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On Quasi-Similarity and Putnam- Fuglede Property of Operators In Hilbert Spaces

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ABSTRACT
The problem of finding conditions under which two given operators say A and B have equal spectra has been considered by several authors. In this paper we show that quasi-similar operators A and B which satisfy the Putnam- Fuglede property have equal spactra and also equal essential spectra.
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1.INTRODUCTION

Let H be a complex Hilbert space and $B(H)$ denote the Banach algebra of all bounded linear operators on H. An operators $A \in B(H)$ is said to be a quasiaffinity if A is both one-one and has dense range.The two operators A and B are said to be similar if there exist an invertible operator S such that $AS = SB$, while A *and* B are said to be quasi-similar if there exist quasi-affinities X *and* Y such that $AX = XB$ *and* $BY = YA$.

The concept of quasi-similarity and equality of spectra for a given pair of operators has been considered by a number of authors among them W.C. Clary [1] who showed that quasi-similar hyponormal operators have equal spectra. J.M. Khalagai and B. Nyamai [6] showed that if A *and* B quasi-similar operators with A dominant and B^* M.hyponormal then A *and* B have equal spectra. B.P.Dugga [4] showed that if A_i, $i = 1,2$ are quasi-similar operators such that U_i is unitary in the polar decomposition $A_i = U_i|A_i|$ then A_1 and A_2 have equal spectra and also equal essential spectra. J.P Williams [8] and [9] showed that there are several cases which imply that two operators A and B have equal essential spectra under quasi-similarity. For example if A and B are both hyponormal or are both partial isometrics or are quasi-normal.

In this paper we prove results on equality of spectra and essential spectra for classes of operators that satisfy the Putnam-Fuglede property.

Theorem *A* *(Putnam-Fuglede)*

Let $A, B \in B(H)$ be normal operators . For any other operator X we have that $AX = XB$ implies $A^*X = XB^*$.

The following result by J.M. Khalagai and B.Nyamai [6] will be required.

Theorem B

Let A,B and X be operators such AX=XB implies $A^*X = XB^*$. If X is either one-one or has dense range then A and B are normal operators.

The following result by R.G. Douglas [2] will also be required for the proofs of our results.

Theorem C

Let $A, B \in B(H)$ be quasi similar normal operators. Then A and$B$ are unitarily equivalent and hence have equal spectra.

2.NOTATION AND TERMINOLOGY

Given an operator $A \in B(H)$ we denote the numerical range of A by $W(A)$. Thus $W(A) = \{< Ax, x >: \|x\| = 1\}$

The spectrum of A is denoted by $\sigma(A)$.Thus $\sigma(A) = \{\lambda \epsilon \mathbb{C} : A - \lambda I$ is not invertible$\}$

Where $\mathbb{C}$ is the complex number field. The commutator of A and B is denoted by $[A, B]$ where $[A, B] = AB - BA$

An operator A is said to be dominant if to each $\boldsymbol{\lambda} \in \not\subset$ there corresponds a number $M_{\boldsymbol{\lambda}} \geq 1$ such that $\|(A - \lambda)^*x\| \| \leq M_\lambda \|(A - \lambda)x \left\|(A - \lambda) \ \forall x \in H \right\|$

M-hypo-normal if $\exists M \geq M_\lambda$ for all $\boldsymbol{\lambda}$ in the definition of dominant operator .i.e $\|(A - \lambda)^*x\| \leq M\|(A - \lambda)x\| \forall x \in H$

Hyponormal if $A^*A \geq AA^*$
Quasinormal if $[A^*A, A] = 0$
P-hypo-normal if for $0 < p \leq 1, (A^*A)^p \geq AA^*)^p$
Normal if $[A, A^*] = 0$
Self ad joint if $A = A^*$
Partial isometris if $A = AA^*A$
Isometry if $A^*A = I$
Unitary if $A^*A = AA^* = I$
Fredholm if its range denoted by ranA is closed and both null space, ker$A$ and kerA^* are finite dimensional.
The essential spectrum of A is denoted by $\sigma_e(A)$.
Thus $\sigma_e(A) = \{ \lambda \in \not\subset : A - \lambda I$ is not fredholm$\}$
The following operator inclusions are proper.
Normal $\subset$ hyponormal $\subset$ p-hypo-normal and Hypo-normal $\subset$ M-hyponormal $\subset$ Dominant.

3.RESULTS

Theorem 1
Let $A, B, X \in B(H)$ be operators such that:
$AX = XB$ where A *and* B satisfy Putnam –Fuglede property and X is a quasi-affinity.
Then we have:
i. $\sigma(A) = \sigma(B)$
ii. $\sigma(AA^*) = \sigma(BB^*)$
iii. $\sigma(A^*A) = \sigma(B^*B)$

Proof

We note that $AX = XB$._____(1) implies $A^*X = X^*B$. On taking adjoints we have
$BX^* = X^*A$___________(2).
Since X is a quasiaffinity it follows from (1) and (2) that A *and* B are quasi-similar. It now follows from both theorem B and C above that A and B are unitarily equivalent normal operators. Hence $\sigma(A) = \sigma(B)$.
Also using the equations $AX = XB,\ A^*X = XB^*, BX^* = X^*A$ and $B^*X^* = X^*A^*$ we get
$A^*AX = A^*XB = XB^*B$____________ (3)
and
$B^*BX^* = B^*X^*A = X^*A^*A$____________(4)
Also $AA^*X = AXB^* = XBB^*$____________(5)
and
$BB^*X^* = BX^*A^* = X^*AA^*$____________(6)
From (3)and (4) A^*A and B^*B are quasi similar positive operators. Hence $\sigma(A^*A) = \sigma(B^*B)$.
Also from (5) and (6) A^*A and B^*B are quasi-similar positive operators.
Hence $\sigma(AA^*) = \sigma(BB^*)$.
We note that from the results by J.P Williams [8] the following corollary is immediate.

Corollary 1
Let A,B, X$\in$B(H) be operators such that AX=XB where X is quasi-affinity then
$\sigma(A) = \sigma(B), \sigma(A^*A) = \sigma(B^*B)$ and $\sigma(AA^*) = \sigma(BB^*)$ under any one of the following conditions:
i. A is dominant and B^* is m-hyponomal
ii. A is dominant and B^* is p-hyponomal
iii. A and B^* are p-hyponomal

Proof
All the classes of operators stated above are known to satisfy Putnam-Fuglede property.(see[3])
Hence results follow from theorem 1 above.
.
Corollary 2
Let A,B, X$\in$B(H) be operators such that
AX=XB
where X is quasi-affinity then
$\sigma(A) = \sigma(B), \sigma(A^*A) = \sigma(B^*B)$ and $\sigma(AA^*) = \sigma(BB^*)$ under any one of the following conditions:

i. A is dominant and B^* is m-hyponomal
ii. A is dominant and B^* is p-hyponomal
iii. A and B^* are p-hyponomal

Proof
We note that all the classes of operators stated above are known to satisfy Putnam-Fuglede property and exist quasi-affinites X and Y such that AX=XB and BY=YA. We also note that in case of part,(iii),if in addition we have that in the polar decomposition A=V$|A|$ and B=V$|B|$.

U and V are unitary (see[4]) then $\sigma_e(A) = \sigma_e(B)$.
For an operator B$\in$B(H) we say that B is consistent in invertibility (with respect to multiplication) or briefly that B is a CI operator if for each A$\in$B(H), AB and BA are invertible or non-invertible together. Thus B is CI operator if $(\sigma AB) = \sigma(BA)$. W.Gong and.Han [5]) proved among other results that an operator B$\in$B(H) is CI operator iff $\sigma(B^*B) = \sigma(BB^*)$.
From this result the following corollary is immediate.

Corollary 3

Let $A, B, X \in B(H)$ be operators such that:
$AX = XB$ where A *and* B satisfy Putnam –Fuglede property and X is a quasi-affinity.
Then we have:
i. $\sigma(A) = \sigma(B)$
ii. $\sigma(AA^*) = \sigma(BB^*)$
iii. $\sigma(A^*A) = \sigma(B^*B)$

Proof
In this case there exists two quasi-affinities say X andY such that AX=XB and BY=YA
Thus the proof of theorem 1 can be traced to yield the result.

Corollary 4

Let $A, B, X \in B(H)$ be operators such that:
$AX = XB$ where A *and* B satisfy Putnam –Fuglede property and X is a quasi-affinity.
Then we have:

i. $\sigma_e(A) = \sigma_e(B)$
ii. $\sigma_e(AA^*) = \sigma_e(BB^*)$
iii. $\sigma_e(A^*A) = \sigma_e(B^*B)$

Corollary 5

Let $A \in B(H)$ be quasi-similar operators which satisfy Putnam-Fuglede property. Then we have that B is a $C.I$ operator whenever A is and conversely.

Proof
We note that for such operators A and B.
$\sigma(AA^*) = \sigma(BB^*)$ and $\sigma(B^*B) = \sigma(A^*A)$.
Thus if A is CI operator then $\sigma(A^*A) = \sigma(AA^*) = \sigma(B^*B) = \sigma(BB^*)$. Hence A is a C.I operator.
In our next result we show that by imposing more stringent conditions on the intertwining we obtain equality of operators rather than equality of their spectra. In so doing we require the following result by I. H Sheth and J.M Khalegai [7]

Theorem D

Let $T, S, X \in B(H)$ be operators such that.
$TX = XS$ and $SX = XT$ with $O \notin W(X)$ where $S - T$ is normal.

Then $S = T$
We have the following result in this direction.

Theorem 2

Let $A, B, X \in B(H)$ be operators such that A and B satisfy Putnam-Fuglede property and $AX = XB$ with X self adjoint and $0 \notin W(X)$. then $A^*A = B^*B$ and $AA^* = BB^*$

Proof

We first note that $AX = XB \Longrightarrow A^*X = XB^*$
Thus taking adjoints we also have $BX = XA$.
Thus using these equations we have: $A^*AX = A^*XB = XB^*B$ and
$$AA^*X = AXB^* = XBB^*$$
Now letting $T = A^*A$ and $S = B^*B$
we have that $S - T = B^*B - A^*A$ is normal.
Hence by theorem D above
$B^*B = A^*A$
Similarly if $T = AA^*$ and $S = BB^*$
then $BB^* - AA^*$ is normal.
Hence $BB^* = AA^*$

REMARKS

1.We note that $A = B$ implies $A^*A = B^*B$ and $AA^* = BB^*$
However the converse is not necessarily true.
2.We also note that in theorem 2 above we can replace the condition that A and B satisfy Putnam- Fuglede theorem by simply requiring that A and B are normal operators in view of theorem B and the fact that $0 \notin W(X)$.

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