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Master Dissertation in Mathematics

Combinatorics of Hurwitz numbers

Research Report in Mathematics, Number 66, 2016

Ignace Ntezimana

August 2016

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

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Abstract

The goal of this dissertation, is to count branched covering of $\mathbb{P}^1$. The formal count was first instigated by A. Hurwitz in his landmark paper of 1981. Hence the numbers associated to the count of branched covering are called Hurwitz numbers. The general idea is to count the number of holomorphic functions to the complex projective line $\mathbb{P}^1$ by fixing some geometrical conditions to guarantee the finite count. It is shown in this thesis that this number is always finite and using combinatorial and representation theoretic techniques we provide some examples.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

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In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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Dedication

This dissertation is dedicated to my parents.

Contents

Acknowledgments

First, I thank God almighty for His strength and grace throughout the time I worked on my dissertation. I especially wish to thank my first supervisor, Jared Ongaro who introduced me to the area of algebraic geometry which I feel to spend the rest of my time discovering the amazing ideas behind this beautiful mathematics. You are not only my supervisor but also my role model. I was also privileged to have James Katende as my second supervisor, his advices, friendship and encouragement contributed greatly to my work.

My sincere gratitude goes to the University of Nairobi especially the School of Mathematics where I was warmly welcomed by Charles Nyandwi and Mwalimu Claudio Achola. Their lifestyle, friendship, mentorship and interactive nature will always be in my good memories. I really appreciate those who have put efforts and ideas to come up with the EAUMP program. Their contributions to the East African mathematical community have impacted greatly to the young mathematicians.

Lastly, I extend my happiness to my family who have contributed to my education up to date, my classmates especially Pauline Achieng' who have been always there to motivate me, my best friend Nsabimana Jean Paul (Maxime) who came to assist in good time. Last but not least, many thanks goes to my fiancé Uwingabo Redampta for her supportive ideas.

Ignace Ntezimana

Nairobi, 2016.

1 Introduction

The aim of this thesis is to compute topological invariants associated to curves called **Hurwitz numbers**. This is a counting problem in algebraic geometry and the branch of mathematics which deal with this kind of problems specifically is called **enumerative algebraic geometry**. Indeed, one of the famous problem in this field is to find all geometrical different surfaces with the same Euler characteristic or genus branched over a set of fixed points on the projective line. It turns out that there is no finite answer to such kind of problem, but we can obtain a finite count if we care to fix some geometrical condition: the degree d of branch maps, branch profile on each of the fixed branch points. The approach to this was instigated by A. Hurwitz in 1891 in counting maps called branch coverings.

The holomorphic function $f : X \to \mathbb{P}^1$ is called a **meromorphic function**. By considering a meromorphic function f of degree d and a point $q \in \mathbb{P}^1$, we have an divisor $f^{-1}(q) = \lambda_1 p_1 + \lambda\lambda_2 p_2 + \ldots + \lambda_n p_n$, where $p_1, \ldots p_n$ are distinct points on X and $\lambda_1, \ldots \lambda_n$ are positive integers summing to d. The set $(\lambda_1, \ldots \lambda_n)$ is called **branch type** of f at a point q. If the branch type of f at q equals to $(1, 1, \ldots, 1)$, then we say that f is not branched over q and if the branch type corresponds to $(2, 1, \ldots, 1)$ at q, we say that q is a **simple** branch point of f. The datum listed above is determined uniquely by Riemann-Hurwitz

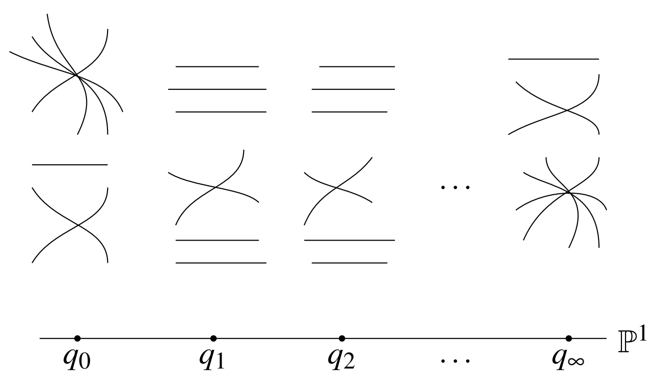


Figure 1. Local Picture of a branched covering map

formula. We need a powerful toolkit of preliminaries material to achieve our goal which we survey it in the following order:

Chapter 2: This chapter is dedicated to topological surfaces; the heart of the chapter is the structure of the Riemann surfaces and maps between them.

Chapter 3: Here we give a detailed view of combinatorics and representations of symmetric group S_d, we describe how partition and representation of S_d are linked. We also survey the theory of irreducible representations of the symmetric group S_d which are fundamental in the computation of Hurwitz numbers.

Chapter 4: Finally, we connect the required materials to calculate the Hurwitz numbers and give examples to make this explicit.

2 Surfaces

Topological surfaces or **surfaces** are compact, connected 2-dimensional real manifolds. If in additional the surface has a complex structure, then it is called a **compact Riemann surface.** For such surfaces their exists a discrete invariant usually denoted by g called (topological) **genus** which uniquely characterize them. In fact, the topological classification of compact Riemann surfaces [RM95], establishes that up to diffeomorphism, there is exactly one compact Riemann surface for each g.

2.1 Topological surfaces

In this section, we will discuss topological surfaces, we start by reminding ourself some basics properties of a topological space.

Definition 2.1.1. *A topological space is a set X together with a collection τ of subsets of X (called open subsets of X) such that*

T1. $\emptyset \in \tau$ *and* $X \in \tau$,

T2. if $U,V \in \tau$ *then* $U \cap V \in \tau$,

T3. if $U_i \in \tau\ \forall i \in I$ *then* $\cup_{i\in I} U_i \in \tau$.

Remark 2.1.2.

*(i) X is called **Hausdorff** if whenever $x, y \in X$ and $x \neq y$ there are open subsets U, V of X such that $x \in V$ and $U \cap V = \emptyset$,*

*(ii) X is called **compact** if every open cover of X has a finite subcover.*

We can now give the precise definition of a topological surface.

Definition 2.1.3. *A **topological surface** (simply called a surface) is a Hausdorff topological space X such that each point $x \in X$ is contained in an open subset U which is homeomorphic to an open subset V of $\mathbb{R}^2$. X is called a **closed** surface if it is compact.*

Note that a surface is also sometimes called a 2-manifold i.e. a manifold of real dimension 2. One way of constructing surfaces is by identification at the boundary of a planar figure. The best way to do this by using identification map. For example, in constructing the torus from the square we define $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$.

Indeed, let $X = [0,1] \times [0,1]$ (or any rectangle). Define $(x_1, y_1) \sim (x_2, y_2)$ if

(a) $(x_1, y_1) = (x_2, y_2)$, with $0 < x_1, y_1 < 1$.

(b) $(x_1, y_1) = (0, y)$ and $(x_2, y_2) = (1, y)$, with $0 < y < 1$.

(c) $(x_1, y_1) = (0, x)$ and $(x_2, y_2) = (x, 1)$, with $0 < x < 1$.

(d) $(x_1, y_1), (x_2, y_2) \in \{(0,0), (1,0), (0,1), (1,1)\}$.

Thus the resulting space $X = [0,1] \times [0,1]/\sim$ is homeomorphic a torus.

The torus is the set of equivalence classes and we give this a topology as follows.

Definition 2.1.4. *Let $\sim$ be an equivalence relation on a topological space X. If $x \in X$ let $[x]_\sim = \{y \in X : y \sim x\}$ be the equivalence class of x and let*

$$X/\sim = \{[x]_\sim : x \in X\}$$

be the set of equivalence classes. Let $\pi : X \to X/\sim$ be the quotient map which sends an element of X to its equivalence class. Then the quotient topology on $X/\sim$ is given by

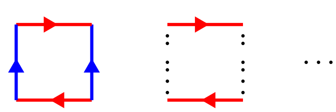
$$\{V \subseteq X/\sim : \pi^{-1}(V) \text{ is an open subset of } X\}.$$

In other words a subset V of $X/\sim$ is an open subset of $X/\sim$ (for the quotient topology) if and only if its inverse image

$$\pi^{-1}(V) = \{x \in X : [x]_\sim \in V\}$$

is an open subset of X.

The following are more examples by identification of a square:



The Möbius band is not closed, as the dotted lines suggest. Here is its rigorous definition:

Definition 2.1.5. *A Möbius band (or Möbius strip) is a surface which is homeomorphic to*

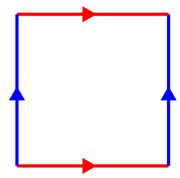
$$(0,1) \times [0,1]/\sim$$

with the quotient topology, where $\sim$ is the equivalence relation given by

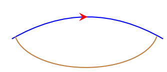
$$(x,y) \sim (s,t) \text{ iff } (x = s \text{ and } y = t) \text{ or } (x = 1 - s \text{ and } \{y,t\} = \{0,1\})$$

2.1.1 Planar models and connected sums

In general, we can construct a closed surface so long as we prescribe the way to identify the sides in pairs. By using arrows to identify sides in pairs systematically: going round clockwise we give each side a letter a say, and when we encounter the side to be identified we call it a if the arrow is in the same clockwise direction and a^{-1} if it is the opposite. For example, instead of



We simply have $aa^{-1}bb^{-1}$ which represent the sphere where the top side is represented by a and the bottom one by b. The same way, projective space is then $abab$,the Klein bottle $abab^{-1}$ and the torus $aba^{-1}b^{-1}$. Note that there are lots of planar models which define the same surfaces, for example the sphere can be defined also by aa^{-1}, a 2-sided polygon.



Similarly the projective plane is aa. We can also get new surfaces by taking more sides, but another nice way to construct new surfaces is to connect two existing surfaces by taking a homeomorphism from the boundary of one disc to the boundary of the other which is called **connected sum** written $A\sharp B$ where A and B are two closed surfaces. For more clarification let us take a look on the following example:

Connected sum with the projective plane P is sometimes called **attaching a cross-cap**. In fact, removing a disc from P gives the Möbius band and we remark that the connected sum $P\sharp P$ is the Klein bottle.

This connected sum give us a chain of new surfaces as the next proposition lists.

Proposition 2.1.6. *The connected sum of a torus T and the projective plane P is homeomorphic to the connected sum of three projective planes.*

2.1.2 The classification of surfaces

Connected surfaces can be classified into the following categories:

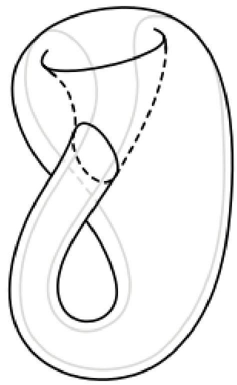


Figure 2. Klein bottle

(i) Connected compact surfaces,

(ii) Connected non compact surfaces.

Our interest in this thesis will be **connected compact surfaces**. Note that we can still classify our surfaces by whether they are orientable or non orientable.

2.1.3 Orientability of surfaces

One way of deciding what a connected sum is in the classification theorem is to check the orientability of the surface.

Definition 2.1.7. *A surface X is orientable if it contains no open subset homeomorphic to a Möbius band.*

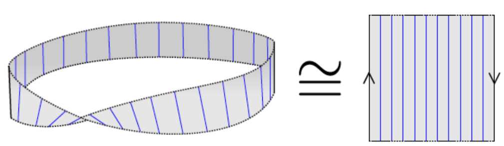


Figure 3. A Möbius strip

Clearly, if X is orientable then any surface homeomorphic to X is also orientable. We saw taking the connected sum with the projective plane means attaching a Möbius band, so the surfaces which are connected sums of projective plane are non-orientable, on other hand the connected sums of tori are orientable.

Another important tool is the **Euler characteristic/Euler number**. The Euler characteristic of a surface is a topological invariant used in the classification of surfaces.

2.1.4 The Euler characteristic

Before we define Euler characteristic for any surface, we first give the following motivations.

Definition 2.1.8. *A triangulation or subdivision of a compact surface X is a partition of X into:*

(i) vertices (these are finitely many point of X),

(ii) edges (finitely many disjoint subsets of X each homeomorphic to the open interval $(0,1)$), and

(iii) faces (finitely many disjoint open subsets of X each homeomorphic to the open disc $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ in $\mathbb{R}^2$, such that:

(a) the faces are the connected components of $X \setminus \{$vertices and edges$\}$,

(b) no edge contains a vertex, and

(c) each edge begins and ends in a vertex(either the same vertex or different vertices), or more precisely, if e is an edge then there are vertices v_0 and v_1 (not necessarily distinct) and a continuous map

$$f : [0,1] \to e \cup \{v_0, v_1\}.$$

which restricts to a homeomorphism from $(0,1)$ to e and satisfies $f(0) = v_0$ and $f(1) = v_1$.

Definition 2.1.9. *The Euler characteristic of a compact surface X with a subdivision is*

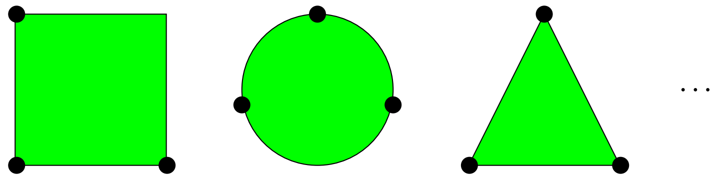
$$\chi(X) = V - E + F,$$

where V is the number of vertices, E is the number of edges and F is the number of faces in the subdivision.

The existence of triangulation for a closed surface guarantee its subdivision for which we have the important fact.

Theorem 2.1.10. *The Euler characteristic of a compact surface is independent of the subdivision*

For a finite polyhedron, the Euler characteristic is simply involves counting of vertices, edges and faces for a given triangulation in terms of topological triangles. Topological triangle look like this:

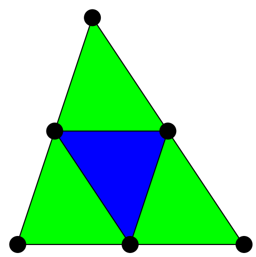


We remark that topological triangles have 3-vertices, 3-edges and 1-face. The point now is how we do counting of vertices edges and faces. Thus, the Euler characteristic of a triangle T is given by

$$\chi(T) = V - E + F = 3 - 3 + 1 = 1$$

Note that subdividing the triangle "**nicely**" does not change the Euler characteristic.

Example 2.1.11.

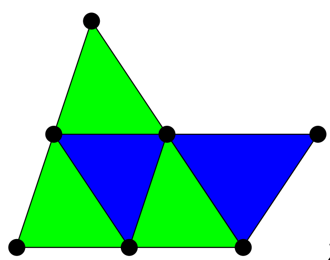


$\chi = 6 - 9 + 4 = 1$

i.e. we are adding 0 *in "clever way"* $3 - (3+3) + 3 = 0$.

The Euler characteristic can give an information about **shape** of a surface because even adding a triangle **nicely** does not change the Euler characteristic.

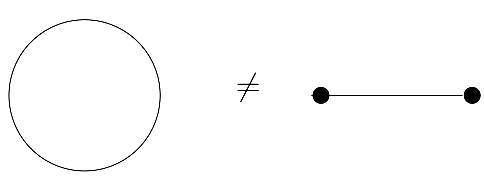
Example 2.1.12.



$\chi = 7 - 11 + 5 = 1$

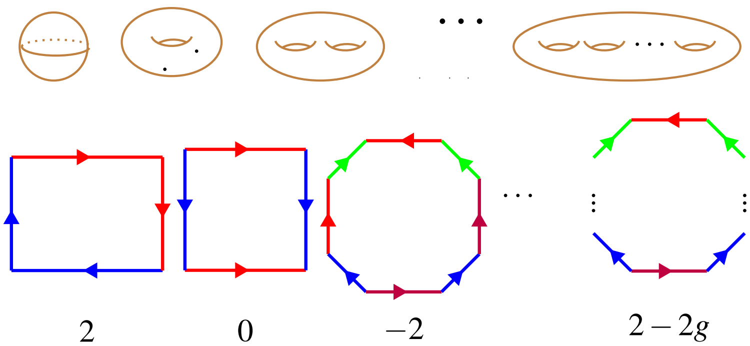
The Euler characteristic is a **topological invariant**. In other words, the Euler characteristic can be used to distinguish between objects. Objects with the same Euler characteristic need not be topologically equivalent.

Example 2.1.13. *All of the following are topologically different*



But they have the same Euler characteristic.

Although, we have seen that the Euler characteristic is **not** a **perfect** topological invariant, it can be used to identify surfaces. Below we indicate the Euler characteristic of all orientable closed connected surfaces:



where g is the **genus** of the the surface. In the rest of this thesis by a **surfaces**, we mean **connected orientable compact surfaces**.

Theorem 2.1.14. *A closed surface is determined up to homeomorphism by its orientability and its Euler characteristic.*

Here we consider connected sums of spaces. Suppose a surface is made up of the union of two spaces X and Y, such that the intersection $X \cap Y$ has a subdivision which is a subset of the subdivisions for X and for Y. Then since V, E and F are just counting the number of elements in a set, we have immediately observe that

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$$

Therefore, for a connected sum, take a closed surface X and remove a disc D to get a space X^0. The disc has Euler characteristic 1 (a polygon has one face, n vertices and n sides) and the boundary circle has Euler characteristics 0 (no face). We then apply the formula,

$$\chi(X) = \chi(X^0 \cup D) = \chi(X^0) + \chi(D) - \chi(X^0 \cap D) = \chi(X^0) + 1$$

To get the connected sum we paste X^0 to Y^0 along the boundary circle so

$$\chi(X \sharp Y) = \chi(X^0) + \chi(Y^0) - \chi(X^0 \cap Y^0) = \chi(X) - 1 + \chi(Y) - 1 - 0 = \chi(X) + \chi(Y) - 2.$$

In particular, $\chi(X \sharp T) = \chi(X) - 2$ this again gives the value $2 - 2g$ for a connected g tori.

We close this discussion with a strong result about classification of surfaces.

Theorem 2.1.15. *A closed, connected surface is either homeomorphic to:*

1. aa^{-1}, *or*
2. $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$, *where g is the genus.*

2.2 Fundamental group

The first and simplest homotopy group is the fundamental group, which records information about loops in a space. To build this group we need the following ingredients.

Definition 2.2.1. *Let $f,g : X \to Y$ be maps of topological spaces. A* ***homotopy*** *between f and g is a continuous function*

$$H : X \times [0,1] \to Y$$

such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

Note that, if a homotopy H exists, we say that f and g are homotopic and write $f \sim g$.

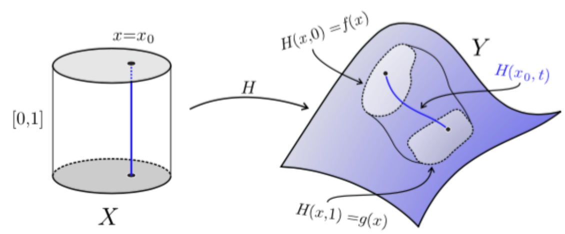


Figure 4. Schematic picture of homotopy of maps

Example 2.2.2. *For any topological space X, any two continuous maps $f,g : X \to \mathbb{R}^n$ are homotopic. This can be seen by considering the straight line homotopy $H : X \times [0,1] \to \mathbb{R}^n$ given by*

$$H(x,t) = (1-t)f(x) + tg(x).$$

Definition 2.2.3. *Two topological spaces X,Y are called* ***homotopy eqivavlent*** *(or simply homotopic) and denoted $X \sim Y$, if there exist maps $f : X \to Y$ such that*

$$g \circ f \sim Id_X,$$

$$f \circ g \sim Id_Y.$$

A topological space which is homotopy equivalent to a point is called **contractible**.

Definition 2.2.4. *Let X be a topological space and $x_0 \in X$. A* ***loop*** *in X with base point x_0 is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$*

Two loops γ, δ with base point x_0 are said to be **homotopic with respect to the base point** if there exists a homotopy $H : [0,1] \times [0,1] \to X$ between γ and δ such that for every $t \in [0,1]$ we have $H(0,1) = H(1,t) = x_0$.

Definition 2.2.5. *Given two loops γ_1, γ_2 in X with base point x_0, we define the loop $\gamma_1 * \gamma_2$ in X with base point x_0 as follows :*

$$\gamma_1 * \gamma_2(s) = \begin{cases} \gamma_1(2s) & \text{if } s \in [0, 1/2] \\ \gamma_2(2s-1) & \text{if } s \in [1/2, 1]. \end{cases}$$

Theorem 2.2.6. *Let X be a topological space and $x_0 \in X$. Then the set of equivalence classes of loops with base point x_0 is a group under the binary operation $*$.*

Definition 2.2.7. *Let X be a topological space and $x_0 \in X$. The **fundamental group of X** with base point x_0, denoted $\pi_1(X, x_0)$ is the group of equivalence of loops based at x_0, with operation induced by concatenation of loops $*$*

Given $f : X \to Y$ a continuous map, we now define a function $\pi_1(f)$ between the fundamental groups:

$$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$
$$[\gamma] \mapsto [f \circ \gamma].$$

This function is called a **functor** Some fews examples of fundamental groups.

Example 2.2.8. ***Circle** any loop on the circle S^1 is homotopic to a loop that travels at constant speed around the circle clock-wise(or counter clock-wise) an integer number times. Furthermore, each loop can be seen as multiple iterations of a loop which travels around only once. Thus*

$$\pi(S^1) = \mathbb{Z}.$$

Example 2.2.9. *Take g circles, label a point on each one, and glue them together at a chosen points. The resulting space is called a flower graph. The loop on such graph are generated by the simple loops which go around each petal once, and there are no relation between them. Thus, if the graph Γ has g petals, then $\pi_1(\Gamma) = \mathbb{F}_g$, the free group on g generators.*

2.3 Riemann surfaces

In this section, we aim to study the classical surfaces called **Riemann surface** which will remain our target in the rest of this work. Riemann surfaces are the particular class of surfaces with complex structure. We follow the ideas of [CM15] for a quick view of Riemann surface.

Definition 2.3.1. *A topological space X is said to be a Riemann surface if it satisfies the following conditions.*

1. *X is a Hausdorff connected topological space.*

2. *For all $x \in X$ there exist a neighborhood $U_x \subset X$ of x and homeomorphisms $\varphi_x : U_x \longrightarrow V_x$ where V_x is on open set in $\mathbb{C}$.*

3. *For any U_x, U_y such that $U_x \cap U_y \neq \emptyset$ the transition function*

$$\tau_{y,x} := \varphi_y \circ \varphi_x{}^{-1} : \varphi_x(U_x \cap U_y) \longrightarrow \varphi_y(U_x \cap U_y)$$

is holomorphic.

Remark 2.3.2. *A Riemann surface is analytic manifold of complex dimension* 1*.*

Example 2.3.3. *The following is a couple of examples of Riemann surfaces.*

1. *Graphs of complex functions $f(z)$. A graph of continuous complex functions give us a class of Riemann surfaces. Let $f(z)$ be a continuous function mapping $\mathbb{C}$ to $\mathbb{C}$. The graph of f is the set $\Gamma_f := (z, f(z)) : z \in \mathbb{C} \subset \mathbb{C} \times \mathbb{C}$. We first note that Γ_f is Hausdorff since $\mathbb{C} \times \mathbb{C}$ is. The atlas of this structure is given by one chart which is all of Γ_f with the local coordinate function which is the first projection map $\phi := \pi_1 \setminus \Gamma_f$ which sends $(z, f(z))$ to z. Given that the above map ϕ is homeomorphic to its image and there is only one chart, the holomorphicity of transition functions is trivially satisfied. Thus Γ_f is a Riemann surface.*

2) *For any $f(x,y) \in \mathbb{C}[x,y]$,the set $V(f) := \{(x,y) \mid f(x,y) = 0\} \subset \mathbb{C}^2$ is called an affine plane curve. We say that $V(f)$ is smooth if there is no $(x_0, y_0) \in V(f)$ such that*

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 = \frac{\partial f}{\partial y}(x_0, y_0).$$

Note that a smooth affine plane curve is a Riemann surface.

2.3.1 Compact Riemann surfaces

Since compactness is a strong constraint on the geometry of surfaces, amongst all Riemann surfaces the compact ones are especially important. We examine some examples of compact Riemann surfaces.

1. Projective line $\mathbb{P}^1$: Let $U_1 = U_2 := \mathbb{C}$ and define $g : U_1 \diagdown \{0\} \longrightarrow U_2 \backslash \{0\}$ by $g(z) = \frac{1}{z}$ where $z \in \mathbb{C}$. For $i = 1, 2$, we denote by $[U_i]$ the image of the set U_i after the identification by g, note that $[U_i]$ is an open set in $\mathbb{P}^1$. Define the local coordinate functions $\varphi_i : [U_i] \longrightarrow U_i$ by $\varphi_i(p) = z_i$, where $z_i \in U_i$ such that $[z_i] = p$. Both φ_1 andφ_1 are homeomorphisms.

   We now consider transition functions. First, start with τ_{21}.The intersection $[U_1] \cap [U_2] = [U_1 \backslash 0] = [U_2 \backslash 0]$ and $\varphi_1([U_1] \cap [U_2]) = \mathbb{C} \setminus \{0\}$. This is the domain of $\tau_{21} = \varphi_2 \circ \varphi_1^{-1}$

and for $z_1 \neq 0$ we have

$$z_1 \longmapsto [z_1] = [z_2 = g(z_1) = \frac{1}{z_1}] \longmapsto z_2 = \frac{1}{z_1}.$$

Since τ_{21} has a pole only at $z_1 = 0$, it is holomorphic on $\mathbb{C} \setminus \{0\}$. Similarly τ_{12} is holomorphic, and thus $\mathbb{P}^1$ is a Riemann surface.

2. Projective curves: Given $P \in \mathbb{C}[X,Y,Z]$ a homogeneous polynomial of degree d, the set

$$V(P) := \{[X:Y:Z] \in \mathbb{P}^2(\mathbb{C}) \mid P(X,Y,Z) = 0\}$$

is called a **plane projective curve** of degree d.

If $\{(X,Y,Z) \in \mathbb{C}^3 | \frac{\partial P}{\partial X} = \frac{\partial P}{\partial Y} = \frac{\partial P}{\partial Z} = 0\} \subseteq \{(0,0,0)\}$, then $V(P)$ is said to be **smooth**. Note that a smooth projective plane curve $V(P)$ is a compact Riemann surface in the following sense. $V(P)$ is compact since it is a closed set in $\mathbb{CP}^2$. To show that $V(P)$ is a Riemann surface, it is sufficient to show that its intersection with any of the coordinate open sets of $\mathbb{P}^2$ is a Riemann surface. Consider without loss of generality the chart

$$U_Z = \{[X:Y:Z] | Z \neq 0\} \subseteq \mathbb{CP}^2$$

with affine coordinates

$$(x,y) = \varphi_Z(X,Y,Z) = (X/Z, Y/Z).$$

The set $\varphi_Z(V(P) \cap U_Z)$ is equal to $V(P)$, where $p(x,y) : P(x,y,1)$ is called the **dehomogenization** of P with respect to Z.
For any $(x,y) \in \mathbb{C}^2$,

$$\begin{aligned} \frac{\partial p}{\partial x}(x,y) =& \frac{\partial P}{\partial X}(x,y,1), \\ \frac{\partial p}{\partial y}(x,y) =& \frac{\partial P}{\partial Y}(x,y,1). \end{aligned}$$

Claim: There can be no $(\bar{x},\bar{y}) \in \mathbb{C}^2$ such that $f(\bar{x},\bar{y}) = \frac{\partial f}{\partial x}(\bar{x},\bar{y}) = \frac{\partial f}{\partial y}(\bar{x},\bar{y}) = 0$. The above claim implies that $V(f)$ is a smooth affine plane curve and therefore a Riemann surface as shown in projective line case in [1], part 2.

2.3.2 Maps of Riemann surfaces

The foundation of this section is to give a quick review of the invariant properties of maps between Riemann surfaces. We begin this section with a couple of definitions.

Definition 2.3.4. *Let X, Y be Riemann surfaces and $f : X \to Y$ be a function. We say that f is* ***holomorphic*** *at $x \in X$ if every choice of charts $\varphi_x, \varphi_{f(x)}$ the function $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ is holomorophic at x.*

Note that we say that f is a **holomorphic map**, if f is holomorhpic on $U = X$ where $U \subset X$ is open. The function $F = \varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ is called a **local expression** for f.

This project involves holomorphic maps $f : X \longrightarrow \mathbb{P}^1$, where X is a compact connected Riemann surface. Such maps are called a **meromorphic functions**.

2.3.3 Meromorphic functions

We first recall that on a (compact) surface X, any continuous real function achieves its maximum at some point. If X is a Riemann surface and f a holomorphic function, then $|f|$ is continuous, assume it has its maximum at x. The maximum modulus principle says that f must be a constant in a neighborhood of x. If X is connected, then f is constant everywhere.

Though there are no nonconstant holomorphic functions on a connected compact Riemann surface X, there exist lots of meromorphic functions on X.

Definition 2.3.5. *A meromorphic function f on a Riemann surface X is a holomorphic map to the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.*

We now provide some few examples of meromorphic functions below.

Example 2.3.6. *A rational functional*

$$f(z) = \frac{p(z)}{q(z)}$$

where p and q are polynomials is a meromorphic function on the Riemann sphere $\mathbb{P}^1$.

For any two polynomials $p(z), q(z) \in \mathbb{C}[z]$ with no common roots, the rational function $f(z) = \frac{p(z)}{q(z)}$ defines a holomorphic map $f : \mathbb{P}^1 \to \mathbb{P}^1$. The following result tell us that such map is a rational function and hence a meromorphic function.

Theorem 2.3.7. *If $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a holomorphic map of Riemann surfaces, then f is a rational function: $f(z) = \frac{p(z)}{q(z)}$, with $p(z), q(z) \in \mathbb{C}[z]$.*

The next example is using the algebraic approach of a meromorphic function on torus.

Example 2.3.8. *Define*

$$\wp(z) = \frac{1}{z^2} + \Sigma_{w\neq 0}(\frac{1}{(z-w)^2} - \frac{1}{w^2})$$

where the sum is over all non-zero $w = mw_1 + nw_2$*. Since for* $2|z| < |w|$

$$|\frac{1}{(z-w)^2} - \frac{1}{w^2}| \leq 10\frac{|z|}{|w|}^3$$

this converges uniformly on compact sets so long as

$$\Sigma_{w\neq 0}\frac{1}{|w|^3} < \infty$$

But $mw_1 + nw_2$ *is never zero if* m, n *are real so we have an estimate*

$$|mw_1 + nw_2| \geq k\sqrt{m^2 + n^2}$$

so by the integral test we have convergence. Because the sum is essentially over all equivalence classes

$$\wp(z + mw_1 + nw_2) = \wp(z)$$

so that this is a meromorphic function on the surface X *called the Weierstrass P-function*

2.4 Branched coverings

In topology, a map is a branched covering if it is a covering map everywhere except for a nowhere-dense set known as the branch set. The general idea of the covering map is the following. Let X, Y be two topological spaces and $p : X \to Y$ be a map, with the following properties:

i). For any $y \in Y$, $p^{-1}(y)$ is a disjoint union points.

ii). There is a nbhd U_y of $y \in Y$ such that $p^{-1} \subseteq X$ is disjoint union of space each homeomorphic to U under p.

In this case Y is called the **base space**, x is **covering space** and p is called **covering map**.

The formal definition of covering map is the following,

Definition 2.4.1. *A* ***covering*** *is a continuous, surjective map* $p : X \to Y$ *such that for every* $y \in Y$ *and each* $x_i \in p^{-1}(y)$ *there exists a neighborhood* U_y *of* y *whose inverse image* $p^{-1}U_y$ *consists of disjoint neighborhood* V_{x_i} *and each restriction of* p *to* V_{x_i} *is a homeomorphism* $p : V_{x_i} \to Y_y$*.*

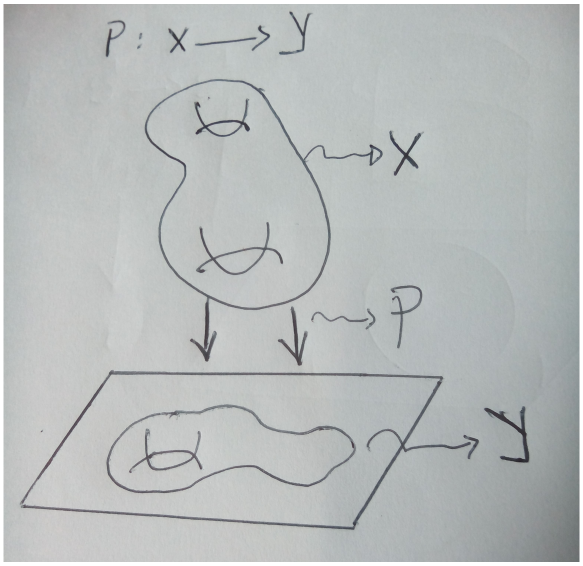
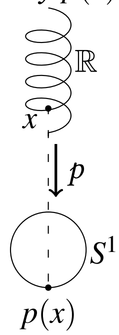


Figure 5. Covering map

For any covering $p : X \to Y$ and $y \in Y$ we have that $p^{-1}(y)$ is a discrete set. Let us look at the more examples.

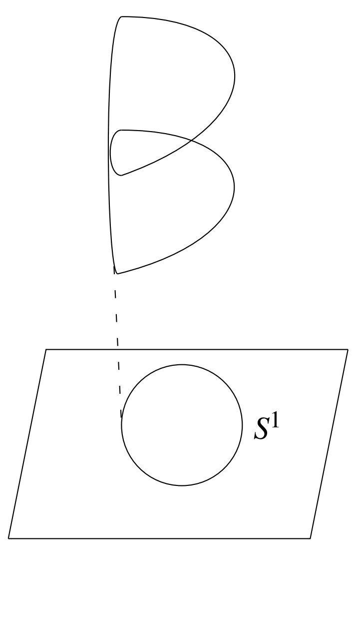
Example 2.4.2.

a). Consider, the map $p : \mathbb{R} \to S^1$ defined by $p(x) = \exp^{2\pi i t}$. i.e. a projection from the line to

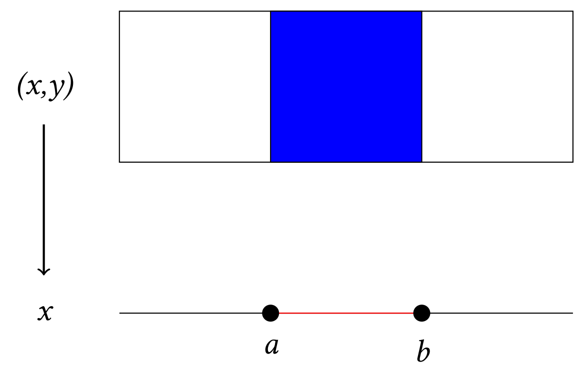


the circle. This map p is a covering.

b). The map $p : S^1 \to S^1$ defined by $p(z) = z^d$ is a covering of degree d.



c). Something which is not a covering map!



2.5 Divisors on Riemann surface

Let X be a Riemann surface. A **divisor** D on X is a formal sum of points p_i on X

$$D = \sum_{i=1}^{n} a_i p_i$$

where $a_i \in \mathbb{Z}$. If $a_i \geq 0$ for all i, we say that D is **effective** and denote it by $D \geq 0$. The divisors on X form an additive group $\mathrm{Div}(X)$.

Consider a non constant mapping of Riemann surfaces $f : X_1 \to X_2$, we can choose a local parameter z at a point $p \in X_1$, and w at $f(p) \in X_2$. In these coordinates f can be written as

$$w = z^n g(z)$$

where n is some integer and the function $g(z)$ is holomorphic in neighborhood of the origin, with $g(z) \neq 0$.

Remark 2.5.1. *The number n is called the **multiplicity** of f at p and is denoted by **mult f**.*

Definition 2.5.2. *The order at $p \in S$ of a meromorphic function $f : X \to \mathbb{P}^1$ is defined as follows:*

$$\mathrm{ord}\, f = \begin{cases} mult\, f & if\, f(p) = 0, \\ -mult\, f & if\, f(p) = \infty. \\ 0 & otherwise \end{cases}.$$

For two regular functions g, h on X , we have

$$\mathrm{ord}(gh) = \mathrm{ord}(g) + \mathrm{ord}(h).$$

For a function $f = g/h$, we define

$$\mathrm{ord}(f) = \mathrm{ord}(g) - \mathrm{ord}(h).$$

If $\mathrm{ord}(f) > 0$, we say that f has a zero along Y. If $\mathrm{ord}(f) < 0$, we say that f has a pole along Y. We also define the divisor associated to f by

$$(f) = \sum_X \mathrm{ord}(f),$$

as well as the divisor of zeros

$$(f)_0 = \sum_X \mathrm{ord}(g),$$

and the divisor of poles

$$(f)_\infty = \sum_X \mathrm{ord}(h),$$

They satisfy

$$(f) = (f)_0 - (f)_\infty.$$

If $D = (f)$ is the associated divisor of a global meromorphic function f, D is called a principal divisor.

2.6 The Riemann-Hurwitz formula

A branched covering $\pi : X \to Y$ between two (compact, connected) Riemann surfaces is a (surjective) holomorphic map (regular morphism). For a general point $q \in Y, \pi^{-1}(q)$ consists of d distinct points. Call d the degree of π. Locally around $p \mapsto q$, if the map is given by

$$z \to w = z^m,$$

where z, w are local coordinates of p, q, respectively, call m the *vanishing order* of π at p and denote it by

$$\mathrm{ord}_p(\pi) = m.$$

If $\operatorname{ord}_p(\pi) > 1$, we say that p is a *ramification point*. If $\pi^{-1}(q)$ contains a ramification point, then q is called a branch point. Define the *pullback*

$$\pi^*(q) = \sum_{p \in \pi^{-1}} (q) \operatorname{ord}_p(\pi) \cdot p.$$

Note that $\pi^*(q)$ is a degree d effective divisor on X.

We can interpret the Riemann-Hurwitz formula from a topological viewpoint. Let $\chi(X)$ denote the topological Euler characteristic of X. If X is a Riemann surface of genus g, take a triangulation of X and suppose the number of $k-$dimensional edges is c_k for $k = 0, 1, 2$. Then we have

$$\chi(X) = c_0 - c_1 + c_2 = 2 - 2g.$$

Proposition 2.6.1. *Let $\pi : X \to Y$ be a degree d branched cover between two Riemann surfaces. Then we have*

$$\chi(X) = d \cdot \chi(Y) - \sum_{p \in X} (\operatorname{ord}_p(\pi) - 1).$$

`Proof.` Take a triangulation of Y such that every branch point is a vertex. Pull it back as a triangulation of X. Note that it pulls back a face to d faces, an edge to d edges and a vertex v to $|\pi^{-1}(v)|$ vertices. Note that if

$$\pi^{-1}(v) = \sum_{i=1}^{k} m_i p_i$$

for distinct points p_i, then $|\pi^{-1}(v)| = m$. In other words, we have

$$|\pi^{-1}(v)| = d - \sum_{p \in \pi^{-1}(v)} (\operatorname{ord}_p(\pi) - 1)$$

The Riemann-Hurwitz formula follows.

□

The numerical version below is usually handy in computations.

Corollary 2.6.2. *Let $\pi : X \to Y$ be a degree d branched covering between two Riemann surfaces of genus g and h, respectively. Then we have*

$$2g-2 = d(2h-2) + \sum_{p\in X}(\mathrm{ord}_p(\pi)-1).$$

In particular, if $g < h$, such branched covers do not exist.

One of the best fruit of Riemann-Hurwitz formula is to determine the genus of a given Riemann surface as it is shown in the following example. Consider an elliptic curve $E = V(P)$, where f is a polynomial of the form:

$$f(X,Y,Z) = Y^2Z - (X-a_1Z)(X-a_2Z)(X-a_3Z)$$

as we have shown early E is a smooth curve which is a Riemann surface, we want to show that **an elliptic curve is a Riemann surface of genus** 1.

`Sketch of the proof.` Consider the affine chart $U_z = \{Z \neq 0\} \subseteq \mathbb{P}^2$, with coordinates $(x,y) = (\frac{X}{Z}, \frac{Y}{Z})$. The restriction of E to this chart is the affine curve E_Z determined by the equation $y^2 = (x-a_1)(x-a_2)(x-a_3)$. The vertical projection map $\pi : (x,y) \mapsto x$ restricts to a holomorphic map $\pi : E_Z \to \mathbb{C}$. For every point x of $\mathbb{C}$ except for $a_i's$ π^{-1} consists of two points which implies that the degree of π is equal to 2. The branch locus for $\tilde{\pi}$ is $B = \{a_1, a_2, a_3, a_4 = \infty\}$; denote $r_1, r_{2,3}, r_4$ the corresponding ramification points. Since for a map of degree two the only non-trivial ramification has differential length equal to one,the Riemann-Hurwitz formula tell us:

$$2g_E - 2 = 2(-2) + \sum_{r_1, r_{2,3}, r_4} 1,$$

which gives $g_E = 1$. □

We now recall the following definitions which we will use time to time.

Definition 2.6.3. *Let $f : X \to Y$ be a degree d non constant holomorphic map of Riemann surfaces.*

i). Given a point $x \in X$, the integer k_x such that there exist a local expression centered at point x of the form $F(z) = z^{k_x}$ is called the **ramification index** *of f at x.*

ii). The quantity $v_x = k_x - 1$ is called the **differential length** *of f at x.*

iii). If a point x has ramification index $k_x = 1$, then we say that f is **unramified** *at x.*

iv). A point x such that $k_x \geq 2$ is called a **ramification point**. *The* **ramification locus** *R is the set of X consisting of all ramification points.*

v). If x is a ramification point, then $f(x) \in Y$ is called a **branch point**. *The* **branch locus** *B is the subset of Y consisting of all branch points.*

Remark 2.6.4.

1). The branch locus is the image of the ramification locus, but the ramification locus is not necessarily the inverse image of the branch locus.

2). The function f is unramified at $x \in X$ i.e.$k_x = 1$ iff for any local expression F of f around x (not necessarily centered at x) we have $F^{'}(\varphi(x)) \neq 0$, i.e. f is locally invertible at x.

The following two results give us the link between ramification locus and branch locus.

Lemma 2.6.5. *The ramification locus R is a discrete subset of X i.e there exist open sets $U_i \subset X$ such that each U_i contains exactly one $x \in R$.*

Compactnesses have been always the strong property of our context. We examine the impact of compact Riemann surface on the maps between them as mentioned in the following results.

Lemma 2.6.6. *If X is a compact Riemann surface and $f : X \to Y$ is a non constant holomorophic map of Riemann surfaces, then the ramification locus is a finite set. Since the branch locus is the image of R via f, it follows that the branch locus is also a finite set.*

2.7 Monodromy representations

Let $f : X \to Y$ be a degree d holomorphic map of connected Riemann surfaces with a branch locus $B = \{b_1, \ldots, b_n\} \subset Y$. Choose a $y_0 \notin B$ and consider a loop $\gamma : [0,1] \to Y \setminus B$ based at y_0 as shown in figure below.

Choosing a preimage $x \in f^{-1}(y_0)$ γ lifts to a path $\tilde{\gamma_x}$ in X starting at x. Since $\gamma(1) = y_0$, the end point of $\tilde{\gamma_x}$ is a preimage of y_0(possibly different from x). We can thus associate to γ a function

$$\sigma_\gamma : f^{-1}(y_0) \to f^{-1}(y_0)$$

defined by $\sigma_\gamma(x) = \tilde{\gamma_x}(1)$.

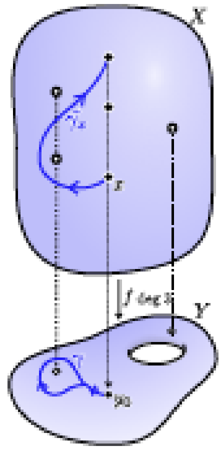


Figure 6. Lifting a generic loop

Definition 2.7.1. *A y_0* ***labeled map*** *is a pair (f, L), where $f : X \to Y$ is a degree d map of Riemann surfaces and $L : f^{-1}(y_0) \to \{1, \dots, d\}$ is a bijection. Note that this forces y_0 to not be a branch point for f. Then L is called a* ***labeling*** *of the inverse images of y_0.*

Remark 2.7.2. *A y_0-labeled map $(f : X \to Y, L)$ gives a group homomorphism*

$$\Phi : \pi_1(Y \setminus B, y_0) \to S_d$$

defined by $\Phi : \gamma \mapsto \sigma_\gamma$. These group homomorphisms are called **monodromy representations**.

Definition 2.7.3. *(Monodromy Representation). Let Y be a connected Riemann surface of genus g and $y_0, b_1, \dots, b_n \in Y$. Let $\lambda_1, \dots, \lambda_n$ be partitions of a positive integer d. A* **monodromy representation** *of type $(g, d, \lambda_1, \dots, \lambda_n)$ is a group homomorphism*

$$\Phi : \pi_1(Y \setminus \{b_1, \dots, b_n\}, y_0) \to S_d$$

such that, if ρ_k is the homotopy class of small loop around b_k, then the permutation $\Phi(\rho_k)$ has cycle type λ_k.

If in addition the subgroup $Im\Phi \leq S_d$ acts transitively on the set $\{1, 2, \dots, d\}$, we say that Φ is a **connected monodromy representation**.

Example 2.7.4. *We now describe the monodromy representations for $Y = \mathbb{P}^1(\mathbb{C})$. Choose a finite subset $B = \{b_1, \dots\} \subset \mathbb{P}^1(\mathbb{C})$. The punctured sphere $\mathbb{P}^1(\mathbb{C}) \setminus B$ is homotopic to a point with $n-1$ loops attached to it. The fundamental group of this space is the free group $\mathbb{F}_{n-1}$ generators $\rho_1, \dots, \rho_{n-1}$, representing loops that wind around each of the first $n-1$ branch points. Thus, for a chosen d, a group homomorphism $\Phi : \pi(\mathbb{P}^1(\mathbb{C}) \setminus B, y_0) \to S_d$ is given by a choice of images $\Phi(\rho_k) \in S_d$ with no restrictions.*

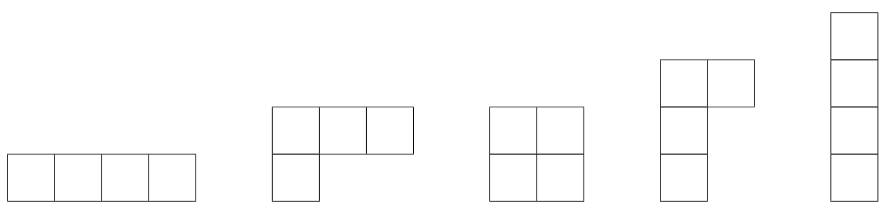
3 Combinatorics and representations of symmetric group

In this chapter, we survey a connection between representations of the symmetric group S_d and combinatorial objects called Young tableaux and then we describe the irreducible representation of the symmetric group S_d. To achieve this, we need to begin this section with a set of some definitions and notations regarding partitions and Young diagrams by considering the ideas of [CM15] and [Ful].

3.1 Partitions

A **partition** of a positive integer d is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots \lambda_l)$ satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_l > 0$ and $d = \lambda_1 + \lambda_2 + \dots + \lambda_l$. We write $\lambda \vdash d$ to denote that λ is a partition of d.

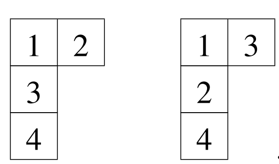
Example 3.1.1. *The positive integer* 4 *has* 5 *partitions which are:* (4), $(3,1)$, $(2,2)$, $(2,1,1)$ *and* $(1,1,1,1)$. *We can also represent partitions using Young diagrams.*



Definition 3.1.2. *A* ***Young diagram*** *is a finite collection of boxes arranged in left-justified rows, with the row sizes weakly decreasing. The Young diagram associated to the partition* $\lambda = (\lambda_1, \lambda_2, \dots \lambda_l)$ *is the one that has* l *rows, and* λ_i *boxes on the* ith *row.*

Definition 3.1.3. *A* ***standard Young tableau*** *is a Young tableaux whose the entries are* $\{1,2,\dots,d\}$ *increasing across each row and each column.*

For instance, $\lambda = (2,1,1)$, the number of standard Young tableaux with this shape is 2.



A nice observation is that if λ denotes our young diagram its conjugate is given by flipping λ over its main diagram and is denoted by $\bar{\lambda}$.

Example 3.1.4.

$$\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad \bar{\lambda} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

3.2 Permutations

We denote the set consisting of the first d positive integers $\{1,2,\cdots,d\}$ by $[d]$. Let S_d be the group of all permutations on $[d]$, that is, the set of all bijections from $[d]$ to itself under composition. The elements of $\sigma \in S_d$ are called **permutations** and the group is called the **symmetric group** on d words while its subgroup is referred to as a **permutation group**. To conform with usually composition of functions, we take a convection that permutation are multiplied from right to left.

For a permutation $\sigma \in S_d$ given by

$$1 \mapsto \sigma(1), \qquad 2 \mapsto \sigma(2), \qquad \cdots, \qquad d \mapsto \sigma(d).$$

We can represent σ in three different ways. First as an array,

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & d \\ \sigma(1) & \sigma(2) & \cdots & \sigma(d) \end{pmatrix}$$

usually called the **two-line notation.** As the top line is fixed, we can simply drop the first row and write the **one-line notation** for the permutation as the as the sequence $\sigma = \begin{pmatrix} \sigma(1) & \sigma(2) & \cdots & \sigma(d) \end{pmatrix}$.

The simplest notation for representation of a permutation is the **cycle notation.** A permutation $\alpha \in S_d$ is a **cycle of length** k or $k-$**cycle** if there exist numbers $i_1, i_2, \cdots, i_k \in [d]$ such that

$$\alpha(i_1) = i_2, \qquad \alpha(i_2) = i_3, \qquad \cdots, \qquad \alpha(i_k) = i_1.$$

We can write α in the form $(i_1, i_2, \cdots, i_k)$. A permutation of length two is called a **transposition** while a cycle of length two is called a **cycle**. The way to view this decomposition is to consider a directed graph representing a permutation $\sigma \in S_d$ with vertex set $[d]$ and arcs $i \in \sigma(i)$ for each $i \in [d]$.

A cycle of length k can be represented in k different ways depedent on the element we chose to be the first in the cycle. Not every permutation is a cycle but its bulding block of a permutation are. If we write one representative for each cycle σ gives a disjoint cycle representation for a permuation.

Example 3.2.1. *Consider the permutation*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 5 & 4 & 3 & 1 \end{pmatrix}$$

Then σ has a cycle of length 3, a cycle of legtht 2 and another of length 1 by looking at the associated directed graph in figure 8.

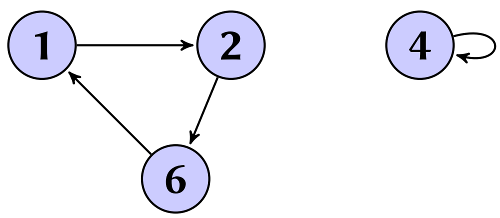


Figure 7. The directed graph of a permutation $\sigma \in S_6$ as $(126)(35)(4)$.

If we fix $\sigma \in S_d$, then σ can be decomposed uniquely into a set of disjoint cycles. Moreover, every permutation can be expressed as a product of transpositions.

The sum of the cycle lengths of σ is equal to d, so the lenghts form a partition of d.

The **cycle type** of σ is an expression of the form

$$1^{m_1} \cdot 2^{m_2} \cdots d^{m_d}$$

where the m_i is the number of $i-$cycles in σ.

We are particularly interested in the products in transpositions in a permutation. A sequence $(a_1,b_1),(a_2,b_2),\cdots,(a_n,b_n)$ such that the product $(a_1,b_1)(a_2,b_2)\cdots(a_n,b_n)$ is equal to σ is called a **transposition factorisarion** of σ of length n. This factorisation is not unique, For instance, $(123)=(12)(13)=(13)(23)$. Given any transposition factorisarion, say

$$\tau_1\tau_2\cdots\tau_n = \sigma \in S_d,$$

we can assign a corresponding decorated graph G_σ to it in the following way. Vertices are labelled with the permuted elements $[d] = \{1, 2, \cdots, d\}$ and edges represent transpositions such that for the transposition $\tau_i = (a, b)$ in the product, we label the edge by the index i. For instance,

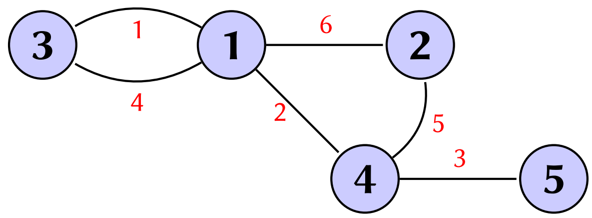


Figure 8. The decorated graph of a factorisation $(1,3)(1,4)(4,5)(1,3)(2,4)(1,2) = (1,5,3,4,2) \in S_6$

If we fix $\sigma \in S_d$, then σ can be decomposed uniquely into a set of disjoint cycles. Moreover, every permutation can be expressed as a product of transpositions. The sum of the cycle lengths of σ is equal to d, so the lenghts form a partition of d. The **cycle type** of σ is an expression of the form

$$1^{m_1} \cdot 2^{m_2} \cdots d^{m_d}$$

where the m_k is the number of $k-$cycles in σ.

Example 3.2.2. *The permutation* $(12)(45)(368) \in S_8$ *has cycle type* $\{3, 2, 2, 1\}$ *also written as* $1 \cdot 2^2 \cdot 3$.

The number of k-cycles in symmetric group S_d is given by

$$\frac{d!}{(d-k)!k}.$$

Theorem 3.2.3. *Every permutation is uniquely expressible as a product of disjoint cycles.*

3.2.1 Conjugate of the Symmetric Group

Finally, we denote the set of all elements conjugate to σ in the symmetric group S_d by C_σ, that is

$$C_\sigma = \{\pi \sigma \pi^{-1} : \pi \in S_d\}.$$

These are called **conjugacy classes** of S_d. An easy fact, two permutations are conjugate if and only if they have the same cycle type.

Lemma 3.2.4. *Let* $\alpha, \tau \in S_d$ *where* α *is the* k*-cycle* $a_1 a_2 \ldots a_k$. *Then* $\tau \alpha \tau^{-1} = (\tau(a_1)\tau(a_2)\ldots\tau(a_k))$.

This result tell us that any conjugate of a cycle is also a cycle of the same length. The following results also asserts that the converse is also true. That is if two cycles have the same length then they are conjugate.

Theorem 3.2.5. *All cycles of the same length in S_d are conjugate.*

As an example, let's describe the conjugacy classes of the symmetric group S_3.

Example 3.2.6. *In S_3, what are the conjugates of (12)? We make a table of $\pi(12)\pi^{-1}$ for all $\pi \in S_3$.*

π	(1)	(12)	(13)	(23)	(123)	(132)
$\pi(12)\pi^{-1}$	(12)	(12)	(23)	(13)	(23)	(13)

From this table, the conjugate of (12) in S_3 is $\{(12),(13),(2,3)\}$. Similarly, you can check that the conjugate of (123) is (132) and the conjugate of (1) is just (1). So S_3 has three conjugacy classes namely:

$$\{(1)\},\{(12),(13),(23)\},\{(123),(132)\}.$$

The following nice result summarize the concept of conjugacy classes in S_d.

Theorem 3.2.7. *The conjugacy classes of any $\pi \in S_d$ are determined by cycle type. That is, if π has cycle type $(k_1,k_2,\ldots,k_l)$, then any conjugate of π has cycle type $(k_1,k_2,\ldots k_l)$ and if ρ is any other element of S_d with cycle type $(k_1,k_2,\ldots k_l)$, then π is conjugate to ρ.*

Thus from theorem 3.2.7, the partitions of d are in one-to-one correspondence with the conjugacy classes of S_d.

3.3 Representations of the symmetric group S_d

The building blocks of a representation of any finite group are the irreducible representations. The number of irreducible representations over the complex numbers is equal to the number of partition of d in the case of the symmetric group. The main ideas in this section, is that the representations of a group S_d over a field $\mathbb{C}$ are the same as modules over the group ring $\mathbb{C}[S_d]$.

3.3.1 Representations

In this work, we define representations in three equivalent ways.

Definition 3.3.1. *A complex (finite dimensional)* ***representation*** ρ *of* S_d *is, equivalently:*

*a). **group action**) A finite dimensional vector space* V *together with a linear action of* S_d*, i.e. a map*

$$\bullet : S_d \times V \to V$$

such that, for every $\sigma, \sigma_1, \sigma_2 \in S_d, v, w \in V, \lambda \in \mathbb{C}$*:*

- $e \cdot v = v,$
- $\sigma_2 \cdot (\sigma_1 \cdot v) = (\sigma_2 \sigma_1) \cdot v,$
- $\sigma \cdot (v + w) = \sigma \cdot v + \sigma \cdot w,$
- $\sigma \cdot \lambda v = \lambda \sigma.$

*b). (**module**) A finitely generated module over the group ring* $\mathbb{C}[S_d]$.

*c). (**homomorphism**) A group homomorphism*

$$\Phi_\rho : S_d \to GL(n, \mathbb{C}).$$

Observe the following:

(i) The dimension of V (or the n in $GL(n, \mathbb{C})$) is called the **diminsion** of the representation ρ.

(ii) A **subrepresentation** $\rho' \leq \rho$ is an invariant subspace (or a $\mathbb{C}[S_d]$ submodule) $U_{\rho'}$ of V_ρ. The 0 vector, and V_ρ itself are trivial examples of sub representations of ρ. A representation ρ that does not contain any non-trivial subrepresentation is called **irreducible**.

Example 3.3.2. *The trivial representation. This is usually defined to be one dimensional vector space* $\mathbb{C}$ *with trivial action of* S_d*:*

$$\sigma \cdot z = z$$

for all $\sigma \in S_d, z \in \mathbb{C}$ *and it is denoted by* ρ_1*. The trivial and sign representations are irreducible because a one dimensional vector space does not have any proper subspace.*

The sign representation is given by the group action on $\mathbb{C}$ as follows:

$$\sigma \cdot z = \begin{cases} z & \text{if } \rho \text{ is an even permutation} \\ -z & \text{if } \rho \text{ is an odd permutation} \end{cases}$$

Definition 3.3.3. *Consider a* d*-dimensional vector space* V *with basis* $\{e_1, \dots e_d\}$*. Define a group action of* S_d *on* V *by extending by linearity the following action on the bases vectors:* $\sigma \cdot e_i = e_{\sigma(i)}$*. This is called a* ***permutation representation*** *which is not irreducible.*

The nice property of the group ring is that it is a module over itself and therefore it is a representation of S_d called the **regular representation**. This is the victory of our target because the regular representation contains **all irreducible** representations of S_d.
The following remark, recalls some few fundamental facts about representations.

Remark 3.3.4.

(i) *Any finite dimensional representation of* S_d *decompose uniquely (up to the order of the factors) as direct sum of irreducible representations.*

(ii) *The number of irreducible representations of* S_d *equals the number of conjugacy classes of* S_d*, which in turn are naturally indexed by partitions of the integer* d*.*

(iii) *Denote by* ρ *an irreducible representation of* S_d*, by* V_ρ *the corresponding vector space, and understand a sum over the index* ρ *to mean the sum over all irreducible representations of* S_d*. Then the regular representation decomposes as*

$$\mathbb{C}[S_d] \cong \bigoplus_{\rho} V_\rho^{\oplus \dim \rho}$$

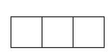
by equating the dimensions on either side of the above equation, we obtain

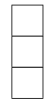
$$d! = \sum_{\rho} (\dim \rho)^2. \tag{1}$$

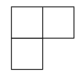
The last formula, help us to describe all irreducible representation of subsets of S_d as in the next example.

Example 3.3.5. *Using the formula in 1, there are three irreducible representations of* S_3*. The two of them are known namely the trivial and the sign representations. Our formula tells that the last one must be two-dimensional. This is called the* ***standard*** *representation.*

It turns out that there is a deep connections between the description of irreducible representations of S_d through Young tableaux. The three irreducible representations of S_3, can be described using Young diagrams as illustrated below.

trivial representation: ρ_1 

sign representation: ρ_{-1} 

standard representation: ρ_s 

The dimension of the irreducible representation corresponding to shape λ is determined by **hook-length formula** denoted by k_λ. In short, if (i,j) denoted the box in row i and column j of the standard Young diagram corresponding to λ; the hook length h_{ij} is the number of boxes directly to the right and directly below i,j including the box (i,j). Thus the **hook-length formula** is given by:

$$k_\lambda = \frac{d!}{\Pi h_\lambda(i,j)}.$$

Example 3.3.6. *The degree of the irreducible representation of S_4 corresponding to partition $\lambda = (2,1,1)$ is*

$$k_{(2,1,1)} = \frac{4!}{4\cdot 2\cdot 1\cdot 1} = 3.$$

3.3.2 Characters

The main key in representation theory is to represent a given group as a matrix, thus the function defined by the trace of that matrix is called a **character**. Characters caries many remarkable information like it can tell us when two representations are equivalent, we can decide whether or not a given representation is irreducible through its character etc. Here is the definition of a character.

Definition 3.3.7. *Let ρ be a representation of S_d. The **character** of ρ is the function*

$$\chi_\rho : S_d \to \mathbb{C}$$

defined as

$$\chi_\rho(\sigma) : trace(\Phi_\rho(\sigma)).$$

Note that the trace of a matrix is a coefficient of the characteristic polynomial of the associated linear transformation, therefore it is **invariant** under conjugation.

Remark 3.3.8. *(1) The character of a representation does not depend on the choice of a basis for V_ρ (which gives rise to the matrices $\Phi_\rho(\sigma)$ e.i if β,β' are bases of V_ρ, then*

$$[\rho]_{\beta'} = T^{-1}[\rho]_\beta T$$

for some invertible matrix T and then

$$trace[\rho]_{\beta'} = trace[\rho]_{\beta}$$

for all $\rho \in S_d$*);*

(2) *Characters are constant along conjugacy classes; functions with this property are called* ***class functions****.*

Example 3.3.9. *We can compute the character of* $Z\mathbb{C}[S_3]$ *and the following is its multiplication table .*

Table 1. The character Table of S_3

S_3	C_e	$C_{(2,1)}$	$C_{(3)}$
ρ_1	1	1	1
ρ_{-1}	1	-1	1
ρ_s	2	0	1

3.3.3 The group ring and class algebra

Definition 3.3.10. *The* ***group ring*** *of the symmetric group* S_d*, denoted* $\mathbb{Z}[S_d]$*, has elements formal* $\mathbb{Z}$*-linear combinations of elements of* S_d*:*

$$\mathbb{Z}[S_d] = \{\sum_{\sigma\in S_d} a_\sigma \sigma | a_\sigma \in \mathbb{Z}\}.$$

In this structure, addition and multiplications are defined as follow:

$$\sum_{\sigma\in S_d} a_\sigma \sigma + \sum_{\sigma\in S_d} b_\sigma \sigma = \sum_{\sigma\in S_d} (a_\sigma + b_\sigma)\sigma$$

and

$$(\sum_{\sigma\in S_d} a_\sigma \sigma)(\sum_{\sigma\in S_d} b_d \tau) = (\sum_{\sigma\in S_d} C_\sigma \lambda)$$

where $C_\lambda \in S_d$ can be given in three different ways:

$$C_\lambda = \sum_{\sigma\tau=\lambda} a_\sigma b_\tau = \sum_{\sigma\in S_d} a_\sigma b_{\sigma^{-1}\lambda} = \sum_{\tau\in S_d} a_{\lambda\tau^{-1}} b_\tau.$$

The following example makes this more clearer.

Example 3.3.11. *For* $d=3$, $x=3(12)+5(123)$ *and* $y=4(13)-6(123)=4(13)+(-6)(123)$ *are elements of* $\mathbb{Z}[S_3]$. *We have*

$$x+y=3(12)+4(13)+(5-6)(123)=3(12)+4(13)-(123)$$

and

$$\begin{aligned} x\cdot y =&(3(12)+5(123))(4(13)-6(123)\\ =&12(12)(13)-18(12)(123)+20(123)(13)-30(123)(123)\\ =&12(132)-18(23)+20(23)-30(132)\\ =&2(23)-18(132). \end{aligned}$$

Remark 3.3.12. *We denote by* $\mathbb{C}[S_d]$ *the set of formal linear combinations of group elements where the coefficients* a_σ *are complex numbers. Together with addition and multiplication defined as above, there is a natural way to multiply elements of* $\mathbb{C}[S_d]$ *by scalars* $t\in\mathbb{C}$ *as follows*

$$t(\sum_{\sigma\in S_d} a_\sigma\sigma)=\sum_{\sigma\in S_d}(ta_\sigma)\sigma,$$

which gives $\mathbb{C}[S_d]$ *also the structure of a vector space, with a natural basis given by* $\sigma\in S_d$. *A set with operations that make it simultaneously a ring and a vector space is called an* ***algebra****, and* $\mathbb{C}[S_d]$ *is called the* ***group algebra*** *of* S_d.

In any group G, the center of the group G denoted $Z(G)$ is the set of elements that commute with every element of G, we introduce a commutative subalgebra of $\mathbb{C}[S_d]$, which plays a central role in our journey.

Definition 3.3.13. *The* ***class algebra*** *of* S_d *is the center of the group ring,*

$$Z\mathbb{C}[S_d]=\left\{x\in\mathbb{C}[S_d]\mid yx=xy\;\forall\, y\in\mathbb{C}[S_d]\right\}.$$

We note some few things for a class algebra in the remark below.

Remark 3.3.14. *For* $\lambda\vdash d$ *(a partition of the positive integer d) denote by* $C_\lambda\in\mathbb{C}[S_d]$ *the sum of all elements of cycle type* λ.

1. C_λ *consists of the sum of all permutations in a particular conjugacy class;*
2. *For any* $\lambda, C_\lambda\in Z\mathbb{C}[S_d]$;
3. *The* C_λ*'s form a basis for* $Z\mathbb{C}[S_d]$ *as a vector space:*

$$Z\mathbb{C}[S_d]=\bigoplus_{\lambda\vdash d}\langle C_\lambda\rangle_{\mathbb{C}}$$

We also remind ourself that the conjugacy class basis is a very natural basis for $Z\mathbb{C}[S_d]$. The following result describe another basis, naturally indexed by the irreducible representations of S_d.

Theorem 3.3.15. *(Maschké). The class algebra $Z\mathbb{C}[S_d]$ is a semi-simple algebra, i.e. there is a basis $\{e_{\rho_1}, \dots e_{\rho_n}\}$ (where the ρ_i's are all irreducible representations of S_d) of idempotent elements. This means:*

$$e_{\rho_i} \cdot e_{\rho_j} = \begin{cases} e_{\rho_i} & \text{if } e_{\rho_i} = e_{\rho_j} \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore the following change of basis formulas hold

$$e_\rho = \frac{\dim \rho}{d!} \sum_\lambda \chi_\rho(\lambda) C_\lambda,$$

$$C_\lambda = |C_\lambda| \sum_\rho \frac{\chi_\rho(\lambda)}{\dim \rho} e_\rho,$$

where the summation index λ denotes all partitions λ of d, and the summation index ρ denotes all irreducible representations of S_d.

Example 3.3.16. *We can now compute the class algebra $Z\mathbb{C}[S_3]$ which is a three dimensional vector space, with basis*

$$C_e = e$$

$$C_{(2,1)} = (12) + (13) + (23)$$

$$C_{(3)} = (123) + (132).$$

We can display the multiplication table for $Z\mathbb{C}[S_3]$ If e_1, e_{-1}, e_s denote the vectors of the

Table 2. Multiplication Table of $Z\mathbb{C}[S_3]$

	C_e	$C_{(2,1)}$	$C_{(3)}$
C_e	C_e	$C_{(2,1)}$	$C_{(3)}$
$C_{(2,1)}$	$C_{(2,1)}$	$3(C_e + C_e)$	$2C_{(2,1)}$
$C_{(3)}$	$C_{(3)}$	$2C_{(2,1)}$	$2C_e + C_{(3)}$

semi-simple basis for $Z\mathbb{C}[S_d]$, the Theorem 3.3.15 give us the change of basis as follows.

$$e_1 = 1/6(C_e + C_{(2,1)} + C_{(3)})$$

$$e_{-1} = 1/6(C_e - C_{(2,1)} + C_{(3)})$$

$$e_s = 1/3(C_e + C_{(2,1)} + C_{(3)})$$

Solving this system of linear equation, we get

$$C_e = e_1 + e_{-1} + e_s,$$

$$C_{(2,1)} = 3e_1 + e_{-1} - 3e_{-1},$$
$$C_{(3)} = 2e_1 + 2e_{-1} - e_s.$$

4 Hurwitz Theory

In chapter 2, we were friends of maps of Riemann surfaces. At this time round, we will introduce the counting problem for maps of Riemann surfaces. This number is always finite and is called **Hurwitz number**, it **count** genus g, degree d covers of $\mathbb{P}^1$ with fixed branch locus and ramification profile. We will follow the treatment given in [CM15].

Definition 4.0.1. *Two holomorphic maps of Riemann surfaces $f: X \to Y$ and $g: \tilde{X} \to Y$ are called **isomorphic** if there is an isomorphism of Riemann surfaces $\phi: X \to \tilde{X}$ such that $f = g \circ \phi$. An **automorphism** of $f: X \to Y$ is an isomorphism $\psi: X \to X$ such that $f = f \circ \psi$. The group of automorphisms of f is denoted $\mathrm{Aut}(f)$.*

We can now give a formulation for counting problem for maps of Riemann surfaces when $Y = \mathbb{P}^1$, the complex projective line.

Definition 4.0.2 (Hurwitz number). *Fix points $b_1, \dots, b_n \in \mathbb{P}^1$ and let $\lambda_1, \dots, \lambda_n$ be partitions of a positive integer d. We define the **Hurwitz number** as*

$$H_g^d(\lambda_1, \dots, \lambda_n) = \sum_{[f]} \frac{1}{|Aut(f)|};$$

this sum runs over each isomorphism class of $f: X \to \mathbb{P}^1$ where

1. *f is a holomorphic map of Riemann surfaces,*
2. *X is connected, compact, and has genus h,*
3. *the branch locus of f is $B = \{b_1, \dots, b_n\}$,*
4. *the ramification profile of f at b_i is λ_i.*

We now focus on **double Hurwitz numbers** $H_g(\mu, \nu)$ (where the partitions λ_i of d are of the form $(2, 1, \dots, 1)$ apart from the branch points at 0 and ∞) by looking at the connections between the following four equivalent definitions as detailed in [PJ13].

Example 4.0.3. *Let $Y = \mathbb{P}^1$ and set $b_1 = 0, b_2 = \infty$. Choose $d > 0$ and let $\lambda_1 = \lambda_2 = (d)$. We compute*

$$H_{0 \xrightarrow{d} 0}((d), (d) = \frac{1}{d}.$$

4.1 Topological definition of double Hurwitz numbers

Definition 4.1.1. *(Hurwitz cover). A* (μ,ν,g)-***Hurwitz cover*** *is a degree* d *map* $f: X \to \mathbb{P}^1$ *from a genus* g *connected Riemann surfaces* X *to* $\mathbb{P}^1$*, satisfying*

1. f *has ramification profile* μ *over* 0 *and* ν *over* ∞,
2. f *has simple ramification over* r *additional fixed points* $p_i \in \mathbb{P}^1$,
3. f *has no other ramification,*
4. *the* m *elements of* $f^{-1}(0)$ *and the* n *elements of* $f^{-1}(\infty)$ *are labeled.*

Definition 4.1.2. *The double Hurwitz number* $H_g(\mu,\nu)$ *is the count of* (μ,ν,g)*-Hurwitz covers, where each cover* f *is counted with weight* $\frac{1}{|\mathrm{Aut}(f)|}$.

4.2 Definition of double Hurwitz numbers in terms of permutations

First, we need to define a *labeled permutation.* If the cycle decomposition of σ has k cycles, then a labeling of σ is a bijection between the cycles and the set $\{1,\dots,k\}$. Therefore, we can talk about ith cycle of a labeled permutation.

Definition 4.2.1. *(Permutations). A* (μ,ν,g)*- monodromy set is an element*

$$(\sigma_0,\tau_1,\dots,\tau_r\sigma_\infty) \in S_d^{r+2},$$

together with a labeling of σ_0 *and* σ_∞*, satisfying :*

1. σ_0 *and* σ_∞ *are permutation of* S_d *with cycle types* μ *and* ν *respectively,*
2. *The* τ_i *are all transpositions,*
3. $\sigma_0 \times (\Pi_{i=1}^r \tau_i) \times \sigma_\infty = 1$,
4. *The group generated by the* τ_i *and* σ_j *acts transitively on* $\{1,\dots,d\}$.

Proposition 4.2.2. $H_g(\mu,\nu)$ *is* $\frac{1}{d!}$ *times the number of* μ,ν,g*-monodromy sets*

Example 4.2.3. *If* $g=1, d=3$*. We show that*

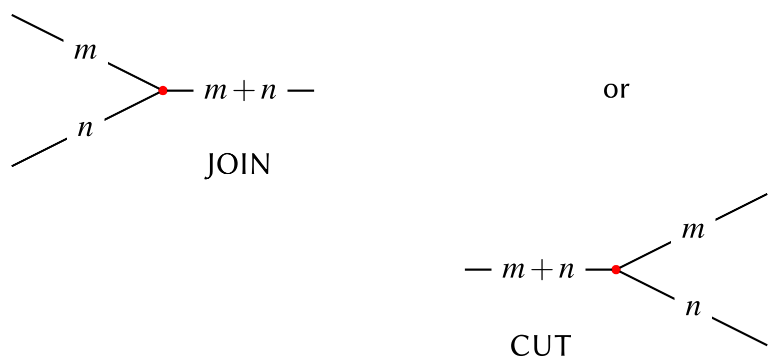
$$H_3^1((2,1),(2,1)) = 40.$$

Note that in this case $g=1, d=3$*, then by Riemann-Hurwitz formula give us* $2g-2+2+2=4$ *points plus two extra ones(i.e 0 and* ∞*). We will have* 6 *permutations,* $\sigma_0,\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_\infty$*.. Hence, by fundamental theorem of enumeration we*

$$H_3^1((2,1),(2,1)) = \frac{(3\times 3\times 3\times 3\times 3\times 1)-3}{3!} = 40.$$

4.2.1 Cut and join equation

A product of a permutation and a transposition leads to the counting branching graphs. The cut and join equations are a collection of recursions among Hurwitz numbers. The construction of the two equation is here below:
Let $\sigma \in S_d$ be a fixed element of cycle type $\eta = (n_1, \dots, n_l)$, written as a composition of disjoint cycles as $\sigma = c_1, \dots, c_l$. Let $\tau = (i, j) \in S_d$ vary among all transpositions. The cycle types of the composite elements $\tau\sigma$ are described below.



Cut: if i, j belong to the same cycle (say c_l), then this cycle gets **cut into two**: $\tau\sigma$ has cycle type $\eta' = (n_1, \dots, n_{n-l}, m', m'')$, with $m' + m'' = n_l$. If $m' \neq m''$, there are n_l transpositions giving rise to an element of cycle type η' . If $m' = m'' = n_l/2$, then there are $n_l/2$.

Join: If i, j belong to different cycles (say c_{l-1} and c_l), then these cycles are, joined: $\tau\sigma$ has cycle type $\eta' = (n_1, \dots, n_{n-l} + n_l)$. There are $n_{l-1}n_l$ transpositions giving rise to cycle type η'

Example 4.2.4. *Let $d = 4$. There are 6 transpositions in S_4. If $\sigma = (12)(34)$ is of cycle type $(2,2)$, then there are 2 transpositions (12) and (34) that, cuts σ to give rise to a transposition and 2×2 transpositions ((13), (14), (23), (24)) that joins σ into a four-cycle.*

More generally, if we know the lengths of the cycles that are being cut and joined, we can count the number of possibilities for τ. There always kl transpositions that join a k cycle and an l cycle into $k+l$ cycle, there are $k+l$ different transpositions that split a $k+l$ cycle into a k cycle and an l cycle when $k \neq l$, and there are k transpositions that split a $2k$ cycle into two cycles of length k.

4.2.2 Branched graphs

We recall some definition and basics facts for Hurwitz numbers

Definition 4.2.5. *Fix $r+s$ points $p_1,\ldots,p_r,q_1,\ldots,q_s$ on $\mathbb{P}^1$, and $\eta_1,\ldots,\eta_r$ partitions of the integer d. The double Hurwitz number:*

$$H_d^g(\eta_1,\ldots,\eta_r) := \textit{weighted number ofdegree } d \textit{ covers } \pi : C \to \mathbb{P}^1$$

such that:

1. *C is a Riemann surfcace of genus g;*
2. *π is unramified over $\mathbb{P}^1 \setminus \{p_1,\ldots,p_r,q_1,\ldots,q_s\}$;*
3. *π ramifies with profile η_i over p_i;*
4. *π has simple ramification over q_i.*

Each cover π is weighted by $1/\mid Aut(\pi)$. We note that this is independent of the locations of the p_i and q_i. For a partition η, let $l(\eta)$ denote the number of parts of η. By the Riemann-Hurwitz formula, we have that

$$2-2g = 2d - dr - s + \sum_{i=1}^{r} l(\eta_i).$$

hence s is determined by g, d and $\eta_1,\ldots,\eta_r$. We note that the use of **Hurwitz number** for the generic case H_d^g when all ramification is simple.

Remark 4.2.6. *A ramified cover is essentially equivalent information to a monodromy representation it induces; thus, an equivalent definition of Hurwitz number counts the number of homomorphisms ϕ from the fundamental group $\Pi_1(\mathbb{P}^1 \setminus \{p_1,\ldots,p_r,q_1,\ldots,q_s\})$ to the symmetric group S_d such that :*

- *i) the image of a loop around p_i has cycle type η_i;*
- *ii) the image of a loop around q_i is a transposition;*
- *iii) the subgroup $\phi(\Pi_1)$ acts transitively on the set $\{1,\ldots,d\}$.*

This number is divided by $|S_d|$, to account both for automorphisms and for different monodromy representations corresponding to the same cover.

Hurwitz numbers and weighted branch graph sums

We want to compute Hurwitz numbers in terms of a weighted sum over graphs, the idea is to start at one of the special points, and count all possible monodromy representations as each transposition gets added until one gets to the second special point with the specified cycle type.

Let us make it clearer, fix g and let $\eta = (n_1, \dots, n_k)$ and $\mu = (m_1, \dots, m_l)$ be two partitions of d. Denote by $s = 2g - 2 + l + k$ the number of non-special branch points, determined by the Riemann-Hurwitz formula.

Definition 4.2.7. *Branched graphs project to the segment $[0, s+1]$ and are constructed according to the following procedure:*

a) Start with k small segments over 0 labeled $n_1, \dots, n_k$. We call these n's the weights of the strands,

b) Over the point 1 create a 3-valent vertex by either joining two strands or splitting one with weight strictly greater than 1. In case of a join, label the new strand with the sum of the weights of the edges joined. In case of a cut, label the two new strands in all possible (positive) ways adding to the weight of the split edge.

c) Consider only one representative for any isomorphism class of labeled graphs.

d) Repeat (b) and (c) for all successive integers up to s.

e) Retain all connected graphs that, terminate with l points of weight $m_1, \dots, m_l$.

Lemma 4.2.8. *The Hurwitz number $H_d^g(\eta, \nu)$ is computed as a weighted sum over monodromy graphs. Each monodromy graph is weighted by the product of the following factors:*

(i) The number $\varepsilon(\eta)$ of elements of S_d of cycle type η,

(ii) $\mid Aut(\eta) \mid$,

(iii) For every vertex, the product of the degrees of edges coming into the vertex from the left,

(iv) A factor of $1/2$ for any balanced fork or wiener.

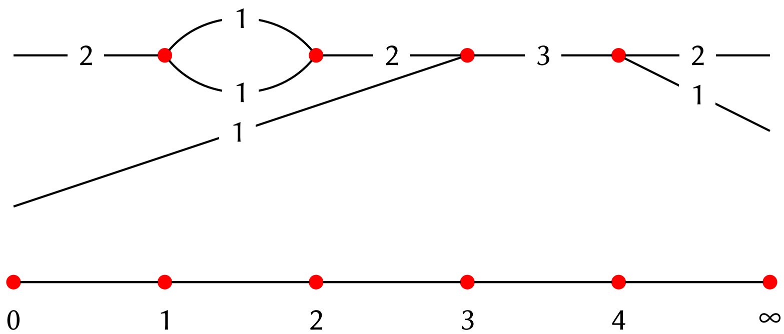
(v) $1/d!$.

We simplify the above lemma in the following formula

$$H_d^g(\eta, \nu) = \sum_{\Gamma} \frac{1}{\mid Aut(\Gamma) \mid} \Pi w(e),$$

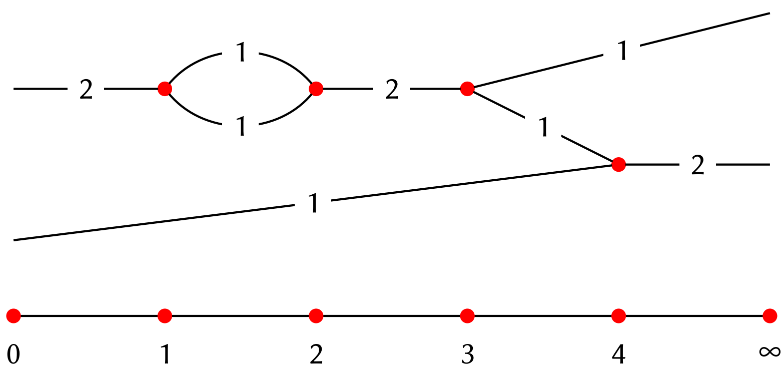
where we take the product of all the interior edge weights; the factors of $1/2$ coming from the balanced forks and wieners amount to the size of the automorphism group of our decorated graphs.

We illustrate this in the following example by finding the corresponding weighted graphs arising from cubic coverings. Note in some cases we are multiplying by 2 to compensate for the new graph obtained by reflecting along the y-axis.



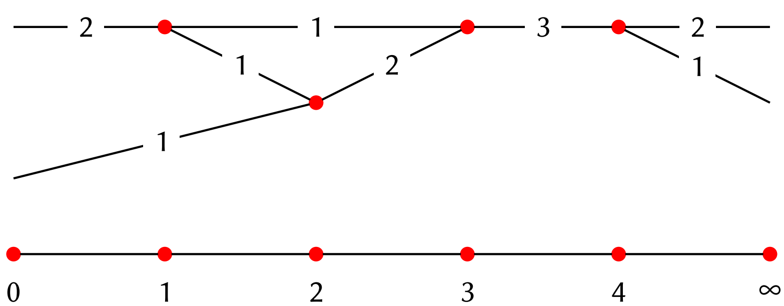
$$\frac{\prod w(e)}{\mathrm{Aut}(G)} = \frac{1{\cdot}1{\cdot}2{\cdot}3}{2} = 3$$

$$\# = 3 \cdot 2$$



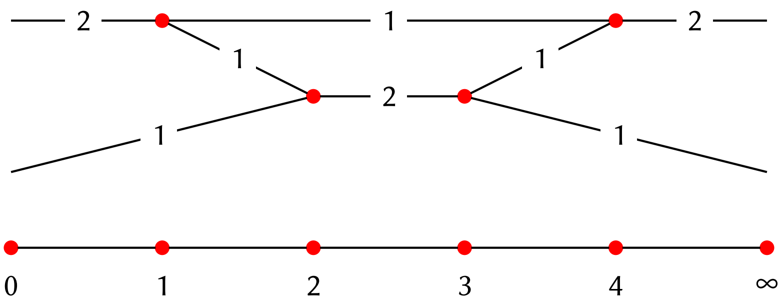
$$\frac{\prod w(e)}{\mathrm{Aut}(G)} = \frac{1{\cdot}1{\cdot}2{\cdot}1}{2} = 1$$

$$\# = 1 \cdot 2$$



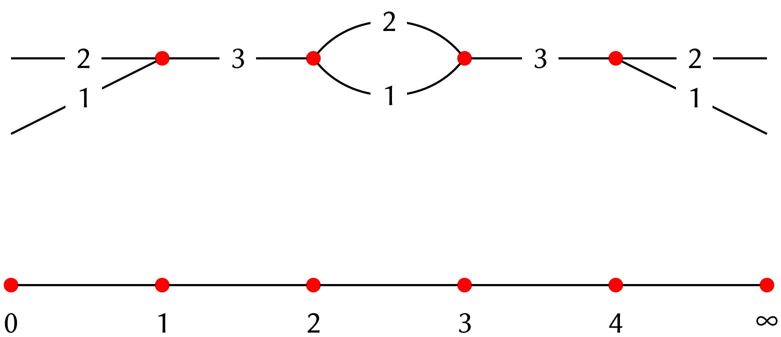
$$\frac{\prod w(e)}{\mathrm{Aut}(G)} = \frac{1{\cdot}1{\cdot}2{\cdot}3}{1} = 6$$

$$\# = 6 \cdot 2$$



$$\frac{\prod w(e)}{\mathrm{Aut}(G)} = \frac{1\cdot 1\cdot 2\cdot 1}{1} = 2$$

$$\# = 2\cdot 1$$



$$\frac{\prod w(e)}{\mathrm{Aut}(G)} = \frac{3\cdot 2\cdot 1\cdot 3}{1} = 18$$

$$\# = 18\cdot 1$$

Thus by the algorithm and the formula, we also find

$$H_3^1((2,1)(2,1)) = 3\cdot 2 + 1\cdot 2 + 6\cdot 2 + 2\cdot 1 + 18\cdot 1 = 40.$$

The representation theory give us another way to compute double Hurwitz numbers. Namely, the character theory of the group ring $Z(\mathbb{C}[S_d])$.

In this case, we use the Burnside character formula.

$$H_d^g(\lambda_1,\cdots,\lambda_m) = \sum_\sigma \left(\frac{\dim\sigma}{d!}\right)^2 \prod_{i=1}^m \frac{|C_{\lambda_i}\chi_\sigma(\lambda_i)|}{\dim\sigma}.$$

For example, let us compute $H_1^0((3),(2,1)^4)$ using the same formula. The existence of a point with full ramification guarantee all the covers to be connected. Thus using the above results on representations of S_3 we get

$$H_1^0((3),(2,1)^4) = \left(\frac{\dim\sigma_1}{3!}\right)^2 \frac{|C_{\lambda_1}|\chi_{\sigma_1}(\lambda_1)}{\dim\sigma_1} \left(\frac{|C_{\lambda_2}|\chi_{\sigma_1}(\lambda_2)}{\dim\sigma_1}\right)^4$$

$$+ \left(\frac{\dim\sigma_{-1}}{3!}\right)^2 |C_{\lambda_1}|\chi_{\sigma_{-1}}(\lambda_1) \left(\frac{|C_{\lambda_2}|\chi_{\sigma_{-1}}(\lambda_2)}{\dim\sigma_{-1}}\right)^4 + \left(\frac{\dim\sigma_s}{3!}\right)^2 |C_{\lambda_1}|\chi_{\sigma_s}(\lambda_1) \left(\frac{|C_{\lambda_2}|\chi_{\sigma_s}(\lambda_2)}{\dim\sigma_s}\right)^4$$

$$= \left(\frac{1}{6}\right)^2 \frac{2\times 1}{1} \left(\frac{3\times 1}{1}\right)^4 + \left(\frac{1}{6}\right)^2 \frac{2\times 1}{1} \left(\frac{3\times(-1)}{1}\right)^4$$

$$+ \left(\frac{1}{6}\right)^2 \frac{2\times 1}{1} \left(\frac{3\times 0}{1}\right)^4 = 9.$$

Observe that the dimensions of the irreducible representations is easily calculated using the hook length formula.

4.3 Future Research

I will be working to establish the needed bridge between Hurwitz numbers and Tautological classes. Hurwitz numbers are purely combinatorial objects which count branched coverings of Riemann surfaces with prescribed monodromy. Tautological classes, on the other hand, are distinguished classes in the intersection ring of the moduli spaces of Riemann surfaces of a given genus, and are thus “geometric.” Localization computations in Gromov-Witten theory provide non-obvious relations between the two.

My project will involve tautological classes in the moduli space of curves and its Hurwitz loci. A natural question is whether there are good formulas to express the class of Hurwitz loci (loci of curve admitting covers with specified ramification data) in terms of standard tautological classes (boundary, ψ, λ, κ). A. Bertram et al have worked on the hyperelliptic case, and have reduced the ratios of certain tautological classes to the pure combinatorics of Hurwitz numbers [ACG]. In particular, I will seek to extend similar computation in hyperelliptic case to the other cases.

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