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FORECASTING MORTALITY RATES AND MODELLING LONGEVITY RISK USING LEE CARTER MODEL

A PROJECT SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMNTS FOR MASTER OF SCIENCE IN ACTUARIAL SCIENCE

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DECLARATION

This report is my own original work and not a duplication of similarly published work of any scholar for academic purposes and has not been submitted to any other institution of higher learning for the award of certificate, diploma or degree or any other award.

I further declare that all materials cited in this report which are not my own have been duly acknowledged.

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ABSTRACT

The retirement benefits provided by annuity providers and pension plans imply are often guaranteed until the death of the pensioners. Trends in mortality/longevity have clearly emerged as a result of increase in life expectancy/ reduction in mortality rates at old age. This has necessitated academicians and actuaries to focus their interest in the field of mortality and longevity risks in particular. The new NSSF Act No. 45 of 2013established a pension fund that is mandatory for all workers in the formal economy as opposed to a provident fund. This exposes the annuity providers to longevity risk among other risks when the scheme members retire.

Appropriate modelling tools or projected life tables are needed for pricing and reserving. In particular the use of stochastic models that allows for various risk causes and components and the relevant impact on portfolio results as opposed to the deterministic models that were only based on the expected present values.

For the purpose of this project, I am using the Lee- Carter Model proposed by Lee and Carter in 1992 to fit mortality rates, forecast mortality trends in an ARIMA framework and then obtain the life expectancy projections. As regards to the longevity risk, I consider the possibility of changing the annuity benefits or calculating the annuity benefits by relating the benefits to the experienced mortality, or to updated mortality forecasts therefore calculating the actuarial present value on annuity .

Table of Contents

ACKNOWLEDGEMENT 3
ABSTRACT 4
CHAPTER 1: INTRODUCTION 7
1.1. Background 7
1.2. Statement of Problem 12
1.3. Objectives 13
1.3.1. General Objective 13
1.3.2. Specific Objectives 13
1.4. Significance of the Study 14
CHAPTER 2: LITERATURE REVIEW 15
2.1. Early Actuarial Models 16
2.2. Stochastic Mortality Models 19
2.2.1. The Lee- Carter Model (1992) 19
2.2.2. Renshaw and Haberman (2003) Model 22
2.2.3. Renshaw and Haberman (2006) Cohort Model 22
2.2.4. Age-Period-Cohort Model 24
2.2.5. P- Splines 24
2.2.6. Cairns, Blake and Dowd Model -1 26
2.2.7. Cairns, Blake and Dowd Model -2 26
2.2.8. Cairns, Blake and Dowd Model -3 27
2.2.9. Cairns, Blake and Dowd Model -4 28
2.3. Model Selection Criteria 28
CHAPTER 3: METHODOLOGY 32
3.1. Basic Building Blocks 34
3.1.1. Initial Rate of Mortality 34
3.1.2. Central Rate of Mortality 34
3.1.3. Instantaneous Force of Mortality 35
3.1.4. Expected Future Lifetime 36
3.1.5. Age Specific Death Rates 37
3.2. The Lee and Carter Model (1992) 38
3.3. Estimation Approaches 40

3.3.1. The Singular Value Decomposition Approach 40
3.3.2. Maximum Likelihood Estimation 44
3.3.3. Weighted Least Squares 45
3.4. Forecasting 47
3.5. Actuarial Present Value 48
CHAPTER 4: DATA ANALYSIS AND RESULTS 50
4.1. Source of Data 50
4.2. Description of software used 51
4.3. Assumptions 51
4.4. General Analysis of the Data Used 52
4.5. Fitting the Model 54
4.6. Model Evaluation 56
4.7. Forecasting 57
4.8. Performing Actuarial Projections 59
CHAPTER 5: CONCLUSIONS AND RECOMMENDATIONS 61
5.1. Conclusions 61
5.2. Recommendations 62
5.2.1. Longevity Risk Management 62
REFERENCES 66
APPENDICES 68
APPENDIX A: Proof of Low Rank Approximation 68
APPENDIX B: R CODES 70
APPENDIX C: Residual Analysis 74
APPENDIX D: Actuarial Projections 77

Figure 1: Log death rates against age from 0 to 110 52
Figure 2 : Log death rates against time from 1933 to 2010 53
Figure 3: Parameter Estimates ax, bx and kt of Lee-Carter Model 55
Figure 4: Projected values of kt 58

CHAPTER 1: INTRODUCTION

1.1. Background

Ageing populations have been recognized as one of the risks in pension schemes in the world. The length in time people are expected to live have increased during the 20th Century as well as the proportion of retired to working people has increased. As long as gains in the life expectancy are foreseeable and taken into account while planning for retirement, they would have a negligible effect on retirement finances or to annuity providers mainly the insurance companies. Regrettably, improvements in mortality and life expectancy are uncertain and therefore results to longevity risk.

Longevity risk is the risk to which pension schemes or annuity providers are exposed to paying out higher amounts of benefits than expected in future. The risk exists due to increasing life expectancy trends among policy holders and pensioners. When Longevity risk is not catered for, it can cause insolvency and cause individuals to lose their hard earned retirement income. Therefore pension schemes and annuity providers need to effectively manage the longevity they are exposed to.

Defined benefit pension plans and annuities which guarantee lifetime benefits for pensioners are the main types of plans exposed to longevity risk. Annuitants receiving income till death may live longer than expected or accounted for in the actuarial calculations to provision of the liabilities in cases of defined benefit pension plan and thus longevity risk. When annuities and insurance benefits are priced and reserved using period-based assumptions, the underestimation of liabilities because of mortality improvement is reduced.

Over the past few years, most companies have established a defined contribution (DC) scheme in order to reduce the risks that are associated with defined benefit (DB) schemes. Both DB and DC plans are meant to provide members with sufficient financial means to be able to retire and maintain a certain and adequate standard of living throughout retirement. Increase in life expectancy due to improved lifestyle and medical advances have meant that people are living longer. In a DB scheme, the risk of increasing longevity is borne by the scheme sponsor therefore companies with Defined Benefit pension plans have paid attention to rising longevity as the rising life expectancy affects the funding costs of DB pension plans. Some companies have specifically a reserve that is purely to take care of longevity risk (longevity reserve).

Even though longevity has no direct effect on the funding costs of DC plans, longevity is much of an issue in DC plans. At retirement, the total fund credit in the DC scheme member's account is converted to a life annuity using an annuity factor. The estimated length of life of the pensioner is normally set by the annuity providers to be equal to the life expectancy of the member's birth cohort. The annuity factor is then divided into the total fund credits to get the total annual pension). Therefore if the life expectancy is underestimated, this will result to annuity providers paying out more than expected.

As life expectancy rises/ mortality rates reduces, the regulators have so far used the increase in retirement age (RA) as a logical counter balance in providing a useful instrument to rebalance lifetime, consumption, saving and investing to derisk the DC pension plans. The retirement age was increased from 55 to 60 for the pension plans in Kenya. In addition, the minimum retirement ages for lecturers in Kenyan republic universities was increased from 60 years to 70 years to take into account longevity improvements, population ageing and the financing of pension.

Summarily, the longevity risk that insurance companies face can be categorized as follows:

I. Risk of reduction in mortality rates beyond expected levels:

In the past few years, mortality rates have decreased for all ages and gender. The rate of decline however has been unstable and therefore it is difficult to estimate what the decline in mortality rates over the next decades will be.

II. Risk of experience assessment error:

This is the risk related to the difference in mortality rates between the overall population and the pension fund-specific population. This is as a result of the fund having certain groups that experience higher or lower mortality than the general population.

III. Risk of random fluctuation in plan experience:

This is the risk that an individual outlives their predicted mortality range purely due to chance. This risk is more relevant if the plan lacks diversification.

In many countries including Kenya, statistical evidence shows that adult mortality has declined and life expectancy increased over the 20th Century and in particular over its last decades. This is the case in the 21st Century. According to United Nations' Department of Economics and Social Affairs, in Kenya adult male mortality rate (per 1000 male adults) was measured at 369.54 in 2011 decreasing from 473.07 in 2002. In addition, the adult female mortality rate (per 1000 female adults) in Kenya was 348.35 as of 2011 decreasing from 441.45 in 2002. Life expectancy was estimated at 52.95 in 2002 and has risen to 60.37 in 2011. Such trend in reduction of mortality and increase in life expectancy exposes insurers and annuity providers to risks in case

they use tables that do not take the mortality trends into account. To mitigate this risk, it is important to use the life table or annuity values that include forecasts of the future mortality trends, the projected tables.

As a result of this uncertainty surrounding future developments in mortality and life expectancy, individuals run the risk of outliving their resources and being forced to reduce their standard of living at old ages. Pension funds and annuity providers mainly the insurance companies on the other hand run the risk that the net present value of their annuity payments will turn out higher than expected as they will have to pay out periodic sum of income that will last for an uncertain life span. The private pension funds and national governments providing retirement benefits as well as financial institutions providing lifetime annuity payments face this longevity risk.

Longevity can be hedged with reinsurance contracts and with longevity derivatives. For instance annuity providers and pension schemes can use a longevity bond which pays coupon that is proportional to the number of survivors in a selected birth cohort. Longevity risk is however not easy to transfer, as it is hard to understand, and therefore to manage. In particular, because of its long-term nature, accurate longevity projections are delicate and modelling the embedded interest rate risk remains a challenge. As to better manage longevity risk, prospective life tables, containing longevity trend projections are used.

Life tables are tables that are used to depict the mortality experience of a population. There are several summary statistics that can be derived using life tables; this includes life expectancy at birth which estimates longevity. There are two types of life tables' i.e. Period and cohort life tables which can be distinguished by the methods used to calculate the age-specific probabilities of death.

Period life tables are life tables in which age-specific probabilities are calculated using the number of deaths and the population size in the current year. This table makes no allowance for later actual or projected changes in mortality. While Cohort life tables are life tables in which age-specific probabilities are calculated using mortality data from a group of individuals born in the same year and followed until all the cohort members are dead. The cohort effects refers to historical factors that are specific to a year of birth e.g. introduction of a new drug or vaccines or to a group of birth years e.g. smoking habits or women's professional activity level.

In using the period life table, the actual longevity of a population is not measured. F. Pelletier et al (1997), investigated mortality in Quebec, Canada during 1800s. Quebec mortality data from 1891 was used to construct a period life table where they estimated female life expectancy to be forty-five years. Contrary to the estimation, the cohort life table showed that the life expectancy of women in Quebec women born in 1891 was fifty one years.

The study of longevity and mortality forecast even more crucial in the present context. Therefore this project assesses how pension schemes, annuity providers mainly the insurance companies and the regulatory framework can access future improvements in mortality and life expectancy. This is by examining and modelling the longevity risk by first fitting and forecast the mortality rates and ensure that the mortality tables used by pension schemes and annuity providers are appropriate or recommend actuarial values. The final section will identify the best practices and discuss the management of longevity risk, putting forward a set of policy options to encourage and facilitate the management of longevity risk.

1.2. Statement of Problem

Longevity has not only increased, the trend has become more uncertain. This has exposed the insurance companies, governments and pension plans to the risk of longer and uncertain post retirement periods. With the decline in mortality rates and increase in life expectancy, national security systems, pension schemes and annuity providers of most developed countries have reconsidered their mortality tables taking into account longevity risks. In recent years, developing countries, including Kenya have experienced the decline in mortality rates and increase in life expectancy. Such trends in mortality reductions and increase in life expectancy especially at retirement age present risks to annuity providers and pension schemes that have priced annuities on the basis of mortality tables that do not take these trends into account and therefore developing countries opt to consider the longevity risk while pricing and reserving annuities.

Earlier actuarial models of forecasting the trends disregarded the stochastic nature of mortality. Therefore, understanding how the future mortality trend using the stochastic models is likely to interest to the actuary in pricing and reserving of annuities. Later several stochastic approaches have been used by demographers and actuaries in forecasting mortality exploring the different ranges of stochastic models. Lee and Carter model is the first stochastic model to consider increased life expectancy has become widely used and several extensions and modifications have been proposed to arrest the main features of mortality intensity.

1.3. Objectives

1.3.1. General Objective

In this project, our main objective is to forecast the mortality rates and then quantifying the longevity risk. The first task is to check and confirm the decreasing trend in mortality rates and increase in life expectancy. We then try to quantify the longevity risk that the pension funds and the annuity providers face as a result of decreasing mortality rates using appropriate forecasting and longevity models keenly focusing on the Lee and Carter (1992) model.

1.3.2. Specific Objectives

In regards to the study, the specific objectives are:

1. Stochastic forecasting of future mortality;
2. Longevity risk measurement; and
3. To highlight and discuss a few options available to manage the longevity risk.

1.4. Significance of the Study

The new NSSF Act No. of 2013 assented to law by the president of the Republic of Kenya on 24th December 2013 established a pension fund that is mandatory and will cover all workers in the formal economy. Under the Act, the pension fund will pay workers monthly pension (annuities). It is therefore important for government and annuity providers in Kenya to properly allow for upward trend in life expectancy and decline in mortality rates and the so called longevity risk.

The project would be important to practitioner and academicians both in the private and public sector by contributing to the existing body of knowledge in the area of mortality forecasting and accessing and quantifying longevity risk. The researcher will also be in a better position to identify a better solution to extreme mortality changes that could affect the financial position of the pension schemes and the annuity providers since omission or miscalculation of the risk could potentially lead to disastrous financial outcome.

The research will be important to practitioner who would like to come up with more reasonably priced products suited for the Kenyan population or any other developing country to enable them manage and transfer longevity risk as it provides the guidance on longevity modelling. This will result to the application of the risk transfer options: buy-out, buy-in and longevity swap used in other countries. Additionally securities such longevity bonds and indexes may be priced ensure that longevity risk is hedged.

CHAPTER 2: LITERATURE REVIEW

Improvement in population longevity is a topic which has become increasingly important in the recent years especially in the 21st Century. Various projects have been made and proposal on different models made to try and explain the causes of decrease in mortality and increase in life expectancy especially when individuals reach the retirement age. Topic on longevity has featured in various actuarial publications such as the British Actuarial Journal (2009). It has also been discussed in various publications such as International Monetary Fund (2012) and World Economic Forum (2010, 2012).

Population projections are normally done with assumptions being factored in order to obtain a realistic projection of the future population projection. The assumptions considered include the expected future rate of fertility, mortality and migration. In this project, the main concern is in fitting and forecasting mortality rates and therefore estimating future life expectancy that is finding the average length of future life and thus models the longevity risks.

Costa D.L. (2005) highlights some of the possible causes of improving longevity at older ages. This includes technological improvement, reduced infectious disease rates, reduced occupational stress as people no longer do manual works, life style changes, rising income. Brockmann et al..(2000) discussed improved lifestyle, medical care and individual economic resources factors as potential determinants of the decline in old age mortality. Increasing longevity has been recognized as a threat to pension funds and annuity providers.

Booth and Tickle (2008) categorized the mortality models into extrapolative models, explanatory model and expectations model. The most successful approach to modelling mortality in recent

decades has been extrapolative model which relies heavily on data which has become more and more reliable in recent years.

2.1. Early Actuarial Models

As noted by Cramer and Wold (1935), the earliest attempt to project mortality is probably due to Swedish astronomer H. Gylden in 1875 where he fitted a straight line to the sequence of general death rates of the Swedish population during the years 1750 to 1870. Mortality trends and their effects on pension annuities were perceived at the beginning of the 20th century. Nordenmark (1906), for example points out that, improvements in mortality must be carefully considered when pricing life annuities and, in particular, cohort mortality should be addressed to avoid underestimation of mortality related liabilities.

The earliest formula was by a French Mathematician De Moivre (1725) who wrote the survival function as $s(x) = 1 - x/\omega$ where ω is the limiting age and deaths are assumed to be uniformly distributed. Later on, British actuary Gompertz (1825) suggested that a law of geometric progression pervades in mortality after a certain age. Even though the model overestimates death rates at ages greater than 80, he observed that for the age grouping of between 20 and 60 years, the force of mortality increased almost exponentially with age hence he proposed the following model:

$$\mu(x) = \alpha \exp(\beta x) \tag{2.1}$$

Where α and β are positive parameters.

Makeham (1860) extended the Gompertz model by adding a constant to give:

$$\mu(x) = \gamma + \alpha \exp(\beta x) \tag{2.2}$$

where all the parameters used are positive real numbers.

The right hand side has two terms, the mortality γ which is independent of age and the mortality $\alpha \exp(\beta x)$ which depends on age.

In order to correct the weakness of Gompertz model, several models were proposed. Thorvald Thiele in 1867 focused on the following model that represents the whole lifespan:

$$\mu(x) = \alpha_1 \exp(-\beta_1 x) + \alpha_2 \exp(-\beta_2 (x-\eta)^2) + \alpha_3 \exp(\beta_3 x) \tag{2.3}$$

The parameters are used positive real numbers.

The first part of the right hand side represents the decreasing mortality at very young ages after the young ones have survived the risks at birth. The second part represents the mortality hump at young-adult ages as a result of accidents or drug abuse that is, it is at the young-adult age that mortality increases due to lifestyle and accidental effects such as excessive drinking, careless driving or drug abuse. The third part represents mortality at adult and old ages. Note that if $\alpha_1 = \beta_1 = \beta_2 = 0$, we obtain a special case of Thiele model know as Makeham law.

W.F. Perks (1932) logistic model is a linear generalization of Gompertz curve gives a relatively good fit to mortality rates over the entire adult range. The model is represented by:

$$\mu(x) = \frac{\alpha \exp(\beta x) + \gamma}{\varepsilon \exp(-\beta x) + \delta \exp(\beta x) + 1} \tag{2.4}$$

All the parameters used are positive real numbers. If we let $\varepsilon = \delta = 0$ we get Makehams law.

Heligman and Pollad (1980) curve also provides a relatively good fit to mortality rates over all ages and the number of parameters is no longer an issue. He proposed the following model:

$$\frac{q_x}{p_x} = A^{(x+b)^c} + D\exp[-E(\ln x - \ln F)^2] + GH^x \tag{2.5}$$

However, studies were conducted to prove practicability of the early actuarial models for instance (Stoto & Arthur, 1983) revealed many errors in the forecasts using the deterministic models. Also, they noted that decline in the old age mortality was also underestimated and increases in life expectancy under projected. Therefore the use of deterministic actuarial models in fitting and forecasting yields wrong forecasts and hence leads to wrong conclusions. Further reviews of earlier contributions to mortality forecasts were provided by Pitacco (2004), (Tuljapurkar & Boe, 1998) and (Wong-Fupuy and Haberman,2004).

Recent advances in the actuarial practices especially in pensions and life mathematics have resulted in proposal of more models for describing and projecting mortality. (Pitacco, Denuit, Haberman, & Oliviera, 2009) carried out a convenient survey and exposition of the models. One of the most important features of the recent models is that they are stochastic as opposed to being deterministic. Stochastic models seem more appealing because they associate a confidence error to each estimate. In addition, the value of annuity or any similar pension product is a non linear

function of future mortality and thus calculations of annuity values should be based upon the entire distribution rather than the expected future mortality.

2.2. Stochastic Mortality Models

Fitting mortality rates and hence longevity risk quantification dynamically continues to be a challenge especially in the developing countries. Earlier development relied on one-factor model proposed by Lee and Carter (1992). However, the Lee and Carter model is widely applied since it has been found to provide fairly accurate estimations and population projections for both the academicians and practitioners. Later on, Renshaw and Haberman and Halzoupoiz (1996) and Renshaw and Heberma (2003) analysed the Lee-Carter model and proposed a new model.

Recently two factor models were proposed and the cohort effect was considered in longevity modeling which Lee and Carter model lacked. For instance, Renshaw and Haberman (2003) applied a cohort effect and later Currie (2006) introduces an age-period-cohort (APC) model. In the most recent proposals Cairns, Blake and Dowd (2006b) allow not only for a cohort effect but also for a quadratic age effect in their CBD model that are found to solve all the problems Lee and Carter model had.

2.2.1. The Lee- Carter Model (1992)

Lee and Carter (1992) came up with a stochastic model where the log of a time series of age specific death rates is the sum of age specific component and a component that is a product of a time varying parameter, however deterministic projections are possible and it works in discrete age or time frame work.

Generally, Lee-Carter express the log of a time series of age-specific death rates $m_{x,t}$ as the sum of an age-specific component $\propto_x$ that is independent on time and a component that is the product of a time- varying parameters k_t reflecting the general level of mortality, and an age specific component β_x that represents the rate of mortality changes at each age.

$$\mu_{x,t} = \ln(m_{x,t}) = \exp(\alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}) \tag{2.6}$$

Since the parameters in the model are not fully identified, Lee and Carter (1992) enforced the constraints $\sum_{X=1}^{\omega} \beta_X = 1, \quad \sum_{t=1}^{n} \kappa_t = 0.$

Lists of approaches have been proposed on how the parameters in the Lee Carter model can be estimated. For instance, in the original paper of Lee and Carter (1992) he used the method of single value decomposition where he assumed that errors in the observed rates compared to those fitted by the model are independent and identically distributed normal variables. The values of k_t were then adjusted slightly to ensure that the total number of deaths that are predicted by the model across all ages are similar to the observed number of deaths across all ages. The Singular Value Decomposition approach has however been replaced by other formal statistical models proposed for example Brouhns et al., (2002) proposed a fitting procedure that takes advantage of the assumption that the death count can be assumed to be a Poisson variable and that the natural logarithms is the canonical link function for the Poisson distribution to use the maximum likelihood estimation in estimating the parameters. This means that in the maximum likelihood estimation method, the errors obtained between the fitted and the observed rates are allowed with varying age unlike that one proposed by Lee and Carter (1992) method.

Lee and Miller (2001) proposed a different method where he expressed that the goodness of fit in the final year in the data set should be looked into keenly. For Lee and Miller, they observed that they observed that modelling objective is to project the mortality rates. Nevertheless, in general the usual statistical procedures aim to fit the historical data so well over all past years.

The advantages of the Lee and Carter (1992) include:

1. Provides a good fit to historical data. Even though the shape of the mortality tables can be complex at the early ages, the α_x age function in the lee-carter allows the model to be used across all ages. In addition, k_t term captures the dominant trend in the evolution of mortality.
2. Simplicity in fitting and projecting. Use of the model provides an easy way of fitting and projecting since the parameters in the model are relatively few in comparison to the other models. The singular value decomposition and Poisson likelihood methods are likewise simple to put into practice.
3. It is easy to project since the linear trend in the k_t's is common in most of the data used. The random walk with drift time series time structure is widely used to give estimates of future central mortality rates.

However, the model has several drawbacks including:

1. The model is a one-factor model which means that the mortality improvements at all ages in the datasets are perfectly correlated. Search results are unrealistic and pauses a problem when looking into how risky the liabilities are based on the central mortality. The β_x age effect in the model is measured as the average improvement at all age x

however β_x is also used in obtaining the level of uncertainty in future mortality rates age x therefore;

$\text{var}\left[\log m_{x,n}/m_t\right] = \beta_x Var\left[\kappa_t^2\right]$.

Historically, the rates of improvement have been lower at the very old ages meaning that the projected future death rates uncertainty will be lower at old ages.

2. Lee- Carter model does not contain any allowance for cohort effects depending on an individual's year of birth. In the recent years, models based on the Lee- Carter model incorporating cohort effects have since been introduced for instance Renshaw and Haberman (2006).

2.2.2. Renshaw and Haberman (2003) Model

Renshaw and Haberman(2003) proposed a multifactor age-period model expressed as:

$$\log m_{x,t} = \beta_x^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\kappa_t^{(3)} \qquad (2.7)$$

Where $\kappa_t^{(2)}$ and $\kappa_t^{(3)}$ are dependent period effects (for example a bivariate random walk).

The model offer significant advantages over the Lee- Carter model including the fact that it is a multi-factor age period. However both did not address the problem caused by the cohort effects.

2.2.3. Renshaw and Haberman (2006) Cohort Model

The Renshaw- Haberman (2006) is an extended version of the Lee- Carter model with an extra parameter that gives the cohort effect expressed as:

$$\log m_{x,t} = \alpha_x + \beta_x^{(1)}\kappa_t + \beta_x^{(0)}\gamma_{t-x} \ . \qquad (2.8)$$

where κ_t is the mortality is index in year t (random period effect) and γ_{t-x} is a random cohort effect that is a function of the years of birth t-x.

The parameters in the model are not fully identified and therefore Renshaw and Haberman enforced constraints are $\sum_{X=1}^{\omega} \beta_x^0 = 1, \quad \sum_{x}^{\omega} \beta_x^{(1)} = 1.$

In their analysis of England and Wales data, Renshaw and Haberman found that there was a significant improvement over the Lee- Carter model. The most noticeable improvement was that an analysis of the standardized residuals revealed very little dependence on the year of birth.

However, the model has several drawbacks:

1. The model lucks robustness. For instance, CMI (2007) discovered that a change in the ranges of ages used to fit the model might result in a qualitatively different set of parameters estimates that are not expected. In his analysis, Cairns et al (2007,2008) found that there is luck or robustness when the range of years used to fit the model was changed.
2. The fitted cohort effect, γ_{t-x} in the model appears to be a deterministic linear or possibly quadratic trend in the year of birth which could mean that the age-cohort effect here is being used purposely to compensate for short of a second age-cohort effect as well as to try to capture the cohort effect in the data. With this drawback, an improvement on the model is to combine the second age –period effect in Renshaw and Haberman (2003) with a simple cohort effect.

2.2.4. Age-Period-Cohort Model

Currie (2006) introduces the simple Age- Period- Cohort (APC) model.

$$\log m_{x,t} = \beta_x^{(1)} + \kappa_t + \gamma_{t-x} \tag{2.9}$$

Without loss of generality in the model, the constraints $\sum_{t=1}^{n} \kappa_t = 0$ and $\sum \gamma_{t-x} = 0$ are imposed.

2.2.5. P- Splines

The P-splines was introduced in Currie et al (2004) in smoothing and projecting central mortality rates in a consistent manner. The model is based on the use of penalized B-splines introduced in Eilers and Marx (1996). Currie et al. (2004) used this approach in smoothing the mortality rates and extract 'shocks' therefore it can be used to derive stress-based scenarios. In this model, the force of mortality is assumed that it can be modeled as a linear combination of smooth functions across age and time to give:

$$\log m_{x,t} = \sum_{ij} \theta^{ij} \beta^{ij}{}_{x,t} \tag{2.10}$$

The use of splines method can lean to functions that are over fitted, hence yielding to mortality surfaces that are unreasonably lumpy. Therefore, to avoid the problem of over-fitting functions P-splines penalizes the roughness in the θ^{ij} by the use of linear and quadratic penalties.

CMI (2006) proved the P-splines approach to be very effective for it produced globally a good fit. Nevertheless, excessive smoothing can lead to systematic over or under estimation of mortality rates (Cairns et al., 2007).

The advantages of the P-splines include:

1. The model generates results for the central mortality rate that is smooth across all the ages and time. This means that it reduces the impact of the random noise from the crude data that is fitted.
2. P-splines gives projections of mortality rates that allow central mortality rates at different ages to change independently based on the observed data.

However, the model is infrequently used for projecting mortality due to its disadvantages.

The disadvantages of the P-splines include:

1. The method can lead to mortality rates that are over fitted resulting to mortality rates that are unreasonably lumpy. This means that the rates will not be a true representation of the true rates.
2. The P-splines fits a deterministic surface to the data and extends this into the future rather than allowing future rates to be generated by a stochastic process.
3. The model does not allow for cohort effects. However, P-splines can be reformulated from an age/period to an age/cohort model if desired but this removes the period effects which are usually felt to be dominant and give rise to problems as some cohorts have limited observations.

Cairns, Blake and Dowd (2006b)

A range of models have been explored given the problems with the preceding model in order to find a model that includes a parsimonious, multifactor age-period structure with a cohort effect that lacked in the previous models.

2.2.6. Cairns, Blake and Dowd Model -1

The model was expressed as:

$$\log itq(t,x) = \beta_x^{(1)}\kappa_t^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} \tag{2.11}$$

Simple parametric assumptions are $\beta_x^{(1)} = 1$, $\beta_x^{(2)} = (x - \bar{x})$

Therefore the method gives $\log itq(t,x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x})$

2.2.7. Cairns, Blake and Dowd Model -2

This is the first generalization of the Cairns, Blake and Dowd that took the cohort effect into consideration and thus gives the formula to:

$$\log itq(t,x) = \beta_x^{(1)}\kappa_t^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\gamma_{t-x}^{(3)} \tag{2.12}$$

We assume simple parametric forms as $\beta_x^{(1)} = 1$, $\beta_x^{(2)} = (x - \bar{x})$, $\beta_x^{(3)} = 1$

Thereby giving the model $\log itq(t,x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)}$.

The drawback with CBD- 2 is that we can move from $\gamma_{t-x}^{(3)}$ to $\tilde{\gamma}_{t-x}^{(3)} = \gamma_{t-x}^{(3)} + \varnothing_1 + \varnothing_2(t-x-\bar{x})$ and with corresponding adjustments to $\kappa_t^{(1)}, \kappa_t^{(2)}$ and there is no impact on the fitted values of the $q(t,x)$. In this model, to avoid arbitrary use of $\varnothing_1$ and $\varnothing_2$ constraints were introduced. The constraints are such that if least squares are used to fit a linear function of t-x to $\gamma_{t-x}^{(3)}$ then the fitted linear function is identically equal to zero.

2.2.8. Cairns, Blake and Dowd Model -3

CBD Model-3 adds a quadratic term into the age effect and still maintains the cohort effects. For constant x_c which is to be estimated, the formula is expressed as:

$$\log itq(t,x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) + \kappa_t^{(2)}((x-\bar{x})^2 - \hat{\sigma}_x^2) + \gamma_{t-x}^{(4)} \qquad (2.13)$$

Where the constant is expressed as $\hat{\sigma}_x^2 = \frac{1}{n_\alpha}\sum_{i=1}^{n}(X_i - \bar{X})^2$ and is the mean of $(x-\bar{x})^2$.

The drawback with CBD-3 is that we can switch from $\gamma_{t-x}^{(4)}$ to $\tilde{\gamma}_{t-x}^{(4)} = \gamma_{t-x}^{(4)} + \varnothing_1 + \varnothing_2(t-x-\bar{x}) + \varnothing_3(t-x-\bar{x})^2$ and with corresponding adjustments to $\kappa_t^{(1)}, \kappa_t^{(2)}, \kappa_t^{(3)}$ and there is no impact on the fitted values of the $q(t,x)$. To avoid the arbitrary use of $\varnothing_1$, $\varnothing_2$ and $\varnothing_3$, constraints are used here which is such that if we use least squares to fit a linear function of t.-x to $\gamma_{t-x}^{(4)}$ then the fitted linear function is identically equal to zero therefore our estimates will be fluctuating around zero and thus no observable up or down systematic curvature.

2.2.9. Cairns, Blake and Dowd Model -4

The impact of the cohort effect $\gamma_{t-x}^{(3)}$ for any specific cohort was assumed to diminish overtime ($\beta_x^{(3)}$ =decreasing with x) instead of remaining constant ($\beta_x^{(3)}$ =constant). Thus the model gives;

$$\log itq_{x,t} = \beta_x^{(1)}\kappa_t^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\gamma_{t-x}^{(3)} \tag{2.14}$$

Where $\beta_x^{(1)} = 1$, $\beta_x^{(2)} = (x - \bar{x})$, $\beta_x^{(3)} = (x_c - x)$ Thus this gives us $\log itq_{x,t} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)}(x_c - x)$

The constraint $\sum_{x,t} \gamma_{t-x}^{(3)} = 0$ is used to avoid the problem of identifiability introduced.

2.3. Model Selection Criteria

Comparison of the stochastic model is advised in order to know if the model is a good one or not. Cairns et al., (2007, 2008) proposed a list of qualities to check and evaluate the models and compare them with the other proposed models.

1. Consistency with historical data

   According to Cairns et al., (2007) a good model should be consistent with historical patterns of mortality. This will therefore give confidence in the use of the forecasted values as opposed to inconsistent one.

   Therefore Cairns et al., (2007) compared different models using the maximum likelihood and using the method that penalize over-parametised models. In their results, they suggested that

improvements in the Lee and Carter (1992) and Cairns, Blake and Dowd (2006b) models can be obtained by incorporating period and cohort effects.

2. Ease of Implementation

A good model should be the one that requires less computing time in that the model can easily be programmed using the available software. All the stochastic models discussed requires some programming in that codes that run should be devised, a good model therefore is the one that is easy to program. If a model will require excessive amounts of computing time, then it should only be used if the model yields an acceptable goodness of fit.

3. Parsimony

Models that are excessively parameterised should be avoided. This is done by the use of the Bayes Information Criterion (BIC) in order to ensure that parameters in the model are only included if the improvement in the fit is significant. Therefore the less the number of parameters the better the model so long as the model has an acceptable goodness of fit..

Each of the models described has a large number of parameters, therefore all are non-parsimonious .All the same, some models are parsimonious in that they have fewer effective parameters to estimate.

4. Transparency

Except for the P-splines, all the model's results are straight forward to analyse and thus deemed to be transparent. The P-splines model is less transparent because its output is smooth surface fitted to historical data and then projected.

5. Sample paths and prediction intervals

According to Cairns et al.,(2008), except for P-splines models, most of the models generate sample paths. This means that an assessment of the uncertainty in future mortality-linked cash flows and pricing of the cash flows is allowed.

6. Uncertainty

The parameters fitted and projected will often be subject to estimation errors because normally, we will have limited data to estimate the parameters. With this in mind, it is wise to include parameter uncertainty into the programming so that we can be in a position to know the impact of the estimation errors. Therefore in their study Cairns et al. (2006b) and in CMI working paper 15(2005) demonstrated that parameter uncertainty forms significant element of the uncertainty in the fitting and forecasting of the future mortality. Any model that does not allow for parameter uncertainty is in danger of significantly underestimating uncertainty in its forecasts.

An additional criterion is that the model is that it should be applicable for a full age range. The annuities providers and pension funds would want to model the mortality rates and their dependencies for the whole portfolio consistently, therefore the model should be applicable for the whole age range.

Some authors have recently sought to identify the similarities amongst stochastic mortality models. For instance, Hunt and Blake (2014b) describe an Age-Period- Cohort model structure which encompasses the vast majority stochastic mortality models. Curie (2014) shows that many common mortality models can be expressed in the standard terminology of generalized linear or non-linear models.

In the previous research, the models are used to fit historical data. The resulting estimates of the time varying parameter is then modeled and forecast as stochastic time series using standard Box- Jenkins methods. From the forecast of the general level of mortality, the age specific rates are derived using the estimated age specific rates are derived using the estimated age effects. Brouhns et al., (2002) resorted to Poisson log-bilinear regression model to build projected life tables.

Among the discussed models, the Lee and Carter model has been widely discussed and used to model the mortality rates and thus quantify the longevity risks involved. For instance Tuljapurkar (1998) and Tuljapurkar and Boe (1998) reviewed the Lee- Carter model and provided recommendations for forecasters. Lee Carter has been found suitable for actuarial applications for several reasons including, the fact that the model has a relatively few parameters that are easy to interpret. In addition, future mortality trends can easily be generated using the stochastic components of the model hence the actuaries are in a position of quantifying the unanticipated mortality improvements using the relevant risks measures. Therefore, in this paper, we have used the Lee- Carter model to forecast the mortality rates and show that indeed the life expectancy has been increasing with time and it's expected to increase in time hence longevity risk.

CHAPTER 3: METHODOLOGY

To model longevity risk, several scholars including suggested securization of the longevity risk and do a valuation methodology by building a mortality index. Cairns et al., (2008b) summarized using specific criterion of the various models that were proposed. Prospective life tables provide a view of the future evolution of the mortality rates. In the past decades, longevity improvement and therefore using the standard life tables will lead to restrictions and underestimation of the real scenario of future mortality when it comes to annuity pricing and reserving. Therefore the use of prospective life tables especially in pricing and reserving annuities will offer a better view of mortality evolution.

Mortality Assumption

These are projections of the expected death rates used to estimate pension obligations and price annuities. Mortality assumptions are based on the mortality tables. In most countries, the insurance and retirement benefits regulator provides a guideline on the mortality rates and assumptions to be used since the assumptions are crucial when it comes to pricing and reserving of annuities.

In the estimation of the life expectancy at birth or at retirement age, one of the key factors considered is the mortality assumption. The life expectancy calculated will then be used to determine the long term obligations of the pension fund and the annuity providers. In event the mortality assumptions are low, the long term liability of the pension fund and the insurance company will be overestimated. On the other hand, if the assumptions are too high, the life expectancy of the pension plan will be underestimated and consequently underestimate the obligations of the pension plan and annuity providers.

Heterogeneity and inter-age dependence

For a given population, the level of heterogeneity differs from any other population. Heterogeneity is as a result of a number of observable factors for example gender, age, occupation and physiological factors or due to features of the living environment such as climate, population and nutritional standards. Pensioners or policy holders that are of higher socio-economic status (assessed by occupation, income or education) have higher life expectancy or tend to experience lower rates of mortality. However, significant difference also exists within the same socio-economic status since generally females experience lower mortality rates compared to males. Longevity patterns and improvements are different from one company to another and from different company to another

Smoothing and closing tables

Age profiles of empirical annual mortality rates are inconsistent at high ages. Therefore actuaries mostly close the mortality tables i.e. extrapolate the shape of the survival functions at high ages from the some exogenous assumptions. In the past, mortality after age 100 was not emphasized since it had a very small impact on residual life expectations (and so annuities) for pensioners. With the recent longevity improvements, this is no longer the case, and it becomes important to have a better view on mortality and longevity risk for high ages since mortality is now improving for those ages.

3.1. Basic Building Blocks

3.1.1. Initial Rate of Mortality

The initial rate of mortality q measures the probability of death over the next year of age or, more generally, over the next rate interval. So the q-type rate applies to the age at the start of the interval.

The rate of mortality q_x is the probability of death over the next year of age for a person aged χ last birthday.

$$q_x = \frac{d_x}{l_x} \tag{3.1}$$

Where;

d_x is the number of deaths over the next year

l_x is the number of people alive at the start of the year

3.1.2. Central Rate of Mortality

m_x is the probability of dying between exact ages χ and $\chi+1$ per person-year lived between exact ages χ and $\chi+1$. Define,

$$L_x = \int_0^1 l_{x+t} dt \tag{3.2}$$

$$L_X = \int_0^1 l_{x+t}\, dt$$

$$L_X = \int_x^{x+1} l_y\, dy$$

$$= \int_x^{\infty} l_y \, dy - \int_{x+1}^{\infty} l_y \, dy$$

$$= T_x - T_{x+1}$$

Central death rate at age x;

$$m_x = \frac{d_x}{L_x} \tag{3.3}$$

$$= \frac{\int_0^1 l_{x+t} \, \mu_{x+t} dt}{\int_0^1 l_{x+t} \, dt}$$

$$= \mu_{x+1/2}$$

3.1.3. Instantaneous Force of Mortality

This is the instantaneous death rate at exact time t for individuals aged x+t at time t. μ_X is the instantaneous rate of mortality. This is the continuous equivalent of the discrete quantity q_x.

$$\mu_X = \lim_{h \to 0+} 1/h \times P\left[T \le x + h / T \succ X\right] \tag{3.3}$$

The probability $P\left[T \le x + h / T \succ X\right]$ is $F_X(h) = {}_h q_x$

$$\mu_X = \lim_{h \to 0+} {}_h q_x / h$$

The small h, we can ignore the limit and write:

$$ {}_hq_x \cong h.\mu_X \text{ for small } h $$

3.1.4. Expected Future Lifetime

This is a measure of the expected time remaining until death.

The Complete Expectation of Life, $\dot{e}_x$

The expected future lifetime after age x is $E\ [T_x]$

$$ \dot{e}_x = \int_0^{\omega-x} t\,{}_tp_x \mu_{x+t} dt \qquad (3.4) $$

$$ \dot{e}_x = \int_0^{w-x} t.\ {}_tp_x\ \mu_{X+t}\ dt $$

$$ = \int_0^{w-x} t.\left(-\frac{\partial y}{\partial x}\ {}_tp_X\right) dt $$

$$ = -\left[t \times {}_tp_x\ \right]_0^{w-x} + \int_0^{w-x} {}_tp_x\ dt $$

$$ = \int_0^{w-x} {}_tp_x\ dt $$

Curtate Expectation of Life, e_x

The curtate future lifetime of a life aged x is,

$$K_x = [T_x]$$

where the square brackets denotes the integer part

The curtate future lifetime K_x of a life aged exactly x is the whole number of years lived after age x.

$$e_x = E[K_X]$$

$$= \sum_{k=1}^{w-x} {}_k p_x$$

3.1.5. Age Specific Death Rates

Age specific death rates is the total number of deaths to residents of a specified age or age group in a specified geographical area divided by the population of the same age or age group in the same geographical area (for a specified time period, usually a calendar year) and multiplied by 100, 000.

$$= \frac{Total\ Deaths\ in\ specified\ age\ group}{Total\ Deathsin\ the\ same\ age\ group} \times 100{,}000$$

3.2. The Lee and Carter Model (1992)

Lee and Carter initially developed their approach specifically for U.S.A mortality data. However, the method has become the leading statistical model of mortality (forecasting) in the demographic literature.

Lee and Carter (1992) suggested a log-bilinear form of the force of mortality $\mu_{x,t}$ as follows:

$$\begin{aligned} &\mu_{x,t} = \ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t} \\ &x = 1, ..., \omega \\ &t = 1, ..., n \end{aligned} \tag{3.5}$$

Where;

α_x describes the age-specific pattern of mortality.

κ_t represents a time-trend index of general mortality model levels, describing the general level of mortality at different times. It captures the most important trend in death rates at all ages. Since mortality is a decreasing function, we can expect this trend to decrease.

β_x shows the decline in mortality at a particular age x when κ_t is changing

$\varepsilon_{x,t}$ is a zero mean Gaussian error $N(0, \sigma^2)$

The coefficients α_x are age specific constants that describe the general shape of the age mortality profile while the index κ_t serves to capture the main temporal level of mortality. Since the parameterization in equation 3.5 is invariant in respect to the transformations:

$$(\beta_x, \kappa_t) \rightarrow (c\beta_x, \kappa_t / c)$$
$$for(\alpha_x, \kappa_t) \rightarrow (\alpha_x - c\beta_x, \kappa_t + c)c \in \square \quad (3.6)$$

Then in order to ensure identifiability of equation 3.5 i.e. there are unique solutions to the model the parameters β_x and κ_t should satisfy the constraints:

$$\sum_{X=1}^{\omega} \beta_X = 1 \qquad \text{and} \qquad \sum_{t=1}^{n} \kappa_t = 0 \quad (3.7)$$

The constraint $\sum_{t=1}^{n} \kappa_t = 0$ implies that by summing over the years t the estimates of parameters α_x are given by the averages of the force of mortality over the time period i.e. $\hat{\alpha}_x = \frac{1}{n}\sum_{t=1}^{n} \mu_{x,t}$ where $\hat{\alpha}_x$ is the average pattern of mortality at age x. An estimate of κ_t is obtained by summing both sides of equation 3.5 over the ages and using $\sum_{X=1}^{\omega} \beta_X = 1$ to obtain $\hat{\kappa}_t = \sum_x (\ln(m_{x,t}) - \hat{\alpha}_x)$.

An estimate for β_x is obtained by differentiating both sides of equation 3.5 with respect to time t to obtain $\hat{\beta}_x = (\partial \ln(m_{x,t}) / \partial t) / (\partial \hat{\kappa} / \partial t)$. Then the parameters β_x captures the relative density of the logarithm of the central death rates to change in the mortality index κ_t . The function β_x moderates the time-dependant element κ_t by age.

We note that all parameters on the right hand side of the equation 3.5 are unobservable. Since they are unobservable, fitting the model using simple methods like ordinary least squares will be

impossible. Lee and Carter (1992) proposed the method of singular value decomposition (SVD) in model fitting. Later on, the maximum likelihood estimation (MLE) was implemented by Wilmoth (1993) and Brouhns et al (2002) and the method of generalized linear models (GLM) was employed by Renshaw and Haberman (2006).

3.3. Estimation Approaches

3.3.1. The Singular Value Decomposition Approach

Lee and Carter used Single Value Decomposition to estimate the parameters of the equation $\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}$ in his first paper.

First the parameter vector α_x is computed as the average overtime of the logarithm of the central death. That is:

$$\hat{\alpha}_x = \frac{1}{n} \sum_{t=1}^{n} \mu_{x,t} \tag{3.8}$$

The Singular Value Decomposition is applied to matrix $y = \ln(m) - \hat{\alpha}$.

To obtain β_x and κ_t, singular value decomposition is applied to the matrix $Y_{x,t} = \left[\ln(m_{x,t}) - \hat{\alpha}_x \right]$.

Theorem of Low Rank Approximation

Low rank approximation problem involves the approximation of a matrix D with another matrix $\hat{D}$, said truncated which has a specific rank r.

If now the approximation is by minimizing Frobenius norm of the difference between D and $\hat{D}$ under the constraint rank $\hat{D} \leq r$ i.e

$$\boldsymbol{Minimize\ over\ \widehat{D} \left\| D - \hat{D} \right\|_F\ subject\ to\ rank\ (\hat{D}) \leq r}$$

We obtain the solution by Singular Value Decomposition of the data matrix to obtain the matrix approximation lemma or Eckart-Young Mirsky (1936). Proof of the low rank approximation theorem is found in appendix A.

Let,

$$D = U\Sigma V^T \in \mathbb{R}^{m \times n}, m \leq n \tag{3.9}$$

be the singular value decomposition of D and partition U, $\Sigma = diag(\sigma_1, ...\sigma_m)$ and V as follows:

$$U = [U_1 U_2], \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \text{ and } V = [V_1 V_2]$$

Where Σ is a $r \times r$, U is $m \times r$ and V_1 is $n \times r$. Then the rank-r matrix obtained from the trancated singular value decomposition is:

$$\hat{D}^* = U_1 \Sigma_1 V_1$$

Is such that

$$\left\| D - \hat{D}^* \right\|_F = \min_{rank(\hat{D}) \leq r} \left\| D - \hat{D} \right\|_F = \sqrt{\sigma_{r+1}^2 + ... + \sigma_m^2}$$

The minimized $\hat{D}^*$ is unique if and only if $\sigma_{r+1} \neq \sigma_r$

Singular Value Decomposition

Denote A, a m×n matrix of rank h. Then there is an m×m orthogonal matrix U, an n×n orthogonal matrix V and a m×n diagonal matrix such that $\boldsymbol{A} = \boldsymbol{USV'}$. Where $V^{'} = (v_{ji})$ is the transpose of matrix v_{ij}.

$$A = \begin{bmatrix} U_{1,1} & \cdots & U_{1,m} \\ \vdots & \ddots & \vdots \\ U_{m,1} & \cdots & U_{m,m} \end{bmatrix} \times \begin{bmatrix} \rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \times \begin{bmatrix} V_{1,1} & \cdots & V_{1,n} \\ \vdots & \ddots & \vdots \\ V_{m,1} & \cdots & V_{n,n} \end{bmatrix}$$

Particularly for $A = [\ln(m_{x,t}) - \propto_x, x = 1, \ldots\ldots X \; and \; t = 1, \ldots\ldots, T$ the rank-1 approximation

$$\hat{Y}_{x,t}^{(1)} = \rho_1 U_{x,1} V_{1,t} = \beta_x^{(1)} K_t^{(1)}$$

To give:

$$\hat{\beta}^{(1)} = (U_{1,1} \;\; U_{2,1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;\; U_{x,1})'$$

$$\hat{K}^{(1)} = \rho_1 \times (V_{1,1} \;\; V_{2,1} \;\; \ldots\ldots V_{i,1})$$

Thus for the Lee and Carter Model, By using the theorem of low rank approximation (first started and approved by Eckart and Young1936)) in the singular value decomposition approach.

The rank h least square approximation is given as:

$$\hat{Y}_{x,t}^{(h)} = \sum_{i=1}^{h} \rho_i \mu_{x,t} \nu_{i,t} = \sum_{i=1}^{h} \beta_x^{(i)} \kappa_t^{(i)} \ , \ h \leq r \tag{3.10}$$

Where $\beta_x^{(i)} \kappa_t^{(i)} = \rho_i \mu_{x,i} \nu_{i,t}$

Then the rank h residuals are

$$\xi_{x,t} = \sum_{i=h+1}^{r} \rho_i \mu_{x,i} \nu_{i,t}$$

The corresponding rank-h least square error is:

$$\xi_n^2 = \sum_{i=h+1}^{r} \rho_i^2$$

This implies that the errors have similar variance.

However, this assumption is violated for mortality data this is because the variance of the log-central death rate is approximately $Var\left[\ln(m_{x,t})\right] \approx \frac{1}{d_{x,t}}$.

The proportion of variance explained by the i[th] term $(\rho_i, \mu_{x,i}, \nu_{i,t})$ of the decomposition $\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}$ is given by $\frac{\rho_i^2}{\sum_{j=1}^{r} \rho_j^2}$ and total variance explained by a rank-h approximation is:

$$\sigma_h^2 = \frac{\sum_{i=1}^{h} \rho_i^2}{\sum_{j=1}^{r} \rho_j^2} \qquad 0 \le \sigma_h^2 \le 1$$

The closer the value is to 1 the better the approximation.

3.3.2. Maximum Likelihood Estimation

Wilmoth (1993) and Alho (2000) proposed using Maximum Likelihood Estimation to find the parameters in the Lee and Carter model (3.1).We use the Poisson approximation of deaths as follows

Based on the Poisson approximation of the number of deaths $D_{x,t}$:

$$D_{x,t} \sim Poisson(E_{x,t} m_{x,t}) \text{ where } m_{x,t} = \exp(\alpha_x + \beta_x \kappa_t) \qquad (3.11)$$

The estimation of the parameters α_x , β_x and κ_t in equation 3.11 takes place using the maximum likelihood method i.e. maximizing the log likelihood of model given by:

$$L(\alpha, \beta, \kappa, D) = \log \prod_{x,t} f(D_{x,t}; \alpha, \beta, \kappa)$$

$$= \log \prod_{x,t} \exp^{-E_{x,t} m_{x,t}} \frac{(E_{x,t} m_{x,t})^{D_{x,t}}}{D_{x,t}!}$$

$$= \sum_{x,t} \left[D_{x,t} (\ln E_{x,t} m_{x,t}) - E_{x,t} \exp(\alpha_x + \beta_x \kappa_t) - \ln(D_{x,t}!) \right]$$

$\alpha = (\alpha_1, \alpha_2, ..., \alpha_M)$

Where $\beta = (\beta_1, \beta_2, ..., \beta_M)$ and there are M data points for each calendar year and N calendar

$\kappa = (\kappa_1, \kappa_2, ..., \kappa_N)$

years of data.

The maximum likelihood estimation allows non-additive heteroscedic (Renshaw and Haberman 2003:255) and avoids the assumption of errors with constant variance present in the SVD approach (Lee and Carter 1992:660). The MLE formulation of the LC model is often referred to as the Poisson log-bilinear model from the paper Brouhns et al. (2002) which provides algorithm to minimize the equation.

3.3.3. Weighted Least Squares

Wilmoth (1993) proposed fitting the Lee Carter model using weighted least squares. Basically we want to estimate the parameters $\alpha_x, \beta_x and \kappa_t$.

The estimation α_x which minimizes the sum of least squares of errors $s = \sum_{x,t} \varepsilon_{x,t}^2$ is the average of $m_{x,t}$ i.e

$$\alpha_x = \frac{1}{n} \sum_t m_{x,t} \tag{3.12}$$

Where n in the total number of calendar years. The difference in matrix is formed as

$z_{x,t} = m_{x,t} - \alpha_x$ and it satisfies $\sum_t \kappa_t = 0$ and $\sum_x (b_x)^2 = 1$

$$Q = \sum_{x,t} (\kappa_t \beta_x - z_{x,t})^2$$

To find the values that minimize Q we introduce the Langrangers multiplier a and b that minimizes:

$$R = Q - a\sum \kappa_t - b\sum \beta_x^2$$

$$\frac{dR}{d\kappa_t} = 2\sum_x \beta_x (\beta_x \kappa_t - z_{x,t}) - a$$

$$\frac{dR}{d\beta_x} = 2\sum_t \kappa_t (\beta_x \kappa_t - z_{x,t}) - 2b$$

$$\frac{a}{2} = \kappa_t \sum_x \beta_x^2 - \sum_x b_x z_{x,t}$$

If we add the sums with respect to t we get that $\alpha = 0$. We then solve for κ_t, β_x from the systems of equation to get;

$$\kappa_t = \sum_x \beta_x z_{x,t}$$

$$\beta_x = \frac{\sum_t \kappa_t z_{x,t}}{\sqrt{\sum_x (\sum_t \kappa_t z_{x,t})^2}}$$

3.4. Forecasting

After estimating the parameters, the second stage involves finding a modified $\kappa_t^{(1)}$ which adjusts the total number of deaths $\sum_x d_{x,t}$ to the estimated number of deaths as follows:

$$\sum_x d_{x,t} = \sum_x E_{x,t} \exp(\hat{\alpha}_x + \sum_i \hat{\beta}_x^{(i)} \kappa_t^{(i)}) \tag{3.13}$$

Where $E_{x,t}$ and $d_{x,t}$ are exposure to risk and actual numbers of death at age x and time t.

Predicting mortality with LC is reduced to forecasting the index κ_t using time series approaches (Brockwell and Davis 1996).

In order to forecast future mortality rates, Lee and Carter assumes that α_x and β_x remains constant over time and the time trend κ_t is intrinsically viewed as a stochastic process . Lee and Carter (1992) suggested the following random walk with drift to model κ_t .

$$\hat{\kappa}_t = \hat{\kappa}_{t-1} + \theta + C\varepsilon_t$$

In which θ is a constant drift term, C is a constant volatility and ε_t is a one dimensional i.i.d N(0,1) error.

An appropriate ARIMA (p,d,q) model for the mortality index κ_t is found by carrying out the standard Box and Jenkins methodology (identification-estimation-diagnosis). In general an

ARIMA (0,1,0) with drift $\hat{\kappa}_t = \hat{\kappa}_{t-1} + \zeta_t \sum_{i=1}^{n} X_i$ is found suitable , though other ARIMA forms provided better fit to some data (Brouhns et. al. 2002).

After having found an appropriate ARIMA model, the variables, the mortality index κ_t can be forecasted. Let $\hat{\kappa}_{t_n+s}$ denote the s-period ahead forecast of the mortality index then in case of the poisson Lee- Carter model, the expected value of future death count is given by:

$$E[D_{x,t_n+s}] = E_{x,t_n+s}\hat{m}_{x,t_n+s}$$

Where E_{x,t_n+s} is the future exposure and $\hat{m}_{x,t_n+s}$ forecasts of future death rates with:

$$\hat{m}_{x,t_n+s} = \exp(\hat{\alpha}_x + \beta_x \hat{\kappa}_{x,t_n+s})$$

Using $\hat{m}_{x,t_n+s}$ we calculate life expectancies and life annuity premiums.

3.5. Actuarial Present Value

The symbol $\ddot{a}_x^{(m)}$ refer to the expected present value of an annuity of 1 per annum payable monthly in advance.

$$\ddot{a}_x^{(m)} = \sum_{t=0}^{\infty} \frac{1}{m} v^{\frac{t}{m}} \, {}_{\frac{t}{m}}p_x \qquad (3.14)$$

We can approximate formula $\ddot{a}_x^{(m)}$ in terms of $\ddot{a}_x$ using either Euler-Maclaurin formula or Woolhouse's formula.

By use of Euler-Maclaurin formula

$$\int_0^{\infty} f(t)dt \approx \sum_{t=0}^{\infty} f(t) - \frac{1}{2}f(0) + \frac{1}{12}f'(0) \qquad (3.15)$$

Woolhouse's formula:

$$\frac{1}{m}\sum_{t=0}^{\infty} f\left(\frac{t}{m}\right) \approx \sum_{t=0}^{\infty} f(t) - \left(\frac{m-1}{2m}\right)f(0) + \left(\frac{m^2-1}{12m^2}\right)f'(0) \qquad (3.16)$$

Assuming that $f(t) \to 0$ and $f'(t) \to 0$ as $t \to \infty$

Using equation 3.16 to fit $f(t) = v^t{}_t p_{x.}$.

$f(t) = v^t{}_t p_{x.} = \exp\left(-\int_0^t (\delta + \mu_{x+r})dr\right)$ then $f'(t) = -(\delta + \mu_{x+t})\exp\left(-\int_0^t (\delta + \mu_{x+r})dr\right)$

Therefore $f(0) = 1$ and $f'(0) = -(\delta + \mu_x)$

This gives

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \left(\frac{m-1}{2m}\right) - \left(\frac{m^2-1}{12m^2}\right)(\mu_x + \delta)$$

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \left(\frac{m-1}{2m}\right)$$

CHAPTER 4: DATA ANALYSIS AND RESULTS

4.1. Source of Data

One of the sources of data used in modelling longevity risk is from the mortality of individual or aggregate pension plans, annuity providers such as the insurance companies. UK's Continuous Mortality Investigation Bureau collects mortality data on insured lives from insurers and data on pensioners lives from pension plans. Even though the regulatory body in Kenya collects this data, it is not easily available as it is never put in public or published every year as in other countries. In addition, the use of this kind of data can result to sampling problems as the data may not be a true representation of the entire population.

Mortality data is also collected and published by government agencies. National mortality data are published for a number of countries in the Human Mortality database (HMD). The Kenya Bureau of Statistics collates the data after every ten years (every census). The entire population data is the most appropriate data since it includes large number of individuals, has low sampling errors.

Our analysis will be based on the U.S.A mortality data downloaded from the Human Mortality Databases (HMD) through demography package dedicated function. The Human Mortality Database began in the year 2000 and was launched in May 2002 after its first phase of development. The database provides a detailed mortality and population data according to sex and year to researchers, policy analysts, students and other stakeholders. Currently, it contains data from 37 countries.

The information that can be obtained by sex, age and time in the HMD includes:

- ✓ Birth counts;

- ✓ Death counts;
- ✓ Population estimate;
- ✓ Population exposed to risk of death(the period & cohort :period data are indexed by the year of deaths; whereas cohort data are indexed by year of births); and
- ✓ Death rates (period and cohort).

4.2. Description of software used

In the analysis, we have used both the R-software and excel. In the R-software, demography and forecast package is used to fit and forecast Lee and Carter model. From the results, we then obtain the future life expectation of different cohorts and by the use of life contingencies package we project the cost of a pension annuity, $\ddot{a}_x^{(m)}$ for specific cohorts.

4.3. Assumptions

In our analysis, we assumed the following:

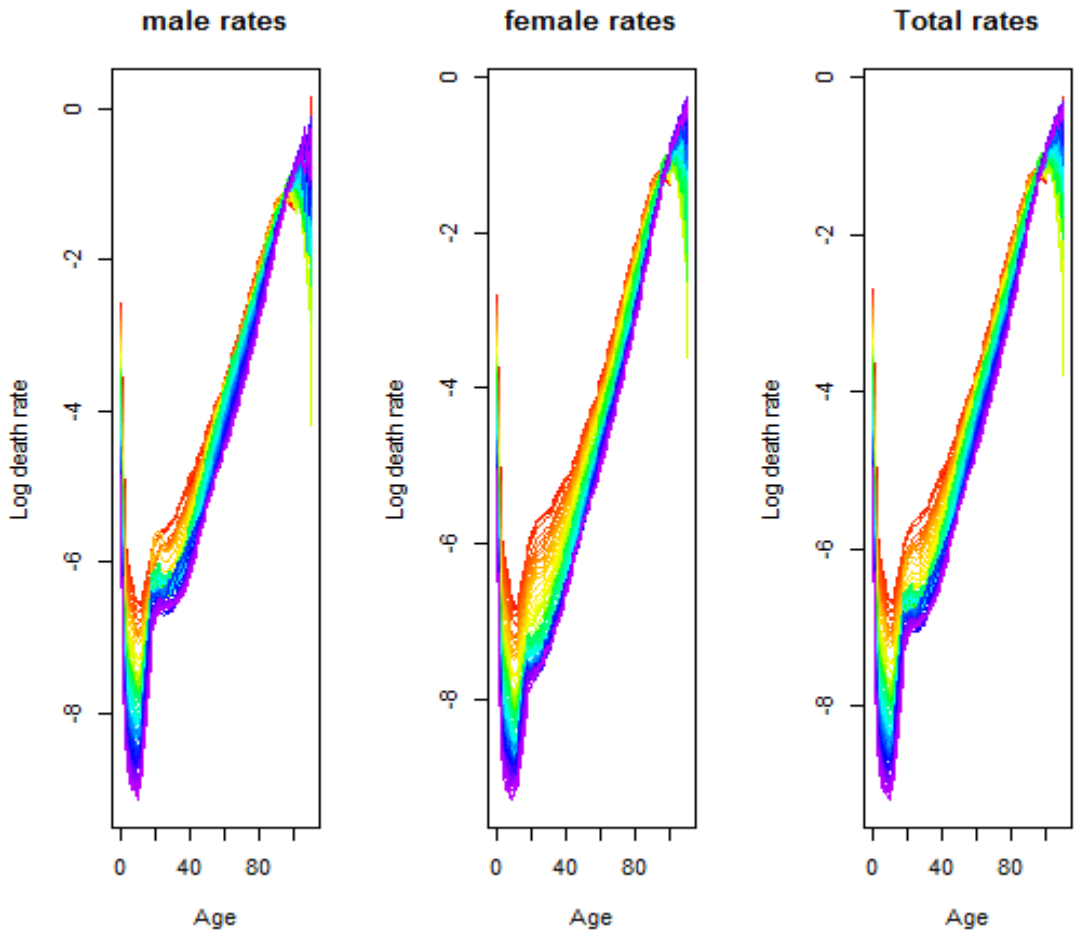
1. The retirement age (x) will be set to 65 regardless of the cohort
2. The pensions are paid monthly. Therefore m, the fractional payments per year will be equal to 12.
3. The present value of annuity of 1 monetary unit will be calculated using an interest rate of 4% and inflation rate of 2%.

Since female mortality is lighter, three data sets were used regarding the male, female and total population.

4.4. General Analysis of the Data Used

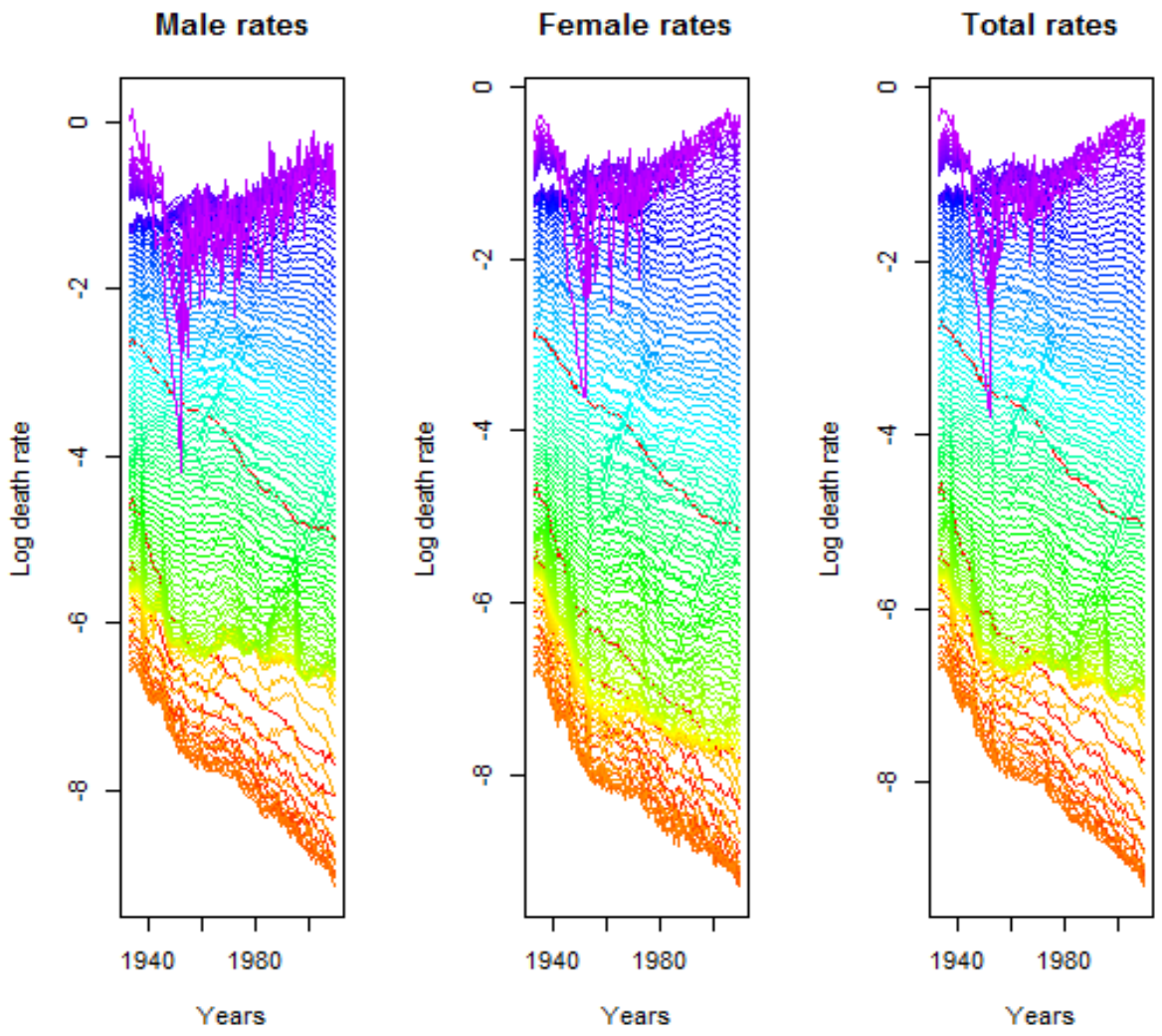
Figure 2 plots the log death rates against age from 0 to 110 of the U.S.A data. The codes used are in the appendix. The plot method is available on **demogdata.**

Figure 1: **Log death rates against age from 0 to 110**



The data confirms that mortality decreases with age. However, the young mortality hump is visible in the age-range (20, 40) probably caused by accident, drug abuse etc. Therefore, even though we note that mortality declines with age, we observe that the decrease has been uneven between the ages of 20 to 40. However, individuals start receiving annuities from the age of 60.

Figure 2 : Log death rates against time from 1933 to 2010



The data confirms that mortality rate has been decreasing with years. Therefore this means that mortality rate in 2010 is quite low compared to the mortality rate in 1933. However, even though we note that mortality declines overtime, we observe that the decrease has been uneven across different ages especially at the age of 20 to 40.

4.5. Fitting the Model

We fit the lee carter model using the **lca** function with singular value decomposition method. The following are the steps in estimation of the parameters using the Singular Value Decomposition approach:

1. $\hat{a}_x = \frac{1}{T}\sum_{t=t_1}^{t_n} \ln(m_{x,t})$
2. A matrix $Z_{x,t}$ is created for estimating b_x and k_t
3. Singular Value Decomposition is applied to matrix $Z_{x,t}$ to decompose the matrix $Z_{x,t}$ into product of three other matrices:

   $ULV' = SDV(Z_{x,t}) = L_1U_{x1}V_{t1} + \ldots\ldots\ldots + L_1U_{xX}V_{tX}$ (4.1)

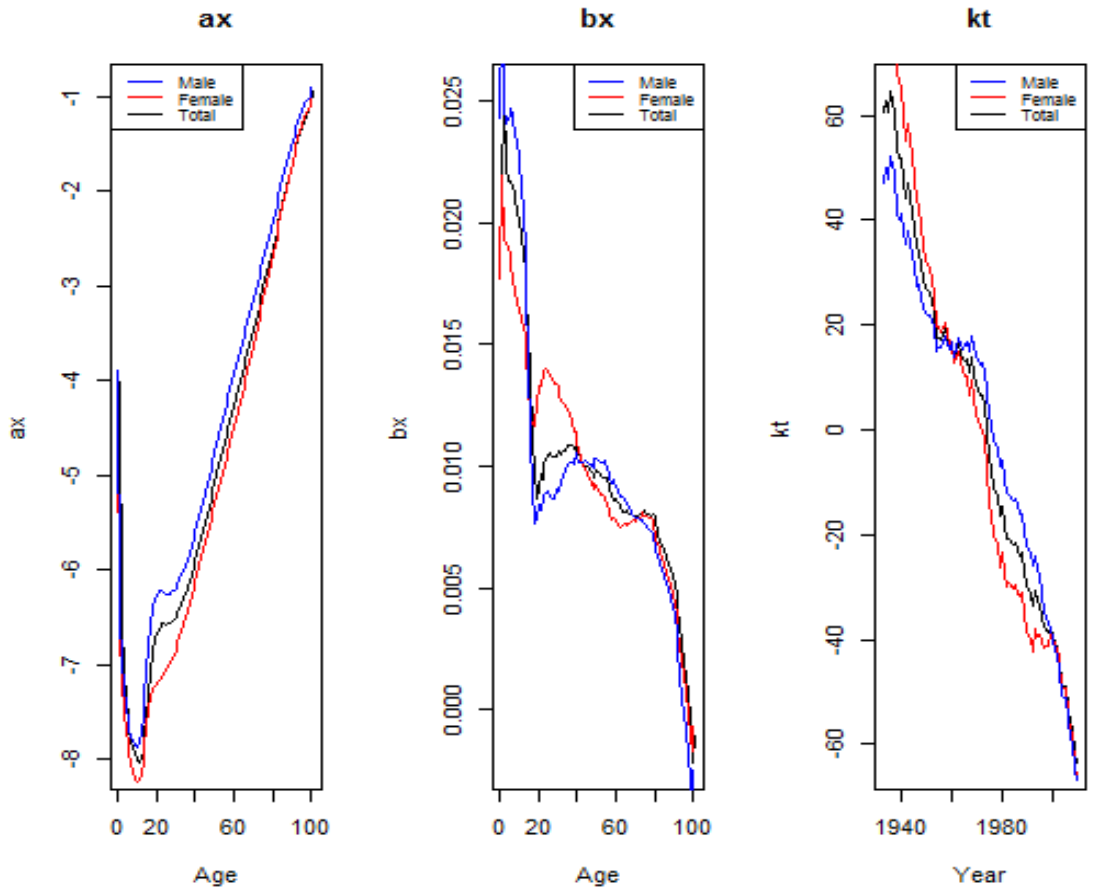
   Where U represents the age component, L represents the singular values and V represents the time component.
4. The first time –component matrix and the first singular values $\hat{k}_t = L_1U_{x1}$ will give the estimated values of $\hat{k}_t$. The first vector of the age components $\hat{b}_x = U_{x1}$will give estimated values of $\hat{b}_x$.
5. Estimation of a new matrix $\hat{Z}_{x,t}$ using the product of the estimated parameters $\hat{b}_x$ and $\hat{k}_t$ to get $\hat{Z}_{x1,t1} = \hat{b}_{x1}\,\hat{k}_{t1}$

$$Z_{x,t} = \begin{bmatrix} \hat{Z}_{x1t1} & \cdots & \hat{Z}_{x1tn} \\ \vdots & \ddots & \vdots \\ \hat{Z}_{xAt1} & \cdots & \hat{Z}_{xAtn} \end{bmatrix} \hat{b}_x \quad (4.2)$$

6. The natural logarithm of the central death rates is then estimated,

   $\ln(m_{x,t}) = \hat{a}_x \widehat{+ \hat{Z}_{x,t}} = \hat{a}_x + \hat{b}_x\hat{k}_t$ (4.3)

Figure 3: Parameter Estimates ax, bx and kt of Lee-Carter Model



From our observations, the average mortality rate grows as the age increases, indicated by the ax pattern except for the hump that is as a result of the accidental. κ_t captures the main trend on logarithmic scale in death rates at all ages and as expected, has a decreasing trend with increment with time. The bx describes the tendency of mortality at age x to change as the general level of mortality kt changes. This indicates that when bx is large for some x, the death rate at age x varies a lot than the general level of mortality and vice versa.

4.6. Model Evaluation

We use the mean percentage error to examine goodness of fit:

$$MAPE = \frac{1}{n}\sum_{i}\frac{\left| y_i - \hat{y}_i \right|}{y_i} * 100\% \qquad (4.4)$$

Female

Lee-Carter analysis

Call: lca(data = usadata, series = "female", max.age = 100)

Percentage variation explained: 96.9%

ERROR MEASURES BASED ON LOG MORTALITY RATES

Averages across ages:

ME	MSE	MPE	MAPE
-0.00007	0.00995	0.00042	0.01606

Averages across years:

IE	ISE	IPE	IAPE
-0.00717	0.98335	0.03723	1.55252

Male

Lee-Carter analysis

Percentage variation explained: 94.3%

ERROR MEASURES BASED ON LOG MORTALITY RATES

Averages across ages:

ME	MSE	MPE	MAPE
0.00501	0.01242	-0.00041	0.01952

Averages across years:

IE	ISE	IPE	IAPE
0.50034	1.22565	-0.04948	1.87779

For males the MAPE of the fitted log death rates is approximately equal to 2% while that for females is approximately equal to 1.6%. Therefore the model fits reasonably well.

Total

Lee-Carter analysis

Percentage variation explained: 96.1%

ERROR MEASURES BASED ON LOG MORTALITY RATES

Averages across ages:

ME	MSE	MPE	MAPE
0.00276	0.00898	-0.00010	0.01633

Averages across years:

IE	ISE	IPE	IAPE
0.27565	0.89149	-0.01477	1.58032

For totals, the MAPE of the fitted log death rates is approximately equal to 1.6333% therefore the model fits reasonably well. Detail results of the analysis are found in Appendix C.

4.7. Forecasting

It is appropriate to have the maximum length of the projection period approximately equal to the length of the fitting period. We use the forecast package to project the future values of κ_t.

The random walk drift model (RWD) for κ_t has been used and the model is as shown below:

$$\hat{k}_t = \hat{k}_{t-1} + \theta + \varepsilon_t \tag{4.5}$$

ε_t is the error term and θ is the drift parameter where:

$$\theta = \frac{\hat{k}_T - \hat{k}_1}{T-1} \tag{4.6}$$

To forecast two periods in time ahead, we substitute $\hat{k}_{t-1}$ moved back in time one period:

$$\hat{k}_t = \hat{k}_{t-1} + \hat{\theta} + \varepsilon_t$$

$$= (\hat{k}_{t-2} + \hat{\theta} + \varepsilon_{t-1}) + \hat{\theta} + \varepsilon_t$$

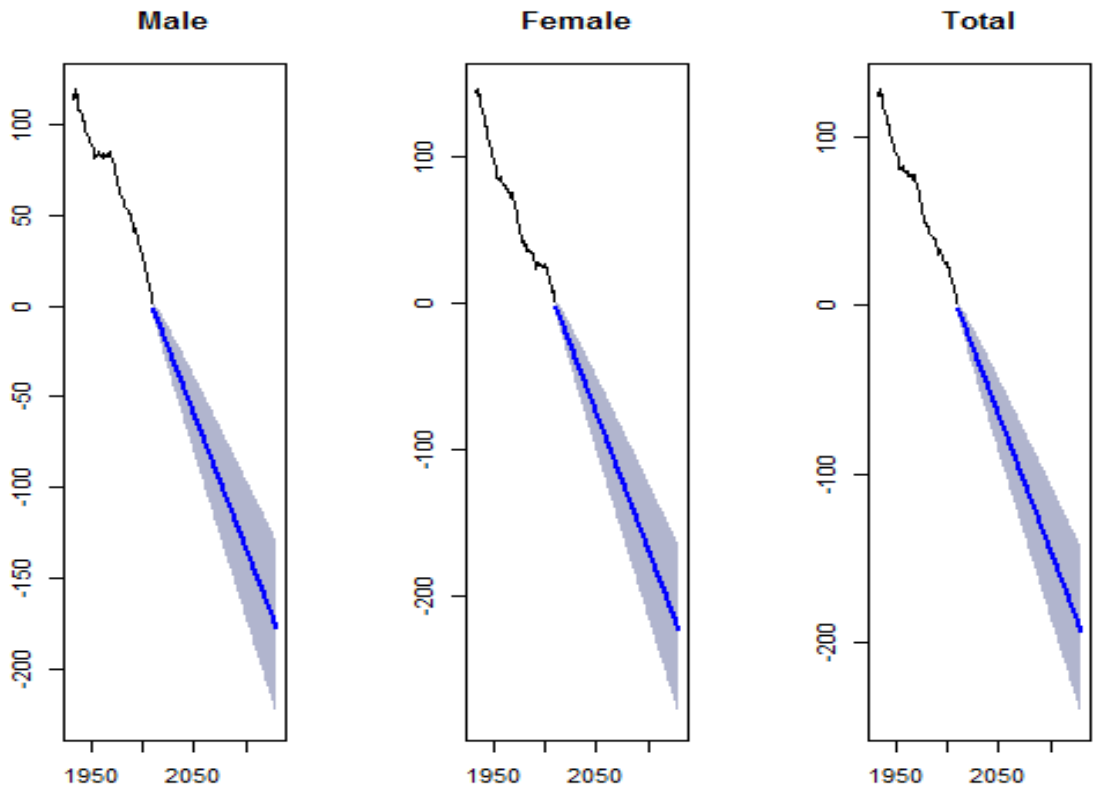
$$= \hat{k}_{t-2} + 2\hat{\theta} + (\varepsilon_{t-1} + \varepsilon_t) \tag{4.7}$$

In order to forecast $\hat{\kappa}_t$ at time $T + (\Delta t)$ where the data available is up to period T, we follow the same procedure and iterate (Δt) times and obtain:

$$\hat{k}_{T+(\Delta t)} = \hat{k}_T + (\Delta t)\hat{\theta} + \sum_{n}^{(\Delta t)} \varepsilon_{T+n-1}$$

$$= \hat{k}_T + (\Delta t)\hat{\theta} + \sqrt{(\Delta t)\varepsilon_t} \tag{4.8}$$

Figure 4: Projected values of kt



We have projected the values of kt, the mortality changes as shown in Figure 4. The Lee- Carter model forecast shows an improvement in mortality rates. This implies that the cost of pension annuities and life insurance is expected to be higher in future as a result of people living longer than expected.

4.8. Performing Actuarial Projections

We use the following to obtain the life expectancy and the actuarial present values:

$$\begin{aligned} \ln \hat{\mu}_{x,t} &= \hat{a}_x + \hat{b}_x \hat{\kappa}_t \\ \hat{p}_{x,t} &= \exp(-\hat{\mu}_{x,t}) \end{aligned} \tag{4.9}$$

We calculate the actuarial present value of $\ddot{a}_{65}^{(12)}$ for the selected cohorts. We have derived values separately for males and females and finally for the total population. See Appendix D for a detailed breakdown of the results. The actuarial present value for the total population is as below:

Total

```
>  }
>  For cohort 1930 of total the e0 is   55.62   and the APV is :   6.10
>  For cohort 1940 of total the e0 is   65.03   and the APV is :   6.53
>  For cohort 1950 of total the e0 is   72.49   and the APV is :   6.98
>  For cohort 1960 of total the e0 is   77.68   and the APV is :   7.48
>  For cohort 1970 of total the e0 is   80.01   and the APV is :   7.60
>  For cohort 1980 of total the e0 is   82.88   and the APV is :   8.20
>  For cohort 1990 of total the e0 is   84.39   and the APV is :   8.58
>  For cohort 2000 of total the e0 is  86.02  and the  APV is :    8.78
>  For cohort 2010 of total the e0 is   89.99   and the APV is :   9.20
```

From the actuarial present value results, we see that the annuities have been increasing with time as a result of the reduction mortality rates and the increase in life expectancy. Therefore the amount that annuity providers should pay to individual should decrease so as to avoid overpaying the annuitants. Longevity risk is arising due to the fact that the general level of mortality change unknown at the time of buying annuities.

CHAPTER 5: CONCLUSIONS AND RECOMMENDATIONS

5.1. Conclusions

We have identified a common trend of mortality changes with age using the standard Lee-Carter model to the U.S. data. In addition, we have estimated the parameters using the Singular Value Decomposition approach and forecasted the values of $\hat{\kappa}_t$ using the ARIMA method. Finally we have forecasted the life expectancies at birth. From the results, we've noted that there is indeed a decrease in mortality rates with age and time. Further, the life expectancy has increased with time which has resulted to an increase in the actuarial present values that are supposed to be used in calculating annuities. We can therefore conclude that the risk that a pensioner will live longer than expected is evident from the results thus longevity risk exists. If the insurance companies and the pension funds do not take the changes in mortality rates into consideration by still using the usual life tables year in year out, they will end up paying more annuities to annuitants than they are supposed to. Therefore insurance companies and annuity providers should reserve for these risks in order to avoid going into liquidation.

We have used US data to fit the Lee-Carter model since the Kenyan Mortality data was not readily available and not up to date. However, according to the World population 2300 published by the United Nations, life expectancy at birth for both developed and developing countries is projected to increase in future the only difference being the rate of increment. We therefore conclude by indicating some directions for future research especially focusing on the Kenyan mortality trend.

The Lee-Carter model has received significant attention in the effort to model mortality rates since 1992. The model has proved to be robust and reaches to good accuracy in its predictions

for forecasts. In addition, for a precise value of the time index κ_t, we can define a complete set of probabilities of death that allows us to find the values of the entire life table. However, we have assumed constant assumption of the parameters which mostly is not the case in practice. In addition the model does not include the cohort effects.

5.2.Recommendations

Based on the conclusion, we have observed clearly that longevity risks exist in pension schemes and for annuity providers. In this regard, we recommend that longevity risk management ideas should be implemented. Governments have responded by reforming the pension systems by encouraging individuals to work for a longer period of time and therefore save more for retirement after the shift from the defined benefit to the defined contribution schemes. Today, the annuities have also evolved into different forms. This includes, commencing the payment immediately or at a later date, the annuities can be fixed or varying with certain underlying factors e.g. inflation or the annuity can be a joint or for a single annuitant depending on the individual preference.

However detailed longevity risk management is as outlined below:

5.2.1. Longevity Risk Management

5.2.1.1.Longevity- Linked Instruments

The annuity providers (insurers) and pension plans may use the different products in place such as longevity bonds and longevity indices in hedging longevity risk. For most of the developing countries, the market for longevity risk is still at the embryonic stage.

Longevity Indices

Longevity index will show the probability of increase of life expectancy for the individuals of a certain age over a period of time. This will enable the transfer of the longevity risk as well as improve the visibility, transparency of the risks. Currently, the existing indices are:

I. Credit Suisse Longevity Index launched in 2005 based on the US data.
II. JP Morgan Lifemetrics index launched in 2007 for the US, England & Wales, Germany and the Netherlands national population data.
III. Goldman Sachs Mortality Index launched in 2007 based on a sample of the US insured population data who are over 65.
IV. Deutsche Borse Xpect Age and Cohort Index launched in 2008 for Germany, Netherlands and England & Wales data.

Longevity Bonds

Longevity bond pays a coupon based on the survivorship of a selected birth cohort. If a higher than expected proportion of the cohort survives, the coupon rate increases in order to offset the provider's cost and so as to hedge against longevity risk.

Longevity Swaps

This is an instrument that offsets the annuity provider risks of their policyholders living longer than expected. The annuity provider makes regular payments to an investment bank based on agreed mortality assumptions and in return, the investment bank will pay out amounts based on the actual pension fund mortality rates. This idea is similar to the one used in interest or inflation swap. There are two types of longevity swaps, namely:-

1) Named lives swaps

   The trustees pay fixed cash flows to the swap provider and in return the swap provider will pay cash flows to the pension scheme based on the actual longevity of the pensioners in the scheme. The trustees will pay a once of cash flow while the swap provider will be paying the cash flows periodically.

2) Population index swaps

   Under the index-based longevity swap, the national population data is used rather than a specific scheme mortality experience. Since the swaps are typically set according to the general population's experience, the index swaps aim to protect the scheme against improvements in longevity assumptions.

5.2.1.2.Re-insurance

The annuity providers and pension plan may re-insure in order to reduce or mitigate the longevity risk therefore the reinsurer will meet part or wholes of the liability due. Cairns et al., (2008b) discussed the possibility of reinsuring in more detail.

5.2.1.3. Asset-Liability Modelling

If the life insurers and the pension plans retain longevity risk as part of their business, then asset liability modelling should be done to ensure that the assets that they hold are sufficient to meet the liability requirements. They may for instance come up with solvency buffers; by this they will reduce the probability of underfunding.

To avoid applying the results of a developed country to a developing country, Kenya, we recommend that annuity providers and the pension plans use the Kenyan mortality data in fitting and forecasting the mortality rates and draw their conclusion with the relevant data Kenyan data.

Since longevity risk is a major concern to annuity providers and pension plans, we recommend that other models to be used by future academicians to fit and forecast the mortality rates and hence measure longevity risks instead of the Lee and Carter model that we have used in this project in order to take care of the limitations of the Lee and Carter model for instance the fact that cohort effects is not taken care of in the Lee and Carter Model. The most appreciated and recommended model is the Cairns-Blake-Dowd (2008) Model (CBD-4).

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APPENDICES

APPENDIX A: Proof of Low Rank Approximation

Proof of low rank approximation

$A = U_n \sum_n V_n^T$

Where U_n and V_n^T are orthogonal matrix, and $\sum_n$ is a diagonal matrix with entries $(\sigma_1 \sigma_2 ..... \sigma_n)$

Such that $(\sigma_n \le \sigma_{n-1} \le ..... \le \sigma_1)$

The best approximation for A is given by

$A^k = \sum_{i=1}^{k} u_i \sigma_i v_i$

To prove A^k is indeed the best approximation i.e. $\|A - A^k\|$ is minimum.

Proof by Contradiction:

Let us suppose $\exists B$ s.t. $\|A-B\|_2^2 < \|A-A^k\|_2^2 = \sigma_{k+1}^2$

rank (B) $\le$ k (Assuming in Low Rank Approximation through dim(null(B)+ rank(B)=N $\rightarrow$ dim(null (B)) $\ge$ n-k.

Let $\omega \in null(B)$

$Let \omega \in null(B)$

$\|(A-B)\omega\|_2 = \|A\omega\|_2 < \sigma_{k+1}$

We know that $\exists(k+1)$ dimension space $(v_1, v_2, ...v_n)$

s.t. $V \in span(v_1, v_2, ..., v_n)$ and $\|AV\|_2 \geq \sigma_{k+1}$

since $n-k+k+1>n$

Therefore by contradiction we get that A^k is the best approximation.

APPENDIX B: R CODES

```
> library(demography)
> library(forecast)
> library(life contingencies)
> usadata<-
  hmd.mx(country="USA",username="ngugiwairimu@gmail.com",password="ngugi",labe
  l="U.S.A")
> summary(usadata)
> par(mfrow=c(1,3))
> plot(usadata,series="male",datatype="rate",main="male rates")
> plot(usadata,series="female",datatype="rate",main="female rates")
> plot(usadata,"total",datatype="rate",main="Total rates")
> par(mfrow=c(1,3))
> plot(usadata,series="male",datatype="rate",plot.type="time",main="Male
  rates",xlab="Years")
> plot(usadata,series="female",datatype="rate",plot.type="time",main="Female
  rates",xlab="Years")
> plot(usadata,series="total",datatype="rate",plot.type="time",main="Total
  rates",xlab="Years")
> usLcaM<-lca(usadata,series="male",max.age=100)
> usLcaF<-lca(usadata,series="female",max.age=100)
> usLcaT<-lca(usadata,series="total",max.age=100)
> usLcaM
> usLcaF
> usLcaT
> par(mfrow=c(1,3))
> plot(usLcaT$ax,main="ax",xlab="Age",ylab="ax",type="l")
> lines(x=usLcaF$age,y=usLcaF$ax,main="ax",col="red")
> lines(x=usLcaM$age,y=usLcaM$ax,main="ax",col="blue")
> legend("topleft",c("Male","Female","Total"),
> cex=0.8,col=c("blue","red","black"),lty=1)
> plot(usLcaT$bx,main="bx",xlab="Age",ylab="bx",type="l")
> lines(x=usLcaF$age,y=usLcaF$bx,main="bx",col="red")
> lines(x=usLcaM$age,y=usLcaM$bx,main="bx",col="blue")
> legend("topright",c("Male","Female","Total"),
> cex=0.8,col=c("blue","red","black"),lty=1)
> plot(usLcaT$kt,main="kt",xlab="Year",ylab="kt",type="l")
```

```
> lines(x=usLcaF$year,y=usLcaF$kt,main="kt",col="red")
> lines(x=usLcaM$year,y=usLcaM$kt,main="kt",col="blue")
> legend("topright",c("Male","Female","Total"),
> cex=0.8,col=c("blue","red","black"),lty=1)
> summary(usLcaF)
> summary(usLcaM)
> summary(usLcaT)
> plot(residuals(usLcaF))
> plot(residuals(usLcaM))
> plot(residuals(usLcaT))
> fM<-forecast(usLcaM,h=120)
> fF<-forecast(usLcaF,h=120)
> fT<-forecast(usLcaT,h=120)
> summary(fM)
> summary(fF)
> summary(fT)
> par(mfrow=c(1,3))
> plot(fM$kt.f,main="Male")
> plot(fF$kt.f,main="Female")
> plot(fT$kt.f,main="Total")
> ratesM<-cbind(usadata$rate$male[1:100,],fM$rate$male[1:100,])
> ratesF<-cbind(usadata$rate$female[1:100,],fF$rate$female[1:100,])
> ratesT<-cbind(usadata$rate$Total[1:100,],fT$rate$total[1:100,])
> par(mfrow=c(1,1))
> plot(seq(min(usadata$year),max(usadata$year)+120),ratesF[65,],col="red",xlab="Years",
  ylab="Death Rates",type="l")
> lines(seq(min(usadata$year),max(usadata$year)+120),ratesM[65,],col="blue",xlab="Year
  s",ylab="Death Rates")
> lines(seq(min(usadata$year),max(usadata$year)+120),ratesT[65,],col="black",xlab="Yea
  rs",ylab="Death Rates")
> lengend("topright",c("Male","Female","Total"),cex=0.8,col=c("blue","red","black"),lty=1
  )



> createActuarialTable<-function(yearOfBirth,rate){
> mxcoh<-rate[1:nrow(rate),(yearOfBirth-min(usadata$year)+1):ncol(rate)]
> cohort.mx<-diag(mxcoh)
> cohort.px=exp(-cohort.mx)
```

```
> #we get projected Px
> fittedPx=cohort.px #add px to table
> px4Completion=seq(from=cohort.px[length(fittedPx)], to=0,length=20)
> totalPx=c(fittedPx,px4Completion[2:length(px4Completion)])
> #create life table
> irate=1.04/1.02-1

> cohortLt=probs2lifetable(probs=totalPx,radix=100000,type="px",
> name=paste("Cohort",yearOfBirth))
> cohortAct=new("actuarialtable",x=cohortLt@x,lx=cohortLt@x,
    o interest=irate,name=cohortLt@name)
> return(cohortAct)
> }


> getAnnuityAPV<-function(yearOfBirth,rate){
> actuarialTable<-createdActuarialTable(yearOfBirth,rate)
> out=axn(actuarialTable,x=65,m=12)
> return(out)
> }


> rate<-ratesM
> for(i in seq(1930,2010,by=10)){
> cat("For cohort",i,"of males e0 at birth is",
> round(exn(createActuarialTable(i,rate)),2),
> "and the APV is : ",round(getAnnuityAPV(i,rate),2),"\n")
> }


> rate<-ratesF
> for(i in seq(1930,2010,by=10)){
> cat("For cohort",i,"of females e0 at birth is",
> round(exn(createActuarialTable(i,rate)),2),
> "and the APV is : ",round(getAnnuityAPV(i,rate),2),"\n")
> }
```

```
> rate<-ratesT
> for(i in seq(1930,2010,by=10)){
> cat("For cohort",i,"of males e0 is",
> round(exn(createActuarialTable(i,rate)),2),
> "and the APV is : ",round(getAnnuityAPV(i,rate),2),"\n")
> }
```

APPENDIX C: Residual Analysis

Female

Lee-Carter analysis

Call: lca(data = usadata, series = "female", max.age = 100)

Adjustment method: dt
Region: U.S.A
Years in fit: 1933 - 2010
Ages in fit: 0 - 100

Percentage variation explained: 96.9%

ERROR MEASURES BASED ON MORTALITY RATES

Averages across ages:

ME	MSE	MPE	MAPE
-0.00006	0.00005	0.00494	0.07025

Averages across years:

IE	ISE	IPE	IAPE
-0.00530	0.00390	0.49530	6.96561

ERROR MEASURES BASED ON LOG MORTALITY RATES

Averages across ages:

ME	MSE	MPE	MAPE
-0.00007	0.00995	0.00042	0.01606

Averages across years:

IE	ISE	IPE	IAPE
-0.00717	0.98335	0.03723	1.55252

Male

Lee-Carter analysis

Call: lca(data = usadata, series = "male", max.age = 100)

Adjustment method: dt
Region: U.S.A

Years in fit: 1933 - 2010
Ages in fit: 0 - 100

Percentage variation explained: 94.3%

ERROR MEASURES BASED ON MORTALITY RATES

Averages across ages:

ME	MSE	MPE	MAPE
0.00000	0.00006	0.01120	0.08008

Averages across years:

IE	ISE	IPE	IAPE
0.00195	0.00501	1.11846	7.93905

ERROR MEASURES BASED ON LOG MORTALITY RATES

Averages across ages:

ME	MSE	MPE	MAPE
0.00501	0.01242	-0.00041	0.01952

Averages across years:

IE	ISE	IPE	IAPE
0.50034	1.22565	-0.04948	1.87779

Total

Lee-Carter analysis

Call: lca(data = usadata, series = "total", max.age = 100)

Adjustment method: dt
Region: U.S.A
Years in fit: 1933 - 2010
Ages in fit: 0 - 100

Percentage variation explained: 96.1%

ERROR MEASURES BASED ON MORTALITY RATES

Averages across ages:

ME	MSE	MPE	MAPE
-0.00002	0.00005	0.00728	0.06803

Averages across years:

IE	ISE	IPE	IAPE
-0.00106	0.00384	0.72981	6.75980

ERROR MEASURES BASED ON LOG MORTALITY RATES

Averages across ages:

ME	MSE	MPE	MAPE
0.00276	0.00898	-0.00010	0.01633

Averages across years:

IE	ISE	IPE	IAPE
0.27565	0.89149	-0.01477	1.58032

APPENDIX D: Actuarial Projections

For Males:

> }
> For cohort 1930 of males the e0 is 59.52 and the APV is : 4.92
> For cohort 1940 of males the e0 is 62.36 and the APV is : 5.48
> For cohort 1950 of males the e0 is 66.32 and the APV is : 6.02
> For cohort 1960 of males the e0 is 76.25 and the APV is : 7.18
> For cohort 1970 of males the e0 is 79.52 and the APV is : 7.52
> For cohort 1980 of males the e0 is 80.21 and the APV is : 7.92
> For cohort 1990 of males the e0 is 82.68 and the APV is : 8.01
> For cohort 2000 of males the e0 is 84.86 and the APV is : 8.46
> For cohort 2010 of males the e0 is 85.98 and the APV is : 8.82

For Females

> }
> For cohort 1930 of females the e0 is 65.72 and the APV is : 6.89
> For cohort 1940 of females the e0 is 69.31 and the APV is : 7.56
> For cohort 1950 of females the e0 is 73.01 and the APV is : 8.03
> For cohort 1960 of females the e0 is 79.83 and the APV is : 8.38
> For cohort 1970 of females the e0 is 83.36 and the APV is : 8.79
> For cohort 1980 of females the e0 is 85.85 and the APV is : 9.11
> For cohort 1990 of females the e0 is 87.38 and the APV is : 9.43
> For cohort 2000 of females the e0 is 89.27 and the APV is : 9.66
> For cohort 2010 of females the e0 is 89.99 and the APV is : 9.84

Total

> }
> For cohort 1930 of total the e0 is 55.62 and the APV is : 6.10
> For cohort 1940 of total the e0 is 65.03 and the APV is : 6.53
> For cohort 1950 of total the e0 is 72.49 and the APV is : 6.98
> For cohort 1960 of total the e0 is 77.68 and the APV is : 7.48
> For cohort 1970 of total the e0 is 80.01 and the APV is : 7.60
> For cohort 1980 of total the e0 is 82.88 and the APV is : 8.20
> For cohort 1990 of total the e0 is 84.39 and the APV is : 8.58
> For cohort 2000 of total the e0 is 86.02 and the APV is : 8.78
> For cohort 2010 of total the e0 is 89.99 and the APV is : 9.20