UNIVERSITY OF NAIROBI
DEPARTMENT OF MATHEMATICS

SOLUTION OF MAGNETO HYDRODYNAMIC FLOW EQUATION BETWEEN CHANNELS BY LAPLACE TRANSFORM METHOD

A Project paper presented in partial fulfillment of the requirements of the course SMA 649: PROJECT IN APPLIED MATHEMATICS.

By George Kimathi

Supervisor: Dr C. B. Singh.

SEPTEMBER, 2001

This is to confirm that the work presented here in is my original work and has not been presented to any other university.

sign...

This is to confirm that this project has been submitted with my approval.

Signature of supervisor ....

(Dr. C. B. Singh) 25/9/01

ACKNOWLEDGMENTS

I wish to thank most sincerely my supervisor, Dr. C. B. Singh, who has contributed immensely to the success of this work, and for the guidance he offered, and the time he spent going through this work.

I would like to thank my wife, who was very encouraging when the going became quite tough, and for the support she offered in many ways. Millie, thanks for your understanding and perseverance.

I cannot forget to thank MR. Francis Nzuki, my only classmate for the many hours we spent together trying to figure out what all the theories and computations were about.

DEDICATION

To Dad, Mum, Brothers and Sister.

TABLE OF CONTENTS

Preamble 6

CHAPTER I

1.1 Introduction 7

1.2 Magnetohydrodynamic equations 9

Chapter two Laplace Transforms 25

2.1 Introduction 25

2.2 Properties of Laplace Integrals 25

2.3 The Convolution Theorem 29

2.4 Computations of some Laplace Transforms 31

2.5 Properties of Laplace Transforms 36

2.6 Inverse Laplace Transforms 40

2.7 Properties of Inverse Laplace Transforms 40

Chapter Three Solution of Magnetohydrodynamic Equation 44

3.1 Two Dimensional Flow 44

3.2 Non-Dimensionalizing 49

3.3 Solution of The Equation 54

Preamble

This project aims at getting the solution of Magnetohydrodynamic flow equation between channels by the Laplace Transform Method under the influence of an applied transverse magnetic field.

In so doing, we shall use the boundary conditions of the flow to convert the boundary value problem into an initial value problem. The initial conditions are then determined, and the problem is subsequently solved.

The solution found is then presented in a graphical manner, where the graph has been generated using the Matlab programme, and an interpretation of the results is discussed.

In chapter one, we have introduced the Magnetohydrodynamic equations from the Navier Stoke's equation of fluid flow.

In chapter two, we have introduced the Laplace integrals and Laplace Transforms, as well as some of their properties. Finally in chapter three, we have solved the MHD steady flow problem under the influence of an applied transverse magnetic field by using the method of Laplace transform. The results have then been presented in form of a graph for some Hartmann numbers.

CHAPTER 1

MAGNETOHYDRODYNAMICS

1.1 INTRODUCTION

Magnetohydrodynamics (MHD) is the study of the macroscopic interaction of electrically conducting fluids with a magnetic field. These fluids include liquid metals and highly ionized gas-like substances called plasmas. MHD has played a role in developing generators and propulsion systems that use conducting fluids. MHD also helps scientists understand electric and magnetic effects around the earth and on the sun. These effects include sunspots, magnetic storms in the earth's magnetic field, and auroras (northern and southern lights).

The principles of MHD have many applications. They are used in MHD generators, which produce electricity from a high-speed stream of plasma. The plasma shoots through a duct in a magnetic field, where it generates current that is drawn off by electrodes. MHD generators provide a highly efficient power source, but they are still in the experimental stage. MHD propulsion systems use electricity from a plasma or another conducting fluid to produce thrust. Such propulsion systems may someday be used to power submarines and space vehicles.

Investigation of steady rectilinear flow of an electrically conducting liquid along a uniform channel and under the action of uniform transverse magnetic field is of importance since electromagnetic measuring devices such as flow meters have been extensively used in the determination of rates of flows of fluids like blood, sodium, and sea water by measuring the potential difference induced in these fluids by motions through transverse magnetic fields.

The principles of MHD are also important in designing experimental fusion reactors. The fuel used in fusion reactors consists of plasma that has been heated many millions of degrees.

Such extremely hot plasma would expand very quickly and would hit the walls of a container. The plasma would then be cooled and would lose energy too quickly for a fusion reaction to occur. Physicists are trying to produce controlled fusion in super-hot plasma with confinement achieved by externally imposed magnetic fields.

In the absence of a magnetic field, a highly ionized gas behaves in most respects like a classical gas, but this behavior is modified in a striking manner when a magnetic field is applied. For example, it is possible to generate waves in an ionized gas having features more in common with electromagnetic waves than with dynamical waves.

An ionized gas is called plasma, when the distance at which the electric field of a charge is shielded by neighboring charges of opposite sign , called Debye Shielding Length, λ_D in the ionized gas is small compared with the representative length of interest.

When the electron mean free path for collisions, l , in the plasma is smaller than the Larmor radius, r_c [that is the radius of the helical path followed by a free electron in the applied magnetic field **B**], the statistical description of a plasma is sufficiently simple that by suitable averaging process, it may be replaced by a continuum representation.

Mironer (1979) has explained that "The continuum model of matter ignores the microscopic molecular structure of matter and smears the mass of the discreet molecules over the entire volume of material. It excludes the occurrence of a hole or void without matter and ignores any interactions between individual molecules. Instead it considers only the statistical average effects on certain gross macroscopic properties of the material."

This is because when the electron Larmor radius is less than the electron mean free path for collisions, the magnetic effects become predominant in the equation describing electron behavior and cause anisotropic electrical conductivity in the plasma.

The anisotropy that occurs is due to the fact that an electric field acts such that it has a component perpendicular to the magnetic field **B.** As well as the helical particle motions produced by the magnetic field, there is also an appreciable general drift of particles in a direction which is perpendicular to the plane determined by the magnetic field and the transverse electric field component.

This then produces the surprising effect that the current that is caused to flow is not as might be anticipated from Ohm's law parallel to the applied field.

The current produced by this anisotropy in the electrical conductivity of the plasma is called the Hall current. If $r_e > l$, the Hall current is negligible and the electrical conductivity of the plasma may be considered to be a scalar.

1.2 MAGNETOHYDRODYNAMICS EQUATIONS

In a continuous fluid, we may attribute density, pressure, velocity (which may have three independent components), temperature, viscosity, and thermal conductivity at any point.

In this paper, we shall have the following assumptions:- we neglect viscosity and thermal conductivity. We take the fluid as incompressible thus density is constant. We assume a steady state of flow in which the velocity at any point does not change with time. Further, we shall assume irrotational motion so that the curl of velocity vector vanishes.

A charged particle in motion suffers forces of three kinds.

I It is repelled by particles of charge similar to its own and attracted by other particles of opposite charge. The force on the particle per unit of its charge due to all the other charges present being the electrostatic field, $\mathbf{E}_s$.

From Coulombs Law, the force **F** on a charge Q due to a single point charge q which is at rest a distance r away is

$$\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{qQ}{r^2}\vec{r} \qquad \text{(Portis 1978)}$$

where ∈ is the permitivity of free space, and $\vec{r}$ is the separation vector from the location of q to the location of Q. If there are several point charges, then the fore **F** becomes

$$F = F1 + F2 + ... = \frac{Q}{4\pi\varepsilon}\left(\frac{q_1}{r_1^2} + \frac{q_2}{r_2^2} + \ldots\right)\vec{r}$$

$$= Q\,\mathbf{E}_s$$

where,

$$\mathbf{E}_s = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\vec{r}$$

Here, **E** is the electric field of the source charges. In fluids in consideration, we assume that the charge is continuous, and thus

$$\mathbf{E}_s = \frac{1}{4\pi\varepsilon}\int\frac{1}{r^2}\vec{r}\,dq$$

and since the charge fills a volume, with charge density of ψ, then

$$dq = \psi dv$$

and

$$\mathbf{E}_s = \frac{1}{4\pi\varepsilon} \int (\psi(v) \frac{1}{r^2} \vec{r}\,) dv$$

Gauss law is an important law that relates the flux of a surface with the net charge q enclosed by the surface.

Flux is a property of any vector field, and it refers to hypothetical surface in an electric field which may be closed or open.

The law, which is proved rigorously, by Grant, I. S., and Phillips, W. R. (1975), for example, states that

$$\int_s \vec{E}.ds = \frac{q}{\varepsilon} dv$$

where s is the surface enclosing a volume element dv

In the limit when dv becomes infinitesimally small, we have

$$\lim_{dv \to 0} \frac{\int_s \vec{E}.ds}{dv} = \frac{q}{\varepsilon} \qquad 1.2.1$$

Take the volume element dv to be a rectangular box with sides parallel to the axes and of lengths dx, dy, and dz.

The flux of **E** out of the sides of the box normal to the x-axis is

$$\left(E_x + \frac{\partial E_x}{\partial x} dx\right) dydz - E_x dydz$$

$$= \frac{\partial E_x}{\partial x} dxdydz$$

There are similar contributions from the sides normal to the y- and z- axes, and the total flux out of the box is

$$\int_s \vec{E}.ds = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dxdydz$$

Thus

$$\lim_{dxdydz \to 0} \frac{\int_s \vec{E}.ds}{dxdydz} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \quad 1.2.2$$

$$= \text{Div } \mathbf{E}$$

Comparing equations 1.2.1 and 1.2.2 reveals that

$$\text{Div E} = \frac{q}{\varepsilon} \quad 1.2.3$$

The curl of $\mathbf{E}_s$ may be found by calculating the line integral of this filed from some point a to some point b i.e.

$$\int_a^b \mathbf{E}_s . d\mathbf{l}$$

Since in spherical coordinates $d\mathbf{l} = dr\,\hat{r} + d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\Phi}$,

we find

$$\mathbf{E}_s . d\mathbf{l} = \mathbf{E}_s\ (dr\hat{r} + d\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi})$$

$$= \left(\frac{1}{4\pi\varepsilon}\int \frac{q}{r^2}\vec{r}dr\right).(dr\hat{r} + d\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi})$$

$$= \frac{1}{4\pi\varepsilon}\frac{q}{r^2}dr$$

This is because $\hat{r}, \hat{\theta}$, and $\hat{\phi}$ are perpendicular unit vectors.

Thus

$$\int_a^b \mathbf{E}_s . d\mathbf{l} = \int_a^b \frac{1}{4\pi\varepsilon}\frac{q}{r^2}dr$$

$$= - \frac{1}{4\pi\varepsilon}\left(\frac{q}{r_b} - \frac{q}{r_a}\right)$$

The integral around a closed path, where $r_a = r_b$ is

$$\oint \mathbf{E}_s.\mathbf{dl} = 0.$$

Using Stokes' Theorem, (which is $\oint \mathbf{E}.\mathrm{da} = \int_s curl E.ds$), we get

$$\nabla \times \mathbf{E}_s = 0.$$

Or

$$\text{Curl } \mathbf{E}_s = 0. \qquad 1.2.4$$

This means that $\mathbf{E}_s$ is irrotational.

Because curl $\mathbf{E}_s = 0$, the line integral of $\mathbf{E}_s$ around any closed loop is independent of the path, and we can define a function

$$\mathbf{V}(\mathrm{r}) = - \int_0^r \mathbf{E}_s . \mathrm{dl}$$

where 0 is some reference point. Thus

$$\mathbf{V}(\mathrm{b}) - \mathbf{V}(\mathrm{a}) = - \int_0^b \mathbf{E}_s . \mathrm{dl} + \int_0^a \mathbf{E}_s . \mathrm{dl}$$

$$= - \int_0^b \mathbf{E}_s . \mathrm{dl} - \int_a^0 \mathbf{E}_s . \mathrm{dl}$$

$$= - \int_a^b \mathbf{E}_s \cdot d\mathbf{l}$$

From the fundamental Theorem on gradients, we have

$$\mathbf{V}(b) - \mathbf{V}(a) = \int_a^b (\nabla \mathbf{V}).d\mathbf{l}.$$

So,

$$\int_a^b (\nabla \mathbf{V}).d\mathbf{l}. = - \int_a^b \mathbf{E}_s \cdot d\mathbf{l}$$

thus,

$$\mathbf{E}_s = - \nabla \mathbf{V}$$

or

$$\mathbf{E}_s = - \text{grad } \mathbf{V} \qquad 1.2.5$$

V is known as the electrostatic potential (in volts).

2 Charged particles moving in a magnetic field produce an electric field, $\mathbf{E}_i$. Let **B** represent an applied magnetic field in Weber/m^2. If the particle is moving with velocity **u** m/s it produces an electric field equal to **u** x **B** . The induced electric field is perpendicular to **u** and **B** .

That is

$$\mathbf{E}_i = \mathbf{u} \times \mathbf{B} \qquad 1.2.6$$

3 If the magnetic field is changing with time, then per unit of its charge, a particle suffers a further force $\mathbf{E}_i$, the induced electric field.

From Faraday's law, a changing magnetic field induces an electric field, and the induced electric filed is equal to the rate of change of the magnetic flux.

The flux rule is

$$\varepsilon = \frac{-d\varphi}{dt}$$

where φ is the flux and ε is the induced e.m.f.

If the e.m.f is equal to the rate of change of flux, then

$$\varepsilon = \oint \mathbf{E}_i.\mathrm{dl} = \frac{-d\varphi}{dt}$$

and so $\mathbf{E}_i$ is related to the change in $\mathbf{B}$ by

$$\oint \mathbf{E}_i.\mathrm{dl} = -\int \frac{\partial}{\partial t}\mathbf{B}.\mathrm{da}$$

Which is Faraday's law.

Stoke's Theorem is an important theorem that relates the line integral around a closed path of any vector field $\mathbf{B}$ to the surface integral of curl $\mathbf{B}$ over the surface defined by the path. It states that

$$\oint \vec{E}.dl = \int_s curl\vec{E}.ds$$

Applying this Theorem, we get

$$\oint \vec{E}_i.dl = \int_s curl\vec{E}_i.ds$$

$$\nabla \times \mathbf{E}_i = -\frac{\partial}{\partial t}\mathbf{B} \qquad 1.2.7$$

As long as $\mathbf{E}_i$ is exclusively due to a changing $\mathbf{B}$ (with q = 0), then

$\nabla.\mathbf{E}_i = 0$. that is

$$\text{div } \mathbf{E}_i = 0. \qquad 1.2.8$$

However, in this study, there is no applied electric field nor is the magnetic field changing. The magnetic field $\mathbf{B}$ of a steady line current is given by the Biot - Savart law, which is

$$\mathbf{B} = \frac{\mu I}{4\pi}\int (dl \times \frac{1}{r^2}\vec{r})dr,$$

where I is the current.

The current density may be obtained by multiplying the current I with dl. If the charge density is $\mathbf{J}$, then

$$\mathbf{B} = \frac{\mu}{4\pi}\int (\mathbf{J} \times \frac{1}{r^2}\vec{r})dr,$$

Thus

$$\nabla.\mathbf{B} = \frac{\mu}{4\pi} \int \nabla.(\mathbf{J} \text{ x } \frac{1}{r^2}\vec{r}\ )\text{dr}.$$

From the properties of divergence, (see appendix) we know that

$$\nabla.(\text{A x B}) = \text{B}.(\nabla \text{ x A}) - \text{A}.(\nabla \text{ x B}),$$

which means that

$$\nabla.(\mathbf{J} \text{ x } \frac{1}{r^2}\vec{r}\ ) = \frac{1}{r^2}\vec{r}\ .\ (\nabla \text{ x } \mathbf{J}\ ) - \mathbf{J}\ .\ (\nabla \text{ x } \frac{1}{r^2}\vec{r}\ )$$

But

$$\mathbf{J} \text{ x } \frac{1}{r^2}\vec{r}\ = \nabla \text{ x } \mathbf{J} = 0$$

This means that

$$\nabla.\ \mathbf{B} = 0.$$

That is

$$\text{Div } \mathbf{B} = 0. \qquad 1.2.9$$

Again,

$$\nabla \text{ x } \mathbf{B} = \frac{\mu}{4\pi} \int \nabla \text{ x } (\mathbf{J} \text{ x } \frac{1}{r^2}\vec{r}\ )\text{dv}$$

And since $\nabla \text{ x (A x B)} = (\text{B}.\nabla)\text{ A} - (\text{A}.\nabla)\text{B} + \text{A}(\nabla.\text{B}) - \text{B}(\nabla.\text{A}),$

we get $$\nabla \times (\mathbf{J} \times \frac{1}{r^2}\vec{r}) = \mathbf{J}(\nabla . \frac{1}{r^2}\vec{r}) - (\mathbf{J}.\nabla)\frac{1}{r^2}\vec{r} .$$

We have dropped terms involving derivatives of **J** because we have assumed a steady fluid flow, thus the current density is constant..

Griffiths (1999) has shown that

$$\nabla . \frac{1}{r^2}\vec{r} = 4\pi\delta^3(r),$$

where δ is the Dirac symbol.

This means that

$$\nabla \times \mathbf{B} = \frac{\mu}{4\pi}\int 4\pi\, \mathbf{J}\, \delta^3\, dv$$

$$= \mu\mathbf{J}$$

Consider an elementary volume dv, with sides dx, dy, and dz, centered on the point (x, y, z). If the charge density varies with time, the charge within the volume element at time t is

$$\rho(x, y, z, t)dv$$

and at time $t + dt$ is

$$\left(\rho + \frac{\partial p}{\partial t}dt\right)dv .$$

The charge flowing into the volume element during the time interval dt over the two sides of area dydz parallel to y-z plane is

$$\left\{ J_x(x - \frac{1}{2}dx, y, z, t) - J_x(x + \frac{1}{2}(dx, y, z, t) \right\} dydzdt$$

$$= \frac{\partial J_x}{\partial x} dxdydt$$

$$= - \frac{\partial J_x}{\partial x} dvdt$$

Where J_x is the x-component of the current density **J** . The total charge flowing into the volume element over all six sides during the time dt is

$$-\left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right) dvdt$$

$$= - div.Jdvdt$$

Since charge is conserved, then this charge must equal the change in the total charge withing the volume element. Hence

$$\frac{\partial p}{\partial t} dvdt = - div.Jdvdt$$

Hence

$$Div.J = -\frac{\partial p}{\partial t}$$

This is called the equation of continuity.

The current density **J** is related to the magnetic field **B** and electric field **E** by the Ampere - Maxwell law which is shown in equation 1.2.11.

In this equation, μ is a constant equal to $4\pi \times 10^{-7}$.

$$\text{curl } \mathbf{B}/\mu = \mathbf{J} + \epsilon_0 \frac{\partial}{\partial t} \mathbf{E} \qquad 1.2.11$$

It shows how the total electric field **E** affects the magnetic field **B**.

We can immediately see that the total electric field **E** suffered by a conducting fluid in motion is

$$\mathbf{E} = \mathbf{E}_s + \mathbf{E}_i \qquad 1.2.12$$

Equation 1.1.2 and 1.1.4 now become

$$\text{Curl } \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} \qquad 1.2.13$$

And

$$\text{div } \mathbf{E} = \frac{q}{\varepsilon_0} \qquad 1.2.14$$

From equations 1.2.4 and 1.2.10, the total force acting on the charged particle therefore is

$$\mathbf{F} = \mathbf{E} + \mathbf{u} \times \mathbf{B} \qquad 1.2.15$$

This force is called the Lorenz force.

If the fluid were at rest, we get from Ohm's law that the current density **J** is proportional to the force per unit charge, that is **F**

$$\mathbf{J} = \sigma\mathbf{F}$$

From equation 1.1.13,

$$\mathbf{F} = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$

thus

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad 1.2.16$$

Where σ is the electrical conductivity of the fluid.

The equation of motion for a viscous incompressible magnetofluid is given by a modification of the Navier-Stokes' equation of fluid flow.

The modified equation become:

$$\rho(\frac{\partial}{\partial t}\mathbf{u} + [\mathbf{u}.\nabla]\,\mathbf{u}) = -\nabla p + (\mathbf{J} \times \mathbf{B}) + \rho\,\upsilon\,\nabla^2\mathbf{u} + \rho\,\mathbf{g} + q\mathbf{E} \qquad 1.2.17$$

Where υ is the kinematic viscosity and ρ is the density of the fluid and g is the force due to gravity.

In the absence of an applied electric field, $q\mathbf{E} = 0$ and the equation then becomes

$$\frac{\partial}{\partial t}\mathbf{u} + [\mathbf{u}.\nabla]\mathbf{u} = -\frac{1}{\rho}(\nabla p) + \frac{1}{\rho}(\mathbf{J} \times \mathbf{B}) + \upsilon\,\nabla^2\mathbf{u} + \mathbf{g}$$

Let the flow be horizontal so that the effects due to the force of gravity are neglected. The equation now becomes

$$\frac{\partial}{\partial t}\mathbf{u} + [\mathbf{u}.\nabla]\,\mathbf{u} = -\frac{1}{\rho}(\nabla p) + \frac{1}{\rho}(\mathbf{J} \times \mathbf{B}) + \upsilon \nabla^2 \mathbf{u} \qquad 1.2.18$$

Because there is no external electric field, $\mathbf{E}_s = 0$ and so

$$\nabla \mathbf{V} = 0. \qquad 1.2.19$$

And

$$\mathbf{J} = \text{curl } \mathbf{B} \qquad 1.2.20$$

Since from equation 1.2.7 div $\mathbf{B} = 0$, we have,

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \kappa \nabla^2 \mathbf{B} \qquad 1.2.21$$

Where κ is the resistivity.

Experimentally, it is found that when a fluid flows over a solid surface, there is no relative motion between the fluid immediately adjacent to the surface and the surface.

The fluid at the surface sticks to the surface, but away from the surface, the fluid moves relative to the surface (Mironer, 1979).

Therefore this observation means that the liquid velocity at the walls of the channel is zero.

We shall measure the length (diameter) of the channel from the middle, such tht the walls are located at y = $\pm\ a$.

Thus the velocity of the liquid at these positions is zero.

We set out to get the solution of equation 1.2.18 using the method of Laplace Transforms.

CHAPTER TWO

LAPLACE TRANSFORMS

2.1 INTRODUCTION

The Laplace transform operator L is very effective in the study of initial value problems involving linear differential equations with constant coefficients. The equation we wish to solve involves boundary conditions that will be changed to initial conditions and solved by this method of Laplace transforms. In this chapter, we discuss the definition of Laplace transform and its basic properties as well as deriving Laplace transforms of some functions.

Let $f(t)$ be any function. The Laplace transform of $f(t)$ denoted by L{ $f(t)$ } is defined by

$$L\{ f(t) \} = \int_0^\infty e^{-st} f(t)dt \qquad 2.1.1$$

If $f_1(t)$ and $f_2(t)$ have Laplace transforms and if c_1 and c_2 are any constants,

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}. \qquad 2.1.2$$

2.2 PROPERTIES OF LAPLACE INTEGRALS

These properties are presented here without proof. Their proofs may be found in Thomson (1957) and in most books dealing with differential equations.

1 If the integral 2.1.1 is convergent at a point s_0, it is convergent at all points s for which $\text{Re}(s - s_0) > 0$. There are three possible cases for the Laplace Integral

1.1 The integral is divergent everywhere

1.2 The integral is convergent everywhere

1.3 There exists a number σ_c such that the integral is convergent for $\operatorname{Re} s > \sigma_c$ and divergent for $\operatorname{Re} s < \sigma_c$. The number σ_c is called the abscissa of convergence of integral 2.1.1.

2 If integral 2.1.1 is absolutely convergent at the point $s_0 = \sigma_0 + i\tau_0$ it is absolutely and uniformly convergent in the half-plane $s \geq s_0$.

3 If 2.1.1 is convergent at the point $s_0 = \sigma_0 + i\tau_0$ and if $Q \geq 0$ and $k \geq 1$ are given constants, the integral is uniformly convergent in the domain Δ given the inequalites

$$\left| s - s_0 \right| \leq k(\sigma - \sigma_0)e^{Q((\sigma - \sigma_0)}, \sigma \geq \sigma_0 \qquad 2.2.1$$

4 If $\sigma_c < \infty$ integral 2.1.1 represents an analytic function if the variable s at all points of the half plane $\operatorname{Re} s > \sigma_c$ and

$$\frac{d^k L\{f(t)\}}{ds^k} \int_0^\infty (-t)^k f(t)e^{-st}dt \qquad 2.2.2$$

An analytic function is a function of a complex variable which possesses a derivative at every point of a region.

5 Let $L\{f_1(t), L\{f_2(t)\}$ be the Laplace transforms of functions $f_1(t)$ and $f_2(t)$. If

both Laplace integrals are convergent at the point s_0 and

$$L\{f_1(s_0 + nl)\} = L\{f_2(s_0 + nl)\},$$

where the constants $l > 0$ and $n = 0,1,2,\ldots,$ then $f_1(t) = f_2(t)$ almost everywhere.

6 If the integral 2.1.1 is convergent at the point

$$s_0 = \sigma_0 + i\tau_0, \sigma > 0, \text{ then}$$

$$\lim_{t \to \infty} e^{-\sigma_0 t} \int_{0^+}^{t} f(u)du = 0$$

that is $\int_{0^+}^{t} f(u)du = 0$ as $t \to \infty$. 2.2.3

A necessary and sufficient condition for convergence of integral 2.1.1 is that for some $\sigma_0 > 0$ and

$t \to \infty$,

$$f_1(t) = \int_{0^+}^{t} f(u)du = 0(e^{\sigma_0 t})$$

that is $\lim_{t \to \infty} e^{-\sigma_0 t} \int_{0^+}^{t} f(u)du = 0$ 2.2.4

Theorem 1: If the integral (2.1.1) has an abscissa of convergence $\sigma_c < \infty$, we have the limit

$$\lim_{w \to \infty} \frac{1}{2\pi i} \int_{\gamma - iw}^{\gamma + iw} L\{f(t)\} \frac{e^{st}}{s} ds = \begin{cases} 0, t < 0 \\ \int_{0^+}^{t} f(u)du, t > 0 \end{cases}$$ 2.2.5

where $\gamma > \sigma_c, \gamma > 0$. Hence for almost all t,

$$f(t) = \frac{d}{dt}\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} L\{f(t)\}\frac{e^{st}}{s}ds, \quad 2.2.6$$

where the integral is understood in the sense of the principle value.

It follows from property 6 that

$$\frac{L\{f(t)\}}{s} = \int_{0'}^{\infty} f_1(t)e^{-st}dt \quad 2.2.7$$

where

$$f_1(t) = \int_{0'}^{t} f(u)du, \sigma > \sigma_c, \sigma > 0, \text{ and } s = \sigma + i\tau .$$

A constant Q exists such that $|f_1(t)| < Qe^{\sigma_0 t} < Qe^{\sigma_0 t}(\sigma > \sigma_c)$ for all t. Hence

$$\left|\frac{L\{f(t)\}}{s}\right| \leq \frac{Q}{\sigma - \sigma_0} \quad 2.2.8$$

Thus if

$L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt, \sigma > \sigma_c$, and $f_1(t) = \int_{0'}^{t} f(u)du$, the Laplace

transform of $f_1 t$ will be $\frac{L\{f(t)\}}{s}$, the Laplace transform being absolutely convergent for

$\sigma > \sigma_c$.

2.3 THE CONVOLUTION THEOREM

Let $a(t)$ and $b(t)$ be functions of a real variable t. The convolution of these functions is the function $c(t)$ given by

$$c(t) = \int_0^t a(t-\tau)b(\tau)d\tau \qquad 2.3.1$$

symbolically written as

$$c(t) = a(t) * b(t) \qquad 2.3.2$$

The operation of obtaining the convolution is called the convolution.

Convolutions are:

1. commutative,

$$a(t) * b(t) = b(t) * a(t). \qquad 2.3.3$$

(2) Associative,

$$(a * b) * c = a * (b * c). \qquad 2.3.4$$

(3) Distributive with respect to addition.

$$[a(t) + b(t)] * c(t) = a(t) * c(t) + b(t) * c(t). \qquad 2.3.5$$

Theorem (Convolution Theorem): If the integrals

$$L\{f_1(t) = \int_0^\infty f_1(t)e^{-st}dt$$

and $$L\{f_2(t)\} = \int_{0^+}^{\infty} f_2(t)e^{-st}dt$$

are absolutely convergent for $\operatorname{Re}s > \sigma_a$, then

$$L\{f(t)\} = L\{f_1(t)\}L\{f_2(t)\} \qquad 2.3.6$$

is the Laplace transform of

$$f(t) = \int_{0^+}^{t} f_1(t-\tau)f_2(\tau)d\tau\ , \qquad 2.3.7$$

and the integral

$$Lf(t) = \int_{0}^{\infty} f(t)e^{-st}dt \qquad 2.3.8$$

is absolutely convergent for $\operatorname{Re}s > \sigma_a$.

f the convolution of functions $a(t)$ and $b(t)$, are continuous for $0 \le t < +\infty$, is identically

zero, at least one of these functions is identically zero.

2.4 COMPUTATIONS OF SOME LAPLACE TRANSFORMS

We may compute the Laplace transforms of some functions as follows

1 For the function e^{pt}, we have

$$L(e^{pt}) = \int_0^\infty e^{-st} e^{pt}\,dt$$

$$= \int_0^\infty e^{-(s-p)t}\,dt$$

For $s \leq k$, the exponent on e is positive or zero, and the integral diverges. For $s > k$ the integral converges, thus

$$L(e^{pt}) = \int_0^\infty e^{-(s-p)t}\,dt$$

$$= \left[\frac{-e^{-(s-p)t}}{s-p}\right]_0^\infty$$

$$= 0 + \frac{1}{s-p}$$

Thus

$$L(e^{pt}) = \frac{1}{s-p}, s > p \qquad 2.4.1$$

for p = 0, we find that

$$L(1) = \frac{1}{s}, \text{ for } s>0. \qquad 2.4.2$$

A sufficient condition for the existence of the Laplace transform of f(t) is evident from 2.4.1. Since the integral

$$\int_0^{\infty} e^{-(s-p)t}\,dt$$

exists for s>a, the Laplace transform exists for all functions f(t) satisfying the inequality

$$\left|e^{-st} f(t)\right| < Ce^{-(s-a)t}$$

where C is a constant. This is to say that f(t) does not grow more rapidly than Ce^{at}, or that f(t) is of exponential order, and that

$$\lim_{t\to\infty} e^{-st} f(t) = 0$$

From elementary calculus,

$$\int e^{ax} \sin mx dx = \frac{e^{ax}(\sin mx - m\cos mx)}{a^2 + m^2} + c. \qquad 2.4.3$$

Therefore, the Laplace transform of (sin pt) is

$$L(\sin pt) = \int_0^\infty e^{-st} \sin pt dt$$

which means that

$$L(\sin pt) = \left[\frac{e^{-st}(-s \sin pt - p \cos pt)}{s^2 + p^2} \right]_0^\infty$$

for positive s, $e^{-st} \to 0$ at $t \to \infty$. Since sin(pt) and cos(pt) are bounded as $t \to \infty$,

the above gives

$$L(\sin pt) = 0 - \frac{1(0 - p)}{s^2 + p^2}$$

$$= \frac{p}{s^2 + p^2}, \text{ for } s>0. \qquad 2.4.4$$

In a similar manner,

$$L(\cos pt) = \frac{s}{s^2 + p^2}, \ s>0. \qquad 2.4.5$$

In partial differential equations, we may find the Laplace transform of

$$L\left(\frac{\partial U}{\partial t} \right)$$

(assuming suitable restrictions) as follows.

$$L\left(\frac{\partial U}{\partial t}\right) = \int_0^\infty e^{-st}\frac{\partial U}{\partial t}dt$$

$$= \lim_{P\to\infty}\int_0^P e^{-st}\frac{\partial U}{\partial t}dt$$

Integrating by parts, we have

$$\lim_{P\to\infty}\int_0^P e^{-st}\frac{\partial U}{\partial t}dt = \lim_{P\to\infty}\left\{e^{-st}U(x,t)\Big|_0^P + s\int_0^P e^{-st}U(x,t)dt\right\}.$$

$$= s\int_0^\infty e^{-st}U(x,t)dt - U(x,0)$$

$$= su(x,s) - U(x,0)$$

$$= su - U(x,0)$$

where $u = u(x,s) = L\{U(x,t)\}$.

To find $L\left\{\frac{\partial^2 U}{\partial t^2}\right\}$, we let $V = \frac{\partial U}{\partial t}$, such that

$L\left\{\frac{\partial^2 U}{\partial t^2}\right\}$ becomes $L\left\{\frac{\partial V}{\partial t}\right\}$.

Thus,

$$L\left\{\frac{\partial^2 U}{\partial t^2}\right\} = L\left\{\frac{\partial V}{\partial t}\right\}$$

$$= sL\{V\} - V(x,0)$$

$$= s[sL\{U\} - U(x,0)] - U_t(x,0)$$

$$= s^2 u - sU(x,0) - U_t(x,0) \qquad 2.4.6$$

where

$$U_t(x.0) = \frac{\partial U}{\partial t}\bigg|_{t=o}$$

and

$$u = u(x,s) = L\{U(x,t)\}$$

We can similarly get the Laplace transform of cosh t in the following way

$$L\{\cosh at\} = \int_0^{\infty} e^{-st} \cosh t dt$$

$$= \int_0^{\infty} e^{-st} \frac{e^{at} + e^{-at}}{2} dt$$

$$= \frac{s}{s^2 - a^2}$$

It is not difficult to show that

$$L\{\sinh at\} = \frac{a}{s^2 - a^2}$$

2.5 PROPERTIES OF LAPLACE TRANSFORMS

For brevity, we shall denote

$$s\int_0^\infty e^{-st} f(t)dt \text{ by } C\{f(t)\} \qquad 2.5.1$$

1 Property of Linearity: -

Let

$$f(t) = \sum_{k=1}^{n} C_k f_k(t) \qquad 2.5.2$$

where C_k are arbitrary constants. Then

$$L[f(t)] = L\left[\sum_{k=1}^{n} C_k f_k(t)\right] = \sum_{k=1}^{n} C_k L[f_k(t)]$$

$$= \sum_{k=1}^{n} C_k L\{f_k(t)\} \qquad 2.5.3$$

Thus

$$L\left[\frac{d}{d\lambda} f(t,\lambda)\right] = L\left[\frac{f(t,\lambda + d\lambda) - f(t,\lambda)}{d\lambda}\right]$$

$$= L\left\{\frac{f(s,\lambda + d\lambda) - f(s,\lambda)}{d\lambda}\right\}$$

$$= \frac{d}{d\lambda} L\{f(s,\lambda)\} \qquad 2.5.4$$

2 Property of Similitude: -

For any constant α , we have

$$L\left\{ f\left(\frac{t}{\alpha}\right)\right\} = \int_0^\infty f\left(\frac{t}{\alpha}\right) e^{-st}\,dt$$

$$= \alpha \int_0^\infty f(\tau) e^{-\alpha st}\,d\tau$$

$$= \alpha L\{f(\alpha s)\} \qquad 2.5.5$$

$$C\left\{ f\left(\frac{t}{\alpha}\right)\right\} = s\int_0^\infty f\left(\frac{t}{\alpha}\right) e^{-st}\,dt$$

$$= \alpha s \int_0^\infty f(\tau) e^{-\alpha s t} d\tau \qquad 2.5.6$$

3 Laplace Transformation of derivatives:-

From integration by parts, we can obtain

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots \qquad 2.5.7$$
$$\ldots - sf^{(n-2)}(0) - sf^{(n-1)}(0).$$

$$C\{f^n(t)\} = s^n C\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots \qquad 2.5.8$$
$$\ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

where n is a positive integer.

4 Differentiation of Laplace Transforms

For a positive integer n,

$$\frac{d^n L\{f(t)\}}{ds^n} = (-1)^n \int_0^\infty t^n f(t) e^{-st} dt$$

$$= (-1) L\{t^n f(t)\} \qquad 2.5.9$$

$$\frac{d^n C\{f(t)\}}{ds^n} = (-1)^n C\{t^n f(t) - n \int_0^t t^{n-1} f(t) dt\} \qquad 2.5.10$$

Laplace Transforms of integrals

$$L\left\{\int_0^t d\tau \int_0^{\tau} d\tau_1 \ldots \int_0^{\tau_{n-2}} f(\tau_{n-1}) d\tau_{n-1}\right\} = \frac{L\{f(t)\}}{s^n} \qquad 2.5.11$$

Where n is a positive integer.

6 Integration of Laplace Transforms

If

$$\int_p^{\infty} L\{f(\tau)\} d\tau$$

is convergent, it is the Laplace transform of $\frac{f(t)}{t}$ that is we have

$$\int_p^{\infty} L\{f(\tau)\} d\tau = L\left\{\frac{f(t)}{t}\right\}. \qquad 2.5.12$$

7 Given any positive τ , assuming that $f(t-\tau) = 0$ for $t < \tau$, we obtain

$$L\{f(t-\tau)\} = \int_{\tau}^{\infty} f(t-\tau) e^{-st} dt$$

$$= \int_0^{\infty} f(u) e^{-s(u+t)} du$$

$$= e^{-s\tau} \int_0^{\infty} f(u) e^{-su} du$$

that is $$L\{f(t-\tau)\} = e^{-st} L\{f(t)\} \qquad 2.5.13$$

2.6 INVERSE LAPLACE TRANSFORMS

If the Laplace transform of a function $F(t)$ is $f(s)$, then $F(t)$ is called the inverse Laplace transform of $f(s)$ and is symbolically written as

$$F(t) = L^{-1}\{f(s)\}$$

Where L^{-1} is called the inverse Laplace transform operator.

2.7 PROPERTIES OF INVERSE LAPLACE TRANSFORMS

1 Linearity Property.

If c_1 and c_2 are any constants while $f_1(s)$ and $f_2(s)$ are the Laplace transforms of $F_1(t)$ and $F_2(t)$ respectively, then

$$L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 L^{-1}\{f_1(s)\} + c_2 L^{-1}\{f_2(s)\}$$

$$= c_1 F_1(t) + c_2 F_2(t)$$

2 First Shifting Property

If

$$L^{-1}\{f(s)\} = F(t),$$

then

$$L^{-1}\{f(s-a)\} = e^{at} F(t)$$

3 Second Shifting Property

If

$$L^{-1}\{f(s)\} = F(t),$$

then

$$** \quad L^{-1}\{e^{-as} f(s)\} = \begin{cases} F(t-a), t>0 \\ 0, t<0 \end{cases}$$

4 Change of scale property

Given that

$$L^{-1}\{f(s)\} = F(t),$$

then

$$L^{-1}\{f(ks)\} = \frac{1}{k} F\left(\frac{t}{k}\right)$$

5 Inverse Laplace Transforms of derivatives

if

$$L^{-1}\{f(s)\} = F(t),$$

then

$$L^{-1}\{f^{(n)}(s)\} = L^{-1}\left\{\frac{d^n}{ds^n} f(s)\right\}$$

$$= (-1)^n t^n F(t)$$

6 Inverse Laplace Transform of Integrals

If

$$L^{-1}\{f(s)\} = F(t),$$

then,

$$L^{-1}\left\{\int_{0}^{\infty} f(u)du\right\} = \frac{F(t)}{t}$$

7 Multiplication by s^n

Let

$$L^{-1}\{f(s)\} = F(t) \text{ and } F(0) = 0,$$

then,

$$L^{-1}\{sf(s)\} = F'(t).$$

This is to say that multiplication by s has the effect of differentiating $F(t)$

If

$$F(0) \neq 0,$$

Then

$$L^{-1}\{sf(s) - F(0)\} = F'(t)$$

8 Division by s

If
$$L^{-1}\{f(s)\} = F(t),$$

Then

$$L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)du$$

That is division by s (or multiplication by $\frac{1}{s}$) has the effect of integrating $F(t)$ from 0 to t.

9 The convolution property

If

$$L^{-1}\{f_1(s)\} = F_1(t) \text{ and } L^{-1}\{f_2(s)\} = F_2(t),$$

then,

$$L^{-1}\{f_1(s)f_2(s)\} = \int_0^t F_1(u)F_2(t-u)du$$

$$= F_1(t) * F_2(t)$$

With this background knowledge, we are now ready to tackle our problem of solving the magnetohydrodynamic equation by this method of Laplace transform.

CHAPTER THREE

SOLUTION OF MAGNETOHYDRODYNAMIC EQUATION

There are several people who have looked at solutions of various types of flows of MHD equation.

Shereclif [1956] has studied the steady motion of electrically conducting fluid in pipes under transverse magnetic field. Singh and Ram [1978] have considered the laminar flow of an electrically conducting fluid through a channel in the presence of a transverse magnetic field, under the influence of a periodic pressure gradient. Simonura [1991] has considered MHD turbulent channel flow under a uniform transverse magnetic field. Kayuzuki [1992] has discussed inertia effects in two dimensional MHD channel flow. Singh [1993] discussed the flow of fluid in a channel under the influence of inclined magnetic field and solved the resulting differential equation by the method for solution of linear differential equations with constant coefficients. Again Singh [1996] considered the MHD unsteady flow of a dusty liquid through a channel under the influence of inclined magnetic field and solved the resulting equation using the Laplace transform method. Singh [1998] discussed the unsteady MHD flow of liquid through a channel under variable pressure gradient, and again solved it using the Laplace transform method.

In the present analysis, we solve the MHD problem for a flow under constant pressure gradient and under the influence of transverse magnetic field using the Laplace transform method.

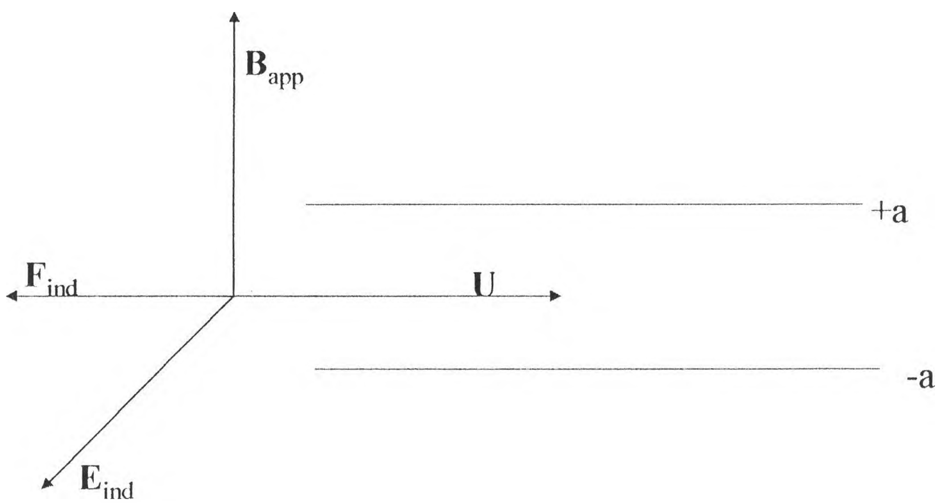
3.1 TWO DIMENSIONAL MAGNETOHYDRODYNAMIC FLOW

For simplicity, we shall assume that the flow is fully developed and thus has achieved a steady state of flow. The equation of continuity for the incompressible flow is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Where u, v, and w are the components of velocity of the fluid in the x, y, and z directions.

We take the flow to be in the x-direction, and the applied magnetic field **B** to be in the y direction as shown in the figure below.



Where **U** is the velocity of the liquid, $\mathbf{E}_{ind}$ is the induced electric current, $\mathbf{F}_{ind}$ the induced force and B_{app} is the applied magnetic field.

We break down Equation 1.2.18 ,and write it in a form that describes flow in each of the three directions as:-

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{F_x}{\rho} \qquad 3.1.2$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \frac{F_y}{\rho} \qquad 3.1.3$$

And

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial w} + v\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \frac{F_z}{\rho} \qquad 3.1.4$$

Where F_x, F_y, and F_z are the component of **J** x **B** in the x, y and z directions respectively.

For simplicity we shall consider a two dimensional flow.

In two dimension, equation 3.1.1 becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad 3.1.5$$

Since the plates are of infinite length, we assume that the flow is only along the x-axis and depend on y.

Thus

$$\frac{\partial u}{\partial x} = 0 \qquad 3.1.6$$

Since we have assumed a steady flow, the flow variables do not depend on time.

Thus equations 3.1.2 to 3.1.4 can now be written as

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial\ u}{\partial y^2}\right) + \frac{F_x}{\rho} \qquad 3.1.7$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{F_y}{\rho} \qquad 3.1.8$$

Now, the flow is in the direction of x only and from the fact that the plates making the channel is of infinite length (hence $\frac{\partial u}{\partial x} = 0$), we have

$$\frac{\partial^2 u}{\partial x^2} = 0$$

so that the equations of motion may now be written as

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right) + \frac{F_x}{\rho} \qquad 3.1.9$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{F_y}{\rho} \qquad 3.1.10$$

There is no component of body force in the y-direction, and F_x = **J** x **B** thus the equations above become

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right) + \frac{\vec{J}x\vec{B}}{\rho} \qquad 3.1.11$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad 3.1.12$$

Equation 3.1.12 implies that the pressure does not depend on y.

We have shown in chapter one that

$$\vec{J} = \sigma\vec{E}$$

and from equation 1.2.4, we find

$$\vec{E} = \vec{U} \times \vec{B}$$

Thus

$$\vec{J} \times \vec{B} = \sigma[(\vec{U} \times \vec{B}) \times \vec{B}]$$

$$= \sigma[(\vec{U}.\vec{B})\vec{B} - (\vec{B}.\vec{B})\vec{U}]$$

Since **U** and **B** are perpendicular ,

$$\vec{U}.\vec{B} = 0$$

giving

$$\vec{J} \times \vec{B} = -B^2\vec{U}$$

Thus

$$\frac{\vec{J}\times\vec{B}}{\rho} = -\frac{\sigma B^2\vec{U}}{\rho} \qquad 3.1.13$$

The equation of motion now reduces to

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma}{\rho}B^2 u \qquad 3.1.14$$

and

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad 3.1.15$$

3.2 NON DIMENSIONALIZING

To simplify equation 3.1.13 further, we reduce the parameters in the equation by introducing the following non-dimensional quantities

$$x' = \frac{x}{a}, y' = \frac{y}{a}, p' = \frac{p}{\rho}\frac{a^2}{\nu^2}, u' = u\frac{a}{\nu} \qquad 3.2.1$$

With these quantities, we see that,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial u'}\frac{\partial u'}{\partial y'}\frac{\partial y'}{\partial y}$$

$$= \frac{v}{a}\frac{\partial u'}{\partial y'}\frac{1}{a}$$

$$= \frac{v}{a^2}\frac{\partial u'}{\partial y'} \qquad 3.2.2$$

Therefore,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right]$$

$$= \frac{\partial}{\partial y'}\left[\frac{v}{a^2}\frac{\partial u'}{\partial y'}\right]\frac{\partial y'}{\partial y}$$

$$= \frac{\partial}{\partial y'}\left[\frac{v}{a^2}\frac{\partial u'}{\partial y'}\right]\frac{1}{a}$$

$$= \frac{v}{a^3}\frac{\partial^2 u'}{\partial y'^2}$$

Again,

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial p'} \frac{\partial p'}{\partial x'} \frac{\partial x'}{\partial x}$$

$$= \frac{\rho v^2}{a^3} \frac{\partial p'}{\partial x'} \qquad 3.2.4$$

Similarly,

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial p'} \frac{\partial p'}{\partial y'} \frac{\partial y'}{\partial y}$$

$$= \frac{\rho v^2}{a^2} \cdot \frac{\partial p'}{\partial y'} \cdot \frac{1}{a}$$

$$= \frac{\rho v^2}{a^3} \cdot \frac{\partial p'}{\partial y'} \qquad 3.2.5$$

Putting these values in equations 3.1.13 and 3.1.14, we get

$$\frac{\partial p'}{\partial y'} = 0 \qquad 3.2.6$$

and

$$0 = -\frac{1}{\rho}\frac{\rho v^2}{a^3}\frac{\partial p'}{\partial x'} + v\frac{v}{a^3}\frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma B^2}{\rho}\frac{v}{a}u'.$$

For convenience, we shall drop the primes and write

$$0 = \frac{v^2}{a^3}\left[-\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2 a^2}{\rho v}u\right]$$

or

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2 a^2}{\rho v}u \qquad 3.2.7$$

We may write the above equation as

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - M^2 u \qquad 3.2.8$$

Where

$$M = Ba\sqrt{\frac{\sigma}{\mu}} \qquad \text{and here, } \mu = \rho v$$

M is known as the Hartmann number. It is directly proportional to B, the magnetic field.

We can differentiate equation 3.2.8 with respect to x and obtain

$$0 = \frac{\partial^2 p}{\partial x^2} \qquad 3.2.9$$

Since p does not depend on y, we may write 3.1.15 as a total derivative thus,

$$0 = \frac{d^2 p}{dx^2} \qquad 3.2.10$$

We can therefore see that

$$\frac{dp}{dx} = \text{constant.} \qquad 3.2.11$$

And we can therefore take the total derivatives of equation 3.2.8 instead of partial derivatives.

Let the constant in equation 3.2.11 be P. Equation 3.2.8 may now be written as

$$0 = P + \frac{d^2 u}{dy^2} - M^2 u$$

or

$$\frac{d^2 u}{dy^2} - M^2 u = -P \qquad 3.2.12$$

We solve this equation using the Laplace transform method .

3.3 SOLUTION OF THE EQUATION

From equation 2.4.6, the Laplace transform of $\frac{d^2u}{dy^2}$ is

$$L\left\{\frac{d^2u}{dy^2}\right\} = s^2\bar{u} - su(0) - u'(0) \tag{2.4.6}$$

while the Laplace transform of $-P$ is $\frac{-P}{s}$

Thus equation 3.2.12 becomes

$$s^2\bar{u} - su(0) - u'(0) - M^2\bar{u} = \frac{-P}{s}$$

or

$$s^2\bar{u} - M^2\bar{u} = \frac{-P}{s} + su(0) + u'(0) \tag{3.3.1}$$

Since we do not know the values of $u(0)$ and $u'(0)$, we let

$$u(0) = c_1 \text{ and } u'(0) = c_2 \tag{3.3.2}$$

Equation 3.3.1 now becomes $s^2\bar{u} - M^2\bar{u} = \frac{-P + c_1 s^2 + c_2 s}{s}$ 3.3.3

which gives

$$\bar{u} = \frac{c_1 s^2 + c_2 s - P}{s(s^2 - M^2)}$$

We may write the equation above as a sum of partial fractions, thus

$$\bar{u} = \frac{c_1 s^2 + c_2 s - P}{s(s^2 - M^2)} = \frac{A}{s} + \frac{B}{s + M} + \frac{D}{s - M} \qquad 3.3.4$$

When this equation is simplified, it yields

$$c_1 s^2 + c_2 s - P = (A + B + D)s^2 + (D - B)Ms - AM^2$$

which gives the values of A, B, and D as

$$A = \frac{P}{M^2}$$
$$B = \frac{c_1}{2} - \frac{c_2}{2M} - \frac{P}{2M^2} \qquad 3.3.5$$
$$D = \frac{c_1}{2} + \frac{c_2}{2M} - \frac{P}{2M^2}$$

When we insert the values above into equation 3.3.4, we obtain

$$\bar{u} = \frac{P}{sM^2} + \frac{c_1}{2(s+M)} - \frac{c_2}{2M(s+M)} - \frac{P}{2M^2(s+M)} +$$

$$+ \frac{c_1}{2(s-M)} + \frac{c_2}{2M(s-M)} - \frac{P}{2M^2(s-M)}$$

$$= \frac{P}{sM^2} + \frac{c_1 s}{s^2 - M^2} + \frac{c_2 M}{M(s^2 - M^2)} - \frac{Ps}{M^2(s^2 - M^2)} \qquad 3.3.6$$

We now get the inverse Laplace transform of the equation above, remembering that

$$L^{-1}\left\{\frac{P}{sM^2}\right\} = \frac{P}{M^2} L^{-1}\left\{\frac{1}{s}\right\} = \frac{P}{M^2}, \qquad 3.3.7$$

$$L^{-1}\left\{\frac{c_1 s}{s^2 - M^2}\right\} = c_1 L^{-1}\left\{\frac{s}{s^2 - M^2}\right\} = c_1 \cosh My \qquad 3.3.8$$

$$L^{-1}\left\{\frac{c_2 M}{M(s^2 - M^2)}\right\} = \frac{c_2}{M} L^{-1}\left\{\frac{M}{s^2 - M^2}\right\} = \frac{c_2}{M} \sinh My \qquad 3.3.9$$

Similarly,

$$L^{-1}\left\{\frac{Ps}{M^2(s^2 - M^2)}\right\} = \frac{P}{M^2} L^{-1}\left\{\frac{s}{s^2 - M^2}\right\} = \frac{P}{M^2} \cosh My \qquad 3.3.10$$

and $$L^{-1}\overline{u} = u \qquad 3.3.11$$

Thus,

$$u = \frac{P}{M^2} + c_1 \cosh My + \frac{c_2}{M}\sinh My - \frac{P}{M^2}\cosh My \qquad 3.3.12$$

From the boundary conditions, we know that

$$u = 0 \text{ When } y = 1 \text{ or } -1$$

Inserting these conditions in equation 3.3.13 consecutively gives,

for y=1,
$$0 = \frac{P}{M^2} + c_1 \cosh M + \frac{c_2}{M}\sinh M - \frac{P}{M^2}\cosh M \qquad 3.3.13$$

and for y=-1, we have

$$0 = \frac{P}{M^2} + c_1 \cosh(-M) + \frac{c_2}{M}\sinh(-M) - \frac{P}{M^2}\cosh(-M) \qquad 3.3.14$$

The definition of cosh M is

$$\cosh M = \frac{e^{M} + e^{-M}}{2}$$

and
$$\sinh M = \frac{e^{M} - e^{-M}}{2}$$

We can therefore see that

$$\cosh(-M) = \frac{e^{M} + e^{M}}{2}$$

$$= \frac{e^{M} + e^{-M}}{2}$$

$$= \cosh M$$

similarly we can easily verify that

$$\sinh(-M) = -\sinh M$$

Thus equation 3.3.14 may now be written as

$$0 = \frac{P}{M^2} + c_1 \cosh M - \frac{c_2}{M} \sinh M - \frac{P}{M^2} \cosh M$$

The simultaneous equations 3.3.13 and 3.3.14 now may be written as

$$0 = \frac{P}{M^2} + c_1 \cosh M + \frac{c_2}{M} \sinh M - \frac{P}{M^2} \cosh M$$

and $$0 = \frac{P}{M^2} + c_1 \cosh M - \frac{c_2}{M} \sinh M - \frac{P}{M^2} \cosh M$$

Adding the two equations together gives

$$0 = \frac{P}{M^2} + c_1 \cosh M - \frac{P}{M^2} \cosh M \qquad 3.3.15$$

which means that

$$c_1 = \frac{P}{M^2} - \frac{P}{M^2 \cosh M} \qquad 3.3.16$$

Substituting this equation in 3.3.13 gives

$$c_2 = 0$$

Thus we find that

$$u = \frac{P}{M^2} + \frac{P}{M^2} \cosh My - \frac{P \cosh My}{M^2 \cosh M} - \frac{P}{M^2} \cosh My$$

or

$$u = \frac{P}{M^2} - \frac{P \cosh My}{M^2 \cosh M}$$

$$= \frac{P}{M^2}\left(1 - \frac{\cosh My}{\cosh M}\right) \qquad 3.3.17$$

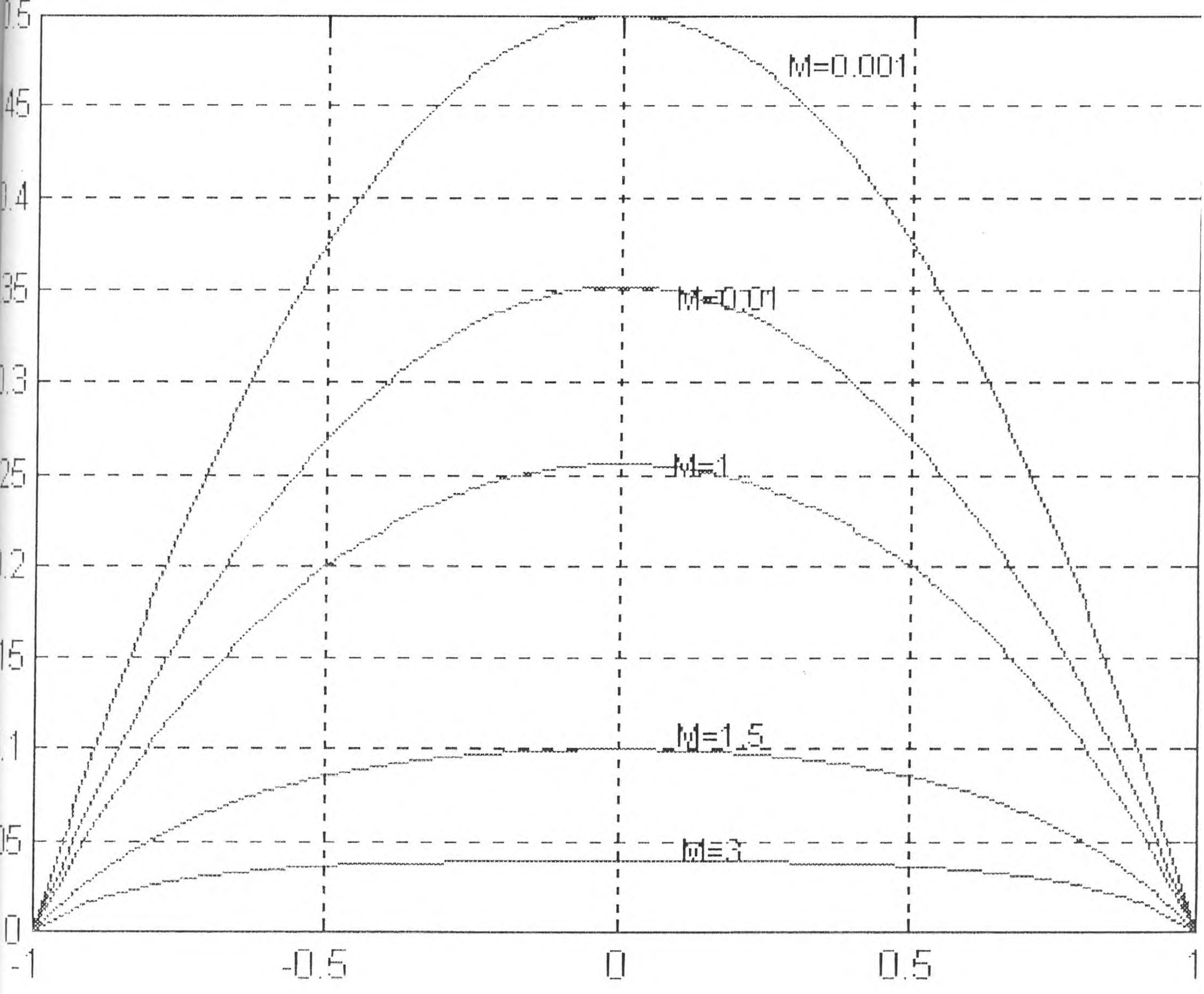
This is the solution we were looking for.

It is clear from the equation that the velocity of the magnetofluid depends on the Hartmann number. The velocity decreases as the Hartmann number increases.

The velocity profile for different Hartman numbers is shown in the diagram below.

Velocity Profile for some selected values of Hartmann Number.

(The value of P has been taken to be 1)



DISCUSSION OF RESULTS AND CONCLUSION

We have successfully managed to use the boundary conditions to find the initial values that we had assumed to be c_1 and c_2

These initial values have been used to solve the initial value problem that was found. Thus the boundary value problem was successfully converted to an initial value problem. This initial value problem has been solved using the Laplace Transform method, and the solution found is similar to the solutions found using other conventional methods (see for example Singh [1993]).

It is clear from the figure on the previous page that as the value of the Hartmann number increases, the velocity of the fluid decreases.

APPENDIX

KEY VECTOR IDENTITIES.

The following vector identities have been adapted from R. Dendy (1993),

Let **A** , **B** and **C** be vectors. Then the following identities hold.

A+**B** = **B** + **A**

A.**B** = **B**.**A**

A X **B** = -**B** X **A**

A (**B** X **C**) = **C** (**A** X **B**)= **B** (**C** X **A**)

A X (**B** X **C**) = **B**(**A**.**C**) - **C**(**A**.**B**)

(**A** X **B**).(**C** X **D**) = (**A**.**C**)(**B**.**D**) - (**A**.**D**)(**B**.**C**)

Key Results from Vector calculus

Operator: - $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

Gradient: $\nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

For a vector field $A(x, y, z)$,

divergence: $\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

curl:
$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Where $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$

$$\nabla \cdot (\nabla \times A) = 0$$

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$$

For two vector fields, $A(x, y, z)$ and $B(x, y, z)$

$$\nabla \cdot (A \times B) = B.(\nabla \times A) - A \cdot (\nabla \times B)$$

$$\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B$$

REFERENCES

1) Ditkin, V. A. And Prudnikov, A. P [1965]. *Integral Transforms and Operational Calculus.* Pergamon Press, Oxford,.

2) Dragos, L [1975]. *Magnetofluid Dynamics.* Abacus Press, Tunbridge Wells.

3) Durney, Carl H. [1969]. *Introduction to Modern Electromagnets* McGraw Hill Book Company, New York.

4) Earl D. Rainville and Bedient P. E. [1981] *Elementary Differential Equations (6th Ed).* Macmillan Publishing Co. Inc, New York.

5) Feraro, V.C.A.[1961]. *An Introduction to Magnetofluid Mechanics.* Oxford University Press, Glasgow.

6) George Vahala [24th June, 2001]. *"Magnetohydrodynamics"* Discovery Channel School, original content provided by World Book Online, http://www.discoveryschool.com/homeworkhelp/worldbook/atozscience/m/338450.html.

7) Grant, I. S. & Phillips, W. R. [1975] *Electromagnetism.* John Willy & Sons Ltd, London

8) Griffiths, David J. [][1999] *Introduction to Electrodynamics.* Prentice Hall, New Jersey.

9) Jeffrey, A. [1966] *Magnetohydrodynamics.* Oliver & Boyd, Edinburgh.

10) Lenhert Bo. [1964] *Dynamics of Charged Particles.* North-Holland Publishing company, Amsterdam.

11) Little, Noel C. [1967] *Magnetohydrodynamics, a Fusion of Old and New.* D. Ban Nostrard Company, Princeton.

12) Mironer A. [1979] *Engineering fluid Mechanics.* Mac Graw Hill Book Company

13) Rendy, Richard. [1993]. *Plasma Physics: An Introductory Course.* Cambridge University Press, Abingdon.

14) Reswill, Robert & Halliday D. [1966] Physics. New Age International (P) Limited, New Delhi.

15) Shercliff, J. [1965] *A Textbook of Magnetohydrodynamics.* Pergamon Press, Oxford

16) Singh C. B. [1993]. *Magnetohydrodynamic Steady Flow of Liquid Between Parallel Plates.* Proceedings of the 1st Conference of the Kenya Mathematical Society, Jomo Kenyatta Foundation.

17) Singh C. B [1996] *Magnetohydrodynamic Unsteady Flow of a Dusty Liquid Through a Channel Under the Influence of Inclined Magnetic Field.* J. International Journal of Biochemiphysics (vol 5 Nos. 1 &2).

18) Singh C.B [1998] *Unsteady Magnetohydrodynamic Flow of Liquid Through a Channel Under Variable Pressure Gradient.* J. Kenya Journal of Sciences

19) Streeter, V. Et al [1975] *Fluid Mechanics.* McGraw-Hill Kogakusha Ltd.

20) Thomson W. [1957] *Laplace Transformation: Theory and Engineering Applications.* Longmans, Green and Co. London.