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ON ALMOST SIMILARITY AND OTHER RELATED EQUIVALENCE RELATIONS OF OPERATORS IN HILBERT SPACES

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Declaration

Declaration by the Candidate

This dissertation is my original work and has not been presented for a degree award in any other University

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Dedication

This project is dedicated to my spouse Mrs. Mercy Chepkemoi Towett and my siblings.

Acknowledgement

I am grateful to the Almighty God who has carried me through for his work is marvelous. My sincere gratitude is also to my supervisor Dr. B.M. Nzimbi for his encouragement, patience, perseverance, understanding and persistent assistance during the whole period of this study. I wish also to acknowledge all my lecturers in the School of Mathematics; Prof. J.M. Khalagai, Prof. G.P. Pokhariyal, Dr. J. K. Mile, Mr. C. Achola, Dr. J.N. Muriuki, Dr. S.W. Luketero, Dr. J.H. Were and Dr. N. Katende who have taught me in my postgraduate level. I will not also forget Dr. K. Moindi for his tireless encouragement throughout my postgraduate programme. Finally, I am grateful to my spouse Mrs. Mercy Chepkemoi Towett, parents and the rest of my family for their spiritual, mental and moral support without which I would have not gone this far.

May God bless you all.

Abstract

In this thesis, we study unitary equivalence, similarity, quasisimilarity, almost similarity and metric equivalence of operators acting on separable Hilbert spaces. We also study the Murray-von Neumann relation of projections and other equivalence relation of operators in Hilbert spaces. We study the relation between equivalence classes of bounded linear operators with respect to different properties such as being self-adjoint, projections, normal, unitary and having specific rank. We will investigate the spectral picture, norms, spectral radii, numerical range, lattice of their invariant subspaces, hyperinvariant subspaces and reducing subspaces of almost similar operators and metrically equivalent operators. Similarly, we characterize near equivalence, Murray-von Neumann equivalence, stable unitary equivalence and stable similarity of operators

Index of Notations

$\mathcal{H}$: Hilbert space over the complex numbers $\mathbb{C}$.

$B(\mathcal{H})$: Banach algebra of bounded linear operators on $\mathcal{H}$.

T^* : the adjoint of T.

$\|T\|$: the operator norm of T.

$\|x\|$: the norm of a vector x.

$< x, y >$: the inner product of x and y on a Hilbert space $\mathcal{H}$.

$\rho(T)$: the resolvent set of an operator T.

$\sigma(T)$: the spectrum of an operator T.

$Ran(T)$: the range of an operator T.

$\omega(T)$: numerical radius of an operator T.

$\ker(T)$: the kernel of an operator T.

$W(T)$: Numerical range of an operator T.

$c.n.n$: completely non-normal.

$c.n.u$: completely non-unitary.

$M \oplus M^{\perp}$: the direct sum of the subspaces M and $M^{\perp}$.

$\{T\}'$:the commutator of T.

$n.h.s$: nontrivial hyperinvariant subspace.

$a.s$: almost similar.

$n.n.d$: non normal decomposition.

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Chapter 1

Preliminaries

1.1 Notations and Terminologies

In this thesis, Hilbert spaces or subspaces will be denoted by capital letters H, H_1, H_2, K, K_1, K_2, and $T, T_1, T_2, S, S_1, A, B$ etc. denotes bounded linear operators where an operator means a bounded linear transformation. We use $B(\mathcal{H})$ to denote the Banach algebra of bounded linear operators on $\mathcal{H}$. $B(\mathcal{H}, K)$ denotes the set of bounded linear transformations from $\mathcal{H}$ to K. We use $G(H, K)$ to denote set of invertible operators from $\mathcal{H}$ to K. For an operator $T \in B(\mathcal{H}), T^*$ denotes the adjoint while $\ker(T), Ran(T), \overline{M}$ and $M^{\perp}$ stands for the kernel of T, range of T, closure of M and orthogonal complement of a closed subspace M of $\mathcal{H}$ respectively. We use $\sigma(T)$ to denote spectrum of $T, \|T\|$, denotes the norm of $T, r(T)$ denotes the spectral radius of T while $W(T)$ denotes the numerical range of T.

In addition, an operator $T \in B(\mathcal{H})$ is said to be:

Unitary if $TT^* = T^*T = I$.

Normal if $TT^* = T^*T$

Self adjoint (or Hermitian) if $T^* = T$.

Skew-adjoint if $T^* = -T$.

Quasinormal if $T(T^*T) = (T^*T)T$ or equivalently if T commutes with T^*T, that is $[T, T^*T] = 0$.

Binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$.

Orthogonal projection if $T^2 = T$ and $T^* = T$.

An involution if $T^2 = I$.

A symmetry if $T = T^* = T^{-1}$ that is, T is a self-adjoint unitary operator.

Isometric if $T^*T = I$.

Partial Isometry if $TT^*T = T$ (equivalently if T^*T is a projection).

A-self-adjoint if $T^* = A^{-1}TA$ where A is self-adjoint invertible operator.

Normaloid if $r(T) = \|T\|$ (equivalently $\|T^n\| = \|T\|^n$).

Hyponormal if $T^*T \geq TT^*$, equivalently if $T^*T - TT^* \geq 0$ (is a positive operator).

Cohyponormal if its adjoint is hyponormal that is, if $TT^* \geq T^*T$.

Subnormal if there exists a Hilbert space K containing $\mathcal{H}$, that is $K \supseteq H$ and a normal operator N acting on K such that $\mathcal{H}$ is N-invariant and T is the restriction of N onto H, that is $T = N/H$. Thus $T \in B(\mathcal{H})$ is subnormal if $\mathcal{H}$ is a subspace of a Hilbert space $K(H$ can be embedded into $K)$ and with respect to the decomposition.
$K = H \oplus H^{\perp}, N = \begin{pmatrix} T & X \\ 0 & Y \end{pmatrix}$ in $B(X)$ for some $X : H^{\perp} \to H$ and $Y : H^{\perp} \to H^{\perp}$ that is T is a part of a normal operator. Note that a part of an operator T is a restriction of it to an invariant subspace.

p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ where $0 < p \leq 1$.

m-hyponormal if $\|(zI - T)^*x\| \leq M\|(zI - T)\|$ for all complex numbers z and for all $x \in M \subset H$ and M a positive number.

Quasihyponormal if $T^{*2}T^2 - (T^*T)^2 \geq 0$ equivalently if $T^*(T^*T - TT^*)T \geq 0$.

Paranormal if $\|Tx\|^2 \leq \|T^2x\|$ for all unit vectors $x \in H$, equivalently if $\|Tx\| \leq \|Tx\|\|x\|$ for every $x \in \mathcal{H}$.

k-quasihyponormal if $T^{*K}(T^*T - TT^*)T^K \geq 0$ for $K \geq 1$ is some integer and every $x \in H$.

p-quasihyponormal if $T^*(T^*T)^P - (TT^*)^PT \geq 0$.

$(p.k)$-quasihyponormal if $T^{*K}(T^*T)^P - (TT^*)^P)T^K \geq 0$ where $0 < P \leq 1$ and K is a positive integer.

Seminormal if it is either hyponormal, equivalently if either T or T^* is hyponormal. Every hyponormal operator is seminormal but the converse is not true.

From the above definition, we have the following inclusions:
Unitary operators $\subset$ Isometric operators $\subset$ Partial Isometries.

Normal $\subsetneq$ Quasinormal $\subsetneq$ Subnormal $\subsetneq$ Hyponormal $\subsetneq$ Seminormal.

1.2 Introduction

The class of almost similar operators was first introduced by Jibril [12]. He defined the class of almost similar operators as follows:
Two operators A and B are said to be almost similar if there exists an invertible operator N such that the following conditions are satisfied:

$$A^*A = N^{-1}(B^*B)N$$

$$A^* + A = N^{-1}(B^* + B)N$$

He proved various results that relate almost similarity and other classes of operators including isometries, normal operators, unitary operators, compact operators and characterization of $\theta-$operators.

Two operators $T \in B(\mathcal{H})$ and $S \in B(\mathcal{H})$ are similar (denoted by $T \approx S$) if there exists an invertible operator $X \in G(H, K)$ such that $XT = SX(i.e.T = X^{-1}SX \ or \ S = XTX^{-1})$.

Similarly $T \in B(\mathcal{H})$ and $S \in B(\mathcal{H})$ are unitarily equivalent (denoted by $T \cong S$) if there is a unitary operator $U \in G(H, K)$ such that $UT = SU$ (i.e. $T = U^*SU$ or equivalently $S = USU^*$).

An operator $T \in B(\mathcal{H})$ is quasiunitary if $T^*T = TT^* = T + T^*$.

T. B. Hoover [8] studied hyperinvariant subspaces and proved that if S and T are quasisimilar operators acting on the Hilbert spaces $\mathcal{H}$ and K respectively and if S has a hyperinvariant subspace, then so does T. If in addition S is normal, then the lattice of hyperinvariant subspaces for T contains a sub-lattice which is lattice isomorphic to the lattice of spectral projections for S.

Hoover [8] has shown some properties of operators that are preserved by quasisimilarity and those that are not. He gave a result to show that quasisimilar normal operators are unitarily equivalent. He also gave an example to show that quasisimilarity preserves neither spectra nor compactness.

Nzimbi et al. [21], have classified those operators where almost similarity implies similarity. If two operators are almost similar and one of them is isometric, then so is the other. Similar results hold true for hermitian, compact, partially isometric and $\theta-$operators. We also note that if $A, B \in B(\mathcal{H})$ are such that A and B are unitarily equivalent, then they are almost similar. Two quasisimilar operators having equivalent quasi-affinities on a finite dimensional Hilbert space, which are unitary, are also almost similar.

Nzimbi et al. [22] introduced the concept of metric equivalence. They further proved that metric equivalence is an equivalence relation. Two operators $A, B \in B(\mathcal{H})$ are said to be metrically-equivalent if $\|Ax\| = \|Bx\|$ for all $x \in \mathcal{H}$, equivalently $A^*A = B^*B$. Of great interest, Nzimbi et al.[22] concretely discussed the spectral picture of metrically equivalent operators.

A contraction on $\mathcal{H}$ is an operator $T \in B(\mathcal{H})$ satisfying $T^*T \leq I$ or equivalently, $\|Tx\| \leq \|x\|$ for all $x \in H$ and a strict contraction if $T^*T < I$ or $\|Tx\| < \|x\|$ for all $x \in \mathcal{H}$.

An operator $T \in B(\mathcal{H})$ is a left shift on $\ell^2(\mathbb{N})$ if $Tx = y$ for all $x = (x_1, x_2, \ldots)$ and$y = (x_2, x_3, \ldots)$ while it is a right shift operator if $Tx = y$ where $x = (x_1, x_2, \ldots)$ and $y = (0, x_1, x_2, \ldots)$.

By a subspace of a Hilbert space $\mathcal{H}$ we mean a closed linear manifold of $\mathcal{H}$ which is also a Hilbert space. If M and N are orthogonal subspaces of a Hilbert space H, then their (orthogonal) direct sum $M \oplus N$ is a subspace of H.

If M is a closed subspace of $\mathcal{H}$ then $\mathcal{H} = M \oplus M^{\perp}$ is the direct sum of decomposition of H. A subspace M of $\mathcal{H}$ is invariant under T if $T(M) \subseteq M$ that is for $x \in M$ then $Tx \in M$.

A subspace M of $\mathcal{H}$ is said to reduce T if both M and $M^{\perp}$ are invariant under T.

The following inclusions are proper:
Reducing subspaces $\subseteq$ Invariant subspaces.
Hyperinvariant subspaces $\subseteq$ Invariant subspaces.

A direct summand of an operator T is the restriction of T to a reducing subspace. An operator is reducible if it is a nontrivial reducing subspace (equivalently, if it has a proper nonzero direct summand). An operator is irreducible if it is not reducible.

An operator T is said to be normaloid if $r(T) = \|T\|$ (equivalently, $\|T^n\| = \|T\|^n$). In a complex Hilbert space H, every normal operator is normaloid and so is every positive operator. Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. The set $\rho(T)$ of all complex numbers λ for which $(\lambda I - T)$ is invertible is called the resolvent set of T, that is $\rho(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) = \{0\}$ and $Ran(\lambda I - T) = H\}$. The complement of the resolvent set $\rho(T)$ denoted by $\sigma(T)$ is called the spectrum of T.

A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of an operator $T \in B(\mathcal{H})$ if there exists a non-zero vector $x \in H$ such that $Tx = \lambda x$, equivalently, if $\ker(\lambda I - T) \neq \{0\}$.

An operator $A \in B(\mathcal{H})$ is called a $\theta-$ operator if $A^* + A$ commutes with A^*A, the class of all $\theta-$operators in $B(\mathcal{H})$ is denoted by θ that is $\theta = \{A \in B(\mathcal{H}) : [A^*A, A^* + A] = 0\}$.

An operator $T \in B(\mathcal{H})$ is Fredholm if it has finite nullity and finite Corank.

Let $A \in B(\mathcal{H})$. A subspace M of $\mathcal{H}$ is said to be hyperinvariant under A or $A-$ hyperinvariant if M is invariant under any operator S that commutes with A.

Adjoint in linear algebra refers to the conjugate transpose.

A lattice is a partially ordered set in which every two elements have a unique supremum and a unique infimum.

In finite dimensional spaces quasisimililarity is the same thing as similarity but it is a weaker relation in finite dimension spaces.

Let $\mathcal{H}$ and K be Hilbert spaces. An affinity from $\mathcal{H}$ to K is a linear one to one and bicontinuous transformation X from $\mathcal{H}$ to K.

An operator $A \in B(\mathcal{H})$ is said to be positive (in symbols $A \geq 0$) if A is self adjoint and $< Ax, x > \geq 0$ for all $x \in H$.

Let T and $S \in B(\mathcal{H})$. Then S is said to be nearly equivalent to T denoted by $S \ominus T$ if and only if there exist an invertible operator $V \in B(\mathcal{H})$ such that $S^*S = V^{-1}T^*TV$. We denote the set of operators S that are nearly equivalent to T by $\xi(T)$.

An operator $X \in B(H, K)$ is called a quasiaffinity if it is injective with dense range. An operator A is said to be a quasiaffine transform of another operator B if there exists a quasiaffinity $X \in B(H, K)$ intertwining A and B (that is $XA = BX$).

Two operators $A \in B(\mathcal{H})$ and $B \in B(K)$ are said to be quasisimilar if they are quasiaffine transforms of each other, that is, if there exists quasiaffinities $X \in B(H, K)$ and $Y \in B(K, H)$ such that $XA = BX$ and $AY = YB$.

Two operators $A \in B(\mathcal{H})$ and $B \in B(K)$ are said to be metrically equivalent if $\|Ax\| = \|Bx\|$ (equivalently, $| < Ax, Ax > |^{\frac{1}{2}} = | < Bx, Bx > |^{\frac{1}{2}}$ for all $x \in H$ that is $A^*A = B^*B$.

Two projections P and Q in $B(\mathcal{H})$ are said to be Murray-von Neumann equivalent denoted by $P \overset{M-v-N}{\sim} Q$ if there exists a partial isometry $V \in B(H)$ such that $V^*V = P$ and $VV^* = Q$.

Two operators A and B are said to be stably similar or power similar denoted by $A \overset{s.s}{\sim} B$ if there is an invertible operator X such that $A^n = X^{-1}B^nX$ for some positive integer n. They are stably unitarily equivalent, denoted by $A \overset{s.u.e}{\sim} B$ if there is a unitary operator U such that $A^n = U^*B^nU$.

Chapter 2

Unitary Equivalence, Similarity, Almost Similarity and Quasisimilarity of Operators

2.1 Unitary Equivalence of Operators

We start off by discussing the following result which shows that unitary equivalence is an equivalence relation.

Theorem 2.1.1. *Unitary equivalence is an equivalence relation.*

Remark 2.1.2. *Unitary equivalence preserves reducing subspaces. That is if A, $B \in B(\mathcal{H})$ such that A is unitarily equivalent to B and there exists a subspace M of $\mathcal{H}$ which reduces A, then M reduces B. If $B \cong C$, for another operator C acting on a Hilbert space, then M also reduces C.*

The natural concept of equivalence between Hilbert-space operators infact is unitary equivalence. However, the weaker form of equivalence will also play an important role throughout this project. The following propositions and auxiliary results will be referred frequently. They deal with parts and direct summands of unitarily equivalent operators. The following are known results by [13].

Proposition 2.1.3. *If an operator $T \in B(\mathcal{H})$ is unitarily equivalent to a part of an operator $L \in B(K)$, then it is a part of an operator unitarily equivalent to L.*

Corollary 2.1.4. *If an operator $T \in B(\mathcal{H})$ is unitarily equivalent to a direct summand of an operator $L \in B(K)$, then it is a direct summand of an operator unitarily equivalent to L.*

Remark 2.1.5. *Recall that a direct summand of an operator T is a part of it whose adjoint also is a part of T^*.*

Corollary 2.1.6. *If an operator $T \in B(\mathcal{H})$ is unitarily equivalent to a direct sum $L \in B(K)$ then it is a direct sum itself with direct summand unitarily equivalent to each direct summand of L (i.e. if $T \cong \oplus_k L_k$, then $T = \oplus_k L_k$, with $T_K \cong \oplus_k L_k$, for each K).*

Corollary 2.1.7. *Every operator unitarily equivalent to a reducible operator is reducible.*

Theorem 2.1.8. *([28], Fuglede-Putnam Theorem) Let $A, B, T \in B(\mathcal{H})$, where A and B are normal and $AT = TB$ then $A^*T = TB^*$.*

Remark 2.1.9. *Note that the hypothesis of 2.1.8 does not imply that $AT^* = T^*B$ even when A and B are self-adjoint and T is normal.*

Example 2.1.10. *Consider* $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Thus a simple computation shows that $AT = TB$ but $AT^* \neq T^*B$.

If $T \in B(\mathcal{H})$ is invertible, then T has a unique polar decomposition $T = UP$, with U an isometry (which is in fact a unitary) and $P \geq 0$. If $T \in B(\mathcal{H})$ is normal, then T has a polar decomposition $T = UP$ in which U and P commute with each other and T.

Theorem 2.1.11. *If $A, B \in B(\mathcal{H})$ are similar normal operators and there exists a unitary operator U then $A = U^*BU$.*

Theorem 2.1.12. *An operator $T \in B(\mathcal{H})$ is quasiunitary if and only if $I - T$ is unitary.*

Proof. Suppose T is quasi-unitary. Then $(I-T)^*(I-T) = (I-T)(I-T^*) = I$. Therefore $I - T$ is unitary.
Conversely, suppose $I - T$ is unitary, then $I - (T + T^*) + T^*T = I - (T^* + T) + TT^* = I$
Simplifying the equation, we have that $T^*T = TT^* = T + T^*$, this implies that
That is $\Omega = (I - T)$ then, $\Omega^*\Omega = \Omega\Omega^* = I$ (that is $\Omega = (I - T)$) is unitary.
Conversely suppose $I - T$ is unitary that is

$(I-T)^*(I-T) = (I-T)(I-T)^* = I$
$I - T - T^* + T^*T = I - T - T^* + TT^* = I$
$-(T+T^*) + T^*T = -(T+T^*) + TT^* = 0$, then
$T^*T = T + T^*$ and $TT^* = T + T^*$.
Thus we have shown that $T^*T = TT^* = T + T^*$ hence $T$is quasi-unitary.

□

Theorem 2.1.13. *[20] Let A and B be normal operators such that $AX = XB$ where X is a quasiaffinity. Then*
(a) $\sigma(A) = \sigma(B)$
(b) $\sigma(A^*A) = \sigma(B^*B)$
(c) $\sigma(AA^*) = \sigma(BB^*)$

Proof. The proof follows from a repeated application of the Fuglede-Putnam Theorem on three pairs of operators in (a), (b) and (c). For full details (see [20], Theorem 1). □

Theorem 2.1.14. *If S and T are unitarily equivalent, then they are unitarily quasi-equivalent.*

Proof. If S and T are unitarily equivalent that is, $S = U^*TU$ then,

$S^*S = U^*T^*UU^*TU$
$= U^*T^*TU.$

□

The converse of 2.1.14 is not generally true. Consider the operators represented by the matrices

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} T = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$$

A simple computation shows that S and T are unitarily quasi-equivalent with the equivalence implemented by the unitary operator $U = I$. However, $\sigma(S) = \{0, 2\} \neq \{0, -2\} = \sigma(T)$. This shows that S and T are not similar and hence cannot be unitarily equivalent.

Theorem 2.1.15. *[13] If S and T are both self-adjoint and unitarily quasi-equivalent operators, then T^2 and S^2 are unitarily equivalent.*

Theorem 2.1.16. *[13] If S and T are unitarily quasi-equivalent and T is skew-adjoint, then S is normal.*

2.2 Similarity of Operators

We recall that two operators $T \in B(\mathcal{H})$ and $S \in B(\mathcal{H})$ are similar (denoted by $T \approx S$) if there exists an invertible operator $X \in G(H, K)$ such that $XT = SX$ (i.e. $T = X^{-1}SX$ or $S = XTX^{-1}$).
Similarly $T \in B(\mathcal{H})$ and $S \in B(\mathcal{H})$ are unitarily equivalent (denoted by $T \cong S$) if there is a unitary operator $U \in G(H, K)$ such that $UT = SU$ (i.e. $T = U^*SU$ or equivalently $S = UTU^*$).

Remark 2.2.1. *It has been shown by [5] that similarity is an equivalence relation on $B(\mathcal{H})$. The natural concept of equivalence between Hilbert space operators is unitary equivalence which is stronger than similarity.*

Lemma 2.2.2. *Suppose that A and B are similar operators on a Hilbert space H, then A and B have the same*
(a) Spectrum
(b) Point spectrum
(c) Approximate point spectrum

Corollary 2.2.3. *If two operators are similar and if one of them has a nontrivial invariant subspace, then so has the other.*

Theorem 2.2.4. *Every similarity transformation $\phi : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\phi(A) = S^{-1}BS$ is an automorphism. That is it maps sums into sums, products into products and scalar multiples into scalar multiples.*

Proof. $\phi(A_1 + A_2) = S^{-1}(B_1 + B_2)S = S^{-1}B_1S + S^{-1}B_2S = \phi(A_1) + \phi(A_2)$.
$\phi(A_1A_2) = S^{-1}(B_1B_2)S = S^{-1}(B_1SS^{-1}B_2)S = (S^{-1}B_1S)(S^{-1}B_2S) = \phi(A_1)\phi(A_2)$.
$\phi(KA) == S^{-1}(KB)S = KS^{-1}BS = K\phi(A)$.

□

Corollary 2.2.5. *The set of similarity operators forms a group.*

Proof. It suffices to prove the composition of similarity operators is a similarity operator. Suppose that $\psi_S(A) = S^{-1}BS$ and $\psi_N(A) = N^{-1}BN$. Then:
$(\psi_S(A))N = N^{-1}(S^{-1}BS)N = N^{-1}S^{-1}BSN = (SN)^{-1}B(SN) = \psi_SN$. □

Theorem 2.2.6. *Let A and B be similar operators. Then dim $(\ker(A)) = dim(\ker(B))$ and $dim(\ker(A^*)) = dim(\ker(B^*))$.*

Note that Theorem 2.2.6 does not necessarily imply equality of kernels. The operators

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \; B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

are similar but $\ker(A) \neq \ker(B)$ however, a simple calculation shows that they have the same nullity.

Proposition 2.2.7. *If T is invertible then T^*T and TT^* are similar.*

Proof. This follows from $TT^* = T(T^*T)T^{-1}$. □

Lemma 2.2.8. *If T is invertible then T^*T and TT^* have the same spectrum.*

Proof. For any $T \in B(\mathcal{H}), \sigma(T^*T)/\{0\} = \sigma(TT^*)/\{0\}$. Since T is invertible, $\{0\} \notin \sigma(T)$ this implies that $\sigma(T^*T) = \sigma(TT^*)$. □

Theorem 2.2.9. *If T^*T and TT^* are similar then, T and T^* are invertible.*

Corollary 2.2.10. *If T^*T and TT^* are similar, then T is normal.*

Proof. This follows from the fact that T and T^* are unitary equivalent. □

Corollary 2.2.11. *Every invertible operator T is nearly normal.*

Theorem 2.2.12. *Let A and B be invertible positive operators. Then $A+B$ is invertible if and only if A is similar to B.*

Proof. Without loss of generality, suppose T is the inverse of $A+B$. Then
$(A+B)T = T(A+B) = I$. That is, $AT + BT = TA + TB = I$.

□

Remark 2.2.13. *Note that the positivity of both operators cannot be dropped. To see this consider*

$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \; B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Clearly $A+B = 0$ and hence not invertible.

It is easy to check that both A and B are not positive operators.

Theorem 2.2.14. *If T commutes with both A and B, then T commutes with $A+B$.*

2.3 Almost Similarity of Operators

Recall that in Jibril [12], two operators A and B are said to be almost similar if there exists an invertible operator N such that the following two conditions are satisfied:

$A^*A = N^{-1}(B^*B)N$
$A^* + A = N^{-1}(B^* + B)N$

Theorem 2.3.1. *[21] theorem 2.1 Almost similarity of operators is an equivalence relation.*

Proposition 2.3.2. *[17] If $A \in B(\mathcal{H})$ and $A \overset{a.s}{\sim} I$ then $A = I$.*

Proposition 2.3.3. *[17] If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and if A is compact then so is B.*

Theorem 2.3.4. *[2] An operator $T \in B(\mathcal{H})$ is hermitian if and only if $(T+T^*)^2 \geq 4T^*T$.*

Proposition 2.3.5. *[17] If $A, B \in B(\mathcal{H})$ such that $\theta \in B$ and $A \overset{a.s}{\sim} B$, then $\theta \in A$.*

Proposition 2.3.6. *[17] If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$, and A is partially isometric then so is B.*

Proposition 2.3.7. *[17] If $A \in B(\mathcal{H})$ is normal then $A \overset{a.s}{\sim} A^*$.*

Remark 2.3.8. *The converse to 2.3.7 is not true in general, for consider*
$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
By matrix computation $A^*A = N^{-1}(AA^*)N$ and $A^* + A = N^{-1}(A + A^*)N$. That is $A \overset{a.s}{\sim} A^*$ although A is not normal.

Proposition 2.3.9. *[2] If $A \in B(\mathcal{H})$ then $A \in \theta$ if and only if $A \overset{a.s}{\sim} B$ for some normal operator B.*

Proposition 2.3.10. *[2] If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and B is hermitian, then A is hermitian.*

Proposition 2.3.11. *[2] If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and if A is hermitian then A and B are unitarily equivalent.*

Proof. By assumption there exists an invertible operator N such that $A^*A = N^{-1}B^*BN$ and $A^* + A = N^{-1}(B^* + B)N$. Since A is hermitian and $A \overset{a.s}{\sim} B$ it follows that B is hermitian. Using this fact, the second equality above becomes $A = N^{-1}BN$. This means that A and B are similar. We have shown that these operators are both hermitian and are therefore normal. $\square$

The following result is an immediate corollary to this result.

Corollary 2.3.12. *If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and if either A or B is hermitian, then A and B are unitarily equivalent.*

Proof. The proof follows easily from 2.3.11. $\square$

Proposition 2.3.13. *[9] (corollary 4.5) if $A, B \in B(\mathcal{H})$ are projection operators such that $A \overset{a.s}{\sim} B$ and $(A + \lambda I) \overset{a.s}{\sim} (B + \lambda I)$ then $\sigma_P(A) = \sigma_P(B)$.*

Note that if A and B are quasisimilar, then A^* and B^* are quasisimilar. This may not be true for other relations. For instance, metric equivalence of A and B does not in general imply metric equivalence of A^* and B^*. It is evident that two unitary operators need not have equal spectra. But if they are quasiaffine or even quasisimilar, then they are unitarily equivalent and hence have equal spectra.
Note that similarity of A and B need not imply similarity of A^*A and B^*B or metric equivalence of A and B (unless both A and B are normal operators).

Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Then A and B are similar non-normal operators, with

$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ implementing similarity. However a simple calculation shows that

$$A^*A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = B^*B.$$

Proposition 2.3.14. *If A is a quasi-unitary operator then A and A^* are almost similar.*

Proof. $A^*A = AA^* = A^* + A$ is equivalent to $A^*A = I^{-1}(AA^*)I = A^* + A = I^{-1}(A + A^*)I$. □

These results can be relaxed to the class of normal operators.

Lemma 2.3.15. *If T is a normal operator, then there exists a unitary operator U such that $T^* = UT$.*

Corollary 2.3.16. *Every normal operator T is almost similar to its adjoint.*

Proof. Since T is normal by 2.3.15 $T^* = UT$ for some unitary operator U. Thus $T^*T = U(TT^*)U^*$ and $T^* + T = UT + T^*U^* = UT^*U^* + UTU^* = U(T + T^*)U^*$. This shows that T is almost unitarily equivalent and hence almost similar to T^*. □

Theorem 2.3.17. *Let $A, B \in B(\mathcal{H})$ and X be an invertible operator. If $XA = BX$ and $XA^* = B^*X$, then A and B are almost similar.*

Proof. A simple calculation shows that $A^*A = X^{-1}(B^*B)X$ and $A^* + A = X^{-1}(B^* + B)X$. □

Corollary 2.3.18. *If A and B are similar normal operators, then they are almost similar.*

Proof. Suppose that $XA = BX$, where X is an invertible operator. Then by the Fuglede-Putnam theorem $XA^* = B^*X$, the rest of the proof follows from Theorem 2.3.17. □

There exists non-normal similar operators that are almost similar.

Example 2.3.19. *The operators $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are almost similar with $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ implementing the almost similarity.*

2.3.1 Almost Similarity and Completely Non-Unitary Operators

The following results are well known.

Lemma 2.3.20. *[15] An operator is a unilateral shift if it is a completely non-unitary isometry.*

Proposition 2.3.21. *Let $A \in B(\mathcal{H})$ be such that $A \overset{a.s}{\sim} T$, where T is an isometry. Then the direct summands of A are isometric.*

Theorem 2.3.22. *[13] (Nagy-Foias-Langer Decomposition Theorem). Let T be a contraction on a Hilbert space $\mathcal{H}$ and let $M = ker(I - A) \cap ker(I - A), M$ is a reducing subspace for T. Moreover, the decomposition $T = C \oplus U$ on $H = M^{\perp} \oplus M$ is such that $C = T|_{M^{\perp}}$ is a c.n.u contraction and $U = T|_M$ is unitary.*

Proposition 2.3.23. *If $A, B \in B(\mathcal{H})$ are contractions such that $A \overset{a.s}{\sim} B$ and B is c.n.u, then A is c.n.u.*

Theorem 2.3.24. *Let P and Q be orthogonal projection operators on a Hilbert space H. Then the following statements are equivalent.*
(a) P and Q are almost similar.
(b) P and Q are similar.

Proof. (a)⇒(b). Suppose that N is an invertible operator such that $P^*P = N^{-1}(Q^*Q)N$ and $P^* + P = N^{-1}(Q^* + Q)N$. Since P and Q are orthogonal projections, a simple computation shows that these two equalities both collapse to the equality $P = N^{-1}QN$.
(b)⇒(a). Suppose $P = N^{-1}QN$ for some invertible operator N and orthogonal projections P and Q in $B(\mathcal{H}.)$ By the idempotence of P and Q we have that $P^2 = N^{-1}Q^2N$. By the self-adjointness of P and Q, we also have $P^*P = N^{-1}(Q^*Q)N$. The other equality follows from the fact that $P^* + P = 2P = 2N^{-1}(Q^* + Q)N$ which upon simplification becomes $P^* + P = N^{-1}(Q^* + Q)N$. □

Remark 2.3.25. *Theorem 2.3.24 says that for orthogonal projection operators of the same rank on a Hilbert space $\mathcal{H}$, the notion of almost similarity coincides with that of similarity. This result may fail for other classes of operators.*

Theorem 2.3.26. *Every almost similarity transformation $\phi : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\phi(A^*A) = S^{-1}(B^*B)S, \phi(A^* + A) = S^{-1}(B^* + B)S$ is an automorphism. That is it maps sums into sums, products into products and scalar multiples into scalar multiples.*

Proof. Suppose that $\phi(A^*A) = S^{-1}(B^*B)S$ and $\phi(C^*C) = S^{-1}(D^*D)S$. Then

$$\phi(A^*A + C^*C) = S^{-1}(B^*B + D^*D)S = S^{-1}(B^*B)S + S^{-1}(D^*D)S = \phi(A^*A) + \phi(C^*C).$$

$$\phi((A^*A)(C^*C)) = S^{-1}((B^*B)(D^*D))S = S^{-1}((B^*B)SS^{-1}(D^*D))S = (S^{-1}(B^*B)S)(S^{-1}(D^*D)S) = \phi(A^*A)\phi(C^*C).$$

$$\phi(kA^*A) = S^{-1}(kB^*B)S = kS^{-1}B^*BS = k\phi(A^*A).$$

□

Proposition 2.3.27. *If A and B are self adjoint operators which are almost similar then $\sigma(A) = \sigma(B)$.*

Proof. Suppose $A^* = A, B^* = B$ and $A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$. A simple calculation shows that $A = N^{-1}BN$. That is A and B are similar and therefore have equal spectrum. □

Remark 2.3.28. *Note that Proposition 2.3.27 is not true in general. Note also that equality of spectra as a set does not generally imply similarity of operators. This is true if and only if the two operators have the same multiplicity.*

Example 2.3.29. *Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Both operators are self-adjoint and a simple computation shows that A and B are almost similar, with $N = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$.*

Now if we let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ then A and B are self-adjoint operators but a simple computation shows that they are not almost similar (and hence, not similar). Clearly $\sigma(A) = \{0, 1\} \neq \{0, -1\} = \sigma(B)$.

It is also evident that almost similarity does not preserve the spectra of operators.

Proposition 2.3.30. *An operator $A \in B(\mathcal{H})$ is isometric if and only if $A \overset{a.s}{\sim} U$ for some unitary operator U.*

Example 2.3.31. *Consider the operator* $T = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}$ *on the two dimensional space* $\mathbb{C}^2$*. Then:* $T^2 = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ *(is an involution) which implies that* $T^{-1} = T$*.*
Thus $T \overset{a.s}{\sim} T^{-1}$*. However* $\|T\| \geq 1$ *which means that* T *is not unitary.*

Proposition 2.3.32. *Let* $A, B \in B(\mathcal{H})$*. Then*

(i) If $A \overset{a.s}{\sim} 0$ *then* $A = 0$*.*

(ii) If $A \overset{a.s}{\sim} B$ *and* B *is isometric, then* A *is isometric.*

2.3.2 Almost Similarity and Lattice of Invariant and Hyperinvariant Subspaces of Operators

Theorem 2.3.33. *If* A *and* B *are similar operators then they have isomorphic lattice of invariant and hyperinvariant subspace. That is,* $Lat(A) \equiv Lat(B)$ *and* $Hyperlat(A) \equiv Hyperlat(B)$*.*

Lemma 2.3.34. *Let* $A \in B(\mathcal{H}), B \in B(K)$ *and* $X \in B(H, K)$ *be such that* $XA = BX$*. Suppose* $M \subset K$ *is a nontrivial invariant subspace for* B*. If* $Ran(X) = K$ *and* $\ker(X) \cap M \neq \{0\}$ *is the inverse image of* M *under* $X, X^{-1}(M)$ *is a nontrivial invariant subspace of* A*.*

If the intertwining operator X is surjective i.e. $XA = BX$ and $Ran(X) = K$, then $X^{-1}(M)$ is a nontrivial subspace for A, thus we have the following corollary:

Corollary 2.3.35. *Take* $T \in B(\mathcal{H}), L \in B(K)$ *and* $X \in B(H, K)$ *such that* $XT = LX$*. Let* $M \subset K$ *be a nontrivial finite-dimensional reducing subspace for* L*. If* $R(X) = K$*, then* $X^{-1}(M^{\perp})$ *is a nontrivial invariant subspace for* T*.*

Theorem 2.3.36. *Let* $A \in B(\mathcal{H})$ *and* $B \in B(K)$ *be self-adjoint operators. Let* $X \in B(H, K)$ *be a quasiaffinity that intertwines* A *and* B *(equivalently* $XA = BX$*). Then* A *and* B *are unitarily equivalent.*

Theorem 2.3.37. *Let $\mathcal{H}$ be an n-dimensional Hilbert space $T \in B(\mathcal{H})$ and $\psi : B(\mathcal{H}) \to B(\mathcal{H})$ be a linear map. Then the following statements are equivalent.*

(a)$Lat(T) \equiv Lat(\psi(T))$ for every $T \in B(\mathcal{H})$.
(b)$Hyperlat(T) \equiv Hyperlat(\psi(T))$ for every $T \in B(\mathcal{H})$.
(c) $Ran(T) \equiv Ran(\psi(T))$ for every $T \in B(\mathcal{H})$.

2.4 Quasisimilarity of Operators

The following results are well known.

Theorem 2.4.1. *If X is a quasi-affinity from $\mathcal{H}$ to K and Y is a quasi-affinity from K to L then:*
(a) YX is a quasi-affinity from $\mathcal{H}$ to K and XY is a quasi-affinity from K to L.

(i) YX is a quasi-affinity from $\mathcal{H}$ to L and XY is a quasi-affinity from L to $\mathcal{H}$
(ii) If $X \in B(\mathcal{H})$ is a quasi-affinity, then X^* is a quasi-affinity.

Theorem 2.4.2. *If A is a quasi-affine transform of B and B is a quasi-affine transform of C, then:*

a) A is a quasi-affine transform of C.
b) B^* is a quasi-affine transform of A^*.

Theorem 2.4.3. *If X is a quasi-affinity from $\mathcal{H}$ to K then $|X| = \sqrt{X^*}X$ is a quasi-affinity on $\mathcal{H}$ (from K to $\mathcal{H}$). Moreover, $X|X|^{-1}$ extends by continuity to a unitary transformation U from $\mathcal{H}$ to K.*

Theorem 2.4.4. *Quasi-similarity is an equivalence relation on the class of operators.*

Theorem 2.4.5. *If $T \in B(\mathcal{H})$ and $S \in B(K)$ are similar operators, then they are quasi-similar.*

Proof. There exists quasi-invertible operator $X \in B(H, K)$ such that $XT = SX$. This implies that $X^{-1}S = TX^{-1}$, where $X^{-1} \in B(K, H)$ which implies that $S \approx T$. □

Theorem 2.4.6. *Quasi-similar hyponormal operators have equal spectra.*

Proof. If S and T are quasi-similar hyponormal operators, then for any complex number $\lambda, S - \lambda I$ and $T - \lambda I$ are also quasi-similar and hyponormal, they are both invertible or both non- invertible. Thus $\sigma(S) = \sigma(T)$. $\square$

Remark 2.4.7. *From the proper inclusion relation, Normal $\subset$ Hyponormal $\subset$ Quasi-hyponormal and if hyponormal operators are replaced by quasi-hyponormal operators, we obtain a similar result, that is $\sigma(S) = \sigma(T)$.*

Chapter 3

Metric Equivalence of Operators

Recall that two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{H})$ are said to metrically equivalent, denoted by $A \sim_M B$, if $\| Ax \| = \| Bx \|$, equivalently $| < Ax, Ax > |^{\frac{1}{2}} = | < Bx, Bx > |^{\frac{1}{2}}$ for all $x \in H$, that is $A^*A = B^*B$.
The numerical radius of T is defined by $r(T) = lim_{n \to \infty} \| T^n \|^{\frac{1}{n}} = sup\{|\lambda| : \lambda \in \sigma(T)\}$. Clearly $r(T) = r(T^*)$. Recall that an operator T is said to be normaloid if $r(T) = \| T \|$ (equivalently $\| T^n \| = \| T \|^n$). In a complex Hilbert space H, every normal operator is normaloid and so is every positive operator.
The following results have been proved in [22].

Theorem 3.0.8. *(Theorem 2.2) A necessary and sufficient condition that an operator $T \in B(\mathcal{H})$ be normal is that $\|Tx\| = \|T^*x\|$ for every $x \in H$.*

Corollary 3.0.9. *(Corollary 2.6) If S and T are metrically equivalent normal operators, then there exists a unitary operator U such that $S = UT$.*

Lemma 3.0.10. *Lemma [2.7] Let S and T be linear operators on a Hilbert space H. If $S \sim_M T$, then*

(i) If T is isometric, then S is also isometric.
(ii)If T is a contraction, then S is also a contraction.
(iii) If T is a partial isometry, then S is also a partial isometry.
(iv) If S and T are positive, then $S = T$.
(v) If S is bounded below, then T is also bounded below. Moreover, S is injective and so is T. If in addition, S has a dense range, then both S and T are invertible.

Example 3.0.11. *Suppose A and B are metrically equivalent contractions with unitary extensions U and V, respectively. Without loss of generality, we let A and B be unitaries. Then $A^*A = B^*B$ and $U^*U = \begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix} = \begin{pmatrix} B^*B & 0 \\ 0 & BB^* \end{pmatrix} = V^*V$, which proves that U and V are metrically equivalent.*

Clearly, if an operator has a unitary extension, then so does every operator unitarily equivalent to it. This is because a unitary operator always has a unitary extension to every larger Hilbert space. This is also true for metrically equivalent operators.

Corollary 3.0.12 ([16], Corollary 2.12)**.** *If $S \in B(\mathcal{H})$ and $T \in B(\mathcal{H})$ are unitarily equivalent then S and T are similar.*

Proposition 3.0.13 ([16], Proposition 2.13)**.** *If S and T are normal operators in a Hilbert space H, then S is unitarily equivalent to T if and only if S is similar to T.*

3.1 Metric Equivalence and Spectral Picture of Operators

We recall that the numerical range $W(T)$ of an operator $T \in B(\mathcal{H})$ is defined as $W(T) = \{\lambda \in \mathbb{C} : \lambda =< Tx, x >, \|x\| = 1\}$ and the numerical radius $w(T)$ of T is defined as $w(T) = Sup\{|\lambda| : \lambda \in W(T)\}$.
The following results were stated and proved in [22].

Theorem 3.1.1. *[Theorem 2.14] If T and S are metrically equivalent operators on H, then $\|S\| = \|T\|$.*

The converse of 3.1.1 is not always true; there exists operators with the same norm which are not metrically equivalent.

Theorem 3.1.2 (Theorem 2.15)**.** *If T and S are metrically equivalent, then $W(|T|) = W(|S|)$.*

Proof. If $\|S\| = \|T\|$ then T^*T is self-adjoint, it is normal and thus $W(T^*T) = \|T\|^2$, also, $W(T^*T) = W(S^*S)$, hence $W(|T|) = W(|S|)$. $\square$

Remark 3.1.3. *Unlike unitarily equivalent operators, metrically equivalent operators S and T need not have equal numerical range. Note that the spectrum of S may be equal to the spectrum of T yet S and T are not metrically equivalent.*

For instance, the operators represented by the matrices
$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in $\mathbb{C}^2$ have the property that $\sigma(S^*S) = \sigma(T^*T)$ and $\sigma(S) = \sigma(T)$ but S and T are not metrically equivalent operators. For example, the unilateral shift and the identity operator on $H = \ell^2$ are metrically equivalent but have unequal spectra.

Theorem 3.1.4. *If S and T are metrically equivalent normaloid operators, then $r(S) = r(T)$.*

Remark 3.1.5. *The converse of Theorem 3.1.4 is not generally true. The operators represented by the matrices*

$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *in* $\mathbb{C}^2$ *have the property that* $r(S) = r(T) = 0$ *but a simple computation shows that S and T are not metrically equivalent. This is because S and T are not normal and hence not normaloid. However, we note that S and T have the same numerical range, which is in a closed disk centered at 0 and of radius $\frac{1}{2}$. Hence, it will be essential to also confirm whether for two almost similarity normaloid operators S and $T, r(S) = r(T)$.*

The following are basic results due to [22].

Proposition 3.1.6. *(Proposition 2.16) Metrically equivalent operators S and T need not have equal spectra.*

Remark 3.1.7. *It is also true that metrically equivalent normal operators S and T need not have equal spectral. Consider, for instance, the operators represented by the matrices*

$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *in* $\mathbb{C}^2$.
A simple computation shows that $\sigma(S) = \{-1, 1\}$ and $\sigma(T) = \{1\}$. It is also clear that $W(S) \neq W(T)$. Thus, the metric equivalence does not preserve numerical range.

Theorem 3.1.8. *(Theorem 2.18) If S and T are metrically equivalent normal operators on H, with polar decomposition $S = U|S|$ and $T = V|T|$, then $|S| = |T|$.*

Theorem 3.1.9. *If S and T are metrically equivalent operators then* $\ker(S) = \ker(T)$.

Proof. $S^*S = T^*T$ implies that $\ker(S^*S) = \ker(T^*T)$, which in turn implies that $\ker(S) = \ker(T)$.

□

Theorem 3.1.10. *(Theorem 2.19) Direct summands of metrically equivalent operators are metrically equivalent.*

Theorem 3.1.11 (Theorem 2.20)**.** *Let $T \in B(\mathcal{H})$. If $N \in B(\mathcal{H})$ is normal and $NT = TN$, then $N^*T = TN^*$*

Theorem 3.1.12 (Theorem 2.21)**.** *Let S and T be metrically equivalent operators on a Hilbert space $\mathcal{H}$ and $ST = TS$. If T is normal, then S is quasinormal.*

Theorem 3.1.13. *If S and T are metrically equivalent operators and S is self-adjoint, then $S = |T|$.*

Proof. $S^*S = T^*T$ and $S^* = S$ implies that $S^2 = T^*T$ and since T^*T is positive it has a positive square root $|T|$. Therefore $S = \sqrt{T^*T} = |T|$. □

Theorem 3.1.14. *Two positive operators S and T are metrically equivalent if and only if $S = T$.*

Proof. We need to show that if $\|Sx\| = \|Tx\|$ for all $x \in H$, then $S = T$. This follows from Theorem 3.1.13.

□

Theorem 3.1.15. *If S and T are metrically equivalent self-adjoint operators then $S = T$ if and only if $|S| = |T|$.*

Example 3.1.16. *The operators*

$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *show that if A and B are not both positive, then* *although $\|Ax\| = \|Bx\| = \|A^*x\| = \|B^*x\|$, for all $x \in H$(that is A and B and A^* and B^* are pairwise metrically equivalent), A and B are not even similar. In this example, B is not positive although it is self-adjoint. Note that $B \leq A$. This shows that metric equivalence does not preserve positivity of operators.*

Theorem 3.1.17. *Let $A, B \in B(\mathcal{H})$. Suppose $A^*A = B^*B, AA^* = BB^*$ and $XB = AX, X^*B = AX^*$, for some quasiaafinity X, then*

(a) A and B are unitarily equivalent.
(b) If X is an injective positive operator, then $A = B$.

Proof. (a) Suppose that $A^*A = B^*B, AA^* = BB^*$ and $XB = AX, X^*B = AX^*$ For some quasi affinity X. Suppose $X = U|X|$ is the polar decomposition of X, where U is a partial isometry and $|X| = \sqrt{X^*X}$ is positive.
Define $W = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$ and $S = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ on $H \oplus H$, since$X$ is quassiaffinity, so is W.
A simple calculation, using $XB = AX$ and $X^*B = AX^*$ we have that

$$S^*S = \begin{pmatrix} BB^* & 0 \\ 0 & A^*A \end{pmatrix} = \begin{pmatrix} AA^* & 0 \\ 0 & B^*B \end{pmatrix} = SS^*$$

and $SW = WS^*$ which means that S and S^* are quasiaffine/quasisimilar normal operators. So by the Fudglede Putnam Theorem, S and S^* are unitarily equivalent. That is there exists unitary operator U such that $S = U^*S^*U$, where U is as in the polar decomposition of X that is

$\begin{pmatrix} 0 & A \\ B^* & 0 \end{pmatrix} = U^* \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} = U$, which shows that $A = U^*BU$.
(b) If X is injective and positive, so is W. Thus, S and S^* are unitarily equivalent positive normal operators. Thus $S = S^*$, which in turn implies that $A = B$. □

3.2 Relationship Between Metric Equivalence of Operators and Other Equivalence Relation

Theorem 3.2.1 ([22] Theorem 2.25). *Let S and T be in $B(\mathcal{H})$. If S and T are unitarily equivalent, then they are metrically equivalent.*

Remark 3.2.2. *The converse of Theorem 3.2.1 is not generally true. Consider the operators in $\mathbb{C}^2$ represented by the matrices*

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

A simple computation shows that $S^*S = T^*T = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ *which means that S and T are metrically equivalent.*
However, $\sigma(S) = \{0, 2\} \neq \{0, -2\} = \sigma\{T\}$. This shows that S and T are not similar and hence cannot be unitarily equivalent.

The following result gives a condition when metric equivalence of operators implies unitary equivalence.

Theorem 3.2.3 ([22], Theorem 2.26)**.** *If S and T are metrically equivalent projections, then they are unitary equivalent.*

Remark 3.2.4. *Note that unitary and metric equivalence are norm-preserving while similarity is not norm-preserving.*

Definition 3.2.5. *Recall that, linear operators S and T acting on a Hilbert space $\mathcal{H}$ are said to be nearly equivalent if there exists a unitary operator U such that $S^*S = U^*T^*TU$ or equivalently if S^*S and T^*T are unitarily equivalent. That is, near equivalence of operators need not imply unitary equivalence of operators and also need not imply similarity of operators.*

Let $T \in B(\mathcal{H})$. We denote by $n_e(T)$ and $m_e(T)$ the classes of operators nearly equivalent to T and metrically equivalent to T, respectively. That is,
$n_e(T) = \{S \in B(\mathcal{H}) : S^*S = U^*T^*TU\}$ and $m_e(T) = \{S \in B(\mathcal{H}) : S^*S = T^*T\}$. Clearly $m_e(T) \subsetneq n_e(T)$.

Theorem 3.2.6 ([22], Theorem 2.30)**.** *Let S and T be metrically equivalent operators in $B(\mathcal{H})$. Then S and T are nearly equivalent if and only if $\|S\| = \|T\|$.*

Theorem 3.2.7. *[22]if $S \in n_e(T)$, then for some unitary operator $U, Sx = TUx$ for all $x \in H$.*

Corollary 3.2.8. *If $S \in n_e(T)$, then $\|S\| = \|T\|$.*

Theorem 3.2.9. *If P is a projection then P^*P is a projection.*

Note that the converse of this result is not true in general. The operator with matrix representation

$P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is not a projection although $P^*P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a projection.

Theorem 3.2.10. *Two projections on a Hilbert space $\mathcal{H}$ are metrically equivalent if and only if they are equal.*

Proof. Let P and Q be projections on a Hilbert space H. If $P^*P = Q^*Q$, then $P = Q$. The converse is trivial. $\square$

Remark 3.2.11. *This result says that two projections on the same Hilbert space $\mathcal{H}$ can only be metrically equivalent if they are equal. The above result is not true in general linear operators. There exists bounded operators which are metrically equivalent but are not equal. Note that two unequal projection operations may be similar. The projections $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are unequal but similar projections. Now consider the projections $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. A simple calculation shows that P and Q are unitarily equivalent (with the unitary operator $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ implementing the equivalence), but are not metrically equivalent.*

Theorem 3.2.12. *Two projections P and Q on a Hilbert space $\mathcal{H}$ are metrically equivalent if and only if $PQ = QP = P = Q$.*

Proof. By Theorem 3.2.10, P = Q. The claim also follows by the fact that every operator commutes with itself. $\square$

Theorem 3.2.13. *If U and V are unitary operators on a Hilbert space H, then they are metrically equivalent.*

Proof. $U^*U = I = V^*V$. $\square$

Theorem 3.2.14. *If two operators T and S are metrically equivalent and one is a partial isometry then the other is an isometry.*

Proof. Without loss of generality suppose T is a partial isometry and metrically equivalent to S. Then $TT^*T = T$ and $T^*T = S^*S$ imply that $T(S^*S - I) = 0$. Thus $S^*S = I$. Note that, T is injective if an only if T^*T is injective. This says that if two operators are metrically equivalent and one is injective, then so is the other. $\square$

Proposition 3.2.15. *If A and B are $Q-$partial isometric equivalent, then they are metrically equivalent.*

Proof. By definition there exists an isometry Q such that $A = QB$. Thus $A^*A = B^*Q^*QB = B^*B$, which establishes the claim. Note that every operator T is $U-$Partial equivalent to $|T|$. This follows easily from the polar decomposition of $T = V|T|$ where V is a partial isometry. $\square$

Question: Does $Q-$equivalence preserve self-adjointness, invertibility, norm, numerical range, etc of operators? How is it is related to other operator equivalence relations?
It is clear that this equivalence preserves invertibility but it does not preserve norm, spectrum, self-adjointness of operators and numerical range of operators.
Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. A simple calculation shows that $A \overset{Q}{\sim} B$.

but $W(A) = [-1, 1] \neq \{1\} = W(B)$. This example also reveals that $Q-$ equivalent operators need not have equal spectra, even if Q is unitary.

Theorem 3.2.16. *If S and T are metrically equivalent operators on a Hilbert space H then $|S| = |T|$.*

Proof. The proof follows from the fact that $S^*S = T^*T$ is the same as $|S|^2 = |T|^2$ and the fact that $|S|^2$ and $|T|^2$ are positive self-adjoined operators and hence have unique square roots. $\square$

Remark 3.2.17. *We remark that metric equivalence need not preserve positivity of operators. To see this, let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly A and B are metrically equivalent, B is positive but A is not. It is also true that metric equivalence does not preserve self-adjointness.*

Note that similarity does not preserve self-adjointness of operators. The operators with matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are similar but A is not self-adjoint although

B is. We note that similar and even unitarily equivalent projections need not be metrically equivalent. The projections $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are similar, with similarity operator $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $P \neq Q$ these operators are not metrically equivalent. We note also that for projection operators on a Hilbert space H, near equivalence coincides with unitary equivalence. This follows from the fact that if P and Q are projection operators on a Hilbert space $\mathcal{H}$ then $P = P^*P = U^*Q^*QU = U^*QU$. This result is not true for general linear operators. To see this, consider the operators $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

These operators are metrically equivalent, nearly equivalent but are not similar and hence are not unitarily equivalent. This example shows also that two unitary operators need not be unitarily equivalent, but are metrically equivalent. Let A be similar to B, then $A = X^{-1}BX$ or $B = XAX^{-1}$, so that $\|A\| \leq \|B\|\|X^{-1}\|\|X\|$ and $\|B\| \leq \|A\|\|X^{-1}\|\|X\|$. Thus $\frac{1}{\|X\|^{-1}\|X\|} \leq \frac{\|A\|}{\|B\|} \leq \|X^{-1}\|\|X\|$.

Lemma 3.2.18. *Metrically equivalent positive operators have equal point spectra.*

Proof. Suppose A and B are metrically equivalent positive operators acting on a Hilbert space H. Suppose that $Ax = \lambda x$, and $By = \beta y, x, y \in H$. Since A and B are positive then $\lambda, \beta \geq 0$ and real.

Therefore $A^*A = B^*B \Leftrightarrow < Ax, Ax >=< Bx, Bx >$

$$\Leftrightarrow \lambda\overline{\lambda} < x, x >= \beta\overline{\beta} < x, x >$$

$$\Leftrightarrow \lambda^2 < x, x >= \beta^2 < x, x >$$

$$\Leftrightarrow \lambda^2 = \beta^2$$

$$\Leftrightarrow \lambda = \beta$$

$$\sigma_p(A) = \sigma_p(B).$$

□

We note that the positivity condition cannot be dropped. Self-adjointness alone is not enough in the conclusion of the above result. In general, metrically equivalent operators may have disjoint spectra.

Corollary 3.2.19. *If A and B are positive metrically equivalent, then $A = B$.*

Chapter 4

Other Equivalence Relations of Operators

4.1 Near Equivalence of Operators

Recall that two bounded linear operators T and S, on a Hilbert space are said to be nearly equivalent if T^*T and S^*S are similar.

Remark 4.1.1. .

1. *For any T in $\xi(T)$, then the positive operator $|S| = (S^*S)^{\frac{1}{2}}$ also belongs to $\xi(T)$.*

2. *For any T and any isometries P and Q (i.e. $P^*P = Q^*Q = I$), $S \in \xi(T)$ if and only if $PS \in \xi(QT) = \xi(T)$.*

3. *S is nearly equivalent to T if and only if there exists a unitary operator U such that $S^*S = U^*T^*TU$. For a similar normal operators are actually unitarily equivalent.*

4. *If S is unitarily equivalent to T i.e. $S = U^*TU$ for a unitary operator to U, then S is nearly equivalent to T, but if S is nearly equivalent to T, then S need not even be similar to T. (Recall that S is similar to T if $S = V^{-1}TV$ for an invertible operator V). The first part of the remark is easy to prove. For the second part, consider*

$$T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}.$$

Then $S^*S = U^*T^*TU$ *if* $U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Hence S is nearly equivalent to T but is not similar (and hence not unitarily equivalent) to T.

As another example (now in the infinite dimensional space ι^2) consider the unilateral shift operator T. Then $T^*T = I$ and hence T is nearly equivalent to I but surely T is not unitarily equivalent to I.

5. *If T is compact, all operators S in $\xi(T)$ are compact.*

6. *$S \in \xi(T)$ if and only if for some unitary operator $U, \|Sx\| = \|TUx\|$ for all $x \in \mathcal{H}$. Consequently, if $S \in \xi(T)$, then $\|S\| = \|T\|$.*

The following results are well known.

Proposition 4.1.2. *Let $A, B \in B(\mathcal{H})$. Then:*

(i) If $A \overset{n.e}{\approx} 0$ then $A = 0$.

(ii)If $A \overset{n.e}{\approx} B$ and B is isometric, then A is isometric.

Theorem 4.1.3. *If A and B are nearly equivalent projections, then they are unitarily equivalent.*

Proposition 4.1.4. *If A and B are nearly equivalent projections, then $A^2 \cong B^2$.*

Proposition 4.1.5. *If $A, B \in B(\mathcal{H})$ such that $A \overset{n.e}{\sim} B$ and A is partially isometric then so is B.*

Theorem 4.1.6. *For a densely defined closed operator from $\mathcal{H}$ into H, the following are equivalent.*

(i) $T = US$ unitary and S positive self-adjoint.
(ii) $T = UN, U$ unitary and N normal.
(iii) $TT^* = UT^*TU^*, U$ unitary. (iv) $dim(N(T)) = dim(N(T^*))$.

Definition 4.1.7. *T is said to be nearly normal if and only if $T^* \in \xi(T), S$ is said to be nearly hyponormal if there exists a unitary operator U such that $\|Sx\| \geq \|S^*Ux\|$ for every $x \in H$.*

Remark 4.1.8. *.*
(i) T is nearly normal if and only if there exists a normal operator N such that $T = UN$ for a unitary operator U.
(ii) An operator S is hyponormal if and only if $S^*S \geq SS^*$. Then it is easily seen that an operator T is nearly hyponormal if and only if there exists a hyponormal operator S such that $T = US$ for a unitary operator U.
(iii) Clearly every normal operator is nearly normal and every hyponormal operator is nearly hyponormal; also every nearly normal operator is nearly hyponormal.

Example 4.1.9. *Nearly normal operator that is not normal.*

Let $\mathcal{H}$ be of dimension 2 and let $T : H \to H$ be defined by the corresponding matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.
Then $TT^* = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $T^*T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. *$T$ is not normal, but $TT^* = U^*T^*TU$ if we take*

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Proposition 4.1.10. *[24] For an operator $T \in B(\mathcal{H})$ its polar decomposition is a n.n.d if and only if T^* is injective.*

Example 4.1.11. *Nearly normal operator is not hyponormal.*

Remark 4.1.12. *In Example 4.1.11 we have an operator that is nearly quasinormal but not quasinormal.*
Choose any non hyponormal operator A. Let $\lambda \notin \sigma(A)$. Take $T = A - \lambda I$. Then $T^*T = A^*A - AA^*$. Hence T is not hyponormal but T being invertible, T is nearly normal.

Definition 4.1.13. *An operator $T \in B(\mathcal{H})$ is said to be nearly quasinormal if and only if T^*T commutes with UT for a unitary operator U.*

Proposition 4.1.14. *[24] For an operator $T \in B(\mathcal{H})$ the following are equivalent:*
(i) T is nearly quasinormal.
(ii)UT is a quasinormal for a unitary operator U.
(iii)T is of the form $T = VQ$ where V is unitary and Q is quasinormal.

Proposition 4.1.15. *[24] The following relations hold: nearly normal $\subset$ nearly quasinormal $\subset$ nearly hyponormal.*

Definition 4.1.16. *An operator $T \in B(\mathcal{H})$ is said to be nearly subnormal if $T = US$ where U is unitary and S is subnormal.*

Corollary 4.1.17. *If T is a partial isometry then all operators in $\xi(T)$ are partial isometries.*

4.2 Murray-von Neumann Relation of Projections

A partial isometry on a Hilbert space $\mathcal{H}$ is an operator V such that

$$\|Vx\| = \begin{cases} \|x\| & x \in (\ker(V))^{\perp} \\ 0 & x \in Ran(V) \end{cases}$$

$(\ker(V))^{\perp}$ is called the initial space of V and $Ran(V)$ is called the final space V.

Definition 4.2.1. *We recall that, two projections P and Q in $B(\mathcal{H})$ are said to be Murray-von Neumann equivalent, denoted by $P \overset{M-v-N}{\sim} Q$ if there exists an operator $V \in B(\mathcal{H})$ such that $V^*V = P$ and $VV^* = Q$.*

We note that such a V is automatically a partial isometry. That is $VV^*V = V$. Thus two projections are Murray-von Neumann equivalent exactly when there is a partial isometry with one projection as the initial space and the other as the final space.
The following results were proved in [19].

Theorem 4.2.2. *Let V be a partial isometry and let P and Q be Murray-von Neumann equivalent projections with respect to V. Then $VP = QV$.*

Proof. Suppose $P = V^*V$ and $Q = VV^*$ for a partial isometry V. Then from the definition and the above remark, we have $V = VV^*V = VP = QV = QVP$. Theorem 4.2.2 also says that if P and Q are Murray-von Neumann equivalent projections with respect to V, then $V^* = PV^* = V^*Q$. A consequence of this result is that $P = V^*QV$. Using Theorem 4.2.2 we have that two projections P and Q are Murray-von Neumann equivalent if there exists a partial isometry V such that $P = V^*QV$.

□

Proposition 4.2.3. *The Murray-von Neumann relation is an equivalence relation on the family $P(B(\mathcal{H}))$ of projections in $B(\mathcal{H})$.*

Proof. Suppose that P and Q are projections such that $P = V^*V$ and $Q = VV^*$, for some partial isometry V. Reflexivity follows easily from Definition 4.2.1, because a projection is also a partial isometry.
Symmetry follows from Definition 4.2.1 since V^* is a partial isometry (with the initial and final spaces interchanged) whenever V is. Now suppose $P, QandR$ are projections and that $P \overset{M-v-N}{\sim} Q$ and $Q \overset{M-v-N}{\sim} R$. Then there exists partial isometries V and W such that $P = V^*V, Q = VV^* = W^*W$ and $R = WW^*$. Now let $Z = WV$. Then using the proof of Theorem 4.2.2, Z is also a partial isometry and that $Z^*Z = V^*W^*WV = V^*QV = V^*V = P$, and $ZZ^* = WVV^*W^* = WQW^* = WW^* = R$ which proves transitivity.

□

Theorem 4.2.4. *Let P and Q be projections such that $P \overset{M-v-N}{\sim} Q$ with an implementing partial isometry V. If V is invertible then P and Q are similar projections.*

Proof. Suppose $P = V^*V$ and $Q = VV^*$. Invertibility of V implies that $V^* = PV^{-1} = V^{-1}Q$. Thus $Q = VV^* = VPV^{-1}$.
The result above also says that P and Q are invertible and hence $P = Q = I$, since the only invertible projection is the identity operator. In addition, we conclude that V is unitary. □

Corollary 4.2.5. *Let P and Q be invertible projections. If $P \overset{M-v-N}{\sim} Q$ with an implementing partial isometry V then $P = Q$.*

Corollary 4.2.5 says that for invertible projections in a Hilbert space $\mathcal{H}$, the notions of unitary equivalence, similarity, quasisimilarity, metric equivalence and Murray-von Neumann equivalence coincide with equality.
The statement is also valid if we assume that V is a normal partial isometry. Note that a normal partial isometry need not be unitary. The operator $V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a normal partial isometry which is a direct sum of a unitary and zero but it is not unitary.

Corollary 4.2.6. *Let P and Q be projections such that $P = V^*V$ and $Q = VV^*$ for some partial isometry V. If V is normal then $P = Q$.*

Remark 4.2.7. *We note that for any orthogonal projections $P, Q \in B(\mathcal{H})$, if $P = V^*V$ and $Q = VV^*$ for some isometry V , the condition of V being either normal, unitary, or invertible and the condition of invertibility of both P and Q all coincide. If any of these conditions is satisfied, then $P = Q$.*

Proof. This follows from $0 = V^*V - VV^* = P - Q$. □

Corollary 4.2.8. *Let P and Q be invertible projections. If $P \overset{M-v-N}{\sim} Q$ with an implementing isometry V then $P \approx Q$ where $\approx$ is any of the equivalence relations: unitary equivalence, metric equivalence, almost similarity etc.*

Proposition 4.2.9. *If P and Q are unitarily equivalent projections, they are Murray-von Neumann equivalent.*

Proof. Suppose that $P = UQU^*$ for some unitary operator U. Since a unitary operator is a partial isometry, the result follows from the proof of Theorem 4.2.4. □

We remark that Murray-von Neumann equivalence does not in general imply unitary equivalence or metric equivalence of projection operators. Let S be a non-unitary isometry, for instance, the unilateral shift on ℓ^2. Then $P = S^*S$ and $SS^* = Q$ are projections. Clearly $P \overset{M-v-N}{\sim} Q$ but P and Q are not unitarily equivalent.
A simple calculation also shows that these projection operators are not metrically equivalent. In finite dimensions, it is clear that Murray-von Neumann equivalence implies similarity. However, this is not true in infinite dimensions. To see this, let $P = S^*S$

and $SS^* = Q$, where S is the unilateral shift operator on ι^2. This example also shows that Murray-von Neumann equivalence does not in general imply metric equivalence of projection operators.

If $A_1, A_2, \ldots, A_n$ are elements in $B(\mathcal{H})$, then $diag(A_1, A_2, ..., A_n)$ denotes the $n \times n$ matrix whose main diagonal consists of the elements $A_1, A_2, \ldots, A_n$. The following result gives a condition when Murray-von Neumann equivalence implies unitary equivalence of operators.

Proposition 4.2.10. *Let P and Q be projections. If $P \overset{M-v-N}{\sim} Q$ then $diag(P, 0)$ is unitarily equivalent to $diag(Q, 0)$.*

Proof. Suppose there is a partial isometry V such that $P = V^*V$ and $Q = VV^*$. Then by Theorem 4.2.4, we have that $V = VP = QV = QVP$. Using this fact, the operators

$U = \begin{pmatrix} V & 1-Q \\ 1-P & V^* \end{pmatrix}$ and $W = \begin{pmatrix} Q & 1-Q \\ 1-Q & Q \end{pmatrix}$ are unitary and hence WU is also unitary. Clearly $WU \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} U^*W^* = W \begin{pmatrix} UPU^* & 0 \\ 0 & 0 \end{pmatrix} W^* = W \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} W^* = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$. □

The following two results are a consequence of Proposition 4.2.9.

Corollary 4.2.11. *Let P and Q be projections. If $P \overset{M-v-N}{\sim} Q$ then $\sigma(P)/\{0\} = \sigma(Q)/\{0\}$.*

This result can be improved as follows.

Corollary 4.2.12. *Let P and Q be Murray-von Neumann equivalent projections. If P and Q are invertible, then $\sigma(P) = \sigma(Q)$.*

Proof. There are several ways to prove this result. One of them is to use the fact that 0 is not contained in the spectra of P and Q. The other is to use Corollary 4.2.5 that is $P = Q$. □

Recall that the range of a projection P on a Hilbert space $\mathcal{H}$ is the set $Ran(P) = \{x \in \mathcal{H} : Px = x\}$ and the null space or kernel of P is $\ker(P) = \{x \in \mathcal{H} : Px = 0\}$. Note that these two sets are algebraic complements of each other: $Ran(P) + Ker(P) = \mathcal{H}$ and $Ran(P) \cap \ker(P) = \{0\}$. Two orthogonal projections P_1 and P_2 on a Hilbert space $\mathcal{H}$ are said to be orthogonal to each other (or mutually orthogonal) if $Ran(P_1) \perp Ran(P_2)$ (which is equivalent to saying that $P_1P_2 = P_2P_1 = 0$).

Theorem 4.2.13. *Two projections P and Q on a Hilbert space $\mathcal{H}$ are Murray-von Neumann equivalent if and only if $dim(Ran(P)) = dim(Ran(Q))$.*

This says that two projections are Murray-von Neumann equivalent if and only if they have the same rank.

Example 4.2.14. *Consider the projections* $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *acting on the Hilbert space $\mathcal{H} = \mathbb{R}^3$. A simple calculation shows that P and Q are Murray-von Neumann equivalent, with the equivalence being implemented by the partial isometry* $V = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

A simple calculation also shows that P and R are Murray-von Neumann equivalent, with the equivalence being implemented by the partial isometry $W = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. *Also Q and R are Murray-von Neumann equivalent, with the equivalence being implemented by the partial isometry* $Z = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

By Theorem 4.2.13, S is not Murray-von Neumann equivalent to either P, Q or R. A simple calculation also shows that P, Q and R are pairwise similar and also pairwise *almost similar. In particular* $X = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 3 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ *implements the similarity and also the almost similarity between* P *and* Q*. A closer look also reveals that* P, Q *and* R *are pairwise unitarily equivalent. The operator* S *is Murray-von Neumann equivalent to, for instance, the operator* $T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$*, with the equivalence being implemented by the partial isometry* $Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Remark 4.2.15. *We note that the partial isometry implementing the Murray-von Neumann equivalence need not be unique. For example in the example above, the partial isometry* $X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *also implements the Murray-von Neumann equivalence between* P *and* R.

The next result gives the relation between the partial isometries implementing a Murray-von Neumann equivalence of P and Q.

Theorem 4.2.16. *Two projections* P *and* Q *are unitarily equivalent if and only if they are Murray von-Neumann equivalent and* $I - P$ *and* $I - Q$ *are Murray-von Neumann equivalent*

Proof. Suppose $Q = UPU^*$, for some unitary operator U. Put $V = UP$ and $W = U(I - P)$. Then $V^*V = PU^*UP = P^2 = P, VV^* = UP^2U^* = UPU^* = Q$ and similarly $W^*W = I - P, WW^* = I - Q$. Thus P and Q are Murray-von Neumann equivalent and $I - P$ and $I - Q$ are Murray-von Neumann equivalent. Conversely suppose P and Q are Murray-von Neumann equivalent and $I - P$ and $I - Q$ are Murray-von Neumann equivalent. Then there exists partial isometries $V : Ran(P) \to Ran(Q)$ and $W : Ran(I - P) \to Ran(I - Q)$ satisfying the above conditions. Now, let $Z = V + W$. Direct calculation shows that Z is unitary with $Z^* = U^*$ and therefore $ZPZ^* = UPU^* = Q$, which proves the claim.

□

4.3 Stable Similarity and Stable Unitary Equivalence

We recall that, two operators A and B are said to be stably similar or power similar denoted by $A \overset{s.s}{\sim} B$ if there is an invertible operators X such that $A^n = X^{-1}B^nX$, for some positive integer n (that is, B^n is similar to A^n). They are stably unitarily equivalent, denoted by $A \overset{s.u.e}{\sim} B$ if there is a unitary operator U such that $A^n = U^*B^nU$.
We note that operators are stably similar / unitary equivalent if they exhibit the same long term behavior.

Theorem 4.3.1. *Stable similarity is an equivalence relation.*

Proof. Reflexivity and symmetry follow easily from the definition: If $A \overset{s.s}{\sim} B$ and $B \overset{s.s}{\sim} C$ then there exists positive integers m, n and invertible operators X and Y such that $A^n = X^{-1}B^nX$ and $B^m = Y^{-1}C^mY$.
Let $s = lcm(n, m)$. Then $s = nr = mt$, for some integer r and t. Hence that is $A^s = A^{nr} = X^{-1}B^{nr}X = X^{-1}B^{mt}X = X^{-1}(Y^{-1}C^{mt}Y)X = (YX)^{-1}C^s(YX)$. This proves that $A \overset{s.s}{\sim} C$. Therefore this relation is transitive.

□

We note also that unitarily equivalent operators are stably unitarily equivalent. The converse of Theorem 4.3.1 is not generally true. The operators $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ are stably similar but not similar. This follows from the fact that $A^2 = B^2 = 0$.

Theorem 4.3.2. *If T is normal then T^*T and TT^* are stably similar.*

Proof. Since T is normal we have $(T^*T)^n = T^{*n}T^n = T^nT^{*n} = (TT^*)^n$. □

Theorem 4.3.3. *If A and B are stably similar then $Ran(A^n)$ is isomorphic to $Ran(B^n)$.*

Theorem 4.3.4. *Let P and Q be projections. The following statements are equivalent:*

(a) P and Q are similar.
(b) P and Q are stably similar.
(c) P and Q are almost similar.

We observe that stable similarity need not preserve spectra, norms and spectra radii of operators. Let S be the unilateral shift on ι^2 and $T = 0$. Then $S^n \to 0$ as $n \to \infty$. Thus S and T are stably similar. But $\|S\| = 1 \neq 0 = \|T\|$.

Theorem 4.3.5. *If S and T are similar then S^n and T^n are similar (in fact using the same inter-twiner operator).*

Question: Is the converse true?

Not true. Example $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let n = 2, $S^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $S^2 = T^2$ hence S^2 is similar to T^2 but S is not similar to T.

Chapter 5

Conclusion

In this project we have seen that almost similarity is an equivalence relation. As a result showing that if $A, B, \in, B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and B is hermitian then A is hermitian, has been discussed. It has also been shown that if $A, B, \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and A is partially isometric then so is B.

In chapter three, it has been observed that if S and T are metrically equivalent operators on a Hilbert space H and $ST = TS$ and if T is normal, then S is quasinormal. It has also been shown that if T and S are metrically equivalent operators on H, then $\|S\| = \|T\|$ but the converse is not always true, there exists operators with the same norm which are not metrically equivalent.

In chapter four it has been observed that if A and B are nearly equivalent projections then they are unitarily equivalent also if A and B are nearly equivalent projections where A and B are self adjoint then $A^2 \cong B^2$.

In this thesis we have managed to show for the first time, that two orthogonal projections P and Q acting on a Hilbert space $\mathcal{H}$ are Murray-von Neumman equivalent if and only if there exists a partial isometry $V \in B(\mathcal{H})$ such that $P = V^*QV$.

Finally, it has been shown that if T is normal then T^*T and TT^* are stably similar. We note also that unitarily equivalent operators are stably unitarily equivalent. The converse of the above results is not generally true.

Future research

In our research we were able to establish that similar orthogonal projection operators are Murray-von Neumman equivalent. It is a conjecture that there may be orthogonal projections which are not similar but are Murray-von Neumman equvalent. This evidently happens if $0 < \|Q - XPX^{-1}\| < \frac{1}{2}$. This is an open problem that we suggest for future research.

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