FRAILTY MODELS

APPLICATIONS IN PENSION SCHEMES

BY

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DECLARATION

This project is my original work and has never been presented in any learning institution for any academic award.

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I56/68701/2011

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DEDICATION
I would like to dedicate this project to my family, who have given me every reason and opportunity to be happy and to all actuaries who have worked to fairly price insurance products in the light of uncertainty.
ACKNOWLEDGEMENTS
First and foremost, I would like to acknowledge the pioneering work of Beard (1959) who is considered a forerunner in mortality modeling allowing for frailty as this formed a basis for this study.

Thank you to my project supervisor, Prof Jam, for providing me with the proper guidance throughout, and for allowing me to work on my own schedule. This project couldn’t be finished without his assistance. I attended all of his classes and they have been a great learning experience.
ABSTRACT

Heterogeneity in a population of assured lives in respect of mortality can be explained by differences among the individuals; some of these are observable, while others, for instance an individual’s attitude towards health and/or all genetic factors having influence on survival are difficult to monitor and measure. This undermines usage of observable risk factors as the only rating factors for life insurance. Insurance companies have not taken proper care of unobservable risk factors possibly due to difficulties inherent in their modeling. This heterogeneity exposes insurers to adverse selection if only the healthiest lives purchase annuities, so standard annuities are priced with a mortality table that assumes above-average longevity. This makes standard annuities expensive for many individuals. To avoid biases in valuation a better understanding of heterogeneity in required.

Frailty models are extensions of the Cox proportional hazards model which is popular in survival studies. In many applications, the study population needs to be considered as a heterogeneous sample. Sometimes, due to lack of knowledge or for economical reasons, some covariates related to the event of interest are not measured. The frailty approach is a statistical modeling method which aims to account for the heterogeneity caused by unmeasured covariates. It does so by adding random effects which act multiplicatively on the hazard.

This study carries out an extensive review of frailty models and is aimed at extending this work by considering other distributions that can be used in modeling. In particular, the non-central gamma distribution is proposed for frailty modeling.
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CHAPTER 1

GENERAL INTRODUCTION

1.1 Background Information

The concept of frailty modeling is based on mixture distributions and survival analysis.

1.1.1 Finite mixtures

Mixture distributions consist of mixing a distribution with another. This can be achieved by taking k different distributions with probability densities \( f_1(x), f_2(x), \ldots, f_k(x) \) with mixing weights: \( w_1, w_2, \ldots, w_k \)

Where \( w_j > 0 \) and \( \sum_{j=1}^{k} w_j = 1 \)

The new density function or mass function is:

\[
f(x) = \sum_{j=1}^{k} w_j f_j(x)
\]

This is a finite mixture (Johnson et.al. 2005 p.344)

Instead of \( k \) different distributions we can have \( k \) different components of a distribution.

1.1.2 Varying parameter and unknown covariates

A mixture distribution also arises when the density/mass function of a random variable depends on a parameter.
Consider a random variable $x$ depending on its parameter $\theta$ then the conditional probability density function can be written as:

$$f(x) = \int f(x|\theta)g(\theta) \, d\theta$$

Johnson (2005, p.345) states that this mixture distribution includes a situation where the source of a random variable is unknown. Thus instead of considering a parameter $\theta$, we consider an unknown covariate.

1.1.3 Survival analysis

Survival analysis is a branch of statistics which deals with time to the occurrence of a given event of interest. For insurers this event could be time to death, ill health or retirement. It’s different from other fields of statistics in that we are observing something that develops dynamically over time and takes censoring into consideration which is partial information about the variable of interest.

Three important functions of time to the event are:

- The survival function, $S(t)$, describes the probability that an individual survives longer than time $t$.
- The probability density function, $f(t)$
- The hazard function, $h(t)$, describes the instantaneous death rate

Estimating the survival function using non-parametric methods such as the Kaplan-Meier technique leads to obtaining the median time to the event under investigation.

Determining factors affecting the hazard, the Cox PH model has been widely used.
1.2 Problem Statement

The Cox PH model is given by:

\[ h_i(t) = h_o(t) e^{\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k} \]

Where:

- \( h_i(t) \) is the hazard function at time \( t \)
- \( h_o(t) \) is the baseline hazard function at time \( t \)
- \( \beta's \) are the regression coefficients.
- \( X's \) are vector of covariates.

One reason why this model is so popular is because of the ease with which technical difficulties such as censoring and truncation are handled. This is due to the appealing interpretation of the hazard function as a risk that changes over time.

The concept allows for the entering of covariates in order to describe their influence and to model different levels of risk for different sub-groups.

However, we may not know all relevant risk factors. It may also be costly to measure them.

Ordinary life tables assume that populations are homogeneous implying that all individuals have the same risk. Yet in reality we have a mixture of individuals with different risks. Frailty models tackle such issues.

In an insurance setting ignoring heterogeneity could lead to model error i.e. using the wrong life tables to price insurance products which could be costly to both the insurer and the insured.
1.3 Research Objectives

Main Objective:

The main objective of the study is to review univariate frailty models.

Specific Objectives:

- Extend the work by considering other distributions that can be used in modeling.
- To construct the hazard function when the insured population is considered to be heterogeneous.
- Propose the non-central gamma distribution to represent heterogeneity in an insurance setting.
1.4 Significance of the study

Risks within an insurance contract are heterogeneous. Therefore, to come up with premiums that are consistent with the insured risk all relevant factors affecting mortality need to be considered. Buhlmann (1970) developed models to accounting for features of the insured risk in non-life insurance which is unknown but relevant to explain overall claim frequencies. In another context, Olivieri (2006) has applied frailty models to pensions and life annuities to account for unobserved heterogeneity.
1.5 Outline
This thesis is organized as follows: Chapter 1 gives a general introduction to survival analysis and mixture distributions, chapter 2 is an introduction to different aspects of frailty modeling and chapter 3 reviews distributions that have been on frailty models. In chapter 4 Gompertz model parameters are estimated using R program and in chapter 5 the proposed model properties are discussed. Chapter 6 describes an application of frailty to life insurance and chapter 7 is on conclusions and recommendation for further research. Finally, the reference materials used in the study are listed.
1.6 Review of the Literature

**Estimating the survival function**
Existing literature on estimating the survival function includes the parametric, semi-parametric and non-parametric methods. Frailty models are extension of Cox-proportional hazard model where the relative risk is replaced with a random variable called the ‘frailty term’.

**1.6.1 Parametric Methods**
For parametric inference, it is necessary to make assumptions about the distribution of failure times. Parametric approaches such as Weibull, lognormal, exponential, etc can be used to estimate the survival function for homogeneous populations. Basically, any distribution of non-negative random variables can be used.

**1.6.2 Non-Parametric Methods**
Non-parametric approaches such as Kaplan-Meier (1958) and Aalen-Nelson (1978) can be used to estimate the survival function when assumption of the failure time distribution is to be avoided. An advantage of non-parametric models is their good fit and their ability to deal with any distribution without any additional assumptions.

Important consideration when estimating the hazard function is to investigate the relation between the survival time and some risk factors (covariates). These risk factors might be fixed variables, or they may change over time. Examples include; age, gender, socio-economic status, education, blood pressure, body mass index, smoking habits, nutrition, physical activity level, heart rate and so forth. Their influence on the survival is of great interest for insurers and can be estimated by statistical models.
The effect of $X_i$ can be either parameterized as proportional hazards (PH) or accelerated failure time (AFT).

PH assumes $h(t_i) = h_o(t_i) \exp \{x_i' \beta \}$ for some baseline hazard $h_o(t_i)$

AFT assumes $S(t_i) = S_o(\exp \{-x_i' \beta \} \cdot t_i)$ for some baseline survival function $S_o(t)$

Parametric survival models assume some function for $h_o(t)$ and hence for $S_o(t)$

1.6.3 Semi-Parametric Methods

Cox proportional hazard model (1972)

A Cox model is a technique for exploring the relationship between the survival time of an individual and several explanatory variables. The hazard function for each individual is proportional to the baseline hazard $h_o(t)$ and thus the hazard is fully determined by the covariate vector. The hazard function for individual $i$ at time (age) $t$ is written as:

$$h_i(t) = h_o(t) e^{\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k}$$

$h_o(t)$ is the baseline hazard function and corresponds to the probability of dying (or reaching an event) when all the explanatory variables are zero. In this model it is left unspecified. $\exp (\beta' X_i)$ is the relative risk of individual $i$, where $x_i = (x_1, \ldots, x_k)$ are vector of covariables and $\beta' = (\beta_1, \ldots, \beta_k)$ are vector of regression coefficients that give the proportional change that can be expected in the hazard, related to changes in the explanatory variables.

Cox (1972) proposed the partial likelihood method to estimate the $\beta$ parameter of this model. The partial likelihood is a product over the uncensored failure times written as:

$$L(\beta) = \prod_{Y_i \text{ uncensored}} \frac{\exp (\beta' X_i)}{\sum_{Y_j \geq Y_i} \exp (\beta' X_j)}$$
Each factor can be interpreted as the conditional probability that individual \( i \) dies at time \( t_i \), given the risk set \( R_i \). The first and second derivatives of the log likelihood of the model can be derived. Parameter estimates can then be obtained by maximizing \( L(\beta) \).

The log partial likelihood is given by

\[
l(\beta) = \log(L(\beta)) = \sum_{Y_i \text{ uncensored}} \{ \beta'X_i - \log \sum_{Y_j \geq Y_i} \exp(\beta'X_j) \}
\]

### 1.6.4 Frailty Models

Vaupel et al. (1979) introduced the term frailty and used it in individual survival models. Clayton (1978) promoted the model by its application to multivariate situations. Ordinary life table methods implicitly assume that the population under study is homogenous. This means that all individuals in that study are subject under the same risk (e.g., risk of death, risk of accident). Basic observation of medical statistics shows that individuals differ greatly. Thus, the study population cannot be assumed to be homogeneous but must be considered as a heterogeneous sample.

A random effect model takes into account the effects of unobserved or unobservable heterogeneity, i.e. an individual’s attitude towards health or some genotypic personal characteristics. Thus, the role of “frailty” is to include all unobservable factors acting on the individual mortality. The random effect denoted by \( Z \) is the term that describes the individual heterogeneity.

### Differential Mortality Models

#### The Multiplicative approach.

Vaupel et al. (1979) described the model as
\[ h(x|z) = zh_0(x) \]

\[ h_o(x) \] is the baseline mortality considered to be a known function of \( x \) that is to be specified. The frailty, \( Z \) is meant to quantify uncertainty associated with the hazard rate which acts in a multiplicative manner.

**The Aalen Additive Model.**

Aalen (1980;1989) described a nonparametric additive hazard model given by

\[ h(x|z) = h_o(x) + \beta Z \quad \beta > 0 \]

This model is useful in dealing with right censored survival data, especially in the presence of time-varying covariates.

**Age Shifting Model.**

\[ h'(x) = h(x + z) \]

The age shift model was proposed by Humphreys (1874). Who argued that the mortality experience of a group of impaired lives accepted for life insurance should have an increased premium rating determined by assuming that the insured’s age is higher than the real current age, hence adopting the “age shift”.

**Constant mortality model.**

\[ h'(x) = h(x) + b, b > 0 \]

Mortality increase is constant and independent of the initial age; such model is consistent with extra-mortality due to accidents (related either to occupation or to extreme sports).
TYPES OF FRAILTY MODELS

1.6.5 Frailty models without observed covariates.

This model is used when only survival data is available for the analysis, or when additional information is of no interest.

\[ h(t,Z) = Z h_o(t) \]

The non-negative random variable \( Z \) is called frailty and \( h_o(t) \) is the baseline hazard. This model is non-identifiable from survival data, since different combinations of \( h_o(t) \) and frailty distributions may produce the same marginal hazard rate \( h(t) \). The model becomes identifiable when the parametric structure of \( h_o(t) \) is fixed and \( Z \) is assumed to belong to some parametric distribution family.

1.6.6 Frailty models with observed covariates.

If covariates are known, they can be included in the analysis; however it is nearly impossible to include all factors. Therefore, the individual hazard function with frailty factor \( Z \) and covariates \( X \) is given by

\[ h(t,x|Z) = Z h_o(t) e^{\beta'x} \]

The conditional survival function is;

\[ s(t,x|Z) = e^{-Z h_o(t) e^{\beta'x}} \]

\( \beta \) is a vector of regression coefficients characterizing the measure of influence of \( X \) on the hazard rate. \( X \) and \( Z \) are assumed to be independent. However, incorporation of unobservable factors (such as frailty) into Cox PH models poses theoretical difficulties in the estimation and inference procedures (Therneau and Grambsch, 2000).
1.6.7 Univariate Frailty Models

This model describes the influence of unobserved covariates in a proportional hazards model for independent lifetimes. The variability can be split into a part that depends on observable risk factors, and is therefore theoretically predictable, and a part that is theoretically unpredictable, even when all relevant information is known. This model has been used by Hougaard (1991) to show that these two sources of variability can explain some unexpected results or gives an alternative explanation of some results.

1.6.8 Bivariate Frailty Models

Bivariate frailty models are used to analyze the effects of dependence between life spans of two related individuals with random effect. This model estimates the impact of dependence on the regression coefficients of the Cox-proportion hazard model (Clayton, 1978; Hougaard, 1995).

The bivariate survival function is given by:

\[ S(t_1, t_2 | Z) = S_1(t_1)^Z S_2(t_2)^Z \]

1.6.9 Multivariate Frailty Models

The aim for multivariate analysis is to account for the dependence in clustered event times. A natural way to model dependence of clustered event times is through the introduction of a cluster-specific random effect - the frailty. This random effect explains the dependence in the sense that had we known the frailty, the events would be independent. This approach can be used for survival times of related individuals like family members, parent-child, twins) or recurrent observations on the same person. Clayton (1978) used this approach to model statistical dependence for clustered events.
The survival function is given by:

\[
S(t_1, t_2, \ldots, t_n) = \int_0^\infty S(t_1 | z, X_1) S(t_2 | z, X_2) \ldots S(t_n | z, X_n) g(z) dz
\]

\[
S(t_1, t_2, \ldots, t_n) = L_Z(H_{0i}(t_i))
\]

### 1.6.10 Shared Frailty Models

This model was introduced by Clayton (1978) it is relevant to event times of related individuals or observations that are clustered into groups such as cities that are assumed to share the same frailty \( Z \). The survival times are assumed to be conditional independent with respect to the shared (common) frailty.

The hazard function for a shared frailty model is given by:

\[
h_{ij}(t | Z_i) = Z_i h_o(t) e^{\beta' x_{ij}} \quad Z_i \text{ is the random effect associated with the } i^{th} \text{ group}
\]

### 1.6.11 Correlated Frailty Models

In correlated frailty models, the frailty of each individual in a pair is defined by a measure of relative risk where two associated random variables are used to characterize the frailty effect for each pair. For example, one random variable is assigned for the husband and one for the wife so that they would no longer be constrained to have a common frailty. These two variables are associated and have a joint distribution. For two individuals in a pair, frailties are not necessarily the same, as they are in the shared frailty model. The hazard of individual \( j \) \((j = 1; 2) \) in pair \( i \) \((i = 1; \ldots; n) \) has the form

\[
h(t) = Z_{ij} h_{oij}(t) e^{\beta' x_{ij}}
\]

\( h_{oij}(t) \) are baseline hazard functions, and \( Z_{ij} \) are unobserved (random) effect or frailty.
Yashin et al. (1993, 1995) introduced the correlated gamma frailty model and applied to related lifetimes.

### 1.6.12 Nested Frailty Models

Nested frailty models account for hierarchical clustering of the data by including two nested random effects that act multiplicatively on the hazard function. Such models are appropriate when observations are clustered at several hierarchical levels such as in geographical areas (Rondeau et al. 2006).

The hazard function is given by:

\[
h_{ijk}(t|Z_i, U_{ij}) = Z_i U_{ij} h_0(t) e^{\beta'X_{ijk}}
\]

The cluster random effect \(Z_i\) and the sub-cluster random effect \(U_{ij}\) are both independently and identically distributed.

### 1.6.13 Joint Frailty Models

Recurrent events across time for subjects in a study may be terminated by loss to follow-up, end-of-study, or a major failure event such as death. Here, the major failure event could be correlated with the recurrent events. Joint frailty models provide a way to study the joint evolution over time of two survival processes by considering the terminal event as informative censoring (Rondeau et al. 2007).

### 1.6.14 Discrete Frailty Models

The distribution of the frailty factor is normally assumed to be continuous. In some cases, it may be appropriate to express heterogeneity as a discrete mixture. Having zero frailty can be interpreted as being immune, and population heterogeneity may be analyzed using discrete frailty models. Continuous frailty distributions do not allow having zero risks. Nickell(1979) used the binary discrete model to account for heterogeneity in
unemployment spell data. In this study however continuous frailty models will be considered.

**Frailty Distributions**

First, neither theory nor data typically provides much guidance for choosing a specific distribution from which to draw the frailty, thus any distribution with positive support and finite mean is suitable to represent the frailty distribution. However, for tractability reasons the choice of distribution is limited to those that provide a closed form expression for the frailty survivor function, density and hazard functions.

The choice of parametric distributions for $Z$ is often a matter of computational convenience and it should be strictly positive support, since negative frailty leads to negative mortality rates. Some of the distributions considered in this study are:

- Gamma distribution Vaupel et.al (1979)
- Inverse-Gaussian distribution Manton et al (1986)
- Positive Stable distribution; Hougaard (1986)
- Compound Poisson distribution (Aalen 1988, 1992)
- Inverse Gamma distribution
- Reciprocal Inverse-Gaussian distribution
- Non Central Gamma distribution

**Baseline Hazard Distributions**

Two different approaches are possible. In the parametric case the baseline hazard is chosen in the class of parametric lifetime distributions. The model also works without any specification of the baseline hazard function. However, there has been no study or
survival experiment, which restricts estimates for the parametric form of the baseline hazard. Baselines with monotone increasing hazards are often used because one is often interested in the life of a device in the period of its life when an aging process is in force. Models with monotone decreasing hazard functions are used less often but can have application in the study of early lifetimes of devices. Constant hazard functions can be used as baseline distributions to which other distributions are compared.

The baseline hazards considered in the study includes:

- The Gompertz model (1825) (Vaupel et.al. 1979)
- The Weibull distribution (Manton and Stellard, 1988),
- Exponential distribution
- Log-logistic distribution
- Lognormal distribution
- Exponential Power distribution
- Pareto distribution

1.7 Applications in Life Insurance

Frailty models are used in life insurance to represent heterogeneity in a population due to unobservable risk factors. Heterogeneity due to observable risk factors is addressed at policy issue during the underwriting process to ensure that each contract is assigned premium consistent with the insured risk. Neglecting such factors may lead to biased valuation of insurance products.
Actuaries have developed models for valuing life insurance that only consider observable risk factors. However, in general insurance models accounting for unobservable risk has been developed to explain overall claim frequency i.e. the Poisson-Gamma model.

1.7.1 Life Insurance
Life insurance contracts with benefits contingent on the lifetime of an individual and whose benefit is stated in advance is considered. Heterogeneity can be classified as emerging from observable risk factors (at issue) i.e. age, sex, health status, profession, smoke habits, sport activities, and so on. Or unobservable risk factors like an individual’s attitude towards risk.

For immediate annuities, the relation of premium and annuity depends on the health of the insured at the time the contract is taken out. However, in deferred annuities (pension schemes), the insurer has to perform some kind of underwriting at the end of the deferment period.

The underwriting process:

The purpose of underwriting is to assign each insured a frailty factor $\tilde{Z}$ as an estimate of $Z$ to determine the pricing mortality rates. These underwriting factors are observable characteristics, such as smoking status, that explain mortality heterogeneity.

Underwriting is done to ensure that premiums and benefits are fairly priced.

The tests carried out during the underwriting process are based on;

- Biological and physiological factors, such as age, gender, genotype;
- Features of the living environment; in particular: climate and pollution, nutrition standards, population density, hygienic and sanitary conditions;
• Occupation, in particular in relation to professional disabilities or exposure to injury, and educational attainment;
• Individual lifestyle, in particular with regard to nutrition, alcohol and drug consumption, smoke, physical activities and pastimes;
• Current health conditions, personal and/or family medical history, civil status, and so on.

This assessment can be performed through proper questions in the application form and as to health conditions through a medical examination.

**Modeling unobservable factors:**

In addition to observable factors, heterogeneity may be caused by unobservable individual-specific factors, referred to as frailty. Frailty models may provide an appropriate description of the age-specific mortality shape, as well as the estimate of parameters of the relevant models according to mortality observed within the portfolio. However, there is no data available that can be linked to the choice of the distribution of Z since it is unobserved

1.7.2 Pension Schemes

Let $x$ be the age at entry (time 0), $N_t$ the number of annuitants at time $t, t \geq 0$; at the valuation time $T, T \geq 0$, the number $N_T$ is known, while $N_t$ is random for $t > T$. The amount of benefit at time $t$ is denoted by $\delta_t$. Assuming a deterministic financial setting, the short interest rate at time $t$ is assumed to be deterministic, thus $\delta_t = \delta$ for any $t$.

The value at time $T$ of one monetary unit at time $t, t > T$, is $e^{-\delta(t-T)}$
In case the population is considered homogeneous the future lifetimes \( \{ T_{i(x+t)} ; i = 1, 2, \ldots, n \} \) are independent and identically distributed.

If the population is heterogeneous, then the future lifetimes are correlated through \( Z_{x+t} \) and can be assumed conditionally independent and identically distributed. This means dependence between survival times is only due to unobservable covariates or frailty.
CRITICAL LITERATURE REVIEW:

2.1 Probability Tools
Some of the common probability tools used in survival analysis that will be used in the study are described below. Let $T_x$ be the future lifetime variable i.e. the remaining duration of life of a person aged $x$, which is a positive real valued variable, having a continuous distribution with finite expectation. Several functions characterize the distribution of $T_x$:

- $f_x(t), t \geq 0$ is the probability density of $T_x$;
- $S_x(t) = P(T_x > t) = \int_t^\infty f_x(x) \, dx = 1 - F_x(t)$ is the survival function, which is sometimes denoted with $P_x$
- $F_x(t) = P(T_x \leq t)$ Expresses the probability of dying within $t$ years for a person age $x$ and is denoted by $q_x$
- $h(t) = \frac{f(t)}{S(t)} = \lim_{\delta t \to 0} \frac{P(t \leq T < t + \delta t | T \geq t)}{\delta t} = \frac{-S(t)/\partial t}{S(t)}$ is the hazard function, which represents the probability that an individual alive at $t$ experiences the event in the next period $\delta t$. (also called the instantaneous death rate)
- $H(t) = \int_0^t h(x) \, dx$ is the cumulative hazard function

2.1.1 Relationships between $f(t), h(t)$ and $s(t)$
Let $h(t)$ denote the hazard function, defined by

$h(t) = \lim_{(dt \to 0)} \frac{pr(t < T \leq t + dt \mid T > t)}{dt}$

$T$ is nonnegative and represents the future lifetime of an individual

$h(t) = \lim_{(dt \to 0)} \frac{pr(t < T \leq t + dt \mid T > t)}{prob(T > t) * dt}$
\[ h(t) = \lim_{dt \to 0} \frac{pr(t < T \leq t + dt)/dt}{prob(T > t)} \]

\[ h(t) = \frac{f(t)}{1 - F(t)} \]

\[ h(t) = \frac{f(t)}{S(t)} \]

By definition;

\[ S(t) = 1 - F(t) \]

\[ f(t) = F'(t) = -S'(t) \]

Substitute in (1) \[ h(t) = \frac{-S'(t)}{S(t)} \]

\[ h(t) = -\frac{d}{dt} \ln(S(t)) \]

\[ -\int h(t) \, dt = \ln s(t) \]

\[ S(t) = \exp(-\int h(t) \, dt) \]

\[ S(t) = \exp(-H(t)) \]

2.1.2 Laplace Transform

The Laplace transform is crucial in this study since it makes computations of the survival and hazard functions from the density function easy.

The Laplace transform of a random variable \( Z \) with density function \( f(z) \) is given by;

\[ L_Z(s) = E[e^{-sZ}] \]

\[ L_Z(s) = \int e^{-sz} f(z) \, dz \]
2.2 FRAILTY MODELS

Frailty models are extensions of the Cox-proportional hazards model. In many cases it is impossible to measure all relevant covariates related to the subject of interest, sometimes because of economical reasons or sometimes the importance of some covariates is still unknown.

The frailty approach aims to account for heterogeneity, caused by unmeasured covariates in the Cox-proportional model which is described by a mixture variable $Z$ called frailty.

The Cox-proportional model is given by;

$$h(t,x) = h_o(t) \exp(\beta'X).$$

The hazard is modified to the frailty model by substituting the relative risk $\exp(\beta'X_i)$ by a random variable $Z$ which represents the unobserved covariates $X_i$ i.e.

$$h(t,Z) = h_o(t) \times Z$$

The frailty $Z$ is then assumed to follow some distribution with positive support and has a multiplicative effect on the baseline hazard function which is common to all individuals.

2.3 THE MULTIPLICATIVE MODEL

This model describes the population as a mixture and assumes that each individual correspond a frailty quantity $Z$, describing the individual’s relative risk. The non-negative quantity $z$ encompasses all other factors affecting mortality other than age.

The hazard at age $x$ conditional on $Z$ is assumed to be $Z \times h_o(x)$

I.e. $h(x|z) = Z \times h_o(x)$ where $h_o(x)$ is the ‘standard hazard function’ corresponding to a ‘standard individual’, conventionally those with frailty $z = 1$
Individuals with $Z > 1$ experience a force of mortality that is proportionally higher than $h(x)$ at all ages. Individuals with $Z < 1$ experience proportionally lower mortality rates. $Z = 1$ correspond to the standard hazard function.

The composition of a cohort with respect to the frailty $Z$ changes as a cohort grows older because the more frail (susceptible) individuals tend to die earlier than the least frail individuals.

Due to the stochastic nature of $Z$, the random effect or frailty model is stochastic.

The survival function of an individual with frailty $Z$ is given by

$$S(t|Z) = \exp(-\int h(t|Z) \, dt)$$

$$= \exp(-\int Zh_o(t) \, dt)$$

$$S(t|Z) = \exp\{ZH_o(t)\}$$

Since, the individual model $S(t|Z)$ is not observable as each individual $Z$ is unobserved; it is ‘integrated out’ by specifying a distribution and obtaining the unconditional survival function.

The survival function of the total population is the mean of individual survival functions with respect to the frailty distribution. It can be viewed as the survival function of a randomly drawn individual, and corresponds to what can actually be observed.

Integrating over the range of frailty variable $Z$ having density $f(z)$, we get marginal survival survival function representing the population as,

$$S(t) = \int S(t|Z) \, f(Z) \, dz$$

$$S(t) = E[S(t|Z)]$$

$$S(t) = E[\exp\{-ZH_o(t)\}]$$
\[ S(t) = L_Z(H_o(t)) \]

\[ f(z) \] is the density of \( Z \) and \( L_Z(s) \) is the Laplace transform of \( Z \).

To obtain the marginal density function \( f(t) \)

Consider the relationship:

\[ h(t|Z) = \frac{f(t|Z)}{s(t|Z)} = Z h_o(t) \]

\[ f(t|Z) = Z h_o(t) S(t|Z) \]

Since, \( f(t|Z) = \frac{f(t,Z)}{f(z)} \)

\[ f(t,Z) = Z h_o(t) S(t|Z) f(z) \]

Also, \( f(t) = \int f(t,Z) f(z) \, dz \)

\[ f(t) = h_o(t) \int Z S(t|Z) f(z) \, dz \quad \text{...............}(2) \]

\[ = h_o(t) E[ZS(t|Z)] \]

\[ f(t) = - h_o(t) L'(H_o(t)) \]

**2.3.1 Model Assumptions.**

- The frailty \( Z \) has a multiplicative effect on the mortality rate of the individuals:
  \[ h(t; Z) = Z h_o(t) \]

- The frailty \( Z_x \) is stationary. i.e. the frailty of an individual keeps constant throughout the whole lifetime span (but the probability distribution does depend on the age, and this justifies the suffix \( x \))

- \( Z \) is distributed independent of age(\( x \)) or time(\( t \))

- \( Z \) has a strictly positive support since negative hazards are impossible.
2.3.2 Individual Vs Population hazards
In frailty modeling the individual hazard rate increases over time while the population hazard obtained by averaging over all the survivors decreases. This is because the population becomes populated by more and more robust individuals as the frail members fail. In a homogeneous assumption, the population hazard is the same thing since all individuals are assumed to be identical. Whereas in a heterogeneous setting, it turns out that the population hazard can fall while the individual hazards all rise.

2.4 DISCRETE FRAILTY MODELS
There are some situations in which a discrete distribution may be appropriate. For example, heterogeneity in lifetime arises because of exposure to damage on a random number of occasions. Having zero frailty can be interpreted as being immune, and population heterogeneity may be analyzed using discrete frailty models. There are two kinds of discrete frailty models in the literature. One kind of discrete frailty model is constructed by separating the frailty into ones with fixed and random numbers of mass points. The second kind of discrete frailty model is based on a fixed number of components and with masses at integers.

CONSTRUCTION
Survival functions for the frailty model with discrete distributions:

The unconditional survival function for the discrete frailty distribution is given by;

\[ S(t) = \sum_{z=0}^{\infty} S(t|Z)P(z) \]

\[ S(t) = E[S(t|Z)] \]

\[ S(t) = E[e^{-ZH(t)}] \]

Z is a discrete random variable with the probability function \( P(Z = z) = P(z) \).
Standard discrete distributions such as Geometric, Poisson and Negative Binomial distributions have been considered by Carolini et.al (2010)

### 2.4.1 GEOMETRIC DISTRIBUTION

If \( Z \) is a geometric-distributed random variable with the probability function

\[
P(z) = p (1 - p)^z, z = 0,1,2...
\]

The unconditional survival function is given by

\[
S(t) = E[e^{-ZH(t)}] 
\]

\[
S(t) = \sum_{z=0}^{\infty} e^{-ZH(t)} * p (1 - p)^z 
\]

\[
S(t) = p \sum_{z=0}^{\infty} ((1 - p)e^{-H(t)})^z 
\]

let \( S_o(t) = e^{-H(t)} \) and \( 1 - p = q \) where \( S_o(t) \) is the baseline survival function.

\[
S(t) = p \sum_{z=0}^{\infty} (qS_o(t))^z 
\]

\[
S(t) = p \{ 1 + qS_o(t) + (qS_o(t)) + (qS_o(t))^2 + (qS_o(t))^3 + \cdots \} 
\]

\[
S(t) = \frac{p}{1 - qS_o(t)} 
\]

### 2.4.2 POISSON DISTRIBUTION

If \( Z \) is a Poisson distributed random variable with parameter \( \mu > 0 \) and probability density function;

\[
P(z) = \frac{\mu^z e^{-\mu}}{Z!} \quad z = 0,1,2,.. 
\]

The unconditional survival function is given by;

\[
S(t) = E[e^{-ZH(t)}] 
\]
\[ S(t) = \sum_{Z=0}^{\infty} e^{-ZH(t)} \frac{\mu^Z e^{-\mu}}{Z!} \]

\[ S(t) = e^{-\mu} \sum_{Z=0}^{\infty} \frac{(e^{-H(t)} \mu)^Z}{Z!} \]

\[ S(t) = e^{-\mu} \cdot e^{(e^{-H(t)} \mu)} \]

\[ S(t) = e^{-\mu} \cdot e^{(S_o(t) \mu)} \]

\[ S(t) = e^{\mu(S_o(t)-1)} \]

2.4.3 NEGATIVE BINOMIAL

Let \( Z \) be a negative binomial-distributed random variable with the probability function

\[ P(z) = \binom{z-1}{k-1} p^k (1 - p)^{z-k}, \text{ } z = k, k + 1, \ldots, \text{where } k > 0 \text{ and } p > 0. \]

The unconditional survival function is given by

\[ S(t) = E[e^{-ZH(t)}] \]

\[ S(t) = \sum_{Z=0}^{\infty} e^{-ZH(t)} \cdot \binom{Z-1}{k-1} p^k (1 - p)^{z-k} \]

\[ S(t) = \left( \frac{p}{q} \right)^k \sum_{Z=0}^{\infty} \binom{Z-1}{k-1} (q e^{-H(t)})^Z \]

\[ S(t) = \left( \frac{p}{q} \right)^k \sum_{Z=0}^{\infty} \binom{Z-1}{k-1} (q S_o(t))^Z \]

Using \((1 - qS_o(t))^{-k} = \sum_{z=0}^{\infty} \binom{z-1}{k-1} (q S_o(t))^{z-k}\)

\[ S(t) = \left( \frac{p}{q} \right)^k \cdot (q S_o(t))^k \cdot (1 - qS_o(t))^{-k} \]

\[ S(t) = \left( \frac{p S_o(t)}{1 - q S_o(t)} \right)^k \]
2.4.4 BINOMIAL DISTRIBUTION

Let $Z$ be a binomial distributed random variable with the probability function

$$P(z) = \binom{n}{z} p^z (1 - p)^{n-z}, z = 0,1,2 \ldots n \text{ where } p > 0.$$ 

The unconditional survival function is given by

$$S(t) = E[e^{-ZH(t)}]$$

$$S(t) = \sum_{z=0}^{n} e^{-ZH(t)} \binom{n}{z} p^z (1 - p)^{n-z}$$

$$S(t) = \sum_{z=0}^{n} (pe^{-H(t)})^z q^{n-z}$$

$$S(t) = (q + pe^{-H(t)})^n$$

$$S(t) = (q + pS_0(t))^n \text{ where } p > 0 \text{ and } p + q = 1$$

However, in this study frailty models with continuous distribution of ‘frailty’ $Z$ will be considered. Since, with continuous frailty it is possible to capture the finest change in unobserved heterogeneity. Unlike the discrete models, continuous models do not allow having zero risks.
CHAPTER 3

CONTINUOUS FRAILTY MODELS

GAMMA MIXTURES

3.1 GAMMA FRAILTY MODEL

Vaupel et al. (1979) suggest a Gamma distribution, due to its mathematical tractability.

From a computational and analytical point of view, it fits well to failure data because it is easy to derive the closed form expressions of unconditional survival, cumulative density and hazard function. This is due to the simplicity of the Laplace transform.

Vaupel Approach (1979)

Construction

Let $Z \sim \Gamma(p, b)$

With shape parameter $p$ and scale parameter $b$. The marginal density of $Z$ is;

$$f(z) = \frac{b^p z^{p-1} e^{-bz}}{\Gamma(p)} ; z > 0, b > 0, p > 0$$

The Laplace transformation is given by;

$$L_Z(s) = \left(\frac{b}{b+s}\right)^p = \left(1 + \frac{s}{b}\right)^{-p}$$

This is required to integrate out the distribution of the unobserved frailty. Once the frailty is integrated out, accounting for unobserved heterogeneity is reduced to estimating the variance of the frailty term. $\delta^2 = \frac{1}{b}$
The mean frailty at birth is

\[
E(z) = -L'(0) = p \times \left(1 + \frac{S}{b}\right)^{-p-1} \times \frac{1}{b} \quad \text{at } s = 0
\]

\[
= \frac{p}{b}
\]

Variance; \( Var(z) = L''(0) - (L'(0))^2 \)

\[
= -p(-p - 1) \times \left(1 + \frac{S}{b}\right)^{-p-2} \times \left(\frac{1}{b}\right)^2 - \left(\frac{p}{b}\right)^2 \quad \text{at } s = 0
\]

\[
= \frac{p}{b^2}
\]

Coefficient of variation; \( cv(z) = \frac{sd}{mean} = \frac{1}{\sqrt{p}} \)

The CV shows that \( p \) plays the role of measuring, in relative terms, the level of heterogeneity in population. If \( p \to \infty \), then \( cv(z) \to 0 \), i.e. the population can be considered homogeneous; for small values of \( p \), on the contrary, the value of \( cv(z) \) is high, expressing a wide dispersion, i.e. heterogeneity in the population.

However, the coefficient of variation is constant and does not change with age. This is a unique property of the gamma distributed frailty, since other assumed forms of frailty usually exhibit a decreasing coefficient of variation, i.e. Inverse Gaussian distributed frailty.

The marginal survival function is given by;

\[
S(x) = L_Z(H_0(x))
\]

\[
S(x) = \left(1 + \frac{H_0(x)}{b}\right)^{-p}
\]

\[
f(x) = -h_0(t) \cdot L_Z'(H_0(x))
\]

\[
f(x) = h_o(t) \left(1 + \frac{H_0(x)}{b}\right)^{-p-1} \frac{p}{b}
\]
\[ h(x) = \frac{f(x)}{s(x)} = h_o(t) \left(1 + \frac{H_o(x)}{b}\right)^{-1} \frac{p}{b} \]

For purposes of identifiability assume the distribution of \( Z \) has mean normalized to one
(i.e. the standard mortality table describes an "average individual") and variance \( \delta^2 = \frac{1}{b} \).

Let \( p = b \) (i.e. one parametric gamma distribution). The hazard becomes,

\[
\begin{align*}
    h(x) &= \frac{h_o(x)}{1 + \frac{H_o(x)}{b}} \\
    h(x) &= \frac{h_o(x)b}{b + H_o(x)}
\end{align*}
\]

**CHOICE OF \( h_o(x) \)**

**3.1 Gompertz – Gamma Frailty Model**

Benjamin Gompertz (1825) idea of “exponential aging”, postulated that \( h(x) \) satisfies the simple differential equation

\[
\frac{dh(x)}{dx} = \beta h(x)
\]

Solving this

\[
\frac{dh(x)}{h(x)} = \beta dx
\]

\[
\int \frac{dh(x)}{h(x)} = \int \beta dx
\]

\[
\ln h(x) = \beta x + c
\]

\[
h(x) = \propto e^{\beta x} \text{ where } \propto = e^c
\]

In words, this implies that a person's probability of dying increases at a constant exponential rate as age increases. Gompertz' Law is often found to be quite accurate (at least as a first approximation) for ages over about 25 or 30.
Hence,

\[ h_o(x) = \alpha e^{\beta x} \alpha > 0 \] represents baseline mortality and \( \beta > 0 \) is the rate of increase of mortality with age.

\[
H_o(x) = \int_0^x h_o(x) \, dt \\
= \int_0^x \alpha e^{\beta t} \, dt \\
= \frac{\alpha}{\beta} (e^{\beta x} - 1)
\]

Using the Gompertz model the hazard becomes

\[
h(x) = \frac{\alpha b e^{\beta x}}{b + \frac{\alpha}{\beta} (e^{\beta x} - 1)}
\]

For \( b = 0.05 \) \( \alpha = 0.0008 \beta = 0.1 \) \( h(x) \) has a log-logistic shape shown below
The hazard can be re-parametized as:

\[
h(x) = \frac{\alpha be^{\beta x}}{b - \frac{\alpha}{\beta} \{1 + \frac{\alpha}{\beta} e^{\beta x}\}}
\]

\[
h(x) = \frac{1}{b - \frac{\alpha}{\beta}} \frac{\alpha be^{\beta x}}{1 + \frac{\alpha}{\beta} e^{\beta x}}
\]

let \( \alpha = \frac{\alpha b}{b - \frac{\alpha}{\beta}} \), \( \delta = \frac{\alpha}{\beta b - \alpha} \)

Thus,

\[
h(x) = \frac{\alpha e^{\beta x}}{1 + \delta e^{\beta x}}
\]

Has a logistic shape and belongs to the Perks family (1932)

### 3.1.2 Weibull-Gamma Frailty Model

Alternatively, \( h_\phi(x) \) can be chosen to follow a weibull \((\lambda, p)\) distribution with probability density function

\[
f(x) = \lambda px^{p-1} \exp(-\lambda x^p) \quad \text{where} \quad p > 0, \lambda > 0 \quad p \text{ is the shape parameter}
\]

The survival function is:

\[
s(x) = \Pr(X > x)
\]

\[
s(x) = \int_x^\infty \lambda pt^{p-1} \exp(-\lambda t^p) \, dt \quad \text{... eqn 1}
\]

let \( z = \lambda t^p \frac{dz}{dt} = \lambda pt^{p-1} \quad \text{substituting in eqn 1}

\[
s(x) = \int_{\lambda x^p}^\infty \exp(-z) \, dz
\]

\[
s(x) = \exp(-\lambda x^p)
\]

\[
h(x) = \frac{f(x)}{s(x)} = \frac{\lambda px^{p-1} \exp(-\lambda x^p)}{\exp(-\lambda x^p)}
\]
The hazard increases and if $p < 1$ the hazard decreases.

The extreme value character of the Weibull distribution makes it appropriate for the distribution of individual time to death, because there are different causes of death which compete with each other.

Using Weibull as the baseline hazard, the hazard function for gamma:

$$h(x) = \frac{h_0(x)b}{b + H_0(x)}$$

becomes

$$h(x) = \frac{\lambda px^{p-1} * b}{b + \lambda x^p}$$

Let $b = 20$ $p = 1.1$ $\lambda = 0.01$ the output is shown below.

3.1.3 Exponential-Gamma Frailty Model
A special case of the weibull distribution is the exponential distribution when the shape parameter is one ($p = 1$)

The Weibull hazard is given by

$$h_0(x) = \lambda px^{p-1}$$

Substituting $p = 1$

$$h_0(x) = \lambda$$
This is the hazard of an exponential distribution which is constant.

The cumulative hazard is given by;

\[ H_o(x) = \int_0^x h_o(t) \, dt \]
\[ = \int_0^x \lambda \, dt \]
\[ H_o(x) = \lambda x \]

Using the Exponential distribution for the baseline hazard \( h_o(x) \);

The hazard for the gamma distribution;

\[ h(x) = \frac{h_o(x) b}{b + H_o(x)} \]

becomes

\[ h(x) = \frac{\lambda b}{b + \lambda x} \]

3.1.4 Log-logistic Gamma Frailty Model
The probability density function for a log-logistic distribution is

\[ f(x) = \left( \frac{\beta}{\alpha} \right) \left( \frac{x}{\alpha} \right)^{\beta-1} \left( 1 + \left( \frac{x}{\alpha} \right)^{\beta} \right)^{-2} \]
\[ x > 0, \beta > 0, \alpha > 0 \]

\[ S(x) = Pr(X > x) \]
\[ S(x) = \int_x^{\infty} \left( \frac{\beta}{\alpha} \right) \left( \frac{u}{\alpha} \right)^{\beta-1} \left( 1 + \left( \frac{u}{\alpha} \right)^{\beta} \right)^{-2} \, du \]

let \( y = \left( \frac{u}{\alpha} \right)^{\beta} \frac{dy}{du} = \left( \frac{\beta}{\alpha} \right) \left( \frac{u}{\alpha} \right)^{\beta-1} \)
\[ S(x) = \int_{\frac{x}{\alpha}}^{\infty} \frac{dy}{(1+y)^2} \]

\[ s(x) = \frac{1}{1 + \left( \frac{x}{\alpha} \right)^\beta} \]

\[ h(x) = \frac{\left( \frac{\beta}{\alpha} \right) \left( \frac{x}{\alpha} \right)^{\beta-1}}{1 + \left( \frac{x}{\alpha} \right)^\beta} \]

Thus

\[ h_o(t) = \frac{\left( \frac{\beta}{\alpha} \right) \left( \frac{t}{\alpha} \right)^{\beta-1}}{1 + \left( \frac{t}{\alpha} \right)^\beta} \]

The cumulative hazard is given by;

\[ H_o(x) = \int_{0}^{x} \frac{\left( \frac{\beta}{\alpha} \right) \left( \frac{t}{\alpha} \right)^{\beta-1}}{1 + \left( \frac{t}{\alpha} \right)^\beta} \, dt \]

\[ H_o(x) = ln\{1 + \left( \frac{x}{\alpha} \right)^{\beta}\} \]

The hazard for the gamma distribution;

\[ h(x) = \frac{h_o(x)}{b + H_o(x)} \]

becomes

\[ h(x) = \frac{\left( \frac{\beta}{\alpha} \right) \left( \frac{x}{\alpha} \right)^{\beta-1}}{1 + \left( \frac{x}{\alpha} \right)^\beta} / \{ 1 + \frac{1}{b} ln \left( 1 + \left( \frac{x}{\alpha} \right)^\beta \right) \} \]

### 3.1.5 Log normal - Gamma Frailty Model

Using the log normal distribution with parameters \( \mu \) and \( \sigma \):

\[ W = ln(x) - N(\mu, \sigma) \]
\[ f(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} \]

\[ F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right) \]

\[ h(x) = \frac{f(x)}{1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)} \]

\[ h(x) = -\frac{d}{dx} \ln (1 - F(x)) \]

\[ H(x) = -\ln(1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)) \]

Thus

\[ h_o(x) = \frac{f(x)}{1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)} \]

The cumulative hazard is given by;

\[ H_o(x) = \int_0^x -\frac{d}{dx} \ln (1 - F(x)) \, dx \]

\[ H_o(x) = -\ln(1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)) \]

The hazard for the gamma distribution;

\[ h(x) = \frac{h_o(x) b}{b + H_o(x)} \]

becomes

\[ h(x) = \frac{\frac{\phi(x)}{1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)}}{1 - \frac{1}{b} \ln(1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right))} \]

### 3.1.6 Exponential power - Gamma Frailty Model

Using the exponential power density with survival function;
\[ S(t) = e^{1-e^{\lambda t^x}} \quad \alpha, \lambda > 0 \]

\[ h(t) = -\frac{d}{dt} \ln s(t) \]

\[ h(t) = -\frac{d}{dt} (1 - e^{\lambda t^x}) \]

\[ h_0(x) = \alpha \lambda x^{x-1} e^{\lambda x^x} \]

\[ H(t) = -\ln(S(t)) \]

\[ H(t) = -\ln(e^{1-e^{4t}}) \]

\[ H_0(x) = e^{\lambda x^x} - 1 \]

The hazard for the gamma distribution:

\[ h(x) = \frac{h_0(x) b}{b + H_0(x)} \]

becomes

\[ h(x) = \frac{\alpha \lambda x^{x-1} e^{\lambda x^x}}{1 + (e^{\lambda x^x} - 1)/b} \]

**3.1.7 Pareto - Gamma Frailty Model**

Using the Pareto distribution with survival function:

\[ S(t) = \frac{\lambda}{t} \quad \alpha > 0, \lambda > 0, t \geq \lambda \]

\[ h(t) = -\frac{d}{dt} \ln \frac{\lambda}{t} \]

\[ h(t) = \frac{t^{-1}}{t} \]

\[ h_0(t) = \frac{-}{t} \]

\[ H(t) = -\ln(S(t)) \]

\[ H_0(x) = -\ln\left(\frac{\lambda}{t}\right) \]
The hazard for the gamma distribution;

\[ h(x) = \frac{h_o(x)b}{b + H_o(x)} \]

becomes

\[ h(x) = \frac{\bar{x}}{1 - \ln(\frac{\lambda}{x})/b} \]

### INVERSE-GAUSSIAN MIXTURES

#### 3.2 INVERSE-GAUSSIAN FRAILTY MODEL

Alternative to the Gamma distribution is the Inverse Gaussian as a frailty distribution introduced by Hougaard (1984). When the inverse Gaussian is used, the variability of \( Z_x \) decreases with age which can be justified by the fact that those with low frailty keep on living.

**Hougaard Approach (1984)**

**Construction**

Let \( Z \sim IG(\mu, \lambda) \)

The probability density function of \( Z \) is

\[ f(z, \mu, \lambda) = \left(\frac{1}{2\pi \mu^3}\right)^{1/2} \exp\left\{ -\frac{(z-\mu)^2}{2\mu^2} \right\} \quad \text{for } z > 0 \quad 0 \mu > 0 \]

Substituting \( \frac{\mu^2}{\beta} \)

\[ f(z, \mu, \lambda) = \mu \left(\frac{1}{2\pi \mu^3}\right)^{1/2} \exp\left\{ -\frac{(z-\mu)^2}{2\mu^2} \right\} \quad \text{for } z > 0 \beta > 0 \mu > 0 \]

The Laplace transform is given by;

\[ L_Z(s) = \exp\left\{ -\frac{\mu}{\beta} [(1 + 2\beta s)^{1/2} - 1] \right\} \]
Mean = \(-L'_z(0) = \mu\)

Variance = \(L''_z(0) - \mu^2\) = \(\mu\beta\)

Coefficient of Variation = \(\frac{\sqrt{\beta}}{\mu}\)

For identifiability reasons the mean is normalized to one. i.e. \(\mu = 1\) thus the variance \(\delta^2 = \beta\)

The Laplace transform becomes

\[ L_z(s) = \exp\left[\frac{1-(1+2s\delta^2)^{1/2}}{\delta^2}\right] \]

The marginal survival function is given by;

\[ S(x) = L_Z(H_0(x)) \]

\[ = \exp\left[\frac{1-(1+2H_0(x)\delta^2)^{1/2}}{\delta^2}\right] \]

\[ f(x) = -h_0(x) L'_Z(H_0(x)) \]

\[ = \frac{h_0(x)}{(1+2H_0(x)\delta^2)^{1/2}} \exp\left[\frac{1-(1+2H_0(x)\delta^2)^{1/2}}{\delta^2}\right] \]

\[ h(x) = \frac{f(x)}{s(x)} = \frac{h_0(x)}{(1 + 2H_0(x)\delta^2)^{1/2}} \]

**CHOICE OF \(h_o(x)\)**

**3.2.1 Gompertz – Inverse Gaussian Frailty Model**

Using Gompertz assumption for the baseline mortality \(h_o(x) = \alpha e^{\beta x}\)

The Inverse Gaussian hazard becomes
\[ h(x) = \frac{\alpha e^{\beta x}}{(1 + 2\frac{\alpha}{\beta}(e^{\beta x} - 1)\delta^2)^{1/2}} \]

I.G-Gompertz hazard

3.2.2 Weibull-Inverse Gaussian Frailty Model
Using Weibull distribution for \( h_0(t) \)

\[ h_o(t) = \lambda px^{p-1} \]
\[ H_o(x) = \lambda x^p \]
\[ h(x) = \frac{h_o(x)}{(1 + 2H_0(x)\delta^2)^{1/2}} \]

becomes,

\[ h(x) = \frac{\lambda px^{p-1}}{(1 + 2\lambda x^p\delta^2)^{1/2}} \]
### 3.2.3 Exponential Inverse-Gaussian Frailty Model
Using the Exponential distribution for the baseline hazard \( h_o(t) \);

\[
h_o(t) = \lambda
\]

The cumulative hazard

\[
H_o(t) = \lambda t
\]

The hazard for the inverse Gaussian distribution;

\[
h(t) = \frac{h_o(t)}{(1 + 2H_o(t)\delta^2)^{1/2}}
\]

becomes,

\[
h(t) = \frac{\lambda}{(1 + 2\lambda t\delta^2)^{1/2}}
\]

### 3.2.4 Log-logistic Inverse-Gaussian Frailty model
Using the Log-logistic distribution for the baseline hazard \( h_o(t) \);

\[
h_o(t) = \left(\frac{\beta}{\alpha}\right) \left(\frac{t}{\alpha}\right)^{\beta-1} \frac{1}{1 + (\frac{t}{\alpha})^\beta}
\]

The cumulative hazard

\[
H_o(x) = ln\{1 + \left(\frac{x}{\alpha}\right)^\beta\}
\]

The hazard for the inverse Gaussian distribution;

\[
h(t) = \frac{h_o(t)}{(1 + 2H_o(t)\delta^2)^{1/2}}
\]

becomes,
\[
\begin{align*}
\frac{(\frac{\beta}{\alpha}) (\frac{t}{\alpha})^{\beta-1}}{1 + (\frac{t}{\alpha})^\beta} & \quad \frac{(1 + 2 \ln(1 + (\frac{t}{\alpha})^\beta) \delta^2)^{1/2}}{
\end{align*}
\]

3.2.5 Log normal – Inverse Gaussian Frailty Model
Using the Log normal distribution for the baseline hazard \( h_o(t) \);

\[
\begin{align*}
h_o(x) &= \frac{h_o(t)}{1 - \Phi(\ln x - \mu)} \\
\end{align*}
\]

The cumulative hazard is given by;

\[
\begin{align*}
H_o(x) &= -\ln(1 - \Phi(\ln x - \mu)) \\
\end{align*}
\]

The hazard for the inverse Gaussian distribution;

\[
\begin{align*}
h(t) &= \frac{h_o(t)}{1 + 2H_o(t)\delta^2)^{1/2}} \\
\end{align*}
\]

becomes,

\[
\begin{align*}
h(t) &= \frac{f(x)}{1 - \Phi(\ln x - \mu)} \\
& \quad \frac{(1 - 2\ln(1 - \Phi(\ln x - \mu))\delta^2)^{1/2}}{
\end{align*}
\]

3.2.6 Exponential power – Inverse Gaussian Frailty Model
Using the exponential power distribution for the baseline hazard \( h_o(t) \);

\[
\begin{align*}
h_o(x) &= \alpha \lambda^{x-1} e^{\lambda x} \\
\end{align*}
\]

The cumulative hazard is given by;

\[
\begin{align*}
H_o(x) &= e^{\lambda x} - 1 \\
\end{align*}
\]

The hazard for the inverse Gaussian distribution;
\[ h(t) = \frac{h_o(t)}{(1 + 2H_o(t)\delta^2)^{1/2}} \]

becomes,

\[ h(t) = \frac{\alpha t^{\alpha-1} e^{\lambda t^\alpha}}{(1 + 2(e^{\lambda t^\alpha} - 1)\delta^2)^{1/2}} \]

**3.2.7 Pareto – Inverse Gaussian Frailty Model**

Using the Pareto distribution for the baseline hazard \( h_o(t) \);

\[ h_o(t) = -\frac{1}{t} \]

The cumulative hazard is given by;

\[ H_o(t) = -\ln\left(\frac{\lambda}{t}\right) \]

The hazard for the inverse Gaussian distribution;

\[ h(t) = \frac{h_o(t)}{(1 + 2H_o(t)\delta^2)^{1/2}} \]

becomes,

\[ h(t) = \frac{\bar{t}}{(1 - 2\ln(\frac{\lambda}{\bar{t}})\delta^2)^{1/2}} \]

**POWER VARIANCE FUNCTIONS**

**3.3 POWER VARIANCE FUNCTION FRAILTY MODEL**

Tweedy (1984) suggested the family of power variance functions that includes the Gamma, Inverse Gaussian and Positive stable distributions and later derived independently by Hougaard (1986).
Tweedie Approach (1984)

Construction

The PVF model is a three parameter family denoted by \( PVF(r, k, \lambda) \).

The Laplace transform is

\[
L(s) = e^{-\frac{k}{r}(\lambda + s)^r - \lambda^r}
\]

The marginal survival function;

\[
S(x) = L_Z(H_0(x)) = e^{-\frac{k}{r}(\lambda + H_0(x))^r - \lambda^r}
\]

\[
f(x) = -h_o(x) L'_Z(H_0(x)) = h_o(x)k(\lambda + H_0(x))^{r-1}e^{-\frac{k}{r}(\lambda + H_0(x))^r - \lambda^r}
\]

\[
h(x) = \frac{f(x)}{s(x)} = h_o(x)k(\lambda + H_0(x))^{r-1}
\]

For identifiability the mean is normalized to one i.e. \( E[Z] = k\lambda^{r-1} = 1 \) this implies that

\[
\text{Var}[Z] = \delta^2 = k(1 - r)\lambda^{r-2} = \frac{1-r}{\lambda}
\]

The resulting hazard becomes;

\[
h(t) = \frac{h_o(t)}{(1 + \frac{\delta^2}{1-r}H_o(t))^{1-r}}
\]

SPECIAL CASES

Case 1

For \( r = 0 \), the hazard
\[ h(x) = \frac{h_o(t)}{(1 + \delta^2 H_o(t))^{1-r}} \]

Becomes

\[ h(x) = \frac{h_o(t)}{(1 + \delta^2 H_o(t))} \]

This is the hazard function for the gamma \( \Gamma(k, \lambda) \) distribution.

**Case 2**

For \( r = 0.5 \),

\[ h(x) = \frac{h_o(t)}{(1 + \delta^2 H_o(t))^{1-r}} \]

Becomes

\[ h(t) = \frac{h_o(t)}{(1 + 2 \cdot \delta^2 H_o(t))^{1/2}} \]

This is the hazard function for the inverse Gaussian distribution.

**Case 3**

For \( r = -1 \),

\[ h(x) = \frac{h_o(t)}{(1 + \delta^2 H_o(t))^{1-r}} \]

Becomes

\[ h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} \cdot \delta^2 H_o(t))^2} \]

This is the hazard function for the Non-central Gamma distribution with shape parameter zero.
**Case 4**

When $\lambda = 0$ ,

$$h(x) = h_0(x) k(\lambda + H_0(x))^{r-1}$$

Becomes

$$h(x) = h_0(x) k(H_0(x))^{r-1}$$

This is the hazard function for the positive stable distribution.

**POSITIVE STABLE MIXTURES**

A random variable $Z$ is said to have a stable distribution if it has the property that a linear combination of two independent copies of the variable has the same distribution.

**3.4 POSITIVE STABLE FRAILTY MODEL**

Hougaard (1986) introduced the Positive Stable model as a frailty distribution. Despite the fact that no closed form expressions exist for the probability density or the survival function, the Laplace transform has a very simple form.

**Hougaard Approach (1986)**

**Construction**

The density function of positive stable law can be represented using infinite series expansion as.

$$f(z, k, r) = -\frac{1}{\pi z} \sum_{c=1}^{\infty} \frac{\Gamma(cr + 1)}{c!} (-kz^{-r})^{c} \sin(rc\pi)$$

This distribution has an infinite mean.
The Laplace transform is a special case of the $PVF(r, k, \lambda)$ Laplace

$$L(s) = e^{-\frac{k}{r}(\lambda s)^r - \lambda r}$$

When $\lambda = 0$

$$L_Z(s) = e^{-ks^r}$$

For identifiability reasons let $k = r$

$$L_Z(s) = e^{-sr} \quad 0 < r < 1$$

The marginal survival function;

$$S(x) = L_Z(H_0(x)) = e^{-H_0(x)^r}$$

$$f(x) = -h_0(x)L'_Z(H_0(x)) = rh_0(x)H_0(x)^{r-1} e^{-H_0(x)^r}$$

The hazard function is

$$h(x) = \frac{f(x)}{S(x)} = rh_0(x)H_0(x)^{r-1}$$

The positive stable distribution is the only frailty distribution which preserves the proportional hazards assumption in the unconditional hazards after integrating out the frailty.

**CHOICE OF $h_0(x)$**

3.4.1 Gompertz - Positive Stable Frailty Model

Using Gompertz assumption for the baseline mortality $h_0(x) = \alpha e^{\beta x}$
The hazard function for Positive Stable

\[ h(x) = rh_0(x)H_0(x)^{r-1} \]

Becomes,

\[ h(x) = r \propto e^{\beta x} \left\{ \frac{\lambda}{\beta} \left( e^{\beta x} - 1 \right) \right\}^{r-1} \]

The power variance family contains members whose relative frailty distribution in survivors becomes less homogeneous with time.

### 3.4.2 Weibull-Positive Stable Frailty Model

Using Weibull distribution for \( h_o(t) \)

\[ h_o(t) = \lambda px^{p-1} \]

\[ H_o(x) = \lambda x^p \]
The hazard for the positive stable distribution;

\[ h(x) = rh_0(x)H_0(x)^{r-1} \]

becomes,

\[ h(x) = r\lambda x^{p-1}(\lambda x)^{r-1} \]

3.4.3 Exponential-Positive Stable Frailty Model
Using the Exponential distribution for \( h_o(t) \);

\[ h_o(t) = \lambda \]

The cumulative hazard

\[ H_o(x) = \lambda x \]

The hazard for the positive stable distribution;

\[ h(x) = rh_0(x)H_0(x)^{r-1} \]

becomes,

\[ h(x) = r\lambda (\lambda x)^{r-1} \]

3.4.4 Log-logistic Positive Stable Frailty Model
Using the Log-logistic distribution for the baseline hazard \( h_o(t) \);

\[ h_o(t) = \left( \frac{\beta}{\alpha} \right) \left( \frac{t}{\alpha} \right)^{\beta-1} \cdot \frac{1}{1 + \left( \frac{t}{\alpha} \right)^{\beta}} \]

The cumulative hazard

\[ H_o(x) = \ln\{1 + \left( \frac{x}{\alpha} \right)^{\beta} \} \]

The hazard for the positive stable distribution;

\[ h(x) = rh_0(x)H_0(x)^{r-1} \]
becomes,
\[ h(x) = r \frac{\left( \frac{\beta}{\alpha} \right) \left( \frac{t}{\alpha} \right)^{\beta-1}}{1 + \left( \frac{t}{\alpha} \right)^\beta} \ln\{1 + \left( \frac{x}{\alpha} \right)^\beta\}^{r^{-1}} \]

3.4.5 Lognormal-Positive Stable Frailty Model

Using the Log normal distribution for the baseline hazard \( h_o(t) \);

\[ h_o(x) = \frac{f(x)}{1 - \Phi\left( \frac{\ln x - \mu}{\sigma} \right)} \]

The cumulative hazard is given by;

\[ H_o(x) = -\ln(1 - \Phi\left( \frac{\ln x - \mu}{\sigma} \right)) \]

The hazard for the positive stable distribution;

\[ h(x) = r h_o(x) H_0(x)^{r^{-1}} \]

becomes,

\[ h(x) = r \frac{f(x)}{1 - \Phi\left( \frac{\ln x - \mu}{\sigma} \right)} \left\{ -\ln(1 - \Phi\left( \frac{\ln x - \mu}{\sigma} \right)) \right\}^{r^{-1}} \]

3.4.6 Exponential power – Positive Stable Frailty Model

Using the exponential power distribution for the baseline hazard \( h_o(t) \);

\[ h_o(x) = \alpha \lambda t^{\alpha-1} e^{\lambda t^\alpha} \]

The cumulative hazard is given by;

\[ H_o(x) = e^{\lambda t^\alpha} - 1 \]

The hazard for the positive stable distribution;

\[ h(x) = r h_o(x) H_0(x)^{r^{-1}} \]
becomes,

\[ h(x) = r \propto \lambda t^{\infty-1} e^{\lambda t^\infty} * (e^{\lambda t^\infty} - 1)^{r-1} \]

### 3.4.7 Pareto – Positive Stable Frailty Model

Using the Pareto distribution for the baseline hazard \( h_0(t) \);

\[ h_0(t) = \frac{r}{t} \]

The cumulative hazard is given by;

\[ H_0(t) = -\ln(\frac{\lambda}{t}) \]

The hazard for the positive stable distribution;

\[ h(x) = r h_0(x) H_0(x)^{r-1} \]

becomes,

\[ h(x) = \frac{r}{t} (-\ln (\frac{\lambda}{t}))^{r-1} \]

### COMPOUND POISSON MIXTURES

#### 3.5 COMPOUND POISSON FRAILTY MODEL

The compound Poisson distribution was introduced by Aalen (1988, 1992) as a frailty distribution.

**Aalen (1988, 1992) approach**

**Construction**

Using the Laplace transform obtained and \( x \)'s \( \sim \) \( \text{gamma}(k, \lambda) \)

\[ L_Z(s) = e^{p(l_\lambda(s) - 1)} \]
\[= e^{p \left( \left(1 + \frac{s}{\lambda} \right)^{-k} - 1 \right)}\]

by reparametization substitute \( p = \frac{-k \lambda^r}{r} \) and \( k = -r \)

\[L_Z(s) = e^{-\frac{k \lambda^r}{r} \left( \left(1 + \frac{s}{\lambda} \right)^r - 1 \right)}\]

\[L_Z(s) = e^{-\frac{k}{r} \left( (\lambda + s)^r - \lambda^r \right)}\]

For \( r \geq 0 \), the power variance function distribution (PVF) is obtained.

For \( r < 0 \) the compound Poisson distribution is obtained, these two subclasses are separated by the gamma distribution \((r = 0)\).

For identifiability assume the mean frailty is normalized to one.

**Mean**

\[L_Z'(s) = k(\lambda + s)^{r-1} e^{-\frac{k}{r} \left( (\lambda + s)^r - \lambda^r \right)} \quad @s=0\]

\[L_Z'(0) = k \lambda^{r-1} = 1\]

**Variance**

\[L_Z''(s) - (L_Z'(0))^2\]

\[K(r - 1) (\lambda + s)^{r-2} e^{-\frac{k}{r} \left( (\lambda + s)^r - \lambda^r \right)} + (k(\lambda + s)^{r-1})^{-2} e^{-\frac{k}{r} \left( (\lambda + s)^r - \lambda^r \right)} - (k \lambda^{r-1})^2 \quad @s = 0\]

\[\delta^2 = K(r - 1) \lambda^{r-2}\]

\[\delta^2 = \frac{(r - 1)}{\lambda}\]
The marginal survival function is given by

\[ S(x) = L_Z(H_0(x)) \]

\[ S(x) = e^{-\frac{k}{e^{\left(\lambda + H_0(x)\right)^r}}(\lambda - \lambda^r)} \]

\[ f(x) = -h_0(t) \; L_Z'(H_0(x)) \]

\[ f(x) = kh_0(t)(\lambda + H_0(x))^{r-1}e^{-\frac{k}{e^{\left(\lambda + H_0(x)\right)^r}}(\lambda - \lambda^r)} \]

\[ h(x) = \frac{f(x)}{s(x)} = kh_0(t)(\lambda + H_0(x))^{r-1} \]

\[ h(x) = h_0(t) \left( 1 + \frac{\delta^2}{r-1}H_0(x) \right)^{r-1} \]

**CHOICE OF \( h_o(x) \)**

3.5.1 **Gompertz –Compound Poisson Frailty Model**

Using Gompertz assumption for the baseline mortality

\[ h_0(x) = \alpha e^{\beta x} \]

\[ H_0(x) = \frac{\alpha}{\beta} \left( e^{\beta x} - 1 \right) \]

The hazard

\[ h(x) = h_0(t) \left( 1 + \frac{\delta^2}{r-1}H_0(x) \right)^{r-1} \]

becomes,

\[ h(x) = \alpha e^{\beta x} \left( 1 + \frac{\delta^2}{r-1} \frac{\alpha}{\beta} \left( e^{\beta x} - 1 \right) \right)^{r-1} \]

RCODE

54
Using r=1.1

gcp=ho*\((1+(0.05/0.1)*Ho)^{0.1}\)

\[
\text{plot}(x, \text{gcp}, \text{main}="\text{CompoundPoisson-Gompertz}\), \text{type}="\text{o}\), \text{xlab}="\text{age}\), \text{ylab}="\text{hazard}\)
\]

3.5.2 Weibull – Compound Poisson Frailty Model
Using Weibull distribution for \(h_o(t)\)

\[
h_o(t) = \lambda px^{p-1}
\]

\[
Ho(x) = \lambda x^p
\]

The hazard

\[
h(x) = h_o(t) \left(1 + \frac{\delta^2}{r-1} H_0(x)\right)^{r-1}
\]
Becomes,

\[ h(x) = \lambda px^{p-1} \left( 1 + \frac{\delta^2}{r-1} \lambda x^p \right)^{r-1} \]

### 3.5.3 Exponential–Compound Poisson Frailty Model

Using the Exponential distribution for \( h_o(t) \);

\[ h_o(t) = \lambda \]

The cumulative hazard

\[ H_o(x) = \lambda x \]

The hazard

\[ h(x) = h_o(t) \left( 1 + \frac{\delta^2}{r-1} H_o(x) \right)^{r-1} \]

Becomes

\[ h(x) = \lambda \left( 1 + \frac{\delta^2}{r-1} \lambda x \right)^{r-1} \]

### 3.5.4 Log-logistic Compound Poisson Frailty model

Using the Log-logistic distribution for the baseline hazard \( h_o(t) \);

\[ h_o(t) = \left( \frac{\beta}{\alpha} \right) \left( \frac{t}{\alpha} \right)^{\beta-1} \frac{1}{1 + \left( \frac{t}{\alpha} \right)^\beta} \]

The cumulative hazard

\[ H_o(x) = \ln \{ 1 + \left( \frac{x}{\alpha} \right)^\beta \} \]
The hazard

\[ h(x) = h_o(t) \left( 1 + \frac{\delta^2}{r-1} H_0(x) \right)^{r-1} \]

Becomes

\[ h(x) = \left( \frac{\beta}{\alpha} \frac{\alpha}{\beta} \right)^{\frac{\beta-1}{\beta}} \left( 1 + \frac{\delta^2}{r-1} \ln \{ 1 + \left( \frac{x}{\alpha} \right)^{\beta} \} \right)^{r-1} \]

### 3.5.5 Lognormal-Compound Poisson Frailty Model

Using the Log normal distribution for the baseline hazard \( h_o(t) \);

\[ h_o(x) = \frac{f(x)}{1 - \Phi \left( \frac{\ln x - \mu}{\sigma} \right)} \]

The cumulative hazard is given by;

\[ H_o(x) = -\ln \left( 1 - \Phi \left( \frac{\ln x - \mu}{\sigma} \right) \right) \]

The hazard

\[ h(x) = h_o(t) \left( 1 + \frac{\delta^2}{r-1} H_0(x) \right)^{r-1} \]

Becomes

\[ h(x) = h_o(t) \left( 1 - \frac{\delta^2}{r-1} \ln \left( 1 - \Phi \left( \frac{\ln x - \mu}{\sigma} \right) \right) \right)^{r-1} \]

### 3.5.6 Exponential Power – Compound Poisson Frailty Model

Using the exponential power distribution for the baseline hazard \( h_o(t) \);

\[ h_o(x) = \propto \lambda t^{\propto -1} e^{\lambda t^\propto} \]

The cumulative hazard is given by;
\[ H_o(x) = e^{\lambda t^\kappa} - 1 \]

The hazard

\[ h(x) = h_o(t) \left( 1 + \frac{\delta^2}{r-1} H_0(x) \right)^{r-1} \]

Becomes

\[ h(x) = \propto \lambda t^{\kappa-1} e^{\lambda t^\kappa} \left( 1 + \frac{\delta^2}{r-1} e^{\lambda t^\kappa} - 1 \right)^{r-1} \]

3.5.7 Pareto – Compound Poisson Frailty Model

Using the Pareto distribution for the baseline hazard \( h_o(t) \);

\[ h_o(t) = \frac{1}{t} \]

The cumulative hazard is given by;

\[ H_o(t) = -\ln\left(\frac{\lambda}{t}\right) \]

The hazard

\[ h(x) = h_o(t) \left( 1 + \frac{\delta^2}{r-1} H_0(x) \right)^{r-1} \]

Becomes

\[ h(x) = \frac{1}{t} \left( 1 - \frac{\delta^2}{r-1} \ln\left(\frac{\lambda}{t}\right) \right)^{r-1} \]

LOG-NORMAL MIXTURES

A log-normal distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed.
3.6 LOG-NORMAL FRAILTY MODEL

McGilchrist and Aisbett (1991) used the Log-normal frailty mixture to model multivariate dependence structures.

**McGilchrist and Aisbett (1991) approach**

**Construction**

Assuming normally distributed random effect $W$ with $E[W] = 0$ and frailty $Z = e^W$

The hazard is given by

$$h(t) = h_0(t)e^W \quad W \sim N(0, \delta^2)$$

If $Z = e^W$ then $Z \sim \text{lognormal}(0, \delta)$

The probability density function of $Z$ is given by

$$f(z) = \frac{1}{z\delta\sqrt{2\pi}} e^{-\frac{(\ln z)^2}{2\delta^2}} \quad z > 0$$

With scale parameter $\delta > 0$

$$E(Z) = \exp(\delta^2/2) \quad Var(Z) = \exp(2\delta^2) - \exp(\delta^2)$$

The Laplace transform has no explicit form but can be approximated i.e. using the LambertW function

$$L_Z(s) \approx \frac{1}{\sqrt{LW(s\delta^2 e^W) + 2LW(s\delta^2 e^W)}} e^{-\frac{LW^2(s\delta^2 e^W) + 2LW(s\delta^2 e^W)}{2\delta^2}}$$

Using the notation $L_Z(s)$ for this approximation, the marginal survival function is given by
\( S(x) = \mathcal{L}_Z(H_o(x)) \)

\[
S(x) \approx \frac{1}{\sqrt{LW(H_o(x)\delta^2 e^\nu)}} e^{-\frac{LW^2(H_o(x)\delta^2 e^\nu)+2LW(H_o(x)\delta^2 e^\nu)}{2\beta^2}}
\]

\[ f(x) = -S'(x) = -\mathcal{L}_Z'(H_o(x)) \]

\[ h(x) = \frac{f(x)}{s(x)} \]

\[ h(x) = \frac{-\mathcal{L}_Z'(H_o(x))}{\mathcal{L}_Z(H_o(x))} \]

**CHOICE OF \( h_o(x) \)**

**3.6.1 Gompertz –Log-normal Frailty Model**

Using Gompertz assumption for the baseline mortality

\[ h_o(x) = \alpha e^{\beta x} \]

\[ H_o(x) = \frac{\alpha}{\beta} (e^{\beta x} - 1) \]

The hazard

\[ h(x) = \frac{-\mathcal{L}_Z'(H_o(x))}{\mathcal{L}_Z(H_o(x))} \]

Becomes

\[ h(x) = \frac{-\mathcal{L}_Z'\left(\frac{\alpha}{\beta}(e^{\beta x} - 1)\right)}{\mathcal{L}_Z\left(\frac{\alpha}{\beta}(e^{\beta x} - 1)\right)} \]
3.6.2 Weibull – Log-normal Frailty Model
Using Weibull distribution for $h_o(t)$

$$h_o(t) = \lambda px^{p-1}$$
$$H_o(x) = \lambda x^p$$

The hazard

$$h(x) = \frac{-L_Z'(H_o(x))}{L_Z(H_o(x))}$$

Becomes

$$h(x) = \frac{-L_Z'(\lambda x^p)}{L_Z(\lambda x^p)}$$

This model has been used by Damgaard et.al (2002) for sire evaluation of longevity with and without genetic interpretations.

3.6.3 Exponential – Log-normal Frailty Model
Using the Exponential distribution for $h_o(t)$;

$$h_o(t) = \lambda$$

The cumulative hazard

$$H_o(x) = \lambda x$$

The hazard

$$h(x) = \frac{-L_Z'(H_o(x))}{L_Z(H_o(x))}$$

Becomes
\[ h(x) = \frac{-L_Z'(\lambda x)}{L_Z(\lambda x)} \]

### 3.6.3 Log-logistic – Log-normal Frailty Model

Using the Log-logistic distribution for the baseline hazard \( h_o(t) \);

\[ h_o(t) = \frac{\beta}{\alpha} \left( \frac{t}{\alpha} \right)^{\beta-1} \] 

\[ 1 + \left( \frac{t}{\alpha} \right)^{\beta} \]

The cumulative hazard

\[ H_o(x) = \ln\{1 + \left( \frac{x}{\alpha} \right)^{\beta} \} \]

The hazard

\[ h(x) = \frac{-L_Z'(H_o(x))}{L_Z(H_o(x))} \]

Becomes

\[ h(x) = \frac{-L_Z' \left( \ln\{1 + \left( \frac{x}{\alpha} \right)^{\beta} \} \right)}{L_Z \left( \ln\{1 + \left( \frac{x}{\alpha} \right)^{\beta} \} \right)} \]

### OTHER FRAILTY MIXTURES

### RECIPROCAL INVERSE GAUSSIAN MIXTURES

#### 3.7 RECIPROCAL INVERSE-GAUSSIAN FRAILTY MODEL

Given that \( Z = \frac{1}{x} \) where \( X \sim IG(\mu, \gamma) \) then \( Z \) is said to be the reciprocal of the IG distribution. The probability density function is given by;
\[(z) = \left(\frac{1}{2\pi z}\right)^2\text{exp}\left\{-\frac{\mu^2}{2z}(1 - \frac{z}{\mu})^2\right\}\]

is the shape parameter and \(\mu\) is the location parameter.

The Laplace transform is given by

\[L_z(s) = \left(1 + \frac{2s}{\mu}\right)^{-1/2}\text{exp}\left\{\mu \left[1 - \left(1 + \frac{2s}{\mu}\right)^{\frac{1}{2}}\right]\right\}\]

For identifiability the mean is normalized to one i.e. \(E[Z] = \mu = 1\)

The Laplace becomes

\[L_z(s) = \left(1 + \frac{2s}{\mu}\right)^{-1/2}\text{exp}\left\{\left[1 - \left(1 + \frac{2s}{\mu}\right)^{\frac{1}{2}}\right]\right\}\]

The marginal survival function is given by

\[S(x) = L_z(H_0(x))\]

\[S(x) = \left(1 + \frac{2H_0(x)}{\mu}\right)^{-1/2}\text{exp}\left\{\left[1 - \left(1 + \frac{2H_0(x)}{\mu}\right)^{\frac{1}{2}}\right]\right\}\]

\[f(x) = -h_o(x)\ L_z'(H_0(x))\]

\[f(x) = \left(\frac{h_o(x)}{S(x)}\right) \left(1 + \frac{2H_0(x)}{\mu}\right)^{-\frac{3}{2}} + h_o(x) \left(1 + \frac{2H_0(x)}{\mu}\right)^{-1}\text{exp}\left\{\left[1 - \left(1 + \frac{2H_0(x)}{\mu}\right)^{\frac{1}{2}}\right]\right\}\]

\[h(x) = \frac{f(x)}{S(x)}\]

\[h(x) = \frac{h_o(x)}{S(x)}\left(1 + \frac{2H_0(x)}{\mu}\right)^{-1} + h_o(x) \left(1 + \frac{2H_0(x)}{\mu}\right)^{-\frac{1}{2}}\]
CHOICE OF $h_o(x)$

3.7.1 Gompertz – Reciprocal Inverse-Gaussian Model

Using Gompertz assumption for the baseline mortality

$$h_o(x) = \alpha e^{\beta x}$$

$$H_o(x) = \frac{\alpha}{\beta} (e^{\beta x} - 1)$$

The hazard

$$h(x) = \frac{h_o(x)}{1 + 2H_o(x)} + h_o(x) \left( 1 + \frac{2H_o(x)}{1 + 2H_o(x)} \right)^{-\frac{1}{2}}$$

Becomes

$$h(x) = \frac{\alpha e^{\beta x}}{1 + \frac{2\alpha}{\beta} (e^{\beta x} - 1)} + \alpha e^{\beta x} \left( 1 + \frac{2\alpha}{\beta} (e^{\beta x} - 1) \right)^{-\frac{1}{2}}$$

GRAPH
3.7.2 Weibull – Reciprocal Inverse-Gaussian Model

Using Weibull distribution for \( h_o(t) \)

\[
    h_o(t) = \lambda px^{p-1}
\]

\[
    H_0(x) = \lambda x^p
\]

The hazard

\[
    h(x) = \frac{h_o(x)}{1 + \frac{2H_0(x)}{x}} + h_o(x) \left(1 + \frac{2H_0(x)}{x}\right)^{-\frac{1}{2}}
\]

Becomes

\[
    h(x) = \frac{\lambda px^{p-1}}{1 + \frac{2\lambda x^p}{x}} + \lambda px^{p-1} \left(1 + \frac{2\lambda x^p}{x}\right)^{-\frac{1}{2}}
\]
3.7.3 Exponential – Reciprocal Inverse-Gaussian Model

Using the Exponential distribution for \( h_o(t) \);

\[ h_o(t) = \lambda \]

The cumulative hazard

\[ H_o(x) = \lambda x \]

The hazard function

\[ h(x) = \frac{h_o(x)}{1 + \frac{2H_o(x)}{1}} + h_o(x) \left( 1 + \frac{2H_o(x)}{1} \right)^{-\frac{1}{2}} \]

Becomes

\[ h(x) = \frac{\lambda}{1 + \frac{2\lambda x}{1}} + \lambda \left( 1 + \frac{2\lambda x}{1} \right)^{-\frac{1}{2}} \]

3.7.4 Log-logistic - Reciprocal Inverse-Gaussian Model

Using the Log-logistic distribution for the baseline hazard \( h_o(t) \);

\[ h_o(t) = \frac{\beta}{\alpha} \left( \frac{t}{\alpha} \right)^{-1} \]

The cumulative hazard

\[ H_o(x) = \ln\left( 1 + \left( \frac{x}{\alpha} \right)^\beta \right) \]

The hazard function

\[ h(x) = \frac{h_o(t)}{1 + \frac{2H_o(x)}{1}} + 1 \cdot \left( 1 + \frac{2H_o(x)}{1} \right)^{-\frac{1}{2}} \]

Becomes
\[ h(x) = \frac{1}{1 + \left( \frac{t}{\alpha} \right)^{\beta}} \left[ 1 + \frac{2\ln(1 + \left( \frac{x}{\alpha} \right)^{\beta})}{1 + \left( \frac{t}{\alpha} \right)^{\beta}} \right]^{-1} + \sqrt{1 + \left( \frac{t}{\alpha} \right)^{\beta}} \]

### 3.7.5 Lognormal - Reciprocal Inverse-Gaussian Model

Using the Log normal distribution for the baseline hazard \( h_0(t) \):

\[ h_o(x) = \frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})} \]

The cumulative hazard is given by;

\[ H_o(x) = -\ln(1 - \Phi(\frac{lnx - \mu}{\sigma})) \]

The hazard function

\[ h(x) = \frac{h_o(t) \times \left[ 1 + \frac{2H_o(x)}{1 + \left( \frac{t}{\alpha} \right)^{\beta}} \right]^{-1} + \sqrt{1 + \left( \frac{t}{\alpha} \right)^{\beta}}}{1 + \frac{2H_o(x)}{1 + \left( \frac{t}{\alpha} \right)^{\beta}}} \]

Becomes

\[ h(x) = \frac{f(x)}{1 - \Phi(\frac{lnx - \mu}{\sigma})} \times \left[ 1 - \frac{2\ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))}{1 - \left( \frac{t}{\alpha} \right)^{\beta}} \right]^{-1} + \sqrt{1 - \frac{2\ln(1 - \Phi(\frac{lnx - \mu}{\sigma}))}{1 - \left( \frac{t}{\alpha} \right)^{\beta}}} \]

### 3.7.6 Exponential power - Reciprocal Inverse-Gaussian Model

Using the exponential power distribution for the baseline hazard \( h_0(t) \):

\[ h_o(x) = \alpha \lambda t^{\alpha-1} e^{\lambda t^\alpha} \]

The cumulative hazard is given by;

\[ H_o(x) = e^{\lambda t^\alpha} - 1 \]

The hazard
\[ h(x) = \frac{h_o(t)}{t} \left( 1 + \frac{2H_o(x)}{2H_o(x)} \right)^{-1} + 1 \left( 1 + \frac{2H_0(x)}{2H_0(x)} \right)^{-\frac{1}{2}} \]

Becomes

\[ h(x) = \frac{\alpha \lambda t^{x-1} e^{\lambda t^x}}{t} \left( 1 + \frac{2e^{\lambda t^x} - 1}{2e^{\lambda t^x} - 1} \right)^{-1} + 1 \left( 1 + \frac{2e^{\lambda t^x} - 1}{2e^{\lambda t^x} - 1} \right)^{-\frac{1}{2}} \]

3.7.7 Pareto - Reciprocal Inverse-Gaussian Model

Using the Pareto distribution for the baseline hazard \( h_o(t) \);

\[ h_o(t) = \frac{r}{t} \]

The cumulative hazard is given by;

\[ H_o(t) = -\ln \left( \frac{r}{t} \right) \]

The hazard

\[ h(x) = \frac{h_o(t)}{t} \left( 1 + \frac{2H_o(x)}{2H_o(x)} \right)^{-1} + 1 \left( 1 + \frac{2H_0(x)}{2H_0(x)} \right)^{-\frac{1}{2}} \]

Becomes

\[ h(x) = \frac{r}{t} \left( 1 - \frac{2\ln (\frac{r}{t})}{2\ln (\frac{r}{t})} \right)^{-1} + 1 \left( 1 - \frac{2\ln (\frac{r}{t})}{2\ln (\frac{r}{t})} \right)^{-\frac{1}{2}} \]
INVERSE GAMMA MIXTURES

3.8 INVERSE GAMMA FRAILITY MODEL

The probability density function is given by

\[ f(z, \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} z^{-\alpha-1} \exp \left( -\frac{\beta}{z} \right) \text{ for } x > 0 \]

\( \beta \) is the shape parameter and \( \alpha \) scale parameter

The Laplace transform is

\[ L_z(s) = \frac{2(\beta s)^\alpha}{\Gamma(\alpha)} K_\alpha(4\beta s) \]

\( K_\alpha(\cdot) \) is the modified Bessel function of the II kind.

\[ E(Z) = \frac{\beta}{\alpha - 1}, \quad Var(Z) = \frac{\beta^2}{(\alpha - 2)(\alpha - 1)^2} \]

For identifiability \( E(Z) = \frac{\beta}{\alpha - 1} = 1 \) and \( \delta^2 = \frac{1}{\alpha - 2} \)

The Laplace transform becomes

\[ L_z(s) = \frac{2(s^{\alpha-1})^{\alpha}}{\Gamma(\alpha)} K_\alpha(4s^{\alpha-1}) \]

The marginal survival function

\[ S(x) = L_z(H_0(x)) \]

\[ S(x) = \frac{2(H_0(x)(\alpha - 1))^{\alpha}}{\Gamma(\alpha)} K_\alpha(4H_0(x)(\alpha - 1)) \]

CHOICE OF \( h_0(x) \)

3.8.1 Gompertz –Inverse Gamma Frailty Model

Using Gompertz assumption for the baseline mortality
\( h_o(x) = \alpha e^{\beta x} \)

\[ H_o(x) = \frac{\alpha}{B} (e^{\beta x} - 1) \]

The survival function becomes,

\[ S(x) = \frac{2((\alpha - 1) \alpha (e^{\beta x} - 1))^{\alpha}}{\Gamma(\alpha) K_\alpha(4(\alpha - 1) \alpha (e^{\beta x} - 1))} \]

### 3.8.2 Weibull –Inverse Gamma Frailty Model

Using Weibull distribution for \( h_o(t) \)

\( h_o(t) = \lambda px^{p-1} \)

\[ H_o(x) = \lambda x^p \]

The survival function becomes

\[ S(x) = \frac{2((\alpha - 1) \lambda x^p)^{\alpha}}{\Gamma(\alpha) K_\alpha(4(\alpha - 1) \lambda x^p)} \]

### 3.8.3 Exponential –Inverse Gamma Frailty Model

Using the Exponential distribution for \( h_o(t) \)

\( h_o(t) = \lambda \)

The cumulative hazard

\[ H_o(x) = \lambda x \]

The survival function becomes

\[ S(x) = \frac{2((\alpha - 1) \lambda x)^{\alpha}}{\Gamma(\alpha) K_\alpha(4(\alpha - 1) \lambda x)} \]
3.8.4 Log-logistic - Inverse Gamma Frailty Model

Using the Log-logistic distribution for the baseline hazard $h_0(t)$;

$$h_o(t) = \frac{\left(\frac{\beta}{\alpha}\right) \left(\frac{t}{\alpha}\right)^{\beta-1}}{1 + \left(\frac{t}{\alpha}\right)^{\beta}}$$

The cumulative hazard

$$H_o(x) = \ln\{1 + \left(\frac{x}{\alpha}\right)^{\beta}\}$$

The survival function becomes

$$S(x) = \frac{2\left((\alpha - 1)\ln\{1 + \left(\frac{x}{\alpha}\right)^{\beta}\}\right)^{\alpha}}{\Gamma(\alpha)K_\alpha(4(\alpha - 1)\ln\{1 + \left(\frac{x}{\alpha}\right)^{\beta}\})}$$

3.8.5 Lognormal - Inverse Gamma Frailty Model

Using the Log normal distribution for the baseline hazard $h_0(t)$;

$$h_o(x) = \frac{f(x)}{1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)}$$

The cumulative hazard is given by;

$$H_o(x) = -\ln(1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right))$$

The survival function becomes

$$S(x) = \frac{2\left(-\left(\alpha - 1\right)\ln\left(1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)\right)\right)^{\alpha}}{\Gamma(\alpha)K_\alpha(-4(\alpha - 1)\ln(1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)))}$$

3.8.6 Exponential Power – Inverse Gamma Model

Using the exponential power distribution for the baseline hazard $h_0(t)$;
The cumulative hazard is given by;

\[ h_o(x) = \propto \lambda t^{\propto-1} e^{\lambda t^{\propto}} \]

The survival function,

\[ S(x) = \frac{2(\beta H_0(x))^{\propto}}{\Gamma(\propto)} K_\propto(4\beta H_0(x)) \]

Becomes

\[ S(x) = \frac{2((\propto-1)(e^{\lambda t^{\propto}} - 1))^{\propto}}{\Gamma(\propto)} K_\propto(4(\propto-1)(e^{\lambda t^{\propto}} - 1)) \]

NON-CENTRAL GAMMA MIXTURES

3.9 NON-CENTRAL GAMMA FRAILTY MODEL

The probability density function for the non-central gamma distribution with Y being a mixing of the distributions of \( X_1, X_2, \ldots, X_N \)

Where \( Xi's \sim Gamma(n,1) \) and \( N \sim poisson(\lambda) \)

Then the density function is a convolution with respective weights \( \frac{e^{-\lambda}(\lambda)^i}{i!} \) i.e

\[ Y = X_1, X_2, \ldots, X_N \]

\[ Prob(Y = j) = \sum_{j=0}^{\infty} prob(X_1, X_2, \ldots, X_j | N = j) prob(N = j) \]
\[ \text{Prob}(Y = j) = \sum_{j=0}^{\infty} \left( \frac{X^{j-1}e^{-X}}{\Gamma(j)} \right)^n \ast \left( \frac{\lambda^j e^{-\lambda}}{j!} \right) \]

\[ \text{Prob}(Y = j) = \sum_{j=0}^{\infty} \left( \frac{X^{n+j-1}e^{-X}}{\Gamma(n+j)} \right) \ast \left( \frac{\lambda^j e^{-\lambda}}{j!} \right) \]

\[ f(x,n,\lambda) = \sum_{j=0}^{\infty} \left( \frac{X^{n+j-1}e^{-X}}{\Gamma(n+j)} \right) \ast \left( \frac{\lambda^j e^{-\lambda}}{j!} \right) \]

Where \( \Gamma(n) \) is the central complete gamma function with \( n > 0 \lambda > 0 x \geq 0 \)

The hazard function is a special case of the three parameter power variance function when \( r = -1 \)

\[ h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} \ast \delta^2 H_o(t))^2} \]

**CHOICE OF \( h_o(x) \)**

**3.9.1 Gompertz - Non central Gamma Frailty Model**

Using Gompertz assumption for the baseline mortality

\[ h_o(x) = \propto e^{\beta x} \]

\[ H_o(x) = \frac{\propto}{\beta} (e^{\beta x} - 1) \]

The hazard function

\[ h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} \ast \delta^2 H_o(t))^2} \]

Becomes,

\[ h(t) = \frac{\propto e^{\beta x}}{(1 + \frac{1}{2} \ast \delta^2 \propto \beta (e^{\beta x} - 1))^2} \]
Justification

According to a study conducted by Olivieri (2001); mortality experience shows an increasing concentration of deaths around the mode of the curve of deaths and the mode moves towards older ages.

The Non-central gamma model can be used to explain the deceleration of the mortality rate at older ages as suggested above. The model further provides insights on the impact of omitted covariates and heterogeneity when estimating mortality rates for a heterogeneous population.

3.9.2 Weibull –Non central Gamma Frailty Model

Using Weibull distribution for $h_o(t)$
\[ h_o(t) = \lambda px^{p-1} \]
\[ H_o(x) = \lambda x^p \]

The hazard function
\[ h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} * \delta^2 H_o(t))^2} \]

Becomes,
\[ h(t) = \frac{\lambda px^{p-1}}{(1 + \frac{1}{2} * \delta^2 \lambda x^p)^2} \]

### 3.9.3 Exponential – Non central Gamma Frailty Model
Using the Exponential distribution for \( h_o(t) \);

\[ h_o(t) = \lambda \]

The cumulative hazard
\[ H_o(x) = \lambda x \]

The hazard function
\[ h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} * \delta^2 H_o(t))^2} \]

Becomes,
\[ h(t) = \frac{\lambda}{(1 + \frac{1}{2} * \delta^2 \lambda x)^2} \]

### 3.9.4 Log-logistic – Non-Central Gamma Frailty Model
Using the Log-logistic distribution for the baseline hazard \( h_o(t) \);
The cumulative hazard

\[ H_o(x) = \ln \{1 + \left( \frac{x}{\alpha} \right)^\beta \} \]

The hazard function

\[ h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} \cdot \delta^2 H_o(t))^2} \]

Becomes,

\[ h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} \cdot \delta^2 \ln \{1 + \left( \frac{x}{\alpha} \right)^\beta \})^2} \]

### 3.9.5 Lognormal – Non-Central Gamma Frailty Model

Using the Log normal distribution for the baseline hazard \( h_o(t) \);

\[ h_o(x) = \frac{f(x)}{1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)} \]

The cumulative hazard is given by;

\[ H_o(x) = -\ln(1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)) \]

The hazard function

\[ h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} \cdot \delta^2 H_o(t))^2} \]

Becomes.
\[ h(t) = \frac{f(x)}{1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right)} \left(1 - \frac{1}{2} \cdot \delta^2 \ln(1 - \Phi\left(\frac{\ln x - \mu}{\sigma}\right))\right)^2 \]

### 3.9.6 Exponential Power – Non central Gamma Model

Using the exponential power distribution for the baseline hazard \( h_o(t) \);

\[ h_o(x) = \propto \lambda t^{\alpha-1} e^{\lambda t^\alpha} \]

The cumulative hazard is given by;

\[ H_o(x) = e^{\lambda t^\alpha} - 1 \]

The hazard function

\[ h(t) = \frac{h_o(t)}{(1 + \frac{1}{2} \cdot \delta^2 H_o(t))^2} \]

Becomes,

\[ h(t) = \frac{\propto \lambda t^{\alpha-1} e^{\lambda t^\alpha}}{(1 + \frac{1}{2} \cdot \delta^2 (e^{\lambda t^\alpha} - 1))^2} \]

### 3.9.7 Exponential Power – Non central Gamma Model

Using the Pareto distribution for the baseline hazard \( h_o(t) \);

\[ h_o(t) = \frac{r}{t} \]

The cumulative hazard is given by;

\[ H_o(t) = -\ln\left(\frac{\lambda^r}{t^r}\right) \]

The hazard becomes,

\[ h(t) = \frac{r}{t(1 - \frac{1}{2} \cdot \delta^2 \ln(\frac{\lambda^r}{t^r}))^2} \]
CHAPTER 4

4.1 PARAMETER ESTIMATION

Model parameters are fixed quantitative values that characterize the model believed to reflect the real world. They have to be estimated either by statistical inference from observations or by expert opinion.

4.1.1 CHOICE OF EXPLANATORY VARIABLES

In order to make comparisons between the Gamma-Gompertz model, the Inverse-Gaussian-Gompertz model and the Non-central Gamma-Gompertz model, it is necessary to estimate and fix the baseline model parameters using insurance based mortality rating. The baseline model has no underwriting.

The Gompertz parameters are estimated using simulation as shown in the R-CODE (appendix 1C)

4.1.2 CHOICE OF THE INSURED'S LEVEL OF HETEROGENEITY

In Butt and Haberman (2002) an insurance application of frailty-based survival model is proposed. In particular, the authors discuss various choices and fit some models to two sets of life insurance mortality data. The obtained results suggest that when life annuities are referred to $\delta^2 = 1/b$ should fall in the range (0.025, 0.05).

Unless otherwise stated in this exercise the insured population will be considered to have heterogeneity level of $\delta^2 = 0.05$

4.1.3 SHARED FRAILTY MODEL

To show relevance of frailty models, the insured population can be grouped into two i.e. insured persons who have an above average life expectancy are clustered in one group
and individual underwriting is only performed for impaired persons.

The hazard function for the \( j^{th} \) insured in the group is defined as:

\[
h(t_j|Z) = Z h_0(t_j) \exp(\beta' x_j) \quad j = 1, \ldots, k
\]

The joint survival function for the \( k \) individuals is given by

\[
S(t_1, \ldots, t_k) = \text{pr}(T_1 > t_1, \ldots, T_k > t_k)
\]

\[
S(t_1, \ldots, t_k) = \int_0^\infty \prod_{j=1}^k \text{pr}(T_j > t_j|Z) g(z) dz
\]

Since \( Z \sim \text{Noncentral Gamma}(b, \lambda) \) assuming shape parameter \( b = 0 \)

\[
S(t_1, \ldots, t_k) = \frac{-1}{e^{1+1/2\delta^2} \sum_{j=1}^k H_0(t)}
\]
CHAPTER 5

APPLICATIONS TO ACTUARIAL SCIENCE

The aims of this exercise are threefold:

- The first aim is to show that when heterogeneity is disregarded the expected residual lifetime is underestimated.
- Secondly, is that neglecting heterogeneity leads to an underestimation of the insurer’s liability.
- Finally, is to show the relevance of the proposed non-central Gamma frailty mixture to reflect an insurer’s mortality rating.

ILLUSTRATION
Consider three hypothetical insurers i.e. insurer x, y and z.

Insurer X assumes the population to be homogeneous and applies the KE 2001-2003 life tables.

Insurer Y assumes the population to be heterogeneous and uses frailty modeling to account for heterogeneity.

Finally, Insurer Z carries out underwriting and adjusts the rates to reflect safety loadings.

In this case data from Jubilee insurance is considered.
**APPROACH 1**
The first approach is to use the KE 2001-2003 as the baseline hazard for the frailty model.

The KE life table was published by the Association of Kenya Insurers and is based on a study conducted between 2005 and 2007. The data compiled was supplied by 18 Kenya based insurance companies.

**CASE 1: Inverse Gaussian Frailty**
The inverse Gaussian frailty mixture is given by:

\[
 h(t) = \frac{h_o(t)}{(1 + 2 \cdot \delta^2 H_o(t))^{1/2}}
\]

The frailty model: \( h(t|Z) = Z \cdot h_o(t) \)

When \( Z = 1 \) the hazard corresponds to an average individual, hence \( h(t|1) = h_o(t) \).

Thus the baseline hazard \( h_o(t) \) can be approximated using the standard life tables

**CASE 2 : Non-central Gamma Frailty**
The inverse Gaussian frailty mixture is given by:

\[
 h(t) = \frac{h_o(t)}{(1 + 1/2 \cdot \delta^2 H_o(t))^2}
\]

Using similar assumptions for the baseline i.e.

\( h_o(t) \sim K.E \ 2001 - 2003 \ life \ table \)

From TABLE 1.1 to 2.1 the output is represented in the graph below.
RESULTS

1. Ignoring heterogeneity leads to an underestimation of life expectancy.

2. The choice of frailty distribution does not have a significance impact on the life expectancy

APPROACH 2
Considering law based assumptions for the baseline hazard. i.e $H_0(t) \sim Gompertz$

Case 1: Non-central gamma gompertz model

Considering $Z \sim Non - central Gamma gompertz$, $H_0(t) \sim Gompertz$

The model is given by:

$$h(x) = \frac{\alpha e^{\beta x}}{(1 + \frac{1}{2} \delta^2 \frac{\alpha}{\beta} (e^{\beta x} - 1))^2}$$
From TABLE 2.1 the output is represented in the graph below.

**RESULTS:**

1. The results shows an underestimation of residual life time when heterogeneity is disregarded

**5.1 PENSION SCHEME**
Pension schemes are (essentially) deferred annuities whose benefits are payable on retirement.
5.1.1 Present Value

These are annuities which commence in $m$ (say) years' time, provided that the annuitant is then active. Thus the present value of amount $b$ payable for a future lifetime $T(x+t)$

$$m|\bar{a}_x = \frac{D_{x+m}}{D_x} \cdot a_{x+m}$$

Where $\frac{D_{x+m}}{D_x}$ is a pure endowment factor and $a_{x+m}$ is an annuity factor at age $x+m$.

For illustration purposes any safety loadings assigned by the insurer is not accounted for since the focus is on the effects of heterogeneity.

From TABLE 3.1 the output is represented in the graph below.

**GRAPH**
RESULTS

1. When heterogeneity is disregarded the expected liability is underestimated.
2. The non-central gamma frailty is a close estimate of the insurer liability.

CHAPTER 6

6.1 SUMMARY TABLE

<table>
<thead>
<tr>
<th>MIXING DISTRIBUTION</th>
<th>BASELINE HAZARD</th>
<th>FRAILTY HAZARD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Gamma Distribution</td>
<td>Gompertz</td>
<td>$h(x) = \frac{\alpha be^{\beta x}}{b + \frac{\alpha}{\beta}(e^{\beta x} - 1)}$</td>
</tr>
<tr>
<td></td>
<td>Weibull</td>
<td>$h(x) = \frac{\lambda px^{p-1} * b}{b + \lambda x^p}$</td>
</tr>
<tr>
<td></td>
<td>Exponential</td>
<td>$h(x) = \frac{\lambda b}{b + \lambda x}$</td>
</tr>
<tr>
<td></td>
<td>Log-logistic</td>
<td>$h(x) = \frac{(\frac{\beta}{\alpha})(\frac{x}{\alpha})^{\beta-1}}{1 + (\frac{x}{\alpha})^{\beta}} / {1 + \frac{1}{b} \ln (1 + (\frac{x}{\alpha})^{\beta})}$</td>
</tr>
<tr>
<td></td>
<td>Log normal</td>
<td>$h(x) = \frac{\Phi(x)}{1 - \Phi(\frac{\ln x - \mu}{\sigma})} / {1 - \frac{1}{b} \ln (1 - \Phi(\frac{\ln x - \mu}{\sigma}))}$</td>
</tr>
<tr>
<td></td>
<td>Exponential Power</td>
<td>$h(x) = \frac{\alpha \lambda x^{x-1}e^{\lambda x}}{1 + (e^{\lambda x} - 1)/b}$</td>
</tr>
<tr>
<td></td>
<td>Pareto</td>
<td>$h(x) = \frac{\bar{x}}{1 - \ln(\frac{\lambda}{\bar{x}})/b}$</td>
</tr>
<tr>
<td>2. Inverse Gaussian Distribution</td>
<td>Gompertz</td>
<td>$h(x) = \frac{\alpha e^{\beta x}}{(1 + 2 \frac{\alpha}{\beta}(e^{\beta x} - 1))^{1/2}}$</td>
</tr>
<tr>
<td>Distribution</td>
<td>Formula</td>
<td></td>
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<tr>
<td>--------------------</td>
<td>-------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>Weibull</td>
<td>( h(x) = \frac{\lambda p x^{p-1}}{(1 + 2 \lambda x p \delta^2)^{1/2}} )</td>
<td></td>
</tr>
<tr>
<td>Exponential</td>
<td>( h(t) = \frac{\lambda}{(1 + 2 \lambda t \delta^2)^{1/2}} )</td>
<td></td>
</tr>
<tr>
<td>Log-logistic</td>
<td>( h(t) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}}{1 + \left(\frac{t}{\alpha}\right)^{\beta}} \frac{1}{(1 + 2ln(1 + \left(\frac{t}{\alpha}\right)^{\beta})\delta^2)^{1/2}} )</td>
<td></td>
</tr>
<tr>
<td>Log normal</td>
<td>( h(t) = \frac{f(x)}{1 - \Phi\left(\frac{ln x - \mu}{\sigma}\right)} \frac{1}{(1 - 2ln(1 - \Phi(\frac{ln x - \mu}{\sigma}))\delta^2)^{1/2}} )</td>
<td></td>
</tr>
<tr>
<td>Exponential Power</td>
<td>( h(t) = \frac{\alpha \lambda t^{\alpha-1}e^{\lambda t^\alpha}}{(1 + 2(e^{\lambda t^\alpha} - 1)\delta^2)^{1/2}} )</td>
<td></td>
</tr>
<tr>
<td>Pareto</td>
<td>( h(t) = \frac{\bar{t}}{(1 - 2ln(\bar{t})\delta^2)^{1/2}} )</td>
<td></td>
</tr>
<tr>
<td>3. Positive Stable Distribution</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gompertz</td>
<td>( h(x) = r \alpha e^{\beta x} \left{\frac{\alpha}{\beta}(e^{\beta x} - 1)\right}^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Weibull</td>
<td>( h(x) = r \lambda p x^{p-1} (\lambda x^p)^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Exponential</td>
<td>( h(x) = r \lambda (\lambda x)^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Log-logistic</td>
<td>( h(x) = r \left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1} \ln{1 + \left(\frac{x}{\alpha}\right)^{\beta}}^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Log normal</td>
<td>( h(x) = r \frac{f(x)}{1 - \Phi\left(\frac{ln x - \mu}{\sigma}\right)} \ast \left{-ln\left(1 - \Phi\left(\frac{ln x - \mu}{\sigma}\right)\right)\right}^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Distribution</td>
<td>Formula</td>
<td></td>
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<tr>
<td>----------------------------</td>
<td>-------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>Exponential Power</td>
<td>( h(x) = r \propto \lambda t^{\alpha-1} e^{\lambda t} \times (e^{\lambda t} - 1)^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Pareto</td>
<td>( h(x) = \frac{r}{t} (-\ln(t))^r )</td>
<td></td>
</tr>
<tr>
<td>4. Log-normal Distribution</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gompertz</td>
<td>( h(x) = -\frac{\lambda x'}{\lambda x} \left( \frac{\alpha}{\beta} (e^{\beta x} - 1) \right) )</td>
<td></td>
</tr>
<tr>
<td>Weibull</td>
<td>( h(x) = -\frac{\lambda x'}{\lambda x} (e^{\lambda x'}) )</td>
<td></td>
</tr>
<tr>
<td>Exponential</td>
<td>( h(x) = -\frac{\lambda x'}{\lambda x} )</td>
<td></td>
</tr>
<tr>
<td>Log-logistic</td>
<td>( h(x) = -\frac{\lambda x'}{\lambda x} \left( \ln \left( 1 + \left( \frac{x}{\alpha} \right) \right) \right) )</td>
<td></td>
</tr>
<tr>
<td>5. Compound Poisson</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gompertz</td>
<td>( h(x) = \alpha e^{\beta x} \left( 1 + \frac{\delta^2}{r-1} \frac{x}{\alpha} (e^{\beta x} - 1) \right)^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Weibull</td>
<td>( h(x) = \lambda px^{p-1} \left( 1 + \frac{\delta^2}{r-1} \lambda p \right)^{p-1} )</td>
<td></td>
</tr>
<tr>
<td>Exponential</td>
<td>( h(x) = \lambda \left( 1 + \frac{\delta^2}{r-1} \lambda x \right)^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Log-logistic</td>
<td>( h(x) = \frac{\left( \frac{x}{\alpha} \right)^{\beta-1}}{1 + \left( \frac{x}{\alpha} \right)^{\beta}} \left( 1 + \frac{\delta^2}{r-1} \ln \left( 1 + \left( \frac{x}{\alpha} \right)^{\beta} \right) \right)^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Log normal</td>
<td>( h(x) = h_0(t) \left( 1 - \frac{\delta^2}{r-1} \ln(1 - \Phi(\frac{lnx-\mu}{\sigma})) \right)^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Exponential Power</td>
<td>( h(x) = \alpha \lambda t^{\alpha-1} e^{\lambda t} \times \left( 1 + \frac{\delta^2}{r-1} e^{\lambda t} - 1 \right)^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Pareto</td>
<td>( h(x) = \frac{1}{t} \left( 1 - \frac{\delta^2}{r-1} \ln(t) \right)^{r-1} )</td>
<td></td>
</tr>
<tr>
<td>Distribution</td>
<td>Formula (Gompertz)</td>
<td>Formula (Weibull)</td>
</tr>
<tr>
<td>-------------------------</td>
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<td>-------------------</td>
</tr>
<tr>
<td>Reciprocal Inverse Gaussian Distribution</td>
<td>$h(x) = \frac{\alpha e^{\beta x}}{1 + 2 \frac{\beta}{\alpha} (e^{\beta x} - 1)}^{-1}$</td>
<td>$h(x) = \frac{\lambda px^{p-1}}{1 + 2 \frac{\lambda x}{p}}^{-1} + \alpha e^{\beta x} \left(1 + 2 \frac{\lambda x}{p} \right)^{-\frac{1}{2}}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$S(x) = \frac{2((\alpha - 1) \alpha (e^{\beta x} - 1))^{\frac{\alpha}{\Gamma(\alpha)}}}{\Gamma(\alpha)} K_\alpha(4(\alpha - 1) \alpha (e^{\beta x} - 1))$</td>
<td>$S(x) = \frac{2((\alpha - 1) \lambda x^{p})^{\frac{\alpha}{\Gamma(\alpha)}}}{\Gamma(\alpha)} K_\alpha(4(\alpha - 1) \lambda x^{p})$</td>
</tr>
<tr>
<td>Log-logistic</td>
<td>$S(x) = \frac{2((\alpha - 1) \lambda x^{p})^{\frac{\alpha}{\Gamma(\alpha)}}}{\Gamma(\alpha)} K_\alpha(4(\alpha - 1) \lambda x^{p})$</td>
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<td>Distribution</td>
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</tr>
<tr>
<td>-----------------------</td>
<td>-------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>Log normal</td>
<td>( S(x) = \frac{2(-\alpha - 1)\ln(1 - \Phi(\frac{\ln x - \mu}{\sigma}))}{\Gamma(\alpha)} \alpha \Phi(\frac{\ln x - \mu}{\sigma}) )</td>
<td></td>
</tr>
<tr>
<td>Exponential Power</td>
<td>( S(x) = \frac{2((\alpha - 1)(e^{\lambda t^\alpha} - 1))^{\alpha}}{\Gamma(\alpha)} K_\alpha((\alpha - 1)(e^{\lambda t^\alpha} - 1)) )</td>
<td></td>
</tr>
<tr>
<td>Non-Central Gamma Distribution</td>
<td>( h(t) = \frac{\alpha e^{\beta x}}{(1 + \frac{1}{2} \delta^2 \alpha (e^{\beta x} - 1))^2} )</td>
<td></td>
</tr>
<tr>
<td>Weibull</td>
<td>( h(t) = \frac{\lambda px^{p-1}}{(1 + \frac{1}{2} \delta^2 \lambda x^p)^2} )</td>
<td></td>
</tr>
<tr>
<td>Exponential</td>
<td>( h(t) = \frac{\lambda}{(1 + \frac{1}{2} \delta^2 \lambda x)^2} )</td>
<td></td>
</tr>
<tr>
<td>Log-logistic</td>
<td>( h(t) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}}{1 + \left(\frac{t}{\alpha}\right)^{\beta}} )</td>
<td></td>
</tr>
<tr>
<td>Log normal</td>
<td>( h(t) = \frac{f(x)}{1 - \Phi(\frac{\ln x - \mu}{\sigma})} )</td>
<td></td>
</tr>
<tr>
<td>Exponential Power</td>
<td>( h(t) = \frac{\alpha \lambda t^{\alpha-1} e^{\lambda t^\alpha}}{(1 + \frac{1}{2} \delta^2 (e^{\beta x} - 1))^2} )</td>
<td></td>
</tr>
<tr>
<td>Pareto</td>
<td>( h(t) = \frac{r^r}{t(t - \frac{1}{2} \delta^2 \ln(\frac{\lambda^r}{t^r}))^2} )</td>
<td></td>
</tr>
</tbody>
</table>
6.2 Model Framework

CONSTRUCTION

CONTINUOUS DISCRETE

MIXING DISTRIBUTION

FRAILTY MIXTURE
- Baseline hazards

SURVIVAL FUNCTION
DENSITY FUNCTION
HAZARD FUNCTION

✔ ESTIMATES
✔ APPLICATIONS
✔ GRAPHS

PROPERTIES

INFINITE DIVISIBILITY IDENTIFIABILITY
6.3 DISCUSSION
The conclusion to be reached from the analyses and discussions is that comparing the standard life tables with the Gamma-Gompertz and Inverse Gaussian model; shows an increase in the insurers expected liability when heterogeneity is considered. That is, assuming the insured to be homogeneous could lead to an underestimation of future liability.

Further, using Non-central Gamma model in estimating future liability by directly adjusting the A.K.I mortality tables shows an increase in longevity risk. The extent of heterogeneity of the insured group determines the level of risk.

A key point to note is that the non-central gamma frailty model as proposed gives better estimate of the insurer rates compared to the gamma frailty model with similar assumptions of the population level of heterogeneity. The correlation coefficient between the non-central gamma and insurers rates is also higher.

Thus, the non-central family distributions is recommended for further study as it gives better estimates for the insurer’s rating.
APPENDIX 1A

LAPLACE TRANSFORMS

1.1 Gamma Distribution

Let $Z \sim \Gamma(p, b)$ be gamma distributed with shape parameter $p$ and scale parameter $b$. The probability density function is given by:

$$f(z) = \frac{b^p z^{p-1} e^{-bz}}{\Gamma(p)}$$

$$L_Z(s) = E[e^{-sZ}]$$

$$= \int_0^\infty e^{-sZ} \frac{b^p z^{p-1} e^{-bz}}{\Gamma(p)} dz$$

$$= \frac{b^p}{\Gamma(p)} \int_0^\infty e^{-Z(s+b)} Z^{p-1} dz$$

let $y = Z \ast (s+b) ; z = \frac{y}{s+b} ; dz = \frac{dy}{s+b}$

$$= \frac{b^p}{\Gamma(p)} \int_0^\infty e^{-y(s+b)p-1} \frac{dy}{s+b}$$

$$= \frac{b^p}{\Gamma(p)(s+b)^p} \int_0^\infty e^{-y} (y)^{p-1} dy$$

$$L_Z(s) = \left( \frac{b}{b+s} \right)^p = (1 + \frac{s}{b})^{-p}$$

1.2 Inverse-Gaussian distribution

Willmot (1986) derived the Laplace from a Generalized Inverse-Gaussian mixing distribution

$$f(z) = \frac{e^{-\alpha z} z^{-\alpha-1} e^{-(z^2+\mu^2)/2\beta z}}{2K_{\alpha}(\mu \beta^{-1})}$$

Where $K_\alpha(.)$ is the modified Bessel function of the III kind

Substituting $\alpha = -\frac{1}{2}$
The Laplace transform is

\[ L_Z(s) = \exp \left\{ -\frac{\mu}{\beta} (1 + 2\beta s)^{1/2} - 1 \right\} \]

### 1.3 Reciprocal Inverse-Gaussian distribution

\[ (z) = \left( \frac{1}{2\pi \lambda} \right)^{1/2} \exp \left\{ -\frac{\mu^2}{2\lambda} \left( 1 - \frac{\lambda}{\mu} \right)^2 \right\} \]

The Laplace transform is given by

\[ L_Z(s) = \left( 1 + \frac{2s}{\lambda} \right)^{-1/2} \exp \left\{ \mu \left[ 1 - \left( 1 + \frac{2s}{\lambda} \right)^{1/2} \right] \right\} \]

### 1.4 Compound Poisson distribution

Let \( N \) be a Poisson random variable with parameter \( p > 0 \) and let \( X_i, i = 1, 2, \ldots \) be i.i.d. random variables, independent of \( N \).

\( Z \sim \) Compound Poisson Distribution (CPD) defined as

\[ Z = \sum_{i=1}^{N} X_i \]

If \( E(X) \) and \( V(X) \) are the common mean and variance of the random variables \( X_i, i = 1, 2, \ldots \) then, the moments of \( Z \) are given by

\[ E(Z) = p * E(X) \]

\[ V(Z) = p \left[ V(X) + [E(X)]^2 \right] \]
The Laplace transform is given by

\[ L_Z(s) = E[e^{-sz}] \]

\[ = E \{ E[e^{-s(x_1+x_2+...+x_N)} | N = n] \} \]

\[ = E \{ E[(e^{-sx} * e^{-sx} * ... * e^{-sx})] \} \]

\[ = E \{ [E(e^{-sx})]^n \} \]

\[ L_Z(s) = F(L_x(s)) \]

\[ L_Z(s) = e^{p(L_x(s)-1)} \]

### 1.5 Power Variance Functions

The PVF model is a three parameter family denoted by \( PVF(r, k, \lambda) \).

The Laplace transform is

\[ L(s) = e^{\frac{k}{r}((\lambda+s)^r-\lambda^r)} \]

**Special Case (\( \lambda=0 \))**

The Laplace transform becomes

\[ L(s) = e^{\frac{k}{r}s^r} \]

This is the Laplace for a Positive Stable distribution.

### 1.6 Log-Normal distribution

If \( W \) is a random variable with a normal distribution, then \( Z = \exp(W) \) has a log-normal distribution.
The probability density function $Z$ is given by

$$f(z) = \frac{1}{z\delta \sqrt{2\pi}} e^{-\frac{(lnz-\mu)^2}{2\delta^2}} \quad z > 0$$

Where $\delta > 0$ is the scale and $-\infty < \mu < \infty$ is the location parameter

The mean and variance are

$$E[Z] = e^{\mu + \delta/2}$$

$$Var[Z] = e^{2\mu + \delta} * e^{\delta^2 - 1}$$

$$CV = (e^{\delta^2 - 1})^{1/2}$$ Only depends on $\delta$

An approximation of the Laplace transform is obtained by

$$L_z(s) = E[e^{-sz}]$$

$$= \int_0^\infty e^{-sz} f(z)dz$$

$$L_z(s) = \int_0^\infty \frac{1}{z\delta \sqrt{2\pi}} e^{-sz-\frac{(lnz-\mu)^2}{2\delta^2}} dz$$

No closed form of the Laplace transform is known, however several approximation methods are possible. i.e.

**The Initiative Approach**

Consider for $k = 0, 1, 2, 3 \ldots$
\[ E[Z^k e^{-SZ}] = \int_0^\infty \frac{Z^{K-1}}{\delta\sqrt{2\pi}} e^{-SZ - \frac{(\ln Z - \mu)^2}{2\delta^2}} \, dz \]

Using change of variable

\[ w = \log z \]

\[ z = e^w \]

\[ \frac{dz}{dw} = e^w \]

\[ E[e^{wk - Se^w}] = \int_0^\infty \frac{e^{w(K-1)}}{\delta\sqrt{2\pi}} e^{-Se^w - \frac{(w-\mu)^2}{2\delta^2}} \, dw \ast e^w \]

\[ = \int_{-\infty}^\infty \frac{1}{\delta\sqrt{2\pi}} e^{-Se^w + kw - \frac{(w-\mu)^2}{2\delta^2}} \, dw \]

Replacing the expression \(-Se^w + kw - \frac{(w-\mu)^2}{2\delta^2}\) by a Taylor approximation of second order around the value \(k\) that maximizes this expression. That is

\[-Se^k[1 + (w - k) + \frac{(w-k)^2}{2}] + kw - \frac{(w-\mu)^2}{2\delta^2}\]

Thus the resulting integral can be explicitly obtained

\[ = \int_{-\infty}^\infty \frac{1}{\delta\sqrt{2\pi}} e^{-Se^k[1+(w-k)+\frac{(w-k)^2}{2}]+kw-\frac{(w-\mu)^2}{2\delta^2}} \, dx \]

The difficulty in the lognormal case is the explicit calculation of the value

\[ k = -LW(s\delta^2 e^{k\delta^2 + \mu} + k\delta^2 + \mu) \]

Where the function \(LW[-e^{-1}, \infty]\) known as the LambertW, is the inverse of
\( f(W) = We^W: \)

In particular, with \( k = 0 \)

\[
L_z(s) \approx \frac{1}{\sqrt{\text{LW}(\delta^2 e^s)}} e^{-\frac{LW^2(\delta^2 e^s)+2LW(\delta^2 e^s)}{2\delta^2}}
\]

### 1.7 Inverse Gamma Distribution

The probability density function is given by

\[
f(z, \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} z^{-\alpha-1} \exp \left( \frac{-\beta}{z} \right) \text{ for } x > 0
\]

Where \( \beta \) is the shape parameter and \( \alpha \) scale parameter

The Laplace transform is given by

\[
L_z(s) = \frac{2(-\beta s)^\alpha}{\Gamma(\alpha)} K_\alpha(-4\beta s)
\]

Where \( K_\alpha(\cdot) \) is the modified Bessel function of the II kind.

**Construction**

The density function for the Gamma distribution is

\[
f(x) = \frac{x^{(k-1)}e^{-x/\beta}}{k\Gamma(k)}
\]

Define the transformation \( Z = g(x) = \frac{1}{x} \)

\[
f(z) = f(g^{-1}(z)) \left| \frac{d}{dz} g^{-1}(z) \right|
\]

\[
f(z) = \frac{(\frac{1}{z})^{(k-1)}e^{(-1/z)} \frac{1}{z^2}}{k\Gamma(k)}
\]

\[
f(z) = \frac{(z)^{(-k-1)}e^{(-1/z)}}{k\Gamma(k)}
\]
Replacing $k = \alpha, -1 = \beta$

$$f(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} z^{-\alpha - 1} \exp \left( \frac{-\beta}{z} \right)$$

1.8 Non Central gamma distribution

Let $\lambda > 0$, let $N; X_1, X_2, \ldots, X_N$ be independent random variables such that $N$ is Poisson distributed with mean $\lambda$. Define $Y = X_1 + \cdots + X_N$ with the convention that $Y = 0$ if $N = 0$; then the Laplace transform is

$$L_Y(s) = E(e^{-sY})$$

$$L_Y(s) = E[e^{-s(X_1 + \cdots + X_N)}]$$

$$L_Y(s) = E\{e^{-sX}\} \text{ since } X_i \text{ are IID}$$

$$L_Y(s) = E\{(L_X(s))^n\} = F(L_X(s))$$

$$L_Y(s) = e^{-\lambda(1-L_X(s))} \text{ where } L_X(s) \text{ is the Laplace transform of } X$$

$Y$ being a mixing of the distributions of $X_1 + \cdots + X_N$ with respective weights $\frac{e^{-\lambda X}}{n!}$ leads to the convolution;

$$f(x, b, \lambda) = \sum_{n=0}^{\infty} e^{-\lambda X} \frac{n!}{\Gamma(b + n)} \left[ \frac{e^{-sx} x^{b+n-1}}{\Gamma(b + n)} \right]$$

The Laplace transform is given by

$$L_X(s) = E(e^{-sx})$$

$$L_X(s) = \int_0^\infty e^{-sx} f(x, b, \lambda) dx$$
\[ L_X(s) = \frac{1}{(1+\lambda s)^b} e^{-\frac{s\alpha(\lambda)^2}{1+\lambda s}} \] with \( \alpha \) being the non-centrality parameter, \( b \) is the shape parameter and \( \lambda \) is the scale parameter

Using shape parameter \( b = 0 \)

\[ L_Y(s) = e^{-\frac{s\alpha(\lambda)^2}{1+\lambda s}} \]

For identifiability the mean is normalized to one

\[ \text{Mean} = a(\lambda)^2 = 1 \]

\[ \text{Var} = 2a(\lambda)^3 : \delta^2 = 2\lambda : \frac{1}{2} \delta^2 = \lambda \]

\[ L_Y(s) = e^{-\frac{s}{1+2\delta^2 s}} \]

The marginal survival function is given by;

\[ S(x) = L_Y(H_o(t)) = e^{-\frac{H_o(t)}{1+1/2\delta^2 H_o(t)}} \]

\[ S(x) = e^{-\frac{H_o(t)}{1+1/2\delta^2 H_o(t)}} \]

\[ f(x) = \frac{h_o(t)}{(1 + 1/2\delta^2 H_o(t))^{-2}} \cdot e^{-\frac{H_o(t)}{1+1/2\delta^2 H_o(t)}} \]

The hazard function is given by;

\[ h(t) = \frac{-s'(x)}{s(x)} = \frac{h_o(t)}{(1 + \frac{1}{2} \delta^2 H_o(t))^2} \]
The Multiplicative Model using Taylor’s Series

The multiplicative model can be described using Taylor’s series expansion as shown

Let \( h(t, z) \) be an individual hazard.

\[
h_t(z) = \lim_{t \to 0} \frac{P(Tx < t \mid Zx = z)}{t}
\]

By Taylor series expansion;

\[
h(t; Z) = h(t; 0) + Z h'(t; 0) + \frac{Z^2}{2!} h''(t; 0) + \frac{Z^3}{3!} h'''(t; 0) + \cdots
\]

\[
h(t; Z) = h(t; 0) + Z h'(t; 0) + o(Z)
\]

The \( o(Z) \) denotes the terms containing \( Z \) of higher order than one. By omitting these terms we get

\[
h(t; Z) = h(t; 0) + Z h'(t; 0)
\]

Making the natural assumption that zero frailty (susceptibility) yields zero mortality it holds that

\[
h(t; Z) = Z h_o(t)
\]

where \( h_o(t) = h'(t; 0) = \frac{d}{dz} h(t; z) \mid_{z=0} \]

Thus the underlying hazard \( h_o(t) \) is a partial derivative of the individual hazard with respect to frailty taken at point \( Z = 0 \). This is the non-frailty hazard.

This shows that a multiplicative frailty model is a rather simplified view of how heterogeneity may act.
APPENDIX 1C

R-CODES

R-Code 1

r-code

alpha = 4.419e-06
beta=0.1
ho=alpha*(exp(beta*x))
Ho=(alpha/beta)*(exp(beta*x)-1)
gig=ho/sqrt(1+2*Ho*0.05)

plot(x,gig,main="I.G-Gompertz hazard",type="o",xlab="age",ylab="hazard")

R-code 2

Using r=1.1, λ=0.01

wcp=(1.1*0.01*age^0.1)*((1+0.5*0.01*age^1.1)^0.1)

plot(age,wcp,main="CompoundPoisson-Weibull",type="o", xlab="age",ylab="hazard")

R-code 3

grig=(ho/1.1)/(1+2*Ho/1.1)+ho/sqrt(1+2*Ho/1.1)

plot(age,grig,main="gompertz-reciprocal IG",type="o",col="blue")

R-code 4

psgf=0.4*alpha*exp(beta*x)*(((alpha/beta)*(exp(beta*x)-1))^(0.4-1))

plot(x,psgf,main="Positive stable-Gompertz",type="o", xlab="age",ylab="hazard")
**R-code 5**

alpha=0.00004 #initial estimate

beta=0.1 #fixed for estimation

ho=alpha*(exp(beta*x))

qho=ho/(1+0.5*ho)

jubileeqx /((1-0.5* jubileeqx)*( exp(beta*x)))#inverse

median(jubileeqx /((1-0.5* jubileeqx)*( exp(beta*x))))

4.419414e-06 #alpha new estimate

**R-code 4**

alpha= 0.00025

beta=0.1

age=20:100

ho=alpha*(exp(beta*age))

Ho=(alpha/beta)*(exp(beta*age)-1)

gngf=ho/((1+0.05*Ho/2)^2)

plot(age,gngf,main="gompertz-non-centralgamma",type="o", xlab="age",ylab="hazard")
### APPENDIX 1D

#### TABLES

**TABLE 1.0**

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### TABLE 1.1

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TABLE 3.0

NON-CENTRAL GAMMA MODEL: \( h(x) = h_0(x)/(1+0.5*\delta^2*h_0(x))^2 \)

\( h_0(x) \sim \text{Gompertz} (\alpha, \beta) \)

the resulting model: \( h(x) = \alpha \exp(\beta x)/(1+0.5*0.05*\delta^2*(\alpha/\beta)\exp(\beta x-1))^2 \)

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COMPARISON GRAPHS

1.1 Heterogeneity effect in the Gamma-Gompertz model:

Graphs for the Gamma-Gompertz model with different values of $b$ is shown when $\alpha = 8 \times 10^{-7}$ $\beta = 0.15$

$$h(x) = \frac{\alpha b e^{\beta x}}{b + \frac{\alpha}{\beta}(e^{\beta x} - 1)}$$

$b = 0.05$ (black) $b = 0.5$ (blue) $b = 50$ (green) $b = 500$ (red)

RESULTS

If $b$ is low, then heterogeneity is strong; the opposite occurs if $b$ is high.

Higher variance implies heterogeneity and lower variance homogeneity
1.2 Comparing the Gamma-Gompertz and Inverse Gaussian-Gompertz

Comparisons can be made with similar parameter estimators for both models.

**RESULTS**

The output shows that the relative frailty distribution among survivors is independent of age for the Gamma, but becomes more homogeneous with time for the Inverse Gaussian. (i.e. concentration of deaths at small age interval)
1.3 Comparing different frailty levels i.e.

\[ z = 1 \text{ (average)}, z = 2 \text{ (more frail)}, z = 0.5 \text{ (less frail)} \]

The frailty model is given by: \( h(t) = Z \ast h_0(t) \), assuming \( h_0(t) \sim \text{gompertz}(\alpha, \beta) \)

**RESULTS**

The output shows that for higher levels of frailty the probability of dying is higher compared to lower frailty levels, i.e. more frail individuals are likely to die earlier that the less frail ones.
REFERENCE


Beard R E (1952), Some further experiments in the use of the incomplete gamma function for the calculation of actuarial functions, Journal of the Institute of Actuaries, 78: 341–353


**Hougaard, P. (1986).** Survival models for heterogeneous populations derived from stable distributions


**Nihal Ata a & Gamze Özel (2010)** Survival functions for the frailty models based on the discrete compound Poisson process


**R Development Core Team: (2008)** The R project for statistical computing. URL: http://www.r-project.org,


