SPECTRUM AND OPERATORS THAT ARE CONSISTENT IN INVERTIBILITY

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By

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DECLARATION

This project is my original work and has not been presented for a degree in any other University.

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DEDICATION

This project is dedicated to my brothers, in particular James Masika and parents Mr. and Mrs. Wabuya who have provided both the financial and moral support that has made me complete my graduate studies.

CONTENTS

INTRODUCTION

Spectral theory owes part of its motivation to the theory of quadratic forms. In early studies of Hilbert spaces (by Hilbert, Hellinger, Toeplitz and others), the objects of chief interest were quadratic forms.

For a complex vector space X, let $\theta : X \times X \to \mathbb{C}$ be a sesquilinear functional. Then the map $\hat{\theta} : X \to \mathbb{C}$ defined by $\hat{\theta}(x) = \theta(x, x)$ for all $x \in H$ is called the quadratic form associated with the given sesquilinear functional θ. However, modern developments are naturally given in terms of bounded linear operators on a Hilbert space.

In the study of operator theory in Hilbert nowadays, spectrum is used as a powerful tool to achieve properties like normality, self-adjointness, similarity, uniticity, hyponormality, e.t.c on operators which at first glance do not seem to behave as such.

To begin with, the study of the spectrum involves majorly the study of eigenvalues of a given operator on a Hilbert space. Hence, as a pre-requisite to what is to be studied, we consider in the first chapter a look into the eigenvalues, eigenvectors and eigenspaces of a given operator, in the same chapter, we look at the spectrum, its several classes and look at how the eigenvalues are used in their definition.

Since the spectrum is related to the numerical range, which is an integral tool in operator theory, the first chapter also involves the basic definition of the numerical range and gives some of its several classes.

The second chapter is dedicated to the study of the spectrum of quasisimilar operator, the aspect of quasisimilarity as introduced by Sz. Nagy and Faios, being a weaker version of similarity of operators, helps in an in-depth study of some aspects of operators that are generalized in the ordinary similarity. The chapter still looks at quasiaffinity of operators as a basis for the definition of quasisimilarity.

Chapter three, is on the study of operators that are consistently invertible, (CI operators), this is a very challenging aspect of operators that has not been widely explored, in this chapter, we look at the definition of CI operators, their spectrum and we give several cases under which an operator is considered to be consistently invertible.

NOTATION AND DEFINITIONS

In this thesis, H will represent a Hilbert space, the capital letters $A, B, C\ldots$ will be as symbols for operators in the Hilbert space H, $B(H)$ will denote the class of bounded operators in H.

Definition

An operator $A \in B(H)$ is said to be: -

Self - adjoint if	$A = A^*$
Normal if	$A^*A = AA^*$
Unitary if	$A^*A = AA^* = I$ i.e. $A^* = A^{-1}$
Isometric if	$A^*A = I$
Hyponormal if	$A^*A \geq AA^*$
p-hyponormal if	$(A^*A)^p \geq (AA^*)^p$ where $0 \leq p \leq 1$
Semi-hyponormal if	$(A^*A)^{\frac{1}{2}} \geq (AA^*)^{\frac{1}{2}}$
M - hyponormal if	$(A-\eta I)(A-\eta I)^* \leq M(A-\eta I)^*(A-\eta I)$
	$\|(A-\eta I)^* x\| \leq M \|(A-\eta I)x\|$ for all complex numbers η, and for all $x \in H$ and M some positive number $(M > 0)$
Quasinormal if	$A \leftrightarrow A^*A$ i.e A commutes with A^*A
Quasihyponormal if	$A^*(A^*A - AA^*)A \geq 0$

Fredholm: - A bounded operator A on a Hilbert space H is said to be Fredholm if the nullspace of A and A^* are finite dimensional and the range of A is closed.

By Atkinson's theorem, a bounded operator A is Fredholm if and only if zero does not belong to the essential spectrum of A.

Weyl: - A bounded operator $A \in B(H)$ is Weyl if $A \in \phi(H)$ and $ind\ A = 0$

The list of notations used in the thesis is as follows: -

$\ker(A)$ - Kernel of A, which is a subspace of H containing all elements that have been mapped to the identity by the operator A.

i.e. $\ker(A) = \{x \in H : Ax = 0\}$.

$\alpha(A)$ - Dimension of the kernel of A.

$\beta(A)$ - Co - dimension of the range of A.

$ind\ A = \alpha(A) - \beta(A)$ - The index of a semi-Fredholm operator A.

$\phi_+(H) = \{A \in B(H) : \alpha(A) < \infty\}$ - The class of upper semi-Fredholm operators.

$\phi_-(H) = \{A \in B(H) : \beta(A) < \infty\}$ - The class of lower semi-Fredholm operators.

Ascent $p = p(A)$ of A is the smallest non-negative integer p such that $\ker A^p = \ker A^{p+1}$ if such an integer does not exist, then $p(A) = \infty$.

Descent $q = q(A)$ of A is the smallest non-negative integer q such that $A^q(H) = A^{q+1}(H)$ if such an integer does not exist, then $q(A) = \infty$.

$B_+(H) = \{A \in \phi_+(H) : \rho(A) < \infty\}$ - Upper semi-Fredholm operators.

$B_-(H) = \{A \in \phi_-(H) : q(A) < \infty\}$ - Lower semi-Fredholm operators.

$B_w(H) = B_+(H) \cap B_-(H)$ - The class of all Browder operators.

CHAPTER ONE

EIGENVALUES. EIGENVECTORS AND EIGENSPACES

The study of eigenvalues and eigenvectors is related to linear equations which arise from steady state problems. Eigenvalues have their greatest significance in dynamic problems, where the solution is changing with time - growing, decaying or oscillating; therefore we can't find it by elimination.

It so happens that many of the problems which arise in modeling physical systems lead directly to eigenvalue problems: for example, in any vibrating system eigenvalues are closely linked with natural vibration frequencies; in control system analysis eigenvalues determine the stability and response of the system; and in quantum physics eigenvalues are connected with the attainable energy levels of the atom.

To explain eigenvalues, I first explain eigenvectors. Almost all vectors when multiplied by an operator (matrix) change direction, there are some exceptions which are in the same direction after being acted upon by an operator and those ones are referred to as ***eigenvectors***.

Let A be an operator in a complex Hilbert space H, suppose x is an arbitrary nonzero vector in the Hilbert space H, then if A acts on x as shown below:-

$$Ax = \lambda x$$

then the nonzero vector x is the eigenvector and the scalar λ, is known as the ***eigenvalue*** of the operator A. Equivalently, $\lambda \in \mathbb{C}$ is an eigenvalue of A if $\ker(A - \lambda I) \neq \{0\}$.

The eigenvalue tells whether the nonzero vector x ∈ H is stretched, shrunk, reversed or left unchanged when multiplied by the operator A.

Firstly, let us consider the case when H is of finite dimension. If A is the identity operator, then every nonzero vector multiplied by it is left unchanged, therefore all vectors are eigenvectors for A = I for the eigenvalue λ = 1 with multiplicity 2.

Consider the set M of all the eigenvectors of A together with the vector 0 (note that 0 is not an eigenvector). M is the set of all vectors $x \in H$ satisfying the equation: -

$$(A-\lambda I)x=0$$

It is a nonzero closed linear subspace of H. Thus M is the ***eigenspace*** of A corresponding to the eigenvalue λ. Alternatively, $\ker(A-\lambda I)$ is the eigenspace of A corresponding to the eigenvalue λ. Evidently M is invariant under A and the restriction of A to M is the operator 'scalar multiplication by λ'. The dimension of the eigenspace is called the geometric multiplicity of λ.

There exists some relationship between eigenvectors and the different eigenvalues as shown in the following theorem;

Theorem 1.1

Eigenvectors corresponding to distinct eigenvalues of A are linearly independent.

The eigenvalues of an arbitrary operator in a Hilbert space form a set in the complex plane, the theorem below shows the relationship between the eigenvalues and the dimension of the given Hilbert space;

Theorem 1.2

If A is an arbitrary operator on a Hilbert space H, then the eigenvalues of A constitute non-empty finite subsets of the complex plane. Furthermore, the number of points in this set does not exceed the dimension n of the space H.

When considering operators in a Hilbert space that are commuting, then theorem 1.4 exhibits how their common eigenvectors are related;

Theorem 1.4

If A and B are commuting compact self-adjoint (that is AB = BA), then they have a complete orthogonal set of common eigenvectors (vectors which are eigenvectors of both A and B)

Proof

Let λ be an eigenvalue of A and S the corresponding eigenspace. For any $x \in S$ we have

$$ABx = BAx = \lambda Bx$$

Thus Bx is an eigenvector of A with eigenvalue λ, unless $Bx = 0$. In any case, $Bx \in S$, and B maps S into itself. Now $B : S \to S$ is a compact self-adjoint operator on S, and the spectral theorem shows that S has a basis consisting of eigenvectors of B; these vectors are also eigenvectors of A because they belong to S. if we take such a basis for each eigenspace of A and put them together, the spectral theorem for A shows that the resulting set is complete.

After discussing the eigenvalues, eigenvectors and eigenspace of operators in the Hilbert space, we now give an example that will show how to find all the above in a given operator.

Example 1

Find the eigenvalues and the corresponding eigenvectors of

$$T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Solution

$$Tx = \lambda x$$

$$(T - \lambda I)x = 0 \Leftrightarrow \begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(4-\lambda) - 4 = 0$$

$$\lambda^2 - 5\lambda = 0 \Leftrightarrow \lambda = 0 \ or \ \lambda = 5$$

eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 5$

For $\lambda_1 = 0$, the eigenvector is given by

$$(T - 0I)x = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 + 2x_2 = 0$$

$$2x_1 + 4x_2 = 0$$

$\Rightarrow x_1 = -2x_2$ therefore if $x_2 = -1$, then $x_1 = 2$

hence the eigenvector corresponding to $\lambda_1 = 0$ is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

We now introduced the spectrum which is closely related to what has been previously discussed and its subclasses.

Let A be a bounded linear operator on a Hilbert space H. The **spectrum** of A, denoted by $\sigma(A)$ is the set given by: -

$$\sigma(A) = \{\lambda \in \mathbb{C} : A-\lambda \text{ is not invertible or is singular}\}$$

Alternatively,

If we consider the set of all $\lambda \in \mathbb{C}$, such that ($A-\lambda$) is invertible and is bounded in H, it constitutes the **regular values** of A called the **resolvent set** of A denoted by $\rho(A)$. The spectrum is defined as the compliment of $\rho(A)$ in H.

i.e. $\sigma(A) = (\rho(A))^c$

The spectrum of an operator A can be decomposed into the following subsets: -

<u>Continuous spectrum</u>

Denoted by $\sigma_c(A)$, i.e. $\lambda \in \sigma_c(A)$ if $\overline{R\ (\lambda I - A)} = H$, and $(\lambda I - A)^{-1}$ exists as a map which but it is unbounded.

<u>Residual spectrum</u>

Denoted by $\sigma_r(A)$, i.e. $\lambda \in \sigma_r(A)$ if $\overline{R\ (\lambda I - A)} \neq H$, but $(\lambda I - A)^{-1}$ exists as a map which may or may not be bounded.

<u>Approximate Point spectrum</u>

Denoted by $\Pi(A)$; a number λ belongs to the approximate point spectrum of A if and only if there exists a sequence of unit vectors $\{x_n\}$ such that $\|(A-\lambda I)x_n\| \to 0$.

Point spectrum

Denoted by $\sigma_p(A)$, i.e. λ ∈ $\sigma_p(A)$ if $(\lambda I - A)^{-1}$ does not exist as a map on R (λI - A), this is an important subset of the approximate point spectrum; it has only the eigenvalues of the given operator. That is, a complex number λ ∈ $\sigma_p(A)$ if and only if there exists a nonzero vector x such that $Ax = \lambda x$.

Compression spectrum

Denoted by $\Gamma(A)$, this is the set of complex numbers λ such that the closure of the range of $(A - \lambda I)$ is a proper subset of H.

i.e. $\lambda \in \Gamma(A)$ if $\overline{R(A - \lambda I)} \subset H$

Essential spectrum

Denoted by $\sigma_e(A)$, this is the set of complex numbers such that $(A - \lambda)$ is not Fredholm.

i.e. $\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm}\}$

Weyl spectrum

Denoted by $\sigma_w(A)$, this is the set of complex numbers such that $(A - \lambda)$ is not Weyl.

i.e. $\sigma_w(A) = \{\lambda \in A : A - \lambda \text{ is not Weyl}\}$

Browder spectrum

Denoted by $\sigma_b(A)$, the set of complex numbers such that $(A-\lambda)$ is not Browder.

i.e. $\sigma_b(A)=\{\lambda\in\mathbb{C}: A-\lambda \text{ is not Browder}\}$

Upper semi - Fredholm spectrum

Denoted by $\sigma_{SF_+}(A)$, is the set of complex numbers such that $(A-\lambda)$ is not upper semi - Fredholm.

Lower semi - Fredholm spectrum

Denoted by $\sigma_{SF_-}(A)$, is the set of complex numbers such that $A-\lambda$ is not lower semi - Fredholm.

Upper semi - Browder spectrum

Denoted by $\sigma_{ub}(A)$, the set of complex numbers such that the operator $A-\lambda$ does not belong to the class of upper semi - Browder operators.

i.e. $\sigma_{ub}(A)=\{\lambda\in\mathbb{C}: A-\lambda\notin B_+(H)\}$

Lower semi - Browder spectrum

Denoted by $\sigma_{lb}(A)$, the set of complex numbers such that the operator $A-\lambda$ does not belong to the class of lower semi - Browder operators.

i.e. $\sigma_{lb}(A)=\{\lambda\in\mathbb{C}: A-\lambda\notin B_-(H)\}$

From the above classes one can come to the conclusion that the spectrum of an operator A is the union of the approximate point spectrum and the compression spectrum. Furthermore, if H is a finite dimensional space then $\sigma(A) = \sigma_p(A)$.

Having discussed the various subclasses of the spectrum, we now give some results on the relationships between the various subclasses of the spectrum.

<u>**Proposition 1.3**</u>

For any operator $A \in B(H)$, $\sigma_c(A) \subset \Pi(A)$

<u>**Lemma 1.4**</u>

The spectrum of any closed operator is closed and the spectrum of a bounded linear operator cannot be empty.

<u>**Lemma 1.5**</u>

For each operator A, the approximate point spectrum is closed.

Proof

We show that the compliment of $\Pi(A)$ is open. Suppose $\lambda_0 \notin \Pi(A)$ then $A-\lambda_0$ is bounded from below; say

$$\| Ax-\lambda_0 x \| \geq \delta \| x \| \quad \forall x$$

Since

$$\| Ax-\lambda_0 x \| \leq \| Ax-\lambda x \| + \| \lambda x-\lambda_0 x \| \ \forall \lambda$$

It follows that

$$(\delta - | \lambda - \lambda_0) \| x \| \leq \| Ax-\lambda x \|$$

This implies that if $|\lambda-\lambda_0|$ is sufficiently small, then $A-\lambda$ is bounded from below.

We now give a proof of the theorem 1.6 which shows that the spectrum of an arbitrary self-adjoint operator is contained in some interval of real numbers.

Theorem 1.6

Suppose $A \in B(H)$ is self adjoint. Then

$\sigma(A) \subseteq [m, M]$

where $m = \inf_{\|x\|=1} \langle Ax, x \rangle$ and $M = \sup_{\|x\|=1} \langle Ax, x \rangle$

Proof

Suppose

$\lambda \notin [m, M]$ and let $d = (\lambda, [m, M])$.

Let $x \in H$ be any unit vector and write $\beta = \langle Ax, x \rangle$. Then

$\langle (A - \beta I)x, x \rangle = \langle x, (A - \beta I)x \rangle = 0$ and

$$\begin{aligned}
\| (A-\lambda I)x \|^2 &= \| \{(A-\beta I)+\beta I-\lambda I\} x \|^2 \\
&= \langle \{(A-\beta I)+\beta I-\lambda I\} x, \{(A-\beta I)+\beta I-\lambda I\} x \rangle \\
&= |\lambda-\beta|^2 \| x \|^2 + (\bar{\beta}-\bar{\lambda})\langle (A-\beta I)x, x \rangle + (\beta-\lambda)\langle x, (A-\beta I)x \rangle + \| (A-\beta I)x \|^2 \\
&\geq |\lambda-\beta|^2 \\
&\geq d^2
\end{aligned}$$

it follows that $\|(A-\lambda I)x\| \geq d \| x \|$. Hence $A-\lambda I$ is one to one and has a closed range.

Further, if $0 \neq z \perp \mathrm{ran}(A-\lambda I)$

then $0 = \langle (A-\lambda I)x, z \rangle = \langle x, (A-\bar{\lambda} I) z \rangle$ for all x and so $(A-\bar{\lambda} I)z = 0$.

But this is impossible, since from above, noting that

$d = dist(\lambda, [m, M]) = dist(\lambda, [m, M])$ we have $\| (A-\bar{\lambda} I)z \| \geq d \| z \|$.

Therefore, ran $(A-\lambda I) = H$, (being both dense and closed)

Therefore, for any $y \in H$, there is a unique $x \in H$ such that $y = (A-\lambda I)x$.

Define $(A-\lambda I)^{-1} y = x$. Then $\| y \| \geq d \| x \|$ so $\|(A-\lambda I)^{-1} y \| = \| x \| \leq \frac{1}{d} \| y \|$,

showing that $(A-\bar{\lambda} I)^{-1} \in B(H)$ (i.e. it is continuous).

Thus $\lambda \notin \sigma(A)$, proving the theorem.

We give the definition of similar operators and show the relationship that exists between their spectra.

Two operators A and B are said to be similar if there exists an invertible operator P such that

$$P^{-1}AP = B$$

Lemma 1.7

Suppose A and B are similar operators on a Hilbert space H,.then A and B have the same

1. Spectrum
2. Point spectrum
3. Approximate point spectrum

Proof

1. If $A-\lambda$ is invertible, then so is $P^{-1}(A-\lambda)P = P^{-1}AP-\lambda$
2. If $Ax=\lambda x$, then $P^{-1}AP(P^{-1}x)=\lambda(P^{-1}x)$
3. If $Ax_n - \lambda x_n \to 0$ where $\| x_n \| = 1$, then $P^{-1}AP(P^{-1}x_n)-\lambda(P^{-1}x_n) = P^{-1}(Ax_n-\lambda x_n) \to 0$. The norms $\| P^{-1}x_n \|$ are bounded from below by $1/\| P \|$ and consequently, division by $\| P^{-1}x_n \|$ does not affect convergence to 0. This implies that $P^{-1}AP\left(\frac{P^{-1}x_n}{\| P^{-1}x_n \|}\right)-\lambda\left(\frac{P^{-1}x_n}{\| P^{-1}x_n \|}\right)\to 0$

Associated with the spectrum is a quantity known as the spectral radius defined as:-

$$r(A)=\sup\{|\lambda|:\lambda\in\sigma(A)\}$$

It is the radius of the smallest closed circular disc in $\mathbb{C}$, with center at 0, which contains $\sigma(A)$.

The spectrum exists in relation to the numerical range of operators, infact the closure of the numerical range always contains the spectrum, and thus we give the definition of the numerical range below;

The numerical range of a linear operator A, on a Hilbert space H, denoted by W(A), is a subset of the complex field, and is defined by

$$W(A) = \{ < Ax, x >: \| x \| = 1 \}$$

It owes part of its motivation to the theory of quadratic forms. It is the range of the restriction to the unit sphere of the quadratic form associated with A.

The image of the unit ball is the union of all the closed segments that join the origin to points of the numerical range; the entire range is the union of all the closed rays from the origin through points of the numerical range.

The numerical range can be divided into the following classes:-

Classical numerical range

This is just the ordinary numerical range of an operator A on a Hilbert space H. It is defined as:-

$$W(A) = \{ < Ax, x >: \| x \| = 1 \}$$

It is considered to be always convex according to the celebrated Toeplitz - Hausdorff theorem.

Toeplitz (1918) proved that the boundary of W(A) is a convex curve, but left open the possibility that it had interior holes. Hausdorff (1919) proved that it did not actually contain any holes.

Spatial numerical range

It is the union of the classical numerical ranges. Suppose $\mathbf{A} = \{ A_1, A_2, ..., A_n \}$.

Then, the spatial numerical range of the given operator **A** is

$$W(\mathbf{A}) = \bigcup_{i=1}^{n} W(A_i)$$

Joint Numerical range

This refers to the set of the numerical range of a set of operator, that is, they can all be self-adjoint, normal etc.

Suppose $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$ is a set of self-adjoint operators, then, the joint numerical range is given as:-

$$W(\mathbf{A}) = \{W(A_1), W(A_2), \ldots, W(A_n)\}$$

Essential numerical range

Let A be a bounded linear operator on a Hilbert space H i.e. $A \in B(H)$ and let K(H) be the set of compact operators on B(H).

Essential numerical range of an operator A is given as:-

$$W(\mathbf{A}) = \overline{\bigcap_{K \in K(H)} \{W(A+K)\}}$$

the intersection being taken over all compact operators K.

Associated with the numerical range is a quantity known as the **numerical radius** which is defined as:-

$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\}$$

Since the spectrum of an operator A on a Hilbert space H is contained in the closure of the numerical range of A, then we observe that

$$r(A) \le w(A)$$

CHAPTER TWO

SPECTRUM AND QUASISIMILARITY OF OPERATORS

Introduction

A natural method for constructing an invariant subspace for an operator on a Hilbert space is to find a second, known operator which is similar in some weak sense to the given operator and then to use this second operator and the weak similarity to construct the desired subspace. One such weak similarity is the notion of quasi-similarity introduced by Sz. Nagy and Foias.

Definition 2.1

Let H be an infinite dimensional complex Hilbert space, and let B(H) denote the algebra of all bounded linear operators on H.

An operator $X \in B(H)$ is a quasi-affinity (or a quasi-invertible operator) if T is injective and has dense range.

An operator $A \in H$ is a quasi-affine transform of an operator B if there exists a quasiaffinity A such that $BX = AX$. A and B are quasi-similar if they are quasi-affine transforms of each other.

Having defined quasi-affinity we give a theorem showing a relation between the aspects of quasi-affinity and invertibility.

Theorem 2.2

Let $A, B, X \in B(H)$ satisfy the equation $AXB = X$. Then we have the following: -

i) If A is left invertible, then A is invertible.
ii) If B is right invertible, then B is invertible.

Proof

Let A_1 be the left inverse of A. i.e. $A_1A = I$.

Then we have $A_1AXB = A_1X$. i.e. $XB = A_1X$.

Hence, $AXB = AA_1X = X$ i.e. $X - AA_1X = 0$

$\Rightarrow (I - AA_1)X = 0$ or $\Rightarrow (I - AA_1) = 0$, since X has dense range.

$\Rightarrow AA_1 = I$

Hence A is invertible.

Similarly, let B_r be the right inverse of B. i.e. $BB_r = I$

Then we have that:

$$
\begin{aligned}
&AXBB_r = XB_r \\
&i.e\ AX = XB_r \\
&i.e\ AXB = XB_rB = X \quad or \qquad X - XB_rB = 0 \\
&i.e\ X(I - B_rB) = 0 \\
&i.e\ I - B_rB = 0, \text{ since X is injective.} \\
&i.e\ B_rB = I
\end{aligned}
$$

Hence B is invertible.

Now consider an operator that is an isometry, we a theorem that indicates the existence of a quasi-affinity shows that the operator is necessarily unitary.

Theorem 2.3

Let $A, B, X \in B(H)$ satisfy $AXB = X$, then

i) If A is an isometry then A is unitary.
ii) If B is a co - isometry then B is unitary.

Theorem 2.4

Quasisimilar hyponormal operators have equal spectra.

Proof

If A and B are quasisimilar hyponormal operators; then for any complex number λ, $A-\lambda I$ and $B-\lambda I$ are also quasisimilar and hyponormal, so they are both invertible or both noninvertible. Thus the spectrum of A is equal to that of B. i.e $\sigma(A)=\sigma(B)$

Corollary 2.5

Let $T_1 \in B(H_1)$ and $T_2 \in B(H_2)$ be injective p - quasihyponormal. If T_1 and T_2 are quasisimilar then they have same spectra and essential spectra.

Proof

Let $T_i = N_i \oplus V_i$ on $H_i = H_{i1} \oplus H_{i2}$ where N_i and V_i are the normal and pure parts of $T_i (i=1,2)$. Since N_1 and N_2 are unitarily equivalent, we have

$$\sigma(N_1)=\sigma(N_2) \qquad and \qquad \sigma_e(N_1)=\sigma_e(N_2)$$

Also, since there exists operators $X_* \in L(H_{22}, H_{12})$ and $Y_* \in L(H_{12}, H_{22})$ having dense ranges such that

$$V_1X_* = X_*V_2 \quad \text{and} \quad Y_*V_1 = V_2Y_*,$$

We have

$$\sigma(V_1)=\sigma(V_2) \quad \text{and} \quad \sigma_e(V_1)=\sigma_e(V_2)$$

Hence we have

$$\sigma(T_1)=\sigma(T_2) \quad \text{and} \quad \sigma_e(T_1)=\sigma_e(T_2)$$

Lemma 2.6

Let $T \in B(H_1)$ be a p - quasihyponormal operator and $N \in B(H_2)$ be a normal operator. If $X \in B(H_1, H_2)$ has a dense range and satisfies $TX = XN$, then T is also a normal operator.

Proof

Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} N_1 & 0 \\ 0 & 0 \end{pmatrix}$

With respect to $H_1 = \overline{R(T)} \oplus \ker(T^*)$ *and* $H_2 = \overline{R(N)} \oplus \ker(N^*)$ respectively. Since $TX = XN$ and X has a dense range, we have $\overline{X(\overline{R(N)})} = \overline{(\overline{R(T)})X}$

If we denote the restriction of X to $\overline{R(N)}$ by X_1, then $X_1 : \overline{R(N)} \to \overline{R(T)}$ has a dense range and for every $x \in \overline{R(N)}$

$$X_1 N_1 x = XNx = TXx = T_1 X_1 x$$

So that $X_1 N_1 = T_1 X_1$. Since T_1 is p - hyponormal, there exists a hyponormal operator $\tilde{T}_1$ corresponding to T_1 and a quasiaffinity Y such that

$$\tilde{T}_1 Y = YT_1 .$$

Thus

$$\tilde{T}_1 YX_1 = YT_1 X_1 = YX_1 N_1$$

Since YX_1 has a dense range, $\tilde{T}_1$ is normal and so T_1 is normal.

Thus the inequality

$$(T_1 {}^* T_1)^p \geq (T_1 T_1 {}^* + T_2 T_2 {}^*)^p \geq (T_1 T_1 {}^*)^p = (T_1 {}^* T_1)^p$$

Implies that $T_2 = 0$. Hence T is normal.

Quasisimilarity of log - hyponormal operators

Let $T \in B(H)$ and let $T = U|T|$ be the polar decomposition of T. Then $R = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ the Aluthge transform of T. If T is log - hyponormal and semi - hyponormal, then R is semi - hyponormal and hyponormal respectively.

Let $R = V|R|$ be the polar decomposition of R and $\tilde{T} = |R|^{\frac{1}{2}} V |R|^{\frac{1}{2}}$. Hence T is log - hyponormal and then $\tilde{T}$ is hyponormal.

Given the definition of log - hyponormal operators, we now give a theorem illustrates the equality of the various subclasses of the spectra for quasi-similar log - hyponormal operators.

Theorem 2.7

If T is a log - hyponormal operator, then $\sigma_*(T) = \sigma_*(\tilde{T})$, where σ_* denotes each of the following; the spectrum, the point spectrum, the approximate point spectrum, the essential spectrum and the Weyl spectrum.

Proof

It is enough to observe that there exists an invertible operator $X = |R|^{\frac{1}{2}} |T|^{\frac{1}{2}}$ such that $T = X^{-1} \tilde{T} X$ (i.e. T is similar to $\tilde{T}$).

But similar operators have isomorphic lattices of invariant subspaces and similarity preserves the spectral picture. Hence the proof follows.

An operator T on a Hilbert space H is spectral, if it has a resolution of the identity much like that of a normal operator. Let E be a σ-homomorphism of the σ-algebra of

Borel subsets of the complex plane onto a σ-algebra of uniformly bounded (in norm) idempotents in $B(H)$ which contains the zero and the identity operators. The map E is a resolution of the identity for the operator T if for every Borel set B in the plane, $E(B)T = TE(B)$, and $\sigma(\mathrm{T} \mid \mathrm{E(B)(H)}) \in \bar{B}$ (the closure of B) where $(\mathrm{T} \mid \mathrm{E(B)(H)})$ denotes the restriction of T to the range of $E(B)$. The operator T is called *a spectral operator* if it has a resolution of the identity.

Spectral operators can be canonically decomposed as follows; if T is spectral then, $T = N + S$ where N, the scalar part is similar to a normal operator, S is quasi-nilpotent ($\sigma(S) = \{0\}$), and N commutes with S, this decomposition is unique. The invertible operator A for which ANA^{-1} is normal transforms the resolution of the identity E of T onto the spectral measure of ANA^{-1}. The spectrum of T is the spectrum of N, and if R is an operator which commutes with T then for every Borel set B, R commutes with E(B) and hence R commutes with N.

Lemma 2.8

Let N_1 and N_2 be normal operators acting of the spaces H and K respectively, and let X be an operator from H to K satisfying $XN_1 = N_2X$. If M denotes the orthogonal compliment in H of the kernel of X, and if L denotes the closure in K of the range of X, then M and L reduce N_1 and N_2 respectively, and $N_1 \mid M$ is unitarily equivalent to $N_2 \mid L$ via the unitary operator $U \mid M$ where $X = UP$ is the polar decomposition of $X (P = (X^* X^{1/2}))$. In particular, if X is quasi-invertible, the N_1 and N_2 are unitarily equivalent.

Williams lemma

Let $N_i \in B(H_i)$ be normal for each $i = 1,2$. If $X \in B(H_2, H_1)$ and $Y \in B(H_1, H_2)$ are injective such that $N_1 X = XN_2$ and $YN_1 = N_2 Y$, if either X or Y is compact then N_1 and N_2 are unitarily equivalent.

Theorem 2.9

Suppose for $i = 1,2$, $T_i = N_i + S_i$ are spectral operators written in their canonical decomposition. If there is a quasi-invertible operator X such that $XT_1 = T_2 X$, then

i. $XS_1 = S_2 X$; $XN_1 = N_2 X$

ii. N_1 *is similar to* N_2

iii. $\sigma(T_1) = \sigma(T_2)$

Proof

There are invertible operators A_i such that for $i = 1,2$, $A_i^{-1} N_i A_i$ is normal. Thus, replacing T_i by $A_i^{-1} T_i A_i$, it suffices to assume that the operators N_i are normal. Consider the following operators acting on the Hilbert space $H \oplus K$:

$$Y = \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ 0 & N \end{bmatrix} + \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$$

Since $XT_1 = T_2 X$, these two operators commute. But T is a spectral operator so Y commutes with the scalar (normal) part of T. It follows that $XN_1 = N_2 X$ and thus $XS_1 = S_2 X$. By lemma 2.4, N_1 and N_2 are unitarily equivalent and since $\sigma(T_i) = \sigma(N_i)$, $\sigma(T_1) = \sigma(T_2)$.

Suppose T_1 and T_2 are spectral operators with resolutions of the identity E_1 and E_2 respectively. We say T_2 is *weakly similar* to T_1 if there is a densely defined closed linear transformation A on H with densely defined inverse such that: -

i) $\langle AT_1, A^{-1}\rangle x = T_2 x$ for every X in the domain of A^{-1} and

ii) For every Borel set B, there is a constant M_B such that

$\| (AE_1(B)A^{-1})x \| \leq M_B \| x \|$

for each x in the domain of A^{-1}.

We now state and prove the following theorem on weakly similar spectral operators.

Theorem 2.10

If T_1 and T_2 are spectral operators with resolutions of the identity E_1 and E_2 respectively, and if X is a quasi-invertible operator such that

$XT_1 = T_2 X$, then T_2 is weakly similar to T_1.

Proof

It is enough to show that for every Borel set B, $XE_1(B)X^{-1}$ is bounded on the domain of X^{-1}, since the operator shall satisfy all the conditions of weak similarity. Let $T_i = N_i + S_i$ be the canonical decomposition of T for $i = 1, 2$, assume N_i is normal and its spectral measure is E_i. Write X in its polar decomposition, $X = UP$ where $P = (X^* X)^{1/2}$ and U is unitary. By theorem 2.5, $XN_1 = N_2 X$ and lemma 2.4, U is unitarily equivalence between N_1 and N_2.

As a consequence, $UE_1(B)U^* = E_2(B)$ for every Borel set B. it follows by the Putnam-Fuglede theorem that

$$N_1X^* = X^*N_2$$

and therefore

$$N_1X^*X = X^*N_2X = X^*XN_1$$

or X^*X commutes with N_1. Thus P commutes with N_1 and hence with $E_1(B)$. If x is in the domain of X^{-1}, then

$$\begin{aligned}(XE_1(B)X^{-1})x &= (UPE_1(B)P^{-1}U^*)x \\ &= (UE_1(B)U^*)x \\ &= E_2(B)x\end{aligned}$$

We now show the relation between the spectra of quasi - similar log - hyponormal and an isometry.

<u>**Theorem 2.11**</u>

Let $T_1 \in B(H_1)$ be log-hyponormal operator and let $T_2 \in B(H_2)$ be an isometry. If T_1 and T_2 are quasisimilar, then T_1 and T_2 are unitarily equivalent unitary operators.

<u>Proof</u>

There exists quasi-affinities X and Y such that $T_1X = XT_2$ and $YT_1 = T_2Y$. Since T_1 is invertible and $YT_1 = T_2Y$, T_2 has a dense range. Hence T_2 is unitary. Thus T_1 and T_2 are unitarily equivalent unitary operators.

<u>**Definition**</u>

An operator $A \in B(H)$ belongs to class A if $|A^2| \geq |A|^2$. Class A was first introduced by Furuta - Ito - Yamazaki as a subclass of paranormal operators which includes the classes of p-hyponormal and log-hyponormal operators.

Having defined class A operators, we introduce a quasi - affinity and show that, quasi - similar class A operators have equal spectra.

Theorem 2.12

Let $T^* \in B(H)$ and $S \in B(H)$ be of class A. If $X \in B(H,K)$ and $XT = SX$, then $\sigma(T^*) = \sigma(S^*)$

Proof

Let $A = \begin{bmatrix} T^* & 0 \\ 0 & S \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}$ on $H \oplus K$. Then A is p-hyponormal (log-hyponormal) operator on $H \oplus K$ that satisfies $BA^* = AB$.

Hence $BA = A^* B$ and therefore $S^* X = XT^*$ which implies $T^* = X^{-1} S^* X$. Thus T^* and S^* are similar, hence $\sigma(T^*) = \sigma(S^*)$.

Below we illustrate results on the several aspects on quasi - similarity among the several classes of operators.

Theorem 2.13

Let $T \in B(H)$ be p-hyponormal or co-hyponormal or log-hyponormal and $X \in B(H)$ be a quasi-invertible self-adjoint operator satisfying the operator equation $T^* X = XT$.
Then $\sigma(T) = \sigma(T^*)$.

Theorem 2.14

Let $T \in B(H)$ be co-hyponormal. If $X \in B(H)$ and $T^*X = XT$, then $\sigma(T^*) = \sigma(T)$

Proof

Let $X = L + iR$ be the Cartesian decomposition of X, we have that $T^*L = LT$ and $T^*V = VT$ by hypothesis. By theorem 2.10 we have that $TL = LT^*$ and $TV = VT^*$ which implies $TX = XT^*$ and $T = XT^*X^{-1}$. Hence $\sigma(T) = \sigma(T^*)$.

CHAPTER THREE

CI OPERATORS

Let H be an infinite-dimensional complex Hilbert space, $B(H)$ be the algebra of all bounded operators on H and $K(H)$ be the compact operator ideal in $B(H)$. For $A\in B(H), \sigma(B)$ and $\sigma_e(B)$ denote the spectrum and essential spectrum respectively.

For an operator $A\in B(H)$, we say A is *consistent in invertibility* (with respect to multiplication) or briefly, a CI operator if, for each $B\in B(H)$, AB and BA are invertible or noninvertible together. By Jacobson's Theorem (for $A, B\in B(H)$, the nonzero elements of $\sigma(AB)$ and $\sigma(BA)$ are the same), A is a CI operator if and only if $\sigma(AB)=\sigma(BA)$ for every operator $B\in B(H)$. Thus if A and B are CI operators, then so is AB. The problem is: which elements in B (H) are CI operators?

Fundamental theorem

Every $A\in B(H)$ must be included in one of the following five cases, and in each of them the problem is definitely answered.

Case 1

If A is invertible, then A is a CI operator.

Proof

It is sufficient to note that for every A in B (H), $AB = B^{-1}(BA)B$

Case 2

If ran A is not closed, then A is a CI operator.

Proof

It follows from $ran\, BA \subseteq ran\, B \subset \overline{ran\, B} \subseteq H$ for every operator A in B(H) that BA is not invertible.

It is now to be proved that, for every A in B(H), AB is also not invertible. If, for some $A \in B(H)$, AB were invertible, the expression $(AB)^{-1}AB = (AB)^{-1}(AB) = I$ indicates that B is bounded from below. Then ran B is closed, which contradicts the assumption.

Case 3

If $\ker A \neq 0$ and $\overline{ran\, A} \subset H$, then A is a CI operator.

Proof

For each B in B(H), $\ker BA \supseteq \ker A \neq 0$ implies that BA is not invertible and $ran\, AB \subseteq \overline{ran\, A} \subset H$ implies that AB is not invertible.

Case 4

If $\ker A = 0$ and $ran\, A = \overline{ran\, A} \subset H$, then A^*A is invertible while AA^* is not invertible, and so A is not a CI operator.

Proof

It follows from $ran\, AA^* \subseteq ran\, A \subset H$ that A^*A is not invertible.

Since A has closed range if and only if 0 is not an accumulation point of the spectrum $\sigma(A^*A)$ of A*A, this together with the fact that A*A is one-to-one and has a dense range implies that A*A is invertible.

Case 5

If $\ker A \neq 0$ and $ran\, A = \overline{ran\, A} = H$, then A*A is not invertible while AA* is invertible, and so A is not a CI operator.

<u>Proof</u>

It follows from $\ker A \neq 0$ and $ran\, A = \overline{ran\, A} = H$, that

$\ker A^* = 0$ and $ran\, A^* = \overline{ran\, A^*} \subset H$. Therefore, by replacing A by A^* in the proof of case 4, we obtain that A^*A is not invertible and AA^* is invertible.

From the above cases, theorem 3.1 below exhibits the different conditions under which an operator is consistent in invertibility.

<u>Theorem 3.1</u>

$B \in B(H)$ is a CI operator if and only if one of the following disjoint cases occurs:-

i. B is invertible
ii. ran B is not closed
iii. $\ker A^* = 0$ and $ran\, A^* = \overline{ran\, A^*} \subset H$.

<u>Corollary 3.2</u>

$B \in B(H)$ is a CI operator if and only if B*B and BB* are invertible or non-invertible together.

i.e. $\sigma(B^*B) = \sigma(BB^*)$

Corollary 3.3

Let $B \in B(H)$. If ker B = 0 =ker B*, then B is a CI operator.

Remark

- From the proofs above we can also see that the CI operators can be classified into two parts: -
    1. There is $B \in B(H)$ such that AB and BA are invertible together (in this case if and only if A is invertible)
    2. For all $B \in B(H)$, AB and BA are always noninvertible (if and only if either ran A is non-closed or ker A $\neq 0$ and $ran\, A = \overline{ranA} \subset H$).
- B is a CI operator if and only if so is B*

By the preceding results, normal, compact and invertible operators are immediate examples of CI operators and so are their products.

By the above remark, we can now state and prove theorem 3.4, which indicates the Cartesian decomposition of normal operators being a CI operator.

Theorem 3.4

If A and B are normal operators and AB* = B*A, then A + iB is a CI operator.

Proof

AB* = B*A implies that (AB*)* = (B*A)* i.e. BA* = A*B

It is enough to show that A + iB is normal.

$$(A+iB)^* = A^* + \bar{i}B^* = A^* - iB^*$$

$$\begin{aligned}(A+iB)^*(A+iB) &= (A^*-iB^*)(A+iB)\\ &= A^*A + iA^*B - iB^*A + B^*B\\ &= A^*A + i(A^*B - B^*A) + B^*B\end{aligned}$$

$$\begin{aligned}(A+iB)(A+iB)^* &= (A+iB)(A^*-iB^*)\\ &= AA^* - iAB^* + iBA^* + BB^*\\ &= AA^* + i(BA^* - AB^*) + BB^*\\ &= AA^* + i(A^*B - B^*A) + BB^*\\ &= A^*A + i(A^*B - B^*A) + BB^*\end{aligned}$$

By use of the fact that both A and B are normal. Hence

$$(A+iB)(A+iB)^* = (A+iB)^*(A+iB)$$

Thus, the operator $A+iB$ is a CI operator.

Having discussed in the previous chapter aspects of quasi - similarity, we use theorem 3.5 to give another example of CI operators.

Theorem 3.5

Let $A, B, X \in B(H)$ satisfying the operator equation $AXB = X$, where X is a quasi-invertible operator. Further, let A and B be quasi-normal operators, then A and B* are CI operators.

Proof

Since A is quasi-normal, $[A^*A, A] = 0$

By hypothesis, AXB = X from which

$$\begin{aligned}AA^*AXB &= AA^*X\\ A^*AAXB &= AA^*X\\ A^*AX &= AA^*X\\ A^*AX - AA^*X &= 0\\ (A^*A - AA^*)X &= 0\end{aligned}$$

Thus $A^*A - AA^* = 0$ since the operator X has dense range

i.e $A^*A = AA^*$

Hence A is a CI operator.

Further if B is quasi-normal, then $[BB^*, B] = 0$ and therefore $BB^*B = BBB^*$

By hypothesis, $AXB = X$, from which

$$\begin{aligned}
&AXBB^*B = XB^*B \\
&AXBBB^* = XB^*B \\
&XBB^* = XB^*B \\
&X(BB^* - B^*B) = 0 \quad i.e. BB^* - B^*B = 0 \\
&\qquad \Rightarrow \qquad BB^* = B^*B
\end{aligned}$$

Hence B^* is a CI operator.

An operator $B \in B(H)$, such that $\| Bx \| \geq \| B^*x \|$ for each $x \in H$ is called *hyponormal*. Obviously, $\ker B \subseteq \ker B^*$.

Alternatively, an operator $B \in B(H)$ is hyponormal if $B^*B \geq BB^*$.

An operator $B \in B(H)$ is p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ where $0 < p \leq 1$.

The below theorem gives conditions which make an hyponormal operator CI.

Theorem 3.6

If $B \in B(H)$ is hyponormal and $ranB$ is closed, then B is a CI operator if and only if

a. $\ker B \neq 0$ or
b. $\ker B^* = 0$

Note that if $ran\, B$ is not closed, then, from theorem 3.1, B is a CI operator.

Proof

The conclusion can be obtained, when $\ker B \neq 0$, from $\ker B^* \supseteq \ker B \neq 0$ and theorem 3.1 (3), and, when $\ker B^* = 0$, from $\ker B \subseteq \ker B^* = 0$ and corollary 3.3.

If B is a CI operator, then one of the two cases (1) and (3) in theorem 3.1 must occur. Case (1) implies $\ker B^* = 0$ and case (3) implies $\ker B \neq 0$.

Theorem 3.7

If $B \in B(H)$ is hyponormal, then B is a CI operator if and only if either: -

i. BB^* is invertible, or
ii. B^*B is non-invertible.

Proof

If BB^* is invertible, then it follows from
$ranB \supseteq ranB^*B = H$ and $\ker B \subseteq \ker B^* = \ker BB^* = 0$ that B is invertible, hence B^*B is invertible. This also leads to that, if B^*B is non-invertible, then so is BB^*.

Remark

Let $B \in B(H)$, then B is M-hyponormal if there exists an $M > 0$ such that

$$\|(B-\lambda)^* x\| \leq M \|(B-\lambda)x\|$$

for all $x \in H$ and all complex numbers λ. The theorems 3.4 and 3.5 remain true for M-hyponormal operator B with the proof unchanged.

If $B \in B(H)$ is an isometry, then B is a CI operator if and only if B is unitary.

Theorem 3.8

Let $T = U|T|$ be p-hyponormal, $\frac{1}{2} \leq p < 1$ and U be unitary. Then, the Aluthge transform given by $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is a CI operator.

Proof

First note that any p-hyponormal operator for $\frac{1}{2}$ – hyponormal by Aluthge,

Hence $(T^*T)^{\frac{1}{2}} \geq (TT^*)^{\frac{1}{2}}$, but

$$T = U|T|$$
$$T^* = |T^*|U^* = U^*U|T|U^* = U^*|T^*|$$

Thus

$(T^*T)^{\frac{1}{2}} \geq (TT^*)^{\frac{1}{2}}$ is equal to

$$U^*|T|U \geq |T| \geq U|T|U^*$$
$$\tilde{T}^* = |T|^{\frac{1}{2}} U^* |T|^{\frac{1}{2}}$$
$$\tilde{T}^*\tilde{T} = |T|^{\frac{1}{2}} U^* |T|^{\frac{1}{2}} |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$$
$$\tilde{T}\tilde{T}^* = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} |T|^{\frac{1}{2}} U^* |T|^{\frac{1}{2}}$$
$$\tilde{T}^*\tilde{T} - \tilde{T}\tilde{T}^* = |T|^{\frac{1}{2}} U^* |T|^{\frac{1}{2}} |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} - |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} |T|^{\frac{1}{2}} U^* |T|^{\frac{1}{2}}$$
$$= |T|^{\frac{1}{2}} \{U^*|T|U - U|T|U^*\} |T|^{\frac{1}{2}} \geq 0$$

Since $U^*|T|U \geq U|T|U^*$ and $|T|^{\frac{1}{2}} \geq 0$.

i.e. $\tilde{T}$ is hyponormal, hence a CI operator.

Corollary 3.9

Let $B \in B(H)$ be a quasi-invertible operator. Then B is a CI operator.

Corollary 3.10

Let $B^* \in B(H)$ be such that $0 \notin W(B^*)$. then B^* is a CI operator.

Proof

If $0 \notin W(B^*)$. Then B^* and B are quasi-invertible. Hence by the above corollary both B^* and B are CI operators.

The next theorem gives the equality of the spectrum of commuting operators.

Theorem 3.11

Let A and B be operators on a Hilbert space H , let B be positive and $P = +\sqrt{B}$ be a projection. Then $\sigma(AB) = \sigma(BA) = \sigma(P^2AP^2)$

Proof

By Jacobson's lemma, the nonzero points of the spectrum of two products AB and BA coincide. If BA is invertible and $0 \neq ran\sigma(B^*)$ then B^{-1} exists and

$$
\begin{aligned}
&AB = B^{-1}B(AB) = B^{-1}(BA)B \quad i.e\ AB\ is\ similar\ to\ BA\ and\ \sigma(AB) = \sigma(BA).\\
&AB = AP^2 = AP = (AP)P^2 = (AP)B\\
&Thus\ AB = (AP)B = (AP)BB^{-1}B = B^{-1}B(AP)B\\
&\qquad\qquad = B^{-1}(BAP)B = B^{-1}(PAP)B
\end{aligned}
$$

i.e. AB is similar to PAP and hence $\sigma(AB) = \sigma(PAP)$. Moreover, $\sigma(AB) = \sigma(P^2AP^2)$ since P is a projection.

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