Variance, Kurtosis and Skewness

Chapter three majored in the construction of Negative Binomial Mixtures with prior distributions that are within the range of [0,1] and are related to beta distribution. The approach was to construct the mixing priors before subjecting them to the mixture. A few properties of these mixing distributions were discussed e.g. the jth moment, mean and variance.

Mixing distributions that are beyond Beta and within the [0,1] domain were considered as the mixing priors in chapter 4. The basic properties of the mixing distributions such as the jth moment, mean and variance were highlighted. The jth moment was key to mixing using the method of moments.

Chapter 5 and 6 were special cases of chapter 3 and 4 respectively by the fact that geometric distribution is also a special case of the Negative Binomial distribution. These chapters talks about the Geometric distribution mixtures with beta generated priors and Beyond Beta priors respectively. This was achieved by considering the first success (r=1).

A new way to conduct the mixing is through letting r be the varying variable instead of x. the parameter p is held constant in the respective mixtures. This was done in chapter 7 with few mixing distributions.

In conclusion, the project actually met its main objective of construction of Negative Binomial mixtures when the mixing distributions come from the probability of success and the number of success as random variable.

The following table is a summary of the project

<u>**TABLED SUMMARY OF THE MIXTURES OF NEGATIVE BINOMIAL**</u>

Table 1: Negative Binomial mixtures with [0,1] domain distribution priors based on Classical Beta

DISTRIBUTIONS BASED ON CLASSICAL BETA AND [0, 1] DOMAIN GENERATED BETA DISTRIBUTIONS		
Mixing Distribution	Methods of mixing	Mixture
Classical beta $g(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}$ $for\ 0 < p < 1; \alpha,\beta > 0$	Explicit	$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\frac{B(r+\alpha, x+\beta)}{B(\alpha,\beta)}$
	Method of moments	$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\frac{B(\alpha+r+k,\beta)}{B(\alpha,\beta)};$ $for\ x = 0,1,2,\ldots; r,\alpha,\beta > 0$
	Recursive relation	a. Using ratio of the conditional distribution $p_r(x) = \frac{r+x-1}{x}\left[p_r(x-1) - \frac{r}{r+x-1}p_{r+1}(x-1)\right]$ b. Using ration of the mixed distribution $f(x) = \frac{(r+x-1)^2}{x(r+x+\alpha+\beta)}f(x-1)$ $for\ \boldsymbol{r},\alpha,\beta > 0, x = 1,2,3\ldots;$ c. Using a dummy function $f(x) = f(x) = \frac{(r+x-1)(x+\beta-1)}{x(r+x-2)}f(x-1)$ $for\ r,\beta,\alpha > 0\ and\ x = 1,2,3,\ldots$
Truncated Beta distribution $g(p) = \frac{p^{a-1}(1-p)^{b-1}}{B_\beta(a,b) - B_\alpha(a,b)}$ $for\ 1 < \alpha < p < \beta < 1; a,b > 0$	Explicit	$f(x) = \binom{r+x-1}{x}\frac{B_\beta(a+r,b+x) - B_\alpha(a+r,b+x)}{B_\beta(a,b) - B_\alpha(a,b)}$ $for\ a,b;\ 1 < \alpha < p < \beta < 1; x = 0,1,2\ldots > 0;$
	Method of moments	$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\frac{B_\beta(a+r+k,b) - B_\alpha(a+r+k,b)}{B_\beta(a,b) - B_\alpha(a,b)}$ $for\ r > 0;\ 1 < \alpha < p < \beta < 1; a,b > 0;\ x = 0,1,2,\ldots$

Arcsine distribution $$g_1(p) = \frac{1}{\pi\sqrt{p(1-p)}}$$ $$for\ 0 < p < 1; a, b > 0 and\ \pi = B\left(\frac{1}{2}, \frac{1}{2}\right)$$	Explicit	$$f(x) = \binom{r+x-1}{x} \frac{B(r+\frac{1}{2}, x+\frac{1}{2})}{\pi}$$ $$for\ r > 0; x = 0,1,2, \dots; and\ \pi = B\left(\frac{1}{2}, \frac{1}{2}\right)$$
	Method of moments	$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B\left(r+k+\frac{1}{2}, \frac{1}{2}\right)}{\pi}$$ $$for\ r, \alpha > 0; x = 0,1,2, \dots, ; and\ \pi = B\left(\frac{1}{2}, \frac{1}{2}\right)$$
	Recursive relation	**1st form** $$f(x) = \frac{(r+x-1)(2x-1)}{2x(r+x)} f(x-1)$$ $$for\ r > 0; x = 1,2, \dots;$$ **2nd form** $$f(x) = \frac{r(1-\ 2x)}{x(2r+1)} f(x-1)$$ $$for\ r > 0; x = 1,2, \dots;$$
Power function distribution $$g(p) = ap^{a-1} \quad for \quad 0 < p < 1;\ \alpha > 0$$	Explicit	$$f(x) = \binom{r+x-1}{x} \alpha\, B(r+\alpha, x+1)$$ $$for\ x = 0,12 \dots. \quad ; a, r > 0$$
	Method of moments	$$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{\alpha}{\alpha+r+k}$$ $$for\ r, \alpha > 0; x = 0,1,2, \dots$$
	Recursive relation	1st form $$f(x) = \frac{r+x-1}{r+\alpha+\ x} f(x-1)$$ $$for\ r > 0; x = 1,2, \dots;$$ 2nd form $$f(x) = \frac{(r+x-1)(r+x-2)}{(r+\alpha)(r-1)} f(x-1)$$ $$for\ r > 0; x = 1,2, \dots;$$

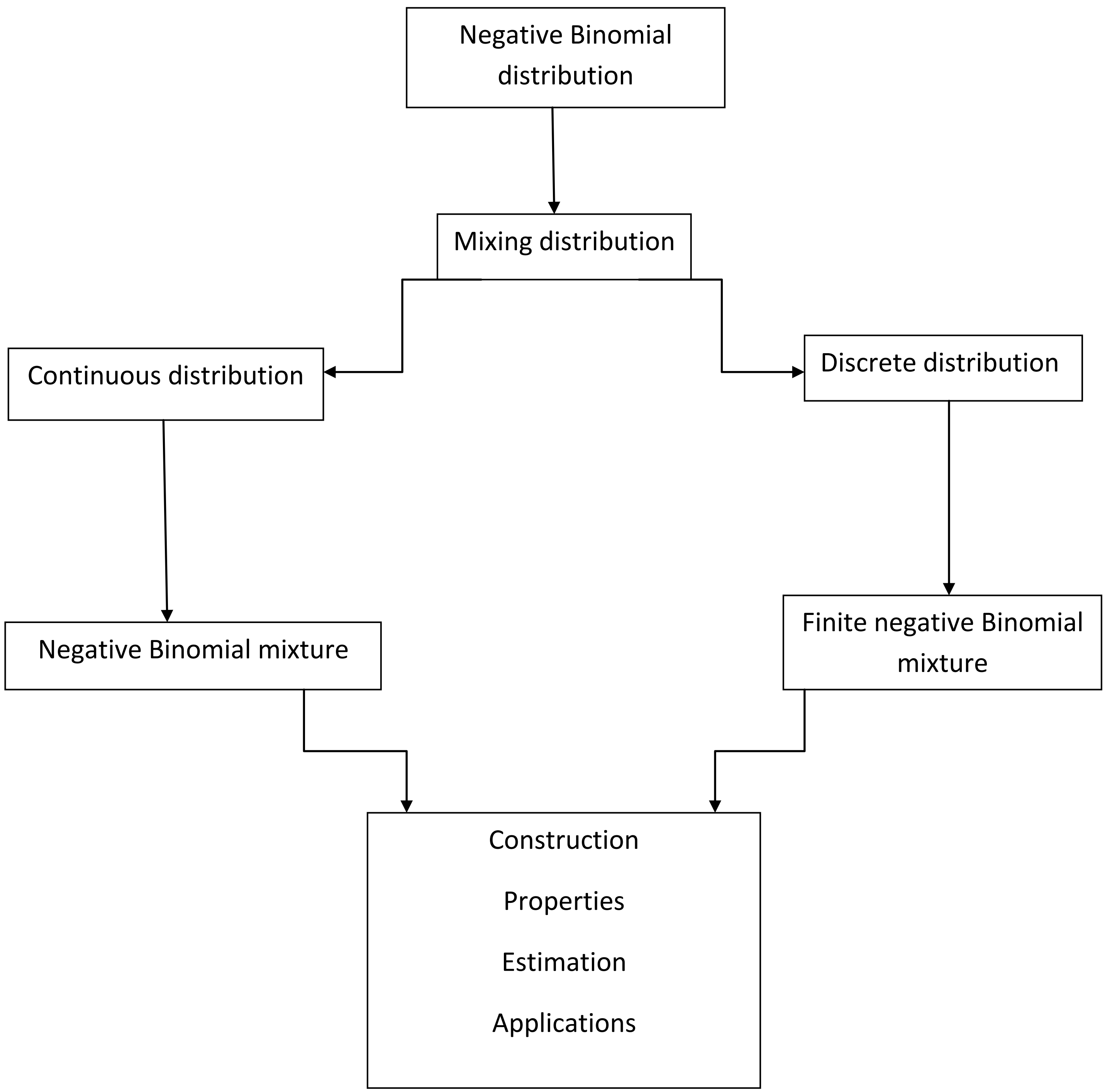
Uniform distri2bution $\boldsymbol{g(p)} = \begin{cases} \mathbf{1} & \mathbf{0} < p < 1 \\ \mathbf{0} & \boldsymbol{elsewhere} \end{cases}$	Explicit	$f(x) = \begin{cases} \frac{r}{(r+x+1)(r+x)} & x = 0,1,2,\dots r-1; r > 0 \\ 0 & elsewhere \end{cases}$
		$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{1}{r+k+1}$ $for\ r > 0; x = 0,1,2,\dots$
	Recursive relation	1st form $f(x) = \frac{r+x-1}{r+x+1} f(x-1)$ 2nd form $f(x) = \frac{r}{r+1} f(x-1)$ $for\ r > 0; x = 1,2,\dots;$
Confluent Hypergeometric distribution $g(p) = \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)_1F_1(a,a+b;\ -\mu)} e^{-p\mu}$ $for\ 0 < p < 1; a,b > 0;\ -\infty < \mu < \infty$	Explicit	$f(x) = \frac{B(r+a,x+b)_1F_1(a+r,a+b+r+x;\ -\mu)}{B(a,b)_1F_1(a,a+b;\ -\mu)} \binom{r+x-1}{x}$ $for\ r,a,b > 0;\ -\infty < \mu < \infty;\ x = 0,1,2,\dots$
	Method of moments	$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B(r+k+a,b)_1F_1(a+r+k,a+r+k+b;\ -\mu)}{B(a,b)_1F_1(a,a+b;\ -\mu)}$ $for\ x = 0,1,2,\dots; a,b > o\ ; r > 0$
Gauss Hypergeometric distribution $g(t) = \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)(1+zt)^{\varepsilon}\,{}_2F_1(a,\varepsilon,a+b;-z)}$ $for\ 0 < t < 1; a,b > 0;\ -\infty < \varepsilon < \infty$	Explicit	$f(x) = \binom{r+x-1}{x} \frac{B(a+r,b+x)_2F_1(a+r,\varepsilon;a+b+r+x;-z)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)}$ $for\ x = 0,1,2\dots; r = 0,1,2\dots.; a,b > 0;\ -\infty < \varepsilon < \infty$
	Method of moments	$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B(j+a,b)_2F_1(r+k+a,\varepsilon,a+b+r+k;-z)}{B(a,b)_2F_1(a,\varepsilon,a+b;-z)}$

Table 2: Negative Binomial mixtures with [0,1] domain Beyond Beta distribution priors

DISTRIBUTIONS BEYOND BETA		
Mixing Distribution	Methods of mixing	Mixture
Kumaraswamy (I) Distribution $g(u) = \begin{cases} b(1-u)^{b-1} & for\ 0<u<1; b>0 \\ 0 & elsewhere \end{cases}$	Explicit	$f(x) = \binom{r+x-1}{x} bB(r+1, x+b)\ \ for\ x = 0,1,2,\dots;\ \ b,r>0$
	Method of moments	$Prob(X=x) = \frac{\Gamma(r+x)}{\Gamma(\mathrm{r})x!} b \sum_{k=0}^{x} (-1)^k \binom{x}{k} B(r+k+1, b)$ $for\ r,b>0; x = 0,1,2,3,\dots$
Truncated Exponential Distribution ($TEX(\lambda, b)$) $g(p) = \begin{cases} \frac{\frac{b}{\lambda} e^{-bp/\lambda}}{[1-e^{-b/\lambda}]} & 0<p<1;\ \ b,\lambda>0 \\ 0 & elsewhere \end{cases}$	Explicit	$f(x) = \frac{b_1F_1\left(r+1, r+x+2; -\frac{b}{\lambda}\right)}{\lambda\left[1-e^{-\frac{b}{\lambda}}\right]} \frac{r}{(r+x+1)(r+x)}$ $for\ x = 0,1,2,..; b,r,\lambda>0$
Truncated Gamma Distribution $g(p) = \begin{cases} \frac{a^b p^{b-1} e^{-ap}}{\gamma(a,b)} & 0<p<1; a,b \\ 0 & elsewhere \end{cases}$	Explicit	$f(x) = \binom{r+x-1}{x} \frac{a^b{}_1F_1(r+b, r+b+x+1; -a)B(r+b, x+1)}{\gamma(a,b)}$ $for\ x = 0,1,2,\dots; r,a,b>0$
	Method of moments	$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)x!} \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{\gamma(a, b+r+k)}{a^{(r+k)}\gamma(a,b)}$ $for\ x = 0,1,2,\dots; r,a,b>0$
Minus Log Distribution $g(p) = \begin{cases} -\log p & 0<p<1 \\ 0 & elsewhere \end{cases}$	Explicit	$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{1}{(r+k+1)^2}$ $for\ x = 0,1,2,\dots r>0$
	Method of moments	$f(x) = \frac{\Gamma(r+x)}{\Gamma(r)(x)!} \sum_{k=0}^{x} (-1)^k \binom{x}{k} \frac{1}{(r+k+1)^2}$ $for\ \ x = 0,1,2,\dots r>0$

Standard Two – Sided Power Distribution $g(p)=\begin{cases} k\left(\frac{p}{\theta}\right)^{k-1} & 0<p<\theta \\ k\left(\frac{1-p}{1-\theta}\right)^{k-1} & \theta<p<1 \end{cases}$	Explicit	$f(x)=\binom{r+x-1}{x}k\left\{\frac{B_\theta(r+k,x+1)}{\theta^{k-1}}+\frac{[B(r+1,x+k)-B_\theta(r+1,x+k)]}{(1-\theta)^{k-1}}\right\}$ $for\ \ x=0,1,2,\dots r>0$
Ogive Distribution $g(p)=\frac{2m(m+1)}{3m+1}p^{\frac{m-1}{2}}+\frac{1-m^2}{3m+1}p^m;$ $0<p>1\,;m>0$	Explicit	$f(x)=\binom{r+x-1}{x}\frac{(m+1)}{3m+1}\left\{2mB\left(\frac{2r+m-1}{2}+1,x+1\right)+(1-m)B(r+m+1,x+1)\right\}$ (4.27) $for\ x=0,1,2,\dots.;r>0;m>0$
Standard Two – Sided Ogive Distribution $g(p)=\begin{cases} \frac{2m(m+1)}{3m+1}\left[\frac{p}{\theta}\right]^{\frac{m-1}{2}}+\frac{1-m^2}{3m+1}\left[\frac{p}{\theta}\right]^m & 0<p<\theta\,; \\ \frac{2m(m+1)}{3m+1}\left[\frac{1-p}{1-\theta}\right]^{\frac{m-1}{2}}+\frac{1-m^2}{3m+1}\left[\frac{1-p}{1-\theta}\right]^m & \theta<p< \end{cases}$	Explicit	$f(x)=\binom{r+x-1}{x}\frac{(m+1)}{3m+1}\left\{\frac{2m}{[\theta]^{\frac{m-1}{2}}}B_\theta\left(\frac{2r+m-1}{2}+1,x+1\right)+\frac{1-m}{\theta^m}B_\theta(r-m+1,x+1)\right.$ $+\frac{2m}{(1-\theta)^{\frac{m-1}{2}}}\left[B\left(r+1,\frac{2x+m-1}{2}+1\right)-\ B_\theta(r+1,\frac{2x+m-1}{2}\right.$ $\left.\left.+1)\right]\frac{1-m}{(1-\theta)^m}[B(r+1,x+m+1)-B_\theta(r+1,x+m+1)]\right\}$ $for\ x=0,1,2,\dots.\ \ ;m>0\,;r>0$
Triangular distribution $g(p)=\begin{cases} \frac{2p}{\theta} & 0<p<\theta \\ \frac{2(1-p)}{1-\theta} & \theta<p<1 \\ 0 & elsewhere \end{cases}$	Explicit	$f(x)=\frac{2\Gamma(r+x)}{x!\,\Gamma r}\left\{\frac{B_\theta(r+2,x+1)}{\theta}+\frac{B(r+1,x+2)-\ B_\theta(r+1,x+2)}{1-\theta}\right\}$ $for\ x=0,1,2,\dots\dots;0<\theta<1,r>0$
	Methods of moments	1st Form $f(x)=\frac{\Gamma(r+x)}{\Gamma(r)!\,(x)!}\sum_{k=0}^{x}\binom{x}{k}(-1)^{x-k}\frac{2(1-\theta^{r+x-k+1})}{(r+x-k+1)(r+x-k+2)(1-\theta)}$ 2nd Form $f(x)=\frac{\Gamma(r+x)}{\Gamma(r)!\,(x)!}\sum_{k=0}^{x}(-1)^k\binom{x}{k}\frac{2}{(1-\theta)}\left[\frac{1-\theta^{r+k+1}}{(r+k+1)(r+k+2)}\right]$

A framework for the mixtures



This is the framework applied in the project. The mixing distributions are classified into two categories, namely, the **continuous and the discrete** mixing distributions. Constructions of these mixing distributions were first carried out before being incorporated in the mixture with the Negative Binomial distribution. Some of the properties, estimations and applications of the new distributions were highlighted.

Recommendations

Negative Binomial mixtures can take other dimensions that were not actually covered in this project. More work can be done on these other scopes. They include:

a. Developing negative Binomial mixtures from the second definition of the Negative Binomial distribution such that

$$f(x) = \int_0^1 \binom{x-1}{r-1} p^r (1-p)^{x-r}\, g(p)dp \qquad x = r, r+1, r+2, \ldots$$

b. Transforming the parameter$p = e^{-\lambda}$so that the new distribution take the format

$$f(x) = \int_0^\infty \binom{r+x-1}{x} e^{-\lambda r} \left(1 - e^{-\lambda}\right)^x g(\lambda)d\lambda$$

$$x = 0,1,2,3 \ldots.; r, \lambda > 0$$

$$f(x) = \int_0^1 \binom{x-1}{r-1} e^{-\lambda r} \left(1 - e^{-\lambda}\right)^{x-r} g(\lambda)d\lambda$$

$$for\ \ x = r, r+1, r+2, \ldots \lambda > 0$$

c. Transforming the parameter $p = 1 - e^{-\lambda}$such that the new distribution takes the form

$$f(x) = \int_0^\infty \binom{r+x-1}{x} e^{-\lambda x} \left(1 - e^{-\lambda}\right)^r g(\lambda)d\lambda$$

$$x = 0,1,2,3 \ldots.; r, \lambda > 0$$

$$f(x) = \int_{0}^{1} \binom{x-1}{r-1} e^{-\lambda(x-r)} \left(1 - e^{-\lambda}\right)^{r} g(\lambda) d\lambda$$

$$for \;\; x = r, r+1, r+2, \dots \lambda > 0$$

We can generate the properties, estimate parameters and verify the identities of the new distributions.

REFERENCE

Xiaohu, L, Yanyan, H., & Xueyan, Z. (2011). The Kumaraswamy Binomial Distribution. *Chinese Journal of Applied Probability and Statistics, Vol 27, No 5, 511 – 521.*

Nadarajah, S., and Kotz, S. (2007) Multitude of Beta distributions with applications, Statistics: *A Journal of Theoretical and Applied Statistics, Vol 41, No 2, 153 – 179.*

Nadarajah, S., Hajebi and M., Rezaei, S. (2012). An Exponential – Negative Binomial Distribution: *Statistical Journal, Vol 11, No 2, 191 – 210.*

Furman, E.(2007) The convolution of the Negative Binomial Random Variables: *Statistics and Probability letters.* Vol 77, 169 – 172.

Ghitany, M. E., Al-Awadhi S. A., and Kalla S. I. (2001): On Hypergeometric Generalized Negative Binomial distribution. *Hindawi Publishing Corp. IJMMS 29:12, 29:12, 727 – 736.*

Gerstenkorn, T. (2004). A Compound of the Generalized Negative Binomial distribution with Generalized Beta distribution. *Central European Journals, CEJM 2 (4), 527 – 537.*

Armero, S., and Bayarri, M. J. (1994). Prior Assessments for Prediction in Queues. *The Statistician 43, 139 – 153.*

Bhattacharya, S. K. (1968). Bayes Approach to Compound Probabilities arising from Truncated Mixing Densities. *Annals of the Institute of Statistical Mathematics, Vol 20, No. 1, 375 -381.*

Wang, Z. (2010). One mixed negative binomial distribution with application. *Journal of Statistical Planning and Inference 141, 1153 – 1160*

Dorp, J. R., and Kotz, S. (2003). Generalization of Two – Sided Power Distribution and their convolution. *Communication in Statistics – Theory and Methods, Vol 32, Issue 9, 1703 – 1723.*

Kotz, S., Johan, and Dorp, V. (2004) - Beyond Beta: Other Continuous families of Distributions with Bounded Support and applications*: World Scientific Publication Company. 308*

Sarabia, J. M. and Gomez-D. (2008). Construction of multivariate distributions: a review of some recent results. *Sort 32 (1), 3 – 36.*

Feller. W.(1957): *An introduction to probability theory and its application, Volume 1. New York: John Wiley and Sons*

Walck, C. (2007): A Hand Book on Statistical Distributions for Experiments*: Internal Report, Universitet Stockholms: SUF – PFY/96-01*