



**UNIVERSITY OF NAIROBI**  
**COLLEGE OF BIOLOGICAL AND PHYSICAL SCIENCES**  
**SCHOOL OF MATHEMATICS**

**ENUMERATION OF RATIONAL CURVES IN COMPLEX  
PROJECTIVE PLANE**

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**A dissertation submitted to the school of mathematics in partial fulfillment  
for a degree of Master of Science in Pure Mathematics**

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# Declaration

## Declaration by the Candidate

I the undersigned declare that this dissertation is my original work and to the best of my knowledge has not been presented for the award of a degree in any other university.

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## Declaration by the Supervisors

This dissertation has been submitted for examination with my approval as the university supervisor.

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# Dedication

This dissertation is dedicated to Sam and Jane.

# Acknowledgement

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# Abstract

This project studies rational curves and their enumeration. First, we show that there are a finite number of rational curves of degree  $d$  passing through  $3d - 1$  general points in the complex projective plane. Finally, we derive the Kontsevich formula which provides a recursive relation for computing this number of rational curves.

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# Chapter 1

## Introduction

Enumerative geometry as a branch of algebraic geometry is mainly concerned with counting curves in varieties satisfying some given conditions. The only requirement is that enough conditions are given to achieve a finite count. These conditions include specifying the genus, degree and intersections.

Enumerative problems have long history with many problems posed by the ancient Greeks. For example, the simplest enumerative question is: how many lines are there in the complex projective plane  $\mathbb{P}^2$  passing through two given distinct points? The answer was given by Euclid as 1. Another old problem also posed by Apollonius [Lei13] thousands of years ago is: how many circles exist that are tangent to a set of three given circles? The answer is 8 and many proofs have been given to this fact.

The curve  $C$  is topologically a sphere with handles on it. The number of such handles is known as genus, denoted by  $g$ . It can be shown [Har77] that the only curve of genus 0 is the projective line  $\mathbb{P}^1$ . This curve is topologically a sphere with no handles on it and is called a rational curve.

This project aims at studying the enumeration of rational curves on the projective plane. That is, we seek the answer to the enumerative problem seeks the answer to the question: how many rational curves of degree  $d$  pass through  $3d - 1$  general points on the complex projective plane  $\mathbb{P}^2$ . We also derive the Kontsevich's formula for counting rational curves.

Notice that we consider  $3d - 1$  general points since it is the right number to obtain a finite number to this enumeration problem. In particular, we have a unique line passing through 2 general points on a plane.



Figure 1.1: 2 general points on a plane

We also have infinitely many lines passing through a general point on a plane.

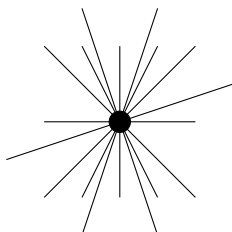


Figure 1.2: 1 general point on a plane

In addition, we have no line passing through 3 general points on a plane.



Figure 1.3: 3 general points on a plane

We denote such a number by  $N(d)$ . Thus  $N(1) = 1$  by Euclid. Apollonius (262BC-190BC) showed there is a unique conic through 5 general points, so  $N(2) = 1$ . Studies on enumeration of rational curves in  $\mathbb{P}^2$  has been essential in the study of Gromov-Witten theory and Quantum Cohomology in algebraic geometry [Pag10]. Following [Zin10], we explain the concept of enumerative geometry and give the introductory formula for the number  $N(d)$  of plane degree  $d$ -rational curves passing through  $3d - 1$  points, counting stable maps.

The outline of the project is as follows:

**Chapter 2:** This chapter gives a brief introduction to complex projective spaces and fundamental understanding of varieties and curves.

**Chapter 3:** Chapter 3 provides a background of this study and specifically discusses the enumerative algebraic geometry of conics.

**Chapter 4:** Here, we introduce the concept of the moduli space of rational curves with  $n$  marked points and their compactification. We also study the intersection theory on these moduli spaces.

**Chapter 5:** Finally, we introduce the concept of moduli space of stable maps and derive the Kontsevich formula that gives the answer to the research question: how many rational curves of degree  $d$  pass through  $3d - 1$  general points on the complex projective plane  $\mathbb{P}^2$ .

# Chapter 2

## General Preliminaries

This chapter gives an introduction to algebraic varieties in the projective space. In particular, we discuss curves in the complex projective plane.

### 2.1 Complex projective spaces

Fix a vector space  $V$  over a field  $k$ , the associated projective space  $\mathbb{P}(V)$  is the set of one-dimensional subspaces of  $V$ .

**Definition 2.1.1.** *The complex projective Space  $\mathbb{P}^n$  of dimension  $n$  is the set of 1-dimensional subspaces of  $\mathbb{C}^{n+1}$ .*

Thus  $\mathbb{P}^n$  denotes the set of lines through the origin in  $\mathbb{C}^{n+1}$  [Got04]. We equivalently say

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$$

where  $(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$  if  $\lambda \in \mathbb{C}^*$ . The space is a smooth  $2n$ -manifold. Let

$$U_i = \{[z_0, \dots, z_n] \in \mathbb{P}^n : z_i \neq 0\} \text{ and,}$$

$\phi_i : \mathbb{C}^n \rightarrow U_i$ , where  $\phi(w_1, \dots, w_n) = [w_1, \dots, w_{i-1}, 1, w_{i+1}, \dots, w_n]$ . The set  $\{(U_i, \phi_i, \mathbb{C}^n)\}$  is the standard atlas for  $\mathbb{P}^n$ . For  $i < j$ , corresponding overlap is given by

$$\begin{aligned} \phi_{ij} : \{(w_1, \dots, w_n) \in \mathbb{C}^n : w_{i+1} \neq 0\} &\rightarrow \{(w_1, \dots, w_n) \in \mathbb{C}^n : w_j \neq 0\} \\ (w_1, \dots, w_n) &\rightarrow \left( \frac{w_1}{w_{i+1}}, \dots, \frac{w_i}{w_{i+1}}, \frac{w_{i+2}}{w_{i+1}}, \dots, \frac{w_j}{w_{i+1}}, w_{i+1}^{-1}, \frac{w_{j+1}}{w_{i+1}}, \dots, \frac{w_n}{w_{i+1}} \right). \end{aligned}$$

The map  $\phi_{ij}$  is holomorphic and so is its inverse.

When  $n = 1$ ,  $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$ , is called the complex **projective line**, and when  $n = 2$ ,  $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^3)$ , is called the complex **projective plane**.

## 2.2 Projective varieties

If  $g \in \mathbb{C}[x_0 \dots x_n]$  is homogenous of degree  $d$ , then for all  $\lambda \in \mathbb{C}$ ,

$$g(\lambda a_0, \dots, \lambda a_n) = \lambda^d g(a_0, \dots, a_n)$$

Let  $I \subset \mathbb{C}[x_0 \dots x_n]$  be a set of homogenous polynomials. The projective zero set or vanishing set of  $I$  is given as:

$$\mathbb{V}(I) = \{p = (a_0, a_1, \dots, a_n) \in \mathbb{P}^n : f(p) = 0 \text{ for all } f \in I\}$$

**Definition 2.2.1.** A subset of the form  $\mathbb{V}(I)$  is a projective algebraic set.

**Definition 2.2.2.** A reduced projective algebraic set is called a **projective variety** or simply a **variety**.

An open subset of a projective algebraic set is called a **quasiaffine variety** and we simply call it a variety.

**Example 2.2.3.** The algebraic set  $\mathbb{V}(xy) \subset \mathbb{P}^2$  is a reducible projective variety, since it is reducible into other components,  $\mathbb{V}(x)$  and  $\mathbb{V}(y)$  in  $\mathbb{P}^2$ .

*Remark:* If  $V$  and  $W$  are projective varieties then so is  $V \cup W$ .

## 2.3 Curves in $\mathbb{P}^2$

From the definition of vanishing set of  $I$ ,  $\mathbb{V}(I) = \{p \in \mathbb{P}^n : f(p) = 0 \text{ for all } f \in I\}$ .  $I$  is a set of polynomials say  $I = \{f_1 \dots f_r\}$ , we then have that

$$\mathbb{V}(f_1 \dots f_r) = \mathbb{V}(I).$$

We recall that  $\mathbb{P}^2$  is the set of 1-dimensional subspaces of  $\mathbb{C}^3$ . Let us look at the definition of curve  $C$  in  $\mathbb{P}^2$ .

**Definition 2.3.1.** Let  $f \in \mathbb{C}[x : y : z]$  be a non-constant homogenous polynomial. The projective curve  $C$  is defined by

$$C := \{[x : y : z] \in \mathbb{P}^2 : f(x, y, z) = 0\}. \quad (2.1)$$

**Example 2.3.2.** Let  $f \in \mathbb{C}[x, y, z]$ , we define  $f(x, y, z) = x^2 + y^2 - z^2$ . The curve defined by  $x^2 + y^2 = z^2$  is a projective curve in  $\mathbb{P}^2$ .

If we define the affine curve  $C$  by:

$$C := \{(x, y) \in \mathbb{C}^2 : h(x, y) = 0\}, \quad (2.2)$$

for  $h \in \mathbb{C}[x, y]$ .

and define the projective curve,  $\bar{C}$  by

$$\bar{C} := \{[x, y, z] \in \mathbb{P}^2 : f(x, y, z) = 0\} \text{ with } f \text{ homogenous.} \quad (2.3)$$

We then view  $\bar{C}$  as a compactification of the affine curve  $C$ .

**Definition 2.3.3.** The **degree** of a projective curve  $C$  in  $\mathbb{P}^2$  defined by a homogenous polynomial  $f \in \mathbb{C}[x, y, z]$ , is the degree  $d$  of  $f$ .

In this case if  $I$  is defined by a single polynomial  $f$  of degree  $d$ ,  $d > 0$ , then  $\mathbb{V}(f)$  is called a **hypersurface** defined by  $f$ . A hypersurface of degree 1 is called a **hyperplane**.

**Example 2.3.4.** Let  $I \subset \mathbb{C}[x, y, z]$  with  $I = \{f_1, f_2, f_3\}$ ,  $f_i$  homogenous for all  $i=1, 2, 3$ . Let  $f_1 = x^2yz + 2zy^2x$  and  $f_2 = 3x - 4y + 5z$ . Now  $\mathbb{V}(f_1) = \mathbb{V}(x^2yz + 2zy^2x)$  is a hypersurface defined by  $f_1$ .  $\mathbb{V}(f_2) = \mathbb{V}(3x - 4y + 5z)$  is a hyperplane since  $f_2$  has degree 1.

**Definition 2.3.5.** The curve  $C$  is called **irreducible** if  $f \in \mathbb{C}[x, y, z]$  is irreducible, that is,  $f$  has no constant polynomial factors other than scalar multiples of itself.

Using the numerical invariant degree, we call a curve of degree 2 in  $\mathbb{P}^2$  a **conic** and a curve of degree 3 in  $\mathbb{P}^2$  a **cubic**.

**Example 2.3.6.** The curve defined by  $ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$  where  $a, b, c, d, e, f \in \mathbb{C}$ , and are not all zeros is a conic.

**Example 2.3.7.** The curve defined by  $ax^3 + by^3 + cz^3 + dx^2z + ex^2z + fy^2z + gxyz = 0$  where  $a, b, c, d, e, f, g \in \mathbb{C}$ , and are not all zeros is a cubic.

The following definitions are given in [Har77].

**Definition 2.3.8.** Let  $Y \subseteq \mathbb{P}^n$  be a variety. A function  $f : Y \rightarrow \mathbb{C}$  is **regular** at a point  $P \in Y$  if there is an open neighborhood  $U$  with  $P \in U \subseteq Y$ , and homogenous polynomials  $g, h \in S = \mathbb{C}[x_0, \dots, x_n]$ , of the same degree, such that  $h$  is nowhere zero on  $U$ , and  $f = g/h$  on  $U$ .  $f$  is regular on  $Y$  if it is regular at every point of  $Y$ .

**Definition 2.3.9.** Let  $X$  and  $Y$  be two varieties, a **morphism**  $\varphi : X \rightarrow Y$  is a continuous map such that for every open set  $V \subseteq Y$ , and for every regular function  $f : V \rightarrow \mathbb{C}$ , the function  $f \circ \varphi : \varphi^{-1}(V) \rightarrow \mathbb{C}$  is regular.

**Definition 2.3.10.** An **isomorphism**  $\varphi : X \rightarrow Y$  of two varieties is a morphism which admits an inverse morphism  $\psi : Y \rightarrow X$  with  $\psi \circ \varphi = I_x$  and  $\varphi \circ \psi = I_y$ .

**Definition 2.3.11.** A **rational map**  $\varphi : X \rightarrow Y$  is an equivalence class of pairs  $\langle U, \varphi_U \rangle$  where  $U$  is a nonempty open subset of  $X$ ,  $\varphi_U$  is a morphism of  $U$  to  $Y$ , and where  $\langle U, \varphi_U \rangle$  and  $\langle V, \varphi_V \rangle$  are equivalent if  $\varphi_U$  and  $\varphi_V$  agree on  $U \cap V$ .

**Lemma 2.3.12.** If  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^n$  is a holomorphic map, there exists homogenous polynomials  $p_0, \dots, p_n$  in two variables such that  $p_0, \dots, p_n$  are of the same degree, have no factor and

$$f([z_0, z_1]) = [p_0(z_0, z_1), \dots, p_n(z_0, z_1)] \text{ for all } [z_0, z_1] \in \mathbb{P}^1. \quad (2.4)$$

Conversely, if  $p_0, \dots, p_n$  are homogenous polynomials in two variables that are of the same degree and have no common factor, the map  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^n$  given by 2.4 is well-defined and holomorphic.

## 2.4 Bezout's Theorem

Consider two projective curves in  $\mathbb{P}^2$ , say  $C$  and  $D$ , which intersect transversely. We determine the number of points of their intersections. The theorem of Bezout gives the number of intersections between such two curves  $C$  and  $D$ .

**Theorem 2.4.1** (Bezout's Theorem, [Kir92]). *If  $C$  and  $D$  are two projective curves of degree  $n$  and  $m$  in  $\mathbb{P}^2$  which have no common component then they have precisely  $nm$  points of intersection counting multiplicity.*

Let  $C$  be a curve in  $\mathbb{P}^2$  and  $f \in \mathbb{C}[x, y, z]$ . A point  $P \in C$  is called a **singular point** or **singularity** of  $C$  if



$$\frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0. \quad (2.5)$$

The set of singular points is denoted by  $\text{sing}(C)$ .

**Definition 2.4.2.** The curve  $C$  is called a **non-singular** or a **smooth curve** if  $\text{sing}(C) = \emptyset$ . Otherwise the curve  $C$  is called a **singular curve**.

**Example 2.4.3.** The curve  $C := (x, y, z) : x^2 + y^2 + z^2 = 1$  is a non-singular curve. Let  $f = x^2 + y^2 + z^2 - 1$ . We check if there exists a point  $P = (x, y, z) \in C$  such that 2.5 is satisfied.

We have  $\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 2z$ . Thus  $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \Rightarrow 2x = 2y = 2z = 0 \Rightarrow x = y = z = 0$ . The point  $P = (0, 0, 0) \notin C$  hence the curve  $C$  is non-singular.

**Definition 2.4.4.** A **node** is a singularity on the curve which is locally complex-analytically isomorphic to a neighborhood of the origin in the zero locus  $xy = 0 \subset \mathbb{C}^2$ . A **nodal curve** is a curve such that every one of its points is either smooth or a node.

## 2.5 Degree-genus formula

The subsets of a curve  $C := \{[x, y, z] \in \mathbb{P}^2 : f(x, y, z) = 0\}$  with  $f$  homogenous, have standard topology. Thus it is possible to investigate non-singular projective curves from the topological point of view.

A non-singular projective curve in  $\mathbb{P}^2$  is topologically a sphere with  $g$  handles. This number  $g$  is called **genus** of a curve.



Figure 2.1: Smooth curves of genus  $g = 0, 1$  and  $2$ .

The relationship between the genus  $g$  of a curve and its degree  $d$  follows from the degree-genus formula, see for example [Kir92].

$$g = \frac{1}{2}(d-1)(d-2). \quad (2.6)$$

*Remark:* We observe that curves of degree 1 and 2 are of genus zero.

For the case of singular curves in  $\mathbb{P}^2$ , to each singular point  $p_i$ , there can be assigned a positive integer  $\delta(p_i)$  such that equation 2.7 holds, for  $p_i, \forall i = 1 \dots r$  are singular points [Kir92].

$$g = \frac{1}{2}(d-1)(d-2) - \sum_{j=1}^r \delta(p_j). \quad (2.7)$$

There is only one curve of genus 0, upto isomorphism.

**Definition 2.5.1.** *A curve of genus 0 is called a **rational curve**.*

# Chapter 3

## Solving Enumerative Problems

This project aims at studying the enumeration of rational curves on the projective plane. Namely complex plane curves of genus 0. A plane conic curve in  $\mathbb{C}^2$  is the set of points  $(x, y) \in \mathbb{C}^2$  satisfying the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

with coefficients not all zero [BKT08]. An example of plane rational curve is plane conic with familiar examples such as circles, ellipses, parabolas and hyperbolas.

A conic is said to be **nondegenerate** if the conic is irreducible. Otherwise a conic is said to be **degenerate** if the polynomial defining the conic factors into a product of linear polynomials. Such a conic is a union of two lines. If the polynomial defining the curve is a square of a linear polynomial, then the conic is thought of as a **double line**. These

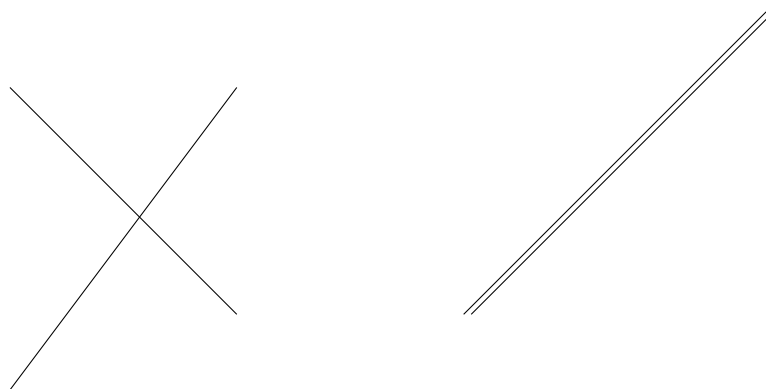


Figure 3.1: An example of a degenerate conic (left) and double line (right).

double lines play key role in counting problems involving conics.

To motivate our study, we will consider Jacob Steiner problem of 1848,: Given five conics in the plane, are there any conics that are tangent to the five conics? If so, how many are they? Steiner's original answer to his problem, 1776, was incorrect. This is because he assumed that the intersection of five subsets corresponding to the five given conics was finite. The correct answer was given in 1859 by de Jonquieres and later Michel Chasles developed a method which determines the number of such conics as 3264.

Any conic is determined the coefficients  $a, b, c, d, e$  and  $f$  of its defining equation  $ax^2 + bxy + cy^2 + dx + ey + f = 0$  but not so uniquely. For example, the equations  $y - x^2 = 0$  and  $4y - 4x^2 = 0$  describes the same curve. If a point  $(a, b, c, d, e, f)$  represents the conic so does  $(\lambda a, \lambda b, \lambda c, \lambda d, \lambda e, \lambda f)$ . The lines in  $\mathbb{R}^6$  form a 5- dimensional projective space,  $\mathbb{C}\mathbb{P}^5$  and so the **parameter space** for conics is  $\mathbb{C}\mathbb{P}^5$ .

### 3.1 Motivation

We introduce a basic counting strategy to count the conics passing through some fixed points and tangent to some fixed lines or conics. For each point  $p$  we form the subset  $H_p \subset \mathbb{C}\mathbb{P}^5$  of conics passing through the point, for each line  $l$  we form the subset  $H_l \subset \mathbb{C}\mathbb{P}^5$  of conics tangent to  $l$ , and for each given nondegenerate conic  $Q$  we form the subset  $H_Q$  of conics tangent to  $Q$ . Points of intersection of all these subsets corresponds to the conics that pass through all the points and are tangent to all the lines and conics. The three hypersurfaces  $H_p, H_l$  and  $H_Q$  are projective algebraic varieties.

For example, any conic passing through  $p(2, 3)$  must satisfy

$$4a + 6b + 9c + 2d + 3e + f = 0$$

a linear condition. So the set of points  $H_p$  is a hyperplane in  $\mathbb{C}\mathbb{P}^5$ .

Similarly, consider a conic tangent to a given line  $L : y = Mx + C$ , for example the line  $y = 0$ . The point of intersection have the form  $(x, 0)$ , where

$$ax^2 + dx + f = 0. \tag{3.1}$$

When the discriminant  $d^2 - 4af = 0$  the equation 3.1 has a double root and the line  $y = 0$  is tangent to the conic. So  $H_l$  is a hypersurface in  $\mathbb{C}\mathbb{P}^5$ . defined by a degree 2 equation.

Lines $l$	0	1	2	3	4	5
Points $p$	5	4	3	2	1	0
Conic solutions	1	2	4	?	?	?

Table 3.1: Number of conics passing through  $l$  lines and  $p$  points.

**Definition 3.1.1.** *The hypersurfaces  $X_1, \dots, X_n$  intersect **transversely** at a point  $p \in X_1 \cap \dots \cap X_n$  when their tangent spaces at  $p$  intersect in the point  $p$  only.*

Counting in the projective space is given by Bezout's theorem.

**Theorem 3.1.2.** *If  $n$  hypersurfaces of degrees  $d_1, d_2, \dots, d_n$  in  $\mathbb{P}^n$  intersect transversally, then the intersection consists of  $d_1 \cdot d_2 \cdots d_n$  points.*

For the case of counting curves passing through five points, there is a bijection between the conics and the points in the intersection of five hypersurfaces  $H_p$  of degree 1. Thus in this case, the five hyperplanes intersect in  $1^5 = 1$  point by Bezout's theorem. Thus there is a unique conic passing through 5 general points in  $\mathbb{P}^2$ . Again by Bezout's theorem, the number of conics which pass through four given points and are tangent to a given line is given by  $1^4 \cdot 2 = 2$  points. Thus we are intersecting four degree 1 hypersurfaces  $H_p$  and one degree 2 hypersurface  $H_l$ .

For the case of three points and two lines, the number of intersections will be  $1^3 \cdot 2^2 = 4$ . For the last three cases, the corresponding hypersurfaces cannot intersect transversally and it involves the double line conics. The only nonreduced conics are the double line conics. We have infinitely double line conics as solutions to this problem.

However, considering the conics passing through one point and tangent to four lines, we have finite number of reduced conics passing through the point and tangent to the four given lines. We would like to ignore the double line solutions and count the number of reduced conics passing through  $p$  and are tangent to  $5 - p$  given lines.

## 3.2 Duality in $\mathbb{P}^2$

We use duality in  $\mathbb{P}^2$  to remove the double lines from our count. Duality allows exchange of lines and points and also transformation of conics into conics. The dual projective plane, denoted  $\tilde{\mathbb{P}}^2$ , is just another copy of  $\mathbb{P}^2$ . Consider the set of all lines in  $\mathbb{P}^2$  defined by the

linear equation  $L : AX + BY + CZ = 0$ . The line can be represented by a point  $[A : B : C]$  in a projective plane. Thus the set of lines in  $\mathbb{P}^2$  is called the **dual projective plane**. By definition, a line  $L$  in  $\mathbb{P}^2$  corresponds to a point in  $\check{\mathbb{P}}^2$ , which we call  $\check{L}$  ('L dual'). Geometrically duality associates lines in  $\check{\mathbb{P}}^2$ , to points in  $\mathbb{P}^2$  and vice versa.

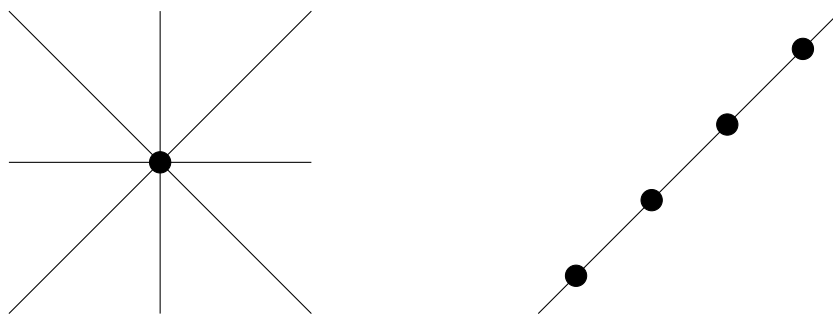


Figure 3.2: Four lines through one point (left) and the their duals (right).

Next we see what duality does to conics. If  $Q$  is a conic in  $\mathbb{P}^2$ , we define the dual curve  $\check{Q}$  in  $\check{\mathbb{P}}^2$  to be the collection of all lines tangent to  $Q$ . This means that  $\check{Q}$ , will contain a point  $\check{L}$  if and only if the corresponding line  $L$  was a tangent to  $Q$ . If a line  $L_1$  is tangent to a nondegenerate conic  $Q$ , then by definition  $\check{L}_1$  is a point on  $\check{Q}$ . If  $p$  is a point  $Q$ , then the line  $\check{p}$  must be tangent to the dual conic  $\check{Q}$  at a unique point.

For instance, if  $Q$  is the conic  $x^2 - yz = 0$  then a line  $Ax + By + Cz = 0$  with  $A \neq 0$  will intersect  $Q$  where

$$A^2x^2 - A^2yz = 0 \text{ and } Ax = -(By + Cz).$$

Therefore,  $B^2y^2 + (2BC - A^2)yz + C^2z^2 = 0$ . The line is a tangent when the discriminant  $A^2(A^2 - 4BC)$  vanishes.

If we have five lines in the plane, then any non-degenerate conic tangent to them will have a dual conic passing through the five dual points. There is only one such dual conic, thus we only have one conic tangent to all five lines. We have 2 conics tangent to four lines and passing through a point in general position. Finally, there are 4 conics tangent to 3 lines and passing through a pair of points in general position.

Lines $l$	0	1	2	3	4	5
Points $p$	5	4	3	2	1	0
Conic solutions	1	2	4	4	2	1

Table 3.2: Complete table for number of conics passing through  $l$  lines and  $p$  points.

### 3.3 Steiner's problem

We now turn to solve the Steiner's problem: 'How many conics are tangent to five given conics?' According to Bezout's theorem, the intersection consists of  $6^5 = 7776$  points, the answer Steiner gave. The intersection of five hypersurfaces  $H_Q$  in  $\mathbb{P}^5$  contains the set of double lines. Duality cannot be used to filter out double lines conics, and here we use the blow up.

**Definition 3.3.1.** *The set of all points in  $\mathbb{P}^5$  corresponding to double lines is called the **Veronese surface**,  $V$ .*

The lines in  $\mathbb{P}^2$  are parametrized by  $\check{\mathbb{P}}^2$ , so the Veronese is 2-dimensional surface. It is the image of the injective map from  $\check{\mathbb{P}}^2$  into  $\mathbb{P}^5$ ;

$$[A : B : C] \mapsto [A^2 : 2AB : B^2 : 2AC : 2BC : C^2].$$

Consider the map defined as  $\delta : \mathbb{P}^5 \rightarrow \mathbb{P}^5$ , which sends a conic to its dual. This map is defined by polynomials and it is called a **rational map**. We take the Veronese surface out of  $\mathbb{P}^5$  and replace it with a four dimensional variety by looking at the graph of  $\delta$  in  $\mathbb{P}^5 \times \mathbb{P}^5$  and closing it up.

**Definition 3.3.2.** *The **blow up** of  $\mathbb{P}^5$  along the Veronese surface, denoted as  $Bl_V \mathbb{P}^5$  is the closure of the graph of  $\delta$  in  $\mathbb{P}^5 \times \mathbb{P}^5$  and closing it up. The **blowing down morphism**  $\pi : Bl_V \mathbb{P}^5 \rightarrow \mathbb{P}^5$  is given by projection onto the first factor.*

In construction of the blow up we have taken the Veronese out of  $\mathbb{P}^5$  and replace it with a two dimensional larger, a four dimensional hypersurface. The hypersurface is called **exceptional divisor** of the blow up, which we denote as  $E$ . The name 'blowing up of the Veronese' can be thought of as inserting a "soda straw" in a bubble of air to stretch it out into a four-dimensional object.

Consider a hypersurface  $Y$  in  $\mathbb{P}^5$  that contains the exceptional divisor. If we remove  $V$

from  $Y$ , and consider the image  $\pi^{-1}(Y/V)$ . This will be isomorphic to  $Y/V$ . Taking the closure  $\overline{\pi^{-1}(Y/V)}$  we get a hypersurface intersecting  $E$  but does not contain it, called **strict transform** of  $Y$ , denoted  $\tilde{Y}$ . To solve Steiner's problem, we will intersect the transforms of the hyperplanes in  $\mathbb{P}^5$ .

### 3.3.1 The Chow ring

The Chow ring is a ring that describes how its subvarieties intersect. The elements of the Chow ring are classes of subvarieties having the same intersection properties. For the case of  $\mathbb{P}^5$ , Bezout's theorem says that given the degree of a hypersurface we can be able to determine its intersection properties. These hyperplanes will be in a particular class, which we denote as  $[H]$ . Addition operation in Chow ring corresponds to the union of two varieties while multiplication corresponds to the intersection of the varieties.

For example, two conics meet in 4 points. We see this in [EH11] as shown in figure 3.3.

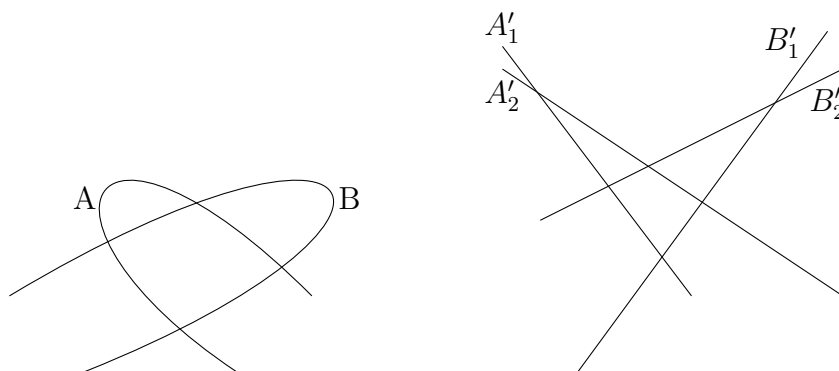


Figure 3.3: Intersection of two conics.

$$[A] = [A'_1] + [A'_2] \text{ and } [B] = [B'_1] + [B'_2].$$

$$\text{So } [A][B] = \sum_{i,j} [A'_i][B'_j] = [4 \text{ points}].$$

Any degree  $d$  hypersurface in  $\mathbb{P}^5$  will be in the class  $d[H]$ . Intersecting five hypersurfaces of the general degree corresponds in the Chow ring to the multiplication

$$d_1[H] \cdot d_2[H] \cdot d_3[H] \cdot d_4[H] \cdot d_5[H] = d_1 \cdot d_2 \cdot d_3 \cdot d_4 \cdot d_5 [H]^5$$

representing  $d_1 \cdot d_2 \cdot d_3 \cdot d_4 \cdot d_5$  points.



Let  $Y$  be a hypersurface of degree  $d$  in  $\mathbb{P}^5$ , then  $[\pi^{-1}(Y)] = d[\tilde{H}]$ . If  $H$  is a hyperplane in  $\mathbb{P}^5$  that does not contain the Veronese, then the strict transform  $\tilde{H}$  is equal to its inverse image  $\pi^{-1}(H)$ . Let  $Y$  be a general degree  $d$  hypersurface in  $\mathbb{P}^5$  containing the Veronese. Then  $[\pi^{-1}(Y)] = \tilde{Y} + n[E]$  for some  $n$ , hence we have

$$\tilde{Y} = d[\tilde{H}] - n[E]$$

We now find the strict transforms which we need to solve our enumerative problem. There are many double line conics passing through a point  $p$  so the hyperplane  $H_p$  of conics through  $p$  does not contain the Veronese. Thus  $[\tilde{H}_p] = [\tilde{H}]$ .

The hypersurface  $H_l$  of conics tangent to  $l$  has degree 2 and its first partial derivatives do not vanish along  $V$ . So

$$[\tilde{H}_l] = 2[\tilde{H}] - [E].$$

The hypersurface  $H_Q$  of conics tangent to  $Q$  has degree 6 and its second partial derivatives do not vanish along  $V$ . So

$$[\tilde{H}_Q] = 6[\tilde{H}] - 2[E].$$

### 3.3.2 Counting curves

We now compute the answer to Steiner's problem by intersecting the transforms in the blow up. We already have the following equations:

$$[\tilde{H}_p] = [\tilde{H}] \tag{3.2}$$

$$[\tilde{H}_l] = 2[\tilde{H}] - [E] \tag{3.3}$$

$$[\tilde{H}_Q] = 6[\tilde{H}] - 2[E] \tag{3.4}$$

From earlier calculations involving conics passing through general points and tangent to the given lines, we learn that in the Chow ring of the blow up,

$$[\tilde{H}_p]^5 = [\tilde{H}_l]^5 = 1.$$

$$[\tilde{H}_p]^4[\tilde{H}_l] = [\tilde{H}_p][\tilde{H}_l]^4 = 2.$$

$$[\tilde{H}_p]^3[\tilde{H}_l]^2 = [\tilde{H}_p]^2[\tilde{H}_l]^3 = 4.$$

We have that

$$[\tilde{H}_Q] = 6[\tilde{H}] - 2[E] = 2[\tilde{H}_p] + 2[\tilde{H}_l]$$

We now compute the answer to Steiner's original problem, 'How many conics are tangent to five given conics?'

$$[\tilde{H}_Q]^5 = (2[\tilde{H}_p] + 2[\tilde{H}_l])^5$$

Expanding this we get

$$\begin{aligned} [\tilde{H}_Q]^5 &= 32([\tilde{H}_p]^5 + 5[\tilde{H}_p]^4[\tilde{H}_l] + 10[\tilde{H}_p]^3[\tilde{H}_l]^2 + 10[\tilde{H}_p]^2[\tilde{H}_l]^3 + 5[\tilde{H}_p][\tilde{H}_l]^4 + [\tilde{H}_l]^5) \\ &= 32(1 + 5(2) + 10(4) + 10(4) + 10(4) + 5(2) + 1) \\ &= 3264. \end{aligned}$$

Thus there are 3264 conics that are tangent to five given conics.

# Chapter 4

## Moduli Space of Rational Curves

In this chapter we discuss Moduli Spaces and their compactification.

### 4.1 Introduction to moduli space

A moduli space is a geometric object which ‘counts’ the equivalence classes of geometric objects given some kind of equivalence. Thus if we have  $n$ -ordered marked points on a rational curve  $\mathbb{P}^1$ , we view the moduli space as parametrising the  $n$ -points on  $\mathbb{P}^1$  up to projective equivalence. We then say that moduli space is the ‘answer’ to the moduli problem, i.e the problem of finding a parameter space of some objects up to some kind of equivalence. We want to particularly have a bijection between the moduli space and the objects under consideration.

**Example 4.1.1.** *We can classify all circles in  $\mathbb{R}^2$  up to equality (i.e circles are equivalent if they have the same set of points). A circle is completely described as an ordered triple  $(x, y, r)$ , where  $(x, y)$  is the center of the circle with radius  $r > 0$ . The moduli space is  $\mathbb{R}^2 \times \mathbb{R}_{>0}$ , the cross product of the real plane and the real line.*

**Example 4.1.2.** *We can also classify conics in  $\mathbb{P}^2$ . A conic is defined by the vanishing set of a polynomial of the form  $ax^2 + by^2 + cz^2 + dxy + exz + fyz$ . The conic is determined by the coefficients of this equation, i.e the 6-tuple  $(a, b, c, d, e, f)$ . Two equations which are scalar multiples of each other determine the same vanishing set. The moduli space is  $\mathbb{C}^6 / \mathbb{C}^* = \mathbb{P}^5$ , where  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ .*

We can be able to pick any  $n$  distinct points on  $\mathbb{P}^1$  thus we have  $n$  - tuple  $(x_1, \dots, x_n)$ . We have an equivalence relation generated by  $(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$  if there is some  $A \in \mathbb{PGL}_2(\mathbb{C})$ .

**Definition 4.1.3.** [Zvo11] A **Stable curve**  $C$  with  $n$  marked points  $x_1, \dots, x_n$  is an algebraic curve which has simple nodes as the only singularities and satisfies the stability condition  $2 - 2g - n < 0$ .

**Definition 4.1.4.** For  $2 - 2g - n < 0$ , the moduli space  $\mathcal{M}_{g,n}$  is the set of isomorphism classes of smooth algebraic curves of genus  $g$  with  $n$ -marked points.

An isomorphism of such  $n$ -pointed smooth rational curves, is an isomorphism of curves that respects the marked points, in order. We consider the case where  $g = 0$ , thus from the stability condition  $2 - 2g - n < 0$  we have that  $n \geq 3$ . This is true since for  $g = 0$ , we have from the stability condition  $2 - n < 0$  thus  $n \geq 3$  since  $n \in \mathbb{N}$ .

**Example 4.1.5.** For  $g = 0, n = 3$ , every rational curve  $(C, x_1, x_2, x_3)$  with three marked points can be identified with  $(\mathbb{P}^1, 0, 1, \infty)$  in a unique way. That is any three marked points  $(x_1, x_2, x_3)$  can be sent to the ordered triple  $(0, 1, \infty)$ . If we have the map  $\varphi : (x_1, x_2, x_3) \rightarrow (0, 1, \infty)$  and  $\psi : (y_1, y_2, y_3) \rightarrow (0, 1, \infty)$  then  $(x_1, x_2, x_3) \sim (y_1, y_2, y_3)$  via the map  $\psi^{-1} \circ \varphi$ . Thus any two ordered three marked points are equivalent and hence  $\mathcal{M}_{0,3} = \text{point}$ .

**Example 4.1.6.** For  $g = 0, n = 4$ , every curve  $(C, x_1, x_2, x_3, x_4)$  can be uniquely determined with  $(\mathbb{P}^1, 0, 1, \infty, t)$ . The number  $t \neq 0, 1, \infty$  is determined by the position of marked points of  $C$ . The number  $t$  is called modulus and hence the term ‘moduli space’. The moduli space  $\mathcal{M}_{0,4}$  is the set of values of  $t$ , i.e  $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ .

In general  $g = 0$  and  $n$  arbitrary chosen, the curve  $(C, x_1, \dots, x_n)$  can be uniquely identified with  $(\mathbb{P}^1, 0, 1, \infty, t_1, \dots, t_{n-3})$ . The moduli space  $\mathcal{M}_{0,n}$  is given by,

$$\mathcal{M}_{0,n} = \{(t_1, \dots, t_{n-3}) \in (\mathbb{P}^1)^{n-3} | t_i \neq 0, 1, \infty, t_i \neq t_j\} \quad (4.1)$$

The dimension of the space  $\mathcal{M}_{0,n}$  is given by

$$\dim \mathcal{M}_{0,n} = n - 3.$$

For example the  $\dim \mathcal{M}_{0,3} = 3 - 3 = 0$ . Actually the dimension of a point is 0. We will draw twigs cartoons to represent rational curves.

**Example 4.1.7.** Rational curves with three marked points and four marked points respectively are drawn as follows:



Figure 4.1: Rational curve with three marked points.



Figure 4.2: Rational curve with four marked points

## 4.2 Compactification of $\mathcal{M}_{0,n}$

The space  $\mathcal{M}_{0,n}$  is in general non-compact. To see this, let us consider an example, for the case  $n = 4$ .

**Example 4.2.1.** *Compact spaces contain limits of families, yet for the family  $C_t = (0, 1, \infty, 0)$  becomes  $(0, 1, \infty, 0)$  as  $t \rightarrow 0$ . This is not in the moduli space since the points are not distinct. For the case  $g = 0$ ,  $n = 4$ , as seen earlier the moduli space  $\mathcal{M}_{0,4}$  is isomorphic to  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ . A point  $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$  encodes that  $(C, x_1, x_2, x_3, x_4) \simeq (\mathbb{P}^1, 0, 1, \infty, t)$ . When  $t \rightarrow 0$ , we obtain the curve  $(\mathbb{P}^1, 0, 1, \infty, 0)$  with four marked points. In this case  $x_1$  and  $x_2$  coincide, i.e  $x_1 = x_2$ . Without changing the curve  $C$  we can change the local coordinate of the curve via the map  $x \mapsto \frac{x}{t}$  and obtain the curve  $(C, x_1, x_2, x_3, x_4) \simeq (\mathbb{P}^1, 0, \frac{1}{t}, \infty, 1)$ . When  $t \rightarrow 0$ , we obtain the curve  $(\mathbb{P}^1, 0, \infty, \infty, 1)$ . In this case we have  $x_2$  and  $x_3$  colliding, i.e  $x_2 = x_3$ .*

Since we cannot prefer one local coordinate to the other, we include both limit curves in the description of this limit. The limit of this family is a nodal curve with one component, in which  $x_2$  and  $x_3$  coincide and another in which  $x_1$  and  $x_4$  coincide. We can think of this as saying that when two marked points approach one another in the limit, the curve sprouts an extra component to receive them. The genus of a stable curve  $C$  is the genus of the surface obtained from  $C$  by smoothening all its nodes.

**Theorem 4.2.2.** *For each  $n \geq 3$ , there is a smooth projective variety  $\overline{\mathcal{M}}_{0,n}$ , that is a moduli space for the stable  $n$ -pointed rational curves. It contains  $\mathcal{M}_{0,n}$  as a dense open subset.*

**Example 4.2.3.**  $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$ , which is a compactification of  $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ .

**Definition 4.2.4.** *The space  $\overline{\mathcal{M}}_{0,n}$  is called the **Deligne-Mumford** compactification of the moduli space  $\mathcal{M}_{0,n}$  of smooth algebraic curves.*

It is not nice when we let the points collide, say two points collide in another copy of  $\mathbb{P}^1$  and we draw it on a new twig. In this case let points 2 and 3 coincide.

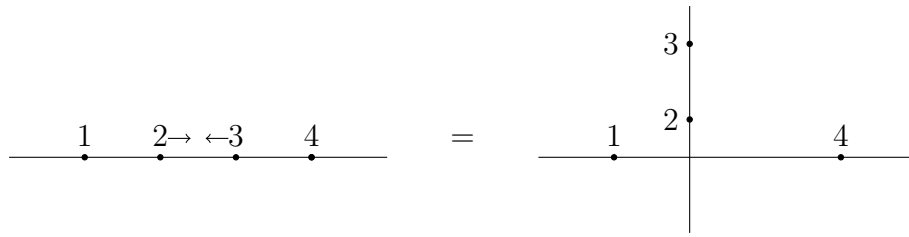


Figure 4.3: Two points colliding.

This is the idea that when two points collide, they pop out on a new twig. In this case we have three points, the two points 1 and 2 together with the node where the twig is attached. We develop the idea of a 'tree' from the 'twigs'.

**Definition 4.2.5.** *A tree of projective lines is a variety which:*

1. *Is a graph in theoretic sense (connected and no loops).*
2. *Has a twig as irreducible component, isomorphic to  $\mathbb{P}^1$ .*
3. *Has no more than two twigs crossing at any point.*

### 4.3 Boundary strata

We have seen that  $\overline{\mathcal{M}}_{0,n}$  contains  $\mathcal{M}_{0,n}$  as a dense subvariety. The boundary is simply the added points in the process of compactification. For example a point in the boundary might correspond to the following tree:

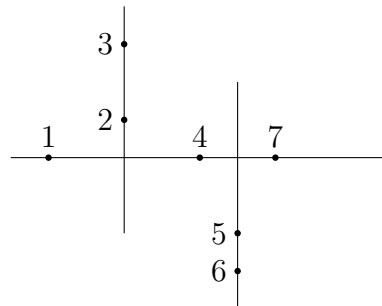


Figure 4.4: A point in the boundary.

We consider the subvariety of all points corresponding to the curves of certain form. We denote this by a tree in brackets, i.e:

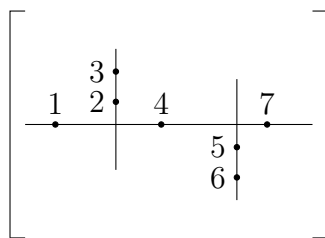


Figure 4.5: A boundary strata.

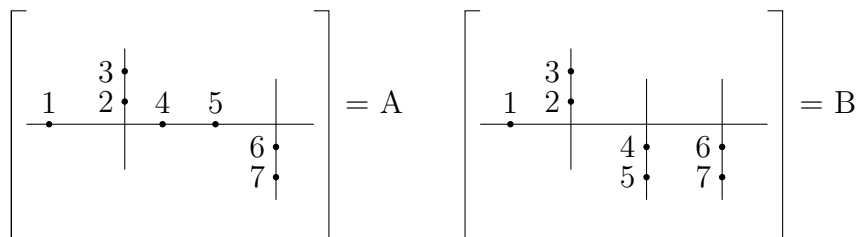
The brackets mean that a whole subset determined by the varying positions of the points. These subsets are called *boundary strata*, since the whole boundary can be expressed as a union of these subvarieties.

We can be able to determine the dimension of these boundary strata. We can view the automorphism of a tree as a bunch of automorphisms of twigs which are ‘glued together’. Thus we can just work out the dimension of the smaller moduli spaces and sum them up counting nodes as marked points on both twigs.

For example, in the case of the stratum above, the stratum corresponds to  $\overline{\mathcal{M}}_{0,5} \times \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3}$ . Thus the stratum has dimension  $2 = 2 + 0 + 0$ , i.e

$$\dim \mathcal{M}_{0,5} = 5 - 3 = 2 \text{ and } \dim \mathcal{M}_{0,3} = 3 - 3 = 0$$

In this example, we are in a space of dimension  $7 - 4 = 3$  and we have the dimension of the stratum as 2. The codimension of a boundary stratum is equal to the number of nodes of the tree. We can get inclusions among the boundary strata. In the example below we have that  $A \supseteq B$ .



The following are the boundary strata for  $\mathcal{M}_{0,4}$ .

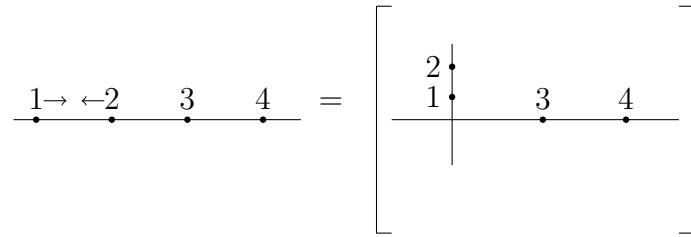


Figure 4.6: Limit curve when points 1 and 2 collide.

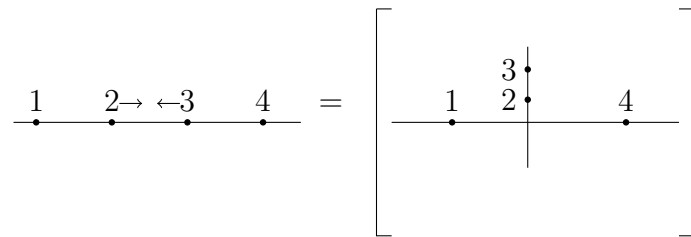


Figure 4.7: Limit curve when points 2 and 3 collide.

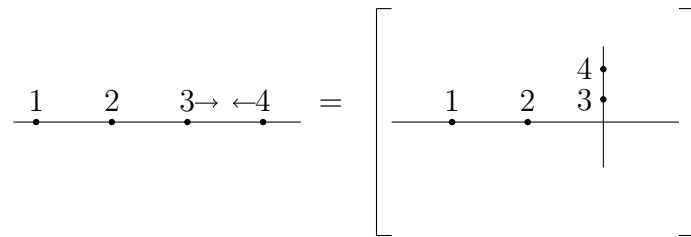


Figure 4.8: Limit curve when points 3 and 4 collide.

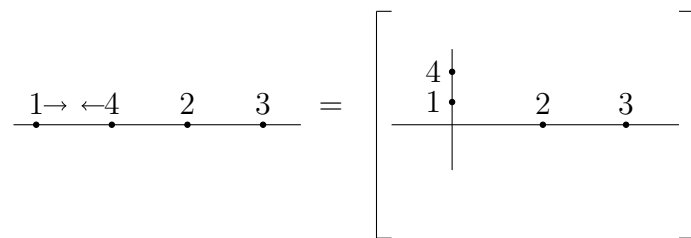


Figure 4.9: Limit curve when points 1 and 4 collide.



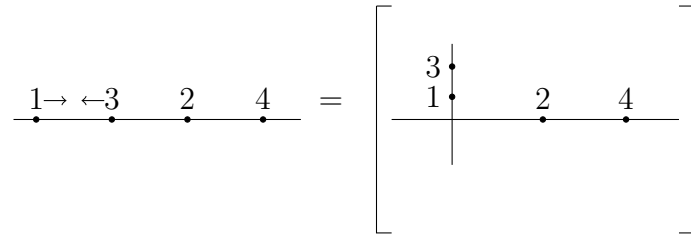


Figure 4.10: Limit curve when points 1 and 3 collide.

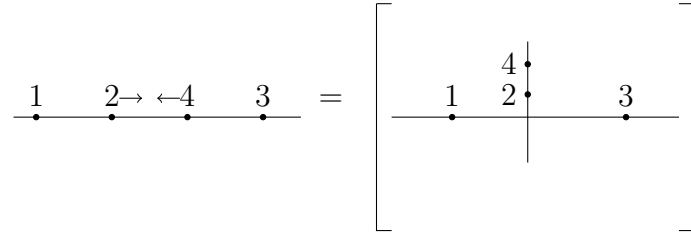


Figure 4.11: Limit curve when points 2 and 4 collide.

#### 4.4 Intersection theory of $\overline{\mathcal{M}}_{0,n}$

To understand the intersection theory of  $\overline{\mathcal{M}}_{0,n}$ , we need to understand the intersections of different boundary strata. Thus we are interested in intersecting the cohomology classes (elements in the boundary). When intersecting two strata we consider the two cases: transverse and non-transverse intersections.

Transverse intersections work out nicely, since the codimensions add up properly. For example, the intersection below is a transverse intersection.

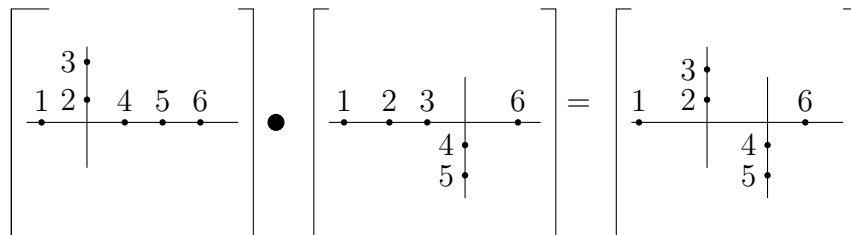


Figure 4.12: An example of a transverse intersection.

However, in non-transverse intersections, codimensions do not add up properly. Intersecting a boundary strata with itself gives the original strata since the one stratum degenerates the other. To help fix the non-transverse intersections, we introduce the  $\psi$ -classes.

**Definition 4.4.1.** For pairwise disjoint  $i, j, k \in \{1, \dots, n\}$ , denote by  $\delta_{i|j,k}$  the set of stable genus 0 curves with a node separating the  $i$ th marked point from the  $j$ th and  $k$ th marked points.

On  $\overline{\mathcal{M}}_{0,n}$  we have  $\psi_i = [\delta_{i|i,j,k}]$ . For example, in the intersection below, the intersection will just be the first stratum. This is because the left boundary stratum is a degeneration of the right one. This is therefore a non-transverse intersection. To 'wobble' to transversality, we first intersect the stratum with higher codimension with  $\psi$ -class at all common nodes of the strata.

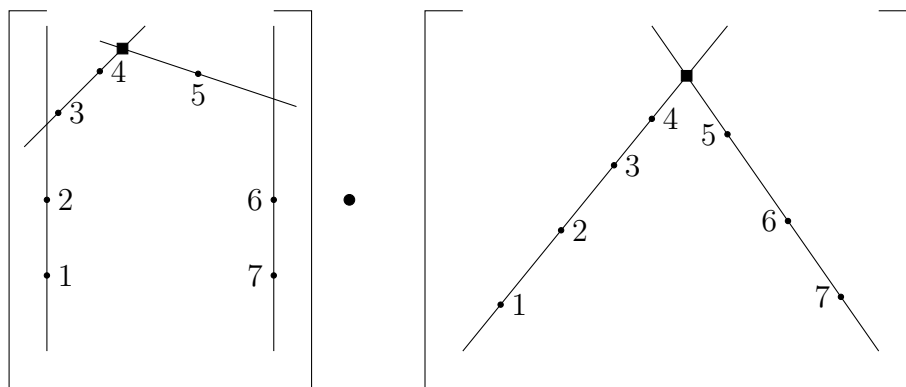
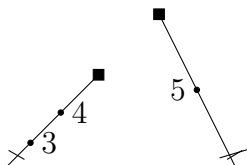


Figure 4.13: An example of a non-transverse intersection

We look at the intersection at the marked node (common node) and separate all the twigs but remembering their attached points. The diagram with the common node will look like this:



We intersect this strata with the  $\psi$ -class,  $\psi_{\blacksquare}$ . Let  $\psi_{\blacksquare}^L$  be the  $\psi$ -class at the mark  $\blacksquare$  on the left twig and  $\psi_{\blacksquare}^R$  be the  $\psi$ -class at the mark  $\blacksquare$  on the right twig.

We have

$$\psi_{\blacksquare}^L = \begin{array}{c} \diagup \\ \bullet 3 \\ \diagdown \end{array} \quad \begin{array}{c} \diagdown \\ \bullet 4 \\ \diagup \end{array} \quad \text{and } \psi_{\blacksquare}^R = 0.$$

The  $\psi$ -class which we intersect is  $\psi_{\blacksquare} = -\psi_{\blacksquare}^L - \psi_{\blacksquare}^R$ . gluing back everything together we find:

$$\psi_{\blacksquare} = \left[ \begin{array}{c} \diagup \\ \bullet 3 \\ \diagdown \end{array} \quad \begin{array}{c} \bullet 4 \\ \text{---} \\ \bullet 5 \\ \diagdown \end{array} \quad \begin{array}{c} \diagdown \\ \bullet 6 \\ \diagup \end{array} \right]$$

$\begin{array}{c} \bullet 2 \\ \text{---} \\ \bullet 1 \end{array} \quad \begin{array}{c} \bullet 7 \end{array}$

The  $\psi$ -class is contained in both strata we are intersecting, thus the intersection is the  $\psi$ -class.

# Chapter 5

## Enumeration of Rational Curves in $\mathbb{P}^2$

### 5.1 Moduli of stable maps

This section introduces moduli spaces of stable maps of curves of genus 0. The moduli space of curves is seen naturally as the moduli space of stable maps of curves. For the case of constant maps, these spaces and the moduli spaces of curves do coincide.

**Definition 5.1.1.** *Let  $(C, x_1, \dots, x_n)$  be a nodal curve with  $n$  points on it, then:*

1. *The curve  $C$  is prestable if the points  $x_i$  are distinct and in the smooth locus of  $C$ .*
2. *The prestable curve  $(C, x_1, \dots, x_n)$  is stable if every geometric genus 0 irreducible component has at least 3 special points on it (where a special point is one of the  $x_i$  or a node).*

**Definition 5.1.2.** *Let  $X$  be a smooth projective variety. A stable map is the datum of  $(C, x_1, \dots, x_n, f)$ , where*

1.  *$(C, x_1, \dots, x_n)$  is a prestable curve.*
2.  *$f : C \rightarrow X$  is a morphism.*
3. *If the map  $f$  contracts an irreducible component  $C_k$  to a point, then if  $C_k$  is of genus 0, it must have three special points on it.*

Let  $\beta \in H_2(X, \mathbb{Z})$ . We say that  $f : C \rightarrow X$  represents a homology class  $\beta$  if  $[C] \in H_2(C, \mathbb{Z})$  is the fundamental class of  $C$  and  $f_*[C] = \beta$ . For example if  $X = \mathbb{P}^2$  and  $f : C \rightarrow \mathbb{P}^2$  is a morphism such that  $f_*[C] = dl$ , where  $l \in H^2(\mathbb{P}^2)$  is the class of a line. We say that  $d$  is the degree of the map  $f$  and write  $d$  for  $dl$ .

**Definition 5.1.3.** *A pointed map of genus  $g$  is a morphism  $f(C, x_1, \dots, x_n) \rightarrow X$  that represents a class  $\beta$  of an  $n$ -pointed smooth curve  $C$ .*

Two pointed maps  $f_1 : (C, x_1, \dots, x_n) \rightarrow X$  and  $f_2 : (C, y_1, \dots, y_n) \rightarrow X$  are called isomorphic if there exists an isomorphism  $\phi : C_1 \rightarrow C_2$  of curves such that  $\phi(x_i) = y_i$  for all  $i$  and  $\phi$  admits the following commutative diagram as seen in [Ong13].

$$\begin{array}{ccc}
 & \phi & \\
 C_1 & \xrightarrow{\quad} & C_2 \\
 f_1 \searrow & & \swarrow f_2 \\
 & X &
 \end{array}$$

We can be able able to determine the space parametrizing isomorphism classes  $[f : (C, x_1, \dots, x_n) \rightarrow X]$  of pointed maps representing a class  $\beta$ , denoted by  $\mathcal{M}_{0,n}(X, \beta)$ .

$$\mathcal{M}_{0,n}(X, \beta) = \{f : C \rightarrow X \mid C \text{ is a prestable curve}\} / \sim .$$

We write  $(C, x_1, \dots, x_n, f)$  as an element in  $\mathcal{M}_{0,n}(X, \beta)$ . The moduli space  $\mathcal{M}_{0,n}(X, \beta)$  of maps is not compact because the maps can degenerate in various ways. There is however a compactification for  $\mathcal{M}_{0,n}(X, \beta)$ , called *Kontsevich compactification*, denoted as  $\overline{\mathcal{M}}_{0,n}(X, \beta)$ . The moduli space  $\overline{\mathcal{M}}_{0,n}(X, \beta)$  has the following properties, see for example [Ong13]:

1. Evaluation maps: There are  $n$  evaluation maps,  $ev_i : \overline{\mathcal{M}}_{0,n}(X, \beta) \rightarrow X$ , defined by:

$$ev_i(C, x_i, f) := f(x_i).$$

2. Forget map;  $\pi : \overline{\mathcal{M}}_{0,n+1}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0,n}(X, \beta)$ .

3. Let  $X$  and  $Y$  be smooth projective varieties. If  $g : X \rightarrow Y$ , then there is an induced map  $f_g : \overline{\mathcal{M}}_{0,n}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0,n}(X, g_*\beta)$ . If  $Y$  is a point, this gives a stabilization map

$$stab : \overline{\mathcal{M}}_{0,n}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0,n}.$$

4. There is a universal map over  $\overline{\mathcal{M}}_{0,n}(X, \beta)$ . If  $n_1 \geq n_2$  and  $\overline{\mathcal{M}}_{0,n_2}(X, \beta)$  exists, then there is a forgetful morphism

$$\overline{\mathcal{M}}_{0,n_1}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0,n_2}(X, \beta).$$

## 5.2 Enumeration of curves

We consider curves of degree  $d > 0$  and seek the answer to the question: How many curves of degree  $d$  pass through  $3d - 1$  points in general position in the plane? We call  $N(d) \in \mathbb{N}$  the answer to such question. A degree  $d$  rational curve in  $\mathbb{P}^2$  is given as the zero locus of an homogenous, degree  $d$  polynomial in 3 variables. The space of such curves is projective of dimension  $\frac{(d+2)(d+1)}{2} - 1$ .

In this space there is zariski open subset  $U^{sm}$  with the resulting curve smooth and of genus  $\frac{(d-1)(d-2)}{2}$ . We have another zariski topology  $U^{nod}$  is made of curves that have at worst nodal singularities. The 0 nodes in this case corresponds to  $U^{sm}$ . The formulae for such a curve will be  $\frac{(d-1)(d-2)}{2} - n$ , where  $n$  is the number of nodes.

The locus of such curves having  $k$ -nodes will have codimension  $k$ . The space of rational curves of degree  $d$  in  $\mathbb{P}^2$  has dimension  $3d - 1$ .

Since the dimension, say  $D$ , is given by

$$D = \frac{(d+2)(d+1)}{2} - 1 = \frac{d^2 + 3d}{2},$$

and the genus of the rational curve having  $k$ - nodes is:

$$\frac{(d-1)(d-2)}{2} - k = 0 \Rightarrow 2k = d^2 - 3d + 2.$$

Thus we have

$$D = \frac{3d + 2n - 2 + 3d}{2} = 3d - 1 + k.$$

If  $d = 1, 2$ , the numbers  $\frac{(d+2)(d+1)}{2} - 1$  and  $3d - 1$  coincide and we have  $g = 0$  for  $d = 1, 2$ . Thus we have  $N(1) = N(2) = 1$ . In general, we first translate our enumerative problem, in an intersection-theoretic problem on a moduli space.

**Definition 5.2.1.** *The moduli space of maps of degree  $d$  from a genus 0 curve is:*

$$\mathcal{M}_{0,n}(\mathbb{P}^2, d) := \{(C, x_1, \dots, x_n, \phi)\},$$

where  $C$  is a smooth genus 0 curve,  $x_i \in C, i = 1, \dots, n$ , are pairwise distinct points of it, and  $\phi : C \rightarrow \mathbb{P}^2$  is a degree  $d$  map.

The dimension of such a space is  $3d - 1 + n$ . There are well defined evaluation maps  $ev_i : \mathcal{M}_{0,n}(\mathbb{P}^2, d) \rightarrow \mathbb{P}^2$  :

$$ev_i(C, x_1, \dots, x_n, \phi) := \phi(x_i).$$

We now translate our enumerative question into the following problem. Let  $P_1, \dots, P_{3d-1}$  be general points in  $\mathbb{P}^2$ . We then compute the number of points

$$N_d := \#\{ev_1^{-1}(P_1) \cap \dots \cap ev_{3d-1}^{-1}(P_{3d-1})\}.$$

It is easy to work with a compactified moduli space. We have the following definition of  $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ .

**Definition 5.2.2.**

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) := \{(C, x_1, \dots, x_n, \phi)\},$$

where  $\{(C, x_1, \dots, x_n, \phi)\}$  is a rational  $n$ -pointed stable map of degree  $d$ .

**Theorem 5.2.3.** *The space  $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$  is a smooth compactification of  $\mathcal{M}_{0,n}(\mathbb{P}^2, d)$ .*

In the view of properties in page 28, we have the following result.

1. An evaluation map:  $ev_i : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \rightarrow \mathbb{P}^2 : ev_i(C, x_i, \phi) := \phi(x_i)$ .
2. A forgetful map that forgets the point  $x_i$  :  $forg_i : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \rightarrow \overline{\mathcal{M}}_{0,n-1}(\mathbb{P}^2, d)$ .
3. A forgetful map that remembers only the rational curve  $(C, x_i)$  :  $f : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \rightarrow \overline{\mathcal{M}}_{0,n}$ .

## 5.3 Preliminary for Kontsevich formula

### 5.3.1 The Grassmanian

The Grassmanian, denoted  $G(k, V)$ , is simply a set of all  $k$ -dimensional linear subspaces in  $V$ . Naturally it is a space parametrizing linear subspaces and is useful in enumerative problems. We get the following definition from [Ran10].

**Definition 5.3.1.** *The Grassmanian  $G(k, V)$  is defined as*

$$G(k, V) := \{W \subset V : W \text{ is a subspace, } \dim W = k\}.$$

To build up on understanding the Grassmanian structure we discuss the wedge product and exterior power of a vector space.

**Wedge product:** Given any two vectors  $v$  and  $w$ , we have  $v \wedge w$  as a 2- multivector or a blade that satisfies

$$v \wedge w = -w \wedge v.$$

We similarly define a  $k$ -multivector as  $v_1 \wedge \dots \wedge v_k$  which negates the product upon interchanging 2 adjacent elements.

**Exterior power of a vector space:** Given a vector space  $V$ , the  $k$ th exterior power of the vector space,  $\bigwedge^k V$  is the span of the  $k$ - blades in  $V$ . I.e  $\bigwedge^k V := \text{span}\{v_1 \wedge \dots \wedge v_k : v_i \in V\}$ .

We need the following definitions:

**Definition 5.3.2.** *A  $K$  multivector  $\omega \in \bigwedge^k V$  is said to be totally decomposable if can be written as  $k$ -blade, i.e  $\omega = \omega_1 \wedge \dots \wedge \omega_k$ .*

**Definition 5.3.3.** *Given a multivector  $\omega \in \bigwedge^k V, v \in V$  is said to be a divisor of  $\omega$  if  $\omega$  can be written as  $\omega = v \wedge \varphi$  such that  $\varphi \in \bigwedge^{k-1} V$ .*

Let  $W \in G(k, V)$  and associate to  $W$  a multivector  $\lambda = v_1 \wedge \dots \wedge v_k$ , where  $\{v_i\}$  span the space  $W$ . We have the map  $\psi : G(k, V) \rightarrow \mathbb{P}(\bigwedge^k V)$ , with  $W = \langle v_1, \dots, v_k \rangle \mapsto [\lambda]$ , where  $[\lambda]$  is the projectivized coordinate of  $\lambda$ , that is, linear subspace generated by  $v_1 \wedge \dots \wedge v_k$ . The map  $\psi$  is known as the plucker map and is an embedding of  $G(k, V)$  into  $\mathbb{P}(\bigwedge^k V)$ . The image  $\psi(G(k, V))$  is the projectivatization of the space of all totally decomposable vectors in  $\bigwedge^k V$ .



Let the coordinates of  $\mathbb{P}^n = \mathbb{P} \wedge^k V$  be plucker coordinates on  $G(k, V)$ . The Grassmanian  $G(k, V)$  is a variety, since it is the intersection of finitely many projective hypersurfaces see for example [Hud07].

### 5.3.2 Almost complex and symplectic structures

Let  $X$  be a smooth manifold. An almost complex structure on  $X$  is a smooth family of linear maps  $J_p : T_p X \rightarrow T_p X$  such that  $J_p(J_p v) = -v$  for all  $v \in T_p X$  and  $p \in X$ . Every complex  $n$ -manifold  $X$  carries a natural almost complex structure  $J$ , defined as follows: [Zin10]

**Definition 5.3.4.** Let  $\{(U_i, \phi_i, U_i^i)\}_{i \in I}$  be the (homolophic) atlas for  $X$ . If  $p \in U_i$  we set

$$J_p = d\phi_i|_{\phi_i^{-1}(p)} \circ i \circ d\phi_i^{-1}|_p.$$

An almost complex structure arising in such a way is called complex or integrable.

Let  $X$  be a smooth manifold. A symplectic form on  $X$  is a closed two-form  $\omega$  on  $X$  which is nondegenerate at every point of  $X$ . I.e  $d\omega = 0$ , and for every point in  $X$  and nonzero tangent vector  $v \in T_p X$ , there exists  $w \in T_p X$  such that  $\omega_p(v, w) \neq 0$ . For example if  $(x_1, y_1, \dots, x_n, y_n)$  are standard coordinates on  $\mathbb{C}^n$ ,

$$\omega \equiv dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n,$$

is a symplectic form on  $\mathbb{C}^n$ . In general, if  $X$  admits a symplectic form, the (real) dimension of  $X$  is even.

### 5.3.3 Tautological line bundle

Let  $\gamma = \{(l; z_0, \dots, z_n) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : (z_0, \dots, z_n) \in l\}$ . We denote by  $\pi : \gamma \rightarrow \mathbb{P}^n$  the projection map. For each  $l \in \mathbb{P}^n$  we have the fiber  $\omega l = \pi^{-1}(l)$  over a point  $l \in \mathbb{P}^n$  is the line  $l$  through the origin in  $\mathbb{C}^n$ .

For each  $i = 0, \dots, n$  let

$$\tilde{U} = \pi^{-1}(U_i) = \{(l; z_0, \dots, z_n) \in \omega : z_i \neq 0\}$$

$$\tilde{\phi}_i : \mathbb{C}^n \times \mathbb{C} \rightarrow \tilde{U}_i, \tilde{U}(w_1, \dots, w_n; \lambda) = (\phi_i(w_1, \dots, w_n); \lambda w_1, \dots, \lambda w_i, \lambda w_{i+1}, \dots, \lambda w_n).$$

The set  $\{(\tilde{U}_i, \tilde{\phi}_i, \mathbb{C}^n) \times \mathbb{C}\}$  is the standard atlas for  $\gamma$ .  $\gamma$  is a complex  $(n+1)$  manifold and  $\gamma \rightarrow \mathbb{P}^n$  is a holomorphic line bundle.

**Lemma 5.3.5.** *If  $p$  is a homogenous polynomial of degree  $d$  in  $n + 1$  variables,  $s_p$  is a holomorphic section of the holomorphic line bundle  $\gamma^{\otimes *d}$ , where  $s_p(l)$  is a map from  $\gamma$  to  $\mathbb{C}$  such that*

$$\{s_p(l)\}(l; z_0, \dots, z_n) = p(z_0, \dots, z_n).$$

*Conversely, if  $s$  is a holomorphic section of  $\gamma^{\otimes *d}$ , then  $s = s_p$  for some homogenous polynomial  $p$  of degree  $d$  in  $n + 1$  variables.*

## 5.4 Kontsevich formula

Let  $x_0, x_1, x_2$  and  $x_3$  be four points in  $\mathbb{P}^2$  given by

$$x_0 = [1 : 0 : 0], x_1 = [0 : 1 : 0], x_3 = [1, 1, 1].$$

We denote by  $H^0(\mathbb{P}^2 : \gamma^{*\otimes 2})$  the space of holomorphic sections of the holomorphic line bundle  $\gamma^{*\otimes 2} \rightarrow \mathbb{P}^2$ , i.e the space of the degree-two homogenous polynomials in 3 variables. Let  $\mathcal{U} = \{([A; B] : [z_0, z_1, z_2] \in \mathbb{P}^1 \times \mathbb{P}^2 : (A - B)z_0z_1 - Az_1z_2 + Bz_0z_2 = 0)\}$ . The space  $\mathcal{U}$  is a compact complex manifold.

Let  $\pi : \mathcal{U} \rightarrow \overline{\mathcal{M}}_{0,4} \equiv \mathbb{P}^1$  denote the projection onto the first component. If  $[A, B] \in \overline{\mathcal{M}}_{0,4}$ , the fiber  $\pi^{-1}([A, B])$  is the conic

$$C_{A,B} = [z_0, z_1, z_2] \in \{\mathbb{P}^2 : (A - B)z_0z_1 - Az_1z_2 + Bz_0z_2 = 0\}.$$

If  $[A, B] \neq [1, 0], [0, 1], [1, 1]$ ,  $C_{A,B}$  is a smooth complex curve of genus zero, i.e a sphere with four marked points. On the other hand, if  $[A, B] = [1, 0], [0, 1], [1, 1]$ ,  $C_{A,B}$  is a union of two lines each containing two marked points and they intersect at a single point. [Zin10] The concept of cross ratio is important in deriving the kontsevich formula. Denote  $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^2, d)$  by  $M_d$ . On  $M_d$ , we have a rational function  $\varphi$ , given by a cross ratio: at a point of  $M_d$  corresponding to a map  $f : (C; p_1, p_2, p_3, p_4) \rightarrow \mathbb{P}^2$ , with  $C$  irreducible, it is the cross ratio of points  $p_1, p_2, p_3, p_4 \in \mathbb{P}^1$ ; i.e in terms of affine coordinate  $z$  on  $\mathbb{P}^1$ ,

$$\varphi = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \text{ where } z_i = z(p_i). \quad (5.1)$$

Cross ratio takes on the values  $[0 : 1], [1 : 0], [1 : 1]$  only when two points coincide.

The Kontsevich recursive formula seeks to answer the question: how many rational curves (finite) pass through  $3d - 1$  points of  $\mathbb{P}^2$ . Such a number is denoted as  $N(d)$  as indicated before.

In 1993, Kontsevich found the recursive formula, given by:

$$N(d) = \sum_{d=1}^{d-1} d_1 d_2 \left[ d_1 d_2 \binom{3d-4}{3d_1-2} - d_2^2 \binom{3d-4}{3d_1-3} \right] N(d_1) N(d_2). \quad (5.2)$$

The next discussion aim at proving equation 5.2.

We introduce a curve  $B \subset M_d$ , which we will make calculations on. Fix a point  $p \in \mathbb{P}^2$  and two lines  $L$  and  $M$  where  $L, M \subset \mathbb{P}^2$ , both passing through point  $p$ . Fix two more general points  $q, r \in \mathbb{P}^2$  and a collection  $\Gamma \subset \mathbb{P}^2$  of  $3d - 4$  general points. We have the locus

$$B = \left\{ f : (C; p_1, p_2, p_3, p_4) \rightarrow \mathbb{P}^2 \mid f(p_1) = q, f(p_2) = r, f(p_3) \in L, f(p_4) \in M \text{ and } \Gamma \in f(C) \right\} \subset M_d.$$

The space of rational curves of degree  $d$  in  $\mathbb{P}^2$  has dimension  $3d - 1$ , and we want curves in our family to pass through  $3d - 2$  points, i.e points  $q, r$  and  $3d - 4$  points of  $\Gamma$ . The locus  $B$  is a curve.

There may be points in  $B$  for which the domain  $C$  of the corresponding map  $f : (C; p_1, p_2, p_3, p_4) \rightarrow \mathbb{P}^2$  is reducible. If the image of  $C$  has components  $D_1, \dots, D_k$  of degrees  $d_1, \dots, d_k$ , then  $D_i$  can contain at most  $3d_i - 2$  points  $\Gamma \cup \{q, r\}$ . Thus

$$3d - 2 \geq \sum_{i=1}^k (3d_i - 1) = 3d - k,$$

whence  $k \leq 2$ , since

$$\sum_{i=1}^k (3d_i - 1) = 3(d_1 + \dots + d_k) - k = 3d - k.$$

In these cases  $C$  will have at most two components. There are only finite points in  $B$  for which the domain  $C$  is reducible.

If  $D = D_1 \cup D_2 \in \mathbb{P}^2$ , with  $D_i$  a rational curve of degree  $d_i$ , and  $\Gamma \cup \{q, r\} \subset D$ , then  $D_i$

must contain exactly  $3d - 1$  points of the  $3d - 2$  points  $\Gamma \cup \{q, r\}$ . The number of such plane curves is thus:

$$\binom{3d - 2}{3d_1 - 1} N(d_1)N(d_2).$$

We now equate the number of zeros and of poles of  $\varphi$  on  $B$ . First we consider the points  $f : (C; p_1, p_2, p_3, p_4) \rightarrow \mathbb{P}^2$  of  $B$  with  $C$  irreducible. Since  $f(p_1) = q$  and  $f(p_2) = r$  are fixed and do not lie on lines  $L$  and  $M$ , for points  $p_i$  to coincide on the curve then we must have  $f(p_3) = f(p_4) = p$  where  $L \cap M = p$ . Such points are zeros of  $\varphi$ : the number of zeros is given by the number of rational plane curves of degree  $d$  through the  $3d - 1$  points  $p, q, r$  and  $\Gamma$ .

The other case is when  $C = C_1 \cup C_2$  is reducible. We can be able to determine the number of zeros and poles of  $\varphi$  coming from points  $f : (C; p_1, p_2, p_3, p_4) \rightarrow \mathbb{P}^2$  of  $B$ . We get a zero of  $\varphi$  at such a point if and only if  $p_1$  and  $p_2$  lie on one component of  $C$  and  $p_3$  and  $p_4$  lie on the other. If the degree of  $C_1$  is  $d_1$  and  $C_2$  is  $d_2$  with  $d_1 + d_2 = d$ , then  $f(C_1)$  must contain  $q, r$  and  $3d_1 - 3$  points of  $\Gamma$ , while  $C_2$  contains  $3d - 4 - (3d_1 - 3) = 3d_2 - 1$  points of  $\Gamma$ .

For any subset of  $3d_1 - 3$  points of  $\Gamma$ , the number of such plane curves is  $N(d_1)N(d_2)$ , and for each such plane curve there are  $d_2$  choices of the point  $p_3 \in C_2 \cap f^{-1}(L)$  and  $d_2$  choices of the point  $p_4 \in C_2 \cap f^{-1}(M)$ . We as well have  $d_1 d_2$  choices of the point  $f(C_1 \cap C_2) \in f(C_1) \cap f(C_2)$ . The total number of zeros of  $\Gamma$  arising in this way are:

$$\sum_{d_1=1}^{d-1} d_1 d_2^3 \binom{3d - 4}{3d_1 - 3} N(d_1)N(d_2). \quad (5.3)$$

We count the poles of  $\Gamma$  similarly. Letting  $p_1$  and  $p_3$  lie on  $C_1$ , and  $p_2$  and  $p_4$  lie on  $C_2$ . Again let the degree of  $C_1$  be  $d_1$  and for  $C_2$  be  $d_2$  with  $d_1 + d_2 = d$ . Then  $f(C_1)$  must contain  $q$  and  $3d_1 - 2$  points of  $\Gamma$ , plus  $r$ . For any subset  $3d - 2$  points of  $\Gamma$ , the number of such plane curves is curve is  $N(d_1)N(d_2)$ , and for each plane curve there are  $d_1$  choices of the point  $p_3 \in C_2 \cap f^{-1}(L)$ , and  $d_2$  choices of the point  $p_4 \in C_2 \cap f^{-1}(M)$ , as well as  $d_1 d_2$  choices of the point  $f(C_1 \cap C_2) \in f(C_1) \cap f(C_2)$ . Thus the total number of poles of  $\Gamma$  is given by:

$$\sum_{d_1=1}^{d-1} d_1^2 d_2^2 \binom{3d - 4}{3d_1 - 2} N(d_1)N(d_2). \quad (5.4)$$

Adding up the poles and zeros we get:

$$N(d) = \sum_{d_1=1}^{d-1} d_1 d_2 \left[ d_1 d_2 \binom{3d-4}{3d_1-2} - d_2^2 \binom{3d-4}{3d_1-3} \right] N(d_1) N(d_2). \quad (5.5)$$

Equation 5.5 is a recursive formula that helps determine  $N(d)$  if we know  $N(d')$  for  $d' < d$ . For example, there is a unique line through two points, and a unique conic through 5 general points, so  $N(1) = N(2) = 1$ . The following examples give calculations for  $N(3)$  and  $N(4)$ .

**Example 5.4.1.** *For the case where  $d = 3$ ,*

$$N(3) = 2 \left[ 2 \binom{5}{1} - 4 \binom{5}{0} \right] + 2 \left[ 2 \binom{5}{4} - 4 \binom{5}{3} \right] = 12.$$

*There are 12 rational cubic curves through 8 general points of  $\mathbb{P}^2$ .*

**Example 5.4.2.** *For the case where  $d = 4$ ,*

$$N(4) = 3 \cdot 12 \left[ 3 \binom{8}{1} - 9 \binom{8}{0} \right] + 4 \left[ 4 \binom{8}{4} - 4 \binom{8}{3} \right] + 3 \cdot 12 \left[ 3 \binom{8}{7} \binom{8}{6} \right] = 620.$$

*Namely, 620 rational quartic curves through 11 general points of  $\mathbb{P}^2$ .*

For  $d = 6$ , we have the number  $N(6) = 26312976$ .

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