Efficiency conditions for multiobjective fractional variational problems

Ştefan Mititelu

Abstract. In this paper we define the notion of normal efficiency solution. Necessary conditions for normal efficient solutions of a class of multiobjective fractional variational problem (MFP) with nonlinear equality and inequality constraints are are established using a parametric approach to relate efficient solutions of fractional problems and a non-fractional problems. Also, the sufficiency of these conditions for efficiency solutions in problem (MFP) is established using the (ρ, b)-quasiinvexity notion. Particularly, for the multiobjective variational problem (MP), similar results are derived.

M.S.C. 2000: 65K10, 90C29, 26B25.
Key words: multiobjective fractional variational problem, efficient solution, quasi-invexity.

1 Introduction

The first result on the necessity of the optimal solutions of the scalar variational problems was established by Valentine [15] in 1937. The papers of Mond and Hanson [9], Bector [1], Mond, Chandra and Husain [11], Mond and Husain [10], Preda [13] developed the duality of the scalar variational problems involving convex and generalized convex functions. Mukherjee and Purnachandra [12], Preda and Gramatovici [14], Mititelu [8] established weak efficiency conditions and developed different types of dualities for multiobjective variational problems generated by various types of generalized convex functions. Kim and Kim [6] used the efficiency property of the nondifferentiable multiobjective variational problems in duality theory. In this paper we introduce the notion of normal efficient solution and establish necessary conditions of normal efficiency for the multiobjective fractional and non-fractional variational problems. For these problems we also establish sufficient efficiency conditions using (ρ, b)-quasiinvexity hypotheses.

Applied Sciences, Vol.10, 2008, pp. 162-175.

2 Notions and statement of the problem

Let $I = [a, b]$ be a real interval and $f = (f_1, \dots, f_p) : I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^p$, $k = (k_1, \dots, k_p) : I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^p$, $g = (g_1, \dots, g_m) : I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^m$, $h = (h_1, \dots, h_q) : I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^q$ be twice differentiable functions.

Consider the vector-valued function $f(t, x, \dot{x})$, where $t \in I$, $x : I \to \mathbf{R}^n$ with derivative $\dot{x}$ with respect to t. f_x and $f_{\dot{x}}$ denote the $p \times n$ matrices of first-order partial derivatives of the components with respect to x and $\dot{x}$, i.e.

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \dots, \frac{\partial f_i}{\partial x_n} \right) \quad \text{and} \quad f_{i\dot{x}} = \left(\frac{\partial f_i}{\partial \dot{x}_1}, \dots, \frac{\partial f_i}{\partial \dot{x}_n} \right), \quad i = 1, 2, \dots, p.$$

Similarly, k_x, g_x, h_x and $k_{\dot{x}}, g_{\dot{x}}, h_{\dot{x}}$ denotes the $p \times n$, $m \times n$, $q \times n$ matrices of the first order of k, g and h respectively, with respect to x and $\dot{x}$.

For any vectors $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ the relations $v = w$, $v < w$, $v \leqq w$, $v \leq w$ are defined as follows

$$v = w \Leftrightarrow v_i = w_i,\ i = \overline{1, n}; \qquad v \leqq w \Leftrightarrow v_i \leqq w_i,\ i = \overline{1, n};$$

$$v < w \Leftrightarrow v_i < w_i,\ i = \overline{1, n}; \qquad v \leq w \Leftrightarrow u \leqq w \text{ and } u \neq v.$$

Denote be X the space of piecewise smooth functions x with the norm $||x|| := ||x||_\infty + ||Dx||_\infty$, where the differential operator D is given by

$$u = Dx \Leftrightarrow x(t) = x(a) + \int_a^t u(s) ds.$$

Therefore, $D = \dfrac{d}{dt}$, except at discontinuities.

The mathematical model of study is the following multiobjective variational problem

$$\text{(MFP)} \begin{cases} \text{Minimize (Pareto)} & \left(\dfrac{\displaystyle\int_a^b f_1(t, x, \dot{x}) dt}{\displaystyle\int_a^b k_1(t, x, \dot{x}) dt}, \dots, \dfrac{\displaystyle\int_a^b f_p(t, x, \dot{x}) dt}{\displaystyle\int_a^b k_p(t, x, \dot{x}) dt} \right) \\ \text{subject to } x(a) = a_0,\ x(b) = b_0, \\ \qquad g(t, x, \dot{x}) \leqq 0,\ h(t, x, \dot{x}) = 0,\ \forall t \in I \end{cases}$$

Assume that $\displaystyle\int_a^b k_i(t, x, \dot{x}) dt > 0$ for all $i = 1, 2, \dots, p$. Let

$$\mathbf{D} = \{ x \in X | x(a) = a_0,\ x(b) = b_0,\ f(t, x, \dot{x}) \leqq 0,\ h(t, x, \dot{x}) = 0,\ \forall t \in I \}$$

be the set of all feasible solutions (domain) of (MFP).

3 Preliminaries: optimality and efficiency for variational problems, (ρ, b)-quasiinvexity

Let $s : I \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ be a scalar twice differentiable function and consider now the following scalar variational problem:

$$\text{(SP)} \qquad \begin{cases} \text{Minimize} \quad \displaystyle\int_a^b s(t, x, \dot{x})dt \\ \text{subject to } x(a) = a_0,\ x(b) = b_0, \\ \qquad\qquad g(t, x, \dot{x}) \leqq 0,\ h(t, x, \dot{x}) = 0,\ \forall t \in I \end{cases}$$

The first result in literature for the necessity of the optimality of x^0 to (SP) was established by Valentine in the following form:

Theorem 3.1 (Necessary Valentine's conditions [15]). *Let x^0 be an optimal solution to* (SP) *and let s, g and h be twice differentiable functions. Then there exists scalar λ and the piecewise smooth functions $\mu^0(t)$ and $\nu^0(t)$ satisfying the following conditions*:

$$\text{(VC)} \qquad \begin{cases} \lambda s_x(t, x^0, \dot{x}^0) + \mu^0(t)' g_x(t, x^0, \dot{x}^0) + \nu^0(t)' h_x(t, x^0, \dot{x}^0) = \\ = \dfrac{\mathrm{d}}{\mathrm{d}t}[\lambda s_{\dot{x}}(t, x^0, \dot{x}^0) + \mu^0(t)' g_{\dot{x}}(t, x^0, \dot{x}^0) + \nu^0(t)' h_{\dot{x}}(t, x^0, \dot{x}^0)] \\ \mu^0(t)' g(t, x^0, \dot{x}^0) = 0,\ \mu^0(t) \geqq 0,\ \forall t \in I,\ [\lambda \geqq 0]. \end{cases}$$

For the relation (VC) there are excluded the values of t corresponding to the corners of $x^0(t)$ where $\mu^0(t)$ and $\nu^0(t)$ cannot vanish.

Definition 3.1. The optimal solution x^0 of (SP) is called *normal* if $\lambda \neq 0$. According to this definition in what follows without loss of generality, we can take $\lambda = 1$.

Consider now the following multiple variational problem

$$\text{(MP)} \qquad \begin{cases} \min \displaystyle\int_a^b f(t, x, \dot{x})dt = \min \left(\int_a^b f_1(t, x, \dot{x})dt, \ldots, \int_a^b f_p(t, x, \dot{x})dt \right) \\ \text{subject to } x(a) = a_0,\ x(b) = b_0, \\ \qquad\qquad g(t, x, \dot{x}) \leqq 0,\ h(t, x, \dot{x}) = 0,\ \forall t \in I \end{cases}$$

The domain of (MP) is also $\mathbf{D}$.

Definition 3.2 [3]. A feasible solution $x^0 \in \mathbf{D}$ is said to be an *efficient solution* (or a *Pareto minimum*) to the problem (MP) if there exists no other feasible solution $x \in \mathbf{D}$ such that

$$\int_a^b f(t, x, \dot{x})dt \leq \int_a^b f(t, x^0, \dot{x}^0)dt.$$

Let $\rho \in \mathbf{R}$ be and the functions $b : X \times X \to [0, \infty)$ and $H(x) = \displaystyle\int_a^b h(t, x, \dot{x})dt$.

Definition 3.3 [13]. The function H is said to be [*strictly*] (ρ, b)-quasiinvex at x^0 if there exist vector functions $\eta : I \times X \times X \to \mathbf{R}^n$ with $\eta(t, x(t), \dot{x}(t)) = 0$ for $x(t) = x^0(t)$ and $\theta : X \times X \to \mathbf{R}^n$ such that for any $x[x \neq x^0]$, $H(x) \leqq H(x^0)$

$$\Rightarrow b(x, x^0) \int_a^b [\eta' h_x(t, x^0, \dot{x}^0, u^0) + (D\eta)' h_{\dot{x}}(t, x^0, \dot{x}^0, u^0)]dt[<] \leqq$$

$$-\rho b(x, x^0)||\theta(x, x^0)||^2.$$

4 Necessary conditions of normal efficiency for the problems (MFP) and (MP)

In this section we establish necessary conditions for the normal efficiency of a point $x^0 \in \mathbf{D}$ in the problem (MFP). Consider the following scalar problem

$$(\mathrm{FP})_i(x^0) \begin{cases} \min\limits_x \dfrac{\displaystyle\int_a^b f_i(t, x, \dot{x})dt}{\displaystyle\int_a^b k_i(t, x, \dot{x})dt} \\ \text{subject to } x(a) = a_0,\ x(b) = b_0, \\ \qquad g(t, x, \dot{x}) \leqq 0,\ h(t, x, \dot{x}) = 0,\ \forall t \in I \\ \dfrac{\displaystyle\int_a^b f_j(t, x, \dot{x})dt}{\displaystyle\int_a^b k_j(t, x, \dot{x})dt} \leqq \dfrac{\displaystyle\int_a^b f_j(t, x^0, \dot{x}^0)dt}{\displaystyle\int_a^b k_j(t, x^0, \dot{x}^0)dt}, j = \overline{1, p},\ j \neq i. \end{cases}$$

Lemma 4.1 (Chankong, Haimes [2]). $x^0 \in \mathbf{D}$ *is an efficient solution to problem* (MFP) *if and only if* x^0 *is an optimal solution to problems* $(\mathrm{FP})_i$, $i = \overline{1, p}$.

We denote

$$R_i^0 = \frac{\displaystyle\int_a^b f_i(t, x^0, \dot{x}^0)dt}{\displaystyle\int_a^b k_i(t, x^0, \dot{x}^0)dt} = \min_{x,u} \frac{\displaystyle\int_a^b f_i(t, x, \dot{x})dt}{\displaystyle\int_a^b k_i(t, x, \dot{x})dt}, \quad i = \overline{1, p},$$

and then, (**Remark 4.1**) the problem $(\mathrm{FP})_i(x^0)$ can be written under the next equivalently form:

$$(\mathrm{FPR})_i \quad \begin{cases} \min\limits_x \dfrac{\displaystyle\int_a^b f_i(t,x,\dot{x})dt}{\displaystyle\int_a^b k_i(t,x,\dot{x})dt} \quad [= R_i^0] \\ \text{subject to } x(a)=a_0,\ x(b)=b_0, \\ \qquad g(t,x,\dot{x}) \leqq 0,\ h(t,x,\dot{x})=0,\ \forall t\in I \\ \displaystyle\int_a^b [f_j(t,x,\dot{x}) - R_j^0 k_j(t,x,\dot{x})]dt \leqq 0,\ j=\overline{1,p},\ j\neq i. \end{cases}$$

Consider now the problem:

$$(\mathrm{SPR})_i \quad \begin{cases} \min\limits_x \displaystyle\int_a^b [f_i(t,x,\dot{x}) - R_i^0 k_i(t,x,\dot{x})]dt \\ \text{subject to } x(a)=a_0,\ x(b)=b_0, \\ \qquad g(t,x,\dot{x}) \leqq 0,\ h(t,x,\dot{x})=0,\ \forall t\in I \\ \displaystyle\int_a^b [f_j(t,x,\dot{x}) - R_j^0 k_j(t,x,\dot{x})]dt \leqq 0,\ j=\overline{1,p},\ j\neq i. \end{cases}$$

Lemma 4.2 (Jagannathan [5]). *$x^0 \in \mathbf{D}$ is optimal to* $(\mathrm{FPR})_i$ *if and only if x^0 is optimal to* $(\mathrm{SPR})_i$.

Lemma 4.3. *If x^0 is a [normal] optimal solution for the scalar problem (SRP)$_i$, then there exist real scalars $\lambda_{ji} \geq 0$ $[\lambda_{ii}=1]$ and the piecewise smooth functions μ_i and ν_i such that*

$$(VF)_i \begin{cases} \displaystyle\sum_{j=1}^p \lambda_{ji}[f_{jx}(t,x^0,\dot{x}^0) - R_j^0 k_{jx}(t,x^0,\dot{x}^0)]+ \\ +\mu_i(t)' g_x(t,x^0,\dot{x}^0) + \nu_i(t)' h_x(t,x^0,\dot{x}^0) = \\ = \dfrac{d}{dt}\{\displaystyle\sum_{j=1}^p \lambda_{ji}[f_{j\dot{x}}(t,x^0,\dot{x}^0) - R_j^0 k_{j\dot{x}}(t,x^0,\dot{x}^0)]+ \\ +\mu_i(t)' g_x(t,x^0,\dot{x}^0) + \nu_i(t)' h_x(t,x^0,\dot{x}^0)\} \\ \mu_i(t)' g(t,x^0,\dot{x}^0) = 0, \quad \mu_i(t) \geqq 0, \quad \forall t\in I \\ \lambda_{ji} \geqq 0, \quad j=\overline{1,p} \quad [\lambda_{ii}=1]. \end{cases}$$

Proof. For $j=\overline{1,p}$, $j\neq i$ we define a twice differentiable function $\omega_j : I\times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$F_j(x) = \int_a^b [f_j(t,x,\dot{x}) - R_j^0 k_j(t,x,\dot{x}) + \omega_j(t,x,\dot{x})]dt = 0,$$

where $\omega_j(t,x,\dot{x}) \geq 0$, $j=\overline{1,p}$, $j \neq i$. Then x^0 is a [normal] optimal solution to $(\mathrm{SPR})_i$ if and only if x^0 is a [normal] optimal solution to the problem

$$\bar{P}_i(x^0)\begin{cases} \text{Maximize } \displaystyle\int_a^b [f_i(t,x,\dot{x}) - R_j^0 k_i(t,x,\dot{x})]dt \\ \text{subject to } \quad x \in D,\ F_j(x)=0, \\ \omega_j(t,x,\dot{x}) \geq 0,\ j=\overline{1,p},\ j \neq i. \end{cases}$$

According to an Euler's theorem, there exist scalars $\lambda_{ji} \geq 0$, $j=\overline{1,p}$ such that x^0 is [normal] optimal to problem

$$\bar{\bar{P}}_i\ (x^0)\begin{cases} \text{Maximize } \displaystyle\int_a^b \{f_i(t,x,\dot{x}) - R_j^0 k_i(t,x,\dot{x}) + \\ +\displaystyle\sum_{j=1, j\neq i}^{p} \lambda_{ji}[f_j(t,x,\dot{x}) - R_j^0 k_j(t,x,\dot{x}) + \omega_j(t,x,\dot{x})]\}dt \\ \text{subject to } \quad x \in D,\ \omega_j(t,x,\dot{x}) \geq 0,\ j=\overline{1,p},\ j \neq i. \end{cases}$$

or equivalently, to

$$\tilde{P}_i(x^0)\begin{cases} \text{Maximize } \displaystyle\int_a^b \{f_i(t,x,\dot{x}) - R_i^0 k_i(t,x,\dot{x}) + \\ +\displaystyle\sum_{j=1, j\neq i}^{p} \lambda_{ji}[f_j(t,x,\dot{x}) - R_j^0 k_j(t,x,\dot{x}) + \omega_j(t,x,\dot{x})]\}dt \\ \text{subject to } \quad x(a)=a_0,\ x(b)=b_0,\ g(t,x,\dot{x}) \leq 0, h(t,x,\dot{x})=0, \\ -\omega_j(t,x,\dot{x}) \geq 0,\ j=\overline{1,p},\ j \neq i. \end{cases}$$

We associate to $\tilde{P}_i(x^0)$ the function

$$V_i(t,x,\dot{x}) = \gamma_i\{f_i(t,x,\dot{x}) - R_i^0 k_i(t,x,\dot{x}) +$$

$$+\sum_{\substack{j=1,\\ j\neq i}}^{p} \lambda_{ji}[f_j(t,x,\dot{x}) - R_j^0 k_j(t,x,\dot{x}) + \omega_j(t,x,\dot{x})]\}dt +$$

$$+\mu(t)'g(t,x,\dot{x}) + \nu(t)'h(t,x,\dot{x}) - \sum_{\substack{j=1,\\ j\neq i}}^{p} \alpha_j(t)\omega_j(t,x,\dot{x}),$$

where $\gamma_i \in \mathbf{R}$ and $\mu : I \to \mathbf{R}^m$, $\nu : I \to \mathbf{R}^q$ and $\alpha_j : I \to \mathbf{R}$, $j=\overline{1,p}$, $j \neq i$ are piecewise smooth functions.

Because x^0 is a [normal] optimal to $\tilde{P}_i(x^0)$ then there are scalars $\gamma_i \geq 0$ [$=1$] and the functions μ_i, ν_j and $\alpha_j \geq 0$, $j=\overline{1,p}$, $j \neq i$ such that the next Valentine's conditions are true

$$(4.1)\qquad \begin{cases} V_{ix}(t,x^0,\dot{x}^0)=\dfrac{d}{dt}V_{i\dot{x}}(t,x^0,\dot{x}^0)\\ \mu_j(t)'g(t,x^0,\dot{x}^0)=0,\ \mu_i(t)\geqq 0,\ \forall t\in I\\ \alpha_j(t)'\omega_j(t,x^0,\dot{x}^0)=0,\ \alpha_j(t)\geqq 0,\ j=\overline{1,p},\ j\neq i,\ \forall t\in I\\ \gamma_i\geq 0[=1],\ \lambda_{ji}\geqq 0, j=\overline{1,p},\ j\neq i. \end{cases}$$

Developed, the first condition of (4.1) can be written so:

$$\gamma_i[f_{ix}(t,x^0,\dot{x}^0)-R_i^0k_{ix}(t,x^0,\dot{x}^0)]+\sum_{\substack{j=1\\ j\neq i}}^{p}\gamma_i\lambda_{ji}[f_{jx}(t,x^0,\dot{x}^0)-R_j^0k_{jx}(t,x^0,\dot{x}^0)]+$$

$$(E)\qquad +\sum_{\substack{j=1\\ j\neq i}}^{p}[\gamma_i\lambda_{ji}-\alpha_j(t)]\omega_{jx}(t,x^0,\dot{x}^0)=\mu_i(t)'g_x(t,x^0,\dot{x}^0)+\nu_i(t)'h_x(t,x^0,\dot{x}^0)=$$

$$\frac{d}{dt}\langle\gamma_i[f_{i\dot{x}}(t,x^0,\dot{x}^0)-R_i^0k_{i\dot{x}}(t,x^0,\dot{x}^0)]+\sum_{\substack{j=1\\ j\neq i}}^{p}\gamma_i\lambda_{ji}[f_{j\dot{x}}(t,x^0,\dot{x}^0)-R_j^0k_{j\dot{x}}(t,x^0,\dot{x}^0)]+$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{p}[\gamma_i\lambda_{ji}-\alpha_j(t)]\omega_{j\dot{x}}(t,x^0,\dot{x}^0)=\mu_i(t)'g_{\dot{x}}(t,x^0,\dot{x}^0)+\nu_i(t)'h_{\dot{x}}(t,x^0,\dot{x}^0)\rangle.$$

(E) is a linear system by n equations and $2(p-1)+m+q$ unknowns (coefficients). Consider $n<2(p-1)+m+q$ and we choose a set of values for these coefficients putting conditions $\gamma_i\lambda_{ji}-\alpha_j=0$, $j=\overline{1,p}$, $j\neq 1$. Then we define $\lambda_{ii}=\gamma_i$ and $\lambda_{ji}\equiv\gamma_i\lambda_{ji}\geqq 0$, $j=\overline{1,p}$, $j\neq 1$ and (E) becomes

$$\lambda_{ii}f_{ix}(t,x^0,\dot{x}^0)-R_j^0k_{ix}(t,x^0,\dot{x}^0)]+$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{p}\lambda_{ji}[f_{jx}(t,x^0,\dot{x}^0)-R_j^0k_{jx}(t,x^0,\dot{x}^0)]+\mu_i(t)'g_x(t,x^0,\dot{x}^0)+$$

$$\nu_l(t)'h_x(t,x^0,\dot{x}^0)=\frac{d}{dt}\langle\lambda_{ii}[f_{ix}(t,x^0,\dot{x}^0)-R_i^0k_{i\dot{x}}(t,x^0,\dot{x}^0)]+$$

$$\sum_{\substack{j=1\\ j\neq i}}^{p}\lambda_{ji}[f_{j\dot{x}}(t,x^0,\dot{x}^0)-R_j^0k_{j\dot{x}}(t,x^0,\dot{x}^0)+\mu_i(t)'g_{\dot{x}}(t,x^0,\dot{x}^0)+\nu_i(t)'h_{\dot{x}}(t,x^0,\dot{x}^0)\rangle.$$

Conditions $\gamma_i\lambda_{ji}-\alpha_j=0$, $j=\overline{1,p}$, $j\neq 1$ corresponding modify the second relation of (4.1). So, from (4.1) there result conditions $(\mathrm{VF})_i$.

Theorem 4.4. *$x^0\in\mathbf{D}$ is an efficient solution to* (MFP) *if and only if it is an optimal solution for each of problems* $(\mathrm{SPR})_i$, $i=\overline{1,p}$.

Proof. x^0 is efficient in (MFP) if and only if it is optimal in the problems $(\text{FPR})_i$, $i = \overline{1,p}$ (according to Lemma 4.1 and Remark 4.1) and also, for each i, x^0 is optimal to $(\text{FPR})_i$ if and only if it is optimal to $(\text{SPR})_i$ (according to Lemma 4.2).

Definition 4.1. $x^0 \in \mathbf{D}$ is a *normal efficient solution* to (MFP) if it is a normal optimal solution at least one of the scalar problems $(\text{FP})_i$, $i = \overline{1,p}$.

Now it follows the main result of this section.

Theorem 4.5 (Necessary efficiency conditions for (MFP)). *Let* $x^0 \in \mathbf{D}$ *be a normal efficient solution to problem* (MFP). *Then there exist* $\lambda^0 \in \mathbf{R}^p$ *and the piecewise smooth functions* $\mu^0 : I \to \mathbf{R}^m$ *and* $\nu^0 : I \to \mathbf{R}^q$ *that satisfy the following conditions*:

$$(\text{MFV}) \qquad \begin{cases} \displaystyle\sum_{i=1}^{p} \lambda_i^0 [f_{ix}(t, x^0, \dot{x}^0) - R_i^0 k_{ix}(t, x^0, \dot{x}^0)] + \mu(t)' g_x(t, x^0, \dot{x}^0) + \\ \displaystyle +\nu(t)' h_x(t, x^0, \dot{x}^0) = \frac{d}{dt}\{\sum_{i=1}^{p} \lambda_i [f_{i\dot{x}}(t, x^0, \dot{x}^0) - R_i^0 h_{i\dot{x}}(t, x^0, \dot{x}^0)] + \\ +\mu(t)' g_{\dot{x}}(t, x^0, \dot{x}^0) + \nu(t)' h_{\dot{x}}(t, x^0, \dot{x}^0)\}, \\ \mu^0(t) g(t, x^0, \dot{x}^0) = 0,\ \mu^0(t) \geqq 0,\ \forall t \in I \\ \lambda^0 \geq 0,\ e'\lambda^0 = 1,\ e' = (1, \dots, 1) \in \mathbf{R}^p. \end{cases}$$

Proof. According to Lemma 4.1 x^0 is an optimal solution for each problem $(\text{SPR})_i$ where $i = \overline{1,p}$. If x^0 is a [normal] optimal solution to $(\text{SPR})_i$, then relations $(\text{VF})_i$ of Lemma 4.3 are true.

Summing over $i = \overline{1,p}$ all the relations of $(\text{VF})_i$ and setting

$$\lambda_j = \sum_{i=1}^{p} \lambda_{ji}, i = \overline{1,p}, \quad \mu(t) = \sum_{i=1}^{p} \mu_i(t), \quad \nu(t) = \sum_{i=1}^{p} \nu_i(t),$$

the next relations are obtained

$$(\text{FV}) \qquad \begin{cases} \displaystyle\sum_{j=1}^{p} \lambda_j [f_{jx}(t, x^0, \dot{x}^0) - R_j^0 k_{jx}(t, x^0, \dot{x}^0)] + \mu(t)' g_x(t, x^0, \dot{x}^0) + \\ \displaystyle +\nu(t)' h_x(t, x^0, \dot{x}^0) = \frac{d}{dt}\{\sum_{j=1}^{p} \lambda_j [f_{j\dot{x}}(t, x^0, \dot{x}^0) - R_j^0 k_{j\dot{x}}(t, x^0, \dot{x}^0)] + \\ +\mu(t)' g_{\dot{x}}(t, x^0, \dot{x}^0) + \nu(t)' h_{\dot{x}}(t, x^0, \dot{x}^0)\}, \\ \mu(t)' g(t, x^0, \dot{x}^0) = 0,\ \mu(t) \geqq 0\ \forall t \in I,\ \lambda_j \geq 0,\ i = \overline{1,p}, [\lambda_j \geqq 1]. \end{cases}$$

By dividing all relations of (FV) by $S = \displaystyle\sum_{j=1}^{p} \lambda_j \geqq 1$ and setting

$$\lambda_j^0 = \lambda_j / S, \quad \mu^0(t) = \mu(t)/S, \quad \nu^0(t) = \nu(t)/S$$

then, from (FV) its result the relations (MFV) (j was replaced by i).

Denote

$$F_i(x^0) = \int_a^b f_i(t, x^0, \dot{x}^0)dt, \quad K_i(x^0) = \int_a^b k_i(t, x^0, \dot{x}^0)dt$$

and we obtain

$$R_i^0 = \frac{F_j(x^0)}{K_i(x^0)}, \quad i = \overline{1,p}. \tag{4.2}$$

By replacing the numbers R_i^0 by (4.2) in the relations $(\text{MFV})_i$ and redefining μ^0 and ν^0 then the following result is obtained:

Theorem 4.6 (Necessary efficiency conditions for (MFP)). *Let x^0 be a normal efficient solution to the problem* (MFP). *Then there exist $\lambda^0 \in \mathbf{R}^n$ and the piecewise smooth functions $\mu^0 : I \to \mathbf{R}^m$ and $\nu^0 : I \to \mathbf{R}^q$ that satisfy*

$$(\text{MFV})_0 \quad \begin{cases} \sum\limits_{i=1}^{p} \lambda_i^0 [K_i(x^0) f_{ix}(t, x^0, \dot{x}^0) - F_i(x^0) k_{ix}(t, x^0, \dot{x}^0)] + \mu^0(t)' g_x(t, x^0, u^0) + \\ + \nu^0(t)' h_x(t, x^0, u^0) = \dfrac{d}{dt} \{ \sum\limits_{i=1}^{p} \lambda_i^0 [K_i(x^0) f_{i\dot{x}}(t, x^0, \dot{x}^0) - \\ - F_i(x^0) k_{i\dot{x}}(t, x^0, \dot{x}^0)] + \mu^0(t)' g_{\dot{x}}(t, x^0, u^0) + \nu^0(t)' h_{\dot{x}}(t, x^0, u^0) \} \\ \mu^0(t)' g(t, x^0, \dot{x}^0) = 0, \ \mu^0(t) \geqq 0, \ \forall t \in I, \ \lambda^0 \geq 0, \ e' \lambda^0 = 1. \end{cases}$$

In Theorem 4.5 or Theorem 4.6 we put $K_i(x) \equiv 1$, $i = \overline{1,p}$ and then there are obtained the following conditions for the of normal efficiency of x^0 in the multiobjective variational problem (MP) :

Theorem 4.7 (Necessary efficiency conditions for (MP)). *Let $x^0 \in \mathbf{D}$ be a normal efficient solution to* (MP). *Then there exist a vector $\lambda^0 \in \mathbf{R}^p$ and the piecewise smooth functions $\mu^0 : I \to \mathbf{R}^m$ and $\nu^0 : I \to \mathbf{R}^q$ that satisfy the following conditions*:

$$(\text{MV}) \quad \begin{cases} \lambda^{0'} f_x(t, x^0, \dot{x}^0) + \mu^0(t)' g_x(t, x^0, \dot{x}^0) + \nu^0(t) h_x(t, x^0, \dot{x}^0) = \\ = \dfrac{d}{dt} [\lambda^{0'} f_{\dot{x}}(t, x^0, \dot{x}^0) + \mu^0(t) g_{\dot{x}}(t, x^0, \dot{x}^0) + \nu^0(t) h_x(t, x^0, \dot{x}^0)] \\ \mu^0(t)' g(t, x^0, \dot{x}^0) = 0, \ \mu_i(t) \geqq 0, \ \forall t \in I, \\ \lambda^0 \geq 0, \ e' \lambda^0 = 1. \end{cases}$$

5 Sufficient conditions of efficiency for the problems (MP) and (MFP)

First, we establish *sufficient efficiency conditions for*(MP).

Theorem 5.1 (Sufficient efficiency for (MP)). *Let x^0, λ^0, μ^0 and ν^0 be satisfying the relations* (MV) *from* Theorem 4.6. *We suppose that exist vector functions η and θ satisfying* Definition 3.3 *and that the following conditions are fulfilled*:

a) *For each* $i = \overline{1,p}$, $\displaystyle\int_a^b f_i(t,x,\dot{x})dt$ *is* (ρ_i^1, b)*-quasiinvex at* x^0 *with respect to* η *and* θ.

b) $\displaystyle\int_a^b \mu^0(t)' g(t,x,\dot{x})dt$ *is* (ρ_2, b)*-quasiinvex at* x^0 *with respect to* η *and* θ.

c) $\displaystyle\int_a^b \nu^0(t)' h(t,x,\dot{x})dt$ *is* (ρ_3, b)*-quasiinvex at* x^0 *with respect to* η *and* θ.

d) *One of the integral of* a)-c) *is strictly* (ρ, b)*-quasiinvex at* x^0

e) $\displaystyle\sum_{i=1}^p \lambda_i^0 \rho_i^1 + \rho_2 + \rho_3 \geqq 0 \quad (\rho_i^1, \rho_2, \rho_3 \in \mathbf{R})$.

Then x^0 *is an efficient solution to* (MP).

Proof. Let us suppose toward a contradiction, that x^0 is not an efficient solution for (MP). For each $i = \overline{1,p}$ there exists x, a feasible solution to (MP), such that:

$$\int_a^b f_i(t,x,\dot{x})dt \leq \int_a^b f_i(t,x^0,\dot{x}^0)dt.$$

The case $x = x^0$ is excluded. According to a) it follows that:

$$b(x,x^0)\int_a^b [\eta(t,x,x^0) f_{ix}(t,x^0,\dot{x}^0) + D\eta(t,x,x^0) f_{i\dot{x}}(t,x^0,\dot{x}^0)]dt \leqq -\rho_i^1 b(x,x^0)||\theta(x,x^0)||^2.$$

Multiplying this inequality by $\lambda_i^0 \geq 0$ and summing over $i = \overline{1,p}$ we obtain

$$b(x,x^0)\int_a^b [\eta(t,x,x^0) \lambda^{0'} f_x(t,x^0,\dot{x}^0) + D\eta(t,x,x^0) \lambda^{0'} f_{\dot{x}}(t,x^0,\dot{x}^0)]dt \leqq -\left(\sum_{i=1}^p \lambda_i^0 \rho_i^1\right) b(x,x^0)||\theta(x,x^0)||^2. \tag{5.1}$$

We have

$$\int_a^b \mu^0(t)' g(t,x,\dot{x})dt \leqq \int_a^b \mu^0(t)' g(t,x^0,\dot{x}^0)dt$$

and taking into account the condition b) it results

$$b(x,x^0)\int_a^b [\eta(t,x,x^0)\mu^0(t)' g_x(t,x^0,\dot{x}^0) + D\eta(t,x,x^0)\mu^0(t)' g_{\dot{x}}(t,x^0,\dot{x}^0)]dt \leqq -\rho_2 b(x,x^0)||\theta(x,x^0)||^2. \tag{5.2}$$

Taking into account the condition c), from

$$\int_a^b \nu^0(t)' h(t,x,\dot{x})dt = \int_a^b \nu^0(t)' h(t,x^0,\dot{x}^0)dt$$

it results

$$b(x,x^0)\int_a^b [\eta(t,x,x^0)\nu^0(t)' h_x(t,x,\dot{x}) + D\eta(t,x,x^0)\nu^0(t)' h_{\dot{x}}(t,x^0,\dot{x}^0)]dt \leqq -\rho_3 b(x,x^0)||\theta(x,x^0)||^2. \tag{5.3}$$

Summing side by side relations (5.1), (5.2) and (5.3) and taking into account d) we obtain
(5.4)

$$b(x,x^0)\int_a^b [\eta(t,x,x^0)[\nu^{0'} f_x(t,x^0,\dot{x}^0)+\mu^0(t)' g_x(t,x^0,\dot{x}^0)+\nu^0(t)' h_x(t,x^0,\dot{x}^0)]dt+$$

$$+b(x,x^0)\int_a^b D\eta(t,x,x^0)[\lambda^{0'} f_{\dot{x}}(t,x^0,\dot{x}^0)+\mu^0(t)' g_{\dot{x}}(t,x^0,\dot{x}^0)+\nu^0(t)' h_{\dot{x}}(t,x^0,\dot{x}^0)]dt<$$

$$<-\left(\sum_{i=1}^p \lambda_i^0\rho_i^1+\rho_2+\rho_3\right) b(x,x^0)||\theta(x,x^0)||^2.$$

¿From (6.4) it results $b(x,x^0)>0$ and then (5.4) becomes

$$\begin{aligned} &\int_a^b \eta(t,x,x^0)[\lambda^{0'} f_x(t,x^0,\dot{x}^0)+\mu^0(t)' g_x(t,x^0,\dot{x}^0)+\nu^0(t)' h_x(t,x^0,\dot{x}^0)]dt+\\ (5.5)\quad &+\int_a^b D\eta(t,x,x^0)[\lambda^{0'} f_{\dot{x}}(t,x^0,\dot{x}^0)+\mu^0(t)' g_{\dot{x}}(t,x^0,\dot{x}^0)+\nu^0(t)' h_{\dot{x}}(t,x^0,\dot{x}^0)]<\\ &<-\left(\sum_{i=1}^p \lambda_i^0\rho_i^0+\rho_2+\rho_3\right)||\theta(x,x^0)||^2. \end{aligned}$$

Using an integration by parts, (5.5) becomes

$$\begin{aligned} &\int_a^b \eta(t,x,x^0)[\lambda^{0'} f_x(t,x^0,\dot{x}^0)+\mu^0(t)' g_x(t,x^0,\dot{x}^0)+\nu^0(t)' h_x(t,x^0,\dot{x}^0)]dt+\\ &+\eta(t,x,x^0)[\lambda^{0'} f_{\dot{x}}(t,x,\dot{x}^0)+\mu^0(t)' g_{\dot{x}}(t,x^0,\dot{x}^0)+\nu^0(t)' h_{\dot{x}}(t,x^0,\dot{x}^0)]|_a^b-\\ (5.6)\quad &-\int_a^b \eta(t,x,x^0)\frac{d}{dt}[\lambda^{0'} f_{\dot{x}}(t,x^0,\dot{x}^0)+\mu^0(t)' g_{\dot{x}}(t,x^0,\dot{x}^0)+\nu^0(t)' h_{\dot{x}}(t,x^0,\dot{x}^0)]dt<\\ &<-\left(\sum_{i=1}^p \lambda_i^0\rho_i^1+\rho_2+\rho_3\right)||\theta(x,x^0)||^2. \end{aligned}$$

Using now the boundary conditions $\eta(a,x(a),\dot{x}^0(a))=0$ (where $x(a)=x^0(a)=a_0$ and $\eta(b,x(b),\dot{x}^0(b)=0)$ (where $x(b)=x^0(b)=b_0$) from (5.6) we obtain

$$\begin{aligned} &\int_a^b \eta(t,x,x^0)\{\lambda^{0'} f_x(t,x^0,\dot{x}^0)+\mu^0(t)' g_x(t,x^0,\dot{x}^0)+\nu^0(t)' h_x(t,x^0,\dot{x}^0)]-\\ (5.7)\quad &-\frac{d}{dt}[\lambda^{0'} f_{\dot{x}}(t,x^0,\dot{x}^0)+\mu^0(t)' g_{\dot{x}}(t,x^0,\dot{x}^0)+\nu^0(t)' h_{\dot{x}}(t,x^0,\dot{x}^0)]\}dt<\\ &<-\left(\sum_{i=1}^p \lambda_i^0\rho_i^1+\rho_2+\rho_3\right)||\theta(x,x^0)||^2. \end{aligned}$$

Taking into account the first relation of (MV), (5.7) becomes

$$0<-\left(\sum_{i=1}^p \lambda_i^0\lambda^{0'}\rho_1+\rho_2+\rho_3\right) b(x,x^0)||\theta(x,x^0)||^2.$$

Having $||\theta(x,x^0)|| \geqq 0$ and the hypothesis e), the obtained inequality $0 < 0$ is false. Therefore x^0 is an efficient solution to (MP).

In what follows we establish *efficiency sufficient conditions for the problem* (MFP).

Theorem 5.2 (Sufficient efficiency for (MFP)). *Let x^0, λ^0, μ^0 and ν^0 be satisfying the relations* (MFV) *from* Theorem 4.4. *Also, we suppose that exist the vector functions η and θ as in* Definition 3.3 *and the following conditions are fulfilled*:

a$'$) *For each $i = \overline{1,p}$, $\int_a^b [f_i(t,x,\dot{x}) - R_j^0 k_i(t,x,\dot{x})]dt$ is (ρ_i^1, b)-quasiinvex at x^0 with respect to η and θ.*

b) $\int_a^b \mu^0(t)' g(t,x,\dot{x})dt$ *is (ρ_2, b)-quasiinvex at x^0 with respect to η and θ.*

c) $\int_a^b \nu^0(t)' h(t,x,\dot{x})dt$ *is (ρ_3, b)-quasiinvex at x^0 with respect to η and θ.*

d$'$) *One of the integral by* a$'$), b) and c) *is strictly (ρ, b)-quasiinvex at x^0 with respect to η and ν ($\rho = \rho_i^1, \rho_2$ or ρ_3, respectively).*

e) $\sum_{i=1}^{p} \lambda_i^0 \rho_i^1 + \rho_3 + \rho_4 \geqq 0$.

Then x^0 is an efficient solution to (MFP).

Proof. It is similarly to those of Theorem 5.1 where, for each $i = \overline{1,p}$, $\int_a^b f_i(t,x,\dot{x})dt$, is replaced by $\int_a^b [f_i(t,x,\dot{x}) - R_i^0 k_i(t,x,\dot{x})]dt$.

Theorem 5.3 (Sufficient efficiency for (MFP). *Let x^0, λ^0, μ^0 and ν^0 be satisfying the relations* (MFV) *from* Theorem 4.5. *Also, assume that exist the vector functions η and θ as in* Definition 3.3 *such that the following conditions are satisfied*:

a$''$) *For each $i = \overline{1,p}$, $\int_a^b [K_i(x^0) f_i(t,x,\dot{x}) - F_i(x^0) k_i(t,x,\dot{x})]dt$ is (ρ_i^1, b)-quasiinvex at x^0 with respect to η and θ.*

b), c) *and* e) *of* Theorem 5.2.

d$''$) *One of the integrals from* a$''$), b) *and* c) *is strictly (ρ, b)-quasiinvex at x^0 with respect to η and θ ($\rho = \rho_i^1, \rho_2$ or ρ_3 respectively).*

Then x^0 is an efficient solution to (MFP).

Proof. It is similarly to those of Theorem 5.1, where the hypothesis a) is replaced by hypothesis a$''$) of this theorem.

If in Theorems 5.1-5.3 the integrals from the hypotheses b) and c) are replaced by the integral $\int_a^b [\mu^0(t)' g(t,x,\dot{x}) + \nu^0(t)' h(t,x,\dot{x})]dt$, then there are the following results:

Corollary 5.1$'$ (Sufficient efficiency conditions (MP)). *Let x^0, λ^0, μ^0 and ν^0 be satisfying the relations* (MV) *from* Theorem 4.6. *We suppose that exist the vector functions η and θ as in* Definition 3.3 *and the following conditions are satisfied*:

a) *For each $i = \overline{1,p}$, $\int_a^b f_i(t,x,\dot{x})dt$ is (ρ_i^1, b)-quasiinvex at x^0 with respect to η and θ.*

b′) $\displaystyle\int_a^b [\mu^0(t)'g(t,x,\dot{x})+\nu^0(t)'h(t,x,\dot{x})]dt$ *is* (ρ_2, b)*-quasiinvex at* x^0 *with respect to* η *and* θ.

d′) *One of the integrals from* a) *and* b′) *is strictly* (ρ, b)*-quasiinvex at* x^0 *with respect to* η *and* θ *or* ($\rho = \rho_i^1$ *or* ρ_2*, respectively).*

e′) $\displaystyle\sum_{i=1}^{p} \lambda_i^0 \rho_i^1 + \rho_2 \geqq 0.$

Then x^0 *is an efficient solution to* (MP).

Corollary 5.2′ (Sufficient efficiency conditions for (MFP). *Let* x^0, λ^0, μ^0 *and* ν^0 *be satisfying the relations* (MFV) *from* Theorem 4.4. *Also, we suppose that exist the vector functions* η *and* θ *as in* Definition 3.3 *such that the following conditions are satisfied*:

a′) *For each* $i = \overline{1,p}$, $\displaystyle\int_a^b [f_i(t,x,\dot{x})-R_i^0 k_i(t,x,\dot{x})]dt$ *is* (ρ_i^1, b)*-quasiinvex at* x^0 *with respect to* η *and* θ.

b′) *and* e′) *from* Corollary 5.1′.

d″) *One of the integrals from* a′) *and* b′) *is strictly* (ρ, b)*-quasiinvex at* x^0 *with respect to* η *and* θ.

Then x^0 *is an efficient solution to* (MFP).

Corollary 5.3′ (Sufficient efficiency conditions for (MFP)). *Let* x^0, λ^0, μ^0 *and* ν^0 *be satisfying the relations* (MFV) *from* Theorem 4.5. *Also, suppose that exist the vector functions* η *and* θ *as in* Definition 3.3 *such that the following conditions are satisfied*:

a″) *For each* $i = \overline{1,p}$, $\displaystyle\int_a^b [K_i(x^0)f_i(t,x,\dot{x})-F_i(x^0)k_i(t,x,\dot{x})]dt$ *is* (ρ_i^1, b)*-quasiinvex at* x^0 *with respect to* η *and* θ.

b′) *and* e′) *from* Corollary 5.1′.

d″) *One of the integrals from* a″) *and* b′) *is strictly* (ρ, b)*-quasiinvex at* x^0 *with respect to* η *and* θ.

Then x^0 *is an efficient solution to* (MFP).

References

[1] C.R. Bector, *Generalized concavity and duality in continuous programming*, Utilitas Math., 25 (1984), 171-190.

[2] V. Chankong, Y.Y. Haimes, *Multiobjective decisions making: theory and methododology*, North Holland, New York, 1983.

[3] M.A. Geoffrion, *Proper efficiency and the theory of vector minimization*, J. Math. Anal. Appl., 149, 6 (1968), 618-630.

[4] S. Gramatovici, *Optimality conditions in multiobjective control problems with geneneralized invexity*, Annals Univ. of Craiova, Math.Comp. Sci. Ser., 32 (2005), 150-157.

[5] R. Jagannathan, *Duality for nonlinear fractional programming*, Z. Oper. Res., 17, (1973), 1-3.

[6] D.S. Kim, A.L. Kim, *Optimality and duality for nondifferentiable multiobjective variational problems*, J. Math. Anal. Appl., 274 (2002), 255-278.

[7] S. Mititelu, *Kuhn-Tucker conditions and duality for multiobjective fractional programs with n-set functions*, 7-th Balkan Conference on Operational Research-Proceedings, 123-132, Constanţa, Romania, May 25-28, 2005, Bucureşti, 2007.

[8] S. Mititelu, *Eficienţă şi dualitate în programarea multiobiectivă a pro-blemelor variaţionale*, 90-93 în "Seminarul Ştiinţific al Catedrei de Matema-tică şi Informatică din UTCB", MatrixRom, Bucureşti, 2005.

[9] B. Mond, M.A. Hanson, *Duality for variational problems*, J. Math. Anal. Appl.,18 (1967), 355-364.

[10] B. Mond, I. Husain, *Sufficient optimality criteria and duality for variational problems with generalized invexity*, J. Austral. Math. Soc. Ser. B, 31 (1989), 106-121.

[11] B. Mond, S. Chandra, I. Husain, *Duality for variational problems*, J. Math. Anal.Appl., 134(1988), 322-328.

[12] R.N. Mukherjee, Ch. Purnachandra Rao, *Mixed type duality for multiobjective variational problems*, J. Math.Anal.Appl., 252 (2000), 571-586.

[13] V. Preda, *On Mond-Weir duality for variational problems*, Rev. Roumanie Math.Pures Appl., 28, 2 (1993), 155-164.

[14] V. Preda, S. Gramatovici, *Some sufficient optimality conditions for a class of multiobjective variational problems*, An. Univ. Bucureşti, Matematică-Informatică, 61, 1 (2002), 33-43.

[15] F.A. Valentine, *The problem of Lagrange with differentiable inequality as added side conditions*, 407-448, in "Contributions to the Calculus of Varia-tions", 1933-37. Univ. of Chicago Press, 1937.

Author's address:

Ştefan Mititelu
Technical University of Civil Engineering
Department of Mathematics and Informatics
124 Lacul Tei , 020396 Bucharest 2, Romania.
E-mail: st_mititelu@yahoo.com