PLANE HURWITZ NUMBERS

School of Mathematics, University of Nairobi

Doctor of Philosophy in Mathematics

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Declaration and Approval

I the undersigned declare that this thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions. To the best of my knowledge, it has not been submitted in support of an application for another degree in other university or other institution of learning.

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In my capacity as advisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge and this thesis has my approval for submission.

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Abstract

The main objects in this thesis are meromorphic functions obtained as projections to a pencil of lines through a point in $\mathbb{P}^2$. The general goal is to understand how a given a meromorphic function $f : X \to \mathbb{P}^1$ can be induced from a composition $X \dashrightarrow C \to \mathbb{P}^1$, where $C \subset \mathbb{P}^2$ is birationally equivalent to the smooth curve X. In particular, it is the desire to characterize meromorphic functions on smooth curves which are obtained in such a way and enumerate such functions.

It is shown in this thesis that any degree d meromorphic function on a smooth projective plane curve $C \subset \mathbb{P}^2$ of degree $d > 4$ is isomorphic to a linear projection from a point $p \in \mathbb{P}^2 \backslash C$ to $\mathbb{P}^1$. Further, a planarity filtration of the small Hurwitz space using the minimal degree of a plane curve is introduced such that a given meromorphic function admits such a composition $X \dashrightarrow C \to \mathbb{P}^1$. Additionally, a notion of plane Hurwitz numbers is introduced.

Publications

The following publications are included in this thesis.

PAPER I: **On formulae for calculating Hurwitz numbers**
J. Ongaro, preprint.

PAPER II: **On a Zeuthen-type problem**
J. Ongaro, submitted.

PAPER III: **Planarity stratification of Hurwitz spaces**
J. Ongaro and B. Shapiro.
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Jared Ongaro

Stockholm, 2014.

To Claudio Achola,

for his mathematical mind, generosity, kindness and his way of life.
Tastes which have since proved highly contagious.

Table of Contents

Introduction

The main objects of interest of this thesis are branched coverings of smooth projective algebraic curves over complex numbers $\mathbb{C}$. The study of branched coverings of curves contributes to curve theory what representation theory of groups gives to abstract groups. More precisely, before the 20th century groups were thought as subsets of the general linear group $GL(n, \mathbb{C})$ or the symmetric group S_n before much later they were defined as abstract objects. Then by means of representation theory, we can concretely study and classify abstract groups on the basis of which maps they admit into $GL(n, \mathbb{C})$. Similarly, algebraic curves were earlier defined as subsets of a n-dimensional projective space $\mathbb{P}^n$ before later they were introduced by Riemann in his revolutionary paper [Rie57], as abstract varieties independent of any particular embedding. Analogously to the study of abstract groups, the problem of studying algebraic curves naturally splits into two directions:

- study of abstract curves, mainly in families called **moduli spaces** of curves;
- representation of abstract curves or study of maps between curves.

Indeed, an intuitive way to study an abstract curve X is to represent it as branched covering over a fixed curve Y; that is using a finite surjective morphism $f : X \longrightarrow Y$. If the target curve Y is well understood, a large amount of information is revealed about the source curve X. As the simplest curve is the projective line $\mathbb{P}^1$, the most fundamental realization is obtained when Y is fixed to be $\mathbb{P}^1$. In other words, this amounts to studying nonconstant meromorphic functions on X, since a morphism $f : X \longrightarrow \mathbb{P}^1$ to the complex projective line $\mathbb{P}^1$ is called a **meromorphic function**. The degree of f is the degree of the morphism $f : X \longrightarrow \mathbb{P}^1$. Given a meromorphic function f of degree d and any point $q \in \mathbb{P}^1$, we have a **branch divisor** $f^{-1}(q) = \mu_1 p_1 + \ldots + \mu_n p_n$, where $p_1, \ldots, p_n$ are distinct points on X and $\mu_1, \ldots, \mu_n$ are positive integers summing to d. In particular, possibly after reordering we can assume $\mu_1 \geq \ldots \geq \mu_n$. The partition $(\mu_1, \ldots, \mu_n) \vdash d$ is called the **branch type** of f at a point q. If the branch type of f at q equals to $(1, 1, \ldots, 1)$, then we say f is not branched over q and if the branch type corresponds to $(2, 1, \ldots, 1)$ at q, we say that q is a **simple** branch point of f. The set of all branch points is called the **branching locus** of f. In this way, every nonconstant meromorphic function on a curve X is a **branched covering**. This set of branch types is called **branch profile** of f.

In the case of a plane curve $C \subset \mathbb{P}^2$, a geometrical method for constructing a branched covering of $\mathbb{P}^1$ by C, is to consider a meromorphic functions arising from projections of C. To achieve this, choose a point $p \in \mathbb{P}^2$ which may or may not be lying on C, identify $\mathbb{P}^1$ with the pencil of lines at p and then project C onto $\mathbb{P}^1$. The finite morphism

$$\pi_p : C \longrightarrow \mathbb{P}^1, \tag{1}$$

obtained by the above projection is the required branched covering of $\mathbb{P}^1$. Points of $\mathbb{P}^1$ where several intersection points in C coincide are branch points of π_p. If $p \in \mathbb{P}^2 \backslash C$, then generically a point of $\mathbb{P}^1$ possesses the same number of distinct intersections points with C as the degree of C. To motivate projections of plane curves from a point in $\mathbb{P}^2$, we will depict the construction of the topological structure of a smooth curve $C \subset \mathbb{P}^2$ based on branch points of (5.1) as given in [Rie57]. It involves cutting the sheets between the branch points and permuting i.e. cross-joining them to obtain the topological picture for the curve.

Consider a smooth plane algebraic curve $C \subset \mathbb{P}^2$. Thus, C is the vanishing set of an irreducible homogeneous polynomial

$$F(x, y, z) = \sum_{i+j+k=d} a_{ijk} x^i y^j z^k, \qquad d \geq 1, \tag{2}$$

where x, y, z represents the standard homogeneous coordinate system in $\mathbb{P}^2$ and $a_{ijk} \in \mathbb{C}$ with simultaneously non-vanishing partial derivatives at all points of $C \subset \mathbb{P}^2$. We shall illustrate by way of examples, how conclusions can be drawn about the topological structure of $C \subset \mathbb{P}^2$. We agree to keep the naive terminology of Riemann of referring to topological operations as *cutting and pasting*. Naturally one can formulate all these in rigorous set-theoretic language, for instance pasting of two spaces is equivalent to passing to the quotient space of disjoint sum in the corresponding quotient topology. However, this standard set-theoretic language helps little for an intuitive understanding of branched structure that we seek.

Example 0.1

Consider a conic $C_1 \subset \mathbb{P}^2$ defined by $y^2 = xz$. The branched covering $\pi_p : C_1 \longrightarrow \mathbb{P}^1$ for $p = [0:1:0]$ has degree 2 with two simple branch points $0 := [1,0]$ and $\infty := [0,1]$. Take 2 copies of $\mathbb{P}^1$(2 equals to $\deg \pi_p$) and slit-cut them along $[0,\infty)$ and glue the opposite sides of the sphere as illustrated in Figure 0.1. Observe that edges to be joined are labeled by the same letters.

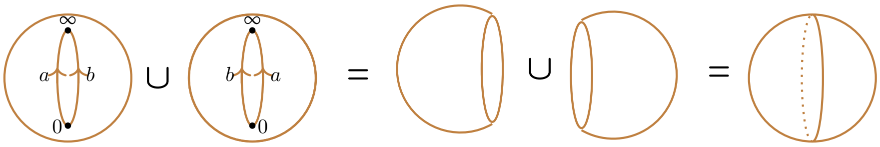


Figure 0.1: Branched structure for $C_1 : y^2 = xz$ over $\mathbb{P}^1$

Conversely, given a branched covering of $\mathbb{P}^1$ of degree 2 with two simple branch points $0, \infty$ in $\mathbb{P}^1$, one can reconstruct the curve by pasting together the spaces and conclude that it is a projective line $\mathbb{P}^1$. Thus, the resulting topological structure for the curve C_1 is a **2-sphere** and it is biholomorphic to the one given by $y^2 = xz$ in $\mathbb{P}^2$.

Example 0.2

The projection of $C_2 \subset \mathbb{P}^2$ defined by $y^2 z = x(x+z)(x-z)$ from the point $p = [0:1:0] \in C_2$ is a branched covering of degree 2 with 4 simple branch points $0 := [1\ ,\ 0]$, $\alpha := [-1\ ,\ 1]$, $\beta := [1:\ 1]$ and $\infty := [0\ :\ 1]$. If we slit-cut the two sheets from 0 to α and from β to ∞, the joining is like shown in Figure 0.2, thus C_2 is topologically a torus.

Figure 0.2: Branched structure for $C_2 : y^2 z = x(x+z)(x-z)$ over $\mathbb{P}^1$

In general, it is possible to construct such maps $\pi_p : C \longrightarrow \mathbb{P}^1$ with a given set of prescribed branching points once we know the branch profile. The Riemann-Hurwitz formula which we will state later, implies that the degree and genus determine the degree of the branch divisor, so we only need to keep track of the degree, genus and branch profiles of branched coverings. Therefore, if we fix the degree, genus and branch profile we are lead to another interesting question of enumeration of branched coverings up to isomorphism which commute with the branched covering maps. We naturally restrict to connected branched coverings as the disconnected ones can be obtained as disjoint union of lower degree connected ones.

To summarize, we are mainly interested in classification and enumeration of nonconstant meromorphic functions $f : X \longrightarrow \mathbb{P}^1$. Hurwitz [Hur91] began the systematic investigation of such pairs (X, f) by constructing a moduli space $\mathscr{H}_{g,d}$ now called the **Hurwitz space**. (Note that branch profile is simply suppressed to avoid notational clutter.) Each point in $\mathscr{H}_{g,d}$ corresponds to an equivalence class of meromorphic functions of degree d on curves of genus g with given branch profile of f, where one identifies two meromorphic functions $f_1 : X_1 \longrightarrow \mathbb{P}^1$ and $f_2 : X_2 \longrightarrow \mathbb{P}^1$ as same if there is an isomorphism of curves $h : X_1 \longrightarrow X_2$ such that $f_1 = f_2 \circ h$. Hurwitz observed that if we fix the degree d of the branched coverings $f : X \longrightarrow \mathbb{P}^1$, the genus g of X and the branch profile, the Hurwitz space $\mathscr{H}_{d,g}$ form a covering space of the space of unordered configurations $Con^w(\mathbb{P}^1)$ of w points in $\mathbb{P}^1$. The degree of the covering map

$$\mathscr{H}_{d,g} \longrightarrow Con^w(\mathbb{P}^1)$$

is called the **Hurwitz number** corresponding to the given branch profile. The fundamental group of $Con^w(\mathbb{P}^1)$ acts on the fibers of the covering and the orbits of this action are known to be in one-one correspondence with the connected components of $\mathscr{H}_{d,g}$.

Generally, the geometry of Hurwitz spaces $\mathscr{H}_{d,g}$ is very complicated. An interesting class of Hurwitz spaces are the so-called **small Hurwitz spaces** $\mathcal{H}_{d,g}$ which consists of meromorphic functions on curves of genus g with only simple branch points. The small Hurwitz spaces play a crucial role in the understanding of the more abstract moduli spaces of curves $\mathcal{M}_{g,d}$ of curves of genus g with d marked points. In particular, in [Hur91] it is shown that the natural map $\mathcal{H}_{g,d} \longrightarrow \mathrm{Sym}^w \mathbb{P}^1 \backslash \Delta$, where Δ is the discriminant hypersurface, assigning a meromorphic function f its branching locus, is a finite étale covering. In this case the degree of map $\mathcal{H}_{g,d} \longrightarrow \mathrm{Sym}^w \mathbb{P}^1 \backslash \Delta$ is called a **simple Hurwitz number**. Furthermore, using a calculation of Lüroth and Clebsch [Cle72], see also §21 of [ACG11] page 857, Hurwitz proved that in this case there is only **one orbit.** In other words, $\mathcal{H}_{g,d}$ is a smooth and connected hence irreducible quasi-projective variety. This result was later generalized to characteristic $p > g + 1$ by Fulton [Ful69]. The natural forgetful map $\pi : \mathcal{H}_{g,d} \longrightarrow \mathcal{M}_{g,d}$ relates the geometry of the Hurwitz space $\mathcal{H}_{g,d}$ to that of the moduli space $\mathcal{M}_{g,d}$. A particularly interesting case is when this map is surjective, which is at least happens as soon as $d \geq 2g - 1$. An immediate consequence is that $\mathcal{M}_{g,d}$ is also irreducible.

It follows that branched coverings of $\mathbb{P}^1$ offer a concrete way to investigate abstract algebraic curves. Thus, one hopes to get an information about an abstract algebraic curve through branched coverings of $\mathbb{P}^1$ or equivalently meromorphic functions on it. As indicated earlier, a geometrically nice way to deduce such information is to consider projections of plane curves. Recall that an abstract complex smooth curve X of genus g can always be embedded into some n-dimensional projective space $\mathbb{P}^n$, $n \geq 3$. More precisely, as proved in Chapter IV Corollary 3.6 of [Har77], every curve can be embedded in $\mathbb{P}^3$ as a smooth curve. In addition to that, Corollary 3.11 of the same Chapter asserts that the image of this embedding in $\mathbb{P}^3$ is birationally equivalent to a plane curve with at most a finite number nodes as singularities. Consequently, we may approach the classification problem of all curves by studying families of curves in $\mathbb{P}^2$ of a fixed degree d and with δ nodes. But this direction is very difficult, in fact it was only rather recently in [Har86] that it was proved that the space parametrizing such curves is an irreducible algebraic variety of dimension $\frac{1}{2}d(d+3) - \delta$.

On the other hand, in view of projections we seek to know how a given meromorphic function on a given smooth curve X of genus g can be realized through projection from a point in $\mathbb{P}^2$. Indeed, we will see in Chapter 6, that every meromorphic function $f : X \longrightarrow \mathbb{P}^1$ can be realized as a projection from a point in $\mathbb{P}^2$. Namely, given a meromorphic $f : X \longrightarrow \mathbb{P}^1$ of degree d and X a smooth curve of genus g, then curve X can be realized as a plane curve C of degree $d + l$ with an ordinary m-fold point at p and at most

$$\delta = \binom{d+l-1}{2} - \binom{l}{2} - g$$

nodes as singularities. Here, the image plane curve C is birationally equivalent to X. Projecting from the point $p \in \mathbb{P}^2$, we obtain that f can be induced from the pencil of lines through p on $C \subset \mathbb{P}^2$.

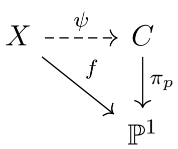


Figure 0.3: Meromorphic functions as projections from a point in $\mathbb{P}^2$

This lead to problem formulation of my research and therefore its goal. More specifically, my general aim is to study how a given meromorphic function $f : X \longrightarrow \mathbb{P}^1$ can be induced from a composition as in Figure 0.3 and enumeration of such functions which yields the notion of plane Hurwitz numbers.

Outline of the thesis

This work is at the intersection of Algebraic Geometry and Combinatorics. For this reason, the general background: where I describe informally the concepts I will use, is made of two parts. I also give room to historical facts, general knowledge and I set up notations. I dedicate chapters 1–3, to introduce and explain the main objects and relevant theory which will be used in this thesis. My main aim is to provide a quick toolkit for results I use in my work. Thus, to shorten the exposition, we will only state most results and indicate appropriate references for details. In summary, the structure of this thesis is as follows:

Chapter 1

In this chapter, we review definitions and develop notation on results about partitions, permutations and representation of the symmetric group.

Chapter 2

Here, we consider complex algebraic curves and we give a brief outline of the many generic facts bordering the theorem of Riemann-Roch. The chapter finishes with a quick introduction to the concept of moduli spaces of curves and moduli space of stable maps.

Chapter 3

Chapter 3 offers a quick review of some fundamental facts and prepare some terminology about branched coverings of curves

Chapter 4

This chapter is dedicated to the survey of various known formulae used in calculating Hurwitz numbers and thus contains little new informations. However, efforts have been made to collect these formulae and present them in an, hopefully coherent manner. In particular, we give a chronological list of most classical formula for counting branched coverings for arbitrary branched types.

Chapter 5

The first step in this investigation will be to show that any degree d meromorphic function on a smooth projective plane curve of degree $d \geq 5$ is isomorphic to a projection from a point $p \in \mathbb{P}^2$ to the pencil of lines through p away from the curve. In addition, we exhibit a 3–dimensional group which acts equivalently keeping the pencil fixed. Finally, we introduce a new notion of plane Hurwitz numbers which has a straight analogy to a special Zeuthen-type problem for calculating characteristic numbers for smooth plane curves.

Chapter 6

Finally, in the last chapter we put the pieces together and generalise some results in chapter 5. First, we easily show that any meromorphic function on a smooth projective curve can be represented as a composition of a birational map of the curve to $\mathbb{P}^2$ and a projection of the image curve from a point $p \in \mathbb{P}^2$ to the pencil of lines through p. Secondly, we introduce a natural stratification of Hurwitz scheme according to the minimal degree of a plane curve such that a given meromorphic function can be represented in this way. We also introduce the corresponding notion of Hurwitz numbers for each strata.

Chapter 1

Combinatorics of the Symmetric Group

Below we adopt notations as found in Chapter 1 of [Mac08]. All definitions and results on the symmetric group represented below are classical, and can be found in most standard texts such as [Sag01] and [GK81].

1.1 Partitions and permutations

The cardinality of a set S will be denoted by $|S|$ unless otherwise specified.

Definition 1.1.1. *A* **partition** μ *of a positive integer* d*, denoted* $\mu \vdash d$*, is a finite, weakly decreasing sequence of positive integers* $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ *called* **parts** *of* μ *such that* $\mu_1 + \mu_2 + \dots + \mu_n = d$.

We usually refer to d as the **size** of μ and denote it by $|\mu|$. The number n of parts of μ is called **length** of μ and is denoted by $\ell(\mu)$.

Example 1.1

There are 5 integer partitions of $d = 4$, namely

$$(4),\ (3\,,1),\ (2\,,2),\ (2\,,1\,,1),\ (1\,,1\,,1\,,1).$$

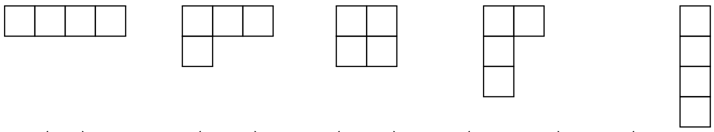
Denote the set consisting of the first d positive integers $\{1, 2, \dots, d\}$ by $[d]$. Let i be an integer in the set $\{1, 2, \dots, d\}$, the **multiplicity** of i in μ which we shall denote by $m_i(\mu)$ is the number of parts μ_j equaling i. We often use exponents to indicate repeated parts, whence a partition μ can be written multiplicatively as $\mu = 1^{m_1(\mu)} \cdot 2^{m_2(\mu)} \dots k^{m_k(\mu)}$ with $|\mu| = \sum_{i=1}^{k} i m_i(\mu)$. For instance, the partition $(2\,,1\,,1) = 1^2 \cdot 2$. The number of permutations of the parts of μ is the quantity

$$|\operatorname{Aut}(\mu)| = \prod_{i=1}^{k} m_i(\mu)!\ .$$

We can also represent partitions pictorially using Young diagrams.

Definition 1.1.2. *A Young diagram is an array of left and top-justified boxes, such that the row sizes are weakly decreasing. The Young diagram corresponding to* $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ *is the one that has* n *rows, and* μ_i *boxes in the* i^{th} *row.*

For instance, the Young diagrams corresponding to the above mentioned partitions of 4 are given below.

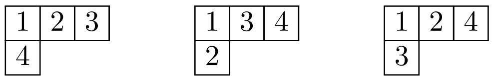


(4) $(3,1)$ $(2,2)$ $(2,1,1)$ $(1,1,1,1)$

A **Young tableau** of shape μ is obtained by filling the boxes of a Young diagram with numbers $[d] = \{1, 2, \ldots, d\}$. A **standard Young tableau** is a Young tableaux whose entries are increasing across each row and each column.

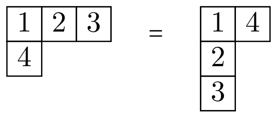
Example 1.2

Consider $\mu = (3,1)$, the number of standard tableaux with this shape is 3.



The conjugate of the Young tableau λ is the reflection of the tableau λ along the main diagonal. This is also a standard Young tableau.

Conjugate of



We will write λ^t to denote the **conjugate** partition of λ.
Let S_d be the group of all permutations on $[d]$, we make the convention that permutations are multiplied from right to left. A permutation $\alpha \in S_d$ is a **cycle of length** k or a $k-$**cycle** if there exist numbers $i_1, i_2, \ldots, i_k \in [d]$ such that

$$\alpha(i_1) = i_2, \qquad \alpha(i_2) = i_3, \qquad \ldots, \qquad \alpha(i_k) = i_1.$$

Thus, we can write α in the form $(i_1, i_2, \ldots, i_k)$. A cycle of length two is called a **transposition**. If we fix $\sigma \in S_d$, then σ can be uniquely decomposed into a product of disjoint cycles. The sum of the cycle lengths of σ is equal to d, so the lenghts form a partition of d. The **cycle type** of σ is an expression of the form

$$1^{m_1} \cdot 2^{m_2} \ldots d^{m_d},$$

where the m_i is the number of $i-$cycles in σ. We denote the set of all elements conjugate to σ in the symmetric group S_d by C_σ, that is

$$C_\sigma = \{\pi\sigma\pi^{-1} : \pi \in S_d\}.$$

Recall that two permutations are conjugate if and only if they have the same cycle type.

1.2 Representations of symmetric groups.

In this section, we review some relevant results on the representation theory of the symmetric group S_d, largely following [FH91] and [Sag01].

There are several equivalent ways of defining representation of groups. Fix a group G and a (finite) $\mathbb{C}$–vector space V. Denote by $GL(V)$ the set of all invertible linear transformations of V to itself, called the **general linear group** of V.

Definition 1.2.1. *A* ***representation*** *of G over $\mathbb{C}$, or simply a $\mathbb{C}$-**representation** of G, is a group homomorphism $\rho : G \longrightarrow GL(V)$.*

We call the dimension of V the **degree of** ρ. Building blocks of any representation of a group are its irreducible representations. In case of a symmetric group S_d these are known to be as many as there are conjugacy classes of the group. Furthermore, it turns out that for a group G(whence S_d), all we need to understand representations are the (irreducible) **characters**, i.e. encoding of the representation $\rho : G \longrightarrow GL(V)$ by a complex-valued function $\chi^\rho : G \longrightarrow \mathbb{C}$ constant on conjugacy classes defined by

$$\chi^\rho(g) = tr(\rho(g)),$$

where tr denotes the trace of the matrix $\rho(g)$ representing $g \in G$.

Each conjugacy class of S_d corresponds to a partition of d and we can use the combinatorial properties of these partitions to explicitly construct the irreducible representations S^λ, from which we can compute the irreducible characters.

Indeed, results in the theory of partitions, Young tableaux and symmetric functions [Mac08] provide not only a straight-forward way of constructing irreducible representations of S_d, but also an explicit formula for computing the corresponding characters. Namely, via the so-called **Murnaghan-Nakayama** rule we have a recursive method to compute the characters. The alternative method of calculating characters is the **Frobenius Formula**. Denote by $\chi^\lambda(C)$ the character of S^λ on the conjugacy class C. Since a conjugacy class C of an element in S_d consists of all permutations of the same cycle type, we use the notation χ^λ_μ to represent the character of S^λ at the conjugacy class of the cycle type μ.

The degree of S^λ is the dimension of the representation S^λ and is denoted by f^λ. There are many methods of computing the degree f^λ. Among them, is the use of the combinatorial fact that f^λ is the number of standard λ-tableaux. Formally, if (i,j) denotes the box in row i and column j of the standard Young diagram corresponding to λ; the hooklength h_{ij} is the number of boxes directly to the right and directly below (i,j) including the box (i,j). In particular, $h_{ij} = \lambda_i - j + \lambda^t_j - i + 1$. For instance, if $\lambda = (3,1)$, the hook length $h_{(2,1)}$ is 2. It can be shown that the dimension of the irreducible representation corresponding to λ is given by the **hook formula**

$$f^\lambda = \frac{d!}{\prod_{(i,j)\in\lambda} h_{ij}}.$$

Example 1.3

The degree of the irreducible representation of S_4 corresponding to partition $\lambda = (3\,,1) \vdash 4$ is the number of standard tableaux which can be calculated as

$$f^{\tiny\begin{array}{|c|c|c|}\hline & & \\ \hline \end{array}} = \frac{4!}{4 \cdot 2 \cdot 1 \cdot 1} = 3.$$

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \vdash d$ and consider the independent formal variables $x = (x_1, x_2, \ldots, x_m)$. The power sum function $p_\mu(x)$ is defined as

$$p_\mu(x) = \prod_{i=1}^{n} (x_1^{\mu_i} + \ldots + x_m^{\mu_i}).$$

Theorem 1.2.1 (Frobenius Character Formula)**.** *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \vdash d$. The character χ_μ^λ is equal to the coefficient of $\prod_{i=1}^n x_i^{\lambda_i+m-i}$ in $\Delta(x) p_\mu(x)$ where $\Delta(x)$ is the Vandermonde determinant*

$$\prod_{i<j} (x_i - x_j) = \det \begin{pmatrix} x_1^{n-1} & x_2^{n-1} & \ldots & x_m^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \ldots & x_m \\ 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Of particular interest, are the irreducible characters evaluated at the conjugancy classes (1^d) and $(1^{d-2} \cdot 2)$. Indeed, the dimension of a representation is the value of the character at the identity element $\mathbb{1} \in S_d$, which has cycle type $\mu = (1^d)$. the cycle type $(1^{d-2} \cdot 2)$ corresponds to transpositions, which we will see later corresponds to simple branch points.

Let $\lambda \vdash d$ be a partition and denote by λ^t the conjugate of λ. Then $l(\lambda) = \lambda_1^t$ is the length of λ. If $\tau \in S_d$ is a transposition then as established by Frobenius, one can show that

$$\binom{d}{2} \cdot \frac{\chi_\mu^\lambda(\tau)}{\chi_\mu^\lambda(\mathbb{1})} = \sum_{i=1}^{l(\lambda)} \left[\binom{\lambda_i}{2} - \binom{\lambda_i^t}{2} \right].$$

This leads to the following relation which we will need in the the computation of generalized simple Hurwitz numbers below.

$$\binom{d}{2} \cdot \frac{\chi_\mu^\lambda(1^{d-2}2)}{\chi_\mu^\lambda(1^d)} = \frac{1}{2} \sum_{i=1}^{n} \mu_i(\mu_i + 1) - \sum_{i=1}^{n} i\mu_i. \qquad (1.1)$$

Chapter 2

Toolbox on Algebraic Curves

In this chapter, we shall recall some definitions and some important results of the classical theory of curves that are useful in this thesis. There are many excellent references for definitions, results and proofs we mention herein, our favourites include [ACGH85, HM98, Mir95, Har77] and a more accessible [Kir98, Gri89].

2.1 Notation, conventions and definitions

The realm of this work is complex algebraic geometry, we fix once and for all the base field to be the field of complex numbers $\mathbb{C}$. By $\mathbb{P}^n$ we denote the $n-$dimensional projective space over $\mathbb{C}$. We agree that by a **variety** we usually mean a reduced algebraic projective variety over $\mathbb{C}$.

As routine, we will write $\mathcal{O}_X$ for sheaf of global holomorphic sections on X. We shall make the identification of invertible sheaves with line bundles and of locally free sheaves with vector bundles. Suppose that $\mathscr{F}$ is a sheaf of vector spaces over a variety X, we set

$$h^i(\mathscr{F}) := \dim \mathbf{H}^i(X, \mathscr{F}) \quad \text{and} \quad \chi(\mathscr{F}) := \sum_{i=0}^{\dim X} (-1)^i h^i(\mathscr{F}),$$

where $\mathbf{H}^i(X, \mathscr{F})$ is the i^{th}-cohomology group and $\chi(\mathscr{F})$ denotes the Euler characteristic of $\mathscr{F}$.

In what follows, the term **curve** means a complete connected variety of dimension 1. We also agree that by a smooth curve we implicitly mean that it is irreducible. A smooth curve is equivalent with having a field extension of **transcendence degree** 1.

Denote by ω_X the **canonical sheaf** on the curve X and by K_X or K the **canonical divisor** class associated to it. Given a curve X, we define its **arithmetic genus** to be

$$p_a(X) := 1 - \chi(\mathcal{O}_X) = 1 - h^0(\mathcal{O}_X) + h^1(\mathcal{O}_X)$$

and its **geometric genus** $p_g(X) = h^0(\omega_X)$ of X as the genus of the normalization of X. It is a beautiful result that for a smooth curve X we have

$$g(X) := p_a(X) = p_g(X).$$

This number is simply called the **genus** of X. Let X be a smooth curve of genus g, we recall that $\deg K_X = 2g - 2$. Throughout, we exploit the equivalence between divisors and line bundles on curves. Thus denote by $h^0(D)$ the dimension of the vector space of meromorphic functions having poles only on D, or equivalently, we will write $r(D)$ for the dimension of the complete linear system $|D| = \mathbb{P}\mathbf{H}^0\big(X, \mathcal{O}(D)\big)$ of effective divisors linearly equivalent to D. The first result we present is the Riemann-Roch theorem.

2.2 Riemann-Roch Theorem

Theorem 2.2.1 (Riemann-Roch Theorem)**.** *Let D be any divisor on a smooth curve X of genus g then*

$$h^0(D) - h^0(K - D) = \deg D - g + 1. \tag{2.1}$$

An effective divisor D on X such that $h^0(K - D) \neq 0$ is called **special**. If $\deg D > 2g - 2$, or, in general, if D is **nonspecial**, we get $\mathbf{H}^0\big(X, \mathcal{O}(K - D)\big) = 0$ so $h^0(D)$ is completely determined in terms of the topological invariants of X and D. However, it is usually special divisors which are relevant to specific geometric problems. Thus at times we may use the geometric version of Riemann-Roch to calculate $r(D)$ for the special divisor $D = p_1 + \ldots p_d$. See [ACGH85], page 12 for more details. For example for a general non-special effective divisor D of degree d we can calculate,

$$r(D) = \begin{cases} 0 & \text{if} \quad 0 \leq d \leq g \\ d - g & \text{if} \quad g \leq d \leq 2g - 2. \end{cases}$$

A projective subspace of a $|D|$ is called a **linear series** or **linear subsystem** on X for a divisor D. A point $p \in X$ common to all divisors in a linear series is called a **base point** and the set of all base points is called the **base locus** of a linear series. Given a linear series there is a simple criterion to check if a point $p \in X$ is its base point. Let's recall this result:

Proposition 2.2.1. *Let D be a divisor on a smooth curve X. Let $r(D)$ be the dimension of the linear series $|D|$. Then the dimension $r(D - p)$ of the linear series $|D - p|$ for any point $p \in X$ is such that*

$$r(D) - r(D - p) = \begin{cases} 0 \\ 1. \end{cases}$$

In particular, p is a base point of $|D|$ if and only if $r(D) - r(D - p) = 0$.

For example for $g \geq 1$, the canonical series $|K| = \mathbb{P}\mathbf{H}^0\big(X, \mathcal{O}(K)\big)$ on X is at least a pencil, i.e. $r(K) \geq 1$. By Riemann-Roch we have $h^0(K - p) = g - 1$ for any $p \in X$ and thus the canonical series is base point free by Proposition 2.2.1.

Definition 2.2.1. *Let X be a smooth curve of genus g. If X admits a finite surjectve morphism $X \longrightarrow \mathbb{P}^1$ of degree 2, we call X* **hyperelliptic** *.*

Let $f : X \longrightarrow Y$ be a nonconstant holomorphic mapping between two smooth curves. For any point $p \in X$ and $p = f(q)$ on Y we can choose coordinates centered at p and q such that we may write f in the normal form

$$w = z^{v_p(f)},$$

where $v_p(f)$ is the **vanishing order** of f at P. Each point $q \in Y$ determines an effective divisor on X of degree d (d is the degree of f) by the *pullback*, i.e the inverse image

$$f^*(q) = \sum_{p \in f^{-1}(q)} v_p(f) \cdot q,$$

whose support is the fiber $f^{-1}(q)$.

Consider a line bundle $\mathscr{L}$ on a curve X. Recall that the degree of $\mathscr{L}$ can be computed by counting zeros and poles of any section of $\mathscr{L}$ not vanishing identically on connected components of X. Moreover, if $f : X \longrightarrow Y$ is a a holomorphic mapping of degree d and $\mathscr{L}$ an invertible sheaf on Y then $\deg_X f^*\mathscr{L} = d \cdot \deg_Y \mathscr{L}$. This leads us to the following well know consequence of Riemann-Roch theorem about holomorphic maps between two smooth curves.

Theorem 2.2.2 (Riemann-Hurwitz formula)**.** *Let $f : X \longrightarrow Y$ be a nonconstant holomorphic map between two smooth curves. Then*

$$K_X \sim f^*K_Y + \sum_{p \in X} \big(v_p(f) - 1\big),$$

where K_X and K_Y are the canonical divisors on X and Y respectively. We shall need the following numerical version of Riemann-Hurwitz formula.

Corollary 2.2.1. *Let $f : X \longrightarrow Y$ be a nonconstant holomorphic map of degree d between two smooth curves of genus g and h respectively. Then*

$$2g - 2 = d(2h - 2) + \sum_{p \in X} \big(v_p(f) - 1\big).$$

Definition 2.2.2. *A* ***node*** *is a singularity on a curve which is locally complex-analytically isomorphic to a neighborhood of the origin in the zero locus $xy = 0 \subset \mathbb{C}^2$. A* **nodal curve** *is a curve such that every one of its points is either smooth or a node.*

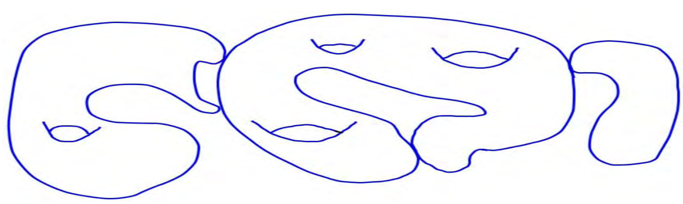


Figure 2.1: A curve with four nodes

Generally, we prefer to work with arithmetic genus since it remains constant in continuous families of curves. However in many other cases we will dwell on geometric genera of curves. In case of a plane curve, its genus can usually be deduced from less complicated calculations and/or explicit formulas.

Genus of plane curves

In this section, we recall the formulas for computing geometric genus of plane curves. In fact, we have the following result, see [Har77], page 393.

Theorem 2.2.3. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d having only ordinary singularities at $p_1, \ldots, p_N$. Suppose the singularities are of multiplicities m_i, at the point p_i. Then then the geometric genus of C is*

$$p_g(C) = \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{N} \binom{m_i}{2}. \tag{2.2}$$

Assuming that the only singularities of an irreducible curve are δ ordinary double points, the theorem yields the degree-genus formula for determining the genus g of the plane curve

$$p_g(C) = \frac{(d-1)(d-2)}{2} - \delta. \tag{2.3}$$

2.3 Moduli spaces of curves

A smooth curve of genus g is topologically a compact (orientable) surface with g handles. Furthermore it is well-known (see for example [Mir95]), that for every genus $g \geq 0$, there exists precisely one such compact topological surface up to diffeomorphism.

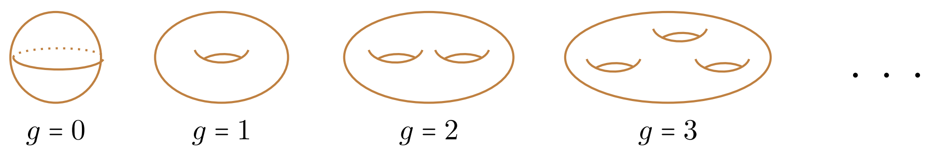


Figure 2.2: Compact Riemann surfaces of genus $g = 0, 1, 2$ and 3.

However, the question about how many different algebraic structures can be introduced to a compact surface with g handles is more complicated. For instance, the projective line $\mathbb{P}^1$ is the only compact surface of genus $g = 0$. On the other hand, for $g \geq 1$ there exist continuous families of non-isomorphic compact Riemann surfaces. Geometrically, this means that besides the discrete topological invariant, which is the genus, algebraic curves have continuous invariants called their **moduli**.

A moduli space is usually a space which parametrizes equivalence classes of geometric objects. So, points of a moduli space correspond to isomorphism classes of the geometric objects of interest. In our case, we are interested in algebraic curves. The arithmetic genus has a property of being constant in families, and therefore in this section (unless otherwise specified) by a genus we will always mean the arithmetic genus.

Definition 2.3.1. *Let $n \geq 0$ be an integer. An n-**pointed curve** is an $n+1$-tuple $(X, p_1, \ldots, p_n)$, where X is a smooth curve and p_i are distinct points on X. The points p_i's are called the **marked points** of X. The genus of $(X, p_1, \ldots, p_n)$ is defined to be the genus of X.*

By definition, a morphism $(X, p_1, \ldots, p_n) \longrightarrow (Y, q_1, \ldots, q_n)$ of smooth pointed curves is a morphism $f : X \longrightarrow Y$ such that $f(p_i) = q_i$ for all i. For non-negative integers (g, n) the set of all smooth $n-$pointed curves of genus g (modulo isomorphism) is denoted by $\mathcal{M}_{g,n}$. In other words,

$$\mathcal{M}_{g,n} := \left\{ (X, p_1, \ldots, p_n) \;\middle|\; \begin{array}{c} X \text{ is a smooth curve of genus } g \text{ with} \\ n \text{ distinct ordered points } p_1, \ldots p_n \end{array} \right\} \Big/ \sim .$$

Here $(X, p_1, \ldots, p_n) \sim (Y, q_1, \ldots, q_n)$ if and only if there exists an isomorphism from X to Y, preserving the marked points. A point in the moduli space $\mathcal{M}_{g,n}$ corresponds to a connected, complete smooth curve X of arithmetic genus g with n marked points $\{p_1, \ldots, p_n\}$.

It is a known fact [GH78], that the moduli space $\mathcal{M}_{g,n}$ exists for each $(g, n) \in \mathbb{N} \times \mathbb{N}$ satisfying the condition $2g - 2 + n > 0$. The space $\mathcal{M}_{g,n}$ is not compact because smooth curves can degenerate. For instance, a family of genus 1 curves given by the family of affine equations $y^2 = x^3 + x^2 + t$ is smooth for $t \neq 0$. At $t = 0$, the curve is singular and can be thought as lying on the boundary of the corresponding moduli space. Thus, to compactify $\mathcal{M}_{g,n}$ we need to allow degenerate curves but with as mild degeneracies as possible so that we can still do meaningful geometry. There are several ways to get good compactifications of $\mathcal{M}_{g,n}$.

One compactification of $\mathcal{M}_{g,n}$ is the Deligne-Mumford compactification. It was first described in [DM69] and is obtained by adding curves with nodes to $\mathcal{M}_{g,n}$. Another compactification is due to D. Schubert [Sch91] which allows cuspidal curves. Still another is the construction of Hassett-Hyeon [Has08] which allows the inclusion of tacnodal curves. Observe that each compactification allows different type of degeneration and therefore is useful in its own situation. However, the most fundamental compactification is due to Deligne-Mumford. Marked points on stable curves are not allowed to come together or to approach nodal points. In this compactification one uses a beautiful concept of **bubbling** when special points tend to collide. Namely, if two smooth marked points approach each other, the curve sprouts off a copy of $\mathbb{P}^1$ with two marked points distributed on it.

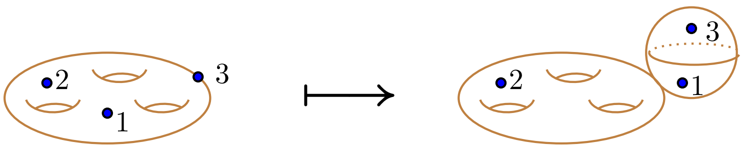


Figure 2.3: Bubbling as the marked point 1 collides with the marked point 3

Similarly, if a marked point approaches a node, we let the limit to sprout another copy of $\mathbb{P}^1$ at the node with the marked point located on it.

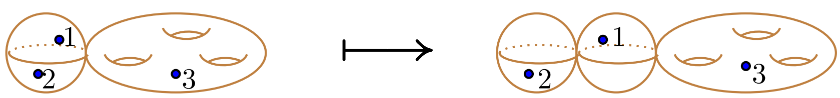


Figure 2.4: Bubbling when the marked point 1 approaches a nodal point

The Mumford-Deligne orbifold $\overline{\mathcal{M}}_{g,n}$ is the space of pointed nodal stable curves of genus g with n marked points together with certain stability condition. To describe points of $\overline{\mathcal{M}}_{g,n}$ formally, we need the following definitions.

Definition 2.3.2. *Let $n \geq 0$ be an integer. An $n-$pointed nodal curve is a tuple $(X, p_1, \ldots, p_n)$, where X is a nodal curve and p_i are distinct smooth points on X. A special point of the $n-$pointed nodal curve means a node branch or a marked point p_i on X.*

The genus of $(X, p_1, \ldots, p_n)$ is defined to be the arithmetic genus of X.

Definition 2.3.3. *A nodal n-pointed curve $(X, p_1 \ldots, p_n)$ is called* **stable** *if for each connected component we have either:*

i. $2g - 2 + n > 0$; *i.e each smooth connected component of X of genus 0 has at least 3 special points while any genus 1 smooth component of X has at least one special point,*

ii. $(X, p_1 \ldots, p_n)$ *has no infinitesimal automorphisms fixing the special points,*

iii. $|\mathrm{Aut}(X, p_1 \ldots, p_n)| < \infty$.

Given a $n-$pointed nodal curve $(X, p_1, \ldots, p_n)$, we can construct its **dual graph** Γ (see [YM99] for precise details) as follows:

- V_Γ = the set of vertices, one for every irreducible component of the curve and labelled by g_v, where g_v is the geometric genus of the corresponding component.
- E_Γ = the set of edges are nodes, there corresponds an edge to each nodal point.
- T_Γ = the set of half edges or tails, one for each marked point or points mapping to nodes, with the same label as the point.

The genus $g(\Gamma)$ of the dual graph Γ is determined by the equation

$$(g(\Gamma) - 1) = \sum_{v \in V_\Gamma} (g_v - 1) + |E_\Gamma|.$$

Observe that the genus of the graph Γ of X equals the arithmetic genus of the curve X. We call the pair $(g(\Gamma), n)$ the type of the dual graph. The valence of a vertex v is the number of edges or tails attached to it, and is denoted by $\deg(v)$. Using the corresponding dual graph we can determine if the curve is stable or not. Namely, the dual graph is stable if and only if for every vertex we have $2g_v - 2 + \deg(v) > 0$.

Figure 2.5: Example of a pointed curve and its corresponding dual graph.

We define

$$\overline{\mathcal{M}}_{g,n} = \left\{ (X, p_1, \ldots, p_n) \;\middle|\; \begin{array}{c} X \text{ is a stable curve of genus } g \text{ with} \\ n \text{ distinct ordered points } p_1, \ldots p_n \end{array} \right\} \Big/ \sim,$$

as a set. Let B be an algebraic variety, recall that a morphism $\pi : \mathcal{C} \longrightarrow B$ is called **flat** if there exists an embedding

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathbb{P}^{\mathbf{N}} \times B \\ {\scriptstyle\pi}\downarrow & \swarrow & \\ B & & \end{array}$$

for some $\mathbf{N} \in \mathbb{N}$ such that $C_b = \pi^{-1}(b) \subseteq \mathbb{P}^N \times \{b\}$ has the same Hilbert polynomial for any point $b \in B$ ([Har77], page 261).

Definition 2.3.4. *Let B be an algebraic variety, a* **family** *$\mathcal{C}$ of $n-$pointed genus g stable curves over B is a flat morphism $\pi : \mathcal{C} \longrightarrow B$ with n sections corresponding to each point p_i such that each geometric fiber $\left(\mathcal{C}_b := \pi^{-1}(b) : p_1(b), \ldots, p_n(b)\right)$ is an $n-$pointed genus g stable curve.*

Theorem 2.3.1 (Deligne-Mumford, [GH78])**.** *For a pair (g, n) of non-negative integers such that $2g - 2 + n > 0$, the set of stable $n-$ pointed curves of genus g is parametrized by compact, complex-analytic orbifold $\overline{\mathcal{M}}_{g,n}$. The space $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is an open Zariski dense subvariety. Moreover, $\overline{\mathcal{M}}_{g,n}$ is connected, irreducible and is endowed with the universal stable curve*

$$\overline{\mathcal{C}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n},$$

and the marked points form n pointwise disjoint sections $\sigma_i : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{C}}_{g,n}$, for all $i = 1, \ldots, n$.

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$. Notice that

$$\dim \overline{\mathcal{M}}_{g,n} = \dim \mathcal{M}_{g,n} = 3g - 3 + n.$$

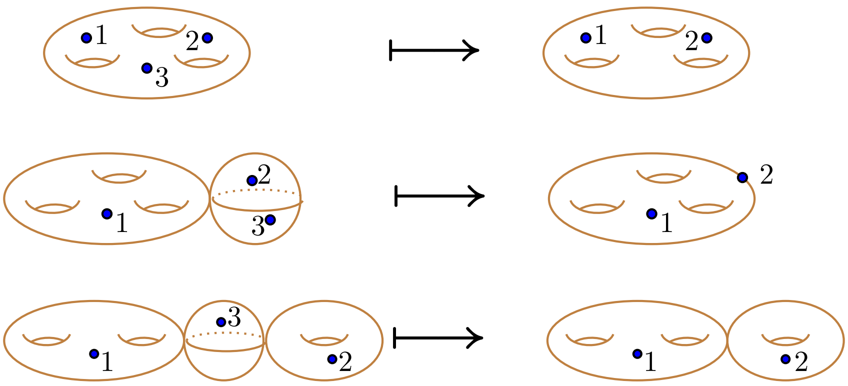
Every curve $(X, p_1 \ldots, p_n)$ in $\mathcal{M}_{g,n}$ is smooth; its dual graph is a corolla with n tails and one vertex of genus g. The locus $\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$ parameterizing singular curves is a sub-orbifold of $\overline{\mathcal{M}}_{g,n}$ of codimension 1 (a normal crossing divisor in the orbifold sense). It is called the **boundary** of $\overline{\mathcal{M}}_{g,n}$ and denoted by $\partial\overline{\mathcal{M}}_{g,n}$. A generic point of $\partial\overline{\mathcal{M}}_{g,n}$ corresponds to a stable curve with only one nodal point. Dual graphs also encode classes of the corresponding strata in $\partial\overline{\mathcal{M}}_{g,n}$ and thus give the stratification of $\overline{\mathcal{M}}_{g,n}$.

2.3.1 Morphisms of moduli spaces

There are some natural morphisms between various moduli spaces of stable pointed curves. Among them we have:

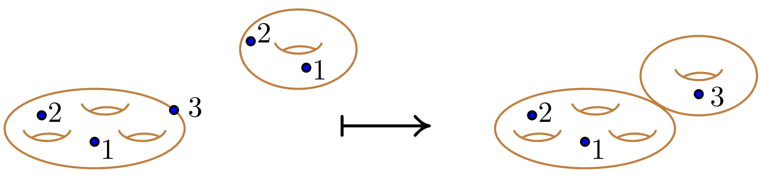
i. The **permutation morphism**: The symmetric group S_n acts naturally on $\overline{\mathcal{M}}_{g,n}$ by permuting the markings of n-pointed curves. This induces an automorphism of $\overline{\mathcal{M}}_{g,n}$ called the permutation morphism.

ii. The **forgetful morphism** $\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ that forgets the $(n+1)$st marked point on a given stable curve of genus g. If the stability of the curve is lost we contract rational unstable components. The forgetful morphism π can be interpreted as the universal curve over $\overline{\mathcal{M}}_{g,n}$.

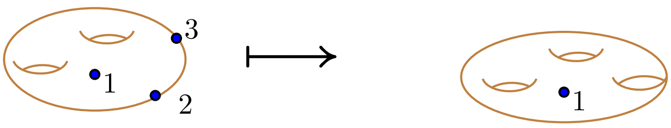


iii. The **gluing morphisms**

(a) The map $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ obtained by gluing the n_1+1-st point with n_2+1-st point of n_1+1-pointed curve of genus g_1 and n_2+1-pointed curve of genus g_2 respectively. This operation gives a stable curve of genus g_1+g_2 with n_1+n_2 marked points.



(b) The map $\overline{\mathcal{M}}_{g,n+2} \longrightarrow \overline{\mathcal{M}}_{g+1,n}$, that glues the points labelled by $n+1$ and $n+2$ of a stable genus g curve with $n+2$ marked points giving rise to a stable n-pointed curve of genus $g+1$.



Moduli space of genus zero curves

For arbitrary pairs (g,n) of nonnegative integers, the corresponding moduli spaces have very big dimensions as well as a complicated geometric structure. However, for small values of g we can explicitly describe the geometry of these spaces. The basic example is when $g=0$. Recall that $\dim \mathcal{M}_{g,n} = 3g-3+n$. Therefore if $g=0$ then $\mathcal{M}_{0,n}$ is well defined for $n \geq 3$. Since $\mathbb{P}^1$ has no moduli, a point in $\mathcal{M}_{0,n}$ corresponds to n distinct ordered points in $\mathbb{P}^1$ up to projective equivalence induced by the action of $\mathbb{P}\mathbf{GL}(2,\mathbb{C})$. The automorphism group $\mathbb{P}\mathbf{GL}(2,\mathbb{C})$ is transitive on triples of points. Fixing three of these points at $(0,1,\infty)$, we still have $n-3$ points which are allowed to vary. Thus,

$$\mathcal{M}_{0,n} \cong (\mathbb{P}^1 \backslash \{0,1,\infty\})^{n-3} \backslash \Delta,$$

where Δ consists of all diagonals, and all point configurations include $0,1$ and ∞. In particular, every smooth curve (X,p_1,p_2,p_3) of genus zero is isomorphic to $(\mathbb{P}^1,0,1,\infty)$. Moreover, since there is no stable 3-pointed nodal curve of genus 0, we conclude that $\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3} = \{\mathbf{pt}\}$. Similarly, every smooth curve (X,p_1,p_2,p_3,p_4) is uniquely identified with $(X,0,1,\infty,\lambda)$ for $\lambda \neq 0,1$, or ∞. The number λ is determined by the position of the marked points on X. Thus one can interpret λ as the cross-ratio of the four points (p_1,p_2,p_3,p_4) given by

$$\lambda = \frac{(p_1-p_4)(p_3-p_2)}{(p_1-p_2)(p_3-p_4)}\,.$$

Therefore, $\mathcal{M}_{0,4}$ coincides with the set of admissible values of λ, i.e. $\mathcal{M}_{0,4} \cong \mathbb{P}^1\backslash\{0,1,\infty\}$. The boundary $\partial\overline{M}_{0,4}$ consists of three smooth points corresponding to nodal curves for $\lambda = 0,1$ and ∞ and thus we have $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$. Intuitively, we can interpret the limit of two colliding points as another $\mathbb{P}^1$ with the two points on it. The corresponding strata in $\overline{\mathcal{M}}_{0,4}$ described by their dual graphs are given in Figure 2.6 and Figure 2.7.

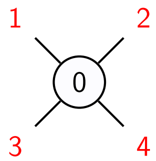


Figure 2.6: Stratum of nonsingular curves in $\overline{\mathcal{M}}_{0,4}$

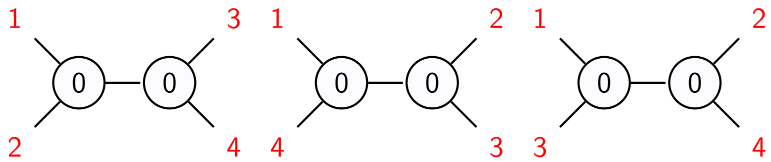


Figure 2.7: Boundary strata

2.3.2 Cohomological classes on $\overline{\mathcal{M}}_{g,n}$

In this subsection, we introduce some cohomological classes on Deligne- Mumford space $\overline{\mathcal{M}}_{g,n}$ of pointed curves and describe various relations among these cohomology classes, top intersection numbers and Hodge integrals. This will enable us to present the EKedahl-Lando-Shapiro-Vainshtein (ELSV) formula which relates Hurwitz numbers to the intersection theory on moduli spaces of curves.

To represent a point in $\overline{\mathcal{M}}_{g,n}$ we often write $[X,p_1,\dots,p_n]$. If ξ is a 0–cycle on $\overline{\mathcal{M}}_{g,n}$ then we define its degree as $\int_{\overline{\mathcal{M}}_{g,n}}\xi$. For general g and n, the cohomology ring of $\mathbf{H}^{\bullet}(\overline{\mathcal{M}}_{g,n})$ or its algebraic counterpart; the Chow ring $A^{\bullet}(\overline{\mathcal{M}}_{g,n})$ (where intersection theory happens) are far from having a complete description. In 1983 D. Mumford defined the cohomological ring $\mathbf{H}^{\bullet}(\overline{\mathcal{M}}_{g,n})$ and the Chow ring $A^{\bullet}(\overline{\mathcal{M}}_{g,n})$ on moduli spaces of stable pointed curves but he emphasized that the subring $R^{\bullet}(\overline{\mathcal{M}}_{g,n}) \subset \mathbf{H}^{2i}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ of the cohomology ring called the **tautological ring**, consists of more geometrically natural classes. In fact, the tautological ring is all we need to get concrete information about the cohomology of $\overline{\mathcal{M}}_{g,n}$ as at present there is no known algebraic class which is not in the tautological ring. The space $\overline{\mathcal{M}}_{g,n}$ has a fundamental class $[\overline{\mathcal{M}}_{g,n}] \in \mathbf{H}_{2(3g-3+n)}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$. So, if we have cohomological classes on $\overline{\mathcal{M}}_{g,n}$ denoted by $\alpha_1,\dots,\alpha_m \in \mathbf{H}^\bullet(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ we can define their **top intersection numbers** to be

$$\int_{\overline{\mathcal{M}}_{g,n}} \alpha_1\dots\alpha_m := \left\langle \alpha_1\dots\alpha_m\ ,[\overline{\mathcal{M}}_{g,n}]\right\rangle \in \mathbb{Q}, \tag{2.4}$$

where $\langle\ ,\ \rangle$ denote the pairing between the cohomology $\mathbf{H}^\bullet(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ and the homology $\mathbf{H}_\bullet(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$. Now we need cohomology classes of $\overline{\mathcal{M}}_{g,n}$ to proceed with intersections. A fundamental way of producing cohomology classes on $\overline{\mathcal{M}}_{g,n}$ is to take the Chern classes of some naturally defined vector bundles and then using the forgetful and gluing morphisms we can pullback and push forward these classes. In addition to cohomological classes defined by natural vector bundles, we do have other natural classes on $\overline{\mathcal{M}}_{g,n}$ coming from the strata. In fact, Keel [Kee92] has shown that in genus 0, the cohomology ring is generated by the fundamental classes of the closure of the strata. All these cohomological classes live in the tautological ring and are called **tautological classes**. The following definitions of the tautological ring $R^\bullet(\overline{\mathcal{M}}_{g,n})$ is due to Faber-Pandharipande and Graber-Vakil.

Definition 2.3.5. *The system of tautological rings $R^\bullet(\overline{\mathcal{M}}_{g,n})$ is the smallest system of*

i. [FP05]: $\mathbb{Q}-$algebras closed under push-forwards by the natural morphisms.

ii. [GV05]: $\mathbb{Q}-$vector spaces closed under push-forwards by the natural morphisms, and which includes all monomials in the $\psi-$classes.

Moreover, it is worth noting that the above two systems are shown to be equivalent in [GV05]. A consequence of this equivalence is that any top intersection class in the tautological ring can be determined only from the top intersections of the $\psi-$classes. The rational cohomology $\mathbf{H}^\bullet(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ of $\overline{\mathcal{M}}_{g,n}$ is an algebra over $\mathbb{Q}$, for any elements $\xi_i \in \mathbf{H}^i(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ and $\xi_j \in \mathbf{H}^j(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ then the product $\xi_i\xi_j \in \mathbf{H}^{i+j}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$.

I. **Boundary classes:** The closure of each codimension 1 stratum D is a divisor in $\overline{\mathcal{M}}_{g,n}$. Denote by $[D] \in \mathbf{H}^2(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ its cohomology class. As was mentioned earlier, these cohomology classes maybe be described using the corresponding dual graphs.

II. $\psi-$ **classes:** (also called the **Witten classes** on $\overline{\mathcal{M}}_{g,n}$). Recall, the forgetful morphism $\pi:\overline{\mathcal{M}}_{g,n+1}\longrightarrow\overline{\mathcal{M}}_{g,n}$ can be identified with the universal curve $\pi:\overline{\mathcal{C}}_{g,n}\longrightarrow\overline{\mathcal{M}}_{g,n}$. This is to each point $[X,p_1,\dots,p_n]$ of $\overline{\mathcal{M}}_{g,n}$ and to each point $p\in X$ we associate a stable pointed curve $[\tilde{X},\tilde{p_1},\dots,\tilde{p}_n,\tilde{p}_{n+1}] \in \overline{\mathcal{M}}_{g,n+1}$ in the following sense:

- If the point $p \in X$ is not a marked or nodal point, then we set the element $[X,p_1,\dots,p_n,p] = [\tilde{X},\tilde{p_1},\dots,\tilde{p}_n]$ with the point p relabelled p_{n+1}.
- If $p = p_i$ for some marked point p_i, then let $\tilde{X}$ be X with a $\mathbb{P}^1$ where $\mathbb{P}^1$ is the bubble at p marked p_i and $p := p_{n+1}$. We will denote this $(n+1)$-pointed stable curve by $\sigma_i([X,p_1,\dots,p_n])$.
- Finally, if p is a nodal point, let $\tilde{X}$ be X with a $\mathbb{P}^1$-bubble at this node labelled by $p := p_{n+1}$.

Now, there is a natural line bundle on $\overline{\mathcal{M}}_{g,n+1} =: \overline{\mathcal{C}}_{g,n}$ whose fiber at the point $[X,p_1,\dots,p_n]$ is the contagent line $T^*_{p_i}X$ at the $i-$th marked point for p_i nonsingular points. We can extend this contagent bundle using the unique line bundle $\mathscr{L} \longrightarrow \overline{\mathcal{M}}_{g,n+1}$ called the **relative dualizing sheaf** of the universal curve. In particular,

$$\mathscr{L} = K_A \otimes \pi^* K_B^{-1},$$

where K_A is the canonical line bundle on $\overline{\mathcal{M}}_{g,n+1}$, and K_B denotes the canonical line bundle on $\overline{\mathcal{M}}_{g,n}$. The sections on $\mathscr{L}$ along a non-singular fiber are exactly the holomorphic 1–forms on the fiber. On the other hand, the sections along a singular fiber are meromorphic 1–forms with at most simple poles allowed at the nodes and the two residues at the preimages of each nodal point through normalization adding up to zero. This way we obtain n holomorphic line bundles $\mathscr{L}_i = \sigma_i^* \mathscr{L}$, one for each of the marked points for the sections $\sigma_i : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n+1}$. We take the first Chern class of the line bundle $\mathscr{L}_i \longrightarrow \overline{\mathcal{M}}_{g,n}$ and define

$$\psi_i = c_1(\mathscr{L}_i) \in \mathbf{H}^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}), \qquad i = 1, \dots, n.$$

The ψ_i-class on $\overline{\mathcal{M}}_{g,n}$ is different from ψ_i on $\overline{\mathcal{M}}_{g,n+1}$. However, using the forgetful morphism we have a relation:

$$\psi_i = \pi^* \psi_i + D_{0,\{i,n+1\}}, \tag{2.5}$$

where $D_{0,\{i,n+1\}}$ is the boundary divisor corresponding to reducible curves with one node, where one component is of genus 0 and contains only the marked points p_i and p_{n+1}.

III. λ–**classes**: The **Hodge bundle** $\mathbb{E}$ is another natural vector bundle on $\overline{\mathcal{M}}_{g,n}$. The Hodge bundle is a rank g– vector bundle $\mathbb{E} \longrightarrow \overline{\mathcal{M}}_{g,n}$ whose fiber over the point $[X, p_1, \dots, p_n]$ is $\mathbf{H}^0(X, \omega_X)$, where ω_X is the dualizing sheaf. More formally, we put $\mathbb{E} = \pi_*(\mathscr{L})$ and define the λ–classes as

$$\lambda_j = c_j(\mathbb{E}) \in \mathbf{H}^{2j}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}), \qquad j = 1, \dots, g \tag{2.6}$$

the j–th Chern class of the Hodge bundle $\mathbb{E} \longrightarrow \overline{\mathcal{M}}_{g,n}$.

The forgetful morphism yields some recurrence relations between the intersection numbers. Consider $\overline{\mathcal{M}}_{g,n}$ where the pair (g, n) satisfies the stability condition $2g - 2 + n > 0$. The simplest integral is over $\overline{\mathcal{M}}_{0,3}$ namely

$$\int_{\overline{\mathcal{M}}_{0,3}} \psi_1^0 \psi_2^0 \psi_3^0 = 1. \tag{2.7}$$

Indeed, since $\overline{\mathcal{M}}_{0,3}$ is a point, (i.e. a unique genus 0 curve with 3 marked points and such a curve has a trivial automorphism group) there is a unique class with nonzero integral which by definition is equal to 1. (It is called the **initial condition** over $\overline{\mathcal{M}}_{0,n}$). The other initial case is the integral

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}, \tag{2.8}$$

which is the **initial condition** case for $\overline{\mathcal{M}}_{1,n}$. We have the following intersection identities for the ψ– classes:

1. **The Dilaton Equation,**

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{m_1} \dots \psi_n^{m_n} \psi_{n+1} = (2g-2+n) \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{m_1} \dots \psi_n^{m_n}, \tag{2.9}$$

2. **The String Equation,**

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{m_1} \dots \psi_n^{m_n} = \sum_{i=1}^{n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{m_1} \dots \psi^{m_i-1} \dots \psi_n^{m_n}. \tag{2.10}$$

In calculation of the intersection numbers it is convenient to adopt the Witten's notation see [Wit91, Ful98]. This notation basically encodes only the symmetry between the ψ–classes and one writes

$$\langle \tau_{m_1} \dots \tau_{m_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{m_1} \dots \psi_n^{m_n}, \tag{2.11}$$

for all intersections of the ψ–classes. Here $\tau_0, \tau_1, \tau_2, \dots$ are commuting formal variables called the **correlation functions**, so that we can write intersection numbers in the form

$$\langle \tau_0^{a_0} \tau_1^{a_1} \tau_2^{a_2} \dots \rangle,$$

with the convention that the product $\langle \tau_{m_1} \dots \tau_{m_n} \rangle = 0$ if $n = 0$ or $m_1 + \dots + m_n \neq 3g-3+n = \dim(\overline{\mathcal{M}}_{g,n})$. Essentially, we have a $\mathbb{Q}$-linear functional

$$\langle \bullet \rangle : \mathbb{Q}[\tau_0, \tau_1, \tau_2, \dots] \longrightarrow \mathbb{Q}.$$

Using correlation functions in Witten's notation the dilaton equation (2.9) and the string equation (2.10) can be written as

$$\text{Dilation Equation:} \quad \langle \tau_{m_1} \dots \tau_{m_n} \tau_1 \rangle_g = (2g-2+n) \langle \tau_{m_1} \dots \tau_{m_n} \rangle_g;$$

$$\text{String Equation:} \quad \langle \tau_0 \tau_{m_1} \dots \tau_{m_n} \rangle_g = \textstyle\sum_{i=1}^{n} \langle \tau_{m_1} \dots \tau_{m_i-1} \dots \tau_{m_n} \rangle_g.$$

As observed by E. Witten due to symmetry the integral (2.11) depends only on the unordered set $\{m_1, \dots, m_n\}$ of non-negative integers. The integral notwithstanding its rationality can be thought as the **intersection number of points** of m_i copies of the divisors ψ_i for all $i = 1, \dots, n$. Moreover, for each set $\{m_1, \dots, m_n\}$ there is at most one g such that the value of the integral is nonzero. It is also important to note that the indices on τ_i have nothing to do with the marked point p_i. It turns out that the value of integral (2.11) can be completely determined using (2.7), and the string equation. Indeed, the symmetric group S_n acts naturally on $\overline{\mathcal{M}}_{g,n}$ and the dimension restriction on the indices we can determine a closed form for $g = 0$ integrals as described in [OP01].

Starting with the simplest integral over $\overline{\mathcal{M}}_{0,n}$, i.e. the initial condition obtained when $n = 3$

and the string equation, we proceed by induction as follows:

$$\begin{aligned} n=3: &\quad \langle\tau_0\tau_0\tau_0\rangle_0 := \langle\tau_0^3\rangle_0 = \int_{\overline{\mathcal{M}}_{0,3}} \psi_1^0\psi_2^0\psi_3^0 = 1. \\ n=4: &\quad \langle\tau_0^3\tau_1\rangle_0 = \langle\tau_0^3\rangle_0 = 1. \\ n=5: &\quad \langle\tau_0^3\tau_1^2\rangle_0 = \langle\tau_0^3\tau_1\rangle_0 + \langle\tau_0^3\tau_1\rangle_0 = 2, \\ &\quad \langle\tau_0^4\tau_2\rangle_0 = \langle\tau_0^3\tau_1\rangle_0 = 1. \end{aligned}$$

In general, we have the following proposition(see [LZ04], p. 254)

Proposition 2.3.1. *Let $m_1 + \ldots + m_n = n-3$. Then*

$$\langle\tau_0\tau_{m_1}\ldots\tau_{m_n}\rangle_0 := \int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{m_1}\ldots\psi_n^{m_n} = \binom{n}{m_1,\ldots,m_n} = \frac{(n-3)!}{m_1!\ldots m_n!}.$$

2.4 Moduli space of stable maps

A natural generalization of the moduli spaces of curves are the moduli spaces of maps of curves. In the case of constant maps these spaces coincide with moduli spaces of curves. If these maps are to $\mathbb{P}^1$, then the spaces coincide with those of meromorphic functions. Following [FP97], we will give a brief account of this spaces.

Definition 2.4.1. *Let X be a smooth projective variety. Let $C = (C, p_1, \ldots, p_n)$ be a n-pointed smooth curve and let $\beta \in \mathbf{H}_2(X,\mathbb{Z})$. We say a map $f : C \longrightarrow X$ represents a homology class β if $[C] \in \mathbf{H}_2(C,\mathbb{Z})$ is the fundamental class of C and that $f_*[C] = \beta$.*

If $X = \mathbb{P}^m$, since $\mathbf{H}_2(\mathbb{P}^m,\mathbb{Z}) \cong \mathbb{Z}[line]$ it follows that $\beta = d[line]$ is the class of a line, we say that d is the degree of the map f and write d for $d[line]$.

Definition 2.4.2. *A pointed map of genus g is a morphism $f : (C, p_1, \ldots, p_n) \longrightarrow X$ that represents a class β of a n-pointed smooth curve C.*

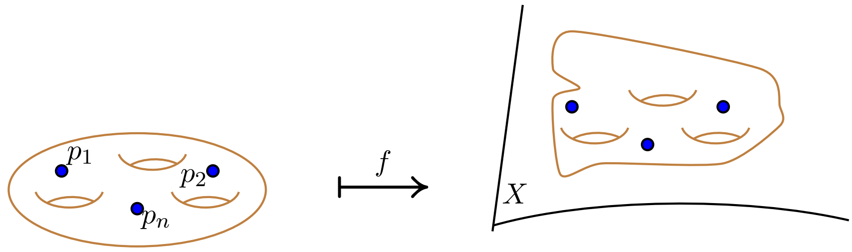
Two pointed maps $f_1 : (C_1, p_1, \ldots, p_n) \longrightarrow X$ and $f_2 : (C_2, q_1, \ldots, q_n) \longrightarrow X$ are called isomorphic if there exists an isomorphism $\phi : C_1 \longrightarrow C_2$ of curves such that $\psi(p_i) = q_i$ for all i and ϕ admits the following commutative diagram:

$$\begin{array}{ccc} C_1 & \xrightarrow{\ \phi\ } & C_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & X & \end{array}$$

The space parametrizing isomorphism classes $[f : (C, p_1, \ldots, p_n) \longrightarrow X]$ of pointed maps representing a class β is denoted by

$$\mathcal{M}_{g,n}(X,\beta) = \left\{ f : C \longrightarrow X \;\middle|\; \begin{array}{c} C \text{ a smooth curve of genus } g \text{ with } n \\ \text{distinct ordered points } p_1, \ldots p_n \end{array} \right\} \Big/ \sim .$$

We use the shorthand $(C, p_1, \ldots, p_n, f)$ for an element in $\mathcal{M}_{g,n}(X,\beta)$. The moduli space $\mathcal{M}_{g,n}(X,\beta)$ of maps is not compact since such maps can degenerate in various ways, but it has a natural compactification by allowing nodal domains. This compactification is credited to M. Kontsevich.



Definition 2.4.3. *Let X be a smooth projective variety and C be a nodal curve with $p_1, \ldots, p_n$ smooth distinct marked points. A pointed map $f : C \longrightarrow X$ such that $f_*[C] = \beta$ where C is a connected nodal curve of arithmetic genus g is called stable if the automorphism group of $(C, p_1, \ldots, p_n, f)$ is finite.*

That is, the morphism $\phi : C \longrightarrow C$ that satisfies $f \circ \phi = f$ and fixes the marked point has a finite automorphism group. Equivalently, if f is constant on irreducible components of C of arithmetic genus 0, then the component has at least 3 special points while all irreducible components of arithmetic genus 1 on which f is constant contains at least 1 special point.

Definition 2.4.4. *The Kontsevich moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is the moduli space of stable maps to X of arithmetic genus g of class $\beta \in \mathbf{H}_2(X,\mathbb{Z})$ written as,*

$$\overline{\mathcal{M}}_{g,n}(X,\beta) = \left\{ f : C \longrightarrow X \;\middle|\; \begin{array}{c} C \text{ a } n\text{-pointed nodal curve of} \\ \text{genus } g,\ \mathrm{Aut}(f) < \infty \text{ and } f_*[C] = \beta \end{array} \right\} \Big/ \sim .$$

Kontsevich moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is known to be a Deligne-Mumford stack. The expected or the virtual dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ denoted by vdim $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is determined by

$$\text{vdim } \overline{\mathcal{M}}_{g,n}(X,\beta) =_\beta c_1(T_X) + (\dim X - 3)(1-g) + n,$$

where $c_1(T_X)$ is the first Chern class of the tangent bundle to X.

Remark 2.4.1. *If $\beta = 0$, the Kontsevich moduli space $\overline{\mathcal{M}}_{g,n}(X,0) = \overline{\mathcal{M}}_{g,n} \times X$. In particular, if X is a point, then $\overline{\mathcal{M}}_{g,n}(X,0) = \overline{\mathcal{M}}_{g,n}$ as earlier claimed.*

However, for $\beta \neq 0$, the space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is not always well-behaved even when X a smooth projective variety as nice as $\mathbb{P}^1$. Indeed $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is possibly reducible, non reduced and may be of impure dimension. For instance, it may contain components whose dimensions exceed the above virtual dimension.

Example 2.1

The moduli space $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$ for $d > 1$ and $g > 0$ consists of two components of different dimensions. In fact, one component consists of generic maps from smooth curves to $\mathbb{P}^1$ which coincides with the small Hurwitz space. Thus it has dimension $2g + 2d - 2$. Another component has dimension $2d + 3g - 3$. The later component consists of generic maps from nodal curves $C_0 \cup C_g$ where C_i has genus i, $C_0 \longrightarrow \mathbb{P}^1$ maps with degree d, while $C_g \longrightarrow \mathbb{P}^1$ is contraction map to a point in $\mathbb{P}^1$.

On the other hand, the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has also known to have well-behaved geometrical properties, which include:

i. The space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is compact and contains a unique open component $\mathcal{M}_{g,n}(X,\beta)$ as a substack (possibly empty), i.e. the coarse moduli of maps of smooth curves. For instance, it follows from stability conditions on maps that $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2,0) = \varnothing$, while the moduli space $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2,3)$ has a open subset of dimension 9 that can be informally be thought as parametrizing smooth cubics in $\mathbb{P}^2$.

ii. The moduli space comes with two natural classes of continuous maps:
   - The **stabilization map** $st : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n}$ which forgets the stable maps on $\overline{\mathcal{M}}_{g,n}(X,\beta)$.
   - For each i in $1 \leq i \leq n$, there are n **evaluation maps**

$$ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \mathrm{Sym}^n X, \text{ given by } (C, p_1, \ldots, p_n, f) \longmapsto f(p_i),$$

iii. There is a universal map over $\overline{\mathcal{M}}_{g,n}(X,\beta)$. If $n_1 \geq n_2$ and $\overline{\mathcal{M}}_{g,n_2}(X,\beta)$ exists then there is a **forgetful morphism**

$$\overline{\mathcal{M}}_{g,n_1}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n_2}(X,\beta).$$

Using the forgetful morphism, we can make an identification of the moduli space $\overline{\mathcal{M}}_{g,n+1}(X,\beta)$ with the universal curve over $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

Cohomological classes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$

The cohomology classes on $\overline{\mathcal{M}}_{g,n}$ can naturally be lifted to $\overline{\mathcal{M}}_{g,n}(X,\beta)$ via the stabilization map. Also using the evaluation maps, cohomology classes can be constructed from that of X. Namely, for the cohomology class $\gamma \in \mathbf{H}^\bullet(X,\mathbb{Q})$ we have its pullback by evaluation which yields $ev^*(\gamma) \in \mathbf{H}^\bullet(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$. More importantly, the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ admits a canonical virtual fundamental class of expected dimension denoted by

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathsf{vir}},$$

which lies in $\mathbf{H}_{2\mathsf{vdim}}(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$ where all intersection invariants of cohomology classes are evaluated. Of course, this is a highly nontrivial fact which follows from the result below.

Theorem 2.4.1 (Behrend-Fantechi)**.** *The Kontsevich moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ carries a natural homology class, i.e. $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in \mathbf{H}_{2\mathsf{vdim}}(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$.*

If D is a enumeratively relevant divisor over $\overline{\mathcal{M}}_{g,n}(X,\beta)$, i.e. a divisor D of degree equal to the vdim $\overline{\mathcal{M}}_{g,n}(X,\beta)$, one can show that the virtual fundamental class behaves as the ordinary fundamental class, so we write

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathsf{vir}} = D \cap [\overline{\mathcal{M}}_{g,n}(X,\beta)],$$

for the degree of this divisor.

Chapter 3

Branched Coverings of Curves

In this chapter, we give a brief account of branched coverings and Hurwitz enumeration problem of branched coverings. Hurwitz enumeration problem is an old but still active research question due to its connections to several modern areas of mathematics and physics.

3.1 Overview remarks

Let X and Y be two smooth curves. Given a **covering map** $f : X \longrightarrow Y$, for each point of $q \in Y$, the number of preimages $f^{-1}(q)$ is the same for each point of Y. Branched coverings relax this requirement, by allowing finitely many points in Y (called branch points) to have less than expected number of distinct preimages. Thus, if we fix the generic number of preimages called the degree of f and genus of X, we hope to obtain only finite number of equivalence classes of such f up to isomorphism. This turns out to be the case, and the number of equivalence classes is called the **Hurwitz number** corresponding to the branched profile. Hurwitz numbers can be computed explicitly for non-complicated branched profiles due to the nice combinatorial interpretations they possess as first observed by A. Hurwitz in [Hur91, Hur02].

Hurwitz numbers connect geometry of curves to combinatorics of the symmetric groups. Riemann-Hurwitz formula tells us that the degree and the genus determine the degree of the **branch divisor** of f, so we only need to keep track of the degree and branch profiles. Indeed, we can encode the local degrees in permutations called monodromy representations whose cycle types correspond to the branch types. Furthermore, an isomorphisms of branched coverings in terms of monodromy representations corresponds to global conjugations. Thus, isomorphic coverings keep the branched profile fixed because conjugation is invariant on cycle types of permutations. In other words, we can construct a one-one correspondence between isomorphisms classes of branched coverings and branched profiles. In addition, Riemann existence theorem ensures that the set of branched profiles determines this isomorphisms class. Thus, we have a bijection between the isomorphisms classes of coverings and a class of elements of the symmetric group on #(degree of f) letters.

The reason why branched coverings have received renewed interests recently, is the existence of a rich geometric structure behind them. These has attracted attention of many mathematicians and physicists alike to the study of branch coverings (alias Hurwitz theory). It turns out that formulae for computing Hurwitz numbers arise in different branches of mathematics including algebraic geometry, combinatorics, representation of symmetric groups, topology of curves, moduli spaces of curves, tropical geometry, Gromov-Witten theory, matrix models and topological string theory.

3.2 Preliminary definitions

In this section, we review branched coverings of curves. Although, branched coverings are interesting more generally, we will later consider branched covering of the projective line $\mathbb{P}^1$ or, equivalently, meromorphic functions on curves. There is a number of books devoted to branched coverings, our favorite being [LZ04].

In what follows, a curve, always means a smooth complex projective algebraic curve.

Definition 3.2.1. *Let X and Y be curves. A surjective continuous map $f : X \longrightarrow Y$ is called a* **covering map** *(or simply a covering) of Y by X if for some discrete set S and for each point $y \in Y$, there exists a neighborhood $U \subset Y$ of y such that the preimage $f^{-1}(U) \subset X$ is homeomorphic to $U \times S$.*

The preimage $f^{-1}(y)$ is called the **fiber** of f over y and if f is a covering then each fiber has the same cardinality. Given an open set $U \subset Y$, we call connected components of the preimage $f^{-1}(U)$ **sheets** of the covering over U. If $d = |f^{-1}(y)|$ is finite, the covering map f is called $d-$ **sheeted**.

Definition 3.2.2. *Given curves X and Y, a* **branched covering** *is a continuous surjective map $f : X \longrightarrow Y$ such that for some finite set $B \subset Y$ the map*

$$f_0 : X \backslash f^{-1}(B) \longrightarrow Y \backslash B$$

is a covering. The set B is called **branch locus** *of f, the points y_j in B are called* **branch points** *of f.*

While counting different coverings we will consider their appropriate equivalent classes. Namely,

Definition 3.2.3. *Two (branched) coverings $f_1 : X_1 \longrightarrow Y$, and $f_2 : X_2 \longrightarrow Y$ are called* **equivalent** *if there exists an isomorphism $h : X_1 \longrightarrow X_2$ such that*

$$\begin{array}{ccc} X_1 & \xrightarrow{\ h\ } & X_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & Y & \end{array}$$

is a commutative diagram. In particular, we are not allowed to act on the base curve by its automorphisms.

Observe that every nonconstant holomorphic map $f : X \longrightarrow Y$ gives rise to a branched covering. Recall that the local behavior of a branched covering at a branch point is well understood. Namely, for appropriate local coordinates z and ω at $p \in X$ and $q = f(p) \in Y$ respectively, f is locally of the form $z \mapsto \omega = z^{\mu_i}$ for some integer $\mu_i \geq 1$. The integer $\mu_i > 1$ is called the **ramification index** of f at p. Additionally $p \in X$ is a ramification point if and only if $\mu_i > 1$.

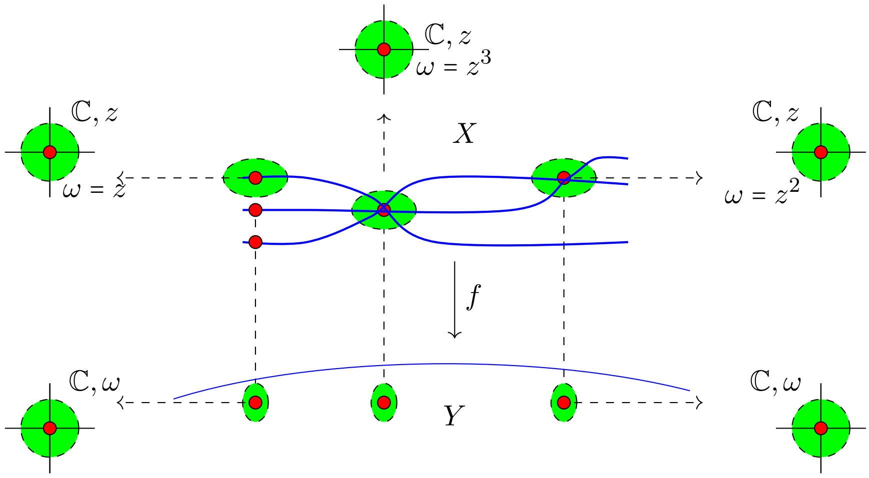


Figure 3.1: Local picture of a branched covering of degree 3.

The following statement about ramification indices is well known, see for example [Har77], Chapter II, Prop. 6.9.

Proposition 3.2.1. *Let $f : X \longrightarrow Y$ be a d–sheeted branched covering (or equivalently a holomorphic map of degree d). Then,*

$$\deg f = \sum_{p \in f^{-1}(q)} \mu_i = [\mathbb{C}(Y) : \mathbb{C}(X)] = \dim_{\mathbb{C}(Y)} \mathbb{C}(X),$$

where $[\mathbb{C}(Y) : \mathbb{C}(X)]$ is the degree of the field extension $\mathbb{C}(Y) \subset \mathbb{C}(X)$.

Let $f : X \longrightarrow Y$ be a branched covering of degree d. For a branch point $y \in Y$ of f, let $f^{-1}(y) := \{x_1, \ldots, x_n\}$ be its fiber with ramification indices $\{\mu_1, \ldots, \mu_n\}$ respectively. We have, $\sum_{i=1}^{n} \mu_i = d$. We can also assume after some reordering of $\{x_1, \ldots, x_n\}$ that $\mu_1 \geq \ldots \geq \mu_n$. The partition $(\mu_1, \ldots, \mu_n) \vdash d$ is called the **branch type** of f at a point y. Now, for each branch point we have an associated branch type. We define the **branch profile** of f with m branch points as a multipartition $\overline{\mu} = (\mu^1, \mu^2, \ldots, \mu^m)$ consisting of partitions $\mu^k \vdash d$, $k = 1, \ldots, m$, and we write $\overline{\mu}_d^m \vDash d$ for the branch profile of f of degree d.

Example 4.1

Any non-constant polynomial or rational function gives a branched covering from $\mathbb{P}^1$ to $\mathbb{P}^1$. For instance, $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ defined by the polynomial $f(z) = z^d$, $d > 0$ gives a d–sheeted branched covering of $\mathbb{P}^1$ which has two branch points 0 and ∞ of index d. Thus the branch profile of f is

$$\overline{\mu}_d^2 = \big((d),\ (d)\big) \vDash d.$$

Definition 3.2.4. *A branched covering $f : X \longrightarrow Y$ of degree d is called* **simple** *if for every branch point q its branch type is of the form $(2, 1, \ldots, 1) \vdash d$.*

In appropriate setting, simple branched coverings are generic among all branched coverings $f : X \longrightarrow Y$. In other words, any branched coverings between curves can be approximated by them; and any branch covering close to a simple branched covering is itself simple. Proofs can be found in [BE79, BE84].

3.3 Monodromy Representations

Let $f : X \longrightarrow Y$ be a branched covering of degree d and let B be the branch locus of f. Take a base point $y_0 \in Y \backslash B$. The preimage $f^{-1}(y_0)$ consists of d distinct points of X. Denote this set by R_d. Let $\gamma : [0,1] \longrightarrow Y \backslash B$ be a path with $\gamma(0) = \gamma(1) = y_0$, i.e. a loop in $Y \backslash B$ with a base point y_0. Since $f_0 : X \backslash f^{-1}(B) \longrightarrow Y \backslash B$ is a covering, then for any point $x \in R_d$, the path-lifting property guarantees the existence of a path $\gamma_x : [0,1] \longrightarrow X \backslash f^{-1}(B)$ with $\gamma_x(0) = x$. The end point $\gamma_x(1)$ belongs to R_d; we denote the lifted path by $\gamma^\sharp(x)$. Moreover, we have a bijection

$$\gamma^\sharp : R_d \longrightarrow R_d$$

satisfying the following:

1. If γ_1 and γ_2 are homotopic as loops in $Y \backslash B$ with base point y_0, then $\gamma_1^\sharp = \gamma_2^\sharp$.

2. If $\gamma_1 \cdot \gamma_2$ is the product of two loops γ_1 and γ_2, then $(\gamma_1 \cdot \gamma_2)^\sharp = \gamma_2^\sharp \circ \gamma_1^\sharp$.

In other words, we get a homomorphism

$$\rho : \pi_1(Y \backslash B, y_0) \longrightarrow S(R_d), \tag{3.1}$$

where $S(R_d)$ is the group of permutations of R_d with the product $fg = g \circ f$. This homomorphism is called the **monodromy representation** of $\pi_1(Y \backslash B, y_0)$ of the branched covering map f, and the image ρ is called the **monodromy group**. We usually fix the identification of $R_d = f^{-1}(y_0)$ with the standard set $\{1, \dots, d\}$. Then (3.1) gives the homomorphism $\rho : \pi_1(Y \backslash B, y_0) \longrightarrow S_d$, where S_d is the symmetric group on d letters. Since the homomorphism (3.1) depends on the identification of R_d with $\{1, \dots, d\}$, a monodromy representation is just determined up to inner-automorphisms of S_d when such an identification is not specified.

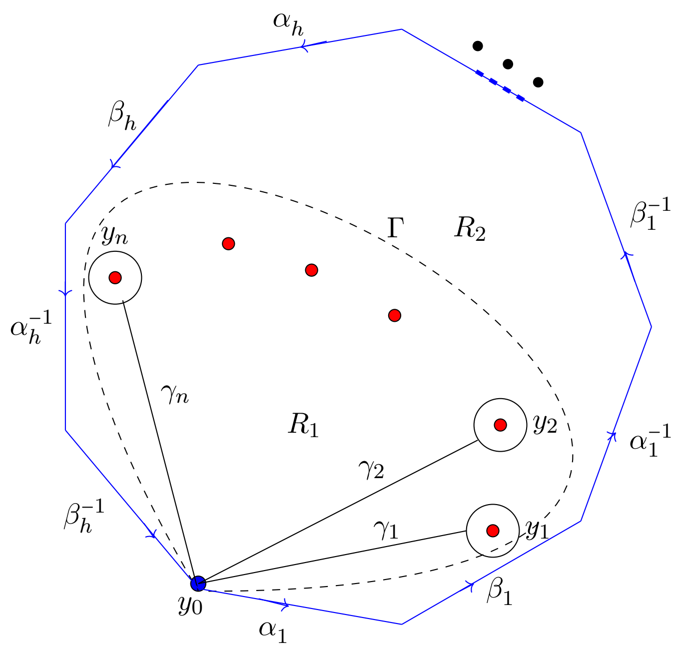


Figure 3.2: The standard representation of closed arcs based at a point y_0.

Given a curve Y of genus h, a branch locus $B = \{y_1, \ldots, y_n\} \subset Y$ and a base point $y_0 \in Y \backslash B$, we choose standard generators for the fundamental group $\pi_1(Y \backslash B, y_0)$ as follows. We fix a counterclockwise orientation for the compact topological surface Y and cut Y along the maximal family $\alpha_1, \beta_1, \ldots, \alpha_h, \beta_h$ of $2h$ simple closed arcs in Y such that $\alpha_i \cap \beta_i$ is a single point of the transverse intersection for each i, and $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \alpha_i \cap \beta_j = \varnothing$ if $i \neq j$ and do not contain any of y_i . Orient each of these arcs so that the orientation of α_i followed by that of β_i corresponds to the orientation of Y at $\alpha_i \cap \beta_i$, and so that the induced orientation on a path representing the commutator $[\alpha_i, \beta_i] = \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$ is the same as in our preceding convention. For $h \geq 1$ we obtain a standard $4h$-polygon with sides $\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \ldots, \alpha_h, \beta_h, \alpha_h^{-1}, \beta_h^{-1}$ with a counterclockwise orientation induced by the orientation of the compact topological surface.

Consider a simple closed arc Γ which begins at y_0 and such that $\Gamma \backslash y_0$ is contained in the interior of the standard $4h$-polygon and which passes through each of the branch points y_i indexed cyclically in the counterclockwise direction. Then Γ divides the standard $4h$-polygon into two regions which are to the left of Γ we call R_1 and R_2 which stay to the right of Γ with respect to the orientation on Y. We choose small nonintersecting disks D_i around y_i with distinct radii. Now we choose a simple arc c_1 which lies inside the region R_1 and which connects y_0 to the boundary ∂D_1. Next, we choose a second simple arc c_2 which lies inside the region R_1 connecting y_0 and boundary ∂D_2 with $c_1 \cap c_2 = y_0$ and lies to the left of c_1. Proceeding this way, we obtain an ordered n–tuple $(c_1, \cdots, c_n)$ of simple arcs whose only common point is y_0. Let γ_i be a closed path beginning at y_0 and traveling on c_i and then on ∂D_i then back to y_0 along c_i.

In particular, we obtain a $(2h+n)$-tuple of closed arcs we call the **standard system of arcs**. The associated homotopy classes yield a standard system of generators for the group $\pi_1(Y\backslash B, y_0)$ which is a quotient group of the free group generated by

$$\Big(\gamma_1, \ldots, \gamma_n, \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h\Big), \tag{3.2}$$

subject to the condition

$$\prod_{i=1}^{n} \gamma_i \prod_{k=1}^{h} [\alpha_k,\ \beta_k] = \mathbb{1}.$$

Given a branched covering $f : X \longrightarrow Y$, with branch locus B and the monodromy representation $\rho : \pi_1(Y\backslash B, y_0) \longrightarrow S_d$, the generators $\gamma_i \in \pi_1(Y\backslash B, y_0)$ correspond to some permutations $\sigma_i \in S_d$. We write $\sigma_i = \rho(\gamma_i)$, $a_k = \rho(\alpha_k)$ and $b_k = a_{h+k} = \rho(\beta_k)$.

Definition 3.3.1. *An ordered sequence $(\sigma_1, \ldots, \sigma_n; a_1, b_1, \ldots, a_h, b_h)$ of permutations in S_d with $\sigma_i \neq \mathbb{1}$ for all i satisfying the condition*

$$\sigma_1 \ldots \sigma_n \cdot [a_1, b_1] \cdot \ldots \cdot [a_h, b_h] = \mathbb{1},$$

is called a **Hurwitz system** *for f corresponding to the standard set of generators (3.2).*

The Riemann-Hurwitz formula provides a necessarily condition but not sufficient for the existence of branched coverings satisfying the branching data. It should be noted that, we have many cases of the data satisfying the Riemann-Hurwitz formula but no corresponding branched coverings. See [PP06, PP08] for details and the references therein.

By connectedness property imposed on X and Y the monodromy group generated by a Hurwitz system is a transitive subgroup of S_d. This leads to the following existence and classification results of branched coverings allowing one to reduce many questions about branched coverings to combinatorial or purely group-theoretic problems.

Theorem 3.3.1 (Existence Theorem)**.** *Let X and Y be curves. Given a d-sheeted covering $f : X \longrightarrow Y$ with branch locus $B = \{y_1, \ldots, y_n\} \subset Y$ there is a homomorphism*

$$\rho : \pi_1(Y\backslash B, *) \longrightarrow S_d,$$

determined up to an inner automorphism (i.e. two homomorphism ρ_1, ρ_2 are equivalent if there exists $\sigma \in S_d$ such that $\rho_2(g) = \sigma\rho_1(g)\sigma^{-1}$ for all $g \in \pi_1(Y\backslash B, *)$).

Conversely, given a monodromy representation

$$\rho : \pi_1(Y\backslash B, *) \longrightarrow S_d,$$

there is a unique branched covering $X \longrightarrow Y$ with branched set contained in B.

Theorem 3.3.2 (Classification theorem)**.** *Two branched coverings of degree d over a given curve Y(equipped with a fixed standard system of arcs) are equivalent if and only if they have Hurwitz systems which are conjugate by an element of S_d.*

The above theorems on existence and classification of branched coverings are two fundamental results proven by A. Hurwitz, see [BE79, Eze78] for modern proofs. The proofs of these results where originally sketched in [Hur91] by using cut and join techniques.

Remark 3.3.1. *Motivated by the path multiplication in $\pi_1(Y\backslash B, *)$, we are adopting the convention that permutations are multiplied from left to right, as opposed to the composition product.*

Obviously, a branched covering $f : X \longrightarrow Y$ is simple if and only if all the permutations $\sigma_1, \dots, \sigma_n$ corresponding to branch points in a Hurwitz system are transpositions. Further, for $Y = \mathbb{P}^1$ we have the uniqueness theorem of Lüroth and Clebsch [Eze78] for the normal form of the Hurwitz system.

Theorem 3.3.3 (Lüroth and Clebsch)**.** *For any simple branched covering $f : X \longrightarrow \mathbb{P}^1$ of degree d, there exists a standard system of arcs such that its Hurwitz system is of the form*

$$\Big(\underbrace{(1,2),(1,2),\dots,(1,2)}_{2g+2},\underbrace{(2,3),(2,3)}_{2},\underbrace{(3,4),(3,4)}_{2},\dots,\underbrace{(d-1,d),(d-1,d)}_{2}\Big). \tag{3.3}$$

Sketch of Proof. The idea of the proof is to consider the switching of the w simple branch points two at a time say y_i and y_{i+1} of the covering while others remain fixed and observe that successful branch points preserve the product $\sigma_i\sigma_{i+1}$ in the monodromy group. This together with a conjugation of the entire sequence by an element of S_d, gives the equivalence map:

$$(\sigma_1, \dots, \sigma_w) \longrightarrow (\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \sigma_{i+1}^{-1}\sigma_i\sigma_{i+1}, \sigma_{i+2}, \dots, \sigma_w).$$

□

3.4 Hurwitz spaces and Hurwitz numbers

In this section, we recall geometric and combinatorial definitions and some basic facts about Hurwitz spaces and numbers. Following §21 of [ACG11], we briefly describe the geometry of Hurwitz spaces in special cases.

3.4.1 Hurwitz spaces

Hurwitz spaces are geometric spaces which parametrize the equivalence classes of branched coverings with some fixed topological and combinatorial data. More specifically, they parametrize branched coverings $f : X \longrightarrow Y$ of degree d over a fixed curve Y with a fixed branch profile $\overline{\mu}_d^w \vDash d$. These spaces were initially introduced by Clebsch [Cle72] and Hurwitz [Hur91] as a tool to study moduli spaces of curves.

Fix a curve Y of genus h, positive integers d, w and a branch profile $\overline{\mu}_d^w \vDash d$. The set of all equivalence classes of branched coverings of degree d with branch profile $\overline{\mu}_d^w \vDash d$ can be given a structure of a moduli space [HM98, Ful69, GHS02], which we denote by $\mathscr{H}_{g,d}^h(\overline{\mu}_d^w)$ and call the **Hurwitz space** associated to $\overline{\mu}_d^w$. Hurwitz spaces are interesting to study in general albeit their geometry is very complicated. On the other hand, we know that any branched covering between curves can be approximated by simple branched coverings [BE79, BE84] whose structure is easier to understand. Thus, an important special class of Hurwitz spaces are the so-called **small Hurwitz spaces** which are moduli spaces of simple branched coverings

$$f : X \longrightarrow Y,$$

of degree d, where Y is a fixed curve of genus h and X a curve of genus g.

Note that the symmetric group S_n acts naturally on the cartesian product $Y^n = \underbrace{Y \times \ldots \times Y}_{n}$ by permuting the factors. The (smooth) quotient variety $\operatorname{Sym}^n Y = Y^n/S_n$ is called the n^{th}-**symmetric product** of Y. Identifying points of $\operatorname{Sym}^n Y$ with the sets of unordered n-tuples of points of Y with possible repetitions, we define the **discriminant locus** $\Delta_n \subset \operatorname{Sym}^n Y$ as the set of all n–tuples which contain less than n distinct points. Fix non-negative integers d, g, h, w related by $w = 2g - 2 - d(2h - 2)$ and choose $B \in \operatorname{Sym}^w Y \backslash \Delta_w$ where $w = |B|$. Now, we define the spaces:

$$\mathscr{H}^h_{g,d,B} := \left\{ f : X \longrightarrow Y \;\middle|\; \begin{array}{c} X \text{ has genus } g \text{ and } f \text{ is a simple branched} \\ \text{covering of degree } d \text{ whose branch locus is } B \end{array} \right\} \Big/ \sim \tag{3.4}$$

and

$$\mathscr{H}^h_{g,d} := \left\{ f : X \longrightarrow Y \;\middle|\; \begin{array}{c} X \text{ has genus } g \text{ and } f \text{ is a simple branched} \\ \text{covering of degree } d \text{ with } w \text{ branch points} \end{array} \right\} \Big/ \sim \quad , \tag{3.5}$$

where $\sim$ denotes the equivalence classes of branched coverings of $f : X \longrightarrow Y$.

Let $\Phi : \mathscr{H}^h_{g,d} \longrightarrow \operatorname{Sym}^w Y \backslash \Delta_w$ be the map assigning to each branched covering its branch locus. If $B \in \operatorname{Sym}^w Y \backslash \Delta_w$, then $\Phi^{-1}(B) = \mathscr{H}^h_{g,d,B}$. Moreover, we can introduce a topology on $\mathscr{H}^h_{g,d}$ in such a way that Φ becomes a topological covering map. In this way, the space $\mathscr{H}^h_{g,d}$ has the structure of a complex variety induced from that on $\operatorname{Sym}^w(Y)$ (and possibly disconnected, if $d > 2$). We have a natural morphism

$$\begin{aligned} \Phi : \mathscr{H}^h_{g,d} &\longrightarrow \operatorname{Sym}^w Y \backslash \Delta_w \\ (f : X \longrightarrow Y) &\longmapsto (Y; \text{branch locus of } f), \end{aligned} \tag{3.6}$$

which we call the **branching morphism**. The map Φ is finite and in fact ètale since $\mathscr{H}^h_{g,d}$ possesses the structure of a variety. This construction was first described in [Hur91], (see also Fulton in [Ful69] for the modern interpretation of Hurwitz's approach). The construction naturally allows an extension to branched coverings which are simple in all but one special point with branch type $\mu = (\mu_1, \ldots, \mu_n) \vdash d$ if the monodromy group is the full symmetric group S_d, see [GHS02].

Now, the existence and classification theorems allow us to reduce questions about the degree of the branch morphism to a combinatorial problem. That is fixing the standard system of closed arcs (3.2) above, the points in the fibers of Φ over B can be identified with the equivalence classes $[\sigma_1, \ldots, \sigma_n; a_1, b_1, \ldots, a_g, b_g]$ of Hurwitz systems modulo inner automorphisms of S_d. Two elements

$$(\sigma_1, \ldots, \sigma_n, a_1, b_1, \ldots, a_g, b_g) \text{ and } (\sigma'_1, \ldots, \sigma'_n, a'_1, b'_1, \ldots, a'_g, b'_g),$$

are considered **equivalent** if there exists an $\pi \in S_d$ such that for all i, k we have $\sigma'_i = \pi^{-1}\sigma_i\pi$, $a'_k = \pi^{-1}a_k\pi$ and also $b'_k = \pi^{-1}b_k\pi$.

An interesting **subclass of small Hurwitz spaces** is $\mathcal{H}_{g,d} := \mathscr{H}^0_{g,d}$, i.e. by the moduli spaces of simple branched covering of projective line. If we choose an affine coordinate in $\mathbb{P}^1$ we can identify $\mathcal{H}_{g,d}$ with the space of meromorphic functions on curves of genus g with $w = 2g+2d-2$ simple branch points. Since branch points can be used as local coordinates on $\mathcal{H}_{g,d}$ this implies that $\dim \mathcal{H}_{g,d} = w$. Using calculations of Lüroth and Clebsch [Cle72], A. Hurwitz in [Hur91] has proved that $\mathcal{H}_{g,d}$ is a connected variety see also §21.11 of [ACG11]. Fixing d, g as above and choosing B in the unordered moduli space $\mathrm{Sym}^w \mathbb{P}^1 \backslash \Delta_w$, we get the ètale branching morphism

$$\begin{aligned} \Phi_0 : \mathcal{H}_{g,d} &\longrightarrow \mathrm{Sym}^w \mathbb{P}^1 \backslash \Delta_w \\ (f : X \longrightarrow \mathbb{P}^1) &\longrightarrow (\mathbb{P}^1; \text{branch locus of } f). \end{aligned} \tag{3.7}$$

The degree of the branching morphism Φ_0 (which is a special case of the single Hurwitz number, see Definition 3.4.1 below) counts the number of non-equivalent simple branched coverings of $\mathbb{P}^1$ with a branch locus B. Recall, that to construct a branched covering of degree d of $\mathbb{P}^1$ with branch locus B, it suffices to specify the monodromy of the d sheets of $X \longrightarrow \mathbb{P}^1$ around each of the branch points (we assume that we fix the system of paths). In other words, we have to specify the Hurwitz system

$$\widetilde{\Phi}_0^{-1}(B) = \left\{ (\sigma_1, \dots, \sigma_w) \in (S_d)^w \;\middle|\; \begin{array}{c} \sigma_i \text{ are transpositions such that} \\ \prod \sigma_i = 1 \text{ and } \langle \sigma_1, \dots, \sigma_w \rangle = S_d \end{array} \right\} \Big/ \sim,$$

where $\sim$ represents all global conjugations. In this form the problem was for the first time formulated by A. Hurwitz. In other words, we need to count sequences of w transpositions which generate a transitive subgroup of S_d whose product equals identity.

For instance, it is immediate to enumerate all degree 3 simple branched coverings for all $g \geq 0$. All we need, is to count sequences of $2g+4$ transpositions with the above properties. Notice that we are free to choose $2g+3$ elements of the sequence as the last transposition is determined by the requirement that the product must be identity. Observe that the product of $2g+3$ transpositions has the same parity as one transposition in S_3. Also, to avoid disconnected coverings we have to avoid choosing the same transpositions $2g+3$ times. Thus, we immediately obtain that the number of simple branched coverings of degree 3 is $\frac{3^{2g+3}-3}{6}$ for all $g \geq 0$ as given on page 17 of [Hur91].

Compactification of Hurwitz spaces

It is clear that the small Hurwitz space $\mathscr{H}^h_{g,d}$ is not compact. It is much easier to calculate the degree of a map if we work with compact spaces. There are different natural ways to compactify Hurwitz spaces. Among them, we can mention the Harris-Mumford compactification [HM82a, HM98] which uses the concept of moduli spaces of admissible coverings. The fundamental idea here, is to forbid branch points to collide; instead as two or more branch points tend to collide, a new component of Y sprouts from the point of collision and these points distribute on it. This way the base curve degenerates to a nodal curve and the covering degenerates into a nodal covering.

Another important compactification is constructed in the proof of ELSV-formula by T. Ekedahl et al. see a rather new notes in [Du12]. The compactification uses an analogue for $\mathcal{H}_{g,d}$ as the space of meromorphic function on X with exactly $d \geq 1$ numbered simple poles and the main feature here, is that the Hurwitz space $\mathcal{H}_{g,d}$ is closely related to the moduli space $\mathcal{M}_{g,d}$ of curves. Namely, for $d \geq 3$ we can associate to a meromorphic function $f : X \longrightarrow \mathbb{P}^1$ the curve $(X : p_1 \dots p_d) \in \mathcal{M}_{g,d}$ if we assume that f is not branched at infinity. Then we have a forgetful morphism

$$\pi : \mathcal{H}_{g,d} \longrightarrow \mathcal{M}_{g,d}, \tag{3.8}$$

determined by the labeling of the poles. The desired compactification $\overline{\mathcal{H}}_{g,d}$ is determined by the projection $\overline{\pi} : \overline{\mathcal{H}}_{g,d} \longrightarrow \overline{\mathcal{M}}_{g,d}$ and the geometry of the fiber. In particular, we define the compactification of $\mathcal{H}_{g,d}$ as a bundle over $\overline{\mathcal{M}}_{g,d}$ whose sections are **stable meromorphic function** on X, where "stable" means that a meromorphic function $f : X \longrightarrow \mathbb{P}^1$ defined on a nodal curve X satisfies the following conditions:

i. f does not have poles at nodal points;

ii. f has a finite group of automorphisms.

3.4.2 Hurwitz Numbers

The number of non-equivalent branched coverings with a given set of branch points and branched profile is called the **Hurwitz number**. The question of determining the Hurwitz number for a given branch profile is called the **Hurwitz enumeration problem**. Hurwitz numbers have both geometric and algebraic interpretations. Geometrically, Hurwitz numbers count the number of holomorphic maps $f : X \longrightarrow Y$ between curves with a fixed branch profile. Using monodromy presentation of branched coverings, we get an equivalent combinatorial descriptions for Hurwitz numbers as counting certain factorizations of permutations.

Definition 3.4.1. *Fix positive integers d, w and a branch profile $\overline{\mu}_d^w \vDash d$. Let $\mathscr{H}_{g,d}^h(\overline{\mu}_d^w)$ be the corresponding Hurwitz space. The degree of the branching morphism*

$$\begin{aligned} \Phi_h : \mathscr{H}_{g,d}^h &\longrightarrow \operatorname{Sym}^w Y \backslash \Delta_w \\ (f : X \longrightarrow Y) &\longmapsto (Y; \textit{branch locus of } f) \end{aligned} \tag{3.9}$$

divided by $|\operatorname{Aut}(\overline{\mu}_d^w)|$ is called the **Hurwitz number** *associated to the profile $\overline{\mu}_d^w \vDash d$.*

We can reinterpret the Hurwitz number in terms of monodromy representations. Fix d, w positive integers and a branch profile $\overline{\mu}_d^w := \left(\mu^1, \dots, \mu^w\right) \vDash d$ for the w branch points. Then a $w-$ tuple $(\sigma_1, \dots, \sigma_w)$ is called **Hurwitz factorization** of type $\overline{\mu}_d^w$ if it satisfies the following

i. For every i the permutation $\sigma_i \in S_d$ has cycle type μ^i,

ii. the product $\sigma_1 \cdots \sigma_w = \mathbf{1}$ in S_d,

iii. $\sigma_1, \cdots, \sigma_w$ generate a transitive subgroup of S_d.

Definition 3.4.2 (Hurwitz number)**.** *Hurwitz number associated to the branch profile $\overline{\mu}_d^w \vDash d$ is the number of Hurwitz factorizations of type $\overline{\mu}_d^w$ divided by $d!$.*

In general, explicit answers to the **Hurwitz enumeration problem** are usually difficult to obtain. One important case when this problem has a rather explicit answer, is when at most one branch point has an arbitrary branch type while all the others are simple. In case of $Y = \mathbb{P}^1$, we usually suppose that the degenerate branch point is at $\infty \in \mathbb{P}^1$. Thus, we are in the situation where all the branch points in $\mathbb{C}$ correspond to transpositions while the permutation at infinity can be described by some partition $\mu = (\mu_1, \ldots, \mu_n) \vdash d$. This leads to the following class of Hurwitz numbers.

Single Hurwitz Numbers

Definition 3.4.3. *The number of equivalence classes of the branched coverings in the above form is called the* **single Hurwitz Number** *and is denoted by* $\mathfrak{h}_{g,\mu}$.

Importantly, to every branched covering we can associate its monodromy data and we obtain equivalent definitions of single Hurwitz numbers in terms of sequences of permutations.

Group theoretic definition

Fix $\sigma \in S_d$, a sequence $(\tau_1, \tau_2, \ldots, \tau_n)$ such that the product $\tau_1\tau_2\ldots\tau_n = \sigma$ is called a **transposition factorization** of σ of length n. Obviously, such a factorization is not unique. However, the number of transpositions in the factorization depends on the cycle type of the permutation σ rather than the permutation itself. Namely, all such transpositions have the same parity as that of σ and there is a minimal such n for which the factorization exists.

Let $\mu = (\mu_1, \ldots, \mu_n) \vdash d$ for $d \geq 1$. Consider an ordered sequence of permutations $(\tau_1, \ldots, \tau_w, \sigma) \in (S_d)^{w+1}$ such that:

i. $\left(\tau_1, \ldots, \tau_w\right)$ are transpositions which generate S_d,

ii. the product $\tau_1 \cdots \tau_w = \sigma$ in S_d whose cycle type is μ.

Definition 3.4.4. *The single Hurwitz number* $\mathfrak{h}_{g,\mu}$ *equals the number of* $w-$*tuples of transpositions as above divided by* $|\mathrm{Aut}(\mu)|$ *where* $\mathrm{Aut}(\mu)$ *denotes the automorphism group of partition that permutes equal parts of* $\mu \vdash d$.

For instance, the number for non-isomorphic branched coverings of degree 3 over $\mathbb{P}^1$ with one complicated branch point can easily be calculated.

Example 4.2

Indeed, we establish that the single Hurwitz number $\mathfrak{h}_{g,(3)\vdash 3} = 3^{2g}$ as follows. Notice that for complicated branch point we can choose freely a 3-cycle in S_3 giving a monodromy of the triple point. The 3-cycle guarantee that we generate S_3. Then we are free to choose cycle for the next $2g+1$ simple branch points, the last is uniquely determined by the fact that the multiplication is identity. So we get $2 \cdot 3^{2g+1}$ elements of S_3. We divide by $3!$ to account for relabelling of the sheets of the branched coverings.

Chapter 4

Formulae for Calculating Hurwitz Numbers

In this chapter, we collect various known formulae in calculating Hurwitz numbers. In other words, formulae for determining the number of connected branched coverings for fixed branched profile over a given connected smooth curve. The chapter finishes with a discussion of the Hurwitz monodromy groups.

4.1 The Hurwitz Formula

In several specific cases A. Hurwitz calculated $\mathfrak{h}_{g,\mu}$ using purely combinatorial methods in 1891 and in terms of irreducible characters of S_n in 1902. In [Hur91] he sketched his solution by using the Riemann existence theorem and also he observed that the calculation $\mathfrak{h}_{g,\mu}$ is a purely group-theoretic problem, but its solution is complicated for arbitrary g and d. On page 17 of [Hur91], Hurwitz found answers for calculating the degree of the map (3.7) for small $d \leq 6$ and any $g \geq 0$. Namely,

$$\begin{aligned}
\mathfrak{h}_{g,(1^2)} &= 1, \\
\mathfrak{h}_{g,(1^3)} &= \frac{1}{3!}(3^{2g+3} - 3), \\
\mathfrak{h}_{g,(1^4)} &= \frac{1}{4!}(2^{2g+4} - 4)(3^{2g+5} - 3), \\
\mathfrak{h}_{g,(1^5)} &= \frac{10^{2g+8}}{7200} - \frac{6^{2g+8}}{288} + \frac{5^{2g+8}}{450} - \frac{4^{2g+8}}{72} + \frac{3^{2g+8}}{18} + \frac{2^{2g+8}}{12} - \frac{5}{9}, \\
\mathfrak{h}_{g,(1^6)} &= \frac{15^{2g+10}}{2\cdot(360)^2} - \frac{10^{2g+10}}{7200} + \frac{9^{2g+10}}{2\cdot(72)^2} - \frac{7^{2g+10}}{2\cdot(24)^2} + \frac{6^{2g+10}}{2\cdot(36)^2} - \frac{5^{2g+10}}{360} + \\
&\quad + \frac{4^{2g+10}}{36} - \frac{19}{324}\cdot 3^{2g+10} - \frac{19}{144}\cdot 2^{2g+10} + \frac{727}{1152}.
\end{aligned} \tag{4.1}$$

Minimal Transposition Factorisation

For genus $g = 0$, the single Hurwitz number $\mathfrak{h}_{0,\mu}$ is equivalent to counting factorisations of a permutation $\sigma \in S_d$ of cycle type $\mu \vdash d$ into a product of transpositions of minimal length divided by $d!$, a result known and published by Hurwitz.

Definition 4.1.1. *Let $\sigma \in S_d$ be a fixed permutation of length m. The sequence $(\tau_1, \dots, \tau_n)$ is called a minimal transitive factorisation of σ into transpositions if the following 3 conditions are satisfied:*

i. **Product cycle type condition:** $\tau_1 \dots \tau_n = \sigma$,

ii. **Minimality condition:** $n := m + d - 2$,

iii. **Transitivity condition:** *The graph G_σ is connected, where G_μ is the graph corresponding to factorisation σ into a product of n transpositions.*

Note that, one needs at least $d-1$ transpositions to build a cycle of length d. Then $n \geq d-1$.

Example 4.1

a. If $\mu = (2) \vdash 2$ and $m = 1$, the only transposition is $(12) = (21)$. Therefore[a] $\mathfrak{h}_{0,\mu} = \frac{1}{2} \cdot 1 = \frac{1}{2}$.

b. If $\mu = (3) \vdash 3$, $m = 2$ there exist 3 transposition factorizations of the three-cycle $(123) = (12)(13) := (23)(21) := (31)(32)$ and we have $3 \cdot 2$ three-cycles in S_3 corresponding to connected trees. Thus $\mathfrak{h}_{0,(3)} = \frac{1}{6}(3 \cdot 2) = 1$.

c. If $\mu = (2,1) \vdash 3$ and $m = 3$ we have 3^3 triples of transpositions but 3 of the triples consist of coinciding transpositions and thus the corresponding graph G_μ is not connected. This implies that the single Hurwitz number $\mathfrak{h}_{0,(2,1)} = \frac{1}{6}(3^3 - 3) = 4$.

[a] *This example also shows that Hurwitz numbers can be rational and not always a positive integer.*

Motivated by enumeration of branched covering of a sphere by a sphere, i.e. genus zero branched coverings of $\mathbb{P}^1$, A. Hurwitz [Hur91] page 21 conjectured and sketched the recurrence proof of the following formula. This conjecture was settled completely only a hundred years later.

Theorem 4.1.1 (Hurwitz Formula). *Let $\sigma \in S_d$ be a permutation of cycle type $\mu = (\mu_1, \mu_2, \dots, \mu_m) \vdash d$. The number of distinct minimal transitive factorizations of σ into transpositions equals*

$$(d + m - 2)! \prod_{i=1}^{m} \frac{\mu_i^{\mu_i}}{\mu_i!} d^{m-3}. \tag{4.2}$$

Many elegant and deep proofs of this formula have appeared in different branches of mathematics. For instance, Strehl [Str96] has reconstructed the proof of Hurwitz using Abelian identities. This proof has been generalized by Golden-Jackson by using of generating functions and partial differential equations combinatorial conditions see [GJ97]. Bousquet-Mélon and Schaffer, [BS00] provided a bijective proof of Theorem 4.1.1 by inclusion-exclusion principle. Geometric proofs include that of Lando-Zvonkine which calculates the degree of LL- (Lyashko-Looijenga) mapping [LZ99] and the ELSV-formula which involves the geometry and cohomology of moduli spaces [ELSV99].

The Hurwitz formula in special cases has been independently rediscovered by many authors. First, for $m = 1$, i.e. $\mu = (d)$ we need to count minimal factorization of a d–cycle. Notice that the product of transpositions $\tau_1 \tau_2 \dots \tau_n$ is a d–cycle if and only if the associated graph of this factorization is a tree. For example, consider the graph corresponding to the transposition factorization

$$\overbrace{(5,9)}^{\tau_1}\ \overbrace{(2,3)}^{\tau_2}\ \overbrace{(6,9)}^{\tau_3}\ \overbrace{(1,5)}^{\tau_4}\ \overbrace{(7,9)}^{\tau_5}\ \overbrace{(8,9)}^{\tau_6}\ \overbrace{(2,4)}^{\tau_7}\ \overbrace{(2,5)}^{\tau_8} = \overbrace{(1,2,3,4,5,6,7,8,9)}^{\sigma\in S_9}.$$

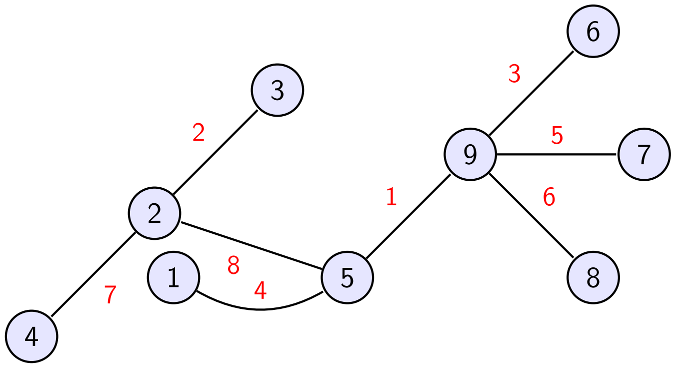


Figure 4.1: The decorated graph of a factorisation 9-cycle $\sigma \in S_9$

The core behind the derivation of this special case is the fact that multiplication of permutation by a transposition can be easily understood; it either **cuts** or **joins** cycles of the permutation. Namely, if $\sigma \in S_d$ has m cycles then the product $(a,b)\cdot\sigma$ has either

1. **Cut**: $m-1$ cycles if a and b are in different cycles of σ.
2. **Join**: $m+1$ cycles if a and b are in same cycle of σ.

Example 4.2

The multiplication of permutation $(1,2,3,4,5) \in S_5$ on the left by $(1,4)$ gives $(1,5)(2,3,4)$. In other words, cuts it into two cycles. On the other hand, multiplication of the permutation $(1,5)(2,3,4)$ on the left by $(1,4)$ joins the two cycles together.

Now, since for $\mu = (d)$ the graph G_μ is a tree, assuming bijective results [Mos89] the corresponding Hurwitz number follows immediately from **Cayley's formula** of 1860 for enumeration of trees. (Observe, the Cayley formula in the language of transpositions, is attributed to the Hungarian mathematician Dénes [Dén59]).

Theorem 4.1.2 (Dénes)**.** *There exist d^{d-2} transposition factorization of an d-cycle into $d-1$ distinct transpositions.*

In the case $m = 2$, V.I. Arnol'd [Arn96] found the corresponding Hurwitz number by using the notion of complex trigonometric polynomials.

Theorem 4.1.3 (Arnol'd)**.** *For a partition $\mu = (\mu_1, \mu_2) \vdash d$ the number of distinct minimal transitive transposition factorizations of σ whose cycle type equals μ is*

$$\mu_1^{\mu_1}\,\mu_2^{\mu_2}\,\frac{(\mu_1+\mu_2-1)!}{(\mu_1-1)!\,(\mu_2-1)!}\,. \tag{4.3}$$

Still another case was settled not that long ago by two physicists M. Crescimanno and W. Taylor.

Theorem 4.1.4 (Crescimano-Taylor)**.** *If $m = d$ means $\mu = (1^d)$ i.e. the factorization of the identity, then the number of distinct minimal transitive factorizations into transpositions*

$$(2d-2)!\, d^{d-3} \tag{4.4}$$

was discovered in [CT95]. (Crescimano-Taylor apparently asked the combinatorialist Richard Stanley who consulted Goulden-Jackson about the result).

Finally, Goulden-Jackson also independently [GJ97, GJ99a] discovered and proved the Hurwitz formula in its complete generality.

4.2 The ELSV Formula

In this section, we formulate the remarkable ELSV formula [ELSV01] following a result of **E**kedahl-**L**ando-**S**hapiro-**V**ainshtein. It provides a strong connection between geometry of moduli spaces and the Hurwitz numbers. In practice it is very difficult to use but it remains one of the most striking results related to Hurwitz enumeration problem. Single Hurwitz numbers turn out to be closely related to the intersection theory on the moduli space of stable curves.

Recall that the Hurwitz number $\mathfrak{h}_{g,\mu}$ is the number of branched coverings of degree d from smooth curves of genus g to $\mathbb{P}^1$ with one branch point (usually taken to be $\infty \in \mathbb{P}^1$) of branched type $\mu \vdash d$ and $w = d + \ell(\mu) + 2g - 2$ other simple branch points.

Theorem 4.2.1 (The ELSV formula)**.** *Suppose that g, n are integers ($g \geq 0, n \geq 1$) such that $2g - 2 + n > 0$, where $n := \ell(\mu)$. Let $\mu = (\mu_1, \dots, \mu_n) \vdash d$ and $\mathrm{Aut}(\mu)$ denote the automorphism group of the partition μ. Then,*

$$\mathfrak{h}_{g,\mu} = \frac{w!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \ldots + (-1)^g \lambda_g}{(1-\mu_1\psi_1)\ldots(1-\mu_n\psi_n)} \tag{4.5}$$

where $\psi_i = c_1(\mathscr{L}_i) \in \mathbf{H}^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is the first Chern class of the contagent line bundle $\mathscr{L}_i \longrightarrow \overline{\mathcal{M}}_{g,n}$ and $\lambda_j = c_j(\mathbb{E}) \in \mathbf{H}^{2j}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is the jth Chern class of the Hodge bundle $\mathbb{E} \longrightarrow \overline{\mathcal{M}}_{g,n}$

$$\frac{1}{1-\mu_i\psi_i} = 1 + \mu_1\psi_1 + \ldots + \ldots \mu_i^i \psi_i^i + \ldots$$

(Observe that the above expansion terminates because $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is nilpotent.)

Notice that the ELSV formula is a polynomial in the variables $\mu_1, \dots, \mu_n$. This fact is stated in the Golden-Jackson polynomiality conjecture [GJ99b] which this formula settles.

Remark 4.2.1. *The ELSV formula is not applicable to coverings of genus 0 with 1 and 2 marked points since the stability condition $2g - 2 + n > 0$ is violated. However, the ELSV formula remains true for these two cases as well*

$$\int_{\overline{\mathcal{M}}_{0,1}} \frac{1}{(1-\mu_1\psi_1)} = \frac{1}{\mu_1^2}, \quad \text{and} \quad \int_{\overline{\mathcal{M}}_{0,2}} \frac{1}{(1-\mu_1\psi_1)(1-\mu_2\psi_2)} = \frac{1}{\mu_1+\mu_2}. \tag{4.6}$$

Apart from the easy combinatorial factor, the ELSV formula involves the integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{m_1} \dots \psi^{m_n} \lambda_1^{k_1} \dots \lambda_g^{k_g}, \tag{4.7}$$

called the **Hodge integrals** which can be reduced to other integrals only involving the ψ-classes. The latter integral are called **descendant integrals** [FP00]. The explicit evaluation of these integrals or computation of the intersection numbers is a difficult task. On the other hand, we can see that using the ELSV formula (4.5) makes it possible to calculate the intersection numbers on $\overline{\mathcal{M}}_{g,n}$ once the single Hurwitz numbers are known.

Applications of the ELSV formula

Although, the ELSV formula (4.5) is hard to use, there is a couple of very well-known cases. These cases are related to Witten's conjecture [Wit91] now known as Kontsevich's theorem [Kon92] which gives a recursive relation for Hodge integrals involving ψ-classes only. In return some of Hodge integrals can be evaluated recursively through string equation and the KdV hierarchy. In particular, we can recover the following well-known cases.

Theorem 4.2.2 (**Hurwitz Formula**[Hur91])**.** *The single Hurwitz Number formula $h_{0,\mu}$ is given by*

$$\mathfrak{h}_{0,\mu} = \frac{(n+d-2)!}{|Aut(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} d^{n-3} \tag{4.8}$$

where $n+d-2$ is the number of simple branch points , cf. Theorem 4.1.1.

Proof. By the ELSV formula and string equation,

$$\begin{aligned}
\mathfrak{h}_{0,\mu} &= \frac{(d+n-2)!}{|Aut(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\mathcal{M}_{0,n}} \frac{1}{(1-\mu_1\psi_1)\dots(1-\mu_n\psi_n)} \\
&= \frac{(d+n-2)!}{|Aut(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \sum_{m_1+\ldots+m_n=n-3} \langle \tau_{m_1} \dots \tau_{m_n} \rangle_0 \cdot \mu_1^{m_1} \dots \mu_n^{m_n} \quad \text{by equation (2.11)} \\
&= \frac{(d+n-2)!}{|Aut(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \sum_{m_1+\ldots+m_n=n-3} \frac{(n-3)!}{m_1!\dots m_n!} \cdot \mu_1^{m_1} \dots \mu_n^{m_n} \quad \text{by Proposition 2.3.1.} \\
&= \frac{(d+n-2)!}{|Aut(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} d^{n-3}.
\end{aligned}$$

□

Moreover, we can recover the classical formulas of Denes, Arnol'd and Crescimano-Taylor, cf. (4.1.2), (4.3) and (4.4) respectively:

Corollary 4.2.1 (Polynomial/Hurwitz's case)**.** *If $\mu = (d)$ then*

$$\mathfrak{h}_{0,\mu} = (d-1)! \frac{d^d}{d!} d^{-2} = d^{d-3}.$$

Corollary 4.2.2 (Rational/Dénes' case)**.** *If* $g = 0$ *and* $\mu = (1^d)$ *then*

$$\mathfrak{h}_{0,\mu} = \frac{(2d-2)!}{d!} d^{d-3}.$$

Corollary 4.2.3 (Arnol'd's case)**.** *If* $g = 0$ *and* $\mu = (\mu_1, \mu_2) \vdash d$ *then*

$$\mathfrak{h}_{0,\mu_2,\mu_2} = \frac{\mu_1^{\mu_1}}{\mu_1!} \cdot \frac{\mu_2^{\mu_2}}{\mu_2!} \cdot (\mu_1 + \mu_2 - 1)!\,.$$

Another well-known case with an explicit generating formula occurs in the computation of genus 1 Hurwitz numbers $\mathfrak{h}_{1,\mu}$. The details can be found in [GJV00]. There has been some progress in calculation of more generalized Hurwitz numbers. For example, there is the ELSV formula for double Hurwitz numbers $\mathfrak{h}_{g,\alpha,\beta}$ which are the Hurwitz numbers for meromorphic functions with two complicated branch points, see [KS03, GJV05].

4.3 The Mednykh Formula

In this section, we consider the general form of Hurwitz enumeration problem. In [Med84], using the original idea of Hurwitz [Hur02] of interpreting branched coverings as irreducible representations of symmetric groups, A. Mednykh has obtained a formula for counting the number of non-equivalent branched coverings with any prescribed branched types and branching data. This result in principle solves the general Hurwitz enumeration problem completely, but the largely untractable sums of irreducible characters of the symmetric group in it results in the fact that this remarkable formula is not very practical for use. However, the formula can be used for prescribed branching data in some very specific cases.

Generalized Hurwitz Enumeration Problem

The generalized Hurwitz enumeration problem has been formulated and solved in [Med84] albeit with a highly intractable for many practical applications generating function. However, the main result in [Med90] shows that for specific cases we can still get explicit information. The main ingredient in Mednykh's solution, is that Hurwitz's original result that the calculation of Hurwitz numbers is reduced by **Riemann existence theorem** to purely algebraic problem, can be lifted to higher genera target curves.

As above, let $f : X \longrightarrow Y$ be a branched covering of degree d of a fixed curve Y of genus h by a curve X of genus g whose branch locus has w branch points. The branch type of f at each of the w branch points is specified by $\mu^p \vdash d$, $p = 1, \ldots, w$. Following Mednykh [Med84], we alternatively use multiplicative notation for partitions and denote by $\mu^p = (1^{t_1^p} \cdots d^{t_d^p}) \vdash d$, $p = 1, \ldots, w$ where t_k^p is the number of branch points of index k for $k = 1, \ldots, d$ to avoid notational clutter. The multipartition $\overline{\mu}_d^w \vDash d$ for the branch profile of f is here denoted by a matrix $\sigma = (\mu_k^p)$, $k = 1, \ldots d$ of size $w \times d$.

The problem of counting equivalence classes of degree d branched covering $f : X \longrightarrow Y$ with the branched profile σ is called the **generalized Hurwitz enumeration problem.**

Let $B = \{y_1, \ldots, y_w\} \subset Y$ be a fixed branching locus. Fixing a standard system of arcs we get from (3.2) the monodromy representation (presentation of the fundamental group $\pi_1(Y\backslash B)$ to the symmetric group S_d). Recall that the presentation of the fundamental group $\pi_1(Y\backslash B)$ is given by

$$\Big(\gamma_1, \ldots, \gamma_w, \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h\Big). \tag{4.9}$$

subject to the condition

$$\prod_{i=1}^{w} \gamma_i \prod_{k=1}^{h} [\alpha_k,\ \beta_k] = \mathbb{1}. \tag{4.10}$$

The **Hurwitz system** of f corresponding to the standard set of generators (4.9) is an ordered sequence

$$(\sigma_1, \ldots, \sigma_w; a_1, b_1, \ldots, a_h, b_h)$$

of permutations in S_d with $\sigma_i \neq \mathbb{1}$ for all $i = 1, \ldots, w$ satisfying the condition

$$\sigma_1 \ldots \sigma_w \cdot [a_1, b_1] \cdot \ldots \cdot [a_h, b_h] = \mathbb{1}.$$

Denote by $(1^{t_1^k} \ldots d^{t_d^k})$ the cycle type of a permutation $\sigma_i \in S_d$. It has t_k cycles of length k, $k = 1, \ldots, d$. Then each branched covering f is uniquely determined by the transitive tuples of the Hurwitz system

$$\mathscr{B}_{h,d,\sigma} = \Big((1^{t_1^1} \ldots d^{t_d^1}), \ldots, (1^{t_1^w} \ldots d^{t_d^w});\ a_1, b_1, \ldots, b_h, b_h\Big) \in (S_d)^{2h+w} \tag{4.11}$$

satisfying the condition

$$\prod_{p=1}^{w} (1^{t_1^p} \ldots d^{t_d^p}) \prod_{i=1}^{h} [a_i, b_i] = \mathbb{1}.$$

Let $\mathscr{T}_{h,d,\sigma} \subset \mathscr{B}_{h,d,\sigma}$ be the subset of those free generators that generate transitive subgroups of S_d. The existence theorem 3.3.1 guarantees a bijection between irreducible branched coverings and **transitive representations** $\mathscr{T}_{h,d,\sigma}$. Moreover, the classification theorem 3.3.2 implies that two coverings are equivalent if only if their corresponding (transitive) Hurwitz systems are conjugate via a permutation of S_d. Hence, the general Hurwitz enumeration problem is reduced to counting the number of orbits in $\mathscr{T}_{d,g,\sigma}$ under the conjugation action of S_d.

Again for simplicity of notation, we write $B_{d,h,\sigma} = |\mathscr{B}_{d,h,\sigma}|$ and $T_{d,h,\sigma} = |\mathscr{T}_{d,h,\sigma}|$. Recall the classical Burnside's orbit-counting formula for the number of orbits under the action of a finite group.

Lemma 4.3.1 (Burnside)**.** *The number of orbits under the action of a finite group on a set X is given by*

$$N = \frac{1}{|G|} \sum_{g \in G} |X_g|, \tag{4.12}$$

where $X_g = \{x \in X : gx = x\}$ In other words, X_g denotes the set of elements in X that are fixed by $g \in G$.

The following result about the number $B_{h,d,\sigma}$ is obtained in [Med84].

Theorem 4.3.1. *The number of elements in* $\mathscr{B}_{d,h,\sigma}$ *(i.e. the number of* **irreducible branched coverings***) is given by the expression*

$$B_{h,d,\sigma} = d! \sum_{\lambda\in\mathscr{B}_d} \left[\prod_{p=1}^{w} \frac{\chi^{\lambda}_{t_1^p \dots t_d^p}}{1^{t_1^p} t_1^p! \cdot 2^{t_2^p} t_2^p! \dots d^{t_d^p} t_d^p!} \right] \left(\frac{d!}{f^\lambda}\right)^{2h+w-2}, \tag{4.13}$$

where $\mathscr{B}_d$ *is the set of all irreducible representations of the symmetric group* S_d *, the symbol* $\chi^{\lambda}_{t_1^p \dots t_d^p}$ *is the character of the permutation* $(1^{t_1} \dots d^{t_d})$ *of the representation* λ *and* $f^\lambda = \deg\lambda$.

Theorem 4.3.2. *The number* $T_{h,d,\sigma}$ *of elements in* $\mathscr{T}_{h,d,\sigma}$*, (i.e. the number of* **reducible branched coverings***) is given by*

$$T_{d,h,\sigma} = \sum_{k=1}^{d} \frac{(-1)^{k+1}}{k} \sum_{\substack{d_1+\dots+d_k=d \\ \sigma_1+\dots+\sigma_k=\sigma}} \binom{d}{d_1,\dots,d_k} B_{h,d_1,\sigma_1} \cdot B_{h,d_2,\sigma_2} \dots B_{h,d_k,\sigma_k}. \tag{4.14}$$

A key ingredient in the Mednykh Formula is the number-theoretic function $\Phi(x,d)$ called **von Sterneck function**. The function $\Phi(x,d)$ defined by the von Sterneck in 1902 is given by the relation

$$\Phi(x,d) = \frac{\varphi(d)}{\varphi(d/(x,d))}\mu(d/(x,d)),$$

where $\varphi(d)$ is the **Euler's phi function** and $\mu(d)$ is the **Möbius function.** Here (x,d) denotes the greatest common divisor (GCD) of x and d are such that $x \geq 0$ and $d > 0$.

Theorem 4.3.3 (The Mednykh Formula)**.** *The number of degree d nonequivalent branched covers of the branched type* $\sigma = (t_s^p)$*, for* $p = 1,\dots,w$ *and* $s = 1,\dots,d$ *is given by*

$$\begin{aligned} N_{h,d,\sigma} = \frac{1}{d} \sum_{\substack{l|v \\ ml=d}} \sum_{\frac{l}{(t,l)}|n|l} \frac{\mu(\frac{l}{n}) n^{(2h-2+w)m+1}}{(m-1)!} \Bigg[\sum_{j^s_{k,p}} T_{h,m,(s_k^p)} \times \\ \times \sum_{x=1}^{n} \prod_{s,k,p} \left[\frac{\Phi(x,s/k)}{n}\right]^{j^s_{k,p}} \prod_{k,p} \binom{s_k^p}{j^1_{k,p},\dots,j^{mn}_{k,p}} \Bigg], \end{aligned} \tag{4.15}$$

where $t := GCD\{t_s^p, p = 1,\dots,w, s = 1,\dots,d\}$, $v := GCD\{st_s^p, p = 1,\dots,w, s = 1,\dots,d\}$, $s_k^p = \sum_{s=1}^{mn} j^s_{k,p}$ and the sum $\sum_{j^s_{k,p}}$ is taken over all collections $j^s_{k,p}$ satisfying the condition

$$\sum_{\substack{1\leq k\leq st_s^p/l \\ s/(s,n)|k|s}} kj^s_{k,p} = \frac{st_s^p}{l}, \qquad p = 1,\dots,w, \quad s = 1,\dots,d\ ,$$

where $j^s_{k,p}$ is nonzero if and only if $1 \leq k \leq st_s^p/l$ and $(s/(s,n))|k|s$. Note that the products $\prod_{s,k,p}$ and $\prod_{k,p}$ range over $s = 1,\dots,mn$, $p = 1,\dots,w$ and with $k = 1,\dots,m$.

Applications of the Mednykh Formula

Clearly, formula (4.15) involves complicated conditional sums and products which makes the above answer to the generalized Hurwitz enumeration problem not very insightful. It is not even immediately clear how to obtain a numerical answer for a given branching data. On the other hand, graph-theoretic techniques [HP73], combinatorics, as well as tools from theoretical physics have enabled mathematicians to rediscover the old formulas and obtain new nice closed-form answers for Hurwitz numbers, see [SSV96, Arn96, GJ97, GJ99a, GJ99b, GJV00, CT95]. Recently, these Hurwitz numbers interpreted as Gromov-Witten invariants for coverings of $\mathbb{P}^1$ with specified branched data in [ELSV99, ELSV01, OP01]. Remarkably, it is possible to obtain some nontrivial results from the Mednykh Formula. In particular, we get **generalized simple Hurwitz numbers** which are the number of simple branched coverings for small degree in generalized case, [MSY04].

Generalized Simple Hurwitz Numbers

In [Med90], using equation (4.15) Mednykh establishes a closed form for the generalized Hurwitz number in the case of branched profile with each of the branched types being $(d) \vdash d$. And in fact, simple Hurwitz numbers for small degrees can be computed in closed forms involving only the genera and the number of branch points of a covering by the expression in (4.15). Below we deduce closed-form formulas for the simple Hurwitz numbers for arbitrary source and target curves for degrees $d = 1, 2, 3, 4$ and 5.

Let $f : X \longrightarrow Y$ be a degree d simple branched covering of a fixed genus h curve by genus g curve. A simple branch point has the branch type $(1^{d-2}, 2)$, and thus the branch profile is described by the matrix $\sigma = (t_s^p)$, for $p = 1, \ldots, w$, and $p = 1, \ldots, d$, where

$$t_s^p = (d-2)\delta_{s,1} + \delta_{s,2}, \quad \text{where } \delta_{m,n} \text{ is the Kronecker symbol.}$$

It follows that for the case of simple Hurwitz numbers, the quantities $t = GCD\{t_s^p\}$ and $v = GCD\{st_s^p\}$ in the formula (4.15) are given by:

$$t = 1 \quad \text{and} \quad v = \begin{cases} 2 & d \quad \text{even} \\ 1 & d \quad \text{odd} \end{cases} .$$

Furthermore, to find the range of the parameters in the first sum (which depends on v) in formula (4.15), we need to determine separately the required conditions for the cases when degree d is odd or even.

Case I: Degree d is odd

Suppose degree d is odd, then $l = n = (t, l) = 1$ and $m = d$. The conditions $(s/(s,n))|k|s$ and

$$\sum_{1 \le k \le st_s^p/l} k j_{k,p}^s = \frac{st_s^p}{l}$$

which determines the collection $\{j_{k,p}^s\}$ as

$$j_{k,p}^s = t_s^p \delta_{k,s}.$$

Since now $\Phi(1,1) = 1$, the equation (4.15) simplifies to

$$N_{h,d,\sigma} = \frac{T_{h,d,(s_k^p)}}{d!} \qquad \text{(for } d \text{ odd)}, \tag{4.16}$$

where

$$s_k^p = \sum_{s=1}^{d} j_{k,p}^s = t_k^p = (d-2)\delta_{k,1} + \delta_{k,2}. \tag{4.17}$$

Case II: Degree d is even

Suppose that d is even. Then $v = 2$ and thus the value of l is either 1 or 2. But if $l = 1$ then all the parameters are just as in the case of d odd. Thus, for $l = 1$ and d even the first sum of $N_{h,d,\sigma}$ is determined by equation (4.16). If $l = 2$ we note that the ranges of the summation are $m = d/2$ and $n = l = 2$. Furthermore, we can see that in this case

$$j_{k,p}^s = \frac{t_1^p}{2}\delta_{s,1}\delta_{k,1} + t_2^p\delta_{s,2}\delta_{k,1},$$

and it then follows that [1]

$$s_k^p =: \bar{s}_k^p = \frac{d}{2}\delta_{k,1}. \tag{4.18}$$

Since the number of simple branch points w is even, and the corresponding von Sterneck numbers are given by

$$\Phi(2,1) = -\Phi(1,2) = \Phi(2,2) = 1.$$

Therefore, we easily see that for $l = 2$ the contribution to $N_{h,d,\sigma}$ is given by

$$\frac{2^{(h-1)d+1}}{(\frac{d}{2}-1)!}\left(\frac{d}{2}\right)^{w-1} T_{h,\frac{d}{2},(\bar{s}_k^p)}. \tag{4.19}$$

Adding up the contributions in both (4.16) and (4.19), we get for all even d the formula

$$N_{h,d,\sigma} = \frac{T_{h,d,(s_k^p)}}{d!} + \frac{2^{(h-1)d+1}}{(\frac{d}{2}-1)!}\left(\frac{d}{2}\right)^{w-1} T_{h,\frac{d}{2},(\bar{s}_k^p)}\,. \tag{4.20}$$

Thus, the difficulty of the problem of computing simple Hurwitz numbers for a given degree d reduces to the calculation of two numbers $T_{h,\frac{d}{2},(\bar{s}_k^p)}$ and $T_{h,d,(s_k^p)}$, with the latter being relevant if the degree d is odd. Using (4.16) and (4.20) we can compute these numbers for low degrees and arbitrary genera g and h. First note, that combining equations (4.13) and (4.14), for $\bar{s}_k^p = d\delta_{k,1}$ we have

$$T_{h,d,(\bar{s}_k^p)} = d!\sum_{k=1}^{d}\frac{(-1)^{k+1}}{k}\sum_{d_1+\ldots+d_k=d}\prod_{i=1}^{k}\left[\sum_{\lambda\in\mathscr{B}_{d_i}}\left(\frac{d_i!}{f^\lambda}\right)^{2h-2}\right], \tag{4.21}$$

where $\mathscr{B}_{d_i}$ is the set of all irreducible representations of the symmetric group S_{d_i} f^λ denotes the degree of the representation λ. In particular, if $h = 0$ then using the fact that the cardinality

[1] *Note we are writing a bar in (4.17) distinguish s_k^p from the one we have in (4.18).*

of a finite group is equal to the sum of squares of the degrees of its irreducible representations, we get

$$T_{0,d,(\bar{s}_k^p)} = \sum_{k=1}^{d} \frac{(-1)^{k+1}}{k} \sum_{d_1+\ldots+d_k=d} \binom{d}{d_1,\ldots,d_k} = \begin{cases} 1, & \text{for } d = 1 \\ 0, & \text{for } d > 1. \end{cases} \tag{4.22}$$

Therefore, the computation of generalized simple Hurwitz numbers reduces to the evaluation of characters of the identity and the transposition elements in the symmetric group S_d. (For details refer to subsection 1.2 containing relevant information on representations of the symmetric group).

In what follows, we denote by $T_{h,d,w}$ the number $T_{h,d,\sigma}$ for simple branched types $\sigma = (t_s^p)$, where $t_s^p = (d-2)\delta_{k,1} + \delta_{k,2}$ for $p = 1,\ldots,w$ and $k = 1,\ldots,d$. Using this notation, let σ be a simple branched type of a covering f. Then, the formula for the number of reducible branched coverings $T_{h,d,\sigma}$ is (4.14) but with the number of irreducible branched coverings simplifies to the form

$$B_{h,d,w} = (d!)^{2h-1}\binom{d}{2}^{w}\left[\sum_{\lambda\in\mathscr{B}_d} \frac{(\chi_2^\lambda)^w}{(f^\lambda)^{2h+w-2}}\right]. \tag{4.23}$$

Generalized simple Hurwitz numbers of degree 1

For $d = 1$, $\sigma = (1)$ which implies

$$N_{h,d,1} = \delta_{g,h}. \qquad \text{where } \delta_{g,h} \text{ is the Kronecker symbol.}$$

Hence, degree 1 simple Hurwitz numbers are equal to $\delta_{g,h}$ for all genera g, h.

Generalized simple Hurwitz numbers of degree 2

In computing the degree 2 simple Hurwitz numbers, we will employ the following lemma.

Lemma 4.3.2. *Let $f : X \longrightarrow Y$ be a branched covering of a prime degree p, with $w \geq 1$ branch points of order p. As before denoting by g and h the genera of curves X and Y respectively, we get that the number $N_{h,p,w}$ of nonequivalent branched coverings is given by the formula*

$$N_{h,p,w} = \frac{1}{p!}T_{h,p,w} + p^{2h-2}\Big[(p-1)^w + (-1)^w(p-1)\Big], \tag{4.24}$$

where

$$T_{h,p,w} = p! \sum_{\lambda\in\mathscr{B}_p} \left(\frac{\chi_p^\lambda}{p}\right)^w \left(\frac{p!}{f^\lambda}\right)^{2h-2+w}.$$

Here $\mathscr{B}_p$ is the set of all irreducible representations of the symmetric group S_p, $f^\lambda = \deg\lambda$ and χ_p^λ is the character of the cycle of length p corresponding to the irreducible representation λ of the group S_p.

Proof. See Corollary 1 on page 1528 of [Med90]. □

For $p = 2$, the symmetric group S_2 has two irreducible representation each of degree 1. The character values of the transposition for both irreducible representations are ± 1. The number of branch points equals $w = 4(1-h) + 2(g-1)$ by the Riemann-Hurwitz formula. Implying formula (4.24), is useful in the case $w \geq 1$ or, in other words, for $g \geq 2h-3$. Thus, it follows that if $g \geq 2h-3$ then

$$N_{h,2,w} = T_{h,2,w} = \begin{cases} 2^{2h} & \text{for } w \text{ is even} \\ 0, & \text{for } w \text{ odd.} \end{cases} \tag{4.25}$$

On the other hand, if $w = 2(g-1) + 4(1-h) = 0$ then either $h = g = 1$ or $h = 1$ and $g > 1$. It is easy to see that in the former case we have 3 equivalence classes of branched coverings and in the latter we get 4.

Generalized simple Hurwitz numbers of degree 3

First note that if $t_k^p = 2\delta_{k,1}\sum_{i=1}^{j}\delta_{p,i} + \delta_{k,2}\sum_{i=j+1}^{w}\delta_{p,i}$, then similarly to $p = 2$ we have

$$B_{h,2,(t_k^p)} = \begin{cases} 2^{2h} & \text{for } j \text{ is even} \\ 0, & \text{for } j \text{ odd.} \end{cases} \tag{4.26}$$

Proposition 4.3.1. *Simple Hurwitz numbers of degree 3 are given by*

$$N_{h,3,w} = 2^{2h-1}(3^{2h-2+w} - 1) = 2^{2h-1}(3^{2g-4h+2} - 1),$$

where $w = 6(1-h) + 2(g-1)$ is the number of branch points.

Proof. There are three partitions of 3 namely, $(3), (2,1)$ and $(1,1,1)$ and consequently there are three irreducible representations of S_3, whose dimensions are 1, 2 and 1 respectively. The corresponding values of their characters on a transposition χ_τ^λ are -1, 0 and 1. Therefore, the quantity $T_{h,3,w}$ receives nonzero contributions only from the partitions (3) and (1^3). The formula follows from easy combinatorial computations. □

Generalized simple Hurwitz numbers of degree 4

Proposition 4.3.2. *The simple Hurwitz numbers of degree 4 are given by*

$$\begin{aligned} N_{h,4,w} =& 2^{2h-1}\left[(3^{2h-2+w}+1)2^{4h-4+w} - 3^{2h-2+w} - 2^{2h-3+w} + 1\right] + \\ &+ 2^{4h-4+w}(2^{2h}-1), \end{aligned} \tag{4.27}$$

where $w = 2(g-1) + 8(1-h)$.

Proof. There are 5 partitions of 4 namely, $(4), (3,1), (2^2), (2,1^2)$ and (1^4) and thus are 5 irreducible representations of S_4, whose dimensions are 1, 3, 3, 2 and 1. To evaluate $T_{h,4,w}$, observe that nonzero contributions come from four partitions of 4 namely $(4), (3,1), (2,2)$, and $(2,1^2)$ whose characters evaluated on transpositions are -1, 1, -1 and 1. The corresponding non-trivial sum involving σ is

$$\sum_{\sigma_1+\sigma_2=\sigma} = B_{h,2,\sigma_1}B_{h,2,\sigma_2} = 2^{4h+w-1},$$

by using equation (4.25). The last term in (4.28) comes from the second term in (4.20) by equation (4.21). □

Generalized simple Hurwitz numbers of degree 5

Proposition 4.3.3. *The simple Hurwitz numbers of degree* 5 *are given by*

$$\begin{aligned} N_{h,5,w} =& 2^{2h-1}\left[2^{2h-2+w} - 2^{4h-4+w} - 1\right] - 2^{2h-1}\cdot 3^{2h-2}\left[1+2^{2h-2+w}+2^{2h-2+2w}\right]+ \\ &+ 2^{2h-1}\cdot 3^{2h-2+w}\left[1-2^{4h-4+w}+2^{2h-2+w}\right] + 2^{6h-5+w}\cdot 3^{2h-2}+ \\ &+ 2^{2h-1}\cdot 3^{3h-2}\cdot 5^{2h-2+w}\left[1+2^{4h-4+w}+2^{2h-2+w}\right], \end{aligned} \tag{4.28}$$

where $w = 2(g-1)+10(1-h)$.

Proof. Here there are 7 partitions of 5: (5), $(4,1$, $(3,2)$, $(3,1^2)$, $(2^2,1)$, $(2,1^3)$, and (1^4). So there are exactly 7 irreducible representations of S_5, whose dimensions are 1, 4, $5,6$, 5, 4 and 1 by the hook length formula. Moreover, the value of its and characters on transpositions are 1, -1, $2,-2$, 1, -1 and 0 respectively. It follows that for $t_k^p = 3\delta_{k,1}\sum_{i=1}^{j}\delta_{p,i} + \big(\delta_{k,1} + \delta_{k,2}\big)\sum_{i=j+1}^{w}\delta_{p,i}$. Then, we obtain

$$B_{h,3,(t_k^p)} = \begin{cases} 2^{2h}\cdot 3^{2h-1+w-j} & \text{for } j < w \text{ is even} \\ 0, & \text{for } j \text{ odd} \\ 2\cdot 3^{2h-1}\left[2^{2h-1}+1\right] & \text{for } j = w. \end{cases} \tag{4.29}$$

Applying formula (4.24) for prime degree for $p = 5$, the proposition is established. ☐

4.4 Generating functions of Hurwitz Numbers

In this section, we want to describe the generating series for the single Hurwitz numbers, giving a recursion for a single Hurwitz number $\mathfrak{h}_{g,\mu}$ in terms of single Hurwitz numbers $h_{g',\mu}$ of lower genera. So far we have restricted ourselves to connected branched coverings since disconnected branched coverings can be recovered as a disjoint union of lower degree connected branched coverings. In the generating function, we consider both connected and disconnected coverings. Observe that we can easily define disconnected single Hurwitz numbers $h^{\bullet}_{g,\mu}$ combinatorially by dropping the condition of transitivity of the action of the monodromy group.

Let $p_1, p_2, p_3, \ldots$ be formal commuting variables and set $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ for $\mu = (\mu_1, \ldots, \mu_n) \vdash d$ and also $p_\mu = p_{\mu_1}\cdots p_{\mu_n}$. Now we introduce the generating functions for connected and disconnected single Hurwitz numbers as

$$\mathbf{H}(t,\mathbf{p}) = \sum_{\substack{g\geq 0\\ d,n\geq 1}}\sum_{\substack{l(\mu)=n\\ \mu\vdash d}} \mathfrak{h}_{g,\mu}\mathbf{p}_\mu\frac{t^w}{w!} \tag{4.30}$$

$$\mathbf{H}^{\bullet}(t,\mathbf{p}) = \sum_{\substack{g\geq 0\\ d,n\geq 1}}\sum_{\substack{l(\mu)=n\\ \mu\vdash d}} \mathfrak{h}^{\bullet}_{g,\mu}\mathbf{p}_\mu\frac{t^w}{w!}, \tag{4.31}$$

where in each case the summation is over all partitions of length n and $w = 2g-2+d+n$ is the number of simple branch points. Here $\mathbf{p} = p_1, p_2, p_3, \cdots$ are parameters that encodes the cycle type of σ. The parameter t counts the number of simple branch points. Since w and μ

recover the genus g, t is thus a **topological parameter**.

Recall that single Hurwitz numbers, coresponds to the case when the simple branch points correspond to transpositions, while the branch point at ∞ corresponds to a permutation with cycle type $\mu = (\mu_1, \dots, \mu_n) \vdash d$. If we consider the process of merging of the last simple branch point say y_w to ∞ then it means multiplication $\tau_w \cdot \sigma \in S_d$ and the result decreases the number of simple branch points w by 1. Equivalently we differentiate the generating function with respect to t. The result of this differentiation is the cut-and -join linear partial differential equation of Goulden-Jackson.

The Cut-and-Join Equation

Hurwitz numbers satisfy combinatorial conditions of partial differential equations (PDEs) called the cut-and-join equation. These PDEs are only useful for very specific branched covering with a given branch profile. In particular, single Hurwitz numbers satisfy a cut-and-join equation of Goulden-Jackson in [GJ97].

The key point for this result is the cut and join recursion for the multiplication of a transposition corresponding to τ_w by a permutation $\sigma \in S_d$ described §4.1. The cut cases can result in disconnected branched covering explaining why the generating functions must involve both connected and disconnected branched coverings of $\mathbb{P}^1$. There is a simple relation [Hur91] between the generating functions in (4.30) and (4.31). Namely,

$$\mathbf{H}^\bullet = \exp(\mathbf{H}), \tag{4.32}$$

where the exponential generating function for single Hurwitz numbers is defined to be

$$\exp\big(\mathbf{H}(t,\mathbf{p})\big) = 1 + \mathbf{H}(t,\mathbf{p}) + \frac{\mathbf{H}(t,\mathbf{p})^2}{2!} + \frac{\mathbf{H}(t,\mathbf{p})^3}{3!} + \cdots$$

and counts disconnected single branched coverings and the power of $\mathbf{H}(t,\mathbf{p})$ is the number of connected components. Then the cut and join recursion takes the following form:

Lemma 4.4.1.

$$\left[\frac{\partial}{\partial t} - \frac{1}{2}\sum_{i,j\geq 1}\Bigg(\underbrace{p_{i+j}\cdot(i\cdot j)\cdot\frac{\partial}{\partial p_i}\cdot\frac{\partial}{\partial p_j}}_{\tau_w\ joins} + \underbrace{p_i\cdot p_j\cdot(i+j)\cdot\frac{\partial}{\partial p_{i+j}}}_{\tau_w\ cuts}\Bigg)\right]\mathbf{H}^\bullet = 0.$$

We immediately deduce the cut-and-join equation of Goulden-Jackson for the generating function $\mathbf{H}(t,\mathbf{p})$ of the number of connected single Hurwitz numbers.

Theorem 4.4.1 (Cut and Join equation, [GJ97]). *The generating function* $\mathbf{H}$ *satisfies the following partial differential equation*

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2}\sum_{i,j} p_{i+j}\cdot(i\cdot j)\cdot\frac{\partial \mathbf{H}}{\partial p_i}\cdot\frac{\partial \mathbf{H}}{\partial p_j} + (i\cdot j)p_{i+j}\cdot\frac{\partial^2 \mathbf{H}}{\partial p_i \partial p_j} + p_i\cdot p_j\cdot(i+j)\cdot\frac{\partial \mathbf{H}}{\partial p_{i+j}}.$$

In other words, $\mathbf{H}$ is the unique formal power series solution of the cut and join partial differential equation above.

The fact that $\mathbf{H}$ satisfies a second order partial equation, is not surprising as more is known to hold. Namely there exists the KP (Kadomtsev-Petviashvili) Hierarchy for Hurwitz numbers. The KP Hierarchy is a completely integrable system of partial differential equations originating from mathematical physics.

4.5 The Hurwitz Monodromy Group

Recall an observation of A. Hurwitz, that if we fix the degree d of the branched coverings $f : C \longrightarrow \mathbb{P}^1$, the number w of branch points and branch types for all the w branch points, the Hurwitz space $\mathscr{H}_{d,g}$ form a covering space of the configuration space $Cov^w(\mathbb{P}^1)$ of w points in $\mathbb{P}^1$. The degree of the covering map

$$\mathscr{H}_{d,g} \longrightarrow Cov^w(\mathbb{P}^1), \tag{4.33}$$

is called the **Hurwitz number** corresponding to the branching data. The fundamental group of $Cov^w(\mathbb{P}^1)$ acts on the fibers of the covering and the orbits of this action are in one-one correspondence with the connected components of $\mathscr{H}_{d,g}$. In general, it is an unsolved problem to determine the image or in other words the monodromy group for branching morphism as described in (4.33) called the **Hurwitz monodromy group**. However, in special cases see [EEHS91] a good description can be obtained. This includes the case of small Hurwitz space $\mathcal{H}_{g,d}$.

The small Hurwitz space $\mathcal{H}_{g,d}$ is an irreducible quasiprojective variety, which comes with a finite śtale covering of $\mathrm{Sym}^w \mathbb{P}^1 \backslash \Delta_w$, where $w = 2g + 2d - 2$. The image of the fundamental group $\pi_1(\mathrm{Sym}^w \mathbb{P}^1 \backslash \Delta_w)$ to the symmetric group S_{h_d}, where h_d =simple hurwitz number is the Hurwitz monodromy group. Directly from the simple Hurwitz formulae, we have an intuitive indication, that the Hurwitz monodromy groups are less than the full symmetric group at least for the first nontrivial cases $d = 3$ and 4, but nothing much we can say for $d > 4$ from the shape of the formulae seen earlier. Indeed, the simple Hurwitz numbers for degree 3 and 4 consists of the factors $\frac{3^n-1}{2}$ and $2^n - 1$. Recall that $\frac{3^n-1}{2}$ is the number of points in the $n - 1$ dimensional projective space over a field with 3 elements and $2^n - 1$ is the number of points in a n dimensional projective space over a field with 2 elements. In fact, one way to compute the simple Hurwitz number for degree 3 branched coverings is to establish a bijection between transpositions $t_1 \ldots, t_w$ in S_3 specifying a branched covering curve X with $w = 2g + 4$ branch points and the projective space of dimension $w - 3$ over $\mathbb{F}_3$. This is easily obtained. Namely, up to conjugation we can assume $t_1 = (1,2)$ and consider the assignment

$$\eta : (1,2) \mapsto 0 \quad (1,3) \mapsto 1, (2,3) \mapsto 2.$$

Let $f^*((1,2))t_2 \ldots t_w) = (\eta(t_2), \ldots, \eta(t_{w-1}))$ we define the map f from the projective points via

$$f(X) = f^*((t_2), \ldots, \eta(t_{w-1})).$$

As an example, if $(1,2)(1,3)(1,2)(2,3)(1,3))$ represent X, $f(X) = (1,0,2,1)$. One then can easily show the map f is well defined from the requirement that the product of the transpositions must be identity, moreover its a bijection. Thus, we can compute the number of degree 3 covering branched over w simple branch points to be $\frac{1}{2}(3^{w-2} - 1)$ which is the number of points in the projective space of dimension $w-1$ a field $\mathbb{F}_3$ with three elements. Thus for $d=3$ or 4, the Hurwitz monodromy groups can be anticipated to have a structure which heavily reflects the geometrical structure of $\mathbb{F}_2$ and $\mathbb{F}_2$ vector spaces. Although, the formulation of the problem is purely of topological nature, interesting results comes algebraically. In that view, one considers the finite extension of function field of $\mathcal{H}_{g,d}$ by that $\mathrm{Sym}^w\,\mathbb{P}^1\backslash\Delta_w$ by regarding the spaces as irreducible quasiprojective varieties. Then if we denote this image by G, it has been calculated in [Coh74] and confirmed in [EEHS91] that

Theorem 4.5.1. *Let $g\geq 0$, then the Hurwitz monodromy group of $\mathcal{H}_{g,3}\longrightarrow \mathrm{Sym}^w\,\mathbb{P}^1\backslash\Delta_w$ is the simple group PSp$(2g+2,\mathbb{F}_3)$.*

Theorem 4.5.2. *The Hurwitz monodromy group G of $\mathscr{H}_{g,4}\longrightarrow \mathrm{Sym}^w\,\mathbb{P}^1\backslash\Delta_w$ fits into the into the following split exact sequence for $g>1$*

$$1\longrightarrow \prod_{\Omega} \mathit{Sp}(2g+2,\mathbb{F}_2)\longrightarrow G\longrightarrow \mathit{PSp}(2g+4,\mathbb{F}_3)\longrightarrow 0, \tag{4.34}$$

where Ω denotes the $2g+3$-dimensional projective space over $\mathbb{F}_3$ and the group Sp$(2g+2,\mathbb{F}_2)$ permutes the factors of the product in the obvious way.

If $g=0$, then the factor $\prod_{\Omega}\mathrm{Sp}(2g+2,\mathbb{F}_2)$ in the sequence (4.34) is the deck $3^{40}:2^{16}$ instead of $(S_3)^{40}$ and the sequence is non split. Similarly for $g=1$, the term $\mathrm{Sp}(2g+2,\mathbb{F}_2)$ is $(A_6)^{168}$ i.e. the direct product of copies of the alternating group instead of $(S_6)^{364}$ and it has not been determined whether the sequence is split or not.

Chapter 5

Functions on Smooth Plane Curves

In this chapter, we show that every degree function on a smooth connected projective curve $C \subset \mathbb{P}^2$ of degree $d > 4$ is isomorphic to a linear projection from a point $p \in \mathbb{P}^2 \setminus C$ to $\mathbb{P}^1$. We will then pose a Zeuthen-type problem for calculating the plane Hurwitz numbers.

5.1 Meromorphic functions on smooth plane curves

Consider $C \subset \mathbb{P}^2$ a plane curve of degree d. A surjective morphism $f : C \longrightarrow \mathbb{P}^1$ is called a *meromorphic function*. More precisely, a meromorphic function f gives a finite morphism to the complex projective line $\mathbb{P}^1$ whose degree d by definition is the degree of the morphism $f : C \longrightarrow \mathbb{P}^1$. Thus for a meromorphic function f and any fixed point $q \in \mathbb{P}^1$ we have the divisor $f^{-1}(q) = \mu_1 p_1 + \ldots + \mu_n p_n$, where $p_1, \ldots, p_n$ are pairwise distinct points on C and $\mu_1, \ldots, \mu_n$ are positive integers summing up to d. In particular, the morphism f is a branched covering of $\mathbb{P}^1$ of branch type $\mu := (\mu_1, \ldots, \mu_n) \vdash d$ at a point q.

Let $C \subset \mathbb{P}^2$ be a plane curve of degree d. An important geometric method for studying C, involves meromorphic functions arising from linear projections of C from a point $p \in \mathbb{P}^2$. For instance, B. Riemann established in his famous work [Rie57], that the topological structure of a smooth curve $C \subset \mathbb{P}^2$ depends entirely on the nature of branch types of the branched covering π_p arising from a linear projection. To construct π_p, we choose a point $p \in \mathbb{P}^2$ which may or may not be lying on C and then identify $\mathbb{P}^1$ with the pencil of lines passing through $p \in \mathbb{P}^2$. If $p \in \mathbb{P}^2 \setminus C$, then a generic line through p meets the curve C in d distinct points.

Definition 5.1.1. *Let $C \subset \mathbb{P}^2$ be a plane curve of degree d. A* **linear projection** *or simply a projection from a point $p \in \mathbb{P}^2 \setminus C$ is a meromorphic function*

$$\pi_p : C \longrightarrow \mathbb{P}^1. \tag{5.1}$$

Notice that the morphism π_p has degree d. In particular, π_p is a branched covering of $\mathbb{P}^1$ and the points of $\mathbb{P}^1$ where several intersection points of the corresponding line with C coincide are the branch points of π_p.

Therefore, it is a basic problem to characterize and enumerate those meromorphic functions f on C which can be realized as linear projections. First, note that in general not all meromorphic functions on a curve $C \subset \mathbb{P}^2$ can be realized as such. However, for $d > 4$ we have the following result which we will prove.

Theorem 5.1.1. *Suppose that $C \subset \mathbb{P}^2$ is a smooth projective plane curve of degree $d > 4$. Then any meromorphic function $f : C \longrightarrow \mathbb{P}^1$ of degree d can be realized as a linear projection $\pi_p : C \longrightarrow \mathbb{P}^1$.*

Recall that, given a smooth curve C, specifying a meromorphic function $f : C \longrightarrow \mathbb{P}^1$ of degree d on C corresponds to identifying an effective degree d divisor D of f such that the linear system $|D|$ has no base points and $\dim |D| \geq 1$. (See for example Arbarello et. al. [ACGH85]).

Definition 5.1.2. *Let $D = p_1 + \ldots + p_d$ be a divisor on a smooth curve C. If $|D|$ has no base point and $\dim |D| = 1$, we say that D* **moves in a linear pencil** *$|D|$. Equivalently, we have a meromorphic function of degree d*

$$f : C \longrightarrow \mathbb{P}^1$$

such that $f^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathscr{L}$, where $\mathscr{L} \cong \mathcal{O}_C(D)$ for $\mathcal{O}_C(D)$ the invertible sheaf over C determined by the divisor D and $h^0(\mathscr{L}) = 2$, so that we may choose a basis say $\{f_0, f_1\}$ for $\mathbf{H}^0(C, \mathscr{L})$ such that $f = [f_0 : f_1]$.

Remark 5.1.1. *The assertion of Theorem 5.1.1 fails if $d = 3$ and $d = 4$. For instance, we have a meromorphic functions of degree 4 on a smooth projective quartic which are never isomorphic to linear projections.*

Example 5.1

If $C \subset \mathbb{P}^2$ is a smooth projective quartic, then there is a meromorphic function on C of degree 4 which is not isomorphic to a linear projection π_p.

Indeed let $D = p_1 + \ldots + p_4$ be a divisor given by any 4 points on C such that no three of them are collinear. The Riemann-Roch's theorem gives

$$h^0(\mathscr{L}) = h^0(D) + 1 - g + h^0(K - D) = 4 + 1 - 3 + h^0(K - D) = 2,$$

since $h^0(K - D) = 0$, (otherwise we will have $K \sim D$, which implies $p_1 + \ldots + p_4 \in |K|$ all lie in a line). Next, recall that an invertible sheaf $\mathscr{L}$ on C is base point free if $h^0(\mathscr{L}) - h^0(\mathscr{L}(-p)) = 1$ for all $p \in C$. By Riemann-Roch, we have

$$\begin{aligned} h^0(\mathscr{L}(-p)) &= \deg(\mathscr{L}(-p)) - g + 1 \\ &= \deg(D - p) - g + 1 + h^0(K - D + p) \\ &= 3 + 1 - 3 + h^0(K - D + p) \\ &= 1 + h^0(K - D + p). \end{aligned}$$

The fact that

$$h^0(\text{ a degree 1 divisor}) = \begin{cases} 0 \\ 1 \\ 2 & \text{which happens only on } \mathbb{P}^1, \end{cases}$$

enables us to deduce that $\deg(K-D+p)=0$. Indeed, the complete linear system $|D-p|\subset|D|$. To see this, assume $|D-p|\subseteq|D|$. Then we have an injection $E\mapsto E+p$. So $p=p_i$ for some i. We can assume $p=p_4$

$$\begin{aligned}h^0(\mathscr{L}(-p)) &= h^0(p_1+p_2+p_3)\\ &= \deg(p_1+p_2+p_3)-g+1+h^0(K-p_1-p_2-p_3)\\ &= 1+h^0(K-p_1-p_2-p_3).\end{aligned}$$

Since $K-p_1-p_2-p_3$ is a degree 1 divisor, then there exists a point q such that $K-p_1-p_2-p_3$ is linearly equivalent to q. In other words, $K\sim q+p_1+p_2+p_3$, not possible since p_1,p_2,p_3 do not lie on a line. So we obtain $h^0(\mathscr{L}(-p))=1=h^0(\mathscr{L})-1$ and we conclude that the linear system $|p_1+p_2+p_3+p_4|$ has no base points. Hence, the four points move in a linear pencil but a meromorphic function specified by this divisor on a smooth quartic cannot be realised as a linear projection as this 4 points are not in a line.

Similarly, a configuration of 3 points p_1,p_2,p_3 not all of them in a line, for the divisor $p_1+p_2+p_3$ on a smooth cubic, provides a counterexample for the case of $d=3$.

Definition 5.1.3. *The finite set $\Gamma=\{p_1,\dots,p_d\}\subset\mathbb{P}^2$ of distinct points* imposes linear independent conditions *on plane curves of degree m if for every point $P\in\Gamma$ there exist plane curves of degree m that contains $\Gamma\setminus P$ and does not contain the point $P\in\Gamma$.*

Consider the subset $\Gamma\subset\mathbb{P}^2$ as a closed zero-dimensional subscheme of $\mathbb{P}^2$. Then we have the standard exact sequence of sheaves

$$0\longrightarrow\mathscr{I}_\Gamma\otimes\mathcal{O}_{\mathbb{P}^2}(m)\longrightarrow\mathcal{O}_{\mathbb{P}^2}(m)\longrightarrow\mathcal{O}_\Gamma(m)\longrightarrow0, \tag{5.2}$$

where $\mathscr{I}_\Gamma\subset\mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf of the zero dimensional variety Γ. Note that $\mathcal{O}_\Gamma(m)\cong\oplus_{i=1}^d\mathcal{O}_{p_i}\cong\mathbb{C}^d$, and that surjectivity of

$$\alpha:\mathbf{H}^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(m))\longrightarrow\mathbf{H}^0(\Gamma,\mathcal{O}_\Gamma(m))$$

exactly means that there is for each p_i, $i=1,\dots,d$ a plane curve of degree m that contains $\Gamma\setminus\{p_i\}$ but not p_i. Hence $\Gamma\subset\mathbb{P}^2$ fails to impose independent conditions on curves of degree m if and only if α is not surjective. Namely if and only if

$$h^0(\mathscr{I}_\Gamma\otimes\mathcal{O}_{\mathbb{P}^2}(m))>h^0(\mathcal{O}_{\mathbb{P}^2}(m))-d=\frac{(m+1)(m+2)}{2}-d.$$

Equivalently since $\mathbf{H}^1(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(m))=0$, we see that Γ fails to impose independent conditions on $|\mathcal{O}_{\mathbb{P}^2}(m)|$ if we have $h^1\big(\mathscr{I}_\Gamma\otimes\mathcal{O}_{\mathbb{P}^2}(m)\big)>0$.

The proof of Theorem 5.1.1 will be derived from the following result.

Theorem 5.1.2. *Let $\Gamma=\{p_1,\dots,p_d\}\subset\mathbb{P}^2$, be any collection of $d\geq5$ distinct points. If Γ fails to impose independent linear conditions on $|\mathcal{O}_{\mathbb{P}^2}(d-3)|$ then at least $d-1$ of the points are collinear.*

To see why the proof of Theorem 5.1.1 follows from that of Theorem 5.1.2, note that we need to specify an effective divisor D of degree d on C such that the linear system $|D|$ has no base points and $\dim|D| \geq 1$, where

$$\dim|D| := h^0(D) - 1. \tag{5.3}$$

In the case the divisor D on C has a linear system as above, we say that D moves.

Now let $D = p_1 + \ldots + p_d$ be a divisor of degree d on a smooth curve $C \subset \mathbb{P}^2$. A criterion for determining when D moves is given by the Riemann-Roch theorem for curves. Denote by H the divisor of a general linear section: i.e. H is a pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ along the inclusion $C \longrightarrow \mathbb{P}^2$. The adjunction formula tells us that,

$$K_C \sim (d-3)H.$$

By the Bézout theorem the degree of the divisor $(d-3)H$ is equal to $d(d-3)$. So we obtain that,

$$2g - 2 = (d-3)d \quad \text{or} \quad g = \frac{(d-1)(d-2)}{2}.$$

The Riemann-Roch formula implies that

$$h^0(D) = d - g + 1 + h^0\big(K_C - D\big),$$

and hence $\dim|D| \geq 1$ if and only if

$$\dim|K_C - D| \geq \frac{(d-1)(d-2)}{2} - d. \tag{5.4}$$

Now the ideal sheaf $\mathscr{I}_C$ of C in $\mathbb{P}^2$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-C)$, and so

$$\mathbf{H}^0(\mathbb{P}^2, \mathscr{I}_C \otimes \mathcal{O}_{\mathbb{P}^2}(d-3)) \cong \mathbf{H}^1(\mathbb{P}^2, \mathscr{I}_C \otimes \mathcal{O}_{\mathbb{P}^2}(d-3)) = 0$$

since $\mathbf{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \cong \mathbf{H}^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) = 0$. Twisting the exact sequence

$$0 \longrightarrow \mathscr{I}_C \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

by $\mathcal{O}_{\mathbb{P}^2}(d-3)$, we find that $\mathbf{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)) \cong \mathbf{H}^0(C, \mathcal{O}_C(d-3))$. Furthermore we have that

$$\mathbf{H}^0(\mathbb{P}^2, \mathscr{I}_\Gamma \otimes \mathcal{O}_{\mathbb{P}^2}(d-3)) = \ker\big(\mathbf{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)) \longrightarrow \mathbf{H}^0(\Gamma, \mathcal{O}_\Gamma(d-3))\big).$$

On the other hand, $K_C \sim (d-3)H$ and $\mathcal{O}_C(D)$ is the ideal of D in C which implies that

$$\mathbf{H}^0(C, \mathcal{O}_C(K_C - D)) = \ker\big(\mathbf{H}^0(C, \mathcal{O}_C(d-3)) \longrightarrow \mathbf{H}^0(\Gamma, \mathcal{O}_\Gamma(d-3))\big),$$

so we find that $h^0(\mathcal{O}_C(K_C - D)) = h^0\big(\mathscr{I}_D \otimes \mathcal{O}_{\mathbb{P}^2}(d-3)\big)$. Hence (5.4) is equivalent to the inequality

$$h^0\big(\mathscr{I}_D \otimes \mathcal{O}_{\mathbb{P}^2}(d-3)\big) > \frac{(d-1)(d-2)}{2} - d. \tag{5.5}$$

In other words, the divisor $D = p_1 + \ldots + p_d$ satisfies $\dim|D| \geq 1$ if and only if the set $\Gamma = \{p_1, \ldots, p_d\}$ fails to impose independent conditions on the canonical linear system $|K_C|$. We will now see that we may use this to derive Theorem 5.1.1 from Theorem 5.1.2.

Proof of Theorem 5.1.1. Now observe that to complete the proof, it suffices to show that either all the d points of D are collinear, or if only the $d-1$ points of D lie on a line then the d-th point is a base point of the linear system $|D|$. In the first case $D \sim H$ and we are done. In the second case, suppose that $D = p_1, \ldots, p_{d-1} + q$, where the points $p_1, \ldots, p_{d-1}$ lie on a line ℓ and $q \notin \ell$. We must show that q is a base point of the linear system $|D|$ or equivalently that we have

$$\dim |p_1 + \ldots + p_{d-1}| = \dim |p_1 + \ldots + p_{d-1} + q|.$$

But as the degree of the divisor $p_1 + \ldots + p_{d-1}$ is equal to $\deg D - 1$, the Riemann-Roch then implies that it is enough to show that the following equality:

$$\dim |K_C - p_1 - \ldots - p_{d-1} - q| = \dim |K_C - p_1 - \ldots - p_{d-1}| - 1 \tag{5.6}$$

holds. Since $\deg C = d$, we can write the divisor cut by C on ℓ as $C \cdot \ell = p_1 + \ldots + p_{d-1} + b$, where $b \neq q$ because $q \notin \ell$. If a curve C_1 of degree $d-3$ passes through $d-1$ collinear points $p_1, \ldots, p_{d-1}$, it must contain ℓ as a component. Thus, the linear system in equation (5.6) on left-hand side

$$|K_C - p_1 - \ldots - p_{d-1} - q| \cong |\mathscr{I}_q \otimes \mathcal{O}_{\mathbb{P}^2}(d-4)|,$$

whereas the linear system on right-hand side in (5.6)

$$|K_C - p_1 - \ldots - p_{d-1}| \cong |\mathcal{O}_{\mathbb{P}^2}(d-4)|$$

which follows from the fact that $\dim |\mathscr{I}_q \otimes \mathcal{O}_{\mathbb{P}^2}(d-4)| = \dim |\mathcal{O}_{\mathbb{P}^2}(d-4)| - 1$. And this implies (5.6), which completes the proof. □

It is worthy to note that if $p_1, \ldots, p_{d-1}$ are distinct points in $\mathbb{P}^2$, then they will always impose independent conditions on curves of degree $d \geq 4$. In particular, the divisor $D = p_1 + \ldots + p_{d-1}$ moves in a linear pencil if and only if the points $p_1, \ldots, p_{d-1}$ lie on a line. It follows that for a smooth plane curve $C \subset \mathbb{P}^2$ of degree d, there is no nonconstant meromorphic function of degree less than $d-1$.

Proof of Theorem 5.1.2

To shorten the proof of theorem 5.1.2, we first reformulate it below in a slightly different but equivalent form.

Theorem 5.1.3. *Let $\Gamma = \{p_0, \ldots, p_d\} \subset \mathbb{P}^2$, be any collection of $d+1 \geq 5$ distinct points. If Γ fails to impose independent linear conditions on $|\mathcal{O}_{\mathbb{P}^2}(d-2)|$ then at least d of the points in Γ are collinear.*

Proof. By assumption there exists at least one point (without loss of generality) say $p_0 \in \Gamma$ such that any curve of degree $d-2$ passing through the points in $\Gamma \setminus p_0$ also passes through p_0. Note that if we have a curve C of degree $n \leq d-2$ that passes through $\Gamma \setminus p_0$, then it follows by assumption that C also must pass through p_0.

Let $p_0, p_1 \ldots, p_j$ be the minimal number of points in Γ lying on a line ℓ containing the point p_0. Rename the remaining points as $q_1, \ldots q_{d-j}$. By construction, any line through a point

$p_i \neq p_0$ and a point q_i, will not pass through p_0. We now construct a curve C being a product of such lines. We let ℓ_i be the line through p_i and q_i if $1 \leq i \leq \min\{j, d-j\}$. For the possible remaining points, we either let ℓ_i denote the lines through p_i and q_1 (if $d-j < i \leq j$) or the line through q_i and p_1 (if $j < i \leq d-j$). The curve

$$C = \ell_1 \dots \ell_n \qquad \text{(where } n = \max\{j, d-j\}\text{)}$$

passes through all the points of $\Gamma \smallsetminus p_0$, but not though p_0.

If we have $2 \leq j \leq d-2$ then we get that the degree $n \leq d-2$, which is a contradiction to our assumption.

If we have $j = 1$, then any line ℓ' through two points $\Gamma \smallsetminus p_0$ would not contain p_0. Observe that, to cover $\Gamma \smallsetminus p_0$, we need at most $n \leq d/2$ lines $\ell'_1, \dots, \ell'_n$ if d is even, and at most $n \leq (d+1)/2$ lines to cover $\Gamma \smallsetminus p_0$, if d is odd. Note that $d \geq 5$ is equivalent to $(d+1)/2 \leq d-2$, and if $d = 4$ then we have that $d/2 \leq d-2$. Hence for any d, in our range, we have the curve

$$C' = \ell'_1 \dots \ell'_n$$

of degree $n \leq d-2$ that passes through all points of $\Gamma \smallsetminus p_0$, but not through p_0. This is impossible by assumption.

Finally, we are left with the only possibility that $j > d-2$. However if $j \geq d-1$, then we have at least $j+1 \leq d$ point $p_0, \dots, p_j$ aligned on the line ℓ. This completes the proof. □

Plane Hurwitz numbers and Zeuthen numbers

Hurwitz numbers [Hur91, OP01] count non-isomorphic meromorphic functions on curves with fixed genus g having a fixed branched profile. On the other hand, Zeuthen numbers [Zeu73] count nodal plane curves of a fixed degree d and geometric genus g passing through a general points and tangent to b general lines in $\mathbb{P}^2$, where $a + b = 3d + g - 1$. There is a class of Zeuthen numbers corresponding to what we call *plane Hurwitz numbers*. Zeuthen numbers have been interpreted by R.Vakil in the context of stable maps as positive degree Gromov-Witten invariants of $\mathbb{P}^2$. Below, following [Vak99], we will sketch a derivation of a class of characteristic numbers of smooth plane curves which correspond to calculating plane Hurwitz numbers.

5.2 Plane Hurwitz Numbers

Generally in calculating Hurwitz numbers, we make no reference to the embedding of curves. For example, one can not expect for instance a branched covering of $\mathbb{P}^1$ whose domain is genus 2 to be planar and smooth, since a smooth plane curve of degree d, has $g = \binom{d-1}{2}$. Additionally, we expect that not all curves of genus $g = \binom{d-1}{2}$ can be embedded in $\mathbb{P}^2$ as smooth curves. For instance, among all smooth curves of genus 3 (for $d = 4$), there are hyperelliptic curves, which are not planar.

Fix $d > 0$; the space parametrizing all degree d algebraic curves in $\mathbb{P}^2$ is a complete system $|\mathcal{O}_{\mathbb{P}^2}(d)|$, which forms a projective space

$$\mathbb{P}(\mathbf{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))) \cong \mathbb{P}^{\mathbf{N}},$$

where $\mathbf{N} = \binom{d+2}{2} - 1 = d(d+3)/2$. In particular, the set of all smooth plane curves of a given degree d is an open subset of $\mathbb{P}^{\mathbf{N}}$. The group $\mathbb{P}\mathbf{GL}(3,\mathbb{C})$ of all projective automorphisms of $\mathbb{P}^2$ acts on $\mathbb{P}^{\mathbf{N}}$ in a natural way. Of particular interest is the subgroup $\mathcal{G}_p \subset \mathbb{P}\mathbf{GL}(3,\mathbb{C})$ fixing p and preserving the pencil of lines through p. Given a smooth curve $C \subset \mathbb{P}^2$, for instance if $p = [0:1:0] \in \mathbb{P}^2 \setminus C$ for some choice of coordinate system of $\mathbb{P}^2$ an element of the group $\mathcal{G}_p$ has the form

$$g = \begin{bmatrix} g_0 & 0 & 0 \\ g_1 & g_2 & g_3 \\ 0 & 0 & g_0 \end{bmatrix} \quad \text{with} \quad g_0 g_2 \neq 0.$$

The group of automorphisms $\mathcal{G}_p$ acts equivalently on $\mathbb{P}^{\mathbf{N}}$ keeping the branching points of the projection $\pi_p : C \longrightarrow \mathbb{P}^1$ fixed. Recall from Definition 3.2.3, that two branched coverings $\pi_p^1 : C_1 \longrightarrow \mathbb{P}^1$ and $\pi_p^2 : C_2 \longrightarrow \mathbb{P}^1$ are called **equivalent** if there exists an isomorphism $g : C_1 \longrightarrow C_2$ such that $\pi_p^2 \circ g = \pi_p^1$. Then we have:

Proposition 5.2.1. *Let $C_1, C_2 \subset \mathbb{P}^2$ be two smooth projective plane curves of the same degree $d > 1$ and not passing through $p \in \mathbb{P}^2$. Two projections $\pi_p^1 : C_1 \longrightarrow \mathbb{P}^1$ and $\pi_p^2 : C_2 \longrightarrow \mathbb{P}^1$ are equivalent if and only if there exists an automorphism $g \in \mathcal{G}_p$ such that $g(C_1) = C_2$.*

Proof. Let $C_1, C_2 \subset \mathbb{P}^2$ be smooth projective curves not passing through $p \in \mathbb{P}^2$. If there exists an automorphism $g \in \mathcal{G}_p$ such that $C_2 = g(C_1)$, then the morphisms π_p and π_p' are equivalent by an isomorphism given by g. For the 'only if ' direction, suppose that π_p^1 and π_p^2 are equivalent and that this equivalence is determined by an isomorphism $g : C_1 \longrightarrow C_2$. For each line $\ell \ni p$ the isomorphism g maps $C_1 \cap \ell$ to $C_2 \cap \ell$; thus, g maps hyperplane sections of C_1 to hyperplane sections of C_2. Since both C_1 and C_2 are embedded in $\mathbb{P}^2$ by complete linear system of hyperplane sections $\mathbf{H}^0(\mathbb{P}^2, \mathcal{O}_{C_i}(1))$, for $i = 1, 2$, this implies that g is induced by projective automorphism $\mathbb{P}\mathbf{GL}(3,\mathbb{C})$. To complete the proof, it only remains to check that $g \in \mathcal{G}_p$; to that end, consider a generic line $\ell \ni p$; this line intersects C_i for $i = 1, 2$ at $d = \deg C_i > 1$ points and this points are mapped by g to d distinct points on ℓ. So $g(\ell) = \ell$ for the generic line and thus for any $\ell \ni p$. If ℓ_1, ℓ_2 containing p then

$$g(p) = g(\ell_1 \cap \ell_2) = g(\ell_1 \cap \ell_2) = g(\ell_1) \cap g(\ell_2) = \ell_1 \cap \ell_2 = p.$$

Hence $g \in \mathcal{G}_p$ as expected and this completes the proof. □

By a *generic projection* of smooth curve $C \subset \mathbb{P}^2$ from a point $p \in \mathbb{P}^2$ which is not on a bitangent line or a flex line we obtain a linear projection $\pi_p : C \longrightarrow \mathbb{P}^1$ with only simple branch points. This leads us to the orbit space parametrizing all generic linear projections. Denote this space of generic linear projections of the set of smooth curves C of degree d by:

$$\mathcal{PH}_d = \left\{ \pi_p : C \longrightarrow \mathbb{P}^1 \;\middle|\; \begin{array}{c} \pi_p \text{ is a simple linear projection from} \\ p \in \mathbb{P}^2 \setminus C \text{ of a smooth curve } C \subset \mathbb{P}^2 \end{array} \right\} \Big/ \sim . \tag{5.7}$$

where $\sim$ is the equivalence of projections from a point $p \in \mathbb{P}^2$ up to the $\mathcal{G}_p$-action.

Note that for $g = \binom{d-1}{2}$, we have a natural inclusion $\mathcal{PH}_d \subseteq \mathcal{H}_{d,g}$ of small Hurwitz spaces for $d > 1$. The information about the dimension of $\mathcal{PH}_d$ is a direct consequence of proposition 5.2.1.

Corollary 5.2.1. *The dimension of the space $\mathcal{PH}_d$ is equal to $\mathbf{N} - 3 = \frac{d(d+3)}{2} - 3$.*

The number of branch points of a generic projection $\pi_p : C \longrightarrow \mathbb{P}^1$ of a smooth curve of degree d from $p \in \mathbb{P}^2 \setminus C$ is determined by the Riemann-Hurwitz formula as $w = d(d-1)$. We refer to the number of 3-dimensional $\mathcal{G}$-orbits with the same set of w tangents lines as the *d-th plane Hurwitz number* and denote it by $\mathfrak{h}_d$. Thus, to compute $\mathfrak{h}_d$ as indicated in (3.7), we need to calculate the degree of the branch morphism

$$\mathcal{PH}_d \longrightarrow \operatorname{Sym}^w \mathbb{P}^1 \setminus \Delta, \tag{5.8}$$

restricted to its image. Notice that by Corollary 5.2.1 the $\dim \mathcal{PH}_d < d(d-1)$ for $d \geq 4$. Next we will give two examples of known plane Hurwitz numbers.

Degree 3-plane Hurwitz Numbers

The first nontrivial case involves projections of smooth plane cubics. The remark following Theorem 5.1.1 asserts that if $d = 3$ not all meromorphic function of degree 3 on smooth plane cubics are realizable as projections. However, degree 3 simple plane Hurwitz numbers coincides with the usually Hurwitz number. Namely, over $w = 6$ pairwise distinct points on the projective line $\mathbb{P}^1$ there are exactly 40 three-dimensional orbits of smooth cubics branched over them, see [Hur91]. To see this, recall that Hurwitz numbers count branched covering up to equivalence, the equivalence of plane Hurwitz with the usual Hurwitz number is a consequence of the fact that every meromorphic function of degree 3 on a smooth cubic is a composition of a group shift of C followed by a linear projection from $p \in \mathbb{P}^2 \setminus C$. This is a well-known consequence of the fact that any smooth plane cubic curve is an abelian group. We give the details below.

Proposition 5.2.2. *Every meromorphic function of degree 3 on a smooth cubic curve $C \in \mathbb{P}^2$ can be represented as a composition of a group shift on C by a fixed point on C with a linear projection from a point $p \in \mathbb{P}^2$.*

Proof. Let C be a smooth projective cubic and let $f : C \longrightarrow \mathbb{P}^1$ be a meromorphic function of degree 3. If we write $f^{-1}(0) = z_1 + z_2 + z_3$, $f^{-1}(\infty) = p_1 + p_2 + p_3$ for the zero divisor and polar divisor of f respectively (where z_i and p_i for all $i = 1, 2, 3$ are not necessarily distinct). The linear equivalence of divisors $f^{-1}(0) \sim f^{-1}(\infty)$ implies the equality

$$p_1 + p_2 + p_3 = z_1 + z_2 + z_3,$$

where "+" denotes the addition from group law on the cubic curve. Fix a point $P_0 \in C$ such that $p_1 + p_2 + p_3 + 3P_0 = 0$ and define

$$Q_i = p_i + P_0, \quad \text{and} \quad R_i = z_i + P_0 \quad \text{for all } i = 1, 2, 3.$$

Then we have

$$Q_1 + Q_2 + Q_3 = p_1 + p_2 + p_3 + 3P_0 = 0$$
$$R_1 + R_2 + R_3 = z_1 + z_2 + z_3 + 3P_0 = 0.$$

In particular, $\{Q_1, Q_2, Q_3\}$ and $\{R_1, R_2, R_3\}$ lie on distinct lines in $\mathbb{P}^2$, since otherwise these sets would be equal and so $f^{-1}(0) = f^{-1}(\infty)$, which is impossible. Denote the lines given by the translates $\{Q_1, Q_2, Q_3\}$ and $\{R_1, R_2, R_3\}$ by $\ell_1 \subset \mathbb{P}^2$ and $\ell_2 \subset \mathbb{P}^2$ respectively. If $l_1(x, y, z)$ and $l_2(x, y, z)$ are equations for the lines ℓ_1 and ℓ_2, the meromorphic function given by composition of the group shift and projection is the quotient l_1/l_2: $f(P - P_0) = \frac{\ell_1(P)}{\ell_2(P)} \iff f(P) = \frac{\ell_1(P+P_0)}{\ell_2(P+P_0)}$, (where $P = (x, y, x)$) after possibly multiplying with a constant using the fact that a meromorphic function without poles will be constant. □

Degree 4-plane Hurwitz Numbers

The case $d = 4$ is more exciting. Note that the space parametrizing projections $\mathcal{PH}_4$ has dimension $\frac{4(4+3)}{2} - 3 = 11$. As branched coverings, this 11-dimensional family $\mathcal{PH}_4$ admits a natural inclusion into the small Hurwitz space $\mathcal{H}_{4,3}$ defined in (6.1.2) which is a smooth irreducible variety of dimension 12. The inclusion $\mathcal{PH}_4 \subset \mathcal{H}_{4,3}$ implies that the branch locus defines an hypersurface $\mathbf{B} \subset \mathrm{Sym}^{12}\,\mathbb{P}^1$. R. Vakil in [Vak01] has computed its degree to be equal to 3762. Moreover, he establishes that there are essentially 120 smooth plane quartic branched over admissible 12 points in $\mathbb{P}^1$. Thus, it follows that the plane Hurwitz number of degree 4 is

$$\mathfrak{h}_4 = 120 \times \frac{(3^{10} - 1)}{2}. \tag{5.9}$$

The corresponding Hurwitz number is known to be equal to $\mathfrak{h}_{3,4} = 255 \times \frac{(3^{10}-1)}{2}$.

5.3 Zeuthen numbers

This notion of plane Hurwitz numbers has a strong analogy to the special case of Zeuthen's classical problem which asks to calculate the number of irreducible plane curves of degree $d > 0$ and geometric genus $g \geq 0$ passing through a general points and b tangent lines in $\mathbb{P}^2$, where $a + b = 3d + g - 1$. More precisely, assuming that the only singularities of an irreducible curve $C \subset \mathbb{P}^2$ are δ nodes, since each node reduces the freedom of the curve by 1, we expect that the set of irreducible degree d curves with δ nodes depends on

$$\dim |\mathcal{O}_{\mathbb{P}^2}(d)| - \delta = \frac{d(d+3)}{2} - \delta = 3d + g - 1$$

parameters. Indeed, for all fixed integers $d > 0$ and $g \geq 0$ as first observed by F. Severi [Sev21] and proved by J. Harris [Har86], the Severi variety $V_{g,\delta}$ parametrizing irreducible plane curves of degree d with δ nodes is a quasiprojective variety of dimension $3d + g - 1$. It follows that for a fixed $d > 0, g \geq 0$, the numbers $N_d(g)$ of curves passing through $3d + g - 1$ general points is finite and does not depend on the generic configuration of points chosen. This $N_d(g)$ number is commonly referred to as *Severi degree* of plane curves. The number $N_d(g)$ can be calculated classically by hand for small d. For instance, if $g = 0$, Euclid postulated that there is 1 curve of degree 1 through 2 points, Apollonius showed that there is a unique conic passing though 5 points in general position and a result of M. Chasles gives 12 rational cubics through 8 points in general position. H. Schubert established that there are 620 rational quartics passing through 11 general points in $\mathbb{P}^2$. In general, Kontsevich [FP97] proved a recursive formula for computing $N_d := N_d(0)$ for all d.

$$N_d = \sum_{\substack{d_i+d_2=d \\ d_1\ ,\ d_2>0}} N_{d_1}N_{d_2}\left(d_1^2\, d_2^2\binom{3d-4}{3d_1-2} - d_1^3\, d_2\binom{3d-1}{3d_1-2}\right). \tag{5.10}$$

Indeed starting from the Euclid's result $N_1 = 1$, from the formula (5.10) we can easily calculate

$$\begin{aligned} N_2 =& N_1^2 = 1, \\ N_3 =& N_1N_2\left(4\binom{5}{1} - 8\binom{5}{0}\right) + N_2N_1\left(4\binom{5}{1} - 2\binom{5}{3}\right) = (20-8)+0 = 12 \\ N_4 =& 620,\ N_5 = 87304,\ N_6 = 26312976,\ \ldots \text{and so on.} \end{aligned}$$

In general, fix integers $d > 0$ and $a, b, g \geq 0$. The number of irreducible curves of geometric genus g and degree d passing through a general points and tangent to b general lines in $\mathbb{P}^2$ is finite provided $a + b = 3d + g - 1$. These numbers are called *characteristic numbers* of plane curves and we denote them by $N_g(a, b)$. The question of calculating characteristic numbers is the *classical problem of Zeuthen* and thus we usually refer to the numbers $N_g(a, b)$ as *Zeuthen Numbers*. In [Zeu73], H.G. Zeuthen calculated the characteristic numbers of smooth curves in $\mathbb{P}^2$ of degree at most 4 and [Vak99] has verified Zeuthen's results by using modern methods on moduli spaces of stable maps, for an exposé see e.g. [FP97].

5.3.1 Homological interpretation of Zeuthen numbers

Let $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)$ be the Kontsevich moduli space of maps to $\mathbb{P}^2$ of fixed degree $d > 0$ and arithmetic genus $g \geq 0$. Consider the open substack of maps of smooth curves $\mathcal{M}_{g,0}(\mathbb{P}^2, d)$. The closure of $\mathcal{M}_{g,0}(\mathbb{P}^2, d)$ is a unique component of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)$ of dimension $3d + g - 1$ we denote by $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger$. The Zeuthen number $N_g(a, b)$ can be interpreted in the language of stable maps.

Let α and β denote the divisors in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger$ representing classes of a point and a line respectively. The characteristic number $N_g(a, b)$ is given by the degree of $\alpha^a\beta^b$ and is denoted by $\alpha^a\beta^b \cap \left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger\right]$. For example, it is known there is a unique smooth cubic through 9 general points, then we will write $\alpha^9 \cap \left[\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2, 3)^\dagger\right] = 1$.

The following existence result is the key point for this interpretation.

Proposition 5.3.1. *There exist two divisors α and β such that the number $N_g(a, d)$ is $\alpha^a\beta^b \cap \left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger\right]$.*

Proof. See [Vak98], Theorem 3.15. □

We finish with an open problem. As above, let $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger$ be the closure of the open substack $\mathcal{M}_{g,0}(\mathbb{P}^2, d)$ of maps of smooth curves of degree d. Among the boundary divisors representing the closure of loci of maps (see [Vak98] for precise descriptions) of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger$, we have a divisor $\mathbb{I}_d$ which is the closure of the locus of degree $d : 1$ maps of smooth curves of degree d into a line in $\mathbb{P}^2$. Such generic maps are necessarily branched at $d(d-1)$ points by Riemann-Hurwitz formula. Thus the divisor $\mathbb{I}_d$ enumerates a special class of Zeuthen numbers whose calculation is related to that of Hurwitz numbers. Namely, the Zeuthen numbers $\beta^{3d+g-2}[\mathbb{I}_d]$ for $g = \binom{d-1}{2}$. For instance, R. Vakil in [Vak99] calculates that $\beta^8[\mathbb{I}_3] = 40 \times 210$ and $\beta^{13}[\mathbb{I}_4] = 120 \cdot 2535$. It makes sense to consider the divisor $\mathbb{I}_d$ up to the $\mathcal{G}_p$-action.

Open problem: Consider the orbit space $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger/\mathcal{G}_p$. Is there a natural homology class

$$\beta \in \mathbf{H}_{2(3d+g-4)}\big(\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger/\mathcal{G}_p, \mathbb{Q})$$

such that $\mathfrak{h}_d = \beta^{3d+g-5} \cap \big[\mathbb{I}_d/\mathcal{G}_p\big]$?

Chapter 6

Planarity Stratification of Hurwitz Spaces

In this chapter, we show that every nonconstant meromorphic function on a nonsingular complex projective algebraic curve can be represented as a composition of a birational map of this curve to $\mathbb{P}^2$ and a projection of the image curve from an appropriate point $p \in \mathbb{P}^2$ to the pencil of lines through p. Then we introduce a natural stratification of Hurwitz spaces according to the minimal degree of a plane curve such that a given meromorphic function can be represented in the above way and calculate the dimensions of these strata. We observe that they are closely related to a family of Severi varieties studied earlier by J. Harris [Har86], Z. Ran [Ra89] and I. Tyomkin [Tyo07].

6.1 Basic definitions and facts

In what follows by a genus $p_g(\mathcal{C})$ of a (singular) curve $\mathcal{C}$ we mean its geometric genus, i.e. the genus of its normalization. We start with the following statement.

Proposition 6.1.1. *Every nonconstant meromorphic function $f : \mathcal{C} \longrightarrow \mathbb{P}^1$ on a smooth complex projective curve $\mathcal{C}$ can be represented as $f = \pi_p \circ \nu$ where $\nu : \mathcal{C} \longrightarrow \mathbb{P}^2$ is a birational mapping of $\mathcal{C}$ to its image and $\pi_p : \nu(\mathcal{C}) \longrightarrow \mathbb{P}^1$ is the projection of the image curve $\nu(\mathcal{C})$ from a point $p \in \mathbb{P}^2$ to the pencil of lines through p.*

Proof. Let $\mathcal{M}(\mathcal{C})$ be the field of meromorphic functions on $\mathcal{C}$. Consider its subfield $\mathbb{C}(f) \subset \mathcal{M}(\mathcal{C})$ of rational expressions in f. Let $\left[\mathcal{M}(\mathcal{C}) : \mathbb{C}(f)\right]$ the dimension of $\mathcal{M}(\mathcal{C})$ over $\mathbb{C}(f)$. Since $\mathcal{C}$ is one-dimensional the field extension $\left[\mathcal{M}(\mathcal{C}) : \mathbb{C}(f)\right]$ is finite. Choose any meromorphic function $g : \mathcal{C} \longrightarrow \mathbb{P}^1$ generating this extension. Removing a point from $\mathbb{P}^1$ and its inverse images under f and g, we get a birational mapping $\mathcal{C} \setminus \{\text{finite set}\} \longrightarrow \mathbb{C}^2$ given by the pair (f, g). Its compactification gives a birational mapping $\nu : \mathcal{C} \longrightarrow \mathbb{P}^2$. Projection "along the second coordinate" gives a presentation of the original meromorphic function $f : \mathcal{C} \longrightarrow \mathbb{P}^1$ as $f = \pi_p \circ \nu$. □

Obviously if ν maps $\mathcal{C}$ birationally on its image and $f = \pi_p \circ \nu$ for some point $p \in \mathbb{P}^2$, then $\deg(\nu(\mathcal{C})) = \deg f$ if and only if $p \notin \nu(\mathcal{C})$ and $\deg(\nu(\mathcal{C})) > \deg f$ if $p \in \nu(\mathcal{C})$.

Definition 6.1.1. *The* **planarity defect** *pdef*(f) *of a meromorphic function* $f:\mathcal{C}\longrightarrow\mathbb{P}^1$ *equals*

$$pdef(f) := \min_{\nu}(\deg(\nu(\mathcal{C})) - \deg(f)$$

such that $f=\pi_p\circ\nu$ *as above.*

We have the following simple observation.

Lemma 6.1.1. *Given* $f:\mathcal{C}\longrightarrow\mathbb{P}^1$, *then pdef*$(f)=0$ *if and only if* $h^0(f^*\mathcal{O}_{\mathbb{P}^1}(1))\geq 3$, *and for almost any point* $p\in\mathcal{C}$ *and any other point* $q\neq p$,

$$h^0\big(f^*\mathcal{O}_{\mathbb{P}^1}(1)-p-q\big)=h^0\big(f^*\mathcal{O}_{\mathbb{P}^1}(1)\big)-2.$$

Proof. Indeed, observe that f determines a linear subsystem in the complete linear system $f^*\mathcal{O}_{\mathbb{P}^1}(1)$. Moreover, if $r_f=h^0\big(f^*\mathcal{O}_{\mathbb{P}^1}(1)\big)\geq 3$ then this linear system defines a map $\phi_f:\mathcal{C}\longrightarrow\mathbb{P}^{r_f-1}$, with $r_f-1\geq 2$. If additionally, sections of $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ separate each generic point on $\mathcal{C}$ from all other points then ϕ_f is birational on the image. The latter condition is made explicit above. Choosing an appropriate 3-dimensional subsystem of $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ including f, we get the required statement. □

Unfortunately, the second condition is not easy to check in concrete situations, see Remark below. We say that a linear system $\mathcal{L}$ on a curve $\mathcal{C}$ is **birationally very ample** if the image of $\mathcal{C}$ in the projectivized space of its sections is birationally equivalent to $\mathcal{C}$, cf. [Ohb97].

The following sufficient condition of the birational very ampleness of $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ is valid.

Lemma 6.1.2. *If* $f:\mathcal{C}\longrightarrow\mathbb{P}^1$ *has at most one complicated branching point, then pdef*$(f)=0$ *if and only if* $h^0\big(f^*\mathcal{O}_{\mathbb{P}^1}(1)\big)\geq 3$. *In particular, under the above assumptions, if* $\deg(f)=d\geq g+2$ *where* g *is the genus of* $\mathcal{C}$ *then pdef*$(f)=0$.

Proof. As in Lemma 6.1.1, the necessary condition for $pdef(f)=0$ is that we have $r_f=h^0\big(f^*\mathcal{O}_{\mathbb{P}^1}(1)\big)\geq 3$. By Riemann-Roch's formula

$$r_f := h^0\big(f^*\mathcal{O}_{\mathbb{P}^1}(1)\big)=d-g+1+h^0\big(K-(f)_\infty\big), \tag{6.1}$$

where $(f)_\infty$ is the pole divisor of f. The linear system $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ determines the mapping $\phi_f:\mathcal{C}\longrightarrow\mathbb{P}^{r_f-1}$. Moreover if $r_f\geq 3$ and f has at most one complicated branching point, then ϕ_f defines a birational mapping of $\mathcal{C}$ on its image $\phi_f(\mathcal{C})$. Indeed, since $r_f\geq 3$ the only thing that we have to exclude is that $\phi_f:\mathcal{C}\longrightarrow\phi_f(\mathcal{C})$ is a non-trivial covering. Assume that $\phi_f:\mathcal{C}\longrightarrow\phi_f(\mathcal{C})$ is a non-trivial covering. Notice that independently of the fact whether ϕ_f is birational on the image or not, $f=\pi_p\circ\phi_f$ where π_p is a projection of $\mathbb{P}^2\setminus p\longrightarrow\mathbb{P}^1$ from some point $p\in\mathbb{P}^2$. Also the map f can be lifted in the standard way to $f=\tilde{\pi}_p\circ\tilde{\phi}_f$ where $\tilde{\phi}_f:\mathcal{C}\longrightarrow\tilde{\phi}_f(\mathcal{C})$ is the standard lift of ϕ_f to the normalization $\tilde{\phi}_f(\mathcal{C})$ of the image $\phi_f(\mathcal{C})$, and $\tilde{\pi}_p$ is the composition of the standard map from the normalization $\tilde{\phi}_f(\mathcal{C})$ to the image curve $\phi_f(\mathcal{C})$ with the projection π_p. Branching points of f are either the images under $\tilde{\pi}_p$ of the branching points of $\tilde{\phi}_f$ or the branching points of $\tilde{\pi}_p$ itself. But each branching point of $\tilde{\pi}_p$ is a non-simple branching point of f. Contradiction. The case when $\phi_f(\mathcal{C})$ is a line in $\mathbb{P}^2$ is obviously impossible due to the dimension of the linear system $f^*\mathcal{O}_{\mathbb{P}^1}(1)$. Finally observe that if $d\geq g+2$ then r_f is at least 3 by Riemann-Roch's formula (6.1). □

Remark 6.1.1. *Observe that for $d \geq g+1$, any curve $\mathcal{C}$ of genus g admits a meromorphic function of degree d with all simple branching points, i.e. the natural map $\mathcal{H}_{g,d} \longrightarrow \mathcal{M}_g$ where $\mathcal{M}_g$ is the moduli space of curves of genus g is surjective, see [Sev21]. Also for $d \geq 2g+1$, no genericity assumptions whatsoever on f are required for birational ampleness since $f^*(\mathcal{O}_{\mathbb{P}^1}(1))$ becomes very ample and defines an embedding $\mathcal{C} \longrightarrow \mathbb{P}^{r_f - 1}$. However in the interval $g+2 \leq d \leq 2g$ this linear system might define a non-trivial covering on the image as shown by the next classical example, see Proposition 5.3 in [Har77]. This circumstance shows that one needs some additional assumption on the branching points to avoid such coverings.*

Example 6.1

Let $\mathcal{C}$ be a hyperelliptic curve of genus $g > 2$ and let $|L| : \mathcal{C} \longrightarrow \mathbb{P}^1$ be the hyperelliptic map. Let s_0 and s_1 be a basis for $\mathbf{H}^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(L))$. Riemann-Roch's formula gives that $h^0(gL) = g+1 < 2g$. Note that there are precisely $\binom{d+n-1}{n-1}$ monomials of degree d in n variables. Therefore there are precisely $d+1$ monomials of degree d in s_0 and s_1. The map $|\mathscr{L}| : \mathcal{C} \longrightarrow \mathbb{P}^1$ is given by $\mathcal{C} \ni p \mapsto [s_0(p) : s_1(p)] \in \mathbb{P}^1$, while the map $|mL| : \mathcal{C} \longrightarrow \mathbb{P}^g$ is given by

$$p \mapsto [s_0(p)^g : s_0(p)^{g-1} s_1(p) : \cdots : s_1(p)^g].$$

But it is now clear that $|mL| : \mathcal{C} \longrightarrow \mathbb{P}^g$ can be factored as $|L| : \mathcal{C} \longrightarrow \mathbb{P}^1$ followed by the Veronese embedding $V : \mathbb{P}^1 \longrightarrow \mathbb{P}^g$. Hence, the image of $\mathcal{C}$ under the map $|mL|$ is a rational normal curve. Now suppose that $m > g$. Then Riemann-Roch's formula gives $h^0(mL) = 2m1 > m+1$. Thus, s_0 and s_1 only generate a subspace of $\mathbf{H}^0(C, \mathcal{O}(mL))$ and the above argument no longer works (which is good since $|mL|$ determines a closed embedding).

We now characterize the vanishing of the planarity defect in different terms. Consider the push-forward sheaf $f_* \mathcal{O}_{\mathcal{C}}$ on $\mathbb{P}^1$. Since f is a finite map of compact curves, $f_* \mathcal{O}_{\mathcal{C}}$ is a vectorbundle on $\mathbb{P}^1$ whose dimension equals $\deg(f)$. By the well-known result of Grothendieck, $f_* \mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathbb{P}^1} \oplus \sum_i \mathcal{O}_{\mathbb{P}^1}(a_i)$, where a_i are integers see e.g. [HM82b]. Observe that all a_i must be negative since $h^0(\mathcal{O}_{\mathcal{C}}) = h^0(f_* \mathcal{O}_{\mathcal{C}}) = 1$.

Proposition 6.1.2. *For any meromorphic function $f : \mathcal{C} \longrightarrow \mathbb{P}^1$ with at most one complicated branching point, its planarity defect pdef(f) vanishes if and only if $a_{max} = -1$, where a_{max} is the maximal of all a_i's in the above notation.*

Proof. Let us show that under our assumptions pdef(f) $= 0 \leftrightarrow a_{max} = -1$. We need to check that $h^0\big(f^* \mathcal{O}_{\mathbb{P}^1}(1)\big) \geq 3$ if and only if $a_{max} = -1$. Consider $f_*(f^* \mathcal{O}_{\mathbb{P}^1}(1))$. Observe that, $h^0\big(f_*(f^* \mathcal{O}_{\mathbb{P}^1}(1)\big) = h^0\big(f^* \mathcal{O}_{\mathbb{P}^2}(1)\big)$ since f is a finite map of compact algebraic curves. Now by projection formula, see Ex. 8.3 in [Har77]

$$f_*\big(f^* \mathcal{O}_{\mathbb{P}^2}(1)\big) = \mathcal{O}_{\mathbb{P}^1}(1) \otimes f_*(\mathcal{O}_{\mathcal{C}}) = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \sum_i \mathcal{O}_{\mathbb{P}^1}(a_i + 1).$$

Since $a_{max} = -1$ then at least one of the terms $\mathcal{O}_{\mathbb{P}^1}(a_i + 1)$ equals $\mathcal{O}$. Therefore

$$h^0\big(f_*(f^* \mathcal{O}_{\mathbb{P}^1}(1)\big) = h^0\big(f^* \mathcal{O}_{\mathbb{P}^1}(1)\big) + \sum_i h^0\big(\mathcal{O}_{\mathbb{P}^1}(a_i + 1)\big) \geq 2 + 1.$$

In fact, $h^0\big(f_*(f^* \mathcal{O}_{\mathbb{P}^1}(1)\big) = 2 +$ the number of indices i such that$a_i = -1$. □

Proposition 6.1.2 shows that there is a connection of the planarity defect with the slope invariants of meromorphic functions and with the Maroni strata, cf. [DP14] and [Pat13]. In fact, the following statement is true.

Proposition 6.1.3. *Given a meromorphic function $f : \mathcal{C} \longrightarrow \mathbb{P}^1$ of degree d, its planarity defect $pdef(f)$ equals $d' - d$ where d' is the minimal degree of a linear system $\mathcal{L}$ such that*

(i) $\mathcal{L}$ is birationally very ample;

(ii) the (effective) divisor of $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ is contained in the (effective) divisor of $\mathcal{L}$.

Proof. If $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ can serve as $\mathcal{L}$ then there is nothing to prove. Otherwise the divisor of $\mathcal{L}$ must be strictly larger than that of $f^*\mathcal{O}_{\mathbb{P}^1}(1)$. In the latter case one can choose a 1-dimensional linear subsystem of $\mathcal{L}$ defining a meromorphic function $g : \mathcal{C} \longrightarrow \mathbb{P}^1$ which is not proportional to f. Consider the map $\psi : \mathcal{C} \longrightarrow \mathbb{C}^2$ given by (f, g) and extending it to the map $\tilde{\psi} : \mathcal{C} \longrightarrow \mathbb{P}^2$ we get the required planarity defect. □

6.1.1 Planarity stratification of small Hurwitz spaces

Recall that the small Hurwitz space of degree d functions of genus g curves is defined as:

$$\mathcal{H}_{g,d} = \left\{ f : \mathcal{C} \longrightarrow \mathbb{P}^1 \;\middle|\; \begin{array}{l} \mathcal{C} \text{ has genus } g \geq 0 \text{ and } f \text{ is a branched covering} \\ \text{of degree } d \geq 2 \text{ with only simple branch points} \end{array} \right\}.$$

Also recall that $\dim \mathcal{H}_{g,d}$ equals the number of branching points of a function from $\mathcal{H}_{g,d}$ and is given by the formula

$$\dim \mathcal{H}_{g,d} = 2d + 2g - 2.$$

Proposition 6.1.1 allows us to introduce the *planarity stratification* of $\mathcal{H}_{g,d}$:

$$\mathcal{H}_{g,d}^{m(g,d)} \subset \mathcal{H}_{g,d}^{m(g,d)+1} \subset \cdots \subset \mathcal{H}_{g,d}^{M(g,d)} = \mathcal{H}_{g,d}, \tag{6.2}$$

where $\mathcal{H}_{g,d}^l$ consists of all meromorphic functions in $\mathcal{H}_{g,d}$ whose planarity defect does not exceed l. We present some information about this stratification.

Proposition 6.1.4. *For any pair (g, d) where $g \geq 0$ and $d \geq 2$,*

$$m(g,d) = \min_{l \geq 0} \binom{d+l-1}{2} - \binom{l}{2} \geq g. \tag{6.3}$$

which gives

$$m(g,d) = \max\left(0, \left\lceil \frac{g - \binom{d-1}{2}}{d-1} \right\rceil\right). \tag{6.4}$$

Moreover the following result holds.

Theorem 6.1.1. *In the above notation, given g, d and $l \geq m(g,d)$, the stratum $\mathcal{H}_{g,d}^l$ is irreducible and its dimension is given by:*

$$\dim \mathcal{H}_{g,d}^l = \min(3d + g + 2l - 4, 2d + 2g - 2). \tag{6.5}$$

The substantial part of the proof of Theorem 6.1.1 consists of the following generalization of the famous result by J. Harris [Har77] showing that the space of plane curves of genus g and degree d where $g \le \binom{d-1}{2}$ is an irreducible variety whose dense subset consists of nodal curves of genus g (irreducibility of Severi's varieties). Fixing as above a point $p \in \mathbb{P}^2$, denote by $S_{g,d,l}$ the variety of reduced irreducible plane curves of degree d having genus g and order l at p, where $g \le \binom{d+l-1}{2} - \binom{l}{2}$. (The order of a plane curve at a given point is the multiplciity of its local intersection at p with a generic line passing through p.) Denote by $W_{g,d,l} \subset S_{g,d,l}$ its subset consisting of curves having an ordinary singularity of order l at p (i.e. transversal intersection of l smooth local branches) and only usual nodes outside p.

Theorem 6.1.2.

(i) $W_{g,d,l}$ *is a smooth manifold of dimensional* $3d + g + 2l - 1$*;*

(ii) $W_{g,d,l}$ *is dense in* $S_{g,d,l}$*;*

(iii) $S_{g,d,l}$ *is irreducible.*

The main result of [Har77] is the proof of the same statement in the basic case $l = 0$. Theorem 6.1.2 follows from already known results of Z. Ran [Ra89] and I. Tyomkin [Tyo07]. We first prove Proposition 6.1.4 and Theorem 6.1.2 and then Theorem 6.1.1.

Lemma 6.1.3. *The genus of a plane curve decreases by at least* $\binom{l}{2}$ *by a singularity of order* l*. Moreover the ordinary singularity of order* l *decreases the genus by exactly* $\binom{l}{2}$*.*

Proof. The following algorithm describes by which number the genus of a plane curve of degree d is decreased due to a singularity of order l.

Step 1. Subtract $\binom{l}{2}$ from $\binom{d-1}{2}$.

Step 2. Blow up the singularity in the plane. The strict transform of the curve will intersect the exceptional divisor at l points (counting multiplicities). If each of these (geometrically distinct) points is smooth on the strict transform then the genus drops by exactly $\binom{l}{2}$.

Step 3. If among the latter points there exist singular we have to repeat the previous step, i.e. if the order of singularity is s then we decrease the genus by $\binom{s}{2}$, then we blow up this point etc. After finitely many such steps the curve becomes smooth. (Further blow-ups will not change the genus). Thus the minimal decrease of genus equals $\binom{l}{2}$.

□

Proof of Proposition 6.1.4. The necessity of (6.3) is obvious. Indeed we need to construct a plane curve of degree $d + l$ such that it has a singularity of order l at p (so that projection from p will be a covering of degree d) and has a genus of normalization equal to g. Having a singularity of order l at p decreases the genus by at least $\binom{l}{2}$ compared to $\binom{d+l-1}{2}$ which is the genus of a smooth curve of degree $d + l$, see Lemma 6.1.3 above. Thus the inequality (6.3) must be satisfied. To show that the least value of l satisfying (6.3) is enough consider first a configuration of l generic lines through p and additionally d lines in $\mathbb{P}^2$ in general position. This curve has genus 0. A slight deformation of this curve by a polynomial vanishing up to order $l+1$ at p will resolve all nodes outside p and given $g = \min_{l \ge 0} \binom{d+l-1}{2} - \binom{l}{2}$.

A more careful deformation will resolve any number of nodes between 0 and $\binom{d}{2}$, see the proof of Theorem 6.1.2 below. The classical case $g \leq \binom{d-1}{2}$ is well presented in [HM98], Appendix E and the general case in [Ra89]. □

We will need some information about the Hirzebruch surfaces and Severi varieties on them. For a given non-negative integer n, let $\Sigma_n = \mathbf{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ be the n^{th}-Hirzebruch surface and let $\kappa : \Sigma_n \longrightarrow \mathbb{P}^1$ be the natural projection. Consider two non-zero sections $(1,0),(0,\sigma) \in \mathbf{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. They define the maps

$$\mathbb{P}^1 \setminus \mathbb{V}(\sigma) \longrightarrow \Sigma_n,$$

where $\mathbb{V}(\sigma)$ is the zero locus of σ. We denote the closures of the images of these maps by L_0 and L_∞, respectively. (It is clear that the homological class of L_∞ is independent of the choice of σ.) The following facts are standard.

Proposition 6.1.5.

(i) The Picard group $\mathrm{Pic}(\Sigma_n) = \mathbf{H}_2(\Sigma_n, \mathbb{Z})$ *is a free abelian group* $\mathbb{Z} \times \mathbb{Z}$ *generated by the classes* F *and* L_∞, *where* F *denotes the fiber of projection* κ. *(Observe that* $L_0 = nF + L_\infty$.*)*

(ii) The intersection form on $\mathrm{Pic}(\Sigma_n)$ *is given by* $F^2 = 0, L_\infty^2 = -n$ *and* $F \cdot L_\infty = 1$.

(iii) Every effective divisor $M \in \mathrm{Div}(\Sigma_n)$ *is linearly equivalent to a linear combination of* F *and* L_∞ *with non-negative coefficients. Moreover, if* M *does not contain* L_∞, *then it is linearly equivalent to a combination of* F *and* L_0 *with non-negative coefficients.*

(iv) The canonical class is given by:

$$K_{\Sigma_n} = -(2L_\infty + (2+n)F) = -(L_0 + L_\infty + 2F).$$

(v) Every smooth curve $\mathcal{C}$ *with the class* $dL_0 + kF$ *has genus* $g(\mathcal{C}) = \frac{(d-1)(dn+2k-2)}{2}$.

Let g, d, k be non-negative integers. We define the Severi variety $V_{g,d,k} \subseteq |\mathcal{O}_{\Sigma_n}(dL_0 + kF)|$ to be the closure of the locus of reduced nodal curves of genus g which do not contain L_∞, and we define $V_{g,d,k}^{irr} \subset V_{g,d,k}$ to be the union of the irreducible components whose generic points correspond to irreducible curves.

The main result of [Tyo07] (see Theorem 3.1 there) is as follows.

Theorem 6.1.3. *For any triple* g, d, k *of non-negative integers, the variety* $V_{g,d,k}^{irr} \subset V_{g,d,k}$ *(if non-empty) is irreducible and of expected dimension.*

PROOF OF THEOREM 6.1.2. Let us first naively count the expected dimension of $S_{g,d,l}$. Indeed, the dimension of the space $S_{g,d,l}$ of plane curves of degree $d+l$ with a singularity at p of order l equals $\frac{(d+l)(d+l+3)}{2} - \binom{l+1}{2}$. The number of nodes on such a curve under the assumptions that it has genus g equals

$$\delta = \binom{d+l-1}{2} - \binom{l}{2} - g. \tag{6.6}$$

Assuming that each node decreases the dimension by 1 we get

$$\exp\dim S_{g,d,l} = 3d + g + 2l - 1.$$

We finish our proof with a reference to Theorem 6.1.3. Indeed, if one blows up the projective plane at a point $p \in \mathbb{P}^2$ then one gets the first Hirzebruch surface Σ_1. Observe that plane curves of degree $d+l$ having a singularity of order l at p will after the blow-up lie in the class $(d+l)L_0 - lL_\infty = dL_0 + lF$. Therefore the above set $W_{g,d,l}$ of irreducible plane curves having the singularity of order l at the point p after this blow-up will transform into the space $V^{irr}_{g,d,l}$ in the above notation. (We consider only the strict transform of each curve disregarding the exceptional divisor.) Thus by the latter Theorem $S_{g,d,l}$ is irreducible and of expected dimension. Another proof of essentially the same result directly in the plane $\mathbb{P}^2$ can be found in [Ra89], see Irreducibility Theorem on p. 122. □

Proof of Theorem 6.1.1. To settle Theorem 6.1.1 we need to prove an analog of Proposition 5.2.1 or a weaker statement that such curves equivalent as coverings do not appear in families of $\mathcal{G}_p$-orbits. If this is true then $\dim \mathcal{H}^l_{g,d} = \dim S_{d,l,g} - 3$. We need the following Proposition.

Let $S(d,l,g)$ be the Severi variety of all plane curves of degree $d+l$, genus g and ordinary singularity of order l at point p. Let $H(g,d)$ be the Hurwitz space of all branched coverings of degree d and genus g. Let $br : S(d,l,g) \longrightarrow H(g,d)$ be the branching morphism sending each plane curve from $S(d,l,g)$ to the branched covering from its normalization to $\mathbb{P}^1$ obtained by projection from the point p.

Proposition 6.1.6. *The dimension of the fiber of the above map at the curve N obtained by normalization of a generic curve $\mathcal{C}$ from $S(d,l,g)$ equals $h^0(N,\mathcal{O}_N(E))$ where E is the divisor of degree $d+2l$ on N obtained as the pull-back of projection point p together with the pull-back of the general line section of $\mathcal{C}$. (For an arbitrary curve $\mathcal{C} \in S(d,l,g)$ the dimension of the fiber is at most $h^0(N,\mathcal{O}_N(E))$.)*

Proof. Let $\pi : \Sigma_1 \longrightarrow \mathbb{P}^2$ be the standard projection of the first Hirzebruch surface Σ_1 obtained by the blow-up of the point p to $\mathbb{P}^2$. We have natural maps

$$\begin{array}{ccc} N & \xrightarrow{h} & \Sigma_1 \\ & {\scriptstyle f}\searrow & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

and exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_N & \longrightarrow & h^*T_{\Sigma_1} & \longrightarrow & N_h & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_N & \longrightarrow & f^*T_{\mathbb{P}^1} & \longrightarrow & N_f & \longrightarrow & 0. \end{array} \qquad (6.7)$$

It is known that $\mathrm{Def}^1(N,h) = H^0(N,N_h)$ and $\mathrm{Def}^1(N,f) = H^0(N,N_f)$ are the tangent spaces to the space of deformations of the pairs (N,h) and (N,f) resp. The first one is the tangent space to the Severi variety if h is an immersion; the second one is the tangent space to the Hurwitz space. The sequence (6.7) implies that the kernels $\alpha : h^*T_{\Sigma_1} \longrightarrow f^*T_{\mathbb{P}^1}$ and $N_h \longrightarrow N_f$ coincide since $h^*T_{\Sigma_1} \twoheadrightarrow f^*T_{\mathbb{P}^1}$. Since the $\mathbb{P}^1$-bundle $\Sigma_1 \longrightarrow \mathbb{P}^1$ admits two non-intersecting sections (the line L and the inverse image of p in Σ_1) then $\ker\alpha = h^*\mathcal{O}_{\Sigma_1}(L + \pi^{-1}(p))$. □

For small number of nodes compared to the degree of the irreducible plane curve Theorem 6.1.1 is immediate from the following fact, see Exercise 20 (iii) of § 1, Appendix A, Chapter 1 of [ACGH85]. (Moreover a stronger statement is valid.) It claims that if the number δ of nodes of an irreducible plane nodal curve $\Gamma \subset \mathbb{P}^2$ of degree d satisfies the inequality $\delta < d-3$ then the linear system g^2_d cut out on Γ by lines is complete and unique on the normalization $\mathcal{C}$ of Γ. This fact immediately implies that under the above assumptions two plane curves whose normalizations are isomorphic will be projectively equivalent. Then for degree at least 4 it will be straight-forward that if the isomorphism of their normalizations is induced by the equivalence of the meromorphic functions obtained by projection from the same point p, then the projective transformation realizing this equivalence belongs to $\mathcal{G}_p$, see the proof of Proposition 5.2.1. In general, one should show that for a generic curve in $S(d,l,g)$, one has $h^0(N, \mathcal{O}_N(E)) = 3$. This fact is also valid and will appear in a forth-coming publication [ST14].

Corollary 6.1.1. *Given g, d as above,*

$$M(g,d) = \max\left(0, \left\lceil \frac{g-d+2}{2} \right\rceil\right). \tag{6.8}$$

In particular, $m(g,d) = M(g,d) = 0$ if and only if $d \geq g+2$.

Proof. From Theorem 6.1.1 is follows that $M(g,d)$ equals the minimal non-negative integer l for which

$$3d + g + 2l - 4 \geq 2d + 2g - 2 \iff 2l \geq g - d + 2.$$

The latter inequality implies that $M(g,d) = \max\left(0, \left\lceil \frac{g-d+2}{2} \right\rceil\right)$. This formula for $M(g,d)$ gives that $M(g,d) = 0$ if and only if $d \geq g+2$. □

Corollary 6.1.2. *The planarity stratification of $\mathcal{H}_{g,d}$ consists of one term in the following two cases. Either $d \geq g+2$ in which case the planarity defect vanishes, or $d = 3$ in which case the planarity defect equals $\lceil \frac{g-1}{2} \rceil$.*

Proof. We have that $\mathcal{H}_{g,d}$ consists of one term if and only if $m(g,d) = M(g,d)$. By Proposition 6.1.4 and Theorem 6.1.1 (unless $M(g,d)$ vanishes which happens if and only if $d \geq g+2$) this corresponds to the case when

$$\left\lceil \frac{g - \binom{d-1}{2}}{d-1} \right\rceil = \left\lceil \frac{g-d+2}{2} \right\rceil.$$

If $d > 3$ then the denominator of the left-hand side is smaller than that of the right-hand side and the numerator of the left-hand side is bigger than that of the right-hand side which means that the equality never holds. For $d = 3$ the left-hand side and the right-hand side coincide giving the planarity defect equal to $\lceil \frac{g-1}{2} \rceil$. □

6.1.2 Stratification of Hurwitz spaces with one complicated branching point

Analogously to the above, given a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n) \vdash d$ of positive integer d, denote by

$$\mathcal{H}_{g,\mu} = \left\{ f : \mathcal{C} \longrightarrow \mathbb{P}^1 \;\middle|\; \begin{array}{l} \mathcal{C} \text{ has genus } g \geq 0 \text{ and } f \text{ has all simple branched points} \\ \text{except at } \infty \text{ whose profile is given by } \mu \vdash \deg f = d \geq 2 \end{array} \right\}$$

the Hurwitz space of all degree d functions on genus g curves with one complicated branching point at ∞ having a given branch type μ. Recall that $\dim \mathcal{H}_{g,\mu}$ equals the number of simple branching points of a function from $\mathcal{H}_{g,\mu}$ and is given by the formula

$$\dim \mathcal{H}_{g,\mu} = 2d + 2g - 2 - \sum_{i=1}^{n} (\mu_i - 1).$$

Proposition 6.1.1 allows us to introduce the *planarity stratification* of $\mathcal{H}_{g,\mu}$:

$$\mathcal{H}_{g,\mu}^{m(g,\mu)} \subset \mathcal{H}_{g,\mu}^{m(g,\mu)+1} \subset \cdots \subset \mathcal{H}_{g,\mu}^{M(g,\mu)} = \mathcal{H}_{g,\mu}. \tag{6.9}$$

Here $\mathcal{H}_{g,\mu}^{l}$ consists of all meromorphic functions in $\mathcal{H}_{g,\mu}$ whose defect does not exceed l.

By Lemma 6.1.2, $M(g,\mu) \leq d+2$.

Proposition 6.1.7. *For any pair $(g, \mu \vdash d)$ where $g \geq 0$ and $d \geq 2$,*

$$m(g,\mu) = \min_{l \geq 0} \binom{d+l-1}{2} - \binom{l}{2} \geq g. \tag{6.10}$$

which gives

$$m(g,\mu) = \left\lceil \frac{g - \binom{d-1}{2}}{d-1} \right\rceil.$$

(Observe that $m(g,\mu) = m(g,d)$ given by (6.3).)

Proof. Since the stratum $\mathcal{H}_{g,u}^{m(g,\mu)}$ should lie at least in $\mathcal{H}_{g,d}^{m(g,d)}$ or, possibly in the higher strata of the planarity stratification of $\mathcal{H}_{g,d}$. Therefore $m(g,\mu)$ is at least equal to the minimal l given by the right-hand side of (6.10). The fact that $m(g,\mu)$ is exactly equal to the minimal l satisfying the latter condition is explained in the proof of Theorem 6.1.4. □

We have the following result above the dimensions of the strata of (6.9).

Theorem 6.1.4. *In the above notation, given g, d and $l \geq m(g,\mu)$, the stratum $\mathcal{H}_{g,\mu}^{l}$ is equidimensional and its dimension is given by:*

$$\dim \mathcal{H}_{g,\mu}^{l} = \min\left(3d + g + 2l - 4 - \sum_{i=1}^{n}(\mu_i - 1), 2d + 2g - 2 - \sum_{i=1}^{n}(\mu_i - 1)\right). \tag{6.11}$$

Proof. Theorem 6.1.4 follows directly from Lemmas 6.1.4 and 6.1.5. □

Fix a flag $p \in L_0 \subset \mathbb{P}^2$, positive integers g, d, l, and a partition $\mu \vdash d$. Consider the locus $V \subset |\mathcal{O}_{\mathbb{P}^2}(d+l)|$ of plane curves $\mathcal{C}$ such that:

(i) $\deg \mathcal{C} = d + l$;

(ii) $\mathcal{C}$ is reduced and irreducible;

(iii) $\text{mult}_p \mathcal{C} = l$;

(iv) $p_g(\mathcal{C}) = g$;

(v) $\kappa^{-1} L_0 = \sum_i \mu_i q_i$ where $\kappa : \tilde{\mathcal{C}} \longrightarrow \mathcal{C}$ is the normalization map.

Again let $\Sigma_1 = Bl_p \mathbb{P}^2$ be the first Hirzebruch surface obtained by the blow-up of $\mathbb{P}^2$ at p. Let $F_0 \subset \Sigma_1$ be the strict transform of L_0, and let F be the class of F_0. Denote by $L \subset \Sigma_1$ the class of the preimage of a general line in $\mathbb{P}^2$, and denote by $E \subset \Sigma_1$ the exceptional divisor. Then V can be identified with the locus of curves $\mathcal{C} \in |\mathcal{O}_{\Sigma_1}((d+l)L - lE| = |\mathcal{O}_{\Sigma_1}(dL + lF)|$ such that i). $\mathcal{C}$ is reduced and irreducible; ii). $p_g(\mathcal{C}) = g$; iii). $\kappa^{-1} F_0 = \sum_i \mu_i q_i$. Let $V_1 \subset V$ be an irreducible component of V.

Lemma 6.1.4. $\dim V_1 \geq \exp\dim := -K_{\Sigma_1} \cdot \mathcal{C} + g - 1 - \sum_{i=1}^n (\mu_i - 1)$.

Proof. Let $o \in V_1$ be a general point, $\mathcal{C}_o$ be the corresponding curve. By [KS12] Lemma A.3 there exists a neighborhood W of $o \in V_1$ over which the family $\mathcal{C}_W \longrightarrow W$ is equinormalizable, i.e. if $\widetilde{\mathcal{C}}_W \longrightarrow \mathcal{C}_W$ is the normalization then $\forall a \in W, (\widetilde{\mathcal{C}}_W)_a \longrightarrow (\mathcal{C}_W)_a = \mathcal{C}_a$ is the normalization. Thus $\dim V_1$ is equal to the dimension of (a component of) the deformation space of $f : \widetilde{\mathcal{C}}_0 \longrightarrow \Sigma_1$ satisfying condition (iii). Notice that condition (iii) has codimension $\leq \sum_{i=1}^n (\mu_i - 1)$ in the space of all deformations of the pair $(\tilde{\mathcal{C}}_0, f_0)$. Thus, it suffices to show that (any component of) $Def(\tilde{\mathcal{C}}_0, f_0)$ has dimension at least $-K_{\Sigma_1} \cdot \mathcal{C} + g - 1$. By the standard deformation theory any component of the latter space has dimension $\geq \dim Def'(\tilde{\mathcal{C}}_0, f_0) - \dim Ob(\tilde{\mathcal{C}}_0, f_0)$. In our case $Def'(\tilde{\mathcal{C}}_0, f_0) = H^0(\tilde{\mathcal{C}}_0, N_{f_0})$ and $Ob(\tilde{\mathcal{C}}_0, f_0) = H^1(\tilde{\mathcal{C}}_0, N_{f_0})$ where N_{f_0} is the normal sheaf of f_0, i.e. $N_{f_0} = Coker(T_{\tilde{\mathcal{C}}_0} \longrightarrow f_0^* T_{\Sigma_1})$. This implies the statement since $h^0(\tilde{\mathcal{C}}_0, N_{f_0}) - h^1(\tilde{\mathcal{C}}_0, N_{f_0}) = \chi(\tilde{\mathcal{C}}_0, N_{f_0}) = -K_{\Sigma_1} \cdot \mathcal{C} + g - 1$ by Riemann-Roch's theorem. □

Lemma 6.1.5. $\dim V_1 \leq \exp\dim V_1$.

Proof. If $\dim V_1 > \exp\dim$ then there exists a configuration of r points on F_0 such that $\{\mathcal{C} \in V_1 | \mathcal{C} \cap F_0 = \text{given configuration}\}$ has dimension greater than $-K_{\Sigma_1} \cdot \mathcal{C} + g - 1 - \sum_{i=1}^n (\mu_i - 1) - n = -K_{\Sigma_1} \cdot \mathcal{C} + g - 1 - F_0 \cdot \mathcal{C}$, which is a contradiction with [Tyo07], Lemma 2.9. □

Corollary 6.1.3. *Given g, μ as above,*

$$M_{g,\mu} = \max\left(0, \left\lceil \frac{g - d + 2}{2} \right\rceil\right). \tag{6.12}$$

In particular, $m_{g,\mu} = M_{g,\mu} = 0$ if and only if $d = \sum_{i=1} \mu_i \geq g + 2$.

Proof. See the proof of Corollary 6.1.1.

□

Stratification (6.2) is (almost) the special case of (6.9) the difference being that one simple branching point is placed at ∞.

Remark. According to the information the authors obtained from I. Tyomkin one can prove that each stratum $\mathcal{H}^l_{g,\mu}$ is irreducible for $g = 0$ and $g = 1$, and hopefully for other genera if $\mu \vdash d$ is not very complicated. Whether $\mathcal{H}^l_{g,\mu}$ is irreducible for an arbitrary partition μ is unknown at present and might be a difficult problem.

6.2 Hurwitz numbers of the planarity stratification and Zeuthen-type problems

Due to irreducibility of strata of (6.2) and equidimensionality of strata of (6.9) we can introduce the corresponding notion of Hurwitz numbers related to these strata. Recall that the *branching morphism*

$$\delta_{g,d} : \mathcal{H}_{g,d} \longrightarrow \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1 \setminus \Delta \tag{6.13}$$

is by definition, the map sending a meromorphic function f to the unordered set of its branching points (which are distinct by definition). Here $\Delta \subset \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1$ is the hypersurface of unordered $(2d+2g-2)$-tuples of points in $\mathbb{P}^1$ where not all of them are pairwise distinct. It is well-known that $\delta_{g,d}$ is a finite covering and its degree $h_{g,d}$ is called the simple Hurwitz number. In particular, for $g=0$ the corresponding Hurwitz number $h_{0,d}$ equals $(2d-2)!d^{d-3}$. In general, however closed formulas for $h_{g,d}$ (as well as for many other Hurwitz numbers) are unknown. Analogously, the *branching morphism*

$$\delta_{g,\mu} : \mathcal{H}_{g,\mu} \longrightarrow \mathrm{Sym}^{w_\mu}\,\mathbb{P}^1 \setminus \Delta \tag{6.14}$$

is, by definition, the map sending a meromorphic function $f \in \mathcal{H}_{g,\mu}$ to the unordered set of its simple branching points (which are distinct by definition). Here $\Delta \subset \mathrm{Sym}^{w_\mu}\,\mathbb{P}^1$ is the hypersurface of unordered w_μ-tuples of points in $\mathbb{C}$ where not all of them are pairwise distinct, where $w_\mu = 2d+2g-2-\sum_{i=1}^n(\mu_i-1)$. It is well-known that $\delta_{g,\mu}$ is a finite covering and its degree $h_{g,\mu}$ is called the single Hurwitz number. In particular, for $g=0$ the corresponding Hurwitz number $h_{0,\mu}$ equals

$$(d+n-2)!\prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!}d^{n-3}.$$

Stratifications (6.2) – (6.9) allow to introduce Hurwitz numbers which take into account these filtrations. Before we introduce this notion in general, let us start with a motivating example.

Example 6.2

Fixing a point $p \in \mathbb{P}^2$, consider the space $S_{d,p}$ of all smooth plane curves of degree d not passing through p. Each such curve defines a branched covering of $\mathbb{P}^1$ of degree d. There exists a three-dimensional group $\mathcal{G}_p \subset \mathbb{P}\mathbf{GL}(3,\mathbb{C})$ of projective transformations preserving p as well as the pencil of lines through p. In other words, each line through p will be mapped to itself. Since $\mathcal{G}_p$ acts (locally) freely on $S_{d,p}$ for $d>1$ and curves from the same orbit define equivalent branched coverings of $\mathbb{P}^1$.

Denote by $\mathfrak{h}_d$ the number of different 3-dimensional orbits of the above action on the space $S_{d,p}$ with the same set of $d(d-1)$ tangent lines (e.g. branching points of the projection). For instance, we established in §5.2 that the numbers $\mathfrak{h}_2 = 1$, $\mathfrak{h}_3 = 40$ are the usual Hurwitz numbers for degree d and genus $\binom{d-1}{2}$. But starting with $d=4$ the situation changes. In the same section, we noticed that so far the only calculated non-trivial example is $d=4$ found in [Vak01], [Vak99] for which $\mathfrak{h}_4 = 120 \times \frac{(3^{10}-1)}{2}$. The numbers $\mathfrak{h}_d$ for $d>4$ are unknown at present.

Observe a straight-forward analogy of the calculation of $\mathfrak{h}_d$ with (a special case) of the classical Zeuthen's problem, see [Zeu73], [Alu92]. Namely, given integers $d \geq 2$ and $a,b,g \geq 0$ such that $a+b=3d+g-1$ define the number $N_g(a,b)$ as the number of smooth curves of degree d passing through a points in general position and tangent to b lines in general position. In [Zeu73] H. G. Zeuthen predicted these numbers for d up to 4. His predictions were rigorously proven only in the 90's, see [Alu92] and references therein. The above problem of calculation of $\mathfrak{h}_d$ is similar to Zeuthen's problem for $b=3d+g-1$. But instead of taking $3d+g-1$ generic lines we should take $(3d+g-1)-3$ generic lines through a given point p and count the number of 3-dimensional orbits under the action of $\mathcal{G}_p$.

Introduce the Hurwitz number $\mathfrak{h}^l_{g,\mu}$ as the degree of the restriction of the morphism $\delta_{g,\mu}$ to the (irreducible component of the) stratum $\mathcal{H}^l_{g,\mu}$ where $m(g,\mu)\le l\le M(g,\mu)$.

Definition 6.2.1 (Generalised plane Hurwitz numbers)**.** *Define the* **plane Hurwitz numbers** *as Hurwitz numbers restricted to irreducible components of the stratum* $\mathcal{H}^l_{g,\mu}$*.*

Notice that by definition, $\mathfrak{h}^{M(g,\mu)}_{g,\mu}=\mathfrak{h}_{g,\mu}$. Also the number $\mathfrak{h}_d$ introduced above equals $\mathfrak{h}^0_{(d-1)(d-2)/2,1^d}$. In our notation we can rewrite the plane Hurwitz numbers for $d\le 4$ as follows:

d	**Stratification of** $\mathcal{H}_{g,d}$	**Plane Hurwitz Numbers**
2	$\mathcal{H}^0_{0,2}=\mathcal{H}_{0,2}$	$\mathfrak{h}^0_{2,1^2}=\mathfrak{h}_{2,1^2}=1$,
3	$\mathcal{H}^0_{1,3}=\mathcal{H}_{1,3}$	$\mathfrak{h}^0_{3,1^3}=\mathfrak{h}_{3,1^2}=40$,
4	$\mathcal{H}^0_{3,4}\subset\mathcal{H}^1_{3,4}=\mathcal{H}_{3,4}$	$\mathfrak{h}^0_{3,1^4}=120\times\frac{3^{10}-1}{2}$, $\mathfrak{h}^1_{3,1^4}=255\times\frac{3^{10}-1}{2}$.

6.3 Final Remarks

I. It would be very interesting to prove/disprove the irreducibility of the strata $\mathcal{H}^l_{g,\mu}$.

II. It is important to develop tools helping for calculation of the Hurwitz numbers of $\mathcal{H}^l_{g,d}$ and/or $\mathcal{H}^l_{g,\mu}$ due to the fact that they are naturally related to Zeuthen-type problems. In the case of the usual single Hurwitz numbers there exists a standard combinatorial approach to the calculation of those which is not always very useful for practical computations but is very important theoretically. Other standard tools for the usual Hurwitz numbers are the cut-and-join equation, see e.g. [GJV09] and the ELSV-formula. It might be possible to find analogs of the latter tools by using an appropriate compactification of the above strata similar to those already existing in the literature.

III. Another approach to the calculation of the Hurwitz strata of the planarity filtration might come from the correspondence theorem in tropical algebraic geometry. Recently in [BBM04] the authors developed some tropical tools for finding the answers to a similar class of Zeuthen-type problems.

IV. Finally, we want to mention a recent preprint [BL13] which gives a criterion when meromorphic functions of degree d on a certain class of plane curves of degree d with only nodes and some additional non-degeneracy assumptions might be realized by a projection from a point outside the curve.

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