Kenya Journal of Sciences Series A Vol. 14 No. 1, 2010

On the Effects of Random Forces in Non-uniformly Distributed Dynamical System

Authors:
Joseph O. Malo
Department of Physics,
University of Nairobi, Kenya

E-mail: jomalo@unobi.ac.ke

ABSTRACT

Interaction of a dynamical system with stochastic source R(t) is studied using parabolic type equation on a circle of length 2L. Greens function for the system is constructed and it is shown that almost all the correlation functions F(xt) are infinitely differentiable. The problem of spectral density of a random force is investigated and it is shown that a single random force for the entire dynamical system is inadequate.

1. Introduction

Stochastic sources play an important role not only in the study of physics, astronomy and chemistry but also in such practical fields of communication; in the theory of random processes [1 - 3]. In the present paper however, attention is focused on the problem of heat transfer in the presence of random forces [4, 5]. Green's function for such a system is constructed by employing Gaussian integrals [6, 7] and variational methods [8]. The scope of the present communication is limited to the consideration of heat transfer only in the x-direction of particles which are initially non-uniformly distributed in the (y-z) plane. The transfer process is studied by the application of an appropriate parabolic differential equation [9, 10].

2. Dynamical System

Let us consider a simple dynamical system consisting of interacting particles. The motion of these particles is given by a heat equation on a circle of length *2L*. Further, we assume that the system is constantly under the influence of action of random sources *R(t)*. The appropriate differential equation describing the time evolution of the particle distribution function *F(xt)* for such a system is a parabolic type equation of the form

$$\frac{\partial F(xt)}{\partial t}+\frac{\partial^2 F(xt)}{\partial x^2}=\delta(x)R(t) \qquad (2.1)$$

The random function *R(t)* – Gaussian white noise – is assumed to be non-correlated in time. Otherwise the state of system represented by equation (2.1) would be given by points in a space much wider than the phase space we are concerned with. On performing Fourier transformation to equation (1) we come to the following equation with respect to the amplitudes.

$$\frac{\partial f_\ell(kt)}{\partial t}+k_\ell^2 f_\ell(kt)=R(t) \qquad (2.2)$$

where

$$k_\ell^2=\pi/L \qquad \ell=1.....N$$

The above equation (2.2) is a system of equation for the amplitude $f_\ell(kt)$ such that on the RHS we have random function *R(t)*. The solution of equation (2.2) at least for one specific amplitude $f_m(kt)$ is well known [11, 12]. It yields Green's function for the problem in the form of a transformed Wiener measure in a space trajectory corresponding to $f_m(kt)$.

$$Y\left(f_m^o;0;f_m^t,m\right)=\int_c \exp\left\{\int_o^t\left(f_m+k_m f_m\right)^2 d\tau+\frac{k_m^2}{2}t\right\}\pi^t_{\tau=0}\frac{df_m(\tau)}{\sqrt{\pi}\,d\tau}. \qquad (2.3)$$

Equation (2.3) holds true only on the assumption that the spectral density of *R(t)* is unity.

From equation (2.1), it follows that

$f_\ell^o+k_\ell^2 f_\ell=f_m^o+k_m^2 f_m$ for any f_ℓ and therefore

$$\int_o^t\left[\,e^{-k_\ell^2}(t-\tau)\,f_m^o(\tau)+k_m^2 f_m(\tau)\,\right]d\tau=f_\ell^t-f_\ell^o e^{-k_\ell^2 t} \qquad (2.4)$$

where $f_\ell^o=f_\ell(o)$

$f_\ell^t=f_\ell(t)$

It is clear that the solution of the basic problem can be obtained by integrating equation (2.3) in a rather selective manner. This is done if integration is carried with respect to only those trajectories corresponding to hypersurface given by equation (2.4).

For $N + 1$ Fourier amplitudes, we obtain Green's function in the form of the following function integral

$$Y\left(f_m^o, f_1^o \ldots\ldots\ldots\ldots f_N^o; f_m^t; f_1^t \ldots\ldots\ldots f_N^t, t\right) = \int_c \exp\left[-\int_o^t f_m + k_m^2 f_m^o{}_{d\tau} + \frac{k_m^2}{2} t\right] \pi \Big|_{\tau=0}^{t} \frac{df_m(\tau)}{\sqrt{\pi}\, d\tau}$$

(2.5)

The problem of constructing Green's functional is frequently met in the study of stochastic processes and is stated in the following manner. Find the particle distribution function $f(t>0)$ at time $t > 0$ given that the dynamical system has distribution $f(t = 0)$ at time $t = 0$. Since the functional under consideration is Gaussian and further the problem is linear, we can apply variation principle methods and integral (2.5) can be given an approximate expression of the form

$$\exp\left\{\int_o^t \left(\overline{f}(\tau) - k^2 \overline{f}\right) d\tau\right\} \tag{2.6}$$

where $\overline{f}(\tau)$ is an extremal of the following variational problem

$$\min \int_o^t (f_m + k_m^2 f_m) d\tau \tag{2.7}$$

with the following conditions

$$f_m^o = f_m(0)$$

$$f_m^t = f_m(t) \tag{2.8}$$

$$\int_o^t e^{-k_j^2(t-\tau)}\left(f_m^o + k_m^2 f_m\right)^2 d\tau = f_j^t - f_j^o e^{-k_j^2 t} = f_j \qquad j = 1, \ldots\ldots, N \tag{2.9}$$

Further we shall proceed using the following representation

$$\frac{1 - e^{-(k_j^2 + k_n^2)^2 t}}{k_j^2 + k_n^2} = k_j^2 + k_n^2 \tag{2.10}$$

$$f_j^t - f_j^o e^{-k_j^2 t} = f_j \tag{2.11}$$

Thus after simple but rather long calculations and rearrangements the power of exponent in equation (2.6) will take the form.

$$-\int_0^t \left(\bar{f}(\tau) - k^2 \bar{f}(\tau)\right) d\tau \Rightarrow -\frac{\det\left(\bar{B}_{j,m}\right)}{\det\left(B_{\ell,n}\right)} = -\frac{\det\left(B_{\ell,m}\right)}{\det\left(B_{\ell,n}\right)} \tag{2.12}$$

where $\bar{B}_{j,m} = B_{j-1,m-1}$

For $\ell, m \neq 1$

$$B_{1,1} = \frac{f_m}{2k_m} \tag{2.13}$$

$$\bar{B}_{m,1} = \bar{B}_{1,m} = \frac{f_j k_j^2 + k_j^2 - 1}{2k_j^2} \tag{2.14}$$

$$= -f_j - 1 \text{ for } m \neq 1$$

$$B_{\ell,m} = \frac{k_j^2 + k_\ell^o k_j^2 + k_n^2}{2k_j^2} - k_\ell^2 + k_n^2 \qquad \ell, n = 1, 2 \ldots N. \tag{2.15}$$

This is a quadratic form which is negatively defined. Taking into account the normalization constant for the Green's function (2.5), we come to the following expression.

$$Y\left(f_m^o, f_1^o \ldots f_n^o; f_m^t, f_1^t, \ldots, f_n^t, t\right) = \frac{\sqrt{\left(A_{\ell,n}\right)}}{\pi^{n+1}} \exp\left\{\frac{\left|\left(B_{\ell,m}\right|\right.}{\left|\left(B_{\ell,n}\right)\right|}\right\} \tag{2.16}$$

It is important to note that the coefficients of the quadratic form are such that $A_{\ell,n}$ are obtainable from any other $A_{j,m}$ if we simply interchange $k_\ell \leftrightarrow k_j$, $k_n \leftrightarrow k_m$.

3. Random Forces

The random force *R(t)* is responsible for removing the system from equilibrium steady state. In order to maintain the system in a nonequilibrium state with or without collision [13], we shall assume the

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presence of a field force that is quite independent of collision and that it has a spectral density acting on each particle in the system.

$$R_w(t) = \sum_{i=1}^{N} y_i(t) \frac{H_i(V)}{\sqrt{2^i i! \pi}} \exp\left(-\frac{V^2}{2}\right) \qquad (3.1)$$

where $y_i(t)$ is the spectral density of the random field and $H_i(V)$ are the Hermitian polynomials.

We note further that the spectral densities are responsible for the random distribution of particles in the system such that detailed balance principle is satisfied in the presence of collision [14]. We thus interpret $R_w(t)$ as a stochastic field or source acting as a thermostat for the dynamical system. In the absence of random forces, our system will automatically go over to the equilibrium state.

Under these conditions we shall have transformed equation (3.1) into Langevin equation of the form

$$\frac{dV(xt)}{dt} + \beta V(xt) = R(t) \qquad (3.2)$$

such that V denotes the velocity of the particle.

The influence of the surrounding medium on the motion of the particle is then split into two parts. One is the systematic part βV representing the dynamical friction experienced by the particle and the second $R(t)$ is characteristic of the Brownian motion [15]. We assume further that the frictional term $-\beta V$ is governed by Stoke's law which states that the frictional force decelerates a spherical particle of radius a and mass m is given by $6\pi a\eta \frac{V}{m}$ where η denotes the coefficient of viscosity of the surrounding fluid and thus

$$\beta = \frac{6\pi a \eta}{m}.$$

The principal assumption for the fluctuating part $R(t)$ is that it is independent of the velocity and it varies extremely rapidly compared to the variation of V. In other words, though $V(t)$ and $V(t+\Delta t)$ may differ by negligible amount, there is no correlation whatsoever between $R(t)$ and $R(t+\Delta t)$.

Equation (3.2) is equivalent to non-linear integro-differential Boltzmann kinetic equation for a system of molecules of a gas with a distribution function $f(\bar{V}t)$ of the form,

$$\frac{df}{dt} + n\int_{\Omega}\int_{vi} g\,\sigma(\theta g)\{f_1' f' - f f_1'\}dV_1 d\Omega = R_w(t) \tag{3.3}$$

where $\sigma(\theta g)$ is the differential cross-section of particle defined by potential of intermolecular interactions, $n = \frac{N}{\Omega}$ is the particle density, $g = V_1 - V_2$ is the relative velocity. This second term on the left is the Boltzmann collision integral.

The kinetic equation represents the evolution in time of a distribution function with the general form.

$$\frac{dF}{dt} = A(F_1/X_1) \tag{3.4}$$

where A is a functional of distribution function.

For single particle distribution function, Bogolubov's hierarchy has the form

$$\frac{dF_1}{dt} = [H_1, F_1] + n\int[\phi_{12}, F_2]dX_2 \tag{3.5}$$

where -ϕ_{12} is the two particle potential, -F_2 is the two particle distribution function such that in homogeneous space

$$F_1(qpt) \equiv f(pt) \tag{3.6}$$

Our problem is therefore to compute the effective probability density of state at time $t > 0$, $W(p_1 t > 0)$ on the assumption that at the initial moment the state $W(p_1 t = 0)$ was certainly realized. We shall apply the method of interchanging variables in a continuum or path integral. The established functional will be expressed as a path integral over a trajectory in phase space with the help of Smolukovsky's expression [16].

In general the Green's function corresponding to equation (3.3) is the solution of some functional equation in variational derivatives. However, in the present communication we shall construct the functional for a simplified linear problem directly from the Langevin equation (3.2).

3.1 A single random source for the dynamical system

Let us assume that the particle of mass *m* is much heavier compared to the gas mole *m/M* and further that the velocity is never very different from the equipartion value such that *v/V* is always of the order of $(M/m)^{1/2}$. Under such conditions, we can approximate the collision integral in the Boltzmann equation (3.3) by a Fokker-Planck differential operator of the form (3.7),

$$I(V)=\frac{d}{dV}(\beta V)+D^2\frac{d^2}{dV^2} \tag{3.7}$$

where *D* is the diffusion coefficient.

On substitution of (3.4) into (3.3) we obtain a linearized Boltzmann equation of the form

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial v}(\beta V)f-D^2\frac{\partial^2 f}{\partial V^2}=R_w(t) \tag{3.8}$$

Now supposing that for equilibrium states or detailed balance, we assume that only one random source operates on the entire dynamical system. Then equation (3.3) becomes an integral of motion of the form

$$\frac{df}{\partial t}=R_w(t) \tag{3.9}$$

such that

$$R_w(t)=\dot{y}(t)\,f(t) \tag{3.10}$$

where $\dot{y}(t)$ is the spectral density of the random source.

We then expand the probability density of state *f*(*Vt*) in series of Hermitian polynomials

$$f(Vt)=\sum_k G_k(t)\frac{H_k(V)\exp\left(-\frac{V^2}{2}\right)}{\sqrt{2^k\,k!\pi}}. \tag{3.11}$$

This will transform equation (3.5) into characteristic equation of the form

$$\dot{V}(t)+\beta V(t)=0 \tag{3.12}$$

whose solution has the general form

$$V(t)=V(0)\exp(-\beta t) \tag{3.13}$$

Thus equation (3.5) will take the form of the following differential equation

$$\frac{\partial f_1}{\partial t}+\beta f_1-D^2\frac{\partial^2 f_1}{\partial v^2}=f_1\ (V_o t)\,\dot{y}(t) \tag{3.14}$$

Now using expression (3.11) of *f(Vt)* in equation (3.14) above, then multiplying the result by

$$H_m(Vo)\exp\left[\frac{V_o^{\,2}}{2}\right]\ /\sqrt{2^m \pi m!} \tag{3.15}$$

and further make use of orthogonality condition for Hermitian polynomials

$$\int H_m(V)H_n(V)\exp\left[-\frac{V^2}{2}\right]dV=\begin{cases}1 & m=n\\ 0 & m\neq n\end{cases} \tag{3.16}$$

We obtain the following system of equations

$$\dot{G}_m+(\beta+m\beta)\,G_m-\left[\frac{3}{2}(m+1)\dot{G}_{m+2}-\frac{5}{2}G_{m-2}\right]\frac{D^2}{4}=\dot{y}(t)\,G_m(t) \tag{3.17}$$

Thus, in terms of diagonal elements only equation (3.17) takes the form

$$\int_o^t\left(2\beta+\beta\,m-\dot{y}\right)dt=ln\frac{G_m^{\tau}}{G_m^{\circ}} \tag{3.18}$$

The Green's function we are seeking must therefore satisfy the following path integral

$$\Gamma\left(G_m^o,0;G_m^t,t\right)=\int_c \exp\left[-\int_0^t\left[\dot{y}\right]^2 d\tau\right]\prod_{\tau=0}^{t}\frac{dy}{\sqrt{\pi}\,d\tau} \tag{3.19}$$

Under the condition that we integrate (3.19) along a trajectory satisfying

$$\Gamma\left(G_m^0,0;G_m^t,t\right)=\int_c\left[\int_0^t[\dot{y}]^2 d\tau\right]\times\delta\left(2\beta t+\beta_m t-y_t+y_0+In\frac{G_m^t}{G_m^0}\right)\prod_{\tau=0}^{t}\frac{dy}{\sqrt{\pi}\,d\tau} \tag{3.20}$$

which in effect is the Green's function for a fixed coefficient. It then follows that Green's function for two different coefficients $n \neq m$, the integral will be a product of δ - functions, i.e.,

$$\delta_n \times \delta_m = \begin{cases} 1 & m-n \\ 0 & m\neq n \end{cases}$$

Putting

$$2\beta t+\beta_m t = C_m$$
$$2\beta t+\beta_n t = C_n \tag{3.21}$$

then $\delta\left(C_m-\int_0^t \dot{y}\,d\tau\right)$ leads to

$$C_m = \beta(2+m)t - In\frac{G_m^1}{G_m^0} \tag{3.22}$$

and

$\delta\left(C_n-\int_0^t \dot{y}\,d\tau\right)$ giving

$$C_n = \beta(2+n)t - \ln\frac{G_n^t}{G_n^0} \tag{3.23}$$

This indicates that $C_m \neq C_n$ unless $m = n$.

This structure of the stochastic source is inadequate for the system. Our next communication will focus on the effects of spectral densities of random forces on the state of motion of a dynamical system [17]. For this we propose that the particles are in an external potential well $P(V_t)$ that can be expanded in terms of Hermitian polynomials such that the coefficients of expansion are the spectral densities of the random fields.

$$\frac{\partial P}{\partial t} = \sum_{j=0}^{\infty} \dot{y}_j(t) \frac{H(V)\exp\left(-\frac{V^2}{2}\right)}{\sqrt{2^j \pi j!}} \tag{3.24}$$

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