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ON REDUCIBILITY AND QUASIREDUCIBILITY OF OPERATORS IN HILBERT SPACES

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Declaration

This dissertation is my original work and has not been presented for a degree award in any other university.

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This dissertation has been submitted for examination with my approval as the university supervisor.

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Dedication

I dedicate this dissertation to my spouse Stephen and my son Alfons.

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The beginning looked so thin, the journey too long and success was therefore beyond my abilities. But I thank my God who has carried me through.
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Abstract

In this dissertation, we study invariant, reducing and hyperinvariant subspaces and how they play a key role in the study of reducibility and quasireducibility of operators. We also consider some equivalence relations and characterize operators in such equivalence relations and investigate which of the equivalence relations preserve reducibility and quasireducibility. The structure and relationship of invariant and hyperinvariant lattices for some classes of operators are investigated. The isomorphic lattices of similar and unitarily equivalent operators are also discussed.

Abbreviations and Symbols

The following are the lists of abbreviations and symbols used in this paper.
$B(H)$: Banach algebra of bounded linear operators
H : Hilbert space over the complex number $\mathbb{C}$
T^* : the adjoint of T
$\|T\|$: the operator norm of T
$\|x\|$: the norm of a vector x
$\rho(T)$: the resolvent set of an operator T
$\sigma(T)$: the spectrum of an operator T
$\sigma_p(T)$: the point spectrum of an operator T
$\sigma_c(T)$: the continuous spectrum of an operator T
$\sigma_R(T)$: the residual spectrum of an operator T
$Ran(T)$: the range of an operator T
$Ker(T)$: the kernel of an operator T
$M \oplus M^{\perp}$: the direct sum of the subspaces M and $M^{\perp}$
$\{T\}'$: the commutant of T
D_T : the self commutator of T
$D(T)$: the domain of an operator T
Lat(T) : lattice of an operator T
HyperLat(T) : hyperlattice of an operator T
Red(T) : set of reducing subspaces of T
WOT : weak operator topology
SOT : strong operator topology
UOT : uniform operator topology

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Chapter 1

PRELIMINARIES

This chapter, summarizes the background required for the rest of the study. Its purpose is fourfold: introduction, notation and terminology and basic results that will be needed in the sequel.

1.1 Introduction

The notion of quasireducibility of operators is the same as reducibility in finite dimensional space, but in infinite dimensional spaces it is much weaker relation. Clary (1978) [3] proved that quasi-similar hyponormal operators have equal spectra. This claim was supported by Douglas(1969),[4], who proved using the Putnam-Fuglede Commutativity Theorem that quasi-similar normal operators are unitarily equivalent and hence have equal spectra. Pearson [1898] gave the first modern definition of a linear operator. In the theory of linear operators, the major problem is that of approximating the various classes of linear operators by operators of comparatively simple structures such as self-adjoint and normal operators. The original model for operator theory is the study of matrices. Toeplitz [1909] found out that every linear operator can be represented by a matrix thus making it easy to analyze it.
In the process of finding normal forms of quadratic functions, Cauchy in [1826] discovered eigenvalues and generalization of square matrices. Cauchy (1826) [2] also proved the spectral theorem for self-adjoint matrices. That is, every real symmetric matrix is diagonalizable. This spectral theorem for Hermitian matrices was later generalized into spectral theorem for normal operators by Neumann (1942), that is every normal operator is diagonalizable.
The concept of quasireducibility of operators was introduced by Kubrusly

in (2002) [15]. He illustrates basic properties and examples in order to situate the class of quasireducible operators in their place. In particular he has shown that every quasinormal operator is quasireducible. The results linked quasireducible operators with the invariant subspace problem; essentially normal quasireducible operators have non trivial invariant subspace. Kubrusly on the other hand stated the elementary facts about quasireducibility and stated that quasireducibility (as reducibility) is preserved under unitary equivalence. He has also shown the relationship between quasireducibility (reducibility) and similarity. Nilpotent operators of different indices are also discussed and related to quasireducibilty and reducibility.
Luo Yi Shi (1987) stated that every operator on a (separable) Hilbert space is the direct integral of irreducible operators but not every operator can be expressed as direct sum of irreducible operators. For instance it can be easily seen that a normal operator is irreducible if and only if it acts on a one-dimensional space and thus it is the direct sum of irreducible operators if and only if it is diagonalizable. Peter Rosenthal (1968) introduced the concept of completely reducible operators. He showed that a bounded linear operator T on a Hilbert space H is completely reducible if whenever M is a reducing subspace of T of dimension greater than one, the operator $T\mid_M$ has a nontrivial reducing subspace. The spectral theorem implies every normal operator is completely reducible. If H is finite dimensional then every completely reducible operator is normal. This is not the case in general.
The study of reducing, invariant and hyperinvariant subspaces plays a vital role in this research project. An operator T with non-trivial reducing subspace or equivalently if it is the direct sum of two operators on nonzero subspace,then T is said to be reducible. The spectral theorem implies every normal operator is completely reducible. If H is finite dimensional then every completely reducible operator is normal. This is not the case in general.
The invariant subspaces of an operator,their classification and description plays a vital role in operator theory. The invariant subspace of a given linear operator T sheds light on the structure of T. When a Hilbert space H is finite dimensional over algebraically closed field, operators acting on H are characterized by the Jordan canonical form which decomposes H into invariant subspaces of T. The problem of invariant subspace is unsolved yet. There are operators without an invariant subspace due to Per Enflo (1976). A concrete example of an operator without invariant subspace was produced in 1985 by Charles Read. On the other hand, invariant subspaces are defined for sets of operators as subspaces invariant for each operator in the set. Let $B(H)$ denote the algebra of linear operators in H and $Lat(T)$ be the family of subspaces invariant under $T \in B\ (H)$.
The knowledge of hyperinvariant subspaces of an operator T gives informa-

tion on the structure of the commutant of T. Hoover (1973) [10] studied hyperinvariant subspaces and proved the result that if S and T are quasisimilar operators on a Hilbert space H and K, respectively and if in addition S is normal, then the lattice of hyperinvariant subspaces for T contains a sublattice which is lattice isomorphic to the lattice of spectral projection for S. Fillmore, Herrero and Longstaff (1977) [6] showed that in a finite dimensional space H, $HyperLat(T)$ is (lattice) generated by those subspaces which are either $Ker_p(T)$ or $Ran_q(T)$, where p and q are polynomials. Herrero(1969)[4] proved that the structure of hyperlattice of an operator is not preserved under quasisimilarity by giving an example of an operator T such that $T^3 = 0$ which is quasisimilar to all Jordan nilpotent operator of order 3 but $HyperLat(T)$ has five elements while such Jordan nilpotent operators have $4, 6$ or 8 elements.

1.2 Notations and Terminologies

Throughout this study Hilbert spaces are nonzero complex and separable. In principle, they may be finite or infinite dimensional. Capital letters H,K etc denotes Hilbert spaces or subspaces of Hilbert spaces and T, S, A, B etc denotes bounded linear operators.

$B(H)$ denotes the Banach algebra of bounded linear operators on H. The subalgebra of all operators generated by an operator $T \in B(H)$ denoted by $W^*(T)$ is called the weakly closed (von Neumann) algebra generated by T. $B(H, K)$ denotes the set of bounded linear operators from H to K equipped with the norm. By an operator we mean a bounded linear transformation (equivalently a continuous linear transformation $T : H \rightarrow K$.

A linear operator is said to be bounded if its domain is the whole vector space. That is , if a linear operator acting on a Hilbert space H has a matrix representation T, then such an operator is bounded if $D(T) = H$ and there exists a positive real number m such that $||Tx|| = m||x||$, for every vector $x \in H$ where $D(T)$ denotes the domain of T.

If T is the matrix representation for a given linear operator on H, and the action of this T to a vector say, x in H is equivalent to multiplying such an x by a number say, λ, that is $Tx = \lambda x$, then x is called an eigenvector of T corresponding to an eigenvalue λ. The spectrum of a linear operator on a finite-dimensional Hilbert space is the set of all its eigenvalues. The set of all

λ such that (λI- T) has a densely defined continuous inverse is the resolvent set of T, denoted by $\rho(T)$. The complement of $\rho(T)$ denoted by $\sigma(T)$ is the spectrum of T. The set of those λ such that (λI - T) has no inverse is the point spectrum denoted by $\sigma_p(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) \neq \{0\}\}$, which is the set of all eigenvalues of T. The set of those λ for which (λI - T) has a densely defined but unbounded inverse is the continuous spectrum denoted by $\sigma_c(T)$. Thus $\sigma_c = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}, \overline{Ran(\lambda I - T)} = H$ and $Ran(\lambda I - T) \neq H$. If ($\lambda I$ - T) has an inverse that is not densely defined, then λ belongs to the residual spectrum denoted by σ_R. That is $\sigma_R = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}, \overline{Ran(\lambda I - T)} \neq H\}$.
Note that, the parts $\sigma_p(T)$, $\sigma_c(T)$, and $\sigma_R(T)$ are pairwise disjoint and $\sigma(T)$ = $\sigma_p(T) \bigcup \sigma_c(T) \bigcup \sigma_R(T)$.
An operator T is quasinilpotent if $r(T) = 0$ where $r(T)$ denotes the spectral radius of T The numerical range $W(T)$ of an operator T is defined as $W(T)$ = $\{\lambda \in \mathbb{C} : \lambda = \langle Tx, x\rangle, ||x|| = 1$. The numerical radius $w(T)$ of T is defined as $w(T) = Sup\{|\lambda| : \lambda \in W(T)\}$. An operator T is spectraloid if $r(T) = w(T)$ and normaloid if $r(T) = ||T||$ or equivalently $w(T) = ||T||$. Thus every normaloid operator is spectraloid.
T^* denotes the adjoint of $T.\{T\}'$ denotes the commutant of T. D_T denotes the self commutator of T. Ker(T), Ran(T), $M^\perp$ stands for Kernel, range and orthogonal complement of a closed subspace M of H, respectively. By a subspace of a Hilbert space H we mean a closed linear manifold of H which also is a Hilbert space.
An operator $X \in B(H, K)$ intertwines $A \in B(H)$ to $B \in B(K)$ if $XA = BX$.If A is densely intertwined to B, then there exists an operator with dense range intertwining A to B.
$\{O\}$, I denotes the null and identity operators,respectively. If M and N are orthogonal (denoted by $M \perp N$) subspaces of H then, their (orthogonal) direct sum $M \oplus N$ is a subspace of H. Two operators T and S are called orthogonal (denoted by $T \perp S$) if $TS = \{0\}$ (zero operator).
For M a closed subspace of H, we have $H = M \oplus M^\perp$ which is called the direct sum decomposition of H. This justifies the notation $M^\perp = H \ominus M$.
An operator $T \in B(H)$ is said to be;
normal if $T^*T = TT^*$
self adjoint (or Hermitian) if $T = T^*$
unitary if $T^*T = TT^* = I$
an isometry if $T^*T = I$
a co-isometry if $TT^* = I$
a partial isometry if $T = TT^*T$
quasinormal if $T(T^*T)= (T^*T)T$
binormal if T^*T and TT^* commutes

subnormal if it has a normal extension,that is if their exist a normal operator N on a Hilbert space K such that H is a subspace of K and the subspace H is invariant under the operator N and the restriction of N on H concides with T. Note that a part of an operator is the restriction of it to an invariant subspace.
hyponormal if $T^* T \geq T T^*$
scalar if it is a scalar multiple of the identity operator i.e $T = \propto T$ where $\propto \in R$ or $\mathbb{C}$.
contraction if $\|T\| \leq 1$ that is $\|Tx\| \leq \|x\|$ for every x in H
Two operators $T \in B(H)$ and $S \in B(K)$ are said to be similar (denoted by $T \simeq S$) if there exists an invertible operator X such that $XT = SX$, (i.e $T = X^{-1} S X$)
Two operators $T \in B(H)$and $S \in B(K)$ are said to be unitarily equivalent if there exists a unitary operator U such that $T = U^* S U$
An operator $T \in B(H, K)$ is said to be quasiinvertible or quasi-affinity if it is an injective operator with dense range i.e $\text{Ker}(X) = \{0\}$ and $\overline{Ran(X)} = H$
An operator $T \in B(H)$ is a quasiaffine transform of another operator $S \in B(K)$ if there exists a quasi invertible operator $X \in B(H, K)$ such that $XT = SX$.
Two operators $T \in B(H)$ and $S \in B(K)$ are said to be quasi similar (denoted by $T \sim S$) if they are quasi-affine transforms of each other, (i.e if there exists quasiaffinities $X \in B(H, K)$ and $Y \in B(K. H)$ such that $XT = SX$ and $YS = TY$).
Two operators A and B in $B[H]$ are said to be metrically equivalent (denoted by $A \sim_m B$) if $\| Ax \| = \| Bx \|$ for all x in H.
An operator P on a Hilbert space H is idempotent if $P = P^2$. If P is idempotent then $Ran(P) = Ker(I - P)$ so that $Ran(P)$ is a subspace of H.
A projection or orthogonal projection is an idempotent operator P such that $Ker(P) \perp Ran(P)$. If P is a projection on H, then the $Ran(P) = Ker(P)^{\perp}$ so that $H = Ker(P) \oplus Ran(P)$. Conversely, if M is a subspace of H, then there exist a unique projection $P : H \to M$ such that $Ran(P) = M$. This is called the projection onto M. Therefore, associated with the decomposition $H = M \oplus M^{\perp}$,there exists a unique projection P on H such that $M = Ran(P)$ and $M^{\perp} = Ker(P)$.
$T^* \in B[K, H]$ stands for the adjoint of $T \in B[H, K]$.Then, $Ker(T^*) = Ran(T)^{\perp}$ so that $\overline{Ran(T^*)} = Ker(T)^{\perp}$.
An operator $T \in B[H, K]$ has an inverse T^{-1} such that $Ran(T) \subseteq K \to H$ on its range (not necessarily bounded but certainly linear if and only if $Ker(T) = \{0\}$.) An important corollary to the Open Map Theorem(" a surjective bounded linear transformation between Banach spaces maps open sets into

open sets") says that such an inverse is bounded if and only if $Ran(T)$ is closed in K (i.e if and only if $Ran(T) = \overline{Ran(T)}$.
$T \in B[H, K]$ is invertible if it has an inverse on $Ran(T) = K$, and such an inverse must be bounded (i.e $T^{-1} \in B[K, H]$).
Let $\Im[H, K]$ denote the class of all invertible operators in $B[H, K]$. If $T \in \Im[H, K]$ then $T^* \in \Im[K, H]$ and $T^{*-1} = T^{-1*} \in \Im[H, K]$. An invertible operator for which $T^{-1} = T^*$ is called unitary or an isomorphism.
A subspace M of a Hilbert space H is a closed linear manifold of H. If T is an operator in H and $T(M) \subseteq M$, then M is invariant for T (or M is T-invariant). An invariant linear subspace for an operator T is linear subspace of H that as a subset of H is invariant for T. It is clear that 0 and the whole space H are invariant subspace for every T in $B(H)$. It is nontrivial if $0 \neq M \neq H$.If M is invariant subspace for T then its orthogonal complement $M^{\perp} = H \ominus M$ is a nontrivial invariant subspace for the adjoint T^* of T.
Let T be an operator on a Hilbert space H and let M be a subspace of H. If M and its orthogonal complement $M^{\perp}$ are both invariant for T (i.e $T(M) \subseteq M$ and $T(M^{\perp}) \subseteq M^{\perp}$) then we say that M reduces T (or M is a reducing subspace for T). If M reduces T and $\{0\} \neq M \neq H$,then M is a nontrivial reducing subspace for T.
An operator T is reducible if it has a nontrivial reducing subspace (equivalently, it it has a proper nonzero direct summand). (That is if there exists a subspace M of H such that M and $M^{\perp}$ are nonzero and T-invariant or equivalently if M is nontrivial and invariant for both T and T^*). Equivalently, an operator T is reducible if and only if there exists a nonscalar operator $L \in \{T\}' \bigcap \{T^*\}'$.Equivalently T is reducible if and only if both T and T^* lie in $\{L\}'$ for some nonscalar operator L.
An operator T is quasireducible if there exist a nonscalar operator L such that, $LT = TL$, rank($(T^*L - LT^*)\, T$)- $(T(T^*L - LT^*)) \leq 1$. In other words , T is quasireducible if there exist a nonscalar L such that either $T^*L - LT^*$ also lies in $\{T\}'$ or the commutator $[(T^*L - LT^*), T]$ is of rank one.
Recall that nonscalar operators exist only on spaces of dimension greater than one and so the concept of quasireducibility (and reducibility) is germane to operators on Hilbert spaces of dimension greater than one.Note also that rank means dimension of range and the only operator with rank zero is the null operator. Since one-dimensional linear space has no nontrivial subspace, we assume that all operators act on a(complex separable) Hilbert space H of dimension greater than one.
An operator T is finite dimensional if $Ran(T)$ is finite dimensional. An operator T is compact if $\{Tx \in K : ||x|| \leq 1\}$ has compact closure in K. An operator T is called essentially normal if it has a compact self-commutator D_T.

The weak operator topology (WOT) on $B(H)$ is the topology generated by collection $\{T \to |\langle T(x), y\rangle| : x, y \in H\}$ of seminorms. Equivalently it is the smallest topology in which all bounded linear operators on a vector space V are continuous. The strong operator topology (SOT) on $B(H)$ is the topology generated by the collection $\{T \to ||\mathrm{T}_n\ ||: x \in H\}$ of seminorms. Equivalently SOT is the topology of pointwise convergence.
The set of all subspaces of a finite-dimensional Hilbert space H is a lattice (with respect to the operations of intersection and span) with zero element $\{0\}$ as the least element and H as the greatest element. The subspace lattice of all invariant, reducing and hyperinvariant subspaces of T is denoted by $Lat(T)$, $RedT$, and $HyperLatT$, respectively.
Let L be either $Lat(T)$ or $HyperLat(T)$. If $\phi : L \to L$ is an isomorphism (automorphism), then $\phi(M_1 \bigvee M_2) = \phi(M_1) \bigvee \phi(M_2)$ and $\phi(M_1 \bigwedge M_2) = \phi(M_1) \bigwedge \phi(M_2)$ for all $M_1, M_2 \in L$ and where $\bigvee$ denotes *span* and $\bigwedge$ denotes *intersection*, $\bigcap$.
Let $\mathfrak{R}$ and $\mathfrak{L}$ be lattices and $\phi : \mathfrak{R} \to \mathfrak{L}$ be a map. We call ϕ an isomorphism if it is one-to-one and onto and $a \leq b$ if and only if $\phi(a) \leq \phi(b)$ for all $a \in \mathfrak{R}$, $b \in \mathfrak{L}$. As we noted earlier that lattices refers to $Lat(T)$ or $Hyperlat(T)$, then we can define isomorphism as follows. Let L_1 and L_2 be lattices of subspaces of H. An isomorphism ϕ is a one-to-one and onto map with the property that if $M_1, M_2 \in L_1$ then $M_1 \subseteq M_2$ if and only if $\phi(M_1) = \phi(M_2)$.
An operator T is *reductive* if all its invariant subspaces are reducing. Equivalently, T is reductive if and only if $Lat(T) = Lat(T^*)$.

1.3 Basic Results

Here are useful results that will be required in the sequel.

Proposition 1.3.1. *([14], Proposition 1.3) If an operator T acting on a Hilbert space H is quasireducible, then*
(a) λT is quasireducible for every $\lambda \in \mathbb{C}$
(b) $\lambda I + T$ is quasireducible for every $\lambda \in \mathbb{C}$
(c) T^* is quasireducible.

Let M be a non empty subset of a Hilbert space H. The orthogonal complement of M denoted by $M^\perp = x \in H : x \perp y$, for every $y \in M$. Thus $M^\perp$ is the set of all those vectors in H which are orthogonal to every vector in M.

Theorem 1.3.1. *(Orthogonal Decomposition Theorem [22]) If M is a closed linear subspace of a Hilbert space H, then $H = M \oplus M^\perp$.*

Proof. Since M is a subspace of H, $M^{\perp}$ is not only closed but also $M \bigcap M^{\perp}$ $= \{0\}$. Hence in order to show that $H = M \oplus M^{\perp}$, it is enough if we verify $H = M + M^{\perp}$ since M and $M^{\perp}$ are closed subspaces of H, $M + M^{\perp}$ is also a closed subspace of H.
Let us take $N = M + M^{\perp}$ or show that $N = H$. From the definition of N we get $M \subset N$ and $M^{\perp} \subset N$. Then we have that $N^{\perp} \subset M$ and $N^{\perp} \subset M^{\perp\perp\perp}$. Hence $N^{\perp} \subset M^{\perp} \bigcap M^{\perp\perp} = \{0\}$. Hence $N^{\perp} = \{0\}$, thus

$$N^{\perp\perp} = \{0\}^{\perp} = H \tag{1.1}$$

since $N = M + M^{\perp}$ is a closed subspace of H, we have

$$N^{\perp\perp} = N \tag{1.2}$$

From (1.1) and (1.2) we have $N = M = M^{\perp} = H$.

Theorem 1.3.2. *Unitary equivalence is an equivalence relation.*

Proof. We show that unitary equivalence is (i) reflexive, (ii) symmetric and (iii) transitive.
(i) Reflexive i.e $T \cong T$.
Let $T \in B[H]$. Then $UT = TU = T = UTU^*$ where U is a unitary operator. Hence $T \cong T$ (without loss of generality) let $U = I$.
(ii) Symmetric, i.e $T \cong S \Rightarrow S \cong T$.
Now suppose that $T \cong S$. We show that $S \cong T$. Let $T \in B[H]$ and $S \in B[K]$. There exist a unitary operator $U \in B[H, K]$ such that

$$T = U^*SU \tag{1.3}$$

Pre-multiplying (1.3) by U and post-multiplying the same by U^*, it gives
$UTU^* = UU^*SUU^*$, i.e
$UTU^* = ISI = S$. Hence $S \cong T$.
(iii) Transitivity i.e if $T \cong S$ and $S \cong V$ then $T \cong V$.
Suppose $T \cong S$ and $S \cong V$. Then there exists unitary operator $U_1 \in \Im(H, K)$ and $U_2 \in \Im(K, J)$ where J is a Hilbert space such that

$$T = U_1^*SU_1 \tag{1.4}$$

and

$$S = U_2^*VU_2 \tag{1.5}$$

Using (1.4) and (1.5) we have that $T = U_1^*(U_2^*VU_2)U_1 = (U_1^*U_2)V(U_2^*U_1) = UVU$.
where $U = U_2U_1$ is a unitary operator since U_1 and U_2 are unitary . Hence $T \cong V$. □

Theorem 1.3.3. *([19], Theorem 2.6) If T_1,$T_2 \in B(H)$ are quasisimilar with quasiaffines X and Y, then $XY \in \{T_1\}'$ and $YX \in \{T_2\}'$.*

Proof. Suppose that T_1 is quasisimilar to T_2 with quasiaffinities X and Y. Then,

$$T_1X = XT_2 \tag{1.6}$$

and

$$T_2Y = YT_1 \tag{1.7}$$

Post-multiplying the equation (1.6) by Y and using the equation (1.7) we have $T_1XY = XT_2Y = XYT_1$ which proves that $XY \in \{T_1\}'$. Post-multiplying equation (1.7) by X and using equation (1.6) we have $T_2YX = YT_1X = YXT_2$ which proves that $YX \in \{T_2\}'$. □

An operator S_+ acting on a Hilbert space H is a unilateral shift if there exists a sequence of (pairwise) orthogonal subspaces $\{H_k\} : k \geq 0\}$ such that $H = \bigoplus_{k=0}^{\infty} H_k$ and S_+ maps each H_k isometrically onto H_{k+1}. According to this definition, $S_+(H_k) = H_{k+1}$ and $S_+ |_{H_k} : H_k \to H_{k+1}$ is an isometry thus surjective isometry and hence unitary. Therefore H_k and H_{k+1} are unitarily equivalent so that dim H_k = dim H_{k+1} for every $k \geq 0$. Such a common dimension is the multiplicity of S_+.

Theorem 1.3.4. *(von Neumann Double Commutant Theorem) Let H be a Hilbert space and $\mathfrak{A} \subseteq B(H)$ be a unital (self-adjoint) $*$-algebra of $B(H)$. Then the following conditions are equivalent;*

(i) $\mathfrak{A} = \{\mathfrak{A}\}'$

(ii) $\mathfrak{A}$ is closed with respect to the weak operator topology (WOT) on $B(H)$

(iii) $\mathfrak{A}$ is closed with respect to the strong operator topology (SOT) on $B(H)$.

Remark 1.3.5. If $\mathfrak{A}$ satisfies either of these three conditions we say that it is a von Neumann algebra.

Theorem 1.3.6. *(Fuglede-Putnam Theorem [18] pg 35) Let $T \in B(H)$. If $N \in B(H)$ is normal and $NT = TN$, then $N^*T = TN^*$.*

Chapter 2

INVARIANT, HYPERINVARIANT AND REDUCING SUBSPACES

2.1 Invariant Subspaces

The lemma below gives sufficient condition for transferring nontrivial invariant subspaces from B to A whenever A is densely intertwined to B.

Lemma 2.1.1. *[14] Let $A \in B(H)$, $B \in B(K)$ and $X \in B(H, K)$ be such that $XA = BX$. Suppose $M \subset K$ is a nontrivial invariant subspace for B. If $\overline{Ran(X)} = K$ and $Ran(X) \bigcap M \neq \{0\}$ the inverse image of M under X, $X^{-1}(M)$ is a nontrivial invariant subspace of A.*

If the intertwining operator X is surjective i.e $XA = BX$ and $Ran(X) = K$, then $X^{-1}(M)$ is a nontrivial subspace for A. Thus we have the following corollary.

Corollary 2.1.1. *[20] If two operators are similar and if one of them has a nontrivial invariant subspace, then so has the other.*

Corollary 2.1.2. *[20] Take $T \in B(H)$, $L \in B(K)$ and $X \in B(H, K)$ such that $XT = LX$. Let $M \subset K$ be a nontrivial finite-dimensional reducing subspace for L. If $Ran(\bar{X}) = K$, then $X^{-1}(M^{\perp})$ is a nontrivial invariant subspace for T.*

In other words, if an operator T is densely intertwined to an operator L that has a nontrivial finite-dimensional reducing subspace, then T has a nontrivial invariant subspace.

Corollary 2.1.3. *If an operator T is a quasiaffine transform of another operator L that has a nontrivial finite-dimensional reducing subspace, then T has a nontrivial invariant subspace.*

Theorem 2.1.1. *([13], Problem 4.3) Let T be an operator on a Hilbert space H and let M be a subspace of H then the following statements are true.*

(a) M is invariant for T if and only if $M^{\perp}$ is invariant for T^{*}.

(b) M is invariant for every operator that commutes with T if and only if $M^{\perp}$ is invariant for every operator that commutes with T^{*}.

Proof. (a) Take an arbitrary y in $M^{\perp}$. If $Tx \in M$ whenever $x \in M$, then $\langle x; T^*y\rangle = \langle Tx; y\rangle = 0$ and therefore $T^*y \perp M$ which means that T^*y lies in $M^{\perp}$. Conversely since this holds for every operator in $B(H)$, it follows that $T^*(M^{\perp}) \subseteq M^{\perp} \Rightarrow T^{**} \subseteq M^{\perp\perp}$. But $T^{**} = T$ and $M^{\perp\perp} = \bar{M} = M$ and hence $T^*(M) \subseteq M^{\perp} \Rightarrow T(M) \subseteq M$. Summing up: M is invariant for T if and only if $M^{\perp}$ is invariant for T^*.

(b) Let $\{T\}'$ be the commutant of T. It is clear that $L \in \{T\}'$ if and only if $L^* \in \{T^*\}'$. Suppose M is invariant for every operator that commutes with T which means that M is L invariant whenever $L \in \{T\}'$. This implies that $M^{\perp}$ is L^*-invariant whenever $L \in \{T\}'$, according to (a). Thus $M^{\perp}$ is invariant for every operator that commutes with T^*, then $M^{\perp\perp} = \bar{M} = M$ is invariant for every operator that commutes with $T^{**} = T$.

□

Remark 2.1.2. Therefore according to the above result an operator in a Hilbert space has a nontrivial invariant subspace if and only if its adjoint has.

Theorem 2.1.3. *(Lomonosov) An operator has a nontrivial invariant subspace if it commutes with a nonscalar operator that commutes with a nonzero compact operator.*

From Theorem 2.1.3 we can say that every nonscalar compact operator has a nontrivial hyperinvariant subspace . Recall that in infinite-dimensional setting the only scalar compact operator is the null operator. In finite-dimensional setting every operator is compact, every operator has a nontrivial invariant subspace and, if it is nonscalar a nontrivial hyperinvariant subspace as well.

Theorem 2.1.4. *([13], Theorem 4.1) Every essentially normal quasireducible operator has nontrivial subspace.*

Proof. According to Lomonosov Theorem, if a nonscalar operator commutes with a nonzero compact operator, then it has a nontrivial hyperinvariant subspace, that is if an operator L is such that rank$(TL - LT) = 1$ for some compact operator T, then L has a nontrivial hyperinvariant subspace. If there exist a nonscalar L such that $LT = TL$, and rank$(TL - LT) \leq 1$ for some nonzero compact operator T,then the above results ensure that T has a nontrivial invariant subspace. This proves the theorem whenever the self-commutator $D_T = [T^*, T]$ is nonzero and compact. If $D_T = 0$, then T is normal and the result holds trivially. □

Proposition 2.1.1. *[14] Let T and L be nonzero operators on a Hilbert space H. If $LT = 0$, then $Ker(L)$ and $Ran(\bar{T})$ are nontrivial invariant subspaces for both T and L.*

A straight forward corollary to proposition 2.1.1 is as follows.

Corollary 2.1.4. *Every nilpotent operator has a nontrivial invariant subspace.*

If M is an invariant subspace for T, then relative to the decompositio $H = M \oplus M^{\perp}$, T can be written as $T = \begin{pmatrix} T\mid_M & X \\ 0 & Y \end{pmatrix}$ for operators $X : M^{\perp} \to M$ and $Y : M' \to M'$ where $T\|M$ denotes the restriction of T to M.
With respect to the decomposition $H = M \oplus M^{\perp}$, the projection onto M (i.e the unique projection $P:H \to M$ such that $Ran(P) = M$) can be written as $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$

Theorem 2.1.5. *Let H be a Hilbert space and M be a closed subspace of H.Let $T \in B\ (H)$ and P be a projection of H onto M. Then M is invariant under T if and only if $PTP = TP$.*

Proof. $\Rightarrow$ Let M be invariant under T and let $x \in H$ then $Px \in M$ (since P is a projection onto M). Thus $TPx \in M$. Since $TPx \in M$ then $P(TPx)) \in M$. Therefore, $P(TPx) = TPx$. Hence $PTP = TP$.
$\Leftarrow$ Conversely, let $PTP = TP$. Let $x \in M$ then, $(PTP)x = (TP)x = T(Px)$. But,$(PTP)x = P(TP)x \in M$ since M is the range of P. Thus $T(Px) \in M$. Thus Tx is in M since $Px = x$. Thus M is invariant under T. □

2.2 Hyperinvariant Subspaces

The knowledge of hyperinvariant subspaces of an operator A in $B(H)$ give information about the structure of the commutant of A, the set of all operators B such that $AB = BA$ as we can see in the results below.

Theorem 2.2.1. *A linear subspace M of a Hilbert space H is hyperinvariant for T in $B(H)$ if it is invariant for every operator in B (H) that commutes with T. In other words a linear manifold (or subspace) M is hyperinvariant for $T \in B(H)$ if M is L- invariant for every L in $B(H)$ such that $LT = TL$.*

Lemma 2.2.1. *([13], Proposition 4.7) Let H and K be Hilbert spaces and suppose $T \in B(H)$ and $S \in B(K)$, $X \in B(H, K)$ and $Y \in B(K, H)$ such that $XT = SX$ and $YS = TY$. If $C \in B(H)$ commutes with T and M is nontrivial hyperinvariant subspace for S with $\overline{Ran(X)} = K$ and $Ker(Y) \bigcap M = \{0\}$ then M_T is a nontrivial hyperinvariant subspace for T.*

Corollary 2.2.1. *([13], proposition 4.8) If two operators are quasisimilar and if one of them has a nontrivial hyperinvariant subspace, then so has the other.*

Proposition 2.2.1. *[19] Let H and K be Hilbert spaces and suppose $T \in B(H)$ and $S \in B(K)$, $X \in B(H, K)$ and $Y \in B(K, H)$ such that $XT = SX$ and $YS = TY$. If $C \in B(H)$ commutes with T and M is a hyperinvariant subspace for S then M is invariant for XCY whenever $C \in \{T\}'$.*

From the above proposition, it is clear that every hyperinvariant linear manifold (or subspace) for T is invariant for $\{T\}'$.

Theorem 2.2.2. *[8] Suppose T is an operator in $B(H)$ and there exists a nonzero compact operator K in $B(H)$ such that the rank of TK - KT is less than or equal to one. Then T has nontrivial hyperinvariant subspaces.*

Theorem 2.2.3. *[19] Let $U \in B(H)$ be a unitary operator. A closed subspace $M \subseteq H$ is hyperinvariant for U if and only if P_M commutes with $\{U\}'$*

Proof. $\Leftarrow$: If A commutes with P_M then $AM = P_M AH \subset P_M H = M$: thus P_M commutes with all these A, then M is hyperinvariant for U.
$\Rightarrow$: Conversely if $M \in$ HyperLat$U$, then $AP_M = P_M A P_M$ for every $A \in \{U\}'$. By Fuglede's Theorem, A^* also commutes with U (i.e $AU = UA \Rightarrow U^*A = U^*AUU^* = U^*UAU^* = AU^*$ and also $A^*U = UA^*$, so we have that $A^*P_M = P_M A^*$ and hence $AP_M = P_M A P_M = (P_M A P_M)^* = P_M A$. $\square$

Proposition 2.2.2. *Let $T \in B(H)$ be quasisimilar to a unitary operator $U \in B(K)$ with implementing quasiaffinities X, Y such that $TX = XU$ and $UY = YT$. If $M \subseteq K$ is a hyperinvariant subspace of T, then M is a hyperinvariant subspace of U.*

Theorem 2.2.4. *([19], Theorem 2.22) Let T be a nilpotent operator such that $T = T_n \oplus T_{n-1} \oplus ... \oplus T_1$, where T_n is such that $T^n = 0$. Then T has 2^n hyperinvariant subspaces.*

2.3 Reducing Subspaces of Operators

If T has nontrivial reducing subspace then the dimension of H is greater than one since a one-dimensional space has no nontrivial subspace. Suppose M reduces T then T can be written with respect to the decomposition $H = M \oplus M^\perp$ as $T = \begin{pmatrix} T\mid_M & 0 \\ 0 & T\mid_{M^\perp} \end{pmatrix}$, where $T\mid_M$ in $B[M]$ is the restriction of T to M and $T\mid_{M^\perp}$ in $B[M^\perp]$ is the restriction of T to $M^\perp$. Conversely $A \in B[M]$ and $D \in B[M^\perp]$, If $T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, on $H = M \oplus M^\perp$ then M reduces T, $A = T\mid_M$ and $D = T\mid_{M^\perp}$. Hence we can write $T = A \oplus D$ and say that T is the orthogonal direct sum of A and D where the operators A and D are referred to as direct summands of T.
Recall that with respect to the decomposition $H = M \oplus M^\perp$,where M and $M^\perp$ are closed subspaces of H, then $M \neq 0$ if and only if $M^\perp \neq H$ and $M^\perp \neq 0$ if and only if $M \neq H$. Thus an operator T on H is reducible if there exist a subspace M of H such that both M and $M^\perp$ are nonzero, not equal to H and are T-invariant.

Theorem 2.3.1. *[14] A subspace M of a Hilbert space H reduces $T \in B[H]$ if and only if M is invariant for both T and T^*.*

Proof. Let M reduce T. Then M and $M^\perp$ are invariant for T by definition. $M^\perp$ invariant for T means that M is invariant for T^*. Hence M is invariant for both T and T^*.
Conversely if M is invariant for T^* it implies that $M^\perp$ is invariant for T. Thus M and $M^\perp$ are invariant for T. Hence M reduces T.

Theorem 2.3.2. *[13] Let H be a Hilbert space and $T \in B[H]$. Let M be a closed subspace of H. Then M reduces T if and only if $TP = PT$ where*

$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ *is the unique projection onto M such that $Ran(P) = M$.*

Proof. Let M reduce T, then M is invariant for both T and T^* (by Theorem 2.3.1).From Theorem 2.1.6 we have that, M is invariant for

$$T \Leftrightarrow PTP = TP \tag{2.1}$$

M is invariant for

$$T^* \Leftrightarrow PT^*P = T^*P \tag{2.2}$$

Taking adjoints in (2.2) we have $P^*TP^* = P^*T$ i.e

$$PTP = PT \tag{2.3}$$

since $P^* = P$. From (2.1) and (2.3) the equality follows, that is
$TP = PT$. □

Proposition 2.3.1. *([13], Proposition 1.1) Let T be an operator on a Hilbert space H of dimension greater than one.The following assertions are equivalent.*

(a) T is reducible

(b) T commutes with a nonscalar projection

(c) T commutes with a nonscalar normal operator

(d) there exist a nonscalar operator that commutes with T and T^*.

Thus, an operator T is reducible if and only if there exist a nonscalar operator L such that $LT = TL$ and $T^*L - LT^* = 0$, that is if and only if there exist a nonscalar operator L in $\{T\}' \bigcap \{T^*\}'$.

Proposition 2.3.2. *Every operator that commutes with a nonscalar normal operator is reducible.*

Proposition 2.3.3. *([13], Proposition 5.2) An operator T is reducible if and only if there exist a nonscalar L such that*

(a) $LT = TL$, $D_TL = LD_T$ and (b) $(T^*D_L - D_LT^*)T = T(T^*D_L - D_LT^*)$

Proof. If T is reducible, then there exist a nonscalar L in $\{T\}' \bigcap \{T^*\}'$. Thus assertions (a) and (b) holds trivially. Conversely take a nonscalar L and set $C = T^*L - LT^*$. Recall that assertion (a) is equivalent to $LT = TL$ and $CT = TC$. Hence , if (a) holds, $D_C = C^*C - CC^* = (L^*T - TL^*)C - C(L^*T - TL^*) = (L^*C - CL^*)T - T(L^*C - CL^*)$.
However as $L^*T^* = T^*L^*$, $L^*C - CL^* = L^* (T^*L - LT^* - (T^*L - LT^*)L^* = T^*D_L - D_LT^*$
Therefore, if assertion (b) also holds, $D_C = 0$ that is C is normal. If C is nonscalar, then T is reducible (since $CT = TC$). If C is scalar,then $C = 0$ (C is a commutator and nonzero commutators are nonscalar) and hence the nonscalar L lies in $\{T\}' \bigcap \{T^*\}'$, that is T is reducible. □

Recall that an operator T is quasireducible if there exist a non scalar L in $\{T\}'$ such that either
$T^*L - LT^*$ lie in $\{T\}'$ (equivalently if the commutator $[(T^*L - LT^*),T]$ has rank zero) or the commutator $[(T^*L - LT^*),T]$ has rank one. This gives the following results.

Theorem 2.3.3. *[14] Every reducible operator is quasireducible.*

Indeed an operator is reducible if and only if there exist a nonscalar L such that $LT = TL$ and $T^*L - LT^* = 0$ which trivially imply $(T^*L - LT^*)T - T(T^*L - LT^*) = 0$

Theorem 2.3.4. *An operator T is quasireducible if and only if there exist a nonscalar L such that $LT = TL$ and rank($D_TL - LD_T$)≤ 1.*

Proof. Let D_T denote the self commutator of T, that is $D_T = T^*T - TT^*$. If $LT = TL$ then $D_TL - LD_T = (T^*L - LT^*)T - T(T^*L - LT^*)$ so that T is quasireducible if and only if there exist a nonscalar L such that $LT = TL$, rank$(D_TL - LD_T) \leq 1$ □

Theorem 2.3.5. *([11], Problem 4.5) Let L and T be operators on a Hilbert space H. If L commutes with both T and T^*, then $Ker(L)$ and $\overline{Ran(L)}$ reduce T.*

Proof. Let H be a Hilbert space and take T and L in $B[H]$. If L commutes with T then $Ker(L)$ and $\overline{Ran(L)}$ are T-invariant. Similary if L commutes with T^*, then $Ker(L)$ and $\overline{Ran(L)}$ are T^*-invariant. Therefore according to theorem 2.3.1 if L commutes with T and with T^* then subspaces on H, $Ker(L)$ and $\overline{Ran(L)}$ reduce T. □

Theorem 2.3.6. *[14] Let T be an operator on a Hilbert space H.*

(a) if T commutes with an orthogonal projection P, then $Ran(P)$ is a reducing subspace for T

(b) T is reducible if and only if it commutes with a nontrivial orthogonal projection.

Proof. Let H space. Take an operator $T \in B[H]$ and an orthogonal projection $P \in B[H]$.

(a) If $PT = TP$ (by Theorem 2.3.2), then it is clear that $Ran(P)$ is T-invariant. Moreover since P is self-adjoint it follows that $PT^* = T^*P$ and hence $Ran(P)$ is T^*-invariant. Therefore $Ran(P)$ reduce T (Theorem 2.3.1).

(b) Recall that $\{0\} \neq Ran(P) \neq H$ if and only if $0 \neq P \neq I$. Thus according to (a), if T commutes with a nontrivial orthogonal projection, then $Ran(P)$ is nontrivial reducing subspace for T; that is T is reducible. Conversely, suppose T is reducible so that there exists a nontrivial subspace M such that both M and $M^\perp$ are T-invariant. Since M is invariant for T, it follows that the nontrivial orthogonal projection P onto M is such that $PTP = TP$ (Theorem 2.1.6). Similary since $M^\perp$ is T-invariant it also follows that the complement projection $E = (I - P)$ into $M^\perp$ is such that $ETE = TE$ and hence $PTP = TP$. Therefore $PT = TP$.

□

Chapter 3

ON SOME EQUIVALENCE RELATIONS OF OPERATORS

In this chapter we consider some equivalence relations and characterize operators in such equivalence classes and which equivalence relations preserve reducibility and quasireducibility.

Remark 3.0.7. Note that we first need the following known results.

Theorem 3.0.8. *([20], Theorem 2.1) If T is a normal operator and S $\in B(H)$ is unitarily equivalent to T, then S is normal.*

Proof. Suppose $S = U^*TU$, where U is unitary and T is normal.Then S^*S $= (U^*T^*U)(U^*TU) = U^*T^*TU = U^*TT^*U = SU^*T^*U = SU^*S^*U = SS^*$. Which proves the claim. □

A necessary and sufficient condition that an operator $T \in B(H)$ be normal is that $||Tx|| = ||T^*x||$ for every $x \in H$.

Corollary 3.0.1. *([20], Corollary 2.3) An operator $T \in B(H)$ is normal if and only if T and T^* are metrically equivalent.*

Theorem 3.0.9. *([20], Theorem 2.11) If $S \in B(H)$ and $T \in B(H)$ are similar then S^* and T^* are similar.*

Corollary 3.0.2. *([20], Corollary 2.12) If $S \in B(H)$ and $T \in B(H)$ are unitarily equivalent, then S and T are similar.*

Proposition 3.0.4. *([20], Proposition 2.13) If S and T are normal operators in a Hilbert space H, then S is unitarily equivalent to T if and only if S is similar to T.*

Two operators are considered to be the same if they are unitarily equivalent since they have the same properties of invertibility, norm and the spectral picture.

Theorem 3.0.10. *([20] Theorem 2.14) If T and S are metrically equivalent operators on H, then $||S|| = ||T||$.*

Proof. The proof follows immediately from $||T||^2 = ||T^*T|| = ||TT^*|| = ||S^*S|| = ||SS^*|| = ||S||^2$. The converse of the above theorem is not always true. There exist operators with the same norm which are not metrically equivalent. □

Corollary 3.0.3. *([20] Corollary 2.6) If S and T are metrically equivalent normal operators, then there exists a unitary operator U such that $S = UT$.*

Theorem 3.0.11. *([20], Theorem 2.26) If S and T are metrically equivalent projections then they are unitarily equivalent.*

Proof. Since S is metrically equivalent to T, from corollary 3.0.3 there exists a unitary operator U such that $S = UT$.This with the fact that both S and T are projections, we have that $S = S^2 = S^*S = T^*T = UTT^*U^* = UT^2U^2$ $= UTU^*$ which shows that S and T are unitarily equivalent. □

Example 3.0.1. Let $S, T \in B(l^2(N))$ be defined as follows; $S(x_1, x_2, x_3...)$ $= (x_1, x_1, x_2, x_3, ...)$ and $T(x_1, x_2, x_3, ...) = (x_0, x_1, x_2, x_3, ...$ A simple computation shows that S and T are not metrically equivalent and hence they are not unitarily equivalent.

Proposition 3.0.5. *([20], Proposition 2.36) If T is a normal operator and S is metrically equivalent to T, then S is normal.*

3.1 Spectral Picture and Equivalence Relations

Remark 3.1.1. (i) If T is finite-dimensional operator (i.e $Ran(T)$ is finite-dimensional), then $\sigma(T) = \sigma_p(T)$ which is finite.

(ii) If T is compact, then $\sigma(T)\backslash\{0\}\subseteq \sigma_p(T)$ which is countable.

(iii) If T is normal, then $\sigma_R(T) = \emptyset$. A unitary operator in $B(H)$ (which is normal) has its spectrum in the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$. A self-adjoint operator (which is also normal), has a real spectrum (i.e it lies in the real line.

(iv) Positive operators have nonnegative real spectra and if P is a nontrivial idempotent operator, then $\sigma(P) = \sigma_p(P) = \{0, 1\}$.

(v) The spectral radius of an operator T is such that $r(T) = Lim_{n\to\infty}$ $||T^n||^{\frac{1}{n}}$ (Beurling formula).

Proposition 3.1.1. *Unitarily equivalent operators have equal numerical range.*

Theorem 3.1.2. *([20], Theorem 2.15) If $S, T \in B(H)$ are metrically equivalent, then $w(|S|) = w(|T|)$.*

Proof. By the theorem 3.0.10, we have that $||S|| = ||T||$. Since T^*T is self-adjoint, it is normal and thus $w(T^*T) = ||T||^2$. Thus $w(T^*T) = w(S^*S)$. Hence $w(|T|) = w(|S|)$. □

Proposition 3.1.2. *([20], Proposition 2.16) If $T, S \in B(H)$ are metrically equivalent operators, then T and S need not to have equal numerical range.*

Proposition 3.1.2 is illustrated below.

Let T and S be operators represented by the matrices, $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathbb{C}^2$.

A simple computation shows that $\sigma(T) = \{-1, 1\}$ and $\sigma(T) = \{1\}$ and $W(T) \neq W(S)$. Therefore, unlike unitarily equivalent operators, metric equivalence operators does not preserve numerical range.

Proposition 3.1.3. *Suppose that A and B are quasireducible (reducible) operators in Hilbert space H. Then the following assertions are true.*

(a) $\sigma(A) = \sigma(B)$

(b) $\sigma_p(A) = \sigma_p(B)$

3.2 Equivalence Relations Preserving Reducibility and Quasireducibility

In this section we investigate different equivalence relations which preserve reducibility and quasireducibility. For example we have that unitary equivalence preserves reducing subspaces, i.e if A,$B \in B[H]$ such that A is unitarily equivalent to B and there exists a subspace M of H which reduces A, then M reduces B [13].

Proposition 3.2.1. *([13],Proposition 1.4) Every operator unitarily equivalent to a reducible (quasireducible) operator is reducible (quasireducible).*

Since unitary equivalence preserves reducibility, (quasireducibility) (Proposition 3.2.1) and numerical range (see Proposition 3.1.1) then we have the following theorem.

Theorem 3.2.1. *If S and T are unitarily equivalent reducible (quasireducible) operators, then $W(S) = W(T)$.*

Proposition 3.2.1 does not hold under similarity.

Remark 3.2.2. Every operator similar to a reducible (quasireducible) operator need not to be reducible (quasireducible).

For instance consider a nonquasireducible operator that is similar to a reducible one.

Example 3.2.1. Set $H = \mathbb{C}$ and identify the operators on $\mathbb{C}^3$ with their matrices with respect to the canonical basis for $\mathbb{C}^3$. Let $T = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$$T^* = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \text{ Then } D_T = T^*T - TT^* = \begin{pmatrix} -2 & -2 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 0 \end{pmatrix}.$$

Any nonscalar L that commutes with T is of the form, $L = \begin{pmatrix} \alpha & \beta & \psi \\ 0 & \alpha - \psi & 0 \\ 0 & -\beta & \alpha - \psi \end{pmatrix}$

where α and ψ cannot be both zero.

$$D_T L = \begin{pmatrix} -2\alpha & -4\beta - \alpha + \psi & -4\psi + 2\alpha \\ -\alpha & 2\alpha - 2\psi & -\alpha \\ 2\psi & 2\beta - \alpha + \psi & 2\psi \end{pmatrix} \text{ and}$$

$$LD_T = \begin{pmatrix} -2\alpha - \beta + 2\psi & -\alpha + 2\beta - \psi & 2\alpha - \beta \\ -\alpha + \psi & 2\alpha - 2\psi & -\alpha + \psi \\ \beta + 2\alpha - 2\psi & -2\beta - \alpha + \psi & \beta \end{pmatrix}.$$

$$\text{Thus } D_T L - LD_T = \begin{pmatrix} \beta - 2\psi & 2\psi - 6\beta & \beta - 4\psi \\ -\psi & 0 & -\psi \\ 2\psi - \beta & 4\beta & 2\psi - \beta \end{pmatrix}$$

and hence rank $D_T L$ - $LD_T \geq 2$ for every nonscalar L that commutes with T. Thus T is not reducible.

Now put $\tilde{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$,W $= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ so that W is invertible and $WT = \tilde{T}W$. Therefore, the reducible $\tilde{T} = 1 \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is similar to T, which is not even quasireducible. Thus (as reducibility) quasireducibility also is not preserved under similarity.

3.3 Operators Enjoying the Property of Reducibility and Quasireducibility

Clearly every normal operator is quasinormal (and also is every isometry). If $T \neq 0$ is quasinormal, then either $(T^*T$ - $TT^*) = 0$ or $(T^*T$ - $TT^*) \neq 0$ and $(T^*T$ - $TT^*)T = 0$.

Proposition 3.3.1. *([13], Proposition 3.1) Every quasinormal operator is quasireducible.*

Proof. We split the proof into four parts.

(a) An normal operator is trivially reducible and hence qusireducible.

(b) A pure isometry(i.e a completely nonunitary isometry) is precisely a unilateral shift. If its multiplicity is greater than one, then it is the direct sum of two unilateral shifts, thus reducible. If it has a multiplicity of one, then it is not reducible but quasireducible. If S_+ is a unilateral shift of multiplicity one, then $(S_+^*S_+ - S_+S_+^*)S_+ - S_+(S_+^*S_+ - S_+S_+^*) = S_+(S_+S_+^* - I)$ is a rank-one operator.

(c) The von Neumann-Wold decomposition says that every isometry is the direct sum of a unilateral shift and a unitary operator (i.e normal isometry) where any of the direct summands may be missing. Thus parts (a) and (b) ensures that every isometry is quasireducible and so is every multiple of an isometry.

(d) An operator T is a multiple of an isometry if and only if the nonnegative operator T^*T is a scalar (an isometry is precisely an operator V such that $V^*V = I$). If T is quasinormal but not a multiple of an isometry, then T^*T is nonscalar normal operator in $\{T\}'$ by the very definition of quasinormality. Thus T is reducible and hence quasireducible.

Theorem 3.3.1. *([13], Proposition 3.2) Every injective unilateral weighted shift whose self-commutator has multiplicity one is not quasireducible.*

Proposition 3.3.2. *([13], Proposition 2.2) Let T be an operator acting on an arbitrary Hilbert space H.If T is a nilpotent operator of index $(n+1)$ for some $n \geq 1$,then either T^n is reducible or T is quasireducible with nilpotence index 2 on a two-dimensional space.*

Proof. : Take a nonzero operator T on H. Since $Ker(T)$ is T-invariant, $T = \begin{pmatrix} 0 & X \\ 0 & Y \end{pmatrix}$ so that $T^n = \begin{pmatrix} 0 & XY^{n-1} \\ 0 & Y^n \end{pmatrix}$ for every $n \geq 1$.
With respect to the decomposition $H = Ker(T) \oplus Ker(T)^{\perp}$ where $X : Ker(T)^{\perp} \to Ker(T)$ and $Y : Ker(T)^{\perp} \to Ker(T)^{\perp}$ are bounded and linear. If $T^{n+1} = TT^n = 0$ is nontrivial (0 is an eigenvalue of T) so that both $Ker(T)$ and $Ker(T)^{\perp}$ are nonzero and $Y^n = 0$,$Y^{n+1} \neq 0$ and $X \neq 0$. Hence

$T^n = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}$ with $Z = XY^{n-1} : Ker(T)^{\perp} \to Ker(T)$.
Therefore with respect with the same decomposition, $H = Ker(T) \oplus Ker(T)^{\perp}$,set

$Q = \begin{pmatrix} ZZ^* & 0 \\ 0 & Z^*Z \end{pmatrix}$, so that $QT^n = T^nQ = \begin{pmatrix} 0 & ZZ^*Z \\ 0 & 0 \end{pmatrix}$ where $Z^* : Ker(T) \to Ker(T)^{\perp}$ is the adjoint of Z.
If the nonnegative Q is nonscalar then T^n is reducible. Supose that Q is scalar. In this case $Z = \lambda^{\frac{1}{2}}U$ for some positive scalar λ and some unitary transformation U so that $Ker(T)$ and $Ker(T)^{\perp}$ are unitarily equivalent and hence dim $Ker(T$ = dim $Ker(T)^{\perp}$.
Now take an arbitrary operator $A : Ker(T) \to Ker(T)$ and set still on $H = Ker(T) \oplus Ker(T)^{\perp}$, N $= \begin{pmatrix} A & 0 \\ 0 & \lambda^{-1}Z^*AZ \end{pmatrix}$ so that, $NT^n = T^nN = \begin{pmatrix} 0 & AZ \\ 0 & 0 \end{pmatrix}$. If dim $Ker(T) \geq 2$ then let A be nonscalar normal operator so that N is nonscalar normal operator as well,and therefore T^n is reducible.
If dim $Ker(T) = 1$, then dim$H = 2$. In this case,we may assume without loss of generality that $T^n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$. This implies that $n = 1$ and hence any nonscalar L that commutes with T is of the form $L = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ with $\beta \neq 0$, which is never normal. Thus T is irreducible. However T is quasireducible because rank $(D_TL - LD_T) = 1$. $\square$

Note that ($n = 1$) every nilpotent operator of index 2 is quasireducible. A nilpotent operator of index 2 acting on a Hilbert space of dimension greater than two is reducible; on a two-dimensional space, it is irreducible but quasireducible.

Remark 3.3.2. Nilpotent operators of higher index are not necessarily quasireducible.

This is illustrated in the example below.

Example 3.3.1. Set $T = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$ be a nilpotent operator of index 3 that is not quasireducible. $D_T = TT^* - T^*T$

$T^*T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ and $TT^* = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, $D_T = \begin{pmatrix} -2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

If L that commutes with T is in the form $L = \begin{pmatrix} \alpha & \beta & \psi \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix}$ so that

$D_TL - LD_T = \begin{pmatrix} \beta & -2\beta - \psi & -2\beta - 4\psi \\ 0 & \alpha & \beta \\ 0 & 0 & \beta \end{pmatrix}$ and hence rank (D_TL - LD_T) ≥ 2 whenever L is nonscalar, then T is not quasireducible.

Finally, we end this chapter by looking at the product and sum of reducible or quasireducible operators. The example below shows that the product and (ordinary) sum of quasireducible operators are not necessarily quasireducible.

Example 3.3.2. Set $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ so that $T^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The operator T^2 is nilpotent of index 2 on $\mathbb{C}^3$ thus reducible and T is quasireducible (since rank(T^2D_T - D_TT^2) $= 1$) so that $I + T$ is quasireducible by proposition 1.3.1 (b). However $T(I + T) = T + T^2$ which is both a product and a sum of quasireducible operators, is not quasireducible.

Question: Is the square of a quasireducible operator quasireducible?
Question: Is T^n quasireducible for every integer $n \geq 1$ whenever T is quasireducible?

Proposition 3.3.3. *Every polynomial of T is not quasireducible whenever T is quasireducible.*

Observe that there exist operators for which all (positive) powers are not quasireducible. For example; $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$ is idempotent and not

quasireducible (actually $D_T L$ - $L D_T$ is full rank for every nonscalar L in $\{T\}'$) and hence every polynomial of T is not quasireducible by proposition 1.3.1 (a) and (b).
Question: If every polynomial of T is reducible (or quasireducible) must T be reducible (or quasireducible)?

Example 3.3.3. Let the operator T be an operator with matrix

$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then T is quasireducible.

Solution 3.3.1. Let $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ A simple computation shows that commutator of T is $\{T\}' = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$:a, b$\in \mathbb{R}\}$ and that $(T^*L - LT^*)T = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$ and $T(T^*L - LT^*) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and therefore $(T^*L - LT^*)T$ - $T(T^*L - LT^*) = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$, T is quasireducible since rank $(T^*L - LT^*)T$ - $T(T^*L - LT^*)$= $1 \leq 1$. However T is irreducible.

Note that a 2×2 matrix is irreducible and so is every normal operator without an eigenvalue i.e $T^*T = TT^*$ and the point spectrum, $\sigma_p(T) = \emptyset$.

Remark 3.3.3. From the results of this chapter we have the following inclusions
Reducible $\subseteq$ *Irreducible* $\subsetneq$ *Quasireducible*

Chapter 4

INVARIANT AND HYPERINVARIANT LATTICES OF SOME OPERATORS

The von Neumann algebra generated by an operator $T \in B(H)$ is used to investigate the structures of invariant and hyperinvariant lattices for some operators. The double commutant theorem relates the closure of a set of bounded operators on a Hilbert space in certain topologies to the bicommutant of that set and thus it gives the connection between the algebraic and topological side of operator theory. The family of invariant and hyperinvariant subspaces is denoted by $\mathrm{Lat}(T)$ and $\mathrm{HyperLat}(T)$, respectively.In this project, the lattices refers to $\mathrm{Lat}(T)$ or $\mathrm{Hyperlat}(T)$.

4.1 Structure of Invariant and Hyperinvariant Subspaces

Note that the lattices $\mathrm{Lat}(T)$ and $\mathrm{Hyperlat}(T)$ have set-theoretic set inclusion ordering $(\subseteq)$ of the power set $P(H)$ as a partial order $\leq$ on them. With this partial order each of $\mathrm{Lat}(T)$ or $\mathrm{Hyperlat}(T)$ is a complete lattices with H as the greatest element and $\{0\}$ as the least . If L_1 and L_2 are complete lattices, we write $L_1 \equiv L_2$ to signify that there is a complete lattice isomorphism of one onto the other.

Lemma 4.1.1. *For every net $\{T_n\} \in B(H)$ we have $\{T_i\}$ converges in the*

WOT to T if and only if $\langle T_n x, y\rangle \to \{T_x, y\}$ for all $x, y \in H$. In this case T is called the weak limit of T_n.

We say that $\{T_n\}$ converges strongly to T which is called the strong limit of $\{T_n\}$ if $||(T_n - T)x|| \to 0$ for every $x \in H$. Furthermore, we say that $\{T_n\}$ converges uniformly to T which we call the uniform limit of $\{T_n\}$ if $||T_n - T||$ $\to 0$.

Remark 4.1.1. (i) Note that uniform convergence implies strong convergence and strong convergence implies weak convergence.

(ii) The WOT is weaker than SOT and SOT is weaker than the UOT.
From the above remaks we have the following lemma.

Lemma 4.1.2. *[19] For a Hilbert space H and a subset A of $B(H)$, the commutant $\{A\}'$ is always strongly closed.*

Remark 4.1.2. The Double Commutant Theorem says that the unital self-adjoint subalgebra $\mathfrak{A}$ of $B(H)$ in the WOT and the SOT are equal,and are equal to the bicommutant $\{\mathfrak{A}\}'$ of $\mathfrak{A}$.

Theorem 4.1.3. *([19], Proposition 2.1) For $T \in B(H)$ and for every $M \in Hyperlat(T)$, P_M belongs $W^*(T)$ where P_M is the (orthogonal) projection of H onto M.*

Proof. By the Double Commutant Theorem, it is enough to show that if $Q = Q^2 = Q^* \in \{W^*(T)\}' = \{T\}' \bigcap \{T^*\}'$, then $P_M Q = QP_M$ or equivalently that $QM \subseteq M$ since $Q \in \{T\}'$ and $M \in Hyperlat(T)$. □

We prove the following theorem by use of this result.

Theorem 4.1.4. *([19], Theorem 2.1) Let $B \in B(H)$. If an operator $A \in B(H)$ is in $W^*(B)$, then $Lat(B) \subseteq Lat(A)$.*

Proof. It is clear that $HyperLat(T) \subseteq Lat(T)$ for any $T \in B(H)$ since T commutes with itself. Since $A \in W^*(B)$, then $QP_M = P_M Q$ where $Q \in \{W^*(B)\}' = \{B\}' \bigcap \{B^*\}'$ is an orthogonal projection in $\{B\}'$ and $M \in Hyperlat(B)$, hence $P_M A P_M = P_M A$ where $P_M \in W^*(A)$ is the orthogonal projection of H onto M. This means that $M \in Hyperlat(B) \subseteq Lat(B) \Rightarrow M \in Lat(A)$. □

Proposition 4.1.1. *([19] Corollary 2.2) If $A \in W^*(B)$ then $Hyperlat(B) \subseteq Hperlat(A)$.*

Proof. This follows from the proof of theorem 4.1.4 and the fact that for any operator $T \in B(H)$, $Hperlat(T) \subseteq Lat(T)$. □

Theorem 4.1.5. *[1](Proposition 2.2) Let T be normal operator in $B(H)$. Then $Hyperlat(T) = \{M \in H : P_M \in W^*(T)\}$*

Theorem 4.1.6. *[17] If $T \in B(H)$ double commutes A and B and $Lat(A) \bigcap Lat(B)$ is trivial, then T is ether zero or quasiaffinity.*

Proof. T doubly commutes the pair A, B implies $TA = BT$ and $TB = AT$. Since $TA = BT$, then $\overline{Ran(T)} \in Lat(B)$ and $Ker(T) \in Lat(A)$. Since $TB = AT$ we deduce that $\overline{Ran(T)} \in Lat(A) \bigcap Lat(B)$ and $Ker(T) \in Lat(A) \bigcap Lat(B)$. If $\overline{Ran(T)} = \{0\}$, then $T = 0$. If $Ran(T) = H$, the $Ker(T) = \{0\}$ and hence T is injective and hence has dense range, hence a quasiaffinity. □

We can strengthen the above result as follows.

Corollary 4.1.1. *([19], Corollary 2.4) If T commutes with A and B and $Lat(A) \bigcap Lat(B)$ is trivial then T is either zero or quasiaffinity.*

Proof. If T commutes A and B, then $TA = AT$ and $TB = BT$. Using the theorem above, we have $\overline{Ran(T)} \in Lat(A) \bigcap Lat(B)$ and $Ker(T) \in Lat(A) \bigcap Lat(B)$. Thus by the same argument either T is zero or quasiaffinity. The triviality of $Lat(A) \bigcap Lat(B)$ follows from the orthogonality of $\overline{Ran(T)}$ and $Ker(T)$. □

Theorem 4.1.7. *[19] Let $A, B \in B(H)$. If $Lat(A) = Lat(B)$, then $Hyperlat(A) = Hyperlat(B)$.*

This theorem says that if A and B in $B(H)$ have the same invariant subspaces, then they have the same hyperinvariant subspaces.

4.2 Lattice Isomorphism

Corollary 4.2.1. *[19] If $B_1 : H_1 \to H_1$ and $B_2 : H_2 \to H_2$ and $Lat(B_1) \equiv Lat(B_2)$, then $Hyperlat(B_1) \equiv Hyperlat(B_2)$ where $\equiv$ denotes isomorphic.*

Proposition 4.2.1. *([19], Proposition 2.5) If A and B are similar operators, they have isomorphic lattice of invariant and hyperinvariant subspace. That is $Lat(A) \equiv Lat(B)$ and $Hyperlat(A) \equiv Lat(B)$.*

Proof. We proof the case of invariant. Suppose $A, B \in B(H)$ such that $A = X^{-1}BX$. Then $XA = BX$ and $AX = XB$. This means that X double commutes the pair (A, B) and by theorem 4.1.7 and the invertibility of X we have that $\overline{Ran(X)} = Ran(X) = H \in Lat(A) \bigcap Lat(B)$ and $Ker(X) = \{0\} \in Lat(A) \bigcap Lat(B)$. We now show that any nontrivial subspace $M \in Lat(A)$ if and only if the subspace $XM = \{Xx : x \in M\} \subseteq H$ is in $Lat(B)$. Let $M \in Lat(A)$ and let $x \in XM$ so that $x = Xy$ for some $y \in M$. Let $Bx = BXy = XAy$ and since $y \in M$, we find that $Bx \in XM$. This means that XM is in $Lat(B)$. □

Conversely we assume that XM is in $Lat(B)$. Then for $y \in M$ we have $BXy \in XM$ and thus $Ay = X^{-1}Xy \in X^{-1}(XM) = M$. Thus $M \in Lat(A)$.
The above proposition shows that there is a natural correspondence between the sets of invariant and hyperinvariant subspaces of similar operators.

Theorem 4.2.1. *([19] Lemma 2.20) Suppose S and T are bounded linear operators with direct sum decompositions $S = S_1 \oplus S_2$ and $T = T_1 \oplus T_2$. If the respective direct summands of S and T are similar, then S and T are hyper-quasisimilar operators and $HyperLat(S) \equiv Hyperlat(T)$.*

Theorem 4.2.2. *([19] Theorem 2.9) If T_1 and T_2 are hyper-quasisimilar then $Hyperlat(T_1) \equiv Hyperlat(T_2)$ and $Lat(T_1) \equiv Lat(T_2)$.*

Proof. Since $T_1 \asymp T_2$, we have quasiaffinities X and Y satisfying $\overline{YXM_1} = M_1$ and $\overline{XYM_2} = M_2$ for every $M_1 \in HyperLat(T_1)$ and $M_2 \in HyperLat(T_2)$ using theorem [4.2.3] $XY \in \{T_1\}$ and $YX \in \{T_2\}$, $M_1 \in HyperLat(T_2)$ for every $M_1 \in HyperLat(T_1$ and $M_2 \in HyperLat(T_1)$ for every $M_2 \in HyperLat(T_2$. This means that every hyperinvariant subspace of T_1 is a hyperinvariant subspace for T_2 and vice versa. □

We now consider the relationship between reductive operators and the lattices of invariant and hyperinvariant subspaces.

Theorem 4.2.3. *([19] Theorem 2.10) Every reductive operator is normal if and only if it has nontrivial invariant subspace.*

Theorem 4.2.4. *([19] Theorem 2.11) If M reduces every operator A in the commutant of T (i.e M is $Hyper-reducing$, then $M \in Lat\{A\}' \bigcap Lat\{A^*\}$).*

Corollary 4.2.2. *([19] corollary 2.12) T is reductive if $Lat(T) \subseteq Lat(T^*)$.*

Remark 4.2.5. Note that an operator may be reducible and fail to be reductive since not every invariant subspace can reduce the operator in question. If the operator is not normal, it also fails to be reductive (see Theorem 4.2.3).

Corollary 4.2.3. *([16],corollary 1) If A is reductive, then every hyperinvariant subspace of A is hyper-reducing (equivalently $Lat(\{A\}') = Lat(\{A^*\}')$).*

Remark 4.2.6. Note that the members of $Lat\{A\}'$ are called the hyperinvariant subspaces of A.

Theorem 4.2.7. *If $T \in B(H)$ is normal, then every hyperinvariant subspace of T is a hyperinvariant subspace of T^*. That is $Hyperlat(T) = Hyperlat(T^*)$.*

Proof. Since T is normal if and only if T^* is normal, the result follows from the fact that if $T^* \in \{T\}'$ then $T \in \{T^*\}'$ and vice versa. □

Theorem 4.2.8. *([19] Theorem 2.29) Let H be n-dimensional Hilbert space, $T \in B(H)$ and $\varphi : B(H) \to B(H)$ be a linear map. Then the following statements are equivalent.*

(a) $Lat(T) \equiv Lat(\phi(T))$ for every $T \in B(H)$

(b) $Hyperlat(T) \equiv Hyperlat(\phi(T))$ for every $T \in B(H)$

(c) $Red(T) \equiv Red(\phi(T))$ for every $T \in B(H)$.

Chapter 5

CONCLUSION AND RECOMMENDATION FOR FURTHER RESEARCH

5.1 Conclusion

Invariant and hyperinvariant subspaces are very important in the study of reducibility and quasireducibility of operators in Hilbert space. Right from the definitions to the structures of reducible and quasireducible operators, we find out that nontrivial invariant and hyperinvariant subspaces are vital. We have also found out that unlike unitary equivalence, reducibility and quasireducibility is not preserved under similarity. Unitary equivalence on the other hand preserves the numerical range of reducible and quasireducible operators while metrically equivalent reducible and quasireducible operators need not to have same numerical range. We have also observed that every nilpotent operator of of index two is quasireducible, on a Hilbert space of dimension greater than two it is reducible and on a two-dimensional space it is reducible but quasireducible. It is now clear that not all quasireducible operators are reducible or irreducible. But every reducible operator is quasireducible.

5.2 Recommendations

There is a gap between reducibility and quasireducibility of operators. For consider the class $\mathfrak{G}$ of all operators for which there exists a nonscalar L such that $LT = TL$, $D_T L = LD_T$. Clearly, $\mathfrak{G}$ includes the class of reducible operators and it is included in the class of all quasireducible operators. That is

Reducible $\subseteq \mathfrak{G} \subset$ *Quasireducible.* Note that the second inclusion is proper i.e there exists quasireducible operators not in $\mathfrak{G}$. However, we failed to show whether metric equivalence preserves reducibility and/or quasireducibility. Coming up with sufficient conditions under which metric equivalence implies reducibility or quasireducibility can be recommended as an area for further research.

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