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OPTION VALUATION USING FAST FOURIER TRANSFORM IN THE CARBON EMISSIONS MARKET

ACTUARIAL SCIENCE

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Declaration

I do hereby declare that this work is based on a study I took, with reference to other people's work, which has been duly recognized/ acknowledged.

I certify that this is my original work which has not been presented in any other institution.

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In the prescence of my supervisors:

Prof.R.O. Simwa __________________________ __________________________

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DEDICATION

I humbly dedicate this work to my family for their support that they accorded in the reign of my study.

ACKNOWLEDGEMENT

I take this opportunity to first give thanks to Almighty God for His unfailing love and care during the entire course and specifically enabling me to complete this project. The production of this project has been made possible by invaluable support of many people. While it is not possible to name all of them, recognition has been given to few. I am greatly indebted to my Supervisor Professor Simwa and Dr. Ivivi for their professional guidance, advice and unlimited patience in reading through my drafts and suggesting workable alternatives. I also acknowledge the support and cooperation given to me by the entire Nairobi University fraternity. Last but not least I wish to thank my family for the support and encouragement they gave me while working on this project.

Abstract

Kyoto Protocol is an international agreement which commits Annex 1 parties by setting international binding carbon emission reduction targets. Carbon emission trading involves the buying and selling of carbon allowances in the event of non-compliance with the emission reduction targets. There is a lot of price volatility in the carbon emissions market. Most traders use option derivatives to deal with the risks. The non-compliance event defines the price process of the carbon allowances. The non-compliance event is modelled using the normal inverse Gaussian distribution and Brownian motion. The carbon price data is fitted in NIG distribution and Brownian motion using MLE. This helps us know which distribution best fits our data set. The results suggest that normal inverse Gaussian model has a better than Brownian motion. A simple analytic expression for the Fourier transforms using NIG and Brownian motion characteristic functions is defined and used to solve the European time option prices. The results suggest that NIG gives a higher option price than Brownian motion.

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Abbreviations

AAU- Assigned amount units

AM92 Actuarial tables- Formulae and Tables of the Faculty of Actuaries and Institute of Actuaries

CO2 – Carbon dioxide

CH4 – Methane

GBM- Geometric Brownian motion

GHG- Greenhouse gas

FFT- Fast Fourier transform

NIG- Normal inverse Gaussian

NOx- Nitrogen oxides

MLE- Moment of Likelihood Estimation

SNIG- Symmetric Normal inverse Gaussian

Sbm- standard Brownian motion

UNFCCC- United Nations Framework Convention on Climate change

Chapter 1. Introduction

1.1 Background

Greenhouse gas is produced from burned fossil fuels. The fossil fuels are a source of energy and so they are popularly used in production and consumption in the world. The developed world are known to be the main source of GHG emissions because of industrialisation and existence of free markets. The emissions are still low in the developing world and emerging markets when compared to the developed world. However, it is reported that the emissions in the developing world and emerging markets, have grown by approximately 42% between 1990 and 2011. GHG emissions have negative impacts on the earth's natural resources. We see increased heat waves, land slides, flooding and shrinking water supplies. People are dying of cancer, water and vector-borne diseases that are being caused by air pollution. The changing weather patterns and extreme events are affecting the agriculture and tourism sectors In order to address these climate-related crisis, the Kyoto Protocol was introduced. The Kyoto Protocol is an international agreement which commits its parties by setting international binding emission reduction targets for a given period of time which is called the compliance phase, The first compliance phase started in the year 2008 and ended in the year 2012. Through this, the Kyoto Protocol helps curb Climate change as its participants have committed to cut GHG emissions. The participants under the Kyoto Protocol are called Annex 1 parties. It was launched in Kyoto, Japan, on 11 December 1997 and started operations on 16^{th} February, 2005.

Binding emissions reduction commitment for participants meant that the space to pollute was limited, and what is scarce and essential commands a price. Greenhouse gas emissions- most prevalently carbon dioxide was now seen to have monetary value because it was considered as an unpriced externality. Moving forward, we will use carbon dioxide emissions instead of GHG emissions because the carbon dioxide is the principle gas in GHG.

Annex 1 parties can cut on GHG emissions and meet their targets, by changing some of the everyday activities that have a great impact on the environment. However, they can also meet their targets through three-based market mechanisms. This implies that they have to invest in a cost effective project that is being launched anywhere in the world, whether in the developed or developing world, that helps remove or reduce GHG emissions in the atmosphere. These mechanisms are International Emissions Trading, Clean Development Mechanism and Joint Implementation.

Each Annex 1 member is given a limit of the amount of carbon dioxide they should emit in the atmosphere, over a 5-year period. The limit is known as cap. The cap is converted to Assigned Amount Units,where one tonne of carbon dioxide is equivalent to one AAU. When a member exceeds the cap, they face a penalty.

The African continent is growing rapidly because of changes in governance, increased incentives in industrialisation and free markets. With this in mind, The Africa continent could become a developed world over the next two decades. The problem with GHG emissions in Africa might not be viewed as a problem now but that might change soon because of where our economy is headed. According to UNFCC, the Kenya's carbon emissions trajectory is quickly changing because the economy is making alot of progress. This progress is resulting to increased demands for energy, thus more burning of fossil fuels. There was an article on Business Daily Africa that said that Carbon trading is expected to start soon in Kenya. Therefore, Kenyan companies with carbon credits would be able to sell to foreign countries or manufacturers. There are plans ongoing to list carbon permits in the Nairobi Securities Exchange. Some companies like Kengen and East Africa Portland Cement already participate in carbon trading.

The options derivatives represent a small but growing percentage of carbon market activity. It emerged in the second half of 2008. The increasing market for options in the carbon market is believed to be driven by the existence of arbitrage in the market. It is therefore necessary to study the price behavior of historic price data and model the price of carbon as well as value carbon options.

1.2 Statement of the problem

Carbon spot prices are volatile and carbon market is facing the risk all the time. They need derivatives for dealing with such risks. Options are mostly preferred due to their simplicity. However the prices are so much volatile that it may be difficult for the option sellers to deal with the risk they have to take. They may decide to ask for high prices in return, which ruin the attractiveness of option derivatives. It is therefore necessary to do an option valuation the in the carbon market.

1.3 General objective

Identify whether NIG distribution is a suitable model for option valuation in the carbon emissions market.

1.3.1 Specific objectives

1. To model the carbon log returns using the NIG distribution and Brownian motion.
2. To develop a simple analytic expression for the Fourier transforms using NIG and Brownian motion characteristic functions.
3. To evaluate European time option price.

1.4 Significance of the study

- Brownian motion has always been considered the best method when it comes to financial modelling. However, this method rules out alot of factors that need to be put into consideration. Unlike Brownian motion, NIG distributions explain the behavior of derivative prices through volatility, kurtosis and skewness.
- It will be an important study to scholars who need information on pricing and valuation of carbon emissions when doing research related to this study.

Chapter 2. Literature review

Carmona and Hinz.(2011) , explains that European call and put options are mostly traded on the carbon emissions markets. He goes ahead to express that it is not evident how option valuation is done in the carbon emissions market. Regardless of the methods of valuations, the option price should be based on an underlying martingale with a binary terminal value. According to the option quotes (in Euros) published on January 2008, the call options had strike prices ranging between 20 Euros and 50 Euros.

Burger uses options to calibrate his processes. He uses data from European Climate Exchange (ECX), because they are available online. He explains that only options expiring in 2008 are available for different strike prices

Chevallier and Benoit. (2014) , argues that there is evidence to show that carbon prices move essentially by jumps and so this should be put into consideration when modelling carbon spot prices. He has even suggest models that could be useful when pricing carbon derivatives whose underlying accomodate a pure jump process. More information about these models can be looked up at Cont and Tankov (2004), the CGMY model (Carr et al. (2002)).

Seifert et al. (2008) argued that traders of carbon emissions need a carbon price model so that it is possible to value any carbon derivatives they decide to use to hedge the risks in the carbon emissions market. The models ease the decision they have to make regarding investments in the carbon markets.One of the most important property of the carbon price model is that it should be a martingale

Hamza et al.(2005) , considers option pricing when the distribution of the underlying price is NIG and variance-Gamma. The paper explains how the underlying with these type of distributions is developed so as to satisfy the martingale property. Afterwards, the option pricing formula is developed using cumulative distribution functions.

Bu.(2007) , models an asset price process using Normal inverse Gaussian, Meixner and Brownian motion distributions. After making necessary comparisons he concluded that all the non-Gaussian Levy processes are more reliable for modelling the asset price process when compared to the Brownian motion Levy process.

Saebo.(2009) argues that Normal inverse Gaussian process can be used in modelling the stocks. He explores the properties of the NIG market model in comparison to empirical findings in the financial markets.

Carr et al.(1998) , describes an expression that prices options efficiently, using the Fast Fourier transform. He assumes that the expression for the fast Fourier transform is well developed if and only if the characteristic function of the risk-neutral density is known analytically.

Bolviken and Benth. (2000) , argue that the family of normal inverse Gaussian distributions is able to portray stochastic phenomena that have heavy tails or are strongly skewed. In addition to that, normal inverse Gaussian distributions are not confined to the positive half axis. Therefore, with the NIG distribution the financial analyst has at its disposal a model that can be adapted to many different shapes while the distribution of sums of independent random variables are still tractable to compute.

Chapter 3. Methodology

3.1 Levy Process

Definition 1.1 A cadlag real valued stochastic process $(K(t))_{t\geq 0}$ such that $K(0) = 0$ is called a Levy Process if it has stationary independent increments and is stochastically continuous.

3.1.1 Brownian motion

A random variable that has Brownian motion is normally distributed with mean μ and variance σ^2 if

$$P(K > k) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_k^\infty e^{-\frac{(u-\mu^2)}{2\sigma^2} du}, \tag{3.1}$$

for all $k \in \mathbb{R}$

Definition 1.2 A real-valued stochastic process $B(t) : t \geq 0$ is called a Brownian motion in $k \in \mathbb{R}$ if the following holds:

1. $B(0) = k$
2. The increments are independent thus, for all times $0 \leq t_1 \leq t_2 ... \leq t_n$ the increments $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), ...B(t_2) - B(t_1)$ are independent random variables.
3. For $t \geq 0$ and $x > 0$ increments $B(t + x) - B(t)$ are normally distributed with expectation zero and variance x.
4. Almost surely, the function $t \mapsto B(t)$ is continuous.

We say $B(t) : t \geq 0$ is a **standard Brownian motion** if $k = 0$. Hence, it follows from part (3) of the definition that B(t) has probability density function given by

$$f_K(k) = \frac{1}{\sqrt{2\pi x}} \int_k^\infty e^{-\frac{(k^2)}{2x} du}, k \in \mathbb{R} \tag{3.2}$$

3.1.2 Normal inverse Gaussian process

The Normal inverse Gaussian process (NIG) is a Levy process $(K(t))_{t\geq 0}$ that has normal inverse Gaussian distributed increments. Specifically,$K(t)$ has a distribution with parameters $\alpha > 0, \mid \beta \mid < \alpha, \delta > 0$ and $\mu \in \mathbb{R}$

The NIG($\alpha, \beta, \delta, \mu$)-distribution has a probability density function

$$f_{NIG(k;\alpha,\beta,\delta,\mu)} = \frac{\alpha\delta}{\pi}\frac{L_1(\alpha\sqrt{\alpha^2-(k-\mu)^2})}{\sqrt{\alpha^2+(k-\mu)^2}}e^{\delta\sqrt{\alpha^2-\beta^2-\beta(k-\mu)}} \tag{3.3}$$

where

$$L_n(z) = \frac{1}{2}\int_0^\infty u^{v-1}e^{-\frac{z}{2}u+\frac{1}{u}du} \tag{3.4}$$

modified Bessel function of the third kind.

The characteristic function is given by,

$$\phi_{NIG}(m) = e^{-\delta(\sqrt{-\alpha^2-(\beta+im)^2}-\sqrt{\alpha^2-\beta^2})}e^{im\mu} \tag{3.5}$$

For NIG distribution we know the population moments as:

$$E[K] = \mu + \frac{\alpha\delta}{\sqrt{\alpha^2-\beta^2}} \tag{3.6}$$

$$Var[K] = \frac{\alpha^2\delta}{(\sqrt{\alpha^2-\beta^2})^2} \tag{3.7}$$

$$skew[K] = \frac{3\beta}{\alpha(\delta\sqrt{\alpha^2-\beta^2})^{\frac{1}{2}}} \tag{3.8}$$

$$Kurt[K] = 3(1+\frac{\alpha^2+4\beta^2}{\delta\alpha^2\sqrt{\alpha^2-\beta^2}}) \tag{3.9}$$

Where $Skew[K]$ and $Kurt[K]$ are the Skewness and Kurtosis of K respectively.

3.1.3 Symmetric NIG

The Symmetric NIG Levy process has symmetric NIG marginals. When the skewness parameter $\beta = 0$, the NIG distribution is symmetric with the following density function:

$$f_{NIG}(k) = \frac{\alpha}{\pi}\frac{L_1(\alpha\delta\sqrt{1+(\frac{k-\mu}{\delta})^2})}{\sqrt{1+(\frac{k-\mu}{\delta})^2}}e^{\alpha\delta} \tag{3.10}$$

It follows from the equations of the mean, variance and kurtosis, that μ is mean, $\frac{\delta}{\alpha}$ is variance and $3 + \frac{3}{\alpha\delta}$ is kurtosis. Let S-NIG($\alpha, 0, \delta, \mu$) denote the symmetric NIG.
The characteristic function for S-NIG($\alpha, 0, \delta, \mu$)

$$\phi(m) = e^{\alpha\delta(1-\sqrt{1+(\frac{m}{\alpha})^2}} e^{im\mu} \tag{3.11}$$

The equation,

$$\phi(m) = e^{\zeta(1-\sqrt{1+(\frac{2v}{\zeta})}} \tag{3.12}$$

is know as the characteristic generator of S-NIG($\alpha, 0, \delta, \mu$), where, $\zeta = \alpha\beta$

3.2 Basic modeling of the compliance

At the beginning of a compliance phase, Annex 1 parties are allocated assigned amount units that are equivalent to the amount of emission cap,H,set for them by the emissions regulator. During the compliance phase, the Annex 1 parties are required to have emissions that do not exceed the emission cap. The parties that emit less than the emission cap have more AAUs than required to offset their emissions, while for the parties that emit more, they require more AAUs to offset their excess emissions. If they don't have the AAUs to offset their emissions they end up paying a penalty Θ per unit exceeding the emissions cap, at the end of the compliance phase.
Equilibrium analysis shows that AAU price at time T is a random variable taking only the values 0 and Θ. This implies that when the market position is long, the AAUs are deemed worthless because banking of allowances is not allowed and so they have a value 0. However, when the market position is short, the AAU price will tend to the penalty level Θ.
The AAU price evolutions, $(S_t)_{t\in[0,T]}$ are assumed to be given by adapted stochastic processes on a filtered probability space $(\Omega, F, (F_t)_{t\in[0,T]}, \mathbb{P})$ on which we fix an equivalent probability measure$\mathbb{Q} \sim \mathbb{P}$ called the spot martingale measure.
We model $(S_t)_{t\in[0,T]}$ with respect to the non-compliance event N. We define N as

$$\begin{aligned} N &= \Upsilon_T > H \\ &= \frac{\Upsilon_T}{H} > 1 \end{aligned}$$

Let $\gamma = \frac{\Upsilon_T}{H}$, therefore, $N = \gamma_T > 1$. γ_T in our case is assumed to be a geometric Brownian motion and an NIG distribution. The density of the AAU price process is thus,

$$S_T \begin{cases} \Theta 1_N, & \gamma > 1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, our price process,

$$S_T = \Theta 1_N \tag{3.13}$$

In order to achieve the martingale property,

$$\begin{aligned} S_t &= \Theta E^{\mathbb{Q}}(1_N | F_t), \qquad t\epsilon[0,T] \\ &= \Theta E^{\mathbb{Q}}(1_{\{\Upsilon_T \geq 1\}} | F_t) \end{aligned} \tag{3.14}$$

To simplify the notation, we consider the normalized futures price process

$$a_t := \frac{S_t}{\Theta} = E^{\mathbb{Q}}(1_{\{\gamma_T \geq 1\}} | F_t), \qquad t\epsilon[0,T] \tag{3.15}$$

The random variable γ_T is modelled by

$$\gamma_T = e^{z_t} \tag{3.16}$$

where, z_t has a geometric Brownian motion or a NIG distribution.

The random variable γ_T modeled by geometric Brownian motion (gbm) is given by

$$\gamma_T = \gamma_0 e^{\int_0^T \sigma_s dB_s - \frac{1}{2}\int_0^T \sigma_s ds} \tag{3.17}$$

Where $B_t =$ standard Brownian motion and $\sigma =$ is the Volatility parameter. Since $\mu_{gbm} = 0$, γ_T is a martingale with respect to the underlying Brownian motion. which is given by,

$$a_t = \Phi\left(\frac{\ln \gamma_0 - \int_0^T \sigma_s ds}{\sqrt{\int_0^T \sigma_s ds}}\right) = \Phi(h) \tag{3.18}$$

martingale, a_t, is a binary terminal value taking only the values 0 and 1, and satisfies $\mathbb{P}\{\lim_{t\to T} a_t \in \{0,1\} = 1$. We introduce the model for γ_T as a NIG levy process

$$\gamma_T = \gamma_0 e^{[\mu_{nig}(t) + Y_t]} \tag{3.19}$$

where, Y_t has a normal inverse Gaussian distribution and its first two moments (expectation and variance) are given by

$$E[Y_1] = \frac{\alpha\delta}{\sqrt{\alpha^2-\beta^2}}$$
$$Var[Y_1] = \frac{\alpha^2\delta}{(\sqrt{\alpha^2-\beta^2})^2}$$

Since we have an infinitely divisible characteristic function, we can define the NIG process $Y_t = \sum_{t=0}^{\infty} Y_1$, which starts at zero. The parameters, α, β of the NIG for Y_t remain unchanged.

The convolution property states that$NIG(\alpha, \beta, \delta_1) * NIG(\alpha, \beta, \delta_2) = NIG(\alpha, \beta, \delta_1 + \delta_2)$. Therefore $Y_t \sim NIG(\alpha, \beta, \delta t)$.

We illustrate the relationship between the parameters in the geometric Brownian motion and normal inverse Gaussian models

$$\mu_{gbm} = \mu_{nig} + \frac{\alpha\delta}{\sqrt{\alpha^2-\beta^2}}$$
$$\sigma^2 = \frac{\alpha^2\delta}{(\sqrt{\alpha^2-\beta^2})^2},$$

The Levy-Khintchine representation of Y_t is

$$Y_t = \mu t + \int_0^t w\overline{N_t}(dw), \quad \mu = E[Y_1], \tag{3.20}$$

where,

$$\overline{N}_t = N_t - \lambda t \tag{3.21}$$

and N_t is a poisson process while $\overline{N}_t$ is a compensated Poisson process. The Levy-Khintchine representation has no Brownian component.Therefore NIG is a pure jump process.

Inorder to achieve the martingale property in our NIG model, $\mu_{nig} = 0$. Hence,

$$E[Y_1] = \frac{\alpha\delta}{\sqrt{\alpha^2-\beta^2}} = 0$$

$E[Y_1] = 0$ when $\beta = 0$. This implies that our model $Y_t \sim NIG(\alpha, \delta t)$ is a symmetric NIG. Therefore,

$$\gamma_T = \gamma_0 e^{(Y_t)} \tag{3.22}$$

Hence,

$$\begin{aligned} a_t &= E^{\mathbb{Q}}[1_{\gamma_T \geq 1}|F_t] \\ &= \mathbb{Q}[\gamma_T \geq 1|F_t] \\ &= \mathbb{Q}[\gamma_0 e^{Y_t} \geq 1|F_t] \end{aligned} \tag{3.23}$$

3.3 Levy Processes with Symmetric Marginal Distributions

A Levy process is fully determined by its initial value, Y_0 , here assumed to be nil, and the distribution of the increment over one unit time interval, Y_1. The distribution of Y_t is infinitely divisible for any t. Hamza, Kais,et al. (2015). Therefore $Y_t = tY_1$ and its characteristic function satisfies

$$E(e^{imY_t}) = [E(e^{imY_1}]^t, \quad u = \mathbb{R} \tag{3.24}$$

The Levy-Khintchine representation is given by,

$$E(e^{imY_1} = e^{\wedge(m)} \tag{3.25}$$

with characteristic component

$$\wedge(m) = i\mu m - \frac{1}{2}c^2m^2 + \int_{\mathbb{R}}(e^{imy} - 1 - imy1_{\{|y|\geq 1\}})v(dy) \tag{3.26}$$

where $\mu = \mathbb{R}$, and v is a Levy measure satisfying $v(0) = 0$ and $\int_{\mathbb{R}}(1 \wedge y^2)v(dy) < \infty$.
The triplet (μ, b, v) is referred to as the characteristic triplet of Y.
We denote by S-NIG(μ, σ^2, ψ) the distribution of Y_1 whose characteristic function is of the form

$$\phi_{Y_1}(m) = e^{im\mu}\psi(\frac{\sigma^2}{2}m^2), \tag{3.27}$$

The function $\psi(m) : [0, \infty]$ is called the characteristic generator. It is unique up to scaling and if chosen such that $\psi'(u) = -1$, yields that μ and σ^2 are the mean and variance of Y_1 respectively.
Let $\mathbb{Q}$ be the natural equivalent martingale measure for Y_t. Then under $\mathbb{Q}$, Y_t remains a symmetric NIG Levy process with characteristic triplet $(\overline{\mu}, \sigma^2, \psi)$ and the distribution of Y_1 becomes S-NIG($\overline{\mu}, \sigma^2, \psi$) where

$$\overline{\mu} = r - \ln\psi(\frac{-\sigma^2}{2}) \tag{3.28}$$

r is the risk free rate.
Now it is easy to see that $\mathbb{Q}$ is also a natural equivalent martingale measure. Indeed, since $-Y_t$ Levy process with P-characteristic triplets $(-\mu, b, v)$ and since the distribution of -Y_1 is $S(-\mu, \sigma^2, \psi)$, $\mathbb{Q}_1$ is chosen so that

$$\overline{\mu}_1 = r + \ln\psi(\frac{-\sigma^2}{2}) \tag{3.29}$$

$\mathbb{Q}_1$ is unique because $\overline{\mu}_1$ is unique

Under $\mathbb{Q}$, Y_t is a symmetric NIG Levy process with marginals from the family $S(\overline{\mu}t, \sigma^2 t, \psi_t)$

Remark: $\psi_t(v) = (\psi(\frac{v}{t}))^t$

Proposition: Denote by F_T the P-distribution function of the standardized variable $\frac{(Y_T - \mu T)}{\sigma\sqrt{T}}$

$$F_T(y) = \mathbb{P}(Y_T \geq \sigma\sqrt{T} + \mu T)$$

Then

$$F_T(y) = \mathbb{Q}(Y_T \leq \sigma\sqrt{T} + \mu T) = \mathbb{Q}_1(Y_T \leq \sigma\sqrt{T} + \overline{\mu}_1 T)$$

We will use the standard normal to approximate the standardized symmetric NIG distribution.

$$\begin{aligned} a_t &= E^{\mathbb{Q}}[1_{\Upsilon_T \geq 1} | F_t] \\ &= \mathbb{Q}[\Upsilon_T \geq 1 | F_t] \\ &= \mathbb{Q}[\Upsilon_0 e^{Y_t} \geq 1 | F_t] \\ &= \Phi(\frac{\ln \Upsilon_0 + (r + \ln \psi(\frac{-\sigma^2}{2}))t}{\sigma\sqrt{t}}) \end{aligned} \tag{3.30}$$

By equation (3.12) and $\zeta = \alpha\beta = \alpha^2\sigma^2$, we obtain

$$\ln \psi(\frac{-\sigma^2}{2}) = \zeta(1 - \sqrt{1 - \frac{\sigma^2}{\zeta}}) = \alpha^2\sigma^2 - \alpha\sigma^2\sqrt{(\alpha^2 - 1)}.$$

Equation (3.30) becomes

$$\Phi(\frac{\ln \Upsilon_0 + (r + \alpha^2\sigma^2 - \alpha\sigma^2\sqrt{(\alpha^2 - 1)})t}{\sigma\sqrt{t}}) = \Phi(g) \tag{3.31}$$

Remark: If the excess emissions have a normal distribution (Normal inverse Gaussian) then the price process automatically becomes a normal distribution (Normal inverse Gaussian).

3.4 Estimation of model parameters

We will need to get a calibration which is better fitted to the carbon price data. By using the NIG model, it means that we have four parameters that need to be calibrated. While for the normal distribution, only two parameters need to be calibrated. I will use the historical approach to estimate the parameters. The maximum likelihood estimation method is widely used to estimate parameters. By maximizing the likelihood function we increase the probability of getting the parameters that will give us the best fit for our data. The principal of MLE states that the desired probability distribution is the one that makes the historical data 'most likely'. Hence, we seek the parameter values that maximizes the likelihood function

$$L(\theta_1, \theta_2, \ldots, \theta_m) = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2, \ldots, \theta_m) \tag{3.32}$$

We obtain the MLE estimate by maximizing the log likelihood function.The log likelihood function given a random sample of size n from a $NIG(\alpha, \beta, \delta, \mu)$ is given by

$$L = -n\ln(\pi) + n\ln(\alpha) + n(\partial\delta - \beta\mu) - \frac{1}{2}\sum_{i=1}^{n}\phi(x_i) + \beta\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}K_1(\partial\alpha\phi(x_i)^{\frac{1}{2}})$$

The log likelihood function of the normal distribution, $N(\mu, \sigma)$ given a sample of size n is given by

$$L = \sigma^{-n}(2\pi)^{-\frac{n}{2}}e^{[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2]}$$

The maximization of the log-likelihood function is done by a numerical optimization algorithm, see Myung (2003).

3.5 Goodness of fit

After estimating our parameters, we need to check how well our NIG model and Brownian motion model fit the historical carbon price data.

3.5.1 QQ-plots

The quantile-quantile plot is a graphical tool that explains if a set of data plausibly comes from some theoretical distribution.

3.5.2 Anderson-Darling test statistic

Anderson-Darling test statistic is defined by:

$$AD = \max\frac{|F_n(k) - F(k)|}{\sqrt{F(k)(1 - F(k))}}$$

Where, $F_n(k)$ is the empirical cumulative distribution function and $F(k)$ is the cumulative distribution function.
A smaller value of AD means that the empirical distribution and fitted distribution are closer.

3.6 Option Pricing

For the valuation otions, we will consider European call options written on futures price $(S_t)_{t\in[0,T]}$. A European call option gives the holder a right, but not an obligation, to buy at the time of maturity T to a fix strike price U. Thus the payoff function is given by

$$\Phi(S_T) = \max(S(T) - U, 0)$$

3.6.1 Risk-neutral Option pricing

We assume that the price D(t) of a risk-free asset satisfies the differential equation

$$dD(t) = rD(t)dt, \;\; r \geq 0$$

The first fundamental theorem of asset pricing states that there is no arbitrage, if and only if a risk-neutral probability measure exists. In this case, risk-neutral probability is a martingale measure $\mathbb{Q}$ which is equivalent to the original probability measure P. However,when using the NIG market model, it is not practical to assume market completeness. Therefore the options are said to be redundant in the NIG market model.

The arbitrage-free value of the option at time t<T can be defined as

$$C_t = e^{-r(T-t)}E^{\mathbb{Q}}[max(S(T) - U, 0)] \tag{3.33}$$

Key Assumptions:

- Options expire at the end of the compliance phase, that is, T=4.
- There is no banking of allowances.

3.6.2 Option pricing using fast Fourier transform

The European call option value is dependent on the asset price process, S_t , with maturity time T and strike price U . We let $u = \ln(u)$ and $s(T) = \ln(S(T))$. $C_T(u)$ will denote the option price and f_T the risk –neutral probability density function of price S_T.
The characteristic function of the density f_T is:

$$\phi_T(m) = \int_{-\infty}^{\infty} e^{ims} f_T(s)ds. \tag{3.34}$$

The link between the option value and the risk-neutral density f_T is given by

$$C_T(u) = \int_{-\infty}^{\infty} e^{-rT}(e^s - e^u)f_T(s)ds. \tag{3.35}$$

Here $C_T(u)$ is not square integrable because when $u \to -\infty$ so that $U \to 0$, we have $C_T \to S(0)$. However, if we consider the modified price $c_T(u)$ given by

$$c_t(u) = e^{\lambda u}C_t(u) \tag{3.36}$$

then we obtain a square integrable function, for a suitable $\lambda > 0$. The value λ affects the speed of convergence.
The fast Fourier of $c_t(u)$ is defined by

$$\varphi_T(v) = \int_{-\infty}^{\infty} e^{ivu}c_T(u)du \tag{3.37}$$

First we develop an analytical expression for $\varphi_T(v)$ in terms of characteristic function, ϕ_T , so that we can obtain call prices using the inverse transform.

$$\begin{aligned} C_T(u) &= \frac{e^{-\lambda u}}{2\Pi}\int_{-\infty}^{\infty} e^{-ivu}\varphi_T(v)d(v) \\ &= \frac{e^{-\lambda u}}{\Pi}\int_{0}^{\infty} e^{-ivu}\varphi_T(v)d(v) \end{aligned} \tag{3.38}$$

Therefore,

$$\begin{aligned} \varphi_T(v) &= \int_{-\infty}^{\infty} e^{ivu}\int_{u}^{\infty} e^{\lambda u}e^{-rT}(e^s - e^u)f_T(s)dsdu. \\ &= \int_{-\infty}^{\infty} e^{-rT}f_T(s)\int_{-\infty}^{s}(e^{s+\lambda u} - e^{(1+\lambda)u})e^{ivu}duds \\ &= \int_{-\infty}^{\infty} e^{-rT}f_T(s)(\frac{e^{(\lambda+1+iv)}}{\lambda+iv} - \frac{e^{(\lambda+1+iv)}}{\lambda+1+iv})ds \\ &= \frac{e^{-rT}\phi_T(v-(\lambda+1)i}{\lambda^2+\lambda-v^2+i(2\lambda+1)v} \end{aligned} \tag{3.39}$$

By substituting (3.39)into(3.38) and performing integration for equation (3.38), we obtain the call values. Since the FFT evaluates the integrand at $\upsilon = 0$, the use of $e^{\lambda u}$ is required and therefore we will use the condition $\varphi(0)$ being finite provided that $\phi_T(-(\lambda+1)i)$ is finite. That way, the modified call value is square integrable. From the definition of the characteristic function, this requires that

$$E[S_T^{\lambda+1}] < \infty \tag{3.40}$$

Carr and Madan (1998) suggest that, one may determine an upper bound on λ from the analytical expression for the characteristic function and the condition (3.40). One quarter of this upper bound serves as a good choice for λ, that is $\lambda \approx 0.75$.
At $\upsilon = 0$, equation (3.38) becomes,

$$\begin{aligned} C_T(u) &= \frac{e^{-\lambda u}}{\Pi}\varphi_T(0) \\ &= \frac{e^{-rT}\phi_T(-(\lambda+1)i}{\Pi(\lambda^2+\lambda)} \end{aligned} \tag{3.41}$$

The characteristic function of the log of S_T , which follows a NIG distribution, is given by

$$\phi_T(u) = e^{\ln\pi\Phi(g)+T((r+\alpha^2\sigma^2-\alpha\sigma^2\sqrt{(\alpha^2-iu)}))} \tag{3.42}$$

$$\phi_T(-(\lambda+1)i) = e^{\ln\pi\Phi(g)+T((r+\alpha^2\sigma^2-\alpha\sigma^2\sqrt{(\alpha^2-(\lambda+1))}))} \tag{3.43}$$

The equation to get the option price of an NIG model is therefore,

$$C_T(u) = \frac{e^{-\lambda u}}{\Pi}\frac{e^{\ln\pi\Phi(g)+T((r+\alpha^2\sigma^2-\alpha\sigma^2\sqrt{(\alpha^2-(\lambda+1))}))}}{\lambda^2+\lambda} \tag{3.44}$$

We now get the expression for the characteristic function of the standard Brownian motion, which is similar to standard normal distribution, and use it to get the option price.

The characteristic function of Brownian motion is given by

$$\phi_T(u) = e^{iu\mu} - \frac{1}{2}\sigma^2u^2 \tag{3.45}$$

The Brownian motion is a type of Levy process. Therefore, under $\mathbb{Q}$, S_t^N (a random variable that is defined by the Brownian motion) is a Levy process with characteristic triplet $(\overline{\mu}, b, v)$ and the distribution of S_t^N becomes $sbm(\overline{\mu}, \sigma, \psi)$, where

$$\overline{\mu} = r - \ln\psi(\frac{-\sigma^2}{2}) \tag{3.46}$$

The standard Brownian motion is $N \sim (0, \int_0^t \sigma^2 ds)$ Therefore

$$\phi_T(u) = e^{-\frac{1}{2}\int_0^t \sigma^2 u^2 ds} \tag{3.47}$$

$$\phi_T(-(\lambda+1)i) = e^{\ln \pi \Phi(h) + T(r - \frac{1}{2}\int_0^T {\sigma^2}_t (\lambda+1)^2 dt)} \tag{3.48}$$

$$C_T(u) = \frac{e^{-\lambda u}}{\Pi} \frac{e^{\ln \pi \Phi(h) + T(r - \frac{1}{2}\int_0^T {\sigma^2}_t (\lambda+1)^2 dt)}}{\lambda^2 + \lambda} \tag{3.49}$$

Remark: We will use AM92 Actuarial tables to get the values of $\Phi(h)$ and $\Phi(g)$ found in equations (3.18) and (3.31) respectively.

Chapter 4. RESULTS & CONCLUSIONS

This chapter deals with fitting data into the models proposed in chapter three as well as determining the option values using FFT.

4.1 Data Description

The historical data I used was collected from the Europe union emissions market. The data was collected during the first compliance period which was roughly between July 2008 and July 2012.
Here is a link to the data: http://www.investing.com/commodities/carbon-emissions-historical-data.

Table 4.1: Descriptive-statistics

N	534
mean	-0.1
sd.deviation	0.45
variance	0.2025
skew	-0.12
kurtosis	-1.36

Data that is normally distributed has values for skewness and kurtosis as 0 and 3, respectively. Our values for skewness and Kurtosis imply that our data is non-Gaussian. The value of the skewness indicates that our data set is left long-tailed. The Kurtosis indicates that our data has a "light tailed" distribution.

4.2 Goodness of fit

In order to determine the appropriate option price model, we need to establish which distribution best fits the data. QQ-plots of the NIG and Brownian motions of the log returns of the carbon price data as shown in figure 4.1 indicates that the data fat negative tails and fat positive tails. However, if you compare both QQ-plots, the Brownian motion QQ-plots has more outliers that the NIG QQ-plot. Hence NIG distribution gives a better fit for our data than the Brownian motion.

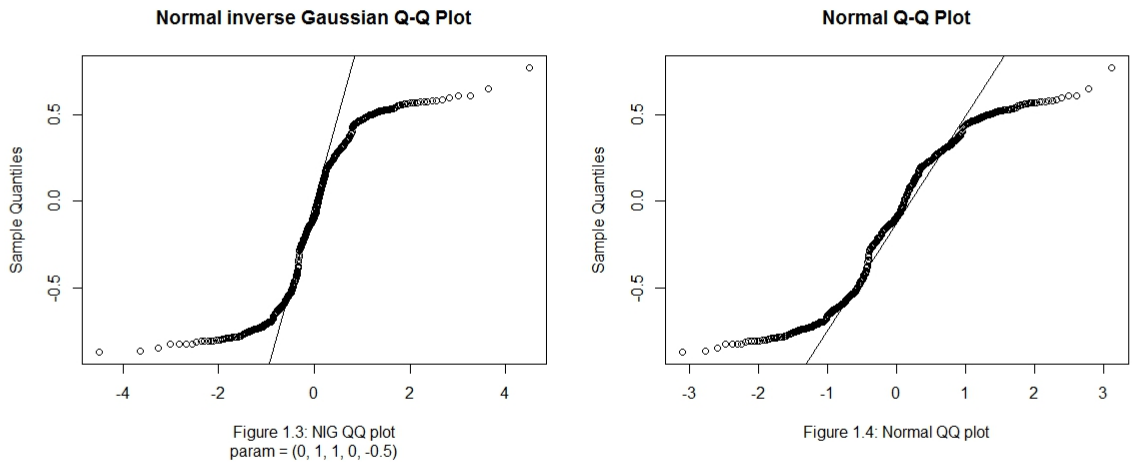

Figure 4.1: QQ-plots

4.2.1 Anderson-Darling (AD) test Statistic

Table 4.2: Anderson-Darling statistics

	AD
NIG	131.7
Normal	208.5

The AD-statistic value imply that the empirical cumulative function and the fitted cumulative function in the NIG distribution are closer than for the Brownian motion.

4.3 Parameter estimation

After fitting the log-returns of carbon emission prices to NIG processes and Brownian motion by maximum likelihood estimation (MLE), the results were as follows:

Table 4.3: Estimated parameters

	μ	σ	β	α
NIG	1.562073917	0.003563	-1.650997	12.358663
Normal	-0.0952422	0.44958767		

4.4 A comparison of the two model results

Suppose the strike price, K, is 20 Euros, 25 Euros or 30 Euros, T=4 (compliance phase),$\Gamma_0 = 5$, and r=0.05. The value of the call option using NIG model and Brownian motion are:

Table 4.4: Option values-NIG model

K	k	$C_T(k)$
20	2.995732	1.2535
25	3.2188	1.06
30	3.4012	0.92

Table 4.5: Option values-Brownian model

K	k	$C_T(k)$
20	2.995732	0.00806
25	3.2188	0.00682
30	3.4012	0.00595

The option prices, (price for unit carbon allowance), in the NIG model are higher than those of the Brownian model with respect to the given strike prices. As the strike price increase, the value of the option decreases in both models.

4.5 Conclusion

The non-compliance event has been modeled using NIG distributions and Brownian motion. Both models satisfy the condition that the allowance price is a random variable taking only the values 0 and π (the penalty coast for non-compliance). The MLE estimated parameters in the NIG model are α and σ , while for the Brownian model the estimated parameter is σ . The NIG distribution suggests a better goodness of fit than the Brownian motion since the AD-statistic for NIG levy process is smaller than for the Brownian motion. Inaddition to that, the Brownian motion QQ-plot has more outliers than the NIG QQ-plot even though the both QQ-plots have fat tails. Using the fast Fourier transform, NIG distribution gives higher option prices than the Brownian motion. However, there is need for adequate data on the options in order to make comparisons with our findings.

4.6 Limitations of the study

1. Carbon emission trading is not yet established in Kenya. Therefore the relevant data cannot be found for the application of the model.
2. The price data for the most recent compliance phase could not be implemented because the phase will end in the year 2020.
3. There is no option quotes data for the final year of the compliance phase. It is therefore not possible to make any comparisons with our findings.

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Appendix:

Appendix 1 :R codes used in Analysis

```
Carbon=read.csv(file.choose())
Carbon
price<-carbon$Price
price
plot(price, xlab="Figure 1.1: Carbon prices ")
plot(log_returns4, type="l", xlab="Figure 2.2: Log returns of carbon prices", ylab="Log returns")
log_returns4<-diff(log(price), lag=364)
log_returns4
par(mfrow=c(1,2))
qqnig(log_returns4, mu = 0, delta = 1, alpha = 1, beta = 0 , xlab="Figure 1.3: NIG QQ plot")
qqnorm(log_returns4, xlab="Figure 1.4: Normal QQ plot")
qqline(log_returns4)
y1<-dnig(log_returns4)
y1
ad.test(y1)
x1<-dnorm(log_returns4)
x1
ad.test(x1)
fit<-fitdist(price, "dnig", method="mle", start=NULL) fit
m<-fit.NIGuv(log_returns4, opt.pars=c(alpha.bar=T, mu=F, sigma=T))
m
gofstat(fit)
fitdistr(log_returns4, "Normal")
```