Hydromagnetic Steady Flow of Liquid Between Two Parallel Infinite Plates Under Applied Pressure Gradient when Upper Plate is Moving with Constant Velocity Under the Influence of Inclined Magnetic Field

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ABSTRACT: *This paper studies steady laminar flow of viscous incompressible fluid between two parallel infinite plates when a constant pressure gradient is imposed on the system and upper plate is also moving with constant velocity and lower plate is held stationary under the influence of inclined magnetic field. The Laplace transform method has been applied to solve governing equation. The analytical expression for fluid velocity at different strengths of magnetic field and at different inclinations has been shown graphically which shows that with the increase of inclination of magnetic field there is a decrease in velocity profile.*

INTRODUCTION

The interaction of two branches, namely electromagnetic theory and fluid mechanics produces magnetohydrodynamics. The basic concept describing magnetohydrodynamic (MHD) phenomena is as follows. Consider an electrically conducting fluid having a velocity vector $\mathbf{V}$. At right angles to this we apply a magnetic field, $\mathbf{B}$. We assume that steady flow conditions have been attained. Because of the interaction of two fields an electric field denoted by $\mathbf{E}$ is induced at right angles to both $\mathbf{V}$ and $\mathbf{B}$. This electric field is given by $\mathbf{E} = \mathbf{V} \times \mathbf{B}$. If we assume that the conducting fluid is isotropic, we can denote its electrical conductivity by the scalar quantity σ. By Ohms law, the density of the current $\mathbf{J}$ induced in the conducting fluid is given by $\mathbf{J} = \sigma \mathbf{E}$, i.e., for stationary condition. Simultaneously occurring with the induced current is the induced electromotive force $\mathbf{F} = \mathbf{J} \times \mathbf{B} \sin \alpha$, where α is angle of inclination of magnetic field with the horizontal.

The laminar flow of an electrically conducting fluid through a channel under uniform transverse magnetic field is important because of the use of MHD generator, the MHD pump and the electromagnetic flow meter. The general model that in normally considered in these studies consists of an infinitely long channel of constant cross – section with a uniform static magnetic field applied transverse to the axis of the channel. The walls of channel are either insulators, conductors or a combination of insulators and conductors depending on the intended application. For example, in the MHD generator and pump, the channel cross-section is normally circular with conducting walls.

Serclif (1956) has studied the steady motion of electrically conducting fluid in pipes under transverse magnetic field. Drake (1965) has considered the flow in a channel due to periodic pressure gradient and solved by the method of separation of variables. Singh and Ram (1977) have considered the laminar flow of an electrically conducting fluid through a channel in the presence of a transverse magnetic field under the influence of a periodic pressure gradient and solved it by Laplace transform. Ram et al. (1984) have considered Hall effects on heat and mass transfer flow through porous medium.

Simonura (1991) has considered magnetohydrodynamic turbulent channel flow under a uniform transverse magnetic field. Kazuyuki (1992) has discussed inertia effects in two dimensional MHD channel flow. Pop and Watanabe (1995) considered the Hall effects on magnetohydrodynamic boundary layer flow over a continuous moving flat plate. Ram (1995) discussed effects of Hall and ion slip currents on free convective heat generating flow in a rotating fluid. Singh (1998) considered unsteady magneto hydrodynamic flow of liquid through a channel under variable pressure gradient and solved it by method of Laplace transform. Singh (2000) considered unsteady flow of liquid through a channel with pressure gradient changing exponentially under the influence of inclined magnetic field and solved it by method of Laplace transform.

In the present paper laminar hydromagnetic steady flow of liquid between two parallel infinite channel is considered when upper plate is moving and lower plate is held stationary and also a constant pressure gradient is imposed on the system.

Governing equations

The equation of continuity for the incompressible fluid flow is given as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

where u, v, and w are the components of velocity of the fluid in the x, y, and z directions.

The equations of motion that describe fluid flow in each of the three directions are given as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{F_x}{\rho} \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \frac{F_y}{\rho} \tag{3}$$

and

There is no component of body force in the y–direction, and $F_x = \vec{J} \times \vec{B} \sin \alpha$, $F_Y = F_Z = 0$ as $v = w = 0$ then equations of motion (9) and (10) become

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\left(\frac{\partial^2 u}{\partial y^2}\right) + \frac{\vec{J} \times \vec{B}}{\rho}\sin \alpha \tag{11}$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} \tag{12}$$

Equation (12) implies that pressure does not depend on y.

We know that

$$\vec{J} = \sigma \vec{E}$$

and

$$\vec{E} = \vec{U} \times \vec{B} \sin \alpha$$

where $\vec{U}$ is fluid velocity along x-axis, the direction of fluid flow. Thus

$$\vec{J} \times \vec{B} \sin \alpha = \sigma\left[(\vec{U} \times \vec{B} \sin \alpha) \times \vec{B} \sin \alpha\right]$$

$$= \sigma\left[(\vec{U} \cdot \vec{B} \sin \alpha)\vec{B} \sin \alpha - (\vec{B} \sin \alpha \cdot \vec{B} \sin \alpha)\vec{U}\right].$$

Since $\vec{U}$ and $\vec{B} \sin \alpha$ are perpendicular vectors, we have

$$\vec{U} \cdot \vec{B} \sin \alpha = 0$$

giving

$$\vec{J} \times \vec{B} \sin \alpha = -\sigma B^2 \vec{U} \sin^2 \alpha.$$

Thus

$$\frac{\vec{J} \times \vec{B}}{\rho}\sin \alpha = -\frac{\sigma B^2 \vec{U}}{\rho}\sin^2 \alpha \tag{13}$$

Using (13) the equation of motion (11) now reduces to

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + v\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{F_z}{\rho} \qquad (4)$$

where F_x, F_y, F_z are components of $\bar{J} \times \bar{B}\sin\alpha$ in the x , y , z directions respectively.

For simplicity we shall consider a two-dimensional flow. In two dimensions, equation (1) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (5)$$

Since the plates are of infinite length, we assume that the flow is only along the x- axis and depends on y. Thus,

$$\frac{\partial u}{\partial x} = 0 \qquad (6)$$

Since we have assumed a steady flow, the flow variables do not depend on time. Thus equations (2) to (4) can now be written as

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{F_x}{\rho} \qquad (7)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{F_y}{\rho} \qquad (8)$$

Using (5) and (6) and the fact that there is no flow in y-direction equations (7) and (8) may now be written as

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\left(\frac{\partial^2 u}{\partial y^2}\right) + \frac{F_x}{\rho} \qquad (9)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{F_y}{\rho} \qquad (10)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\frac{\partial^2 u}{\partial y^2} - \frac{\sigma}{\rho}B^2 u \sin^2\alpha \qquad (14)$$

NON DIMENSIONLIZING

To simplify equation (13) further, we reduce the parameters in the equation by introducing the following non–dimensional quantities Singh (1993);

$$x' = \frac{x}{a} \; , \; y' = \frac{y}{a}, \; p' = \frac{p\,a^2}{\rho\,v^2} \; , \; u' = u\frac{a}{v} .$$

With these quantities, we see that,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial u'}\frac{\partial u'}{\partial y'}\frac{\partial y'}{\partial y}$$

$$= \frac{v}{a^2}\frac{\partial u'}{\partial y'} \qquad 15)$$

Therefore

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right]$$

$$= \frac{\partial}{\partial y'}\left[\frac{v}{a^2}\frac{\partial u'}{\partial y'}\right]\frac{\partial y'}{\partial y}$$

$$= \frac{\partial}{\partial y'}\left[\frac{v}{a^2}\frac{\partial u'}{\partial y'}\right]\frac{1}{a}$$

$$= \frac{v}{a^3}\frac{\partial^2 u'}{\partial y'^2} . \qquad (16)$$

Again

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial p'}\frac{\partial p'}{\partial x'}\frac{\partial x'}{\partial x}$$

$$= \frac{\rho v^2}{a^3}\frac{\partial p'}{\partial x'} \qquad (17)$$

Similarly,

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial p'} \frac{\partial p'}{\partial y'} \frac{\partial y'}{\partial y}$$

$$= \frac{\rho v^2}{a^2} \cdot \frac{\partial p'}{\partial y'} \cdot \frac{1}{a}$$

$$= \frac{\rho v^2}{a^3} \cdot \frac{\partial p'}{\partial y'} \qquad (18)$$

Putting these values in (12) and (14), we get

$$\frac{\partial p'}{\partial y'} = 0 \qquad (19)$$

and

$$0 = -\frac{1}{\rho} \frac{\rho v^2}{a^3} \frac{\partial p'}{\partial x'} + v \frac{v}{a^3} \frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma B^2}{\rho} \frac{v}{a} u' \sin^2 \alpha \qquad (20)$$

For the convenience, we shall drop the primes and write

$$0 = \frac{v}{a^3} \left[-\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2 a^2}{\rho v} u \sin^2 \alpha \right]$$

or

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2 a^2}{\rho v} u \sin^2 \alpha \qquad (21)$$

We may write above equation as

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - M^2 \sin^2 \alpha u \qquad (22)$$

where

$$M \sin \alpha = Ba \sqrt{\frac{\sigma}{\mu}} \sin \alpha = M_*$$

where $\mu = \rho v$ and M_* is known as the Hartmann number. It is directly proportional to the magnetic field B.

We can differentiate equation (22) with respect to x and obtain

$$0 = \frac{\partial^2 p}{\partial x^2} \tag{23}$$

Since p does not depend on y (23) may be put as a total derivative. Thus,

$$0 = \frac{d^2 p}{dx^2} \tag{24}$$

We can therefore see that

$$\frac{dp}{dx} = -P \quad \text{(a constant)} \tag{25}$$

And we can take ordinary derivative of the equation of motion instead of partial derivative. Therefore we have;

$$\frac{d^2 u}{dy^2} - M^2 \sin^2 \alpha u = -P \tag{26}$$

We now solve this equation by using method of solution of ordinary differential equations with constant coefficients.

Solution of the Equation

The auxiliary equation of equation (26) is given as

$$D^2 - M^2 \sin^2 \alpha = 0$$

or $\quad D = \pm M \sin \alpha$

and particular integral

$$P.I. = \frac{-P}{D^2 - M^2 \sin^2 \alpha} = \frac{P}{M^2 \sin^2 \alpha} \cdot$$

Giving solution for (26) as

$$u = c_1 e^{-M \sin \alpha y} + c_2 e^{M \sin \alpha y} + \frac{P}{(M \sin \alpha)^2} \qquad (27)$$

which is to be solved subject to boundary conditions

$u = 0$ when $y = -1$ and

$u = U$ when $y = +1$

Now

$u = 0$ when $y = -1$ then from (27)

$$0 = c_1 e^{M \sin \alpha} + c_2 e^{-M \sin \alpha} + \frac{P}{(M \sin \alpha)^2} \qquad (28)$$

and

$u = U$ when $y = +1$ then from (27)

$$U = c_1 e^{-M \sin \alpha} + c_2 e^{M \sin \alpha} + \frac{P}{(M \sin \alpha)^2} \qquad (29)$$

Multiplying equation (28) by $e^{-M \sin \alpha}$ and (29) by $e^{M \sin \alpha}$ we get

$$0 = c_1 + c_2 e^{-2M \sin \alpha} + \frac{P}{(M \sin \alpha)^2} e^{-M \sin \alpha} \qquad (30)$$

$$U e^{M \sin \alpha} = c_1 + c_2 e^{2M \sin \alpha} + \frac{P}{(M \sin \alpha)^2} e^{M \sin \alpha} \qquad (31)$$

Solving (30) and (31) we get

$$c_2 = \frac{Ue^{M\sin\alpha}}{e^{2M\sin\alpha} - e^{-2M\sin\alpha}} - \frac{P}{(M\sin\alpha)^2}\left[\frac{e^{M\sin\alpha} - e^{-M\sin\alpha}}{e^{2M\sin\alpha} - e^{-2M\sin\alpha}}\right]$$

or

$$c_2 = \frac{Ue^{M\sin\alpha}}{e^{2M\sin\alpha} - e^{-2M\sin\alpha}} - \frac{P}{(M\sin\alpha)^2}\left[\frac{1}{e^{M\sin\alpha} + e^{-M\sin\alpha}}\right]$$

Again multiplying equation (30) by $e^{M\sin\alpha}$ and (31) by $e^{-M\sin\alpha}$ we get

$$0 = c_1 e^{M\sin\alpha} + c_2 e^{-M\sin\alpha} + \frac{P}{(M\sin\alpha)^2} \tag{32}$$

$$U = c_1 e^{-M\sin\alpha} + c_2 e^{M\sin\alpha} + \frac{P}{(M\sin\alpha)^2} \tag{33}$$

Solving (32) and (33) we get

$$c_1 = -\frac{Ue^{-M\sin\alpha}}{e^{2M\sin\alpha} - e^{-2M\sin\alpha}} - \frac{P}{(M\sin\alpha)^2}\left[\frac{e^{M\sin\alpha} - e^{-M\sin\alpha}}{e^{2M\sin\alpha} - e^{-2M\sin\alpha}}\right]$$

or

$$c_1 = \frac{Ue^{M\sin\alpha}}{e^{2M\sin\alpha} - e^{-2M\sin\alpha}} - \frac{P}{(M\sin\alpha)^2}\left[\frac{1}{e^{M\sin\alpha} + e^{-M\sin\alpha}}\right]$$

Putting values of c_1 and c_2 from above in (27) we get

$$u = \left[-\frac{Ue^{-M\sin\alpha}}{e^{2M\sin\alpha} - e^{-2M\sin\alpha}} - \frac{P}{(M\sin\alpha)^2\left(e^{M\sin\alpha} + e^{-M\sin\alpha}\right)}\right]e^{-M\sin\alpha y}$$

$$+ \left[\frac{Ue^{M\sin\alpha}}{e^{2M\sin\alpha} - e^{-2M\sin\alpha}} - \frac{P}{(M\sin\alpha)^2\left(e^{M\sin\alpha} + e^{-M\sin\alpha}\right)}\right]e^{(M\sin\alpha)y} + \frac{P}{(M\sin\alpha)^2}$$

or

$$u = U\left[\frac{e^{(M\sin\alpha)(1+y)} - e^{-(M\sin\alpha)(1+y)}}{e^{2M\sin\alpha} - e^{-2M\sin\alpha}}\right] - \frac{P}{(M\sin\alpha)^2}\left[\frac{e^{(M\sin\alpha)y} + e^{-(M\sin\alpha)y}}{e^{M\sin\alpha} + e^{-M\sin\alpha}}\right] + \frac{P}{(M\sin\alpha)^2}$$

$$u = \frac{U \sinh((M \sin\alpha)(1+y))}{\sinh(2M \sin\alpha)} + \frac{P}{(M \sin\alpha)^2}\left[1 - \frac{\cosh(M \sin\alpha)y}{\cosh(M \sin\alpha)}\right] \qquad (34)$$

As velocity U of upper plate is constant and from (25) P is non–dimensional pressure which is also constant we may take in (34)

$$U = P$$

and get from (34)

$$\frac{u}{U} = \frac{\sinh((M \sin\alpha)(1+y))}{\sinh(2M \sin\alpha)} + \frac{1}{(M \sin\alpha)^2}\left[1 - \frac{\cosh(M \sin\alpha)y}{\cosh(M \sin\alpha)}\right] \qquad (35)$$

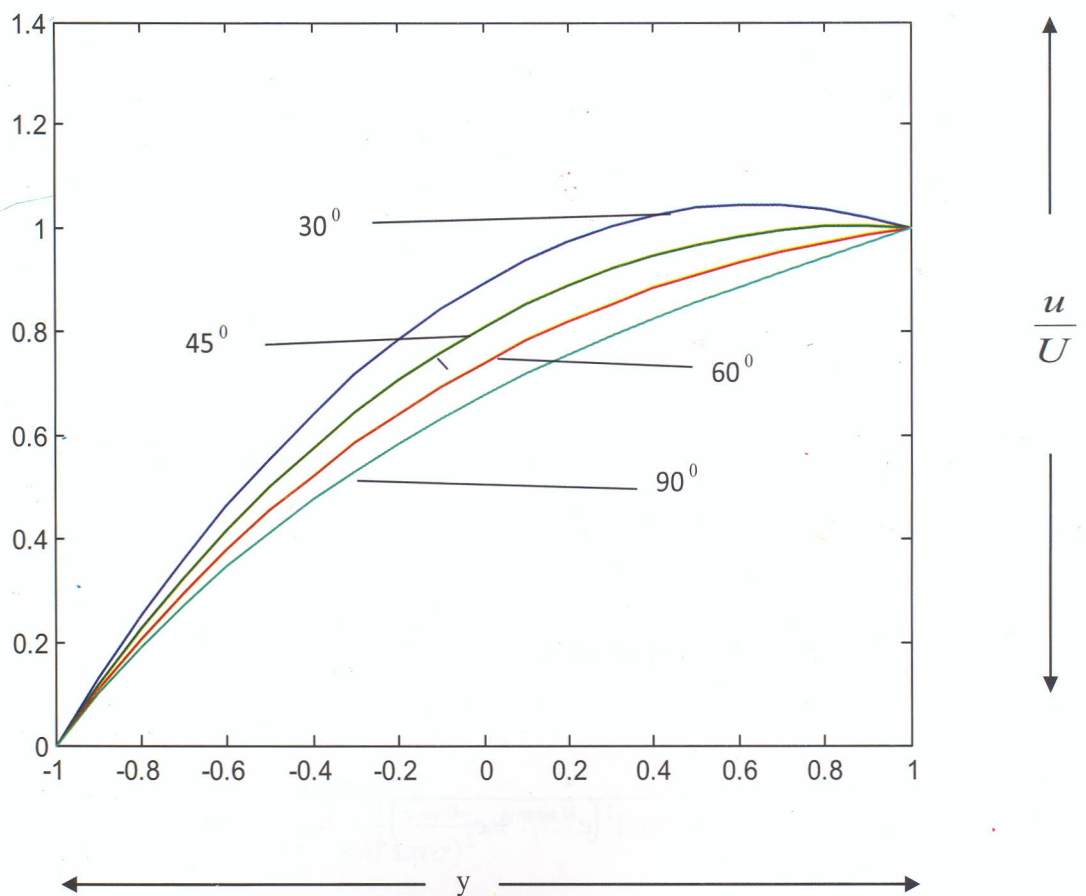The resulting figures drawn from equation (35), by using MATLAB, are shown below.
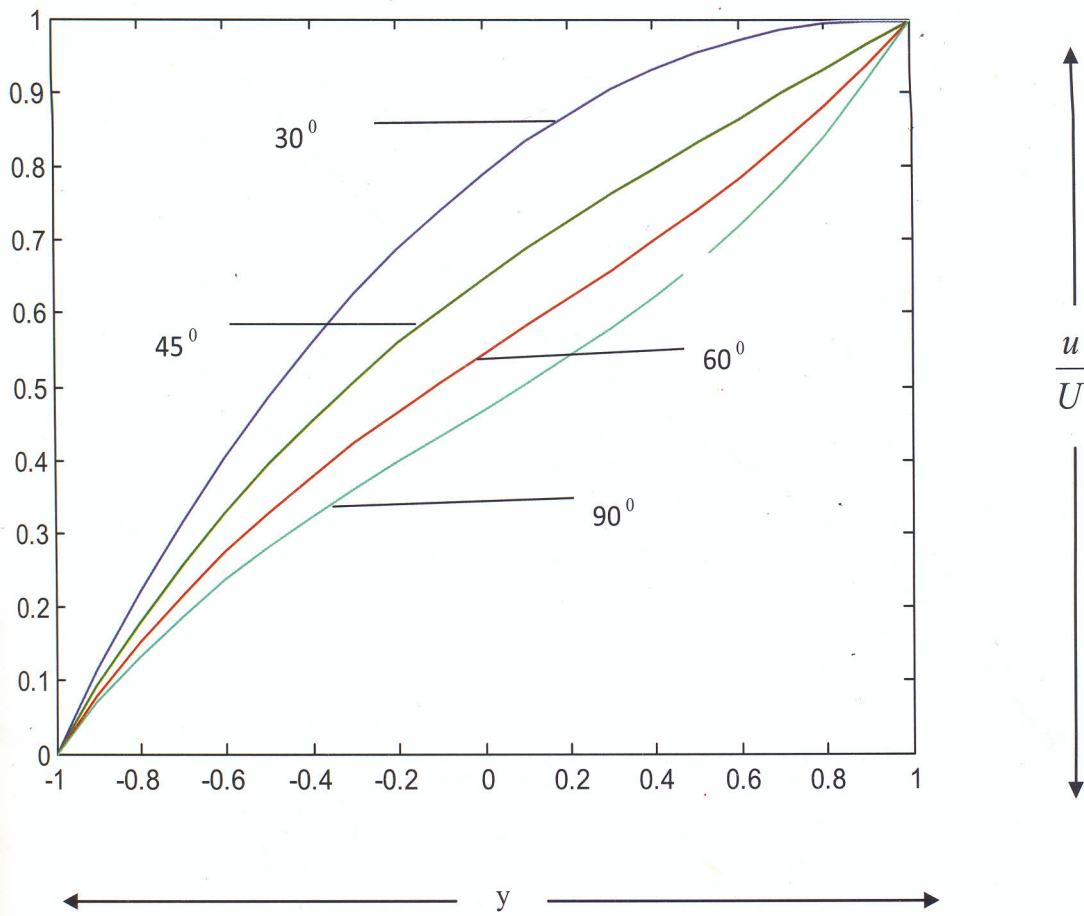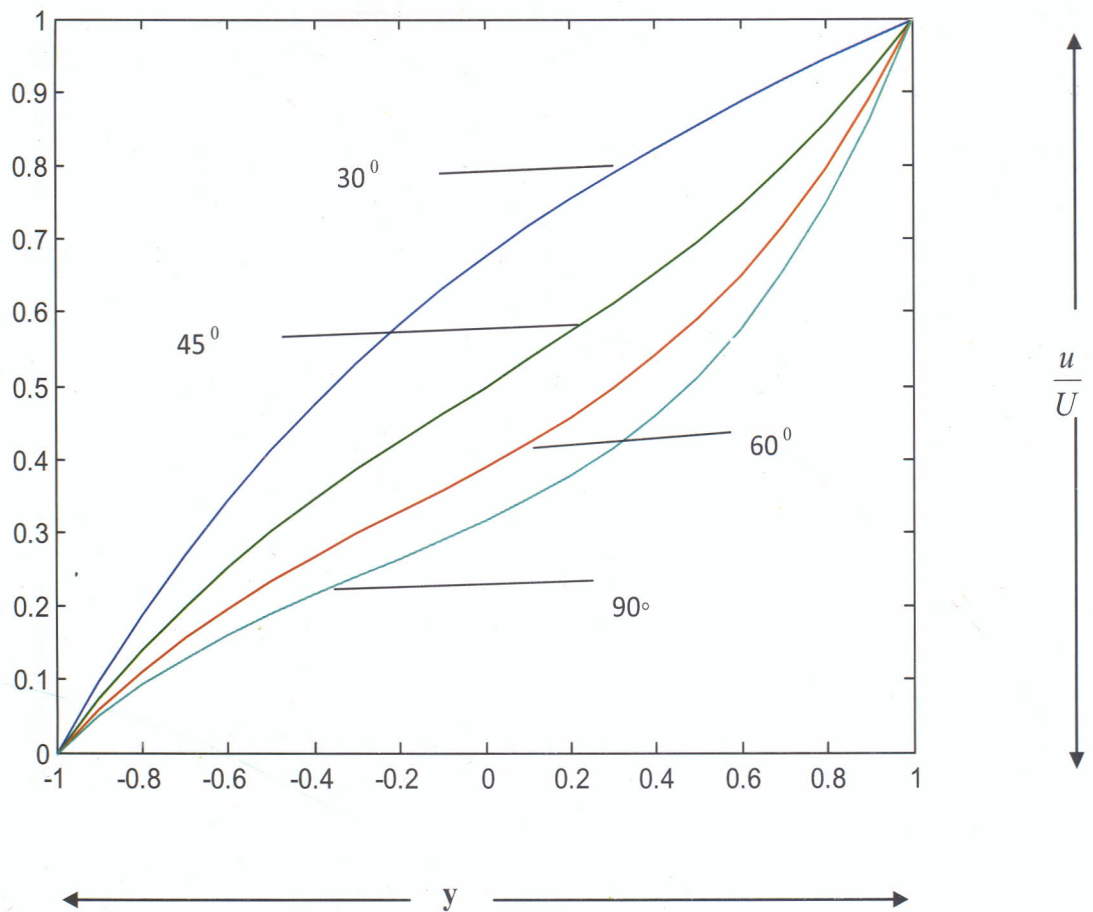

Figure 1: $M = 1.0$

Figure 2: *M=1.5*

Figure 3: *M = 2.0*

DISCUSSION OF RESULTS AND APPLICATIONS

The problem has been solved by the method of solution of linear differential equations with constant coefficients. An analytical expression for the velocity of fluid particle has been obtained. It is evident from equation (34) that if we take applied pressure gradient $P = 0$ then we get problem of Singh (2007) as a particular case of present problem. Figures 1 to fig 3 are drawn for $M = 1$, $M = 1.5$, and $M = 2$ respectively at the inclinations of $30^0, 45^0, 60^0$ and 90^0 with the horizontal. It is clear from the figures that velocity decreases as the strength of magnetic field is increased. Also evident is the fact that with the increase of inclination of magnetic field, there is a decrease in the velocity profile. Again velocity profiles at 90^0 gives us the steady hydromagnetic flow of viscous incompressible fluid under applied pressure gradient and when upper plate is also moving with constant velocity under the influence of transverse magnetic field as a particular case of present problem. It is evident from (34) that when $U = 0$ then the problem of magnetohydrodynamic steady flow of liquid between parallel plates Singh (1993) can be obtained as a particular case of this problem. The results obtained here can be applied to the

designs and operations of MHD generator, MHD pump, electromagnetic flow meter, and to crude oil purification.

REFERENCES

Attia, H. A., (2005). Effect of ion slip on the MHD flow of a dusty fluid with heat transfer under exponential decaying pressure gradient. *Central European Journal of Physics*, **3**(4), October, 484- 504

Drake, D. G., (1965). On the flow in a channel due to periodic pressure gradient. *Quart. J. Mech. Appl. Math.* **18**, 1-10

Kazuyuki, U. (1992). Inertia effect in two dimensional MHD flow under a travelling sine wave magnetic field, *Phys. Fluids* **3** (12), 3107 – 3116

Pop, I. and Watanabe, T. (1995). Hall effects on magnetohydrodynamic boundary layer flow over a continuous moving flat plate. *Acta. Mech.* **108**, 35-47

Ram, P. C., Singh, C. B., and Singh, U. (1984). Hall effects on heat and mass transfer flow through porous medium, *Astrophysics and space science* **100**, 45-51

Ram, P.C., (1995). Effects of Hall and ion slip currents on free convective heat generating flow in a rotating fluid. *Int. J. Energy Res.* **19**, 371 – 376

Sercliff, J. A. (1956). Entry of conducting and non-conducting fluids in pipes. *Proc. Camb. Phil. Soc.* **52**, 573 – 583

Simonura, Y. (1991). Large eddy simulation of magnetohydrodynamic turbulent channel flows under uniform magnetic field. *Phys. Fluids*, 3(12), 3098-3106

Singh, C.B. and Ram P. C. (1978). Unsteady magnetohydrodynamic flow through a channel. *Journal of Scientific Research (B.H.U.)* Vol XXVIII, 2

Singh, C.B. (1993). Magnetohydrodynamic Steady flow of liquid between parallel plates. *Proceedings of the 1st Conference of the Kenya Mathematical Society*, pp 24-26

Singh, C.B. (1998). Unsteady magneto hydrodynamic flow of liquid through a channel under variable pressure gradient. *K.J.S. Series A*, 11(1), 69 – 78

Singh, C.B. (2000). Unsteady flow of liquid through a channel with pressure gradient changing exponentially under the influence of inclined magnetic field, *International Journal of BioChemiPhysic,* **10** 37 – 40

Singh, C. B. (2007). Hydromagnetic steady flow of viscous incompressible fluid between two parallel infinite plates under the influence of inclined magnetic field, K.J.S. Series A, **12**(1)