TRANSITION PROBABILITIES BASED ON KOLMOGOROV EQUATIONS FOR PURE BIRTH PROCESSES

By **Mutothya Nicholas Mwilu**

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DECLARATION

I declare that this is my original work and has not been presented for an award of a degree in any other university

Signature ………………………… Date ………………..

Mutothya Nicholas Mwilu

This thesis is submitted for examination with my approval as the university supervisor

Signature ………………………………… Date……………………………..

Prof J.A.M Ottieno

EXECUTIVE SUMMARY

The aim of this thesis is to identify probability distributions emerging by solving the Kolmogorov forward differential equation for continuous time non homogenous pure birth process. This equation is $\frac{\partial}{\partial t} p_{k,k+n}(s,t) + \lambda_{k+n}(t) p_{k,k+n}(s,t) = \lambda_{k+n-1}(t)\, p_{k,k+n-1}(s,t)$.

This equation has been solved for four different processes. i.e. Poisson Process $(\lambda_n = \lambda)$, Simple Birth process $(\lambda_n = n\lambda)$, in Simple Birth process with immigration $(\lambda_n = n\lambda + v)$ and the Polya process $\left(\lambda_n = \left(\frac{1 + an}{1 + \lambda at}\right)\lambda\right)$.

Three different methods have been applied in solving the forward Kolmogorov equation above with the initial conditions being that $p_{k\,k}(s,s) = 1$ and $p_{k,k+n}(s,s) = 0$ for $n > 0$.

These methods are:

(1) The integrating factor technique
(2) the Lagrange's Method
(3) The generator Matrix technique.

The results from the three Methods were similar.

In addition, the first passage time arising from the solution of the basic difference differential equations has also been derived for each of the four processes.

From the Poisson process, we found the distribution of the increments to be a Poisson distribution with parameter $\lambda(t - s)$

$p_{k,k+n}(s,t) = e^{-\lambda(t-s)} \frac{\left[\lambda(t - s)\right]^n}{n!}$; $n = 0, 1, 2, ...$ This is independent the initial state and depends on the length of the time interval, thus for a Poisson processes the increments are independent and stationary. The first passage distribution was

$f_n(t) = \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n - 1)!} = \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1}$ for $t > 0$; $n = 1, 2, 3,.....$ which is gamma (n,λ)

From the Simple birth process, we found the distribution of the increments to be Negative binomial distribution with $p = e^{-\lambda(t-s)}$ and $q = 1 - e^{-\lambda(t-s)}$

$p_{k,k+n}(s,t) = \binom{k+n-1}{n} e^{-\lambda k(t-s)}\left[1 - e^{-\lambda(t-s)}\right]^n$. It depends on the length of the time interval $t - s$ and on k and is thus stationary and not independent.

The first passage distribution is $f_n(t) = \lambda(n-1)e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-2}$; $t > 0,\ n = 2, 3, \ldots$ which is an exponentiated exponential distribution with parameters λ and $n - 1$

From the Simple birth process with immigration, we found the distribution of the increments to be Negative binomial distribution with $p = e^{-\lambda(t-s)}$ and $q = 1 - e^{-\lambda(t-s)}$

$p_{k,k+n}(s,t) = \binom{k+\frac{v}{\lambda}+n-1}{n} e^{-\left(k+\frac{v}{\lambda}+n\right)(t-s)\lambda}\left(1-e^{-\lambda(t-s)}\right)^n$. It depends on the length of the time interval $t - s$ and on k and is thus stationary and not independent.

The first passage distribution is

$$f_n(t) = \frac{1}{\left(\frac{n_0\lambda + v - \lambda}{\lambda}\right)!} e^{-(n_0\lambda + v - \lambda)t}\left(1 - e^{-\lambda t}\right)^{n-n_0}\left\{(n\lambda + v - \lambda)e^{-\lambda t} - (n_0\lambda + v - \lambda)\right\}$$

From the Polya process, we found the distribution of the increments to be Negative binomial distribution with $p = \frac{1+\lambda as}{1+\lambda at}$ and $q = 1 - \frac{1+\lambda as}{1+\lambda at} = \frac{\lambda a(t-s)}{1+\lambda at}$.

$p_{k,k+j}(s,t) = \binom{j+k+\frac{1}{a}-1}{j}\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\left(\frac{\lambda a(t-s)}{1+\lambda at}\right)^j$ It depends on the length of the time interval $t - s$ and on k and is thus stationary and not independent. The first passage distribution is $f_n(t) = \frac{\Gamma\left(n+\frac{1}{a}\right)}{\Gamma\left(\frac{1}{a}\right)\Gamma(n)} \times \frac{\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}} t^{n-1}}{\left(t + \frac{1}{\lambda a}\right)^{n+\frac{1}{a}}}$, $\quad t > 0,\ n > 0,\ \frac{1}{a} > 0,\ \frac{1}{\lambda a} > 0$ which is a generalized Pareto distribution.

ACKNOWLEDGEMENT

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 Background Information

Consider a case of infectious disease transmission. When an infected person makes contact with susceptible individual there is potential transmission. A susceptible person makes the potential transition to infected person through a contact with infected persons. A newly infected person then makes further transitions to other infected states with the possibility of removal. The removal stage represents death caused by the diseases or eventual recovery. Those individual who have recovered reenters the susceptible stage. The average number of people that an infected person infect during his or her infectious period is the reproductive number and the mean number of affected individuals is the outbreak size. These transitions, for both susceptible and infected persons evolve over time. Such complex process can be described mathematically through what is called the continuous time stochastic processes.

Banks loan money to customers on trust that they will payback. The financial circumstance for the customer servicing loans may remain change within the loan period. If the financial status improves or remains stable the customer repays the loan without default and the loan status is said to be always current. When customers makes a full and final settlement before the end of the loan period attrition is said to have occurred. If the customer is financially constrained he defaults the loan. A customer who has default makes further transitions to other higher default states, early delinquency, mild delinquency and hard delinquency with the possibility paying the loan to fall back to current or lower delinquency status. When the customer defaults for a very long period, the loan is considered irrecoverable and bank write it off as bad debt and customer is removed from the loan book to written off book. In addition some customers die before clearing the loan. This process can be modeled using continuous time non homogenous Markov process with discrete states.

When pollen grains are suspended in water they move erratically. This erratic movement of the pollen grain is thought to be due to bombardment of the water molecules that surround the pollen grains. These bombardments occur many times in each small interval of time, they are independent of each other and the impact of a single hit is very small compared to the total effect. This suggests that the motion of pollen grains can studied as a continuous time stochastic process. A botanist, Brown was the first to observe it in 1927 and named the motion 'Brownian motion'.

1.2 Stochastic Processes

Definition and classifications

A stochastic process is a family of random variables $\{X(t);\ t > 0\}$ on a probability space with t ranging over a suitable parameter set T (t often represents time). The **state space** of the process is a set S in which possible values of each $X(t)$ lie. This $X(t)$ can be either discrete or continuous. The set of t is called **parameter space** T. The parameter t can also be either discrete or continuous. The parameter space is said to be discrete if the set T is countable otherwise it is continuous.

Thus, we have the following four classifications of stochastic processes.

1. Discrete Parameter and Discrete state
2. Discrete Parameter and Continuous state
3. Continuous Parameter and Discrete state
4. Continuous Parameter and Continuous state

Further, stochastic processes are broadly described according to the nature of dependence relationships existing among the members of the family. Some of the relationships are characterized by

(i) Stationary process

A process $\{X(t),\ t \in T\}$ is said to be a stationary if different observations on time intervals of the same length have the same distribution.

i.e. For any $s, t \in T$, $X(t + s) - X(t)$ has the same distribution as $X(s) - X(0)$.

(ii) Markov Processes

Markov process is a stochastic process whose dynamic behavior is such that probability distribution for its future development depends only on its present state, but not on the past history of the process or the manner in which the present state was reached. This property is referred to as the Markov or the memory less property.

A **Markov chain** is a discrete time space Markov process with discrete state space. **Markov jump process** is a continuous time space Markov process with discrete state space.

A Markov process is said to be **time homogenous** if the behavior of the system does not depend on when it is observed. In a time homogenous Markov process the transition rates between states are independent of the time at which the transition occurs. On the hand Markov process is said to be time non homogenous if the behavior of the system is depended on the time when it is observed.

1.3 Pure Birth Processes

Pure birth process is a continuous time, discrete state Markov process. Specifically, we deal with a family of random variables $\{X(t); 0 < t < \infty\}$ where the possible values of $X(t)$ are non negative integers. $X(t)$ represents the population size at time t and the transitions are limited to birth. When a birth occurs, the process goes from state n to state $n+1$. If no birth occurs, the process remains at the current state. The process cannot move from higher state to a lower state since there is no death. The birth process is characterized by the birth rate λ_n which varies according to the state n of the system.

The birth death process is a continuous time markov process where the states are the population size and the transition are restricted to birth and death. When a birth occurs the process jumps to the immediate higher state. Similarly when a death occurs the process jumps to the immediate lower state. In this process birth and death are assumed to be independent of each other. This process is characterized by birth rate and death rate which vary according to the state of the system. Pure birth process is birth death process with death rate equal to zero for all states.

1.4 Literature Review

The negative binomial arises from several stochastic processes. The time - homogeneous birth and immigration process with zero initial population was first obtained by McKendrick (1914).

The non - homogeneous process with zero initial population known as the Polya process was developed by Lundberg (1940) in the context of risk theory.

Other stochastic processes that lead to negative binomial distribution include the simple birth process with non – zero initial population size (Yule, 1925; Furry, 1937).

Kendall (1948) considered non homogeneous birth – and – death process with zero death rate. He also worked on the simple birth – death – and immigration process with zero initial population (Kendall 1949).

A remarkable new derivation as the solution of the simple birth – and – immigration process was given by Mckendrick (1914).

Kendall (1948) formed Lagrange's equation from the differential difference equations for a distribution of the population, and via auxiliary equation, obtained a complete solution of the equations governing the generalized birth and death process in which the birth rate and death rates may be any specified function of time.

Karlin and Mcgregor (1958) expressed transitional probabilities of birth and death processs (BDPs) in terms of a sequence of orthogonal polynomials and spectra Measures. Birth rate and death rate uniquely determine the unique measure on the real axis with respect to the sequence of orthogonal polynomials. Their work gave valuable insights about the existence of unique solution of a given process.

Gani and swift (2008) attempted to derive equations for the probability generating function of the Poisson, pure birth and death process subject to mass movement immigration and emigration. He considered mass movement immigration and emigration as positive and negative mass movements. The resulting probability generating functions turned out to be a product of the probability generating functions of the original processes modified by immigration process.

Simple birth process was originally introduced by Yule (1924) to model new species evolution and by Furry (1937) to model particle creation.

Devaraj and Fu (2008) developed a non-homogenous Markov chain to describe deterioration on bridge elements. They showed that the new model could better predict bridge element deterioration trends.

Wolter et al (1998) presented and compared three different uniformization based algorithms; base algorithm, correction algorithm and interval splitting algorithm, for numerically computing transient state distribution for non homogenous model. They investigated the numerical solution of the transient state probability vector of non homogenous markov process by algorithms that are based on unifomization by Grassmann(1991) and Jensen (1953). The three algorithms numerically solved the integral equations that constituted the uniformization equation as derived by Dijk(1992).

1.5 Problem Statement and Objectives of the Study

Problem Statement

Most of the literatures concerning pure birth processes concentrate on deriving and solving basic difference differential equations for time homogeneous processes and rarely do they present non homogenous birth processes. The few literatures that have presented the non homogeneous birth processes only sketch them in outline form or scatter details in different sections. In the analysis, emphasis is often laid on the steady state solutions while transient or time dependent analysis has received less attention. The assumptions required to derive steady state solutions are seldom satisfied in analysis of real systems. Some systems never approach equilibrium and for such systems steady states measures of performance do not make sense. In additions, systems with finite time horizon of operations, steady state results are inappropriate. Generally, in real life situation, knowledge on the time dependent behavior of the system is needed rather than the easily obtained steady state solution.

Objectives

The aim of this thesis is to derive and solve differential equations for continuous time non homogenous pure birth process by applying four alternative approaches; Iteration method, Kolmogorov equations and solving directly the Lagrange partial differential equation for the generating function by means of auxiliary equations, Laplace transform method and the generator matrix approach. It is hoped that this will demonstrate to statistics scholars the variety of mathematical tools that can be used to solve non homogenous pure birth processes. At the same time it is also hoped that students of mathematics will find in this processes interesting application of standard mathematical methods.

1.6 Areas of application

Pure birth and death processes play a fundamental role in the theory and applications that embrace population growth. Examples include the spread of new infections in cases of a disease where each new infection is considered as a birth.

Pure birth and death processes have a lot of application in the following areas.

(a) Biological field

When an infected person makes contact with susceptible individual there is potential transmission. A susceptible person makes the potential transition to infected person through a contact with infected persons. A newly infected person then makes further transitions to other infected states with the possibility of removal. The removal stage represents death caused by the diseases or eventual recovery. Those individual who have recovered reenters the susceptible stage. The average number of people that an infected person infect during his or her infectious period is the reproductive number and the mean number of affected individuals is the outbreak size. These transitions, for both susceptible and infected persons evolve over time. Such complex process can be described mathematically through what is called the continuous time stochastic processes.

(b) Radioactivity

Radioactive atoms are unstable and disintegrate stochastically. Each of the new atoms is also unstable. By the emission of radioactive particles these new atoms pass through a number of physical states with specified decay rates from one state to the adjacent. Thus radioactive transformation can be modeled as birth process.

(c) Communication

Suppose that calls arrive at a single channel telephone exchange such that successive calls arrivals are independent exponential random variables. Suppose that a connection is realized if the incoming call finds an idle channel. If the channel is busy, then the incoming call joins the queue. When the caller is through, the next caller is connected. Assuming that the successive service times are independent exponential variables, the number of callers in the system at time t is described by a birth and death process.

CHAPTER TWO

TRANSITION PROBABILITIES FOR A GENERAL PURE BIRTH PROCESS

2.1 Introduction

A **stochastic process** $\{N_t : t > 0\}$ is a collection of random variables, indexed by the variable t (which often represents time).

A **counting process** is a stochastic process in which N_t must be a non – negative integer and for $t > s,\ N_t \geq N_s$.

Our interest is in the variable $N_t - N_s$. We refer to this variable as an increment in the interval (s, t). Consider the set of events $\{N_j\}$ where N_j is the j^{th} state. The probability of moving from state N_j to state N_k is denoted by p_{jk} and is called the transitional probability jk. Further, $p_{jk} = \text{Prob}\left(N_k = k | N_j = j\right)$

Definition 1

A stochastic process has **stationary increment** if the distribution of $N_t - N_s$ for $t > s$ depend only on the length of the interval.

Remark

One consequence of stationary increments is that the process does not change over time. Some researchers call this process time homogeneous also.

Definition 2

A stochastic process has **independent increments** if increments for any set of disjoint intervals are independent.

We shall examine counting processes that are **Markovian**. A loose definition is that for $t > s$, the distribution $N_t - N_s$ given N_s is the same as if any of the values $N_s, N_{u_1}, N_{u_2}, \ldots$ with all $u_1, u_2, \ldots \leq s$ were given.

For a Markovian counting process, the probabilities of greatest interest are the transitional probabilities given by

$$p_{k,\, k+n}(s, t) = \text{Prob}\{N_t - N_s = n / N_t = k\} \qquad \textbf{(2.1)}$$

for $0 \leq s < t < \infty$.

The marginal distribution of the increment $N_t - N_s$ may be obtained by an application of the law of total probability.

That is

$$\text{Prob}\{N_t - N_s = n\} = \sum_{k=0} \text{Prob}\{N_t - N_s = n / N_s = k\} \text{Prob}\{N_s = k\}$$

$$= \sum_{k=0} p_{k,\,k+n}(s,\,t)\; p_k(s) \qquad \textbf{(2.2)}$$

2.2 Chapman-Kolmogorov Equations

"Feller (1968), states that the transition probabilities of time homogenous Markov process satisfy the Chapman – Kolmogorov equation.

$$p_{in}(s + t) = \sum_{v} p_{iv}(s) p_{vn}(t) \qquad \textbf{(2.3)}$$

The transitional probabilities of time non-homogenous Markov process satisfies the Chapman-Kolmogorov equation, namely

$$p_{in}(\tau, t) = \sum_{v} p_{iv}(\tau, s) p_{vn}(s, t) \qquad \textbf{(2.4)}$$

and is valid for $\tau < s < t$.

This relation expresses the fact that a transition from the state E_i at epoch(time) τ to E_n at epoch t occurs via some state E_v at the intermediate epoch s, and for Markov process the probability $P_{vn}(s, t)$ of the transition from E_v to E_n is independent of the previous state E_i.

The transition probabilities of Markov processes with countably many states are therefore solutions of the Chapman-Kolmogorov identity **(2.4)** satisfying the side conditions

$$p_{ik}(\tau, t) \geq 0, \sum_{v} p_{ik}(\tau, s) = 1\text{"} \qquad \textbf{(2.5)}$$

2.2.1 Time Homogeneous Markov Process

Proof of formulae (2.3)

The transition probability

$p_{in}(s+t)$ = the probability of moving from state E_i at time 0 to state E_j at time s + t, passing via some state E_v in time s.

$$= \sum_v \text{Prob}\{X(s+t)=j, X(t)=v / X(0)=i\}$$

$$= \sum_v \frac{\text{Prob}\{X(s+t)=j, X(t)=v, X(0)=i\}}{\text{Prob}\{X(0)=i\}}$$

$$= \sum_v \frac{\text{Prob}\{X(s+t)=j / X(t)=v, X(0)=i\}\text{Prob}\{X(t)=v / X(0)=i\}\text{Prob}\{X(0)=i\}}{\text{Prob}\{X(0)=i\}}*$$

Therefore

$$p_{in}(s+t) = \sum_v \text{Prob}\{X(s+t)=j / X(t)=v, X(0)=i\}\text{Prob}\{X(t)=v / X(0)=i\}$$

Because of Markov property,

$$p_{in}(s+t) = \sum_v \text{Prob}\{X(s+t)=j / X(t)=v\}\text{Prob}\{X(t)=v / X(0)=i\}$$

The equation above can also be written in the form

$$p_{in}(s+t) = \sum_v p_{vj}(t)p_{iv}(t)$$

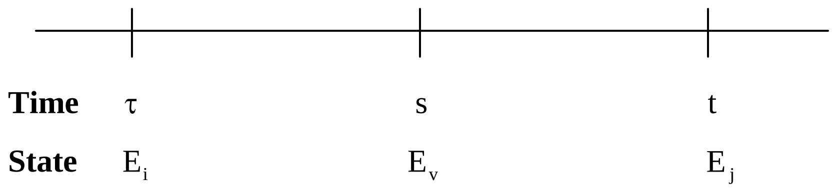
$$= \sum_v p_{iv}(t)\, p_{vj}(t)*$$

2.2.2 Time Non-homogeneous Markov Process

Proof of formula (2.4)

Consider a transition from the state E_i at epoch (time) τ to state E_j at epoch t occurs via some state E_v at the intermediate epoch s.

Diagrammatically,



$$p_{in}(\tau,t) = \sum_v \text{Prob}\{X(\tau,t)=n,\ X(\tau,s)=v / X(\tau)=i\}$$

$$= \sum_v \frac{\text{Prob}\{X(\tau,t)=n,\ X(\tau,s)=v,\ X(\tau)=i\}}{\text{Prob}\{X(\tau)=i\}}$$

$$= \sum_v \frac{\text{Prob}\{X(\tau,t)=n / X(\tau,s)=v,\ X(\tau)=i\}\text{Prob}\{X(\tau,s)=v,\ X(\tau)=i\}}{\text{Prob}\{X(\tau)=i\}}$$

$$= \sum_v \text{Prob}\{X(\tau,t)=n / X(\tau,s)=v,\ X(\tau)=i\}\text{Prob}\{X(\tau,s)=v / X(\tau)=i\}$$

Therefore

$$p_{in}(\tau,t) = \sum_v \text{Prob}\{X(\tau,t)=n / X(\tau,s)=v\}\text{Prob}\{X(\tau,s)=v / X(\tau)=i\}$$

$$= \sum_v P_{vn}(s,t)P_{iv}(\tau,s) = \sum_v P_{iv}(\tau,s)P_{vn}(s,t)$$

2.3 Forward and Backward Kolmogorov Differential Equations

2.3.1 Forward Kolmogorov Differential Equation

Using notations by Klugman, Panjer and Withalt (2008) replace i by k and n by $k + n$.

Therefore,

$$p_{k,k+n}(\tau,t) = \sum_{v} p_{kv}(\tau,s)\, p_{v,k+n}(s,t)$$

Next, replace τ by s, s by t and t by $t + h$. Thus we have

$$p_{k,k+n}(s,t+h) = \sum_{v} p_{kv}(s,t)\, p_{v,k+n}(t,t+h)$$

Finally, replace v by k + j

$$p_{k,k+n}(s,t+h) = \sum_{j=0}^{n} p_{k,k+j}(s,t)\, p_{k+j,k+n}(t,t+h) \qquad \textbf{(2.6)}$$

This can also be re-written as

$$p_{k,k+n}(s,t+h) = \sum_{j=0}^{n-2} p_{k,k+j}(s,t)\, p_{k+j,k+n}(t,t+h) + p_{k,k+n-1}(s,t)\, p_{k+n-1,k+n}(t,t+h) + p_{k,k+n}(s,t)\, p_{k+n,k+n}(t,t+h) \qquad \textbf{(2.7)}$$

But for a pure birth process

$$p_{k,k+1}(t,t+h) = \lambda_k(t)h + o(h); \qquad k = 0, 1, 2, \ldots \qquad \textbf{(2.8)}$$

$$p_{kk}(t,t+h) = 1 - \lambda_k(t)h + o(h); \qquad k = 0, 1, 2, \ldots \qquad \textbf{(2.9)}$$

and

$$p_{k,k+j}(t,t+h) = o(h); \qquad j = 2, 3, \ldots, n \qquad \textbf{(2.10)}$$

Apply **(2.8)**, **(2.9)** and **(2.10)** in **(2.7)** to get

$$p_{k,k+n}(s,t+h) = o(h)\sum_{j=0}^{n-2} p_{k,k+j}(s,t) + \left[\lambda_{k+n-1}(t)h + o(h)\right] p_{k,k+n-1}(s,t) + \left[1 - \lambda_{k+n-1}(t)h + o(h)\right] p_{k\,k+n}(s,t)$$

Therefore,

$$\lim_{h\to 0}\frac{p_{k\,k+n}(s,t+h)-p_{k\,k+n}(s,t)}{h} =$$

$$\lim_{h\to 0}\frac{o(h)}{h}\sum_{j=0}^{n-2}p_{k,k+j}(s,t) + \lim_{h\to 0}\left[\frac{\lambda_{k+n-1}(t)h+o(h)}{h}\right]p_{k,k+n-1}(s,t) + \lim_{h\to 0}\left[\frac{-\lambda_{k+n}(t)h+o(h)}{h}\right]p_{k,k+n}(s,t)$$

i.e.,

$$\frac{\partial}{\partial t}p_{k,k+n}(s,t) + \lambda_{k+n}(t)p_{k,k+n}(s,t) = \lambda_{k+n-1}(t)\,p_{k,k+n-1}(s,t) \qquad \textbf{(2.11)}$$

where

$$p_{k,k-1}(s,t) = 0 \qquad \textbf{(2.12)}$$

In general $\frac{\partial}{\partial t}p_{ij}(s,t) + \lambda_j(t)p_{ij}(s,t) = \lambda_{j-1}(t)\,p_{i\,j-1}(s,t)$ where $p_{j\,j-1}(s,t) = 0$.

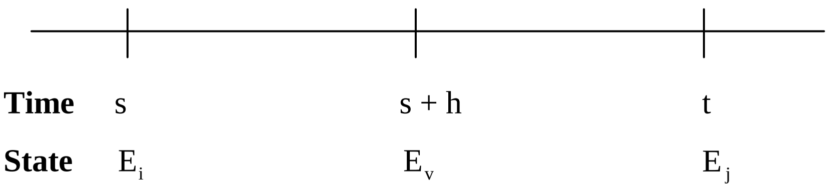
For this non-homogeneous birth process, the initial conditions are:

$$p_{k\,k}(s,s) = 1 \text{ and } p_{k,k+n}(s,s) = 0 \text{ for } n > 0. \qquad \textbf{(2.13)}$$

2.3.2 Backward Kolmogorov Differential Equations

Backward Kolmogorov Differential Equations

Consider the following diagram



$$p_{i\,j}(s,t) = \sum_k p_{i\,k}(s,s+h)p_{k\,j}(s+h,t)$$

But for a pure birth process

$$p_{i,i+1}(s,s+h) = \lambda_i(s)h + o(h)$$

$$p_{i\,i}(s,s+h) = \left[1 - \lambda_i(s).h + o(h)\right]$$

And $p_{i\,k}(s,s+h) = o(h)$; for $k \geq i + 2$

Therefore,

$$p_{ij}(s,t) = p_{ii}(s,s+h)p_{ij}(s+h,t)+p_{i,i}(s,s+h)p_{i+1,j}(s+h,t) + \sum_{k=i+2}^{j} p_{ik}(s,s+h)p_{kj}(s+h,t)$$

$$= \left[1 - \lambda_i(s).h + o(h)\right]p_{ij}(s+h,t)+\left[\lambda_i(s)h + o(h)\right]p_{i+1,j}(s+h,t) + \sum_{k=i+2}^{j} o(h)p_{kj}(s+h,t)$$

Therefore,

$$p_{ij}(s,t) - p_{ij}(s+h,t) = \left[- \lambda_i(s).h + o(h)\right]p_{ij}(s+h,t)+\left[\lambda_i(s)h + o(h)\right]p_{i+1,j}(s+h,t) + \sum_{k=i+2}^{j} o(h)p_{kj}(s+h,t)$$

$$\frac{\partial}{\partial s}p_{ij}(s,t) = \lim_{h\to 0}\frac{p_{ij}(s,t) - p_{ij}(s+h,t)}{h}$$

$$= - \lambda_i(s)p_{ij}(s,t)+\lambda_i(s)p_{i+1,j}(s,t) \qquad \textbf{(2.14)}$$

NB. The duration (s,t) > duration$(s + h,t)$

This is the Kolmogorov Backward Differential equation.

2.4 Methods of Solving Continuous Time Non-homogenous Forward Kolmogorov Differential Equations

Equation **(2.11)** can be solved using the following techniques. In Chapter three, we shall use the integrating factor method. In chapter four, we shall use the Lagrange's Method. In chapter five, we shall use the Matrix method. Highlights of the key steps in each of these three methods are given below. Some comments on the Laplace Method have also been given even though it has not been used.

2.4.1 Integrating Factor Technique

Let a differential equation be of the form

$$\frac{dy}{dx} + ay = R \qquad \textbf{(2.15)}$$

where y is a function of x and R is a constant or a function of x.

To solve for y in this kind of differential equation, given the initial conditions, we use the following procedure.

Procedure

- Let the integrating factor IF $= e^{\int a\,dx}$
- Multiply both sides of the equation (of the form **(2.15)**) you want to solve with the integrating factor.
- Express the new equation in the form $\frac{d}{dx}\left[e^{\int a\,dx} \times y\right] = R \times e^{\int a\,dx}$
- Integrate both sides with respect to x. Use the initial condition to find the constant of integration.
- Find y.

2.4.2 Lagrange's Method

Let P, Q and R be functions of x, y and z. suppose we have an equation of the form

$$P\frac{dz}{dx} + Q\frac{dz}{dy} = R \qquad \textbf{(2.16)}$$

subject to some appropriate boundary conditions. Such an equation is called a linear partial differential equation. The Lagrange method of solving this kind of equation is described below. The **probability generating function** (pgf) is one of the major analytical tools used to work with stochastic processes on discrete state spaces.

Definition

Let X be a non negative integer valued random variable such that $P[X = k] = p_k$ k= 0,1,2,…is the probability mass function (pmf). Then the probability generating function is given by

$$G(s) = E\left[S^X\right] = \sum_{k=0}^{\infty} p_k s^k$$

If we are able to obtain the pgf of a stochastic process, then we can find the pmf of the process. The pmf p_k is the coefficient of s^k

Procedure

- Define $G(s,t,z) = \sum_{n=0}^{\infty} p_{jk}(s,t) z^n = \sum_{k=j}^{\infty} p_{jk}(s,t) z^n$. Consequently define $\frac{\partial}{\partial s} G(s,t,z)$ and $\frac{\partial}{\partial z} G(s,t,z)$.

- Multiply both sides of the second differential equation by s^n and sum over n, and taking advantage of the initial conditions, write the resulting equation in terms of the definitions above.
- Summarize the results by a single Lagrange Partial differential equation for a generation function of the form $P\frac{\partial}{\partial t}G(s,t,z) + Q\frac{\partial}{\partial z}G(s,t,z) = R$.
- Form the resulting auxiliary equations from the equation above. It will be of the form;

$$\frac{\partial t}{P} = \frac{\partial z}{Q} = \frac{\partial G(s,t,z)}{R}$$

- We can form three subsidiary equations from the equation above. These are

$$\frac{\partial t}{P} = \frac{\partial z}{Q} \qquad (i)$$

$$\frac{\partial t}{P} = \frac{\partial G(s,t,z)}{R} \qquad (ii)$$

$$\frac{\partial z}{Q} = \frac{\partial G(s,t,z)}{R} \qquad (iii)$$

- Consider any two equations and solve them
- Solutions of the two considered subsidiary equations are in the form

$$U(s,t,z) = \text{Constant} \qquad \textbf{(2.17)}$$

and

$$V(s,t,z) = \text{Constant} \qquad \textbf{(2.18)}$$

- The most general solution of **(2.16)** is now given by

$$u = \psi(v)$$

where ψ is an arbitrary function. The precise form of this function is determined when the boundary conditions have been inserted.

2.4.3 Laplace Transforms

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt \qquad \textbf{(2.19)}$$

Using integration by parts

$$\int v\, du = uv - \int u\, dv \qquad \textbf{(2.20)}$$

Let $v = e^{-st} \Rightarrow \frac{dv}{dt} = -se^{-st}$

Also, let $du = f'(t) \quad \Rightarrow \quad u = f(t)$

Substituting in equation **(2.20)**

$$\int_0^\infty e^{-st} f'(t)\, dt = \left[f(t).e^{-st}\right]_0^\infty - \int_0^\infty f(t).-se^{-st}\, dt$$

$$= \left[f(t).e^{-st}\right]_0^\infty + s\int_0^\infty f(t).\, e^{-st}\, dt$$

$$\int_0^\infty e^{-st} f'(t)\, dt = \left[(0) - (f(0))\right] + s\int_0^\infty f(t).\; e^{-st}\, dt$$

$$= s\int_0^\infty f(t).\; e^{-st}\, dt - f(0)$$

Thus

$$L\left[f'(t)\right] = sL\left[f(t)\right] - f(0)$$

Substituting $f(t)$ with $p_{ij}(t)$, the equation above becomes

$$L\left[p'_{ij}(t)\right] = sL\left[p_{ij}(t)\right] - p_{ij}(0) \tag{2.21}$$

Procedure

- Take the Laplace transform of the second of the two basic difference equations.
- Apply the relation $L\left[p'_{ij}(t)\right] = sL\left[p_{ij}(t)\right] - p_{ij}(0)$ to replace $L\left[p'_{ij}(t)\right]$ and simplify leaving $L\left[p_{ij}(t)\right]$ as the subject of the formula.
- Starting with the conditions at $t = 0$, generate a recursive relation and use it to generalize for $L\left[p_{ij}(t)\right]$.
- Find the Laplace inverse of the $L\left[p_{ij}(t)\right]$ so got.

2.4.4 Generator Matrix Method

- Express the differential equation in Matrix form.
- Obtain the eigen values and the corresponding eigen vectors.
- If Λ has distinct eigen values $\mu_1, \mu_2, ..., \mu_k$, say, then $\Lambda = UDV$ where $V = U^{-1}$ and

  $D = \text{diag}\left(\mu_1, \mu_2, ..., \mu_k\right)$ and the ith column of U is the right eigen vector associated with μ_i.
- Work out $p(t) = U\ \text{diag}\left(e^{\mu_1 t}, ..., e^{\mu_k t}\right)\ V$
- Substitute λ_n in the equation

$$p_{j\,j+n}(s,t) = \left\{\prod_{k=0}^{n-1}(j+k)\lambda_n\right\}\left\{\sum_{v=0}^{n}\frac{e^{-(j+v)\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq v}}^{n}(j+k)\lambda_n - (j+v)\lambda_n}\right\}$$ and simplify.

CHAPTER THREE

TRANSITIONAL PROBABILITIES BASED ON THE INTEGRATING FACTOR TECHNIQUE

3.1 Introduction

In this chapter, we shall solve equation **(2.11)** using the integrating factor technique. We shall solve this equation for all the four pure birth processes starting with Poisson Process, then the Simple Birth process, the Simple Birth Process with immigration and finally the Polya Process. By putting n = 0 in equation **(2.11)**, we shall find $p_{k\,k}(s,t)$. Putting other higher values of n in the same equation will generate a recursive relation. Each generated equation will be solved using the integrating factor technique. We shall then generalize each case by induction.

3.2 Determining $p_{k\,k}(s,t)$

When n = 0, equation **(2.11)** becomes

$$\frac{\partial}{\partial t} p_{k\,k}(s,t) + \lambda_k(t) p_{k\,k}(s,t) = \lambda_{k-1}(t)\, p_{k,k-1}(s,t)$$

But from **(2.12)**, $p_{k,k-1}(s,t) = 0.$ Therefore

$$\frac{\partial}{\partial t} p_{k\,k}(s,t) + \lambda_k(t) p_{k\,k}(s,t) = 0$$

$$\Rightarrow$$

$$\frac{1}{p_{k\,k}(s,t)} \frac{d}{dt} p_{k\,k}(s,t) = -\lambda_k(t)$$

$$\Rightarrow$$

$$\frac{\partial}{\partial t} \log P_{k\,k}(s,t) = -\lambda_k(t)$$

Therefore,

$$\log p_{k\,k}(s,t)\Big|_s^t = -\int_s^t \lambda_k(x)\, dx$$

Equivalently

$$\log \frac{p_{k\,k}(s,t)}{p_{k\,k}(s,s)} = -\int_s^t \lambda_k(x)\, dx$$

Therefore

$$\Rightarrow$$

$$\log\ p_{k\,k}(s,t) = -\int_s^t \lambda_k(x)\,dx$$

$$\Rightarrow$$

$$p_{k\,k}(s,t) = e^{-\int_s^t \lambda_k(x)\,dx} \qquad \textbf{(3.1)}$$

For $n > 0,$ use the integrating factor $e^{-\int_s^t \lambda_{k+n}(t)\,dt}$ in equation **(2.11)**.

We shall now look for special cases.

3.3 Special Cases of Non-Homogeneous Markov Processes.

3.3.1 Poisson Process

$\lambda_k = \lambda$ for all k.

$$p_{k\,k}(s,t) = e^{-\int_s^t \lambda\,dx} = e^{-\lambda(t-s)} \qquad \textbf{(3.2)}$$

For $n > 0,$ the differential equation for the transition probabilities is

$$\frac{\partial}{\partial t} p_{k,k+n}(s,t) + \lambda p_{k,k+n}(s,t) = \lambda\ p_{k,k+n-1}(s,t) \qquad \textbf{(3.3)}$$

When n = 1,

$$\frac{\partial}{\partial t} p_{k,k+1}(s,t) + \lambda p_{k,k+1}(s,t) = \lambda P_{k\,k}(s,t) = \lambda e^{-\lambda(t-s)}$$

$\text{Integrating factor} = e^{\int \lambda\,dt} = e^{\lambda t}$

Therefore,

$$e^{\lambda t}\frac{\partial}{\partial t} p_{k,k+1}(s,t) + \lambda e^{\lambda t} p_{k,k+1}(s,t) = \lambda e^{\lambda t} e^{-\lambda(t-s)}$$

$$\Rightarrow$$

$$\frac{\partial}{\partial t}\left[e^{\lambda t}\, p_{k,k+1}(s,t)\right] = \lambda e^{\lambda s}$$

Therefore,

$$\int_s^t \partial\left\{e^{\lambda y}\, p_{k,k+1}(s,y)\right\} = \int_s^t \lambda\, e^{\lambda s}\, \partial y$$

$$\Rightarrow$$

$$e^{\lambda y}\, p_{k,k+1}(s,y)\Big|_s^t = \int_s^t \lambda\, e^{\lambda s}\, \partial y$$

Therefore

$$e^{\lambda t}\, p_{k,k+1}(s,t) - e^{\lambda s}\, p_{k,k+1}(s,s) = \lambda e^{\lambda s}(t-s)$$

$$\Rightarrow$$

$$e^{\lambda t}\, p_{k,k+1}(s,t) - 0 = \lambda(t-s)e^{\lambda s}$$

Equivalently

$$p_{k,k+1}(s,t) = \lambda(t-s)e^{-\lambda(t-s)} \qquad \textbf{(3.4)}$$

When n = 2, equation **(2.11)** becomes

$$\frac{\partial}{\partial t} p_{k,k+2}(s,t) + \lambda p_{k,k+2}(s,t) = \lambda^2 (t-s)\, e^{-\lambda(t-s)}$$

Integrating Factor $= e^{\lambda t}$. Multiplying the above equation by the Integrating factor, we have

$$e^{\lambda t} \frac{\partial}{\partial t} p_{k,k+2}(s,t) + \lambda e^{\lambda t} p_{k,k+2}(s,t) = \lambda^2 (t-s) e^{\lambda t}\, e^{-\lambda(t-s)}$$

$$\frac{\partial}{\partial t}\left[e^{\lambda t} p_{k,k+2}(s,t)\right] = \lambda^2 (t-s) e^{\lambda s}$$

Integrating both sides

$$e^{\lambda y} p_{k,k+2}(s,y)\Big|_s^t = \int_s^t \lambda^2 (y-s) e^{\lambda s}\, dy$$

Therefore,

$$e^{\lambda t} p_{k,k+2}(s,t) - e^{\lambda s} p_{k,k+2}(s,s) = \lambda^2 e^{\lambda s}\left[\frac{y^2}{2} - sy\right]_s^t$$

$$= \lambda^2 e^{\lambda s}\left[\left(\frac{t^2}{2} - st\right) - \left(\frac{s^2}{2} - s^2\right)\right]$$

$$e^{\lambda t}p_{k,k+2}(s,t) - e^{\lambda s}p_{k,k+2}(s,s) = \lambda^2 e^{\lambda s}\left[\frac{t^2}{2} - st - \frac{s^2}{2} + s^2\right]$$

$$= \frac{\lambda^2 e^{\lambda s}}{2}\left[t^2 - 2st + s^2\right]$$

$$= \frac{\lambda^2 e^{\lambda s}}{2}(t - s)^2$$

Equivalently, using **(2.21)**

$$e^{\lambda t}p_{k,k+2}(s,t) = \frac{\left[\lambda(t - s)\right]^2 e^{\lambda s}}{2!}$$

Therefore,

$$p_{k,k+2}(s,t) = e^{-\lambda(t-s)}\frac{\left[\lambda(t - s)\right]^2}{2!} \tag{3.5}$$

By induction, assume

$$p_{k,k+n-1}(s,t) = e^{-\lambda(t-s)}\frac{\left[\lambda(t - s)\right]^{n-1}}{(n - 1)!}$$

Then equation **(2.11)** becomes

$$\frac{\partial}{\partial t}p_{k,k+n}(s,t) + \lambda p_{k,k+n}(s,t) = \lambda e^{-\lambda(t-s)}\frac{\left[\lambda(t - s)\right]^{n-1}}{(n - 1)!}$$

Integrating factor $= e^{\lambda t}$

Multiplying the above equation by the integrating factor, we get

$$e^{\lambda t}\frac{\partial}{\partial t}p_{k,k+n}(s,t) + \lambda e^{\lambda t}p_{k,k+n}(s,t) = \lambda e^{-\lambda(t-s)}e^{\lambda t}\frac{\left[\lambda(t - s)\right]^{n-1}}{(n - 1)!}$$

$$\frac{\partial}{\partial t}\left[e^{\lambda t}p_{k,k+n}(s,t)\right] = \lambda e^{\lambda s}\frac{\left[\lambda(t - s)\right]^{n-1}}{(n - 1)!}$$

Integrating both sides

$$e^{\lambda t}p_{k,k+n}(s,t)\Big|_s^t = \frac{\lambda^n e^{\lambda s}}{(n - 1)!}\int_s^t (y - s)^{n-1}\partial y$$

Let $u = y - s \quad \Rightarrow \quad \frac{du}{dy} = 1$ or equivalently $du = dy$. When $y = s,\ u = 0$. When $y = t,\ u = t - s$.

Then

$$e^{\lambda t}p_{k,k+n}(s,y)\Big|_s^t = \frac{\lambda^n e^{\lambda s}}{(n-1)!}\int_0^{t-s} u^{n-1}\,du$$

$$= \frac{\lambda^n e^{\lambda s}}{(n-1)!}\left[\frac{u^n}{n}\right]_0^{t-s}$$

$$= \frac{\lambda^n e^{\lambda s}}{n(n-1)!}(t-s)^n$$

$$= e^{\lambda s}\frac{\left[\lambda(t-s)\right]^n}{n!}$$

Therefore,

$$e^{\lambda t}p_{k,k+n}(s,t) - e^{\lambda t}p_{k,k+n}(s,s) = e^{\lambda s}\frac{\left[\lambda(t-s)\right]^n}{n!}$$

$$e^{\lambda t}p_{k,k+n}(s,t) = e^{\lambda s}\frac{\left[\lambda(t-s)\right]^n}{n!}$$

Therefore

$$\mathbf{p_{k,k+n}(s,t) = e^{-\lambda(t-s)}\frac{\left[\lambda(t-s)\right]^n}{n!};\ n = 0, 1, 2, \ldots} \qquad \textbf{(3.6)}$$

This is independent the initial state and depends on the length of the time interval, thus for a Poisson processes the increments are independent and stationary.

Remarks;

Formula **(3.6)** does not depend on k, which reflects the fact that the increments are independent.

From **(2.2)**, the marginal distribution of the increment $N_t - N_s$ is thus given by

$$\text{Prob}\{N_t - N_s = n\} = \sum_{k=0} p_{k,k+n}(s,t)\,p_k(s)$$

$$= \sum_{k=0} e^{-\lambda(t-s)}\frac{\left[\lambda(t-s)\right]^n}{n!}p_k(s)$$

$$= e^{-\lambda(t-s)}\frac{\left[\lambda(t-s)\right]^n}{n!}\sum_{k=0} p_k(s)$$

Therefore,

$$\text{Prob}\{N_t - N_s = n\} = e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^n}{n!}; \; n = 0, 1, 2, ... \tag{3.7}$$

Which depends only on $t - s$. Therefore, the process is stationary.

Thus, the poisson process is a homogeneous birth process with stationary and independent increments.

Note that

$$\text{Prob}\{N_t - N_0 = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

Since N_t is the number of events that have occurred up to time t, it is convenient to define $N_0 = 0.$

Therefore,

$$\text{Prob}\{N_t = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

i.e.

$$\mathbf{p_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}}; \; n = 0, 1, 2, ... \tag{3.8}$$

N_t has a Poisson distribution with mean λt.

3.3.2 Simple Birth Process

$\lambda_n = k\lambda$, for $k = 0, 1, 2, ...$

For $n = 0$, **(2.11)** becomes **(3.1)**. Therefore,

$$p_{k\,k}(s,t) = e^{-\int_s^t k\lambda \, dx} = e^{-k\lambda(t-s)} \tag{3.9}$$

For $n > 0$, **(2.11)** becomes

$$\frac{\partial}{\partial t} p_{k,k+n}(s,t) + (k+n)\lambda p_{k,k+n}(s,t) = (k+n-1)\lambda \, p_{k,k+n-1}(s,t) \tag{3.10}$$

For $n = 1$,

$$\frac{\partial}{\partial t} p_{k,k+1}(s,t) + (k+1)\lambda p_{k,k+1}(s,t) = k\lambda \, p_{k\,k}(s,t) = k\lambda e^{-k\lambda(t-s)}$$

Integrating factor $= e^{\int (k+1)\lambda dt} = e^{(k+1)\lambda t}$

Multiplying the equation above by the integrating factor, we have

$$e^{(k+1)\lambda t}\frac{\partial}{\partial t}p_{k,k+1}(s,t) + (k+1)e^{(k+1)\lambda t}\lambda p_{k,k+1}(s,t) = k\lambda e^{(k+1)\lambda t}e^{-k\lambda(t-s)}$$

$$\text{R.H.S} = k\lambda e^{(k+1)\lambda t}e^{-k\lambda(t-s)} = k\lambda \times e^{k\lambda t}\times e^{\lambda t}\times e^{-k\lambda t}\times e^{k\lambda s} = k\lambda \times e^{\lambda t}\times e^{k\lambda s} = k\lambda \times e^{\lambda(t+ks)}$$

Therefore,

$$\frac{\partial}{\partial t}\left[e^{(k+1)\lambda t}p_{k,k+1}(s,t)\right] = k\lambda e^{\lambda t}e^{k\lambda s}$$

Integrating both sides with respect to t, we have

$$e^{(k+1)\lambda y}p_{k,k+1}(s,y)\Big|_s^t = k\lambda e^{\lambda ks}\int_s^t e^{\lambda y}dy$$

$$= k\lambda e^{\lambda ks}\left[\frac{e^{\lambda y}}{\lambda}\right]_s^t$$

$$= \frac{k\lambda e^{\lambda ks}}{\lambda}\left[e^{\lambda t} - e^{\lambda s}\right]$$

$$= ke^{\lambda ks}\left[e^{\lambda t} - e^{\lambda s}\right]$$

Therefore,

$$p_{k,k+1}(s,t) = ke^{\lambda ks}\left[e^{\lambda t-(k+1)\lambda t} - e^{\lambda s-(k+1)\lambda t}\right]$$

$$= ke^{\lambda ks}\left[e^{-k\lambda t} - e^{\lambda s-k\lambda t-\lambda t}\right]$$

$$p_{k,k+1}(s,t) = ke^{\lambda ks}e^{-k\lambda t}\left[1 - e^{\lambda s-\lambda t}\right]$$

$$= ke^{\lambda k(s-t)}\left[1 - e^{\lambda s-\lambda t}\right]$$

$$= ke^{\lambda k(s-t)}\left[1 - e^{-\lambda(t-s)}\right]$$

$$= ke^{-\lambda k(t-s)}\left[1 - e^{-\lambda(t-s)}\right]$$

$$= \binom{k}{1}e^{-\lambda k(t-s)}\left[1 - e^{-\lambda(t-s)}\right] \quad \textbf{(3.11)}$$

For $n = 2$

$$\frac{\partial}{\partial t} p_{k,k+2}(s,t) + (k+2)\lambda p_{k,k+2}(s,t) = (k+1)\lambda p_{k,k+1}(s,t) = (k+1)\lambda\left[ke^{-\lambda k(t-s)}\left(1 - e^{-\lambda(t-s)}\right)\right]$$

Integrating factor $= e^{\int \lambda(k+2)dt} = e^{\lambda(k+2)t}$

Multiplying the above equation by the integrating factor, we have

$$e^{\lambda(k+2)t}\frac{\partial}{\partial t} p_{k,k+2}(s,t) + \lambda(k+2)e^{\lambda(k+2)t} p_{k,k+2}(s,t) = \lambda k(k+1)e^{\lambda(k+2)t}e^{-\lambda k(t-s)}\left(1 - e^{-\lambda(t-s)}\right)$$

This can also be written as,

$$\frac{\partial}{\partial t}\left[e^{\lambda(k+2)t}p_{k,k+2}(s,t)\right] = \lambda k(k+1)e^{\lambda kt + 2\lambda t - \lambda kt + \lambda ks}\left(1 - e^{-\lambda(t-s)}\right)$$

$$= \lambda k(k+1)e^{2\lambda t + \lambda ks}\left(1 - e^{-\lambda(t-s)}\right)$$

$$= \lambda k(k+1)e^{\lambda ks}\left(e^{2\lambda t} - e^{\lambda s}e^{\lambda t}\right)$$

Integrating both sides with respect to t, we have

$$e^{\lambda(k+2)t}p_{k,k+2}(s,t) = \lambda k(k+1)e^{\lambda ks}\int_s^t\left(e^{2\lambda y} - e^{\lambda s}e^{\lambda y}\right)dy$$

$$= \lambda k(k+1)e^{\lambda ks}\left[\frac{e^{2\lambda y}}{2\lambda} - \frac{e^{\lambda s}e^{\lambda y}}{\lambda}\right]_s^t$$

$$= k(k+1)e^{\lambda ks}\left[\frac{e^{2\lambda t}}{2} - e^{\lambda s}e^{\lambda t} - \frac{e^{2\lambda s}}{2} + e^{\lambda s}e^{\lambda s}\right]$$

Therefore,

$$p_{k\,k+2}(s,t) = k(k+1)e^{\lambda ks}\left[\frac{1}{2}e^{2\lambda t - k\lambda t - 2\lambda t} - e^{\lambda s + \lambda t - k\lambda t - 2\lambda t} - \frac{1}{2}e^{2\lambda s - \lambda kt - 2\lambda t} + e^{2\lambda s - k\lambda t - 2\lambda t}\right]$$

Simplifying,

$$p_{k\,k+2}(s,t) = k(k+1)e^{\lambda ks}e^{-k\lambda t}\left[\frac{1}{2} - e^{-\lambda(t-s)} - \frac{1}{2}e^{-2\lambda(t-s)} + e^{-2\lambda(t-s)}\right]$$

$$= k(k+1)e^{\lambda ks}e^{-k\lambda t}\left[\frac{1}{2} - e^{-\lambda(t-s)} + \frac{1}{2}e^{-2\lambda(t-s)}\right]$$

$$= \frac{k(k+1)}{2}e^{-\lambda k(t-s)}\left[e^{-2\lambda(t-s)} - 2e^{-\lambda(t-s)} + 1\right]$$

Simplifying, we have

$$p_{k\,k+2}(s,t) = \frac{k(k+1)}{2} e^{-\lambda k(t-s)} \left[1 - e^{-\lambda(t-s)}\right]^2$$

Equivalently,

$$p_{k,k+2}(s,t) = \binom{(k+1)}{2} e^{-\lambda k(t-s)} \left[1 - e^{-\lambda(t-s)}\right]^2 \qquad \textbf{(3.12)}$$

Put $n = 3$, then **(2.11)** becomes

$$\frac{\partial}{\partial t} p_{k,k+3}(s,t) + \lambda_{k+3}(t) p_{k,k+3}(s,t) = \lambda_{k+2}(t)\, p_{k,k+2}(s,t)$$

$$\frac{\partial}{\partial t} p_{k,k+3}(s,t) + (k+3)\lambda\, p_{k,k+3}(s,t) = (k+2)\lambda\, p_{k,k+2}(s,t)$$

Integrating factor $= \exp\left[\int (k+3)\lambda\, dt\right] = \exp\{(k+3)\lambda t\}$

Multiplying the above equation by the integrating factor, we have

$$e^{(k+3)\lambda t} \frac{\partial}{\partial t} p_{k,k+3}(s,t) + (k+3)\lambda e^{(k+3)\lambda t}\, p_{k,k+3}(s,t) = (k+2)\lambda \binom{k+1}{2} e^{(k+3)\lambda t}\, e^{-\lambda k(t-s)} \left[1 - e^{-\lambda(t-s)}\right]^2$$

This can also be written as

$$\frac{\partial}{\partial t}\left[e^{(k+3)\lambda t}\, p_{k,k+3}(s,t)\right] = 3\lambda \binom{k+2}{3} e^{k\lambda t + 3\lambda t - \lambda kt + \lambda ks} \left[1 - e^{-\lambda(t-s)}\right]^2$$

$$\frac{\partial}{\partial t}\left[e^{(k+3)\lambda t}\, p_{k,k+3}(s,t)\right] = 3\lambda \binom{k+2}{3} e^{3\lambda t + \lambda ks} \left[1 - e^{-\lambda(t-s)}\right]^2$$

Integrating both sides with respect to t, we have

$$e^{(k+3)\lambda y}\, p_{k,k+3}(s,y)\Big|_s^t = 3\lambda \binom{k+2}{3} e^{\lambda ks} \int_s^t e^{3\lambda y} \left[1 - e^{-\lambda(y-s)}\right]^2 dy$$

$$= 3\lambda \binom{k+2}{3} e^{\lambda ks} \int_s^t e^{\lambda y} \left[e^{\lambda y} - e^{\lambda s}\right]^2 dy$$

Put $u = e^{\lambda y} - e^{\lambda s} \Rightarrow \frac{du}{dy} = \lambda e^{\lambda y} \Rightarrow du = \lambda e^{\lambda y} dy$. When $y = s$, $u = 0$. When $y = t$, $u = e^{\lambda t} - e^{\lambda s}$.

Therefore

$$e^{(k+3)\lambda y} p_{k,k+3}(s,y)\Big|_s^t = 3\binom{k+2}{3} e^{\lambda ks}\int_s^t \lambda e^{\lambda y}\left[e^{\lambda y} - e^{\lambda s}\right]^2 dy$$

$$= 3\binom{k+2}{3} e^{\lambda ks}\int_0^{e^{\lambda t} - e^{\lambda s}} \lambda e^{\lambda y} u^2 \frac{du}{\lambda e^{\lambda y}}$$

$$= 3\binom{k+2}{3} e^{\lambda ks}\int_0^{e^{\lambda t} - e^{\lambda s}} u^2\, du$$

$$= 3\binom{k+2}{3} e^{\lambda ks}\left[\frac{u^3}{3}\right]_0^{e^{\lambda t} - e^{\lambda s}}$$

$$= \binom{k+2}{3} e^{\lambda ks}\left[e^{\lambda t} - e^{\lambda s}\right]^3$$

$$= \binom{k+2}{3} e^{\lambda ks}\left[e^{\lambda t}\left(1 - e^{-\lambda(t-s)}\right)\right]^3$$

$$= \binom{k+2}{3} e^{3\lambda t} e^{\lambda ks}\left[1 - e^{-\lambda(t-s)}\right]^3$$

But $p_{k,k+3}(s,s) = 0.$ Therefore,

$$e^{(k+3)\lambda t} p_{k,k+3}(s,t) = \binom{k+2}{3} e^{3\lambda t} e^{\lambda ks}\left[1 - e^{-\lambda(t-s)}\right]^3$$

The above equation can also be written in the form

$$p_{k,k+3}(s,t) = \binom{k+2}{3} e^{3\lambda t} e^{\lambda ks} e^{-k\lambda t} e^{-3\lambda t}\left[1 - e^{-\lambda(t-s)}\right]^3$$

$$= \binom{k+2}{3} e^{-\lambda k(t-s)}\left[1 - e^{-\lambda(t-s)}\right]^3 \qquad \textbf{(3.13)}$$

By induction, assume

$$p_{k,k+n-1}(s,t) = \binom{k+n-2}{n-1} e^{-\lambda k(t-s)}\left[1 - e^{-\lambda(t-s)}\right]^{n-1} \text{ to be true.} \qquad \textbf{(3.14)}$$

Then equation **(2.11)** becomes

$$\frac{\partial}{\partial t} p_{k,k+n}(s,t) + \lambda_{k+n}(t) p_{k,k+n}(s,t) = \lambda_{k+n-1}(t)\binom{k+n-2}{n-1} e^{-\lambda k(t-s)}\left[1 - e^{-\lambda(t-s)}\right]^{n-1}$$

Substituting for $\lambda_{k+n}(t)$, we have

$$\frac{\partial}{\partial t} p_{k,k+n}(s,t) + (k+n)\lambda p_{k\,k+n}(s,t) = (k+n-1)\lambda \binom{k+n-2}{n-1} e^{-\lambda k(t-s)}\left[1-e^{-\lambda(t-s)}\right]^{n-1}$$

$$= \binom{k+n-1}{n} n\lambda e^{-\lambda k(t-s)}\left[1-e^{-\lambda(t-s)}\right]^{n-1}$$

Integrating Factor $= e^{(k+n)\lambda t}$

Multiplying the equation above with the integrating factor, we have

$$e^{(k+n)\lambda t}\frac{\partial}{\partial t} p_{k\,k+n}(s,t) + (k+n)\lambda e^{(k+n)\lambda t} p_{k\,k+n}(s,t) = \binom{k+n-1}{n} n\lambda e^{-\lambda k(t-s)} e^{(k+n)\lambda t}\left[1-e^{-\lambda(t-s)}\right]^{n-1}$$

Equivalently,

$$\frac{\partial}{\partial t}\left[e^{(k+n)\lambda t} p_{k,k+n}(s,t)\right] = \binom{k+n-1}{n} n\lambda e^{k\lambda t + n\lambda t - \lambda kt + \lambda ks}\left[1-e^{-\lambda(t-s)}\right]^{n-1}$$

$$= \binom{k+n-1}{n} n\lambda e^{\lambda ks} e^{n\lambda t}\left[1-e^{-\lambda t}e^{\lambda s}\right]^{n-1}$$

$$= n\binom{k+n-1}{n}\lambda e^{\lambda ks} e^{n\lambda t}\left[1-\frac{e^{\lambda s}}{e^{\lambda t}}\right]^{n-1}$$

$$= n\binom{k+n-1}{n}\lambda e^{\lambda ks} e^{n\lambda t} e^{-\lambda t(n-1)}\left[e^{\lambda t}-e^{\lambda s}\right]^{n-1}$$

$$= n\binom{k+n-1}{n}\lambda e^{\lambda ks + n\lambda t - \lambda tn + \lambda t}\left[e^{\lambda t}-e^{\lambda s}\right]^{n-1}$$

$$= n e^{\lambda ks}\binom{k+n-1}{n}\lambda e^{\lambda t}\left[e^{\lambda t}-e^{\lambda s}\right]^{n-1}$$

Integrating both sides with respect to t, we have

$$e^{(k+n)\lambda y} p_{k,k+n}(s,y)\Big|_s^t = n e^{\lambda ks}\binom{k+n-1}{n}\int_s^t \lambda e^{\lambda t}\left[e^{\lambda y}-e^{\lambda s}\right]^{n-1} dy$$

Put $u = e^{\lambda y} - e^{\lambda s} \Rightarrow \frac{du}{dy} = \lambda e^{\lambda y}$ or $dy = \frac{du}{\lambda e^{\lambda y}}$. Further, when $y = s$, $u = 0$ and when $y = t$, $u = e^{\lambda t} - e^{\lambda s}$.

Therefore,

$$e^{(k+n)\lambda y} p_{k,k+n}(s,y)\Big|_s^t = n e^{\lambda ks} \binom{k+n-1}{n} \int_0^{e^{\lambda t}-e^{\lambda s}} \lambda e^{\lambda t} u^{n-1} \frac{du}{\lambda e^{\lambda t}}$$

$$e^{(k+n)\lambda y} p_{k,k+n}(s,y)\Big|_s^t = n e^{\lambda ks} \binom{k+n-1}{n} \int_0^{e^{\lambda t}-e^{\lambda s}} u^{n-1} du$$

$$= n e^{\lambda ks} \binom{k+n-1}{n} \left[\frac{u^n}{n}\right]_0^{e^{\lambda t}-e^{\lambda s}}$$

$$= e^{\lambda ks} \binom{k+n-1}{n} \left[e^{\lambda t} - e^{\lambda s}\right]^n$$

$$= e^{\lambda ks} \binom{k+n-1}{n} e^{n\lambda t} \left[1 - e^{-\lambda(t-s)}\right]^n$$

Therefore,

$$e^{(k+n)\lambda y} p_{k,k+n}(s,y)\Big|_s^t = e^{\lambda ks} \binom{k+n-1}{n} e^{n\lambda t} \left[1 - e^{-\lambda(t-s)}\right]^n$$

$\Rightarrow$

$$e^{(k+n)\lambda t} p_{k,k+n}(s,t) - e^{(k+n)\lambda s} p_{k,k+n}(s,s) = e^{\lambda ks} \binom{k+n-1}{n} e^{n\lambda t} \left[1 - e^{-\lambda(t-s)}\right]^n$$

$\Rightarrow$

$$e^{(k+n)\lambda t} p_{k,k+n}(s,t) = e^{\lambda ks} \binom{k+n-1}{n} e^{n\lambda t} \left[1 - e^{-\lambda(t-s)}\right]^n$$

Therefore,

$$\mathbf{p_{k,k+n}(s,t) = \binom{k+n-1}{n} e^{-\lambda k(t-s)} \left[1 - e^{-\lambda(t-s)}\right]^n} \qquad \textbf{(3.15)}$$

This is a Negative binomial distribution with $p = e^{-\lambda(t-s)}$ and $q = 1 - e^{-\lambda(t-s)}$

It depends on the length of the time interval $t - s$ and on k and is thus stationary and not independent.

3.3.3 Simple Birth Process With immigration

For simple birth with immigration $\lambda_k = k\lambda + v$

$$p_{kk}(s,t) = e^{-\int_s^t (k\lambda + v)dx}$$

$$= e^{-(k\lambda + v)x\big|_s^t}$$

$$= e^{-(k\lambda + v)(t-s)}$$

For $n > 0$, **(2.11)** becomes

$$\frac{\partial}{\partial t} p_{k,k+n}(s,t) + \left[(k+n)\lambda + v\right] p_{k,k+n}(s,t) = \left[(k+n-1)\lambda + v\right] p_{k,k+n-1}(s,t) \quad \textbf{(3.16)}$$

When $n = 1$, equation **(3.16)** becomes

$$\frac{\partial}{\partial t} p_{k,k+1}(s,t) + \left[(k+1)\lambda + v\right] p_{k,k+1}(s,t) = [k\lambda + v] p_{kk}(s,t)$$

$$\frac{\partial}{\partial t} p_{k,k+1}(s,t) + \left[(k+1)\lambda + v\right] p_{k,k+1}(s,t) = [k\lambda + v] e^{-(k\lambda + v)(t-s)} \quad \textbf{(3.17)}$$

To integrate equation **(3.17)**,

$$\text{Integrating factor} = e^{\int\left[(k+1)\lambda + v\right]dt}$$

$$= e^{\left[(k+1)\lambda + v\right]t}$$

Multiplying equation **(3.17)** with the integrating factor, we have

$$e^{\left[(k+1)\lambda + v\right]t} \frac{\partial}{\partial t} p_{k\,k+1}(s,t) + \left[(k+1)\lambda + v\right] e^{\left[(k+1)\lambda + v\right]t} p_{k\,k+1}(s,t) = [k\lambda + v] e^{-(k\lambda + v)(t-s)} e^{\left[(k+1)\lambda + v\right]t}$$

Equivalently,

$$\frac{\partial}{\partial t}\left\{e^{\left[(k+1)\lambda + v\right]t} p_{k\,k+1}(s,t)\right\} = [k\lambda + v] e^{(k\lambda + v)s} e^{\lambda t}$$

$$e^{\left[(k+1)\lambda + v\right]y} p_{k\,k+1}(s,y)\Big|_s^t = [k\lambda + v] e^{(k\lambda + v)s} \int_s^t e^{\lambda y}\, dy$$

$$e^{\left[(k+1)\lambda + v\right]y} p_{k\,k+1}(s,y)\Big|_s^t = [k\lambda + v] e^{(k\lambda + v)s} \frac{e^{\lambda y}}{\lambda}\Bigg|_s^t$$

From **(2.13)**, $p_{k\,k+1}(s,s) = 0$. Therefore,

$$e^{[(k+1)\lambda+v]t}p_{k\,k+1}(s,t) = \left(\frac{k\lambda+v}{\lambda}\right)e^{(k\lambda+v)s}\left(e^{\lambda t} - e^{\lambda s}\right)$$

Therefore,

$$p_{k\,k+1}(s,t) = \left(\frac{k\lambda+v}{\lambda}\right)e^{(k\lambda+v)s}\left(e^{\lambda t} - e^{\lambda s}\right)e^{-[(k+1)\lambda+v]t}$$

$$= \left(\frac{k\lambda+v}{\lambda}\right)e^{(k\lambda+v)s}\left(e^{\lambda t} - e^{\lambda s}\right)e^{-[(k+1)\lambda+v]t}$$

$$= \left(k + \frac{v}{\lambda}\right)e^{(k\lambda+v)(s-t)}e^{-\lambda t}\left(e^{\lambda t} - e^{\lambda s}\right)$$

$$= \left(k + \frac{v}{\lambda}\right)e^{-(k\lambda+v)(t-s)}e^{-\lambda t}\left(1 - e^{-\lambda(t-s)}\right)$$

$$= \left(k + \frac{v}{\lambda}\right)e^{-\lambda\left(k+\frac{v}{\lambda}\right)(t-s)}e^{-\lambda t}\left(1 - e^{-\lambda(t-s)}\right) \qquad \textbf{(3.18)}$$

When n = 2,

$$\frac{\partial}{\partial t}p_{k\,k+2}(s,t) + \left[(k+2)\lambda + v\right]p_{k\,k+2}(s,t) = \left[(k+1)\lambda + v\right]p_{k\,k+1}(s,t)$$

$$= \left[(k+1)\lambda + v\right]\left(k + \frac{v}{\lambda}\right)e^{-\lambda\left(k+\frac{v}{\lambda}\right)(t-s)}e^{-\lambda t}\left(1 - e^{-\lambda(t-s)}\right)$$

$$\text{Integrating factor} = e^{\int[(k+2)\lambda+v]dt}$$

$$= e^{[(k+2)\lambda+v]t}$$

Multiplying equation **(3.18)** by the integrating factor, we have

$$e^{[(k+2)\lambda+v]t}\frac{\partial}{\partial t}p_{k,k+2}(s,t)+e^{[(k+2)\lambda+v]t}\left[(k+2)\lambda+v\right]p_{k,k+2}(s,t)=\left[(k+1)\lambda+v\right]\left(k+\frac{v}{\lambda}\right)e^{-\lambda\left(k+\frac{v}{\lambda}\right)(t-s)}e^{-\lambda t}e^{[(k+2)\lambda+v]t}\left(1-e^{-\lambda(t-s)}\right)$$

Equivalently,

$$\frac{\partial}{\partial t}\left\{e^{[(k+2)\lambda+v]t}p_{k,k+2}(s,t)\right\} = \left(k+\frac{v}{\lambda}\right)\left[(k+1)\lambda+v\right]e^{(k\lambda+v)s}e^{2\lambda t}\left(1-e^{-\lambda(t-s)}\right)$$

Integrating both sides with respect to t, we have

$$\left\{e^{[(k+2)\lambda+v]t}p_{k\,k+2}(s,t)\right\}\Big|_s^t = \left(k+\frac{v}{\lambda}\right)\left[(k+1)\lambda+v\right]e^{(k\lambda+v)s}\int_s^t\left(e^{2\lambda y}-e^{\lambda s}e^{\lambda y}\right)dy$$

$$=\left(k+\frac{v}{\lambda}\right)\left[(k+1)\lambda+v\right]e^{(k\lambda+v)s}\left\{\left[\frac{e^{2\lambda y}}{2\lambda}-\frac{e^{\lambda s}e^{\lambda y}}{\lambda}\right]_s^t\right\}$$

$$=\left(k+\frac{v}{\lambda}\right)\left[(k+1)\lambda+v\right]e^{(k\lambda+v)s}\left\{\left[\frac{e^{2\lambda t}}{2\lambda}-\frac{e^{\lambda s}e^{\lambda t}}{\lambda}\right]-\left[\frac{e^{2\lambda s}}{2\lambda}-\frac{e^{\lambda s}e^{\lambda s}}{\lambda}\right]\right\}$$

$$=\left(k+\frac{v}{\lambda}\right)\left[(k+1)\lambda+v\right]e^{(k\lambda+v)s}\left\{\frac{e^{2\lambda t}}{2\lambda}-\frac{e^{\lambda s}e^{\lambda t}}{\lambda}-\frac{e^{2\lambda s}}{2\lambda}+\frac{e^{2\lambda s}}{\lambda}\right\}$$

$$=\left(k+\frac{v}{\lambda}\right)\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}e^{(k\lambda+v)s}\left\{e^{2\lambda t}-2e^{\lambda(s+t)}-e^{2\lambda s}+2e^{2\lambda s}\right\}$$

$$=\left(k+\frac{v}{\lambda}\right)\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}e^{(k\lambda+v)s}\left\{e^{2\lambda t}-2e^{\lambda(s+t)}+e^{2\lambda s}\right\}$$

Now, from **(2.13)**, $p_{k,k+1}(s,s)=0$. Therefore,

$$e^{[(k+2)\lambda+v]t}p_{k,k+2}(s,t)=\left(k+\frac{v}{\lambda}\right)\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}e^{(k\lambda+v)s}\left\{e^{2\lambda t}-2e^{\lambda(s+t)}+e^{2\lambda s}\right\}$$

Equivalently,

$$p_{k,k+2}(s,t)=\frac{\left(k+\frac{v}{\lambda}\right)}{1}\,\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\,e^{(k\lambda+v)s}e^{-[(k+2)\lambda+v]t}\left\{e^{2\lambda t}-2e^{\lambda(s+t)}+e^{2\lambda s}\right\}$$

$$=\frac{\left(k+\frac{v}{\lambda}\right)}{1}\,\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\,e^{-(k\lambda+v)(t-s)}e^{-2\lambda t}\left\{e^{2\lambda t}-2e^{\lambda(s+t)}+e^{2\lambda s}\right\}$$

$$=\frac{\left(k+\frac{v}{\lambda}\right)}{1}\,\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\,e^{-(k\lambda+v)(t-s)}\left\{1-2e^{-\lambda(t-s)}+e^{-2\lambda(t-s)}\right\}$$

$$=\frac{\left(k+\frac{v}{\lambda}\right)}{1}\,\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\,e^{-(k\lambda+v)(t-s)}\left(1-e^{-\lambda(t-s)}\right)^2$$

Therefore,

$$p_{k,k+2}(s,t) = \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2} e^{-\lambda\left(k+\frac{v}{\lambda}\right)(t-s)}\left(1-e^{-\lambda(t-s)}\right)^2 \qquad \textbf{(3.19)}$$

When n = 3, equation **(3.16)** becomes

$$\frac{d}{dt}p_{k,k+3}(s,t)+\left[(k+3)\lambda+v\right]p_{k,k+3}(s,t) = \left[(k+2)\lambda+v\right]p_{k,k+2}(s,t)$$

$$= \left[(k+2)\lambda+v\right]\frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2} e^{-\lambda\left(k+\frac{v}{\lambda}\right)(t-s)}\left(1-e^{-\lambda(t-s)}\right)^2$$

Integrating this equation,

$$\text{Integrating factor} = e^{\int\left[(k+3)\lambda+v\right]dt}$$

$$= e^{\left[(k+3)\lambda+v\right]t}$$

$$e^{\left[(k+3)\lambda+v\right]t}\frac{\partial}{\partial t}p_{k,k+3}(s,t)+\left[(k+3)\lambda+v\right]e^{\left[(k+3)\lambda+v\right]t}p_{k,k+3}(s,t)=\left[(k+2)\lambda+v\right]\frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}e^{\left[(k+3)\lambda+v\right]t}\; e^{-\lambda\left(k+\frac{v}{\lambda}\right)(t-s)}\left(1-e^{-\lambda(t-s)}\right)^2$$

$$\frac{\partial}{\partial t}\left[e^{\left[(k+3)\lambda+v\right]t}p_{k,k+3}(s,t)\right] = \left[(k+2)\lambda+v\right]\frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}e^{\left[(k+3)\lambda+v\right]t}\; e^{-\lambda\left(k+\frac{v}{\lambda}\right)(t-s)}\left(1-e^{-\lambda(t-s)}\right)^2$$

$$= \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\left[(k+2)\lambda+v\right]e^{(k\lambda+v)s}\; e^{3\lambda t}\left(1-2e^{-\lambda(t-s)}+e^{-2\lambda(t-s)}\right)$$

$$= \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\left[(k+2)\lambda+v\right]e^{(k\lambda+v)s}\left(e^{3\lambda t}-2e^{\lambda s}e^{2\lambda t}+e^{2\lambda s}e^{\lambda t}\right)$$

Integrating, we have

$$e^{\left[(k+3)\lambda+v\right]y}p_{k,k+3}(s,y)\Big|_s^t = \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\left[(k+2)\lambda+v\right]e^{(k\lambda+v)s}\left[\frac{e^{3\lambda y}}{3\lambda}-\frac{2e^{\lambda s}e^{2\lambda y}}{2\lambda}+\frac{e^{2\lambda s}e^{\lambda y}}{\lambda}\right]_s^t$$

$$= \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\frac{\left[\left(k+\frac{v}{\lambda}\right)+2\right]}{3}e^{(k\lambda+v)s}\left[e^{3\lambda y}-3e^{\lambda s}e^{2\lambda y}+3e^{2\lambda s}e^{\lambda y}\right]_s^t$$

Therefore,

$$e^{\left[(k+3)\lambda+v\right]y}p_{k,k+3}(s,y)\Big|_s^t = \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\frac{\left[\left(k+\frac{v}{\lambda}\right)+2\right]}{3}e^{(k\lambda+v)s}\left[\left(e^{3\lambda x}-3e^{\lambda s}e^{2\lambda x}+3e^{2\lambda s}e^{\lambda x}\right)-\left(e^{3\lambda s}-3e^{\lambda s}e^{2\lambda s}+3e^{2\lambda s}e^{\lambda s}\right)\right]$$

$$= \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\frac{\left[\left(k+\frac{v}{\lambda}\right)+2\right]}{3}e^{(k\lambda+v)s}\left[e^{3\lambda x}-3e^{\lambda s}e^{2\lambda x}+3e^{2\lambda s}e^{\lambda x}-e^{3\lambda s}+3e^{3\lambda s}-3e^{3\lambda s}\right]$$

Simplifying,

$$e^{\left[(k+3)\lambda+v\right]t} p_{k,k+3}(s,t)\Big|_s^t = \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\frac{\left[\left(k+\frac{v}{\lambda}\right)+2\right]}{3} e^{(k\lambda+v)s}\left[e^{3\lambda x} - 3e^{\lambda s}e^{2\lambda x} + 3e^{2\lambda s}e^{\lambda x} - e^{3\lambda s}\right]$$

Now, from **(2.13)**, $P_{k,k+1}(s,s) = 0.$ Therefore,

$$e^{\left[(k+3)\lambda+v\right]t} p_{k,k+3}(s,t) = \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\frac{\left[\left(k+\frac{v}{\lambda}\right)+2\right]}{3} e^{(k\lambda+v)s}\left[e^{3\lambda x} - 3e^{\lambda s}e^{2\lambda x} + 3e^{2\lambda s}e^{\lambda x} - e^{3\lambda s}\right]$$

Equivalently,

$$\begin{aligned}
p_{k,k+3}(s,t) &= \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\frac{\left[\left(k+\frac{v}{\lambda}\right)+2\right]}{3} e^{(k\lambda+v)s}\, e^{-\left[(k+3)\lambda+v\right]t}\left[1 - 3e^{\lambda s}e^{-\lambda t} + 3e^{2\lambda s}e^{-2\lambda x} - e^{3\lambda s}e^{-3\lambda t}\right] \\
&= \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\frac{\left[\left(k+\frac{v}{\lambda}\right)+2\right]}{3} e^{-(k\lambda+v)(t-s)}\left[1 - 3e^{-\lambda(t-s)} + 3e^{-2\lambda(t-s)} - e^{3\lambda(t-s)}\right] \\
&= \frac{\left(k+\frac{v}{\lambda}\right)}{1}\frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]}{2}\frac{\left[\left(k+\frac{v}{\lambda}\right)+2\right]}{3} e^{-(k\lambda+v)(t-s)}\left(1 - e^{-\lambda(t-s)}\right)^3 \\
&= \frac{\left(k+\frac{v}{\lambda}\right)\left[\left(k+\frac{v}{\lambda}\right)+1\right]\left[\left(k+\frac{v}{\lambda}\right)+2\right]}{3!} e^{-(k\lambda+v)(t-s)}\left(1 - e^{-\lambda(t-s)}\right)^3 \\
&= \frac{\left[\left(k+\frac{v}{\lambda}\right)+1\right]\left[\left(k+\frac{v}{\lambda}\right)+2\right]!}{3!\left(k+\frac{v}{\lambda} - 1\right)!} e^{-(k\lambda+v)(t-s)}\left(1 - e^{-\lambda(t-s)}\right)^3 \\
&= \binom{k+\frac{v}{\lambda}+2}{3} e^{-(k\lambda+v)(t-s)}\left(1 - e^{-\lambda(t-s)}\right)^3
\end{aligned} \quad \textbf{(3.20)}$$

By induction, assume

$$p_{k,k+n-1}(s,t) = \binom{k+\frac{v}{\lambda}+n-2}{n-1} e^{-(k\lambda+v)(t-s)}\left(1 - e^{-\lambda(t-s)}\right)^{n-1} \quad \textbf{(3.21)}$$

From **(3.16)**, we have for $n > 0$,

$$\begin{aligned}
\frac{\partial}{\partial t} p_{k,k+n}(s,t) + \left[(k+n)\lambda+v\right]p_{k,k+n}(s,t) &= \left[(k+n-1)\lambda+v\right]p_{k,k+n-1}(s,t) \\
&= \lambda\left[k+\frac{v}{\lambda}+n-1\right]\binom{k+\frac{v}{\lambda}+n-2}{n-1} e^{-(k\lambda+v)(t-s)}\left(1-e^{-\lambda(t-s)}\right)^{n-1}
\end{aligned}$$

Integrating factor $= e^{\int[(k+n)\lambda+v]dt}$

$= e^{[(k+n)\lambda+v]t}$

Thus,

$$\frac{\partial}{\partial t}\left[p_{k\,k+n}(s,t)e^{[(k+n)\lambda+v]t}\right] = [(k+n-1)\lambda+v]\binom{k+\frac{v}{\lambda}+n-2}{n-1}e^{[(k+n)\lambda+v]\lambda t}e^{-(k\lambda+v)(t-s)}\left(1-e^{-\lambda(t-s)}\right)^{n-1}$$

$$= \lambda\left(k+\tfrac{v}{\lambda}+n-1\right)\binom{k+\frac{v}{\lambda}+n-2}{n-1}e^{\left(k+\frac{v}{\lambda}+n\right)\lambda s}e^{n\lambda t}\left(1-\frac{e^{\lambda s}}{e^{\lambda t}}\right)^{n-1}$$

$$= \lambda\left(k+\tfrac{v}{\lambda}+n-1\right)\binom{k+\frac{v}{\lambda}+n-2}{n-1}e^{\left(k+\frac{v}{\lambda}+n\right)\lambda s}e^{n\lambda t}e^{-(n-1)\lambda t}\left(e^{\lambda t}-e^{\lambda s}\right)^{n-1}$$

$$= \lambda\left(k+\tfrac{v}{\lambda}+n-1\right)\binom{k+\frac{v}{\lambda}+n-2}{n-1}e^{\left(k+\frac{v}{\lambda}+n\right)\lambda s}e^{\lambda t}\left(e^{\lambda t}-e^{\lambda s}\right)^{n-1}$$

Integrating both sides, we have

$$p_{k,k+n}(s,y)\,e^{[(k+n)\lambda+v]y}\Big|_s^t = \lambda\left(k+\tfrac{v}{\lambda}+n-1\right)\binom{k+\frac{v}{\lambda}+n-2}{n-1}e^{\left(k+\frac{v}{\lambda}+n\right)\lambda s}\int_s^t e^{\lambda y}\left(e^{\lambda y}-e^{\lambda s}\right)^{n-1}dy$$

Let $e^{\lambda y}-e^{\lambda s}=u$. Then $\frac{du}{dy}=\lambda e^{\lambda y}$ or $dy=\frac{du}{\lambda}e^{-\lambda y}$. Also, when $y=s,\ u=e^{\lambda s}-e^{\lambda s}=0$ and when $y=t,\ u=e^{\lambda t}-e^{\lambda s}$.

Therefore,

$$e^{[(k+n)\lambda+v]t}p_{k,k+n}(s,t) = \lambda\left(k+\tfrac{v}{\lambda}+n-1\right)\binom{k+\frac{v}{\lambda}+n-2}{n-1}e^{\left(k+\frac{v}{\lambda}+n\right)\lambda s}\int_0^{e^{\lambda t}-e^{\lambda s}} e^{\lambda y}u^{n-1}\frac{e^{-\lambda y}du}{\lambda}$$

$$= \left(k+\tfrac{v}{\lambda}+n-1\right)\binom{k+\frac{v}{\lambda}+n-2}{n-1}e^{\left(k+\frac{v}{\lambda}+n\right)\lambda s}\int_0^{e^{\lambda t}-e^{\lambda s}} u^{n-1}du$$

$$= \left(k+\tfrac{v}{\lambda}+n-1\right)\binom{k+\frac{v}{\lambda}+n-2}{n-1}e^{\left(k+\frac{v}{\lambda}+n\right)\lambda s}\left[\frac{u^n}{n}\right]_0^{e^{\lambda t}-e^{\lambda s}}$$

$$= \left(k+\tfrac{v}{\lambda}+n-1\right)\binom{k+\frac{v}{\lambda}+n-2}{n-1}\frac{e^{\left(k+\frac{v}{\lambda}+n\right)\lambda s}}{n}\left[\left(e^{\lambda t}-e^{\lambda s}\right)^n-0\right]$$

$$= \left(k+\tfrac{v}{\lambda}+n-1\right)\binom{k+\frac{v}{\lambda}+n-2}{n-1}\frac{e^{\left(k+\frac{v}{\lambda}+n\right)\lambda s}}{n}\left(e^{\lambda t}-e^{\lambda s}\right)^n$$

Equivalently,

$$p_{k,k+n}(s,t) = \left(k+\tfrac{v}{\lambda}+n-1\right)\frac{\left(k+\tfrac{v}{\lambda}+n-2\right)!}{(n-1)!\left(k+\tfrac{v}{\lambda}-1\right)!}\frac{e^{\left(k+\frac{v}{\lambda}+n\right)\lambda s}}{n}e^{-\left[(k+n)\lambda+v\right]t}\left(e^{\lambda t}-e^{\lambda s}\right)^n$$

$$= \frac{\left(k+\tfrac{v}{\lambda}+n-1\right)!}{n!\left(k+\tfrac{v}{\lambda}-1\right)!}e^{-\left(k+\frac{v}{\lambda}+n\right)(t-s)\lambda}\left(e^{\lambda t}-e^{\lambda s}\right)^n$$

$$= \binom{k+\tfrac{v}{\lambda}+n-1}{n}e^{-\left(k+\frac{v}{\lambda}+n\right)(t-s)\lambda}\left(1-e^{-\lambda(t-s)}\right)^n$$

Therefore,

$$\mathbf{p_{k,k+n}(s,t) = \binom{k+\tfrac{v}{\lambda}+n-1}{n}e^{-\left(k+\frac{v}{\lambda}+n\right)(t-s)\lambda}\left(1-e^{-\lambda(t-s)}\right)^n} \qquad \textbf{(3.22)}$$

This is a Negative binomial distribution with $p = e^{-\lambda(t-s)}$ and $q = 1 - e^{-\lambda(t-s)}$

It depends on the length of the time interval $t - s$ and on k and is thus stationary and not independent.

3.3.4 Polya Process

For the Polya process, $\lambda_k = \left(\dfrac{1+ak}{1+\lambda at}\right)\lambda$

From equation **(3.1)**, we have

$$p_{k\,k}(s,t) = e^{-\int_s^t\left(\frac{1+ak}{1+\lambda ax}\right)\lambda\,dx}$$

$$= e^{-(1+ak)\lambda\int_s^t\left(\frac{1}{1+\lambda ax}\right)dx}$$

Let $u = 1+\lambda ax \;\Rightarrow\; \dfrac{du}{dx} = \lambda a$ or $\dfrac{du}{\lambda a} = dx$

Limits

$x = s \;\Rightarrow\; u = 1+\lambda as$

$x = t \;\Rightarrow\; u = 1+\lambda at$

Therefore,

$$p_{k\,k}(s,t) = e^{-\left(\frac{1+ak}{a\lambda}\right)\lambda\int_{1+\lambda as}^{1+\lambda at}\frac{du}{u}}$$

$$p_{kk}(s,t) = e^{-\left(\frac{1+ak}{a\lambda}\right)\lambda \ln u\Big|_{1+\lambda as}^{1+\lambda at}}$$

$$= e^{-\left(\frac{1}{a}+k\right)\ln\left(\frac{1+\lambda at}{1+\lambda as}\right)}$$

$$= e^{\ln\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}}$$

$$= \left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}$$

For $n \geq 1$, equation **(2.11)** becomes

$$\frac{\partial}{\partial t}p_{k\,k+n}(s,t) + \left[\frac{1+a(k+n)}{1+\lambda at}\right]\lambda p_{k\,k+n}(s,t) = \left[\frac{1+a(k+n-1)}{1+\lambda at}\right]\lambda p_{k\,k+n-1}(s,t) \qquad \textbf{(3.23)}$$

For $n = 1$

$$\frac{\partial}{\partial t}p_{k\,k+1}(s,t) + \left[\frac{1+a(k+1)}{1+\lambda at}\right]\lambda p_{k\,k+1}(s,t) = \left[\frac{1+ak}{1+\lambda at}\right]\lambda\, p_{kk}(s,t)$$

$$= \left(\frac{1+ak}{1+\lambda at}\right)\lambda\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}$$

$$= \left(\frac{1+ak}{1+\lambda at}\right)\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\lambda$$

Integrating factor $= e^{\int \lambda\left(\frac{1+a(k+1)}{1+\lambda at}\right)dt}$

$$= (1+\lambda at)^{\left(k+\frac{1}{a}+1\right)}$$

Multiply the equation above by the integrating factor, we obtain

$$\frac{\partial}{\partial t}\left[(1+\lambda at)^{\left(k+\frac{1}{a}+1\right)}p_{k\,k+1}(s,t)\right] = (1+\lambda at)^{\left(k+\frac{1}{a}+1\right)}\left(\frac{1+ak}{1+\lambda at}\right)\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\lambda$$

$$= (1+ak)(1+\lambda as)^{k+\frac{1}{a}}\lambda$$

Integrating, we have

$$(1+\lambda ay)^{\left(k+\frac{1}{a}+1\right)}p_{k\,k+1}(s,y)\Big|_s^t = (1+ak)(1+\lambda as)^{k+\frac{1}{a}}\lambda y\Big|_s^t$$

$$(1+\lambda at)^{\left(k+\frac{1}{a}+1\right)}p_{k\,k+1}(s,t) = (1+ak)(1+\lambda as)^{k+\frac{1}{a}}\lambda(t-s)$$

Therefore,

$$p_{k\,k+1}(s,t) = \frac{(1 + ak)(1 + \lambda as)^{k + \frac{1}{a}} \lambda}{(1 + \lambda at)^{\left(k + \frac{1}{a} + 1\right)}}(t - s)$$

$$= \frac{\left(k + \frac{1}{a}\right)(1 + \lambda as)^{k + \frac{1}{a}} \lambda}{(1 + \lambda at)^{\left(k + \frac{1}{a} + 1\right)}}(t - s)$$

$$p_{k\,k+1}(s,t) = \left(k + \tfrac{1}{a}\right)\left(\frac{1 + \lambda as}{1 + \lambda at}\right)^{k + \frac{1}{a}} \frac{a\lambda(t - s)}{1 + \lambda at} \qquad \textbf{(3.24)}$$

When n = 2

$$\frac{\partial}{\partial t} p_{k\,k+2}(s,t) + \left[\frac{1 + a(k + 2)}{1 + \lambda at}\right]\lambda\, p_{k\,k+2}(s,t) = \left[\frac{1 + a(k + 1)}{1 + \lambda at}\right]\lambda\, p_{k\,k+1}(s,t)$$

$$= \left(\frac{k + \frac{1}{a} + 1}{1 + \lambda at}\right) a\lambda\left(k + \tfrac{1}{a}\right)\left(\frac{1 + \lambda as}{1 + \lambda at}\right)^{k + \frac{1}{a}} \frac{a\lambda(t - s)}{1 + \lambda at}$$

$$= \left(k + \tfrac{1}{a}\right)\left(k + \tfrac{1}{a} + 1\right)(1 + \lambda as)^{k + \frac{1}{a}} \frac{(a\lambda)^2 (t - s)}{(1 + \lambda at)^{k + \frac{1}{a} + 2}}$$

$$\text{Integrating factor} = e^{\int \left[\frac{1 + a(k + 2)}{1 + \lambda at}\right]\lambda\, dt}$$

$$= (1 + \lambda at)^{k + \frac{1}{a} + 2}$$

Multiply equation the above with the integrating factor, we have

$$\frac{\partial}{\partial t}\left[(1 + \lambda at)^{k + \frac{1}{a} + 2} p_{k\,k+2}(s,t)\right] = \left(k + \tfrac{1}{a}\right)\left(k + \tfrac{1}{a} + 1\right)(1 + \lambda at)^{k + \frac{1}{a} + 2}(1 + \lambda as)^{k + \frac{1}{a}} \frac{(a\lambda)^2 (t - s)}{(1 + \lambda at)^{k + \frac{1}{a} + 2}}$$

$$= \left(k + \tfrac{1}{a}\right)\left(k + \tfrac{1}{a} + 1\right)(1 + \lambda as)^{k + \frac{1}{a}} (a\lambda)^2 (t - s)$$

Integrating, we have

$$(1 + \lambda ay)^{k + \frac{1}{a} + 2} p_{k\,k+2}(s,y)\Big|_s^t = \left(k + \tfrac{1}{a}\right)\left(k + \tfrac{1}{a} + 1\right)(1 + \lambda as)^{k + \frac{1}{a}} (a\lambda)^2 \int_s^t (y - s)\,dy$$

$$\left(1+\lambda at\right)^{k+\frac{1}{a}+2} p_{k\,k+2}\left(s,t\right)=\left(k+\tfrac{1}{a}\right)\left(k+\tfrac{1}{a}+1\right)\left(1+\lambda as\right)^{k+\frac{1}{a}}\left(a\lambda\right)^{2}\left[\frac{y^{2}}{2}-sy\right]_{s}^{t}$$

$$=\left(k+\tfrac{1}{a}\right)\left(k+\tfrac{1}{a}+1\right)\left(1+\lambda as\right)^{k+\frac{1}{a}}\left(a\lambda\right)^{2}\left\{\left(\frac{t^{2}}{2}-st\right)-\left(\frac{s^{2}}{2}-s^{2}\right)\right\}$$

$$=\left(k+\tfrac{1}{a}\right)\left(k+\tfrac{1}{a}+1\right)\left(1+\lambda as\right)^{k+\frac{1}{a}}\left(a\lambda\right)^{2}\left\{\frac{t^{2}}{2}-st+\frac{s^{2}}{2}\right\}$$

$$=\left(k+\tfrac{1}{a}\right)\left(k+\tfrac{1}{a}+1\right)\left(1+\lambda as\right)^{k+\frac{1}{a}}\frac{\left(a\lambda\right)^{2}}{2}\left\{t^{2}-2st+s^{2}\right\}$$

$$=\left(k+\tfrac{1}{a}\right)\left(k+\tfrac{1}{a}+1\right)\left(1+\lambda as\right)^{k+\frac{1}{a}}\frac{\left(a\lambda\right)^{2}}{2}\left(t-s\right)^{2}$$

Therefore,

$$p_{k,k+2}\left(s,t\right)=\frac{\left(k+\frac{1}{a}\right)\left(k+\frac{1}{a}+1\right)\left(1+\lambda as\right)^{k+\frac{1}{a}}}{\left(1+\lambda at\right)^{k+\frac{1}{a}+2}}\frac{\left(a\lambda\right)^{2}}{2}\left(t-s\right)^{2}$$

$$=\frac{\left(k+\frac{1}{a}\right)\left(k+\frac{1}{a}+1\right)}{2}\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\left(\frac{\lambda a\left(t-s\right)}{1+\lambda at}\right)^{2}$$

$$=\frac{\left(k+\frac{1}{a}+1\right)!}{\left(k+\frac{1}{a}-1\right)!2!}\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\left(\frac{\lambda a\left(t-s\right)}{1+\lambda at}\right)^{2}$$

$$=\binom{k+\frac{1}{a}+1}{2}\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\left(\frac{\lambda a\left(t-s\right)}{1+\lambda at}\right)^{2} \qquad \textbf{(3.25)}$$

For n = 3

$$\frac{\partial}{\partial t}p_{k,k+3}\left(s,t\right)+\left[\frac{1+a\left(k+3\right)}{1+\lambda at}\right]\lambda p_{k\,k+3}\left(s,t\right)=\left(\frac{1+a\left(k+2\right)}{1+\lambda at}\right)\lambda\; p_{k\,k+2}\left(s,t\right)$$

$$\frac{\partial}{\partial t}p_{k,k+3}\left(s,t\right)+\left[\frac{1+a\left(k+3\right)}{1+\lambda at}\right]\lambda p_{k\,k+3}\left(s,t\right)=\left(\frac{1+a\left(k+2\right)}{1+\lambda at}\right)\lambda\binom{k+\frac{1}{a}+1}{2}\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\left(\frac{\lambda a\left(t-s\right)}{1+\lambda at}\right)^{2}$$

$$=\frac{\left(k+\frac{1}{a}\right)\left(k+\frac{1}{a}+1\right)\left(k+\frac{1}{a}+2\right)\left(1+\lambda as\right)^{k+\frac{1}{a}}}{\left(1+\lambda at\right)^{k+\frac{1}{a}+3}}\frac{\left(\lambda a\right)^{3}}{2}\left(t-s\right)^{2}$$

Integrating factor $= e^{\int\left[\frac{1+a(k+3)}{1+\lambda at}\right]\lambda\,dt}$

$$= (1+\lambda at)^{\left(k+\frac{1}{a}+3\right)}$$

Multiplying the equation above by the integrating factor, we obtain

$$\frac{\partial}{\partial t}\left[(1+\lambda at)^{\left(k+\frac{1}{a}+3\right)} p_{k,k+3}(s,t)\right] = (1+\lambda at)^{\left(k+\frac{1}{a}+3\right)} \frac{\left(k+\frac{1}{a}\right)\left(k+\frac{1}{a}+1\right)\left(k+\frac{1}{a}+2\right)(1+\lambda as)^{k+\frac{1}{a}}}{(1+\lambda at)^{k+\frac{1}{a}+3}} \frac{(\lambda a)^3}{2}(t-s)^2$$

$$= \left(k+\tfrac{1}{a}\right)\left(k+\tfrac{1}{a}+1\right)\left(k+\tfrac{1}{a}+2\right)(1+\lambda as)^{k+\frac{1}{a}} \frac{(\lambda a)^3}{2}(t-s)^2$$

Integrating, we have

$$(1+\lambda ay)^{\left(k+\frac{1}{a}+3\right)} p_{k,k+3}(s,y)\Big|_s^t = \left(k+\tfrac{1}{a}\right)\left(k+\tfrac{1}{a}+1\right)\left(k+\tfrac{1}{a}+2\right)(1+\lambda as)^{k+\frac{1}{a}} \frac{(\lambda a)^3}{2}\int_s^t (y-s)^2 dy$$

We wish to integrate $\int_s^t (y-s)^2 dy$

Let $u = y - s \Rightarrow \frac{du}{dy} = 1$

Limits

When $y=s,\ u=0$ and when $y=t,\ u=t-s$

Therefore,

$$\int_s^t (y-s)^2 dy = \int_0^{t-s} u^2\, du = \left[\frac{u^3}{3}\right]_0^{t-s} = \left[\frac{(t-s)^3}{3} - 0\right]$$

Thus,

$$(1+\lambda ay)^{\left(k+\frac{1}{a}+3\right)} p_{k,k+3}(s,y)\Big|_s^t = \left(k+\tfrac{1}{a}\right)\left(k+\tfrac{1}{a}+1\right)\left(k+\tfrac{1}{a}+2\right)(1+\lambda as)^{k+\frac{1}{a}} \frac{(\lambda a)^3}{2}\times\frac{(t-s)^3}{3}$$

Therefore,

$$(1+\lambda at)^{\left(k+\frac{1}{a}+3\right)} p_{k,k+3}(s,t) = \frac{\left(k+\frac{1}{a}\right)\left(k+\frac{1}{a}+1\right)\left(k+\frac{1}{a}+2\right)(\lambda a)^3}{2}(1+\lambda as)^{k+\frac{1}{a}}\times\frac{(t-s)^3}{3}$$

Re arranging

$$p_{k,k+3}(s,t) = \frac{\left(k+\frac{1}{a}\right)\left(k+\frac{1}{a}+1\right)\left(k+\frac{1}{a}+2\right)(\lambda a)^3}{2.3}\times\frac{(1+\lambda as)^{k+\frac{1}{a}}(t-s)^3}{(1+\lambda at)^{\left(k+\frac{1}{a}+3\right)}}$$

$$p_{k,k+3}(s,t) = \frac{\left(k+\frac{1}{a}\right)\left(k+\frac{1}{a}+1\right)\left(k+\frac{1}{a}+2\right)}{3!}\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\left(\frac{\lambda a(t-s)}{1+\lambda at}\right)^{3}$$

$$= \frac{\left(k+\frac{1}{a}+2\right)!}{\left(k+\frac{1}{a}-1\right)!3!}\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\left(\frac{\lambda a(t-s)}{1+\lambda at}\right)^{3}$$

Therefore,

$$p_{k,k+3}(s,t) = \binom{k+\frac{1}{a}+2}{3}\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\left(\frac{\lambda a(t-s)}{1+\lambda at}\right)^{3} \qquad \textbf{(3.26)}$$

By induction we assume that when n = j – 1

$$p_{k,k+j-1}(s,t) = \binom{k+\frac{1}{a}+(j-2)}{j-1}\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\left(\frac{\lambda a(t-s)}{1+\lambda at}\right)^{j-1} \text{ is true.} \qquad \textbf{(3.27)}$$

When n = j,

$$\frac{\partial}{\partial t}p_{k,k+j}(s,t) + \left[\frac{1+a(k+j)}{1+\lambda at}\right]\lambda p_{k,k+n}(s,t) = \left[\frac{1+a(k+j-1)}{1+\lambda at}\right]\lambda p_{k,k+j-1}(s,t)$$

Equivalently,

$$\frac{\partial}{\partial t}p_{k,k+j}(s,t)+\left[\frac{1+a(k+j)}{1+\lambda at}\right]\lambda p_{k,k+n}(s,t) = \left[\frac{1+a(k+j-1)}{1+\lambda at}\right]\lambda\binom{k+\frac{1}{a}+(j-2)}{j-1}\left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}}\left(\frac{\lambda a(t-s)}{1+\lambda at}\right)^{j-1}$$

$$= \frac{\left(k+\frac{1}{a}+(j-2)\right)!}{\left(k+\frac{1}{a}-1\right)!(j-1)!}\left(1+a(k+j-1)\right)\frac{(a\lambda)^{j}(1+\lambda as)^{k+\frac{1}{a}}}{(1+\lambda at)^{k+\frac{1}{a}+j}}(t-s)^{j-1}$$

$$= \frac{\left(k+\frac{1}{a}+(j-1)\right)!}{\left(k+\frac{1}{a}-1\right)!(j-1)!}(a\lambda)^{j}\frac{(1+\lambda as)^{k+\frac{1}{a}}}{(1+\lambda at)^{k+\frac{1}{a}+j}}(t-s)^{j-1}$$

Now we solve the equation above by use of the integrating factor

$$\text{Integrating factor} = e^{\int\left[\frac{1+a(k+j)}{1+\lambda at}\right]\lambda dt}$$

$$= (1+\lambda at)^{\left(k+\frac{1}{a}+j\right)}$$

Multiplying the equation above by the integrating factor, we have

$$\frac{\partial}{\partial t}\left[(1+\lambda at)^{\left(k+\frac{1}{a}+j\right)} p_{k,k+j}(s,t)\right] = (1+\lambda at)^{\left(k+\frac{1}{a}+j\right)} \frac{\left(k+\frac{1}{a}+(j-1)\right)!}{\left(k+\frac{1}{a}-1\right)!(j-1)!} \frac{(1+\lambda as)^{k+\frac{1}{a}}}{(1+\lambda at)^{k+\frac{1}{a}+j}} (a\lambda)^j \ (t-s)^{j-1}$$

$$= \frac{\left(k+\frac{1}{a}+(j-1)\right)!}{\left(k+\frac{1}{a}-1\right)!(j-1)!} (1+\lambda as)^{k+\frac{1}{a}} (a\lambda)^j \ (t-s)^{j-1}$$

Integrating, we have

$$(1+\lambda ay)^{\left(k+\frac{1}{a}+j\right)} p_{k,k+j}(s,y)\Big|_s^t = \frac{\left(k+\frac{1}{a}+(j-1)\right)!}{\left(k+\frac{1}{a}-1\right)!(j-1)!} (1+\lambda as)^{k+\frac{1}{a}} (a\lambda)^j \int_s^t (y-s)^{j-1} dy$$

We wish to integrate $\int_s^t (y-s)^{j-1} dy$

Let $u = y - s \ \Rightarrow \ \frac{du}{dy} = 1$

Limits

When $y=s, \ u=0$ and when $y=t, \ u=t-s$

Therefore,

$$\int_s^t (y-s)^{j-1} dy = \int_0^{t-s} u^{j-1} du = \left[\frac{u^j}{j}\right]_0^{t-s} = \left[\frac{(t-s)^j}{j} - 0\right]$$

Thus,

$$(1+\lambda at)^{\left(k+\frac{1}{a}+j\right)} p_{k,k+j}(s,t) = \frac{\left(k+\frac{1}{a}+(j-1)\right)!}{\left(k+\frac{1}{a}-1\right)!(j-1)!} (1+\lambda as)^{k+\frac{1}{a}} (a\lambda)^j \ \frac{(t-s)^j}{j}$$

Equivalently,

$$p_{k\ k+j}(s,t) = \frac{\left(k+\frac{1}{a}+(j-1)\right)!}{\left(k+\frac{1}{a}-1\right)!\,j!} \ \frac{(1+\lambda as)^{k+\frac{1}{a}}}{(1+\lambda at)^{\left(k+\frac{1}{a}+j\right)}} (a\lambda)^j \ \frac{(t-s)^j}{j}$$

$$= \binom{k+\frac{1}{a}+(j-1)}{j} \left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}} \left(\frac{\lambda a(t-s)}{1+\lambda at}\right)^j$$

Therefore,

$$p_{k\ k+j}(s,t) = \binom{k+\frac{1}{a}+j-1}{j} \left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}} \left(\frac{\lambda a(t-s)}{1+\lambda at}\right)^j$$

$$= \binom{j+k+\frac{1}{a}-1}{j} \left(\frac{1+\lambda as}{1+\lambda at}\right)^{k+\frac{1}{a}} \left(\frac{\lambda a(t-s)}{1+\lambda at}\right)^j$$

Further,

Let $P = \dfrac{1 + \lambda as}{1 + \lambda at}$

Then,

$$\begin{aligned} q &= 1 - P \\ &= 1 - \left(\frac{1 + \lambda as}{1 + \lambda at}\right) \\ &= \frac{1 + \lambda at - 1 - \lambda as}{1 + \lambda at} \\ &= \frac{\lambda at - \lambda as}{1 + \lambda at} \\ &= \frac{\lambda a(t - s)}{1 + \lambda at} \end{aligned}$$

Thus,

$$\mathbf{p_{k,k+j}(s,t) = \binom{j + k + \frac{1}{a} - 1}{j}\left(\frac{1 + \lambda as}{1 + \lambda at}\right)^{k+\frac{1}{a}}\left(\frac{\lambda a(t - s)}{1 + \lambda at}\right)^{j}}, \ j = 0, 1, 2, 3, \ldots, \qquad \mathbf{(3.28)}$$

is a negative binomial distribution with $p = \dfrac{1 + \lambda as}{1 + \lambda at}$ and $q = \dfrac{\lambda a(t - s)}{1 + \lambda at}$. It depends on the length of the time interval $t - s$ and on k and is thus stationary and not independent.

CHAPTER FOUR

Probability Distributions for Transitional Probabilities Using Langranges Method

4.1 Introduction

In this chapter, we shall solve equation **(2.11)** using the Lagranges method. We shall solve this equation for all the four pure birth processes starting with Poisson Process, then the Simple Birth process, the Simple Birth Process with immigration and finally the Polya Process. The general definitions for this Method are given in the first section.

4.2 General Definitions

Now, from **(2.11)**, we know that

$$\frac{\partial}{\partial t} p_{k,k+n}(s,t) + \lambda_{k+n}(t) p_{k,k+n}(s,t) = \lambda_{k+n-1}(t)\, p_{k,k+n-1}(s,t).$$

Let $p_{ij}(s,t) = P_{k,k+n}(s,t)$. Then, equation **(2.11)** becomes

$$\frac{\partial}{\partial t}\left\{p_{ij}(s,t)\right\} = -\lambda_j(t) p_{ij}(s,t) + \lambda_{j-1}(t) p_{i,j-1}(s,t) \qquad \textbf{(4.1)}$$

Define transition Probability generating function

$$G(s,t,z) = \sum_{j=0}^{\infty} p_{ij}(s,t) z^j$$

$$= \sum_{j=i}^{\infty} p_{ij}(s,t) z^j \qquad (\text{Since } p_{ij}(s,t) = 0 \ \forall \ j < i)$$

$$\frac{\partial}{\partial t}\left[G(s,t,z)\right] = \sum_{j=i}^{\infty} z^j \frac{\partial}{\partial t}\left\{p_{ij}(s,t)\right\}$$

$$\frac{\partial}{\partial z}\left[G(s,t,z)\right] = \sum_{j=i}^{\infty} k\, p_{ij}(s,t) z^{k-1} \ \Rightarrow z\left\{\frac{\partial}{\partial z}\left[G(s,t,z)\right]\right\} = \sum_{j=i}^{\infty} k\, p_{ij}(s,t) z^j$$

Boundary Condition

$$p_{ii}(s,s) = 1,\ p_{ij}(s,s) = 0 \ \forall \ j \neq i$$

$$G(s,s,z) = \sum_{j=i} p_{ij}(s,s) z^j = z^i$$

Multiplying equation **(4.1)** by z^j and summing over j, we obtain

$$\sum_{j=i}^{\infty}\left\{\frac{\partial}{\partial t}\left\{p_{ij}(s,t)\right\}\right\} z^j = -\sum_{j=i}^{\infty} \lambda_j(t) p_{ij}(s,t) z^j + \sum_{j=i}^{\infty} \lambda_{j-1}(t) p_{i,j-1}(s,t) z^j$$

Thus

$$\sum_{j=i}^{\infty}\left\{\frac{\partial}{\partial t}\left[p_{ij}(s,t)\right]\right\}z^{j} = -\sum_{j=i}^{\infty}\lambda_{j}(t)p_{ij}(s,t)z^{j} + z\sum_{j=i}^{\infty}\lambda_{j-1}(t)p_{i,j-1}(s,t)z^{j-1} \quad \textbf{(4.2)}$$

4.3 Poisson Process

When $\lambda_j(t) = \lambda,$ equation **(4.2)** becomes

$$\sum_{j=i}^{\infty}\frac{\partial}{\partial t}\left[p_{ij}(s,t)\right]z^{j} = -\sum_{j=i}^{\infty}\lambda p_{ij}(s,t)z^{j} + z\sum_{j=i}^{\infty}\lambda p_{i,j-1}(s,t)z^{j-1}$$

$$= \sum_{j=i}^{\infty}(z-1)\lambda p_{ij}(s,t)z^{j}$$

$$\sum_{j=i}^{\infty}\frac{\partial}{\partial t}\left[p_{ij}(s,t)\right]z^{j} = \sum_{j=i}^{\infty}(z-1)\lambda p_{ij}(s,t)z^{j}$$

$$\sum_{j=i}^{\infty}\frac{\partial}{\partial t}\left[p_{ij}(s,t)\right]z^{j} = (z-1)\lambda\sum_{j=i}^{\infty}p_{ij}(s,t)z^{j} \quad \textbf{(4.3)}$$

Therefore,

$$\sum_{j=i}^{\infty}\frac{\partial}{\partial t}\left[p_{ij}(s,t)\right]z^{j} = (z-1)\lambda G(s,t,z)$$

$$\frac{\partial}{\partial t}\sum_{j=i}^{\infty}\left[p_{ij}(s,t)\right]z^{j} = (z-1)\lambda G(s,t,z)$$

$$\frac{\frac{\partial}{\partial t}\sum_{j=i}^{\infty}\left[p_{ij}(s,t)\right]z^{j}}{G(s,t,z)} = (z-1)\lambda$$

$$\frac{\frac{\partial}{\partial t}\left[G(s,t,z)\right]}{G(s,t,z)} = (z-1)\lambda$$

$$\partial\log G(s,t,z) = (z-1)\lambda$$

Integrating with respect to t

$$\int\partial\log G(s,t,z) = \int(z-1)\lambda\,\partial x$$

$$\log G(s,t,z) = (z-1)\lambda t + c$$

Taking exponential both sides, we have

$$G(s,t,z) = e^{(z-1)\lambda t + c}$$

$$= k\,e^{(z-1)\lambda t}$$

$$G(s,t,z) = k\,e^{(z-1)\lambda t}$$

When t = s

$$G(s,s,z) = k\,e^{(z-1)\lambda s}$$

But

$$G(s,s,z) = \sum_{j=i} p_{ij}(s,s) z^j = z^i$$

Thus

$$z^i = ke^{(z-1)\lambda s}$$

$$k = z^i e^{-(z-1)\lambda s}$$

$$G(s,t,z) = z^i e^{-(z-1)\lambda s} e^{(z-1)\lambda t}$$

$$= z^i e^{(z-1)\lambda(t-s)} \qquad \textbf{(4.4)}$$

$p_{jk}(s,t)$ is the coefficient of z^n on $G(s,t,z)$.

$$G(s,t,z) = z^j e^{(z-1)\lambda(t-s)}$$

$$= z^j e^{z\lambda(t-s) - \lambda(t-s)}$$

$$= z^j e^{-\lambda(t-s)} e^{\lambda(t-s)z}$$

Therefore,

$$G(s,t,z) = z^i e^{-\lambda(t-s)} \sum_{n=0}^{\infty} \frac{(\lambda(t-s)z)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda(t-s)}(\lambda(t-s))^n z^n z^i}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda(t-s)}(\lambda(t-s))^n}{n!} z^{n+i}$$

Let k = n + i. Then

$p_{jk}(s,t) = \dfrac{e^{-\lambda(t-s)}(\lambda(t-s))^n}{n!}$ which is a Poisson distribution with parameter $\lambda(t-s)$.

Thus

$$\mathbf{p_{jk}(s,t) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^n}{n!}} \qquad n = 0, 1, 2,\ldots \qquad \textbf{(4.5)}$$

4.4 Simple Birth

When $\lambda_j(t) = j\lambda$, equation **(4.2)** becomes

$$\sum_{j=i}^{\infty}\frac{\partial}{\partial t}\left[p_{ij}(s,t)\right]z^j = -\sum_{j=i}^{\infty} j\lambda p_{ij}(s,t)z^j + z\sum_{j=i}^{\infty}(j-1)\lambda p_{ij}(s,t)z^{j-1}$$

$$= -z\sum_{j=i}^{\infty} j\lambda p_{ij}(s,t)z^{j-1} + z^2\sum_{j=i}^{\infty}(j-1)\lambda p_{ij}(s,t)z^{j-2} \qquad \textbf{(4.6)}$$

Therefore,

$$\sum_{j=i}^{\infty}\frac{\partial}{\partial t}\left\{p_{ij}(s,t)\right\}z^j = \sum_{j=i}^{\infty} z(z-1)j\lambda p_{ij}(s,t)z^{j-1}$$

$$\frac{\partial}{\partial t}\sum_{j=i}^{\infty} p_{ij}(s,t)z^j = z(z-1)\lambda\sum_{j=i}^{\infty} k\,p_{ij}(s,t)z^{j-1}$$

$$\frac{\partial}{\partial t}G(s,t,z) = z(z-1)\lambda\frac{\partial}{\partial z}G(s,t,z)$$

$$\frac{\partial}{\partial t}G(s,t,z) - z(z-1)\lambda\frac{\partial}{\partial z}G(s,t,z) = 0$$

The auxiliary equation

$$\frac{\partial t}{1} = \frac{\partial z}{-z(z-1)\lambda} = \frac{\partial G(s,t,z)}{0} \qquad \textbf{(4.7)}$$

Taking the first two equations, we have

$$\frac{\partial t}{1} = \frac{\partial z}{-z(z-1)\lambda}$$

$$-\lambda\partial t = \frac{\partial z}{z(z-1)}$$

Integrating, we obtain

$$-\lambda t = \int \frac{\partial z}{z(z-1)}$$

$$= -\int \frac{\partial z}{z} + \int \frac{\partial z}{z-1}$$

$$= -\ln z + \ln(z-1) + c$$

$$c = -\lambda t + \ln z - \ln(z-1)$$

$$= -\lambda t + \ln\left(\frac{z}{z-1}\right)$$

Taking another pair, we have

$$\frac{\partial t}{1} = \frac{\partial G(s,t,z)}{0}$$

$$0\,\partial t = \partial G(s,t,z)$$

Integrating both sides,

$$\int 0\,\partial t = \int \partial G(s,t,z)$$

$$c = G(s,t,z)$$

Therefore,

$$G(s,t,z) = \psi(u) \text{ where } u = e^{-\lambda t + \ln\left(\frac{z}{z-1}\right)}$$

Then

$$u = e^{-\lambda t}\frac{z}{z-1}$$

$$u(z-1) = z e^{-\lambda t}$$

$$uz - u = z e^{-\lambda t}$$

$$uz - z e^{-\lambda t} = u$$

$$z\left(u - e^{-\lambda t}\right) = u$$

$$z = \frac{u}{u - e^{-\lambda t}}$$

Using boundary condition with t = s, we have

$$G(s,s,z) = z^i$$

$$= \left[\frac{u}{u - e^{-\lambda s}}\right]^i$$

Now,

$$G(s,t,z) = \left[\frac{\dfrac{z e^{-\lambda t}}{z - 1}}{\dfrac{z e^{-\lambda t}}{z - 1} - e^{-\lambda s}}\right]^i$$

$$= \left[\frac{z e^{-\lambda t}}{z e^{-\lambda t} - (z - 1) e^{-\lambda s}}\right]^i$$

$$= \left[\frac{z e^{-\lambda t}}{z\left(e^{-\lambda t} - e^{-\lambda s}\right) + e^{-\lambda s}}\right]^i$$

$$= \left[\frac{z e^{-\lambda t} e^{\lambda s}}{z\left(e^{-\lambda(t-s)} - 1\right) + 1}\right]^i$$

$$= \left[\frac{z e^{-\lambda(t-s)}}{1 - z\left(1 - e^{-\lambda(t-s)}\right)}\right]^i \quad \textbf{(4.8)}$$

Let $p = e^{-\lambda(t-s)}$ and $q = 1 - e^{-\lambda(t-s)}$

$P_{j,k}(s,t)$ is the coefficient of z^n on $G(s,t,z)$.

$$G(s,t,z) = \left[\frac{zp}{1 - zq}\right]^{i}$$

$$= (zp)^{i}(1 - zq)^{-i}$$

$$= (zp)^{i} \sum_{n}^{\infty} \binom{-j}{n} (-zq)^{n}$$

$$= z^{i} p^{i} \sum_{n}^{\infty} (-1)^{n} \binom{j + n - 1}{n} (-1)^{n} (zq)^{n}$$

Therefore,

$$G(s,t,z) = \sum_{n}^{\infty} (-1)^{2n} \binom{j + n - 1}{n} p^{i} q^{n} z^{i} z^{n}$$

$$= \sum_{n}^{\infty} \binom{j + n - 1}{n} p^{i} q^{n} z^{i + n}$$

Let $k = i + n$

$n = 0, 1, 2, \dots$

$$P_{ij}(s,t) = \binom{j + n - 1}{n} p^{i} q^{n}$$

$$= \binom{j + n - 1}{n} \left(e^{-\lambda(t - s)}\right)^{i} \left(1 - e^{-\lambda(t - s)}\right)^{n}$$

Thus

$$\mathbf{p_{ij}(s,t) = \binom{i + n - 1}{n} \left(e^{-\lambda(t - s)}\right)^{i} \left(1 - e^{-\lambda(t - s)}\right)^{n}} \qquad \textbf{(4.9)}$$

Where $k = i + n$ and $n = 0, 1, 2, \dots$

which is negative binomial distribution.

4.5 Simple Birth with Immigration

In this case $\lambda_j(t) = j\lambda + v$

Equation **(4.2)** now becomes

$$\sum_{j=i}^{\infty}\frac{\partial}{\partial t}\left[p_{ij}(s,t)\right]z^j = -\sum_{j=i}^{\infty}(j\lambda + v)p_{ij}(s,t)z^j + z\sum_{j=i+1}^{\infty}\left[(j-1)\lambda + v\right]p_{ij}(s,t)z^{j-1}$$

$$= -\lambda\sum_{j=i}^{\infty}k\,p_{ij}(s,t)z^j - v\sum_{j=i}^{\infty}p_{ij}(s,t)z^j + z\lambda\sum_{j=i+1}^{\infty}(j-1)p_{ij}(s,t)z^{j-1} + zv\sum_{j=i+1}^{\infty}p_{ij}(s,t)z^{j-1}$$

$$= -\lambda z\sum_{j=i}^{\infty}k\,p_{ij}(s,t)z^{j-1} - vG(s,t,z) + z^2\lambda\sum_{j=i+1}^{\infty}(j-1)p_{ij}(s,t)z^{j-2} + zvG(s,t,z)$$

$$= \left(z^2\lambda - \lambda z\right)\sum_{j=i}^{\infty}j\,p_{ij}(s,t)z^{k-1} + (zv - v)\,G(s,t,z)$$

Therefore,

$$\frac{\partial}{\partial t}\sum_{j=i}^{\infty}P_{ij}(s,t)z^j = \lambda z(z-1)\frac{\partial}{\partial z}G(s,t,z) + v(z-1)\,G(s,t,z)$$

$$\frac{\partial}{\partial t}G(s,t,z) - \lambda z(z-1)\frac{\partial}{\partial z}G(s,t,z) = v(z-1)\,G(s,t,z) \qquad \textbf{(4.10)}$$

Auxiliary equations

$$\frac{\partial t}{1} = \frac{\partial z}{-\lambda z(z-1)} = \frac{\partial G(s,t,z)}{v(z-1)\,G(s,t,z)} \qquad \textbf{(4.11)}$$

Selecting the first two,

$$\frac{\partial t}{1} = \frac{\partial z}{-\lambda z(z-1)}$$

$$-\lambda \partial t = \frac{\partial z}{z(z-1)}$$

$$-\lambda \partial t = -\frac{1}{z}\partial z + \frac{1}{z-1}\partial z$$

Integrating both sides

$$-\lambda t = -\int \frac{1}{z}\partial z + \int \frac{1}{z-1}\partial z$$

$$-\lambda t = -\ln z + \ln(z-1) + c$$

$$-\lambda t + \ln z - \ln(z-1) = c$$

$$c = -\lambda t + \ln\left(\frac{z}{z-1}\right) \qquad \text{(i)}$$

Taking the last two, we have

$$\frac{\partial z}{-\lambda z(z-1)} = \frac{\partial G(s,t,z)}{v(z-1)\,G(s,t,z)}$$

Multiply both sides by $v(z-1)$

$$-\frac{v}{\lambda}\frac{\partial z}{z} = \frac{\partial G(s,t,z)}{G(s,t,z)}$$

$$-\frac{v}{\lambda}\frac{\partial z}{z} = \partial \ln G(s,t,z)$$

Integrating both sides, e have

$$-\frac{v}{\lambda}\int \frac{\partial z}{z} = \int \partial \ln G(s,t,z)$$

$$-\frac{v}{\lambda}\ln z + c = \ln G(s,t,z)$$

$$G(s,t,z) = e^{\ln z^{-\frac{v}{\lambda}} + c} = e^{\ln z^{-\frac{v}{\lambda}}} e^{c} = z^{-\frac{v}{\lambda}}.k$$

$$k = z^{\frac{v}{\lambda}}\,G(s,t,z) \qquad \text{(ii)}$$

From (i), let

$$u = e^{-\lambda t + \ln\left(\frac{z}{z-1}\right)} = e^{-\lambda t} e^{\ln\left(\frac{z}{z-1}\right)} = e^{-\lambda t}\left(\frac{z}{z-1}\right)$$

From (i) and (ii),

$$z^{\frac{v}{\lambda}} G(s,t,z) = \psi(u)$$

$$G(s,t,z) = z^{-\frac{v}{\lambda}} \psi(u)$$

Boundary condition t = s.

$$G(s,s,z) = z^{i} = z^{-\frac{v}{\lambda}} \psi(u)$$

$$\psi(u) = z^{i + \frac{v}{\lambda}}$$

$$u = e^{-\lambda s} \frac{z}{z-1}$$

$$u(z-1) = e^{-\lambda s} z$$

$$uz - u = e^{-\lambda s} z \quad \Rightarrow \quad z\left(u - e^{-\lambda s}\right) = u$$

$$z = \frac{u}{u - e^{-\lambda s}}$$

Therefore,

$$\psi(u) = \left(\frac{u}{u - e^{-\lambda s}}\right)^{i + \frac{v}{\lambda}}$$

$$G(s,t,z) = z^{-\frac{v}{\lambda}} \left(\frac{e^{-\lambda t} \dfrac{z}{z - 1}}{e^{-\lambda t} \dfrac{z}{z - 1} - e^{-\lambda s}} \right)^{i + \frac{v}{\lambda}}$$

$$= z^{-\frac{v}{\lambda}} \left[\frac{ze^{-\lambda t}}{ze^{-\lambda t} - (z - 1)e^{-\lambda s}} \right]^{i + \frac{v}{\lambda}}$$

$$= z^{-\frac{v}{\lambda}}.z^{i + \frac{v}{\lambda}} \left[\frac{e^{-\lambda s}}{z\left(e^{-\lambda t} - e^{-\lambda s}\right) + e^{-\lambda s}} \right]^{i + \frac{v}{\lambda}}$$

$$= z^{i} \left[\frac{e^{-\lambda s}}{e^{-\lambda s} + z\left(e^{-\lambda t} - e^{-\lambda s}\right)} \right]^{i + \frac{v}{\lambda}}$$

$$= z^{i} \left[\frac{e^{-\lambda(t - s)}}{1 + z\left(e^{-\lambda(t - s)} - 1\right)} \right]^{i + \frac{v}{\lambda}}$$

$$= z^{i} \left[\frac{e^{-\lambda(t - s)}}{1 - z\left(1 - e^{-\lambda(t - s)}\right)} \right]^{i + \frac{v}{\lambda}} \qquad \textbf{(4.12)}$$

Let $p = e^{-\lambda(t-s)}$ and $q = 1 - p = 1 - e^{-\lambda(t-s)}$. Then

$$G(s,t,z) = z^{i} \left[\frac{p}{1 - qz} \right]^{i + \frac{v}{\lambda}}$$

$p_{jk}(s,t)$ is the coefficient of z^n of $G(s,t,z)$.

$$G(s,t,z) = z^{i} p^{i + \frac{v}{\lambda}} (1 - qz)^{-\left(j + \frac{v}{\lambda}\right)}$$

$$= z^{i} p^{i + \frac{v}{\lambda}} \sum_{n=0}^{\infty} \binom{-\left(i + \frac{v}{\lambda}\right)}{n} (-qz)^{n}$$

$$G(s,t,z) = z^i p^{i+\frac{v}{\lambda}} \sum_{n=0}^{\infty} (-1)^n \binom{i+\frac{v}{\lambda}+n-1}{n} (-1)^n (qz)^n$$

$$= \sum_{n=0}^{\infty} (-1)^{2n} \binom{i+\frac{v}{\lambda}+n-1}{n} p^{i+\frac{v}{\lambda}} q^n z^{i+n}$$

$$= \sum_{n=0}^{\infty} \binom{i+\frac{v}{\lambda}+n-1}{n} p^{i+\frac{v}{\lambda}} q^n z^{i+n}$$

$k = i + n : n = 0,1,2,\ldots$

$$p_{ij}(s,t) = p_{i,i+n}(s,t) = \binom{i+\frac{v}{\lambda}+n-1}{n} p^{i+\frac{v}{\lambda}} q^n, \quad n = 0,1,2,\ldots$$

Thus

$$p_{ij}(s,t) = \binom{i+\frac{v}{\lambda}+n-1}{n} \left(e^{-\lambda(t-s)}\right)^{i+\frac{v}{\lambda}} \left(1 - e^{-\lambda(t-s)}\right)^n$$

Equivalently,

$$\mathbf{p_{ij}(s,t) = \binom{i+\frac{v}{\lambda}+n-1}{n} e^{-(t-s)\lambda\left(i+\frac{v}{\lambda}\right)} \left(1 - e^{-\lambda(t-s)}\right)^n} \qquad \textbf{(4.13)}$$

This is a negative binomial distribution function.

4.6 Polya process

In this case $\lambda_j(t) = \left[\dfrac{1 + aj}{1 + \lambda at}\right]\lambda$

Equation **(4.2)** now becomes

$$\sum_{j=i}^{\infty}\frac{\partial}{\partial t}\left[p_{ij}(s,t)\right]z^{i} = -\sum_{j=i}^{\infty}\left[\frac{1 + aj}{1 + \lambda at}\right]\lambda p_{ij}(s,t)z^{j} + z\sum_{j=i+1}^{\infty}\left[\frac{1 + a(j - 1)}{1 + \lambda at}\right]\lambda p_{i,j-1}(s,t)z^{j-1}$$

$$= -\left[\frac{\lambda}{1 + \lambda at}\right]\left\{\sum_{j=i}^{\infty}(1 + aj)p_{ij}(s,t)z^{j} - z\sum_{j=i+1}^{\infty}\left(1 + a(j - 1)\right)p_{i,j-1}(s,t)z^{j-1}\right\}$$

Expanding, we have,

$$\sum_{j=i}^{\infty}\frac{\partial}{\partial t}\left[p_{ij}(s,t)\right]z^{j} = -\left[\frac{\lambda}{1+\lambda at}\right]\left\{\sum_{j=i}^{\infty}p_{ij}(s,t)z^{j} + a\sum_{j=i}^{\infty}k\,p_{ij}(s,t)z^{j} - z\sum_{j=i+1}^{\infty}p_{i,j-1}(s,t)z^{j-1} - za\sum_{j=i+1}^{\infty}(j-1)p_{i,j-1}(s,t)z^{j-1}\right\}$$

Equivalently,

$$\frac{\partial}{\partial t}G(s,t,z) = -\left[\frac{\lambda}{1+\lambda at}\right]\left\{G(s,t,z) + az\frac{\partial}{\partial z}G(s,t,z) - zG(s,t,z) - z^{2}a\frac{\partial}{\partial z}G(s,t,z)\right\}$$

$$\left(\frac{1+\lambda at}{\lambda}\right)\frac{\partial}{\partial t}G(s,t,z) = \left\{-G(s,t,z) - az\frac{\partial}{\partial z}G(s,t,z) + zG(s,t,z) + z^{2}a\frac{\partial}{\partial z}G(s,t,z)\right\}$$

$$= (z - 1)G(s,t,z) + za(z - 1)\frac{\partial}{\partial z}G(s,t,z)$$

Re writing the above equation, we have

$$\left(\frac{1+\lambda at}{\lambda}\right)\frac{\partial}{\partial t}G(s,t,z) - za(z - 1)\frac{\partial}{\partial z}G(s,t,z) = (z - 1)G(s,t,z) \qquad \textbf{(4.14)}$$

Solving using Lagrange's equation

Auxiliary equations

$$\frac{\lambda\,\partial t}{1+\lambda at} = \frac{\partial z}{-za(z - 1)} = \frac{\partial G(s,t,z)}{(z - 1)G(s,t,z)} \qquad \textbf{(4.15)}$$

Taking the first two equations, we have

$$\frac{\lambda\,\partial t}{1+\lambda at} = \frac{\partial z}{-za(z - 1)}$$

$$\frac{-\lambda a\,\partial t}{1+\lambda at} = \frac{\partial z}{z(z - 1)} = -\frac{\partial z}{z} + \frac{\partial z}{z - 1}$$

Integrating both sides, we have

$$-\lambda a\int\frac{\partial t}{1+\lambda at} = -\int\frac{\partial z}{z} + \int\frac{\partial z}{z-1}$$

$$-\ln(1+\lambda at) = -\ln z + \ln(z-1) + c$$

$$-\ln(1+\lambda at) + \ln z - \ln(z-1) = c$$

$$\ln\left[\frac{z}{(1+\lambda at)(z-1)}\right] = c$$

$$\frac{z}{(1+\lambda at)(z-1)} = u(t) \qquad \text{(i)}$$

Taking the last two, we have

$$\frac{\partial z}{-za(z-1)} = \frac{\partial G(s,t,z)}{(z-1)G(s,t,z)}$$

Multiplying both sides by (z – 1), we have

$$-\frac{1}{a}\frac{\partial z}{z} = \frac{\partial G(s,t,z)}{G(s,t,z)} = \partial \ln G(s,t,z)$$

Integrating,

$$-\frac{1}{a}\int\frac{\partial z}{z} = \int\partial \ln G(s,t,z)$$

$$-\frac{1}{a}\ln z + c = \ln G(s,t,z)$$

$$e^{\ln z^{-\frac{1}{a}}} e^{c} = G(s,t,z)$$

$$k z^{-\frac{1}{a}} = G(s,t,z)$$

$$k = z^{\frac{1}{a}} G(s,t,z) \qquad \text{(ii)}$$

From (i) and (ii), we have

$$z^{\frac{1}{a}} G(s,t,z) = \psi(u(t))$$

$$G(s,t,z) = \psi(u(t)) z^{-\frac{1}{a}}$$

Using the boundary condition t = s, we have

$$G(s,s,z) = \psi(u(s))z^{-\frac{1}{a}}$$

$$z^{i} = \psi(u(s))z^{-\frac{1}{a}}$$

$$\psi(u(s)) = z^{i+\frac{1}{a}}$$

From (i)

$$\frac{z}{(1+\lambda at)(z-1)} = u(t)$$

$$u(t)(z-1) = \frac{z}{(1+\lambda at)}$$

$$zu(t) - u(t) = \frac{z}{(1+\lambda at)}$$

$$zu(t) - \frac{z}{(1+\lambda at)} = u(t)$$

Therefore,

$$z = \frac{u(t)(1+\lambda at)}{u(t)(1+\lambda at)-1}$$

Thus,

$$\psi(u(s)) = \left(\frac{u(s)(1+\lambda at)}{u(s)(1+\lambda at)-1}\right)^{i+\frac{1}{a}}$$

$$G(s,s,z) = \left(\frac{u(s)(1+\lambda at)}{u(s)(1+\lambda at)-1}\right)^{i+\frac{1}{a}} z^{-\frac{1}{a}}$$

Therefore,

$$G(s,t,z) = \left(\frac{u(t)(1+\lambda as)}{u(t)(1+\lambda as) - 1} \right)^{i+\frac{1}{a}} z^{-\frac{1}{a}}$$

$$= \left(\frac{\frac{z}{(1+\lambda at)(z-1)}(1+\lambda as)}{\frac{z}{(1+\lambda at)(z-1)}(1+\lambda as) - 1} \right)^{i+\frac{1}{a}} z^{-\frac{1}{a}}$$

$$= \left(\frac{z(1+\lambda as)}{z(1+\lambda as) - (1+\lambda at)(z-1)} \right)^{i+\frac{1}{a}} z^{-\frac{1}{a}}$$

$$G(s,t,z) = \left(\frac{\left(\frac{1+\lambda as}{1+\lambda at}\right)}{z\left[\left(\frac{1+\lambda as}{(1+\lambda at)}\right) - 1\right] + 1} \right)^{i+\frac{1}{a}} z^{i}$$

$$= \left(\frac{\left(\frac{1+\lambda as}{1+\lambda at}\right)}{1 - z\left[\left(1 - \frac{1+\lambda as}{(1+\lambda at)}\right)\right]} \right)^{i+\frac{1}{a}} z^{i}$$

$$= \left(\frac{\left(\frac{1+\lambda as}{1+\lambda at}\right)}{1 - z\left[\left(\frac{1+\lambda at - 1 - \lambda as}{1+\lambda at}\right)\right]} \right)^{i+\frac{1}{a}} z^{i}$$

Therefore

$$G(s,t,z) = \left(\frac{\left(\frac{1+\lambda as}{1+\lambda at}\right)}{1 - \frac{\lambda a(t-s)}{1+\lambda at} z} \right)^{i+\frac{1}{a}} z^{i} \qquad \textbf{(4.16)}$$

Let $p = \dfrac{1+\lambda as}{1+\lambda at}$

$$q = 1 - p = 1 - \frac{1+\lambda as}{1+\lambda at} = \frac{\lambda a(t-s)}{1+\lambda at}$$

$P_{j,k}(s,t)$ is the coefficient of z^n on $G(s,t,z)$

$$G(s,t,z) = \left(\frac{p}{1 - qz}\right)^{i+\frac{1}{a}} z^i$$

$$= p^{i+\frac{1}{a}} z^i (1 - qz)^{-\left(i+\frac{1}{a}\right)}$$

$$= p^{i+\frac{1}{a}} z^i \sum_{n=0}^{\infty} \binom{-(i + \frac{1}{a})}{n} (- qz)^n$$

$$= p^{i+\frac{1}{a}} z^i \sum_{n=0}^{\infty} (-1)^n \binom{i + \frac{1}{a} + n - 1}{n} (-1)^n (qz)^n$$

$$G(s,t,z) = p^{i+\frac{1}{a}} z^i \sum_{n=0}^{\infty} \binom{i + \frac{1}{a} + n - 1}{n} q^n z^n$$

$$= \sum_{n=0}^{\infty} \binom{i + \frac{1}{a} + n - 1}{n} p^{i+\frac{1}{a}} q^n z^{n+i}$$

$k = i + n,\ n = 0,1,2,3,\ldots$

$$p_{ij}(s,t) = p_{i,\,i+n}(s,t) = \binom{i + \frac{1}{a} + n - 1}{n} p^{i+\frac{1}{a}} q^n$$

Thus

$$\mathbf{p_{ij}(s,t) = p_{i,i+n}(s,t) = \binom{i + \frac{1}{a} + n - 1}{n} \left(\frac{1 + \lambda as}{1 + \lambda at}\right)^{i+\frac{1}{a}} \left(\frac{\lambda a(t - s)}{1 + \lambda at}\right)^{n}} \qquad \textbf{(4.17)}$$

for n = 0, 1, 2, …

Which is negative binomial distribution.

CHAPTER FIVE

INFINITESIMAL GENERATOR

5.1 Introduction

Now, we wish to solve equation **(2.11)** below which was derived in Chapter two. This time, we will use a matrix method.

$$\frac{\partial}{\partial t} p_{k,k+n}(s,t) + \lambda_{k+n}(t) p_{k,k+n}(s,t) = \lambda_{k+n-1}(t)\, p_{k,k+n-1}(s,t)$$

For k = 0, 1, 2, 3, .. equation **(2.11)** becomes

k = 0

$$\frac{\partial}{\partial t} P_{jj}(s,t) = -\lambda_j P_{jj}(s,t) + \lambda_{j-1} p_{jj-1}(s,t)$$

$$= -\lambda_j P_{jj}(s,t)$$

k = 1

$$\frac{\partial}{\partial t} P_{jj+1}(s,t) = -\lambda_{j+1} P_{jj+1}(s,t) + \lambda_j p_{jj}(s,t)$$

$$= \lambda_j P_{jj}(s,t) - \lambda_{j+1} p_{jj+1}(s,t)$$

k = 2

$$\frac{\partial}{\partial t} P_{jj+2}(s,t) = -\lambda_{j+2} P_{jj+2}(s,t) + \lambda_{j+1} p_{jj+1}(s,t)$$

$$= \lambda_{j+1} P_{jj+1}(s,t) - \lambda_{j+2} p_{jj+2}(s,t)$$

k = 3

$$\frac{\partial}{\partial t} P_{jj+3}(s,t) = -\lambda_{j+3} P_{jj+3}(s,t) + \lambda_{j+2} p_{jj+2}(s,t)$$

$$= \lambda_{j+2} p_{jj+2}(s,t) - \lambda_{j+3} P_{jj+3}(s,t)$$

k = 4

$$\frac{\partial}{\partial t} P_{jj+4}(s,t) = -\lambda_{j+4} P_{jj+4}(s,t) + \lambda_{j+3} p_{jj+3}(s,t)$$

$$= \lambda_{j+3} p_{jj+3}(s,t) - \lambda_{j+4} P_{jj+4}(s,t)$$

k = n

$$\frac{\partial}{\partial t} P_{jj+n}(s,t) = -\lambda_{j+n} P_{jj+n}(s,t) + \lambda_{j+n-1} p_{jj+n-1}(s,t)$$

$$= \lambda_{j+n-1} p_{jj+n-1}(s,t) - \lambda_{j+n} P_{jj+n}(s,t)$$

These equations forms a matrix expressed as follows

$$\begin{bmatrix} \frac{\partial}{\partial t} p_{jj}(s,t) \\ \frac{\partial}{\partial t} p_{j,j+1}(s,t) \\ \frac{\partial}{\partial t} p_{j,j+2}(s,t) \\ \frac{\partial}{\partial t} p_{j,j+3}(s,t) \\ \frac{\partial}{\partial t} p_{j,j+4}(s,t) \\ . \\ . \\ \frac{\partial}{\partial t} p_{j,j+k}(s,t) \\ . \end{bmatrix} = \begin{bmatrix} -\lambda_j & 0 & 0 & 0 & 0 & . & . & 0 & . \\ \lambda_j & -\lambda_{j+1} & 0 & 0 & 0 & . & . & 0 & . \\ 0 & \lambda_{j+1} & -\lambda_{j+2} & 0 & 0 & . & . & 0 & . \\ 0 & 0 & \lambda_{j+2} & -\lambda_{j+3} & 0 & . & . & 0 & . \\ 0 & 0 & 0 & \lambda_{j+3} & -\lambda_{j+4} & . & . & 0 & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & \lambda_{j+n-1} & -\lambda_{j+n} & . \\ . & . & . & . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} p_{jj}(s,t) \\ p_{jj+1}(s,t) \\ p_{jj+2}(s,t) \\ p_{jj+3}(s,t) \\ p_{jj+4}(s,t) \\ . \\ . \\ p_{jj+n}(s,t) \\ . \end{bmatrix}$$

(5.1)

5.2 Solving Matrix Equation

$$\frac{\partial}{\partial t} p(s,t) = \Lambda p(s,t)$$

$$\frac{\frac{\partial}{\partial t} p(s,t)}{p(s,t)} = \Lambda$$

$$\partial \log p(s,t) = \Lambda dt$$

Integrating both sides, we have

$$\int_s^t \partial \log p(s,y) dy = \int_s^t \Lambda dy$$

$$\log p(s,y)\Big|_s^t = \Lambda y\Big|_s^t$$

$$\log p(s,t) - \log p(s,s) = \Lambda(t - s)$$

$$\log p(s,s) = I$$

$$\log p(s,t) = \Lambda(t - s)$$

Taking the exponential of both sides

$$p(s,t) = e^{\Lambda(t-s)}$$

Therefore, from Taylor expansion,

$$p(s,t) = \sum_{k=0}^{\infty} \frac{\left(\Lambda(t-s)\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{\Lambda^k (t-s)^k}{k!}$$

$$= I + \sum_{n=1}^{\infty} \frac{\left(\Lambda(t-s)\right)^n}{n!}$$

If all the eigen values μ_i of Λ are distinct, then

$$p(s,t) = e^{\Lambda(t-s)} = U \operatorname{diag} e^{\mu_i(t-s)} V \qquad \textbf{(5.2)}$$

Obtaining the eigen values of Λ,

$$|\Lambda - \mu I| = 0$$

$$\begin{vmatrix} -\lambda_j - \mu & 0 & 0 & 0 & 0 & . & 0 & 0 & . \\ \lambda_j & -\lambda_{j+1} - \mu & 0 & 0 & 0 & . & 0 & 0 & . \\ 0 & \lambda_{j+1} & -\lambda_{j+2} - \mu & 0 & 0 & . & 0 & 0 & . \\ 0 & 0 & \lambda_{j+2} & -\lambda_{j+3} - \mu & 0 & . & 0 & 0 & . \\ 0 & 0 & 0 & \lambda_{j+3} & -\lambda_{j+3} - \mu & . & 0 & 0 & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & \lambda_{j+n-1} & -\lambda_{j+n} - \mu & . \\ . & . & . & . & . & . & . & . & . \end{vmatrix} = 0$$

Taking $(n+1)\times(n+1)$ matrix, we have

$$\left(-\lambda_j-\mu\right)\left(-\lambda_{j+1}-\mu\right)\left(-\lambda_{j+2}-\mu\right)\left(-\lambda_{j+3}-\mu\right)\left(-\lambda_{j+4}-\mu\right)\ldots\left(-\lambda_{j+n}-\mu\right)=0$$

$$(-1)^{n+1}\left(\lambda_j+\mu\right)\left(\lambda_{j+1}+\mu\right)\left(\lambda_{j+2}+\mu\right)\left(\lambda_{j+3}+\mu\right)\left(\lambda_{j+4}+\mu\right)\ldots\left(\lambda_{j+n}+\mu\right)=0$$

$$\left(\lambda_j+\mu\right)\left(\lambda_{j+1}+\mu\right)\left(\lambda_{j+2}+\mu\right)\left(\lambda_{j+3}+\mu\right)\left(\lambda_{j+4}+\mu\right)\ldots\left(\lambda_{j+n}+\mu\right)=0$$

Solving, we obtain

$$\mu=-\lambda_j \text{ or } \mu=-\lambda_{j+1} \text{ or } \mu=-\lambda_{j+2} \text{ or } \ldots \text{ or } \mu=-\lambda_{j+n}$$

Assume that λ_{j+i} i = 0, 1, 2, …, n are distinct. We obtain n + 1 distinct eigen values. Next we obtain eigenvectors corresponding to n + 1 eigen values. Let

$$z=\begin{bmatrix} z_j \\ z_{j+1} \\ z_{j+2} \\ . \\ . \\ z_{j+n} \end{bmatrix}$$

The eigen vector e_{j+i} for $\mu=-\lambda_{j+i}$ i = 0, 1, 2, … is given by

$$\begin{bmatrix} -\lambda_j & 0 & 0 & 0 & 0 & . & . & 0 \\ \lambda_j & -\lambda_{j+1} & 0 & 0 & 0 & . & . & 0 \\ 0 & \lambda_{j+1} & -\lambda_{j+2} & 0 & 0 & . & . & 0 \\ 0 & 0 & \lambda_{j+2} & -\lambda_{j+3} & 0 & . & . & 0 \\ 0 & 0 & 0 & \lambda_{j+3} & -\lambda_{j+4} & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & -\lambda_{j+k} \end{bmatrix}\times\begin{bmatrix} z_j \\ z_{j+1} \\ z_{j+2} \\ z_{j+3} \\ z_{j+4} \\ . \\ . \\ z_{j+n} \end{bmatrix}=-\lambda_{j+i}\begin{bmatrix} z_j \\ z_{j+1} \\ z_{j+2} \\ z_{j+3} \\ z_{j+4} \\ . \\ . \\ z_{j+n} \end{bmatrix}$$

e_j

$$\lambda_j z_j - \lambda_{j+k} z_{j+1} = -\lambda_j z_{j+1}$$

$$z_{j+1} = \frac{\lambda_j}{\lambda_{j+1} - \lambda_j} z_j$$

Let $z_j = 1$

$$z_{j+1} = \frac{\lambda_j}{\lambda_{j+1} - \lambda_j}$$

$$\lambda_{j+1} z_{j+1} - \lambda_{j+2} z_{j+2} = -\lambda_j z_{j+2}$$

$$z_{j+2} = \frac{\lambda_{j+1}}{\lambda_{j+2} - \lambda_j} z_{j+1}$$

$$= \frac{\lambda_j}{\lambda_{j+1} - \lambda_j} \frac{\lambda_{j+1}}{\lambda_{j+2} - \lambda_j}$$

$$\lambda_{j+2} z_{j+2} - \lambda_{j+3} z_{j+3} = -\lambda_j z_{j+3}$$

$$z_{j+3} = \frac{\lambda_{j+2}}{\lambda_{j+3} - \lambda_j} z_{j+2}$$

$$= \frac{\lambda_j \lambda_{j+1} \lambda_{j+2}}{\left(\lambda_{j+1} - \lambda_j\right)\left(\lambda_{j+2} - \lambda_j\right)\left(\lambda_{j+3} - \lambda_j\right)}$$

Similarly

$$z_{j+n} = \frac{\lambda_j \lambda_{j+1} \lambda_{j+2} \ldots \lambda_{j+n-1}}{\left(\lambda_{j+1} - \lambda_j\right)\left(\lambda_{j+2} - \lambda_j\right)\ldots\left(\lambda_{j+n} - \lambda_j\right)}$$

$$= \prod_{k=1}^{n} \left(\frac{\lambda_{j+k-1}}{\lambda_{j+k} - \lambda_j} \right)$$

The eigen vector e_j

$$e_j = \begin{bmatrix} 1 \\ \dfrac{\lambda_j}{\lambda_{j+1} - \lambda_j} \\ \dfrac{\lambda_j \lambda_{j+1}}{\left(\lambda_{j+1} - \lambda_j\right)\left(\lambda_{j+2} - \lambda_j\right)} \\ . \\ . \\ \prod_{k=1}^{n} \dfrac{\lambda_{j+k-1}}{\left(\lambda_{j+k} - \lambda_j\right)} \end{bmatrix}$$

$$e_{j+1} = \begin{bmatrix} 0 \\ 1 \\ \dfrac{\lambda_{j+1}}{\lambda_{j+2} - \lambda_{j+1}} \\ \dfrac{\lambda_{j+1} \lambda_{j+2}}{\left(\lambda_{j+2} - \lambda_{j+1}\right)\left(\lambda_{j+2} - \lambda_j\right)} \\ . \\ \prod_{k=2}^{n} \dfrac{\lambda_{j+k-1}}{\left(\lambda_{j+k} - \lambda_{j+1}\right)} \end{bmatrix}$$

$$e_{j+2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \dfrac{\lambda_{j+2}}{\lambda_{j+3} - \lambda_{j+2}} \\ \dfrac{\lambda_{j+2}\lambda_{j+3}}{\left(\lambda_{j+3} - \lambda_{j+2}\right)\left(\lambda_{j+4} - \lambda_{j+2}\right)} \\ . \\ . \\ \prod_{k=2}^{n} \dfrac{\lambda_{j+k-1}}{\left(\lambda_{j+k} - \lambda_{j+2}\right)} \end{bmatrix}$$

Let

$u = \left(e_j \quad e_{j+1} \quad e_{j+2} \quad . \quad . \quad . \quad e_{j+n}\right)$ then

$$u = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ \dfrac{\lambda_j}{\lambda_{j+1} - \lambda_j} & 1 & 0 & . & . & . & 0 \\ \dfrac{\lambda_j\lambda_{j+1}}{\left(\lambda_{j+1} - \lambda_j\right)\left(\lambda_{j+2} - \lambda_j\right)} & \dfrac{\lambda_{j+1}}{\lambda_{j+2} - \lambda_{j+1}} & 1 & . & . & . & 0 \\ \dfrac{\lambda_j\lambda_{j+1}\lambda_{j+2}}{\left(\lambda_{j+1} - \lambda_{j+k}\right)\left(\lambda_{j+2} - \lambda_j\right)\left(\lambda_{j+3} - \lambda_j\right)} & \dfrac{\lambda_{j+1}\lambda_{j+2}}{\left(\lambda_{j+2} - \lambda_{j+1}\right)\left(\lambda_{j+2} - \lambda_j\right)} & \dfrac{\lambda_{j+2}}{\lambda_{j+3} - \lambda_{j+2}} & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \prod_{k=1}^{n} \dfrac{\lambda_{j+k-1}}{\left(\lambda_{j+k} - \lambda_j\right)} & \prod_{k=2}^{n} \dfrac{\lambda_{j+k-1}}{\left(\lambda_{j+k} - \lambda_{j+1}\right)} & \prod_{k=2}^{n} \dfrac{\lambda_{j+k-1}}{\left(\lambda_{j+k} - \lambda_{j+2}\right)} & . & . & . & 1 \end{bmatrix}$$

(6.3)

$$P(s,t) = Ue^{\text{dag}\Lambda(t-s)}U^{-1}p_{j,j+k}(s,s)$$

$$p_{j,j+k}(s,s) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$

The inverse of matrix U is given below

$$U^{-1} = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ \dfrac{-\lambda_j}{\lambda_{j+1}-\lambda_j} & 1 & 0 & . & . & . & 0 \\ \dfrac{\lambda_j\lambda_{j+1}}{(\lambda_{j+2}-\lambda_j)(\lambda_{j+2}-\lambda_{j+1})} & \dfrac{-\lambda_{j+1}}{\lambda_{j+2}-\lambda_{j+1}} & 1 & . & . & . & 0 \\ \dfrac{-\lambda_j\lambda_{j+1}\lambda_{j+2}}{(\lambda_{j+3}-\lambda_j)(\lambda_{j+3}-\lambda_{j+1})(\lambda_{j+3}-\lambda_{j+2})} & \dfrac{\lambda_{j+1}\lambda_{j+2}}{(\lambda_{j+2}-\lambda_{j+1})(\lambda_{j+3}-\lambda_{j+1})} & \dfrac{-\lambda_{j+2}}{(\lambda_{j+3}-\lambda_{j+2})} & . & . & . & 0 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ (-1)^n \prod_{j=0}^{n-1} \dfrac{\lambda_{j+k}}{(\lambda_{j+n}-\lambda_{j+k})} & (-1)^{n-1} \prod_{k=0}^{n-2} \dfrac{\lambda_{j+k-1}}{(\lambda_{j+k+2}-\lambda_{j+1})} & (-1)^{n-2} \prod_{k=0}^{n-3} \dfrac{\lambda_{j+k+2}}{(\lambda_{j+k+3}-\lambda_{j+2})} & . & . & . & 1 \end{bmatrix}$$

(5.4)

$$e^{\text{diag}\Lambda(t-s)} = \begin{bmatrix} e^{-\lambda_{j+k}(t-s)} & 0 & 0 & 0 & . & . & 0 \\ 0 & e^{-\lambda_{j+k}(t-s)} & 0 & 0 & . & . & 0 \\ 0 & 0 & e^{-\lambda_{j+k}(t-s)} & 0 & . & . & 0 \\ 0 & 0 & 0 & e^{-\lambda_{j+k}(t-s)} & . & . & 0 \\ . & . & . & . & & & . \\ . & . & . & . & & & . \\ 0 & 0 & 0 & 0 & . & . & e^{-\lambda_{j+k}(t-s)} \end{bmatrix}$$

(5.5)

$$U^{-1}\, p_{jj+k}(s,s) = \begin{bmatrix} 1 \\ \dfrac{-\lambda_j}{\lambda_{j+1}-\lambda_j} \\ \dfrac{\lambda_j\ \lambda_{j+1}}{(\lambda_{j+2}-\lambda_j)(\lambda_{j+2}-\lambda_{j+1})} \\ \dfrac{-\lambda_j\ \lambda_{j+1}\ \lambda_{j+2}}{(\lambda_{j+3}-\lambda_j)(\lambda_{j+3}-\lambda_{j+1})(\lambda_{j+3}-\lambda_{j+2})} \\ . \\ . \\ (-1)^n \prod_{n=0}^{n-1} \dfrac{\lambda_{j+k}}{(\lambda_{j+n}-\lambda_{j+k})} \end{bmatrix}$$

Next

$$e^{\operatorname{diag}\Lambda(t-s)}U^{-1}\, p_{jj+k}(s,s) = \begin{bmatrix} e^{-\lambda_j(t-s)} \\ \dfrac{-\lambda_j}{\lambda_{j+1}-\lambda_j} e^{-\lambda_{j+1}(t-s)} \\ \dfrac{\lambda_j\ \lambda_{j+1}}{(\lambda_{j+2}-\lambda_j)(\lambda_{j+2}-\lambda_{j+1})} e^{-\lambda_{j+2}(t-s)} \\ \dfrac{-\lambda_j\ \lambda_{j+1}\ \lambda_{j+2}}{(\lambda_{j+3}-\lambda_j)(\lambda_{j+3}-\lambda_{j+1})(\lambda_{j+3}-\lambda_{j+2})} e^{-\lambda_{j+3}(t-s)} \\ . \\ . \\ (-1)^n \prod_{n=0}^{n-1} \dfrac{\lambda_{j+k}}{(\lambda_{j+n}-\lambda_{j+k})} e^{-\lambda_{j+n}(t-s)} \end{bmatrix}$$

$$P(s,t) = \begin{bmatrix} e^{-\lambda_j(t-s)} \\ \dfrac{\lambda_j e^{-\lambda_j(t-s)}}{\lambda_{j+1}-\lambda_j} - \dfrac{\lambda_i e^{-\lambda_j(t-s)}}{\lambda_{j+1}-\lambda_j} \\ \dfrac{\lambda_j \lambda_{j+1} e^{-\lambda_j(t-s)}}{(\lambda_{j+1}-\lambda_j)(\lambda_{j+2}-\lambda_j)} + \dfrac{\lambda_j e^{-\lambda_j(t-s)}}{\lambda_{j+1}-\lambda_j} \cdot \dfrac{\lambda_{j+1}}{\lambda_{j+2}-\lambda_{j+1}} + \dfrac{\lambda_j \lambda_{j+1} e^{-\lambda_j(t-s)}}{(\lambda_{j+2}-\lambda_j)(\lambda_{j+2}-\lambda_{j+1})} \\ \dfrac{\lambda_j \lambda_{j+1} \lambda_{j+2} e^{-\lambda_j(t-s)}}{(\lambda_{j+1}-\lambda_j)(\lambda_{j+2}-\lambda_j)(\lambda_{j+3}-\lambda_j)} - \dfrac{\lambda_j \lambda_{j+1} \lambda_{j+2} e^{-\lambda_j(t-s)}}{(\lambda_{j+1}-\lambda_j)(\lambda_{j+2}-\lambda_{j+1})(\lambda_{j+3}-\lambda_{j+1})} + \dfrac{\lambda_j \lambda_{j+1} \lambda_{j+2} e^{-\lambda_j(t-s)}}{(\lambda_{j+2}-\lambda_j)(\lambda_{j+2}-\lambda_{j+1})(\lambda_{j+3}-\lambda_{i+2})} - \dfrac{\lambda_j \lambda_{j+1} \lambda_{j+2} e^{-\lambda_j(t-s)}}{(\lambda_{j+3}-\lambda_j)(\lambda_{j+3}-\lambda_{j+1})(\lambda_{j+3}-\lambda_{j+2})} \\ . \\ . \\ \prod_{k=1}^{n} \dfrac{\lambda_{j+k-1} e^{-\lambda_j(t-s)}}{(\lambda_{j+k}-\lambda_j)} - \prod_{k=2}^{n} \dfrac{\lambda_{j+k-1}}{\lambda_{j+k}-\lambda_{j+1}} \cdot \dfrac{\lambda_j e^{-\lambda_{j+1}(t-s)}}{(\lambda_{j+1}-\lambda_j)} + \prod_{k=3}^{n} \dfrac{\lambda_{i+j-1}}{(\lambda_{i+j+1}-\lambda_{i+1})} \cdot \dfrac{\lambda_j}{(\lambda_{j+2}-\lambda_j)} \cdot \dfrac{\lambda_{j+1} e^{-\lambda_{j+2}(t-s)}}{(\lambda_{j+2}-\lambda_{j+1})} + \dots + (-1)^n \prod_{k=0}^{n-1} \dfrac{\lambda_{j+k} e^{-\lambda_{j+n}(t-s)}}{(\lambda_{j+n}-\lambda_{j+k})} \end{bmatrix}$$

Therefore,

$$p_{jj+n}(s,t) = \left\{ \prod_{k=1}^{n} \frac{\lambda_{j+k-1} e^{-\lambda_j (t-s)}}{\left(\lambda_{j+k} - \lambda_j\right)} + \frac{\prod_{k=1}^{n} \lambda_{j+k-1} e^{-\lambda_{j+1}(t-s)}}{\prod_{\substack{k=0 \\ k\neq 1}}^{n} \left(\lambda_{j+k} - \lambda_{j+1}\right)} + \frac{\prod_{k=1}^{n} \lambda_{j+k-1} e^{-\lambda_{j+2}(t-s)}}{\prod_{\substack{k=0 \\ k\neq 2}}^{n} \left(\lambda_{j+k} - \lambda_{j+2}\right)} + \ldots + \frac{\prod_{k=0}^{n-1} \lambda_{j+k} e^{-\lambda_{j+n}(t-s)}}{\prod_{k=0}^{n-1} \left(\lambda_{j+k} - \lambda_{j+n}\right)} \right\}$$

$$= \left\{ \frac{\prod_{k=0}^{n-1} \lambda_{j+k} e^{-\lambda_j (t-s)}}{\prod_{k=1}^{n} \left(\lambda_{j+k} - \lambda_j\right)} + \frac{\prod_{k=0}^{n-1} \lambda_{j+k} e^{-\lambda_{j+1}(t-s)}}{\prod_{\substack{k=0 \\ k\neq 1}}^{n} \left(\lambda_{j+k} - \lambda_{j+1}\right)} + \frac{\prod_{k=1}^{n} \lambda_{j+k} e^{-\lambda_{j+2}(t-s)}}{\prod_{\substack{k=0 \\ k\neq 2}}^{n} \left(\lambda_{j+k} - \lambda_{j+2}\right)} + \ldots + \frac{\prod_{k=0}^{n-1} \lambda_{j+k} e^{-\lambda_{j+n}(t-s)}}{\prod_{k=0}^{n-1} \left(\lambda_{j+k} - \lambda_{j+n}\right)} \right\}$$

$$= \prod_{k=0}^{n-1} \lambda_{j+k} \left\{ \frac{e^{-\lambda_j (t-s)}}{\prod_{k=1}^{n} \left(\lambda_{j+k} - \lambda_j\right)} + \frac{e^{-\lambda_j (t-s)}}{\prod_{\substack{k=0 \\ k\neq 1}}^{n} \left(\lambda_{j+k} - \lambda_{j+1}\right)} + \frac{e^{-\lambda_j (t-s)}}{\prod_{\substack{k=0 \\ k\neq 2}}^{n} \left(\lambda_{j+k} - \lambda_{j+2}\right)} + \ldots + \frac{e^{-\lambda_j (t-s)}}{\prod_{k=0}^{n-1} \left(\lambda_{j+k} - \lambda_{j+n}\right)} \right\}$$

$$= \prod_{k=0}^{n-1} \lambda_{j+k} \left\{ \sum_{v=0}^{n} \frac{e^{-\lambda_{j+v}(t-s)}}{\prod_{\substack{k=0 \\ k\neq v}}^{n} \left(\lambda_{j+k} - \lambda_{j+v}\right)} \right\}$$

5.3 Simple Birth Process

$$\lambda_{j+i} = (j+i)\lambda$$

$$p_{j\,j+n}(s,t) = \prod_{k=0}^{n-1}(j+k)\lambda \left\{ \sum_{v=0}^{n} \frac{e^{-(j+v)\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq v}}^{n} (j+k)\lambda - (j+v)\lambda} \right\}$$

$$= \lambda^n \prod_{k=0}^{n-1}(j+k) \left\{ \sum_{v=0}^{n} \frac{e^{-j\lambda(t-s)} e^{-v\lambda(t-s)}}{\lambda^n \prod_{\substack{k=0\\k\neq v}}^{n} (k-v)} \right\}$$

$$= e^{-j\lambda(t-s)} \prod_{k=0}^{n-1}(j+k)\lambda \left\{ \sum_{v=0}^{n} \frac{e^{-v\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq v}}^{n} (k-v)} \right\}$$

$$= e^{-j\lambda(t-s)} \frac{(j+n-1)!}{(j-1)!} \left\{ \frac{1}{\prod_{k=1}^{n} k} + \frac{e^{-\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq 1}}^{n} (k-1)} + \frac{e^{-2\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq 2}}^{n} (k-2)} + \frac{e^{-3\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq 3}}^{n} (k-3)} + \dots + \frac{e^{-n\lambda(t-s)}}{\prod_{k=0}^{n-1} (k-n)} \right\}$$

Therefore

$$p_{j\,j+n}(s,t) = e^{-j\lambda(t-s)} \frac{(j+n-1)!}{(j-1)!} \left\{ \frac{1}{n!} + (-1)^1 \frac{e^{-\lambda(t-s)}}{1.(n-1)!} + \frac{e^{-2\lambda(t-s)}}{2!(n-2)!} + (-1)^3 \frac{e^{-3\lambda(t-s)}}{3!(n-3)!} + \frac{e^{-4\lambda(t-s)}}{4!(n-4)!} + \dots + \right.$$

$$\left. (-1)^k \frac{e^{-k\lambda(t-s)}}{k!(n-k)!} + \dots + \frac{(-1)^n e^{-n\lambda(t-s)}}{n!} \right\}$$

Therefore,

$$p_{j\,j+n}(s,t) = e^{-j\lambda(t-s)} \frac{(j+n-1)!}{(j-1)!} \sum_{k=0}^{n} (-1)^k \frac{e^{-k\lambda(t-s)}}{k!(n-k)!}$$

Simplifying,

$$p_{j\,j+n}(s,t) = e^{-j\lambda(t-s)} \frac{(j+n-1)!}{(j-1)!} \frac{n!}{n!} \sum_{k=0}^{n} (-1)^k \frac{e^{-k\lambda(t-s)}}{k!(n-k)!}$$

$$= e^{-j\lambda(t-s)} \frac{(j+n-1)!}{n!\,(j-1)!} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \left(e^{-\lambda(t-s)}\right)^k$$

$$= e^{-j\lambda(t-s)} \binom{j+n-1}{n} \sum_{k=0}^{n} \binom{n}{k} \left(-e^{-\lambda(t-s)}\right)^k$$

$$= \binom{j+n-1}{n} e^{-j\lambda(t-s)} \left(1 - e^{-\lambda(t-s)}\right)^n$$

Therefore

$$p_{j\,j+n}(s,t) = \binom{j+n-1}{n} \left(e^{-\lambda(t-s)}\right)^j \left(1 - e^{-\lambda(t-s)}\right)^n \quad n = 0, 1, 2, 3, \ldots\ldots \qquad \textbf{(6.8)}$$

5.4 Simple Birth with Immigration

Now, recall that

$$p_{jj+n}(s,t) = \prod_{k=0}^{n-1} \lambda_{j+k} \left\{ \sum_{i=0}^{n} \frac{e^{-\lambda_{j+i}(t-s)}}{\prod_{\substack{k=0\\k\neq i}}^{n} \left(\lambda_{j+k} - \lambda_{j+i}\right)} \right\}$$

In this case $\lambda_j = j\lambda + v$

$$p_{jj+n}(s,t) = \prod_{k=0}^{n-1} \left((j+k)\lambda + v\right) \left\{ \sum_{i=0}^{n} \frac{e^{-((j+i)\lambda + v)(t-s)}}{\prod_{\substack{k=0\\k\neq i}}^{n} \left(\left((j+k)\lambda + v\right) - \left((j+i)\lambda + v\right)\right)} \right\}$$

$$= \prod_{k=0}^{n-1} \left((j+k)\lambda + v\right) \left\{ \sum_{i=0}^{n} \frac{e^{-((j+i)\lambda + v)(t-s)}}{\prod_{\substack{k=0\\k\neq i}}^{n} (k-i)\lambda} \right\}$$

$$= e^{-(j\lambda + v)(t-s)} \prod_{k=0}^{n-1} \left((j+k)\lambda + v\right) \left\{ \sum_{i=0}^{n} \frac{e^{-i\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq i}}^{n} (k-i)\lambda} \right\}$$

$$= e^{-(j\lambda + v)(t-s)} \prod_{k=0}^{n-1} \left(j + k + \frac{v}{\lambda} \right)\lambda \left\{ \sum_{i=1}^{n} \frac{e^{-i\lambda(t-s)}}{\lambda^n \prod_{\substack{k=0\\k\neq i}}^{n} (k-i)} \right\}$$

$$= e^{-(j\lambda + v)(t-s)} \lambda^n \prod_{k=0}^{n-1} \left(j + k + \frac{v}{\lambda} \right) \left\{ \sum_{i=1}^{n} \frac{e^{-i\lambda(t-s)}}{\lambda^n \prod_{\substack{k=0\\k\neq i}}^{n} (k-i)} \right\}$$

$$p_{jj+n}(s,t) = e^{-(j\lambda + v)(t-s)} \prod_{k=0}^{n-1}\left(j + \frac{v}{\lambda} + k\right)\left\{ \sum_{i=1}^{n} \frac{e^{-i\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq i}}^{n}(k-i)} \right\}$$

Expanding the expression inside the summation sign,

$$p_{jj+n}(s,t) = e^{-(j\lambda+v)(t-s)} \prod_{k=0}^{n-1}\left(j+\tfrac{v}{\lambda}+k\right)\left\{ \frac{e^{-0\lambda(t-s)}}{\prod_{k=1}^{n}(k-0)} + \frac{e^{-\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq 1}}^{n}(k-1)} + \frac{e^{-2\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq 2}}^{n}(k-2)} + \ldots + \frac{e^{-r\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq r}}^{n}(k-r)} + \ldots + \frac{e^{-n\lambda(t-s)}}{\prod_{k=0}^{n-1}(k-n)} \right\}$$

$$= e^{-(j\lambda+v)(t-s)} \prod_{k=0}^{n-1}\left(j+\tfrac{v}{\lambda}+k\right)\left\{ \frac{1}{n!} + \frac{e^{-\lambda(t-s)}}{(-1)(n-1)!} + \frac{e^{-2\lambda(t-s)}}{(-1)(-2)(n-2)} + \ldots + \frac{e^{-r\lambda(t-s)}}{(-1)^r r!(n-r)} + \ldots + \frac{e^{-n\lambda(t-s)}}{(-1)^n n!} \right\}$$

$$= e^{-(j\lambda+v)(t-s)} \prod_{k=0}^{n-1}\left(j+\tfrac{v}{\lambda}+k\right)\left\{ \frac{1}{n!} + \frac{(-1)e^{-\lambda(t-s)}}{(n-1)!} + \frac{(-1)^2 e^{-2\lambda(t-s)}}{2!(n-2)} + \ldots + \frac{(-1)^r e^{-r\lambda(t-s)}}{r!(n-r)} + \ldots + \frac{(-1)^n e^{-n\lambda(t-s)}}{n!} \right\}$$

Therefore,

$$p_{jj+n}(s,t) = e^{-(j\lambda+v)(t-s)} \prod_{k=0}^{n-1}\left(j+\tfrac{v}{\lambda}+k\right) \sum_{r=0}^{n} \frac{(-1)^r e^{-r\lambda(t-s)}}{r!(n-r)}$$

$$= e^{-(j\lambda+v)(t-s)} \frac{\left(j+\tfrac{v}{\lambda}+n-1\right)!}{\left(j+\tfrac{v}{\lambda}-1\right)!} \sum_{r=0}^{n} \frac{(-1)^r e^{-r\lambda(t-s)}}{r!(n-r)}$$

$$= e^{-(j\lambda+v)(t-s)} \frac{\left(j+\tfrac{v}{\lambda}+n-1\right)!}{\left(j+\tfrac{v}{\lambda}-1\right)!} \sum_{r=0}^{n} \frac{(-1)^r e^{-r\lambda(t-s)}}{r!(n-r)} \frac{n!}{n!}$$

$$= e^{-(j\lambda+v)(t-s)} \frac{\left(j+\tfrac{v}{\lambda}+n-1\right)!}{\left(j+\tfrac{v}{\lambda}-1\right)!\,n!} \sum_{r=0}^{n} \frac{(-1)^r n! e^{-r\lambda(t-s)}}{r!(n-r)}$$

$$= e^{-(j\lambda+v)(t-s)} \binom{j+\tfrac{v}{\lambda}+n-1}{n} \sum_{r=0}^{n} \frac{n!\left(-e^{-\lambda(t-s)}\right)^r}{r!(n-r)}$$

$$= e^{-(j\lambda+v)(t-s)} \binom{j+\tfrac{v}{\lambda}+n-1}{n} \sum_{r=0}^{n} \binom{n}{r}\left(-e^{-\lambda(t-s)}\right)^r$$

$$= \binom{j+\tfrac{v}{\lambda}+n-1}{n} e^{-(j\lambda+v)(t-s)}\left(1-e^{-\lambda(t-s)}\right)^n$$

Therefore

$$p_{jj+n}(s,t) = \binom{\left(j+\frac{v}{\lambda}\right)+n-1}{n} e^{-\left(j+\frac{v}{\lambda}\right)\lambda(t-s)}\left(1-e^{-\lambda(t-s)}\right)^{n}$$

Equivalently,

$$\mathbf{p_{jj+n}(s,t) = \binom{\left(j+\frac{v}{\lambda}\right)+n-1}{n}\left(e^{-\lambda(t-s)}\right)^{\left(j+\frac{v}{\lambda}\right)}\left(1-e^{-\lambda(t-s)}\right)^{n}} \quad n = 0,1,2,\ldots \qquad \textbf{(5.9)}$$

5.5 Polya Process

In this case, $\lambda_j = \lambda\left(\frac{1 + aj}{1 + \lambda at}\right)$

$$p_{j\,j+n}(s,t) = \prod_{k=0}^{n-1}\lambda_{j+k}\left\{\sum_{v=0}^{n}\frac{e^{-\lambda_{j+v}(t-s)}}{\prod_{\substack{k=0\\k\neq v}}^{n}\left(\lambda_{j+k} - \lambda_{j+v}\right)}\right\}$$

Substituting, we have

$$p_{j\,j+n}(s,t) = \prod_{k=0}^{n-1}\lambda\left(\frac{1 + a(j+k)}{1 + \lambda at}\right)\left\{\sum_{v=0}^{n}\frac{e^{-\left(\frac{1+a(j+v)}{1+\lambda at}\right)\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq v}}^{n}\left(\lambda\left[\frac{1 + a(j+k)}{1 + \lambda at}\right] - \lambda\left[\frac{1 + a(j+k)}{1 + \lambda at}\right]\right)}\right\}$$

$$= \prod_{k=0}^{n-1}\lambda\left(\frac{1 + a(j+k)}{1 + \lambda at}\right)\left\{\sum_{v=0}^{n}\frac{e^{-\left(\frac{1+a(j+v)}{1+\lambda at}\right)\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq v}}^{n}\lambda\left(\frac{1 + aj + ak - 1 - aj - av}{1 + \lambda at}\right)}\right\}$$

$$= \prod_{k=0}^{n-1}\lambda\left(\frac{1 + a(j+k)}{1 + \lambda at}\right)\left\{\sum_{v=0}^{n}\frac{e^{-\left(\frac{1+a(j+v)}{1+\lambda at}\right)\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq v}}^{n}\frac{\lambda a}{1 + \lambda at}(k - v)}\right\}$$

$$= \left(\frac{\lambda}{1 + \lambda at}\right)^{n}\prod_{k=0}^{n-1}\left(1 + a(j+k)\right)\left\{\sum_{v=0}^{n}\frac{e^{-\left(\frac{1+a(j+v)}{1+\lambda at}\right)\lambda(t-s)}}{\left(\frac{\lambda a}{1 + \lambda at}\right)^{n}\prod_{\substack{k=0\\k\neq v}}^{n}(k - v)}\right\}$$

$$= \prod_{k=0}^{n-1}\left(1 + a(j+k)\right)\left\{\sum_{v=0}^{n}\frac{e^{-\left(\frac{1+aj+av}{1+\lambda at}\right)\lambda(t-s)}}{a^{n}\prod_{\substack{k=0\\k\neq v}}^{n}(k - v)}\right\}$$

$$p_{jj+n}(s,t) = a^n \prod_{k=0}^{n-1}\left(\tfrac{1}{a} + j + k\right)\left\{\sum_{v=0}^{n} \frac{e^{-\left(\frac{1+aj+av}{1+\lambda at}\right)\lambda(t-s)}}{a^n \prod_{\substack{k=0\\k\neq v}}^{n}(k-v)}\right\}$$

$$= e^{-\left(\frac{1+aj}{1+\lambda at}\right)\lambda(t-s)} \prod_{k=0}^{n-1}\left(j + \tfrac{1}{a} + k\right)\left\{\sum_{v=0}^{n} \frac{e^{-\left(\frac{av}{1+\lambda at}\right)\lambda(t-s)}}{\prod_{\substack{k=0\\k\neq v}}^{n}(k-v)}\right\}$$

But $\prod_{\substack{k=0\\k\neq v}}^{n}(k-v) = (-1)^v\, v!\,(n-v)$ and $\prod_{k=0}^{n-1}\left(j+\tfrac{1}{a}+k\right) = \frac{\left(j+\frac{1}{a}+n-1\right)!}{\left(j+\frac{1}{a}-1\right)!}$

$$p_{jj+n}(s,t) = \frac{\left(j+\frac{1}{a}+n-1\right)!}{\left(j+\frac{1}{a}-1\right)!}\, e^{-\left(\frac{1+aj}{1+\lambda at}\right)\lambda(t-s)} \left\{\sum_{v=0}^{n} \frac{(-1)^v\, e^{-\left(\frac{av}{1+\lambda at}\right)\lambda(t-s)}}{v!\,(n-v)!}\right\}$$

$$= \frac{\left(j+\frac{1}{a}+n-1\right)!}{\left(j+\frac{1}{a}-1\right)!}\, e^{-\left(\frac{1+aj}{1+\lambda at}\right)\lambda(t-s)} \left\{\sum_{v=0}^{n} \frac{\left(-e^{-\left(\frac{a}{1+\lambda at}\right)\lambda(t-s)}\right)^v}{v!\,(n-v)!}\right\}$$

$$= \frac{\left(j+\frac{1}{a}+n-1\right)!}{\left(j+\frac{1}{a}-1\right)!}\, e^{-\left(\frac{1+aj}{1+\lambda at}\right)\lambda(t-s)} \left\{\sum_{v=0}^{n} \frac{\left(-e^{-\left(\frac{av}{1+\lambda at}\right)\lambda(t-s)}\right)^v}{v!\,(n-v)!}\times\frac{n!}{n!}\right\}$$

$$= \frac{\left(j+\frac{1}{a}+n-1\right)!}{\left(j+\frac{1}{a}-1\right)!\, n!}\, e^{-\left(\frac{1+aj}{1+\lambda at}\right)\lambda(t-s)} \left\{\sum_{v=0}^{n} \left(-e^{-\left(\frac{av}{1+\lambda at}\right)\lambda(t-s)}\right)^v \times\frac{n!}{v!\,(n-v)!}\right\}$$

$$= \binom{j+\frac{1}{a}+n-1}{n} e^{-\left(\frac{1+aj}{1+\lambda at}\right)\lambda(t-s)} \left\{\sum_{v=0}^{n} \binom{n}{v}\left(-e^{-\left(\frac{av}{1+\lambda at}\right)\lambda(t-s)}\right)^v\right\}$$

$$= \binom{j+\frac{1}{a}+n-1}{n} e^{-\left(\frac{1+aj}{1+\lambda at}\right)\lambda(t-s)} \left\{\sum_{v=0}^{n} \binom{n}{v}\left(-e^{-\left(\frac{av}{1+\lambda at}\right)\lambda(t-s)}\right)^v\right\}$$

$$p_{j\,j+n}(s,t) = \binom{j + \frac{1}{a} + n - 1}{n} e^{-\left(\frac{1+aj}{1+\lambda at}\right)\lambda(t-s)} \left(1 - e^{-\left(\frac{a}{1+\lambda at}\right)\lambda(t-s)}\right)^{n}$$

$$= \binom{j + \frac{1}{a} + n - 1}{n} e^{-\left(\frac{1+aj}{1+\lambda at}\right)\lambda(t-s)} \left(1 - e^{-\frac{a\lambda(t-s)}{1+\lambda at}}\right)^{n}$$

Therefore,

$$\mathbf{p_{j\,j+n}(s,t) = \binom{j + \frac{1}{a} + n - 1}{n} e^{-\left(\frac{1+aj}{1+\lambda at}\right)\lambda(t-s)} \left(1 - e^{-\frac{a\lambda(t-s)}{1+\lambda at}}\right)^{n}} \qquad \textbf{(5.10)}$$

This is a negative binomial distribution.

CHAPTER SIX

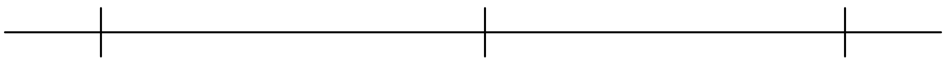
FIRST PASSAGE DISTRIBUTIONS FOR PURE BIRTH PROCESSES

6.1Derivation of $F_n(t)$ from $p_n(t)$

Let $p_n(t)$ = Prob $(X(t) = n)$ and $F_n(t)$ = Prob $(T_n \leq t)$ where

$X(t)$ = the population at time t and T_n = the first time the population size is n.

Consider the following diagram



Time	0	T_n	t
Size	X(0)	$X(T_n) = n$	X(t)

For a birth process

$$T_n < t \Rightarrow X(T_n) \leq X(t), \text{ ie } n \leq X(t)$$

$$T_n = t \Rightarrow X(T_n) = X(t), \text{ i.e } n = X(t)$$

$$\therefore T_n \leq t \Rightarrow X(t) \geq n$$

$$\therefore \text{Prob } (T_n \leq t) = \text{Prob } [X(t) \geq n]$$

i.e

$$F_n(t) = 1 - \text{Prob } [X(t) < n]$$

$$= 1 - \text{Prob } [X(t) \leq n - 1]$$

$$= 1 - \sum_{j=0}^{n-1} P_j(t)$$

Thus

$$F_n(t) = 1 - \sum_{j=0}^{n-1} p_j(t) \qquad \textbf{(6.1)}$$

Next,

$$f(t) = \frac{d}{dt}[F_n(t)]$$

6.2 Poisson Process

6.2.1 Determining $F_n(t)$ for Poisson Process

$$p_j(t) = \frac{e^{-\lambda t}(\lambda t)^j}{j!};\ j = 0, 1, 2, 3, ...$$

Therefore

$$F_n(t) = 1 - \sum_{j=0}^{n-1} e^{-\lambda t}\frac{(\lambda t)^j}{j!}$$

$$= 1 - e^{-\lambda t}\sum_{j=0}^{n-1}\frac{(\lambda t)^j}{j!}$$

But $\frac{d}{dt}\left[F_n(t)\right] = f_n(t)$. Therefore,

$$f_n(t) = -\sum_{j=0}^{n-1}\frac{d}{dt}\left[e^{-\lambda t}\frac{(\lambda t)^j}{j!}\right]$$

$$= -\left\{\sum_{j=0}^{n-1} -\lambda\, e^{-\lambda t}\frac{(\lambda t)^j}{j!} + \sum_{j=0}^{n-1}\lambda j\frac{(\lambda t)^{j-1}}{j!}e^{-\lambda t}\right\}$$

$$= \lambda\sum_{j=0}^{n-1} e^{-\lambda t}\frac{(\lambda t)^j}{j!} - \lambda\sum_{j=1}^{n-1}\frac{j(\lambda t)^{j-1}}{j!}e^{-\lambda t}$$

$$= \lambda\sum_{j=0}^{n-1} e^{-\lambda t}\frac{(\lambda t)^j}{j!} - \lambda\sum_{j=1}^{n-1}\frac{(\lambda t)^{j-1}}{(j-1)!}e^{-\lambda t}$$

$$= \lambda e^{-\lambda t}\left\{\sum_{j=0}^{n-1}\frac{(\lambda t)^j}{j!} - \sum_{j=1}^{n-1}\frac{(\lambda t)^{j-1}}{(j-1)!}\right\}$$

Expanding the expression inside the summation sign, we get

$$f_n(t) = \lambda e^{-\lambda t}\left\{1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + ... + \frac{(\lambda t)^{n-2}}{(n-2)!} + \frac{(\lambda t)^{n-1}}{(n-1)!} - 1 - \frac{\lambda t}{1!} - \frac{(\lambda t)^n}{2!} - .... - \frac{(\lambda t)^{n-2}}{(n-2)!}\right\}$$

Therefore

$$f_n(t) = \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} = \frac{\lambda^n}{\Gamma n}e^{-\lambda t}t^{n-1} \text{ for } t > 0;\ n = 1, 2, 3,..... \qquad \textbf{(6.2)}$$

Which is gamma (n, λ)

6.2.2 Mean and Variance of $f_n(t)$ for the Poisson Process

Mean

$$E(T_n) = \int_0^\infty t \, \frac{\lambda^n}{\lceil n} e^{-\lambda t} \, t^{n-1} \, dt$$

$$= \frac{\lambda^n}{\lceil n} \int_0^\infty e^{-\lambda t} \, t^n \, dt$$

Let $y = \lambda t \Rightarrow t = \frac{y}{\lambda}$ and $dt = \frac{dy}{\lambda}$.

Therefore,

$$E(T_n) = \frac{\lambda^n}{\lceil n} \int_0^\infty e^{-y} \left(\frac{y}{\lambda}\right)^{n-1} \frac{dy}{\lambda}$$

$$= \frac{\lambda^n}{\lceil n} \times \frac{1}{\lambda^{n+1}} \int_0^\infty e^{-y} \, y^n \, dy$$

$$= \frac{1}{\lambda \lceil n} \times \lceil n+1 = \frac{n \lceil n}{\lambda \lceil n}$$

$$= \frac{n}{\lambda} \qquad \textbf{(6.3)}$$

Variance

$$E(T_n^2) = \int_0^\infty t^2 \, \frac{\lambda^n}{\lceil n} e^{-\lambda t} \, t^{n-1} \, dt$$

$$= \frac{\lambda^n}{\lceil n} \int_0^\infty e^{-\lambda t} \, t^{n+1} \, dt$$

$$= \frac{\lambda^n}{\lceil n} \int_0^\infty e^{-y} \left(\frac{y}{\lambda}\right)^{n+1} \frac{dy}{\lambda}$$

$$= \frac{\lambda^n}{\lceil n} \times \frac{1}{\lambda^{n+2}} \int_0^\infty e^{-y} \, y^{n+1} \, dy$$

$$= \frac{1}{\lambda^2 \lceil n} \times \lceil n+2 = \frac{1}{\lambda^2 \lceil n} \times (n+1) \times n \times \lceil n = \frac{n(n+1)}{\lambda^2}$$

But $\text{Var } T = E(T_n^2) - \left[E(T)\right]^2$. Thus,

$$\text{Var } T = \frac{n(n+1)}{\lambda^2} - \frac{n^2}{\lambda^2}$$

Simplifying,

$$\text{Var } T = \frac{n}{\lambda^2} \tag{6.4}$$

6.3 Simple Birth Process

6.3.1 Determining $f_n(t)$ for the Simple Birth Process

Initial conditions: when t = 0, X(0) = 0

$P_j(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{j-1}$

Therefore,

$$F_n(t) = 1 - \sum_{j=0}^{n-1} P_j(t) = 1 - \sum_{j=0}^{n-1} e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{j-1}$$

$$= 1 - e^{-\lambda t} \sum_{j=0}^{n-1}\left(1 - e^{-\lambda t}\right)^{j-1}$$

$$= 1 - e^{-\lambda t}\left\{1 + \left(1 - e^{-\lambda t}\right) + \left(1 - e^{-\lambda t}\right)^2 + \left(1 - e^{-\lambda t}\right)^3 + ... + \left(1 - e^{-\lambda t}\right)^{n-2}\right\}$$

$$= 1 - e^{-\lambda t}\left[\frac{1 - \left(1 - e^{-\lambda t}\right)^{n-1}}{1 - \left(1 - e^{-\lambda t}\right)}\right]$$

$$= 1 - e^{-\lambda t}\left[\frac{1 - \left(1 - e^{-\lambda t}\right)^{n-1}}{e^{-\lambda t}}\right]$$

$$= 1 - \left\{1 - \left(1 - e^{-\lambda t}\right)^{n-1}\right\}$$

$$= \left(1 - e^{-\lambda t}\right)^{n-1}$$

Differentiating both sides with respect to t, we get

$$f_n(t) = (n - 1)\left(1 - e^{-\lambda t}\right)^{n-2}\left(\lambda e^{-\lambda t}\right)$$

$$= \lambda(n - 1)e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-2}; \quad t > 0,\ n = 2, 3, 4, ...$$

Definition of Exponentiated Exponential Distribution

Let $F(x) = \left[G(x)\right]^{\alpha}$ where $G(x) =$ the old or parent cdf and $F(x) =$ the new cdf. $\alpha > 0$.

Then

$f(x) = \dfrac{dF}{dx} = \alpha\left[G(x)\right]^{\alpha - 1} g(x)$ is called exponentiated pdf.

If $g(x) = \lambda e^{-\lambda x}$ (exponential pdf), then $G(x) = 1 - e^{-\lambda t}$

$$\therefore\ f(x) = \alpha\left[1 - e^{-\lambda t}\right]^{\alpha - 1} \lambda e^{-\lambda t}$$

$$= \alpha\lambda\left[1 - e^{-\lambda t}\right]^{\alpha - 1} e^{-\lambda t}$$

is called exponentiated exponential pdf or generalized exponential pdf with parameters λ and α.

Put $\alpha = n - 1,$ then we have

$$\therefore f(x) = \lambda(n - 1)e^{-\lambda t}\left[1 - e^{-\lambda t}\right]^{n - 2}$$

Thus

$$\mathbf{f_n(x) = \lambda(n - 1)e^{-\lambda t}\left[1 - e^{-\lambda t}\right]^{n - 2}\ ,\ t > 0,\ n > 1} \qquad \textbf{(6.5)}$$

is an exponentiated exponential distribution with parameters λ and $n - 1$.

Initial Condition: $\boldsymbol{X(0) = n_0}$

$$F_n(t) = 1 - \sum_{j=n_0}^{n-1} P_j(t)$$

$$P_j(t) = \binom{j-1}{n_0 - 1} e^{-n_0\lambda t}\left(1 - e^{-n_0\lambda t}\right) \quad j = n_0,\ n_0 + 1,\ n_0 + 2,\ ....$$

$$f_n(t) = \frac{d}{dt}F_n(t) = \frac{d}{dt}\left[1 - \sum_{j=n_0}^{n-1} P_j(t)\right] = -\frac{d}{dt}\sum_{j=n_0}^{n-1} P_j(t)$$

$$\sum_{j=n_0}^{\infty} P_j(t) = \sum_{j=n_0}^{\infty}\binom{j-1}{n_0 - 1} e^{-n_0\lambda t}\left(1 - e^{-n_0\lambda t}\right)^{j - n_0} = 1$$

$$\frac{d}{dt}\left[\sum_{j=n_0}^{\infty}\binom{j-1}{n_0 - 1} e^{-n_0\lambda t}\left(1 - e^{-n_0\lambda t}\right)^{j - n_0}\right] = 0$$

$$\frac{d}{dt}\left[\sum_{j=n_0}^{n-1}\binom{j-1}{n_0 - 1} e^{-n_0\lambda t}\left(1 - e^{-n_0\lambda t}\right)^{j - n_0} + \sum_{j=n}^{\infty}\binom{j-1}{n_0 - 1} e^{-n_0\lambda t}\left(1 - e^{-n_0\lambda t}\right)^{j - n_0}\right] = 0$$

$$\frac{d}{dt}\left[\sum_{j=n_0}^{n-1}\binom{j-1}{n_0 - 1} e^{-n_0\lambda t}\left(1 - e^{-n_0\lambda t}\right)^{j - n_0}\right] = -\frac{d}{dt}\left[\sum_{j=n}^{\infty}\binom{j-1}{n_0 - 1} e^{-n_0\lambda t}\left(1 - e^{-n_0\lambda t}\right)^{j - n_0}\right]$$

But $j = n_0 + k$ where $k = 0, 1, 2, \ldots..$

$$f_n(t) = \frac{d}{dt}\left[\sum_{n_0+k=n}^{\infty}\binom{n_0 + k - 1}{n_0 - 1} e^{-n_0\lambda t}\left(1 - e^{-n_0\lambda t}\right)^{k}\right] \qquad \textbf{(6.6)}$$

When $n_0 = 1$

$$f_n(t) = \frac{d}{dt}\left[\sum_{k=n-1}^{\infty}\binom{k}{0}e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{k}\right]$$

$$= \frac{d}{dt}\left[\sum_{k=n-1}^{\infty} e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{k}\right]$$

$$= \frac{d}{dt}\left[e^{-\lambda t}\sum_{k=n-1}^{\infty}\left(1-e^{-\lambda t}\right)^{k}\right]$$

$$= \frac{d}{dt}\left[e^{-\lambda t}\left(\left(1-e^{-\lambda t}\right)^{n-1} + \left(1-e^{-\lambda t}\right)^{n} + \left(1-e^{-\lambda t}\right)^{n+1} + \left(1-e^{-\lambda t}\right)^{n+2} + ...\right)\right]$$

$$= \frac{d}{dt}\left[e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1}\left(1 + \left(1-e^{-\lambda t}\right) + \left(1-e^{-\lambda t}\right)^{2} + \left(1-e^{-\lambda t}\right)^{3} + ...\right)\right]$$

$$= \frac{d}{dt}\left[e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1}\frac{1}{1-\left(1-e^{-\lambda t}\right)}\right]$$

$$= \frac{d}{dt}\left[e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1}\frac{1}{e^{-\lambda t}}\right]$$

$$= \frac{d}{dt}\left[\left(1-e^{-\lambda t}\right)^{n-1}\right]$$

$$= (n-1)\left(1-e^{-\lambda t}\right)^{n-2}(-\lambda)\left(-e^{-\lambda t}\right)$$

$$= (n-1)\lambda\, e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-2} \qquad \textbf{(6.7)}$$

6.3.2 Mean and Variance of $f_n(t)$ for the Simple Birth Process

Initial Condition: X(0) = 0

Mean

$$E(T_n) = \int_0^{\infty} t\, f_n(t)\, dt = \int_0^{\infty} t\,\lambda(n-1)e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-2} dt$$

$$= \lambda(n-1)\int_0^{\infty} t\, e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-2} dt$$

$$= \lambda(n-1)\int_0^{\infty} t\, e^{-\lambda t}\sum_{k=0}^{n-2}\binom{n-2}{k}\left(-e^{-\lambda t}\right)^{k} dt$$

$$= \lambda(n-1)\sum_{k=0}^{n-2}(-1)^{k}\binom{n-2}{k}\int_0^{\infty} t\, e^{-\lambda t-\lambda tk} dt$$

$$E(T_n) = \lambda(n-1)\sum_{k=0}^{n-2}\left\{(-1)^k \binom{n-2}{k}\int_0^\infty t\ e^{-(k+1)\lambda t} dt\right\}$$

Put $y = (k+1)\lambda t \Rightarrow t = \dfrac{y}{(k+1)\lambda}$ and $dt = \dfrac{dy}{(k+1)\lambda}$

$$E(T_n) = \lambda(n-1)\int_0^\infty t\ e^{-\lambda t}\sum_{k=0}^{n-2}\binom{n-2}{k}\left(-e^{-\lambda t}\right)^k dt$$

$$= \lambda(n-1)\sum_{k=0}^{n-2}(-1)^k \binom{n-2}{k}\int_0^\infty t\ e^{-\lambda t-\lambda t k} dt$$

$$= \lambda(n-1)\sum_{k=0}^{n-2}\left\{(-1)^k \binom{n-2}{k}\int_0^\infty t\ e^{-(k+1)\lambda t} dt\right\}$$

Put $y = (k+1)\lambda t \quad \Rightarrow \quad t = \dfrac{y}{(k+1)\lambda}$ and $dt = \dfrac{dy}{(k+1)\lambda}$

$$E(T_n) = \lambda(n-1)\sum_{k=0}^{n-2}\left\{(-1)^k \binom{n-2}{k}\int_0^\infty t\ e^{-y} \times \frac{y}{(k+1)\lambda}\ \frac{dy}{(k+1)\lambda}\right\}$$

$$= \lambda(n-1)\sum_{k=0}^{n-2}\left\{(-1)^k \binom{n-2}{k}\frac{1}{\left[(k+1)\lambda\right]^2}\int_0^\infty t\ e^{-y} \times y^{2-1}\ dy\right\}$$

$$= \lambda(n-1)\sum_{k=0}^{n-2}\left\{(-1)^k \binom{n-2}{k}\frac{1}{\left[(k+1)\lambda\right]^2} \times \overline{|2}\right\}$$

$$= \lambda(n-1)\sum_{k=0}^{n-2}\left\{(-1)^k \binom{n-2}{k}\frac{1}{(k+1)^2\lambda^2}\right\}$$

$$= (n-1)\sum_{k=0}^{n-2}\left\{(-1)^k \binom{n-2}{k}\frac{1}{(k+1)^2\lambda}\right\}$$

$$= \frac{1}{\lambda}\sum_{k=0}^{n-2}(-1)^k (n-1)\binom{n-2}{k}\frac{1}{(k+1)^2}$$

$$= \frac{1}{\lambda}\sum_{k=0}^{n-2}(-1)^k (n-1) \times \frac{(n-2)!}{k!(n-k-2)!} \times \frac{1}{(k+1)^2}$$

$$= \frac{1}{\lambda}\sum_{k=0}^{n-2}(-1)^k \times \frac{(n-1)!}{k!\left[(n-1)-(k-1)\right]!} \times \frac{1}{(k+1)^2}$$

$$E(T_n) = \frac{1}{\lambda}\sum_{k=0}^{n-2}(-1)^k \times \frac{(n-1)!}{(k+1)!\left[(n-1)-(k-1)\right]!} \times \frac{1}{(k+1)}$$

$$= \frac{1}{\lambda}\sum_{k=0}^{n-2}\frac{(-1)^k}{(k+1)} \times \binom{n-1}{k+1}$$

Put $k+1 = j$

$$E(T_n) = \frac{1}{\lambda}\sum_{j=1}^{n-1}\frac{(-1)^k}{j} \times \binom{n-1}{j}$$

$$= \frac{1}{\lambda}\sum_{j=1}^{n-1}\frac{(-1)^k(-1)^2}{j} \times \binom{n-1}{j}$$

$$= \frac{1}{\lambda}\sum_{j=1}^{n-1}\frac{(-1)^{k+2}}{j} \times \binom{n-1}{j}$$

$$= \frac{1}{\lambda}\sum_{j=1}^{n-1}\frac{(-1)^{j+1}}{j} \times \binom{n-1}{j}$$

$$= \frac{1}{\lambda}\sum_{j=1}^{n-1}(-1)^{j+1} \times \binom{n-1}{j} \times \frac{1}{j}$$

Next, we wish to proof that

$$\sum_{j=1}^{j-1}(-1)^{j+1} \times \binom{n-1}{j} \times \frac{1}{j} = \sum_{j=1}^{j-1}\frac{1}{j}$$

Proof

From the identity

$$1 + x + x^2 + ... + x^{n-1} = \frac{1-x^n}{1-x} \qquad \text{i.e}$$

$$\sum_{k=0}^{n-1}x^k = \frac{1-x^n}{1-x}$$

If we put $x = 1 - t$ we have

$$\sum_{k=0}^{n-1}(1-t)^k = \frac{1-(1-t)^n}{1-(1-t)}$$

$$= \frac{\left[1-(1-t)^n\right]}{t}$$

$$\sum_{k=0}^{n-1}(1-t)^k = \left[1-(1-t)^n\right] t^{-1} \qquad 0 \le t \le 1$$

Integrate this identity w.r.t *t*. Thus

$$\int_0^1 \sum_{k=0}^{n-1}(1-t)^k \, dt = \int_0^1 \left[1-(1-t)^n\right] t^{-1} \, dt$$

$$\text{L.H.S} = \sum_{k=0}^{n-1}\int_0^1 (1-t)^k \, dt$$

$$\text{Put } u = 1-t \qquad \Rightarrow \qquad du = -dt$$

$$\text{L.H.S} = \sum_{k=0}^{n-1} \int_1^0 u^k . -du = \sum_{k=0}^{n-1} \int_0^1 u^k . du$$

$$= \sum_{k=0}^{n-1} \left[\frac{u^{k+1}}{k+1}\right]_0^1 = \sum_{k=0}^{n-1} \frac{1}{k+1}$$

$$\text{R.H.S} = \int_0^1 \left[1-(1-t)^n\right] t^{-1} \, dt$$

$$= \int_0^1 \left\{1 - \sum_{k=0}^{n} \binom{n}{k}(-t)^k\right\} t^{-1} \, dt$$

$$\text{R.H.S} = \int_0^1 \left\{1 - \left[1 + \sum_{k=1}^{n} \binom{n}{k}(-t)^k\right]\right\} t^{-1} \, dt$$

$$= \int_0^1 \left\{-\sum_{k=1}^{n} \binom{n}{k}(-1)^k (t)^k\right\} t^{-1} \, dt$$

$$= \int_0^1 \left\{\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} t^{k-1}\right\} dt$$

$$= \left\{\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \int_0^1 t^{k-1} \, dt\right\}$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \left[\frac{t^k}{k}\right]_0^1$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \times \frac{1}{k}$$

$$\therefore \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \times \frac{1}{k} = \sum_{k=0}^{n-1} \frac{1}{k+1} = \sum_{k=1}^{n} \frac{1}{k}$$

Hence $E\left(T_n\right) = \frac{1}{l}\sum_{j=1}^{n-1}\frac{1}{j}$ **(6.8)**

End of proof

Variance

Next

$$E\left(T_n^2\right) = \int_0^\infty t^2\,\lambda\left(n-1\right)e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-2}dt$$

$$= \lambda\left(n-1\right)\int_0^\infty t^2 e^{-\lambda t}\sum_{k=0}^{n-2}\binom{n-2}{k}\left(-e^{-\lambda t}\right)^k dt$$

$$= \lambda\left(n-1\right)\int_0^\infty t^2 e^{-\lambda t}\sum_{k=0}^{n-2}\left(-1\right)^k\binom{n-2}{k}e^{-\lambda t k}dt$$

$$= \lambda\left(n-1\right)\int_0^\infty\sum_{k=0}^{n-2}e^{-\lambda t\left(k+1\right)}\;t^2\binom{n-2}{k}\left(-1\right)^k dt$$

$$= \frac{2\left(n-1\right)}{\lambda^2}\sum_{k=0}^{n-2}\left(-1\right)^k\binom{n-2}{k}\frac{1}{\left(k+1\right)^3}$$

$$= \frac{2}{\lambda^2}\sum_{k=0}^{n-2}\left(-1\right)^k\frac{\left(n-1\right)\left(n-2\right)!}{k!\left(n-2-k\right)!}\times\frac{1}{\left(k+1\right)^3}$$

$$= \frac{2}{\lambda^2}\sum_{k=0}^{n-2}\left(-1\right)^k\frac{\left(n-1\right)!}{\left(k+1\right)!\left[\left(n-1\right)-\left(k+1\right)\right]!}\times\frac{1}{\left(k+1\right)^2}$$

$$= \frac{2}{\lambda^2}\sum_{k=0}^{n-2}\left(-1\right)^k\binom{n-1}{k+1}\times\frac{1}{\left(k+1\right)^2}$$

$$= \lambda\left(n-1\right)\sum_{k=0}^{n-2}\left\{\left(-1\right)^k\binom{n-2}{k}\int_0^\infty e^{-\lambda t\left(k+1\right)}\,t^2\,dt\right\}$$

Put $y = \left(k+1\right)\lambda t \quad\Rightarrow\quad t = \frac{y}{\left(k+1\right)\lambda}$ and $dt = \frac{dy}{\left(k+1\right)\lambda}$

$$E\left(T_n^2\right) = \lambda\left(n-1\right)\sum_{k=0}^{n-2}\left\{\left(-1\right)^k\binom{n-2}{k}\int_0^\infty e^{-y}\left[\frac{y}{\left(k+1\right)\lambda}\right]^2\frac{dy}{\left(k+1\right)\lambda}\right\}$$

$$= \lambda\left(n-1\right)\sum_{k=0}^{n-2}\left(-1\right)^k\binom{n-2}{k}\frac{1}{\left[\left(k+1\right)\lambda\right]^3}\int_0^\infty e^{-y}y^2\,dy$$

$$= \lambda\left(n-1\right)\sum_{k=0}^{n-2}\left(-1\right)^k\binom{n-2}{k}\frac{1}{\left[\left(k+1\right)\lambda\right]^3}\int_0^\infty e^{-y}y^{3-1}\,dy$$

$$E\left(T_n^2\right) = \frac{2\lambda\left(n-1\right)}{\lambda^3}\sum_{k=0}^{n-2}\left(-1\right)^k\binom{n-2}{k}\frac{1}{\left(k+1\right)^3}$$

$$= \frac{2}{\lambda^2}\sum_{k=0}^{n-2}\left(-1\right)^{k+2}\binom{n-1}{k+1}\times\frac{1}{\left(k+1\right)^2}$$

Put $k+1=j$

$$E\left(T_n^2\right) = \frac{2}{\lambda^2}\sum_{j=0}^{n-1}\left(-1\right)^{j+1}\binom{n-1}{j}\times\frac{1}{j^2}$$

6.4 Pure Birth Process with Immigration

6.4.1 Determining $F_n(t)$ for Pure Birth Process with Immigration

Initial Condition: $X(0) = n_0$

Now, for simple birth with immigration,

$$P_j(t) = P_{n_0+k}(t) = \binom{k+m-1}{k}\left(e^{-\lambda t}\right)^m\left(1-e^{-\lambda t}\right)^k$$

$$F_n(t) = 1 - \sum_{j=0}^{n-1} P_j(t)$$

$$= 1 - \sum_{n_0+k=0}^{n-1}\binom{k+m-1}{k}\left(e^{-\lambda t}\right)^m\left(1-e^{-\lambda t}\right)^k$$

$$\frac{d}{dt}\sum_{j=0}^{\infty} P_j(t) = 0$$

$$\frac{d}{dt}\sum_{n_0+k=0}^{\infty}\binom{k+m-1}{k}\left(e^{-\lambda t}\right)^m\left(1-e^{-\lambda t}\right)^k = \frac{d}{dt}\left[\sum_{n_0+k=0}^{n-1}\binom{k+m-1}{k}\left(e^{-\lambda t}\right)^m\left(1-e^{-\lambda t}\right)^k + \sum_{n_0+k=n}^{\infty}\binom{k+m-1}{k}\left(e^{-\lambda t}\right)^m\left(1-e^{-\lambda t}\right)^k\right] = 0$$

$$\frac{d}{dt}\left[\sum_{n_0+k=0}^{n-1}\binom{k+m-1}{k}\left(e^{-\lambda t}\right)^m\left(1-e^{-\lambda t}\right)^k\right] = -\frac{d}{dt}\left[\sum_{n_0+k=n}^{\infty}\binom{k+m-1}{k}\left(e^{-\lambda t}\right)^m\left(1-e^{-\lambda t}\right)^k\right]\Bigg]$$

Therefore

$$f_n(t) = \frac{d}{dt}F_n(t) = -\frac{d}{dt}\sum_{n_0+k=0}^{n-1}\binom{k+m-1}{k}\left(e^{-\lambda t}\right)^m\left(1-e^{-\lambda t}\right)^k$$

$$= \frac{d}{dt}\sum_{n_0+k=n}^{\infty}\binom{k+m-1}{k}\left(e^{-\lambda t}\right)^m\left(1-e^{-\lambda t}\right)^k$$

$$f_n(t) = \frac{d}{dt}\left\{e^{-\lambda t m}\sum_{n_0+k-n=0}^{\infty}\binom{k+m-1}{k}\left(1-e^{-\lambda t}\right)^k\right\}$$

Let $n_0 + k - n = r \quad \Rightarrow \quad k = r + n - n_0$

$$f_n(t) = \frac{d}{dt}\left\{e^{-\lambda t m}\sum_{r=0}^{\infty}\binom{r+n-n_0+m-1}{r+n-n_0}\left(1-e^{-\lambda t}\right)^{r+n-n_0}\right\}$$

$$= \frac{d}{dt}\left\{\left(e^{-\lambda t}\right)^m\left(1-e^{-\lambda t}\right)^{n-n_0}\sum_{r=0}^{\infty}\binom{r+n-n_0+m-1}{r+n-n_0}\left(1-e^{-\lambda t}\right)^r\right\}$$

$$f_n(t) = \frac{d}{dt}\left\{\left(e^{-\lambda t}\right)^m \left(1 - e^{-\lambda t}\right)^{n-n_0} \sum_{r=0}^{\infty} \frac{(r + n - n_0 + m - 1)!}{(r + n - n_0)!(m-1)!} \left(1 - e^{-\lambda t}\right)^r\right\}$$

$$= \frac{d}{dt}\left\{\frac{1}{(m-1)!}\left(e^{-\lambda t}\right)^m \left(1 - e^{-\lambda t}\right)^{n-n_0} \sum_{r=0}^{\infty} \frac{(r + n - n_0 + m - 1)!}{(r + n - n_0)!} \left(1 - e^{-\lambda t}\right)^r\right\}$$

$$= \frac{d}{dt}\left\{\frac{1}{(m-1)!}\left(e^{-\lambda t}\right)^m \left(1 - e^{-\lambda t}\right)^{n-n_0} \sum_{r=0}^{\infty} \frac{(n - n_0 + m - 2)!}{(n - n_0 - 1)!} \prod_{i=0}^{r} \frac{(i + n - n_0 + m - 1)}{(i + n - n_0)} \left(1 - e^{-\lambda t}\right)^r\right\}$$

$$= \frac{d}{dt}\left\{\frac{(n - n_0 + m - 2)!}{(m-1)!(n - n_0 - 1)!}\left(e^{-\lambda t}\right)^m \left(1 - e^{-\lambda t}\right)^{n-n_0} \sum_{r=0}^{\infty} \prod_{i=0}^{r} \frac{(i + n - n_0 + m - 1)}{(i + n - n_0)} \left(1 - e^{-\lambda t}\right)^r\right\}$$

$$= \frac{d}{dt}\left\{\frac{(n - n_0 + m - 2)!}{(m-1)!(n - n_0 - 1)!}\left(e^{-\lambda t}\right)^m \left(1 - e^{-\lambda t}\right)^{n-n_0} \sum_{r=0}^{\infty} \prod_{i=0}^{\infty} \frac{(i + n - n_0 + m - 1)}{(i + n - n_0)} \left(1 - e^{-\lambda t}\right)^r\right\}$$

Now,

$$\prod_{i=0}^{\infty} \frac{i + n - n_0 + m - 1}{i + n - n_0} = \frac{n + m - n_0 - 1}{n - n_0} \cdot \frac{n + m - n_0}{n - n_0 + 1} \cdot \frac{n + m - n_0 + 1}{n - n_0 + 2} \cdots\cdots$$

$$\frac{n + 2m - n_0 - 3}{n - n_0 + m - 2} \cdot \frac{n + 2m - n_0 - 2}{n - n_0 + m - 1} \cdot \frac{n + m - n_0 - 1}{n - n_0 + m} \cdot \frac{n + 2m - n_0}{n - n_0 + m + 1} \cdots$$

$$= \frac{1}{(n - n_0)(n - n_0 + 1)(n - n_0 + 2)\cdots(n - n_0 + m - 2)}$$

$$= \frac{(n - n_0 - 1)!}{(n - n_0 + m - 2)!}$$

$$f_n(t) = \frac{d}{dt}\left\{\frac{(n + m - n_0 - 2)!\,(n - n_0 - 1)!}{(m-1)!(n - n_0 - 1)!(n - n_0 + m - 2)!}\left(e^{-\lambda t}\right)^m \left(1 - e^{-\lambda t}\right)^{n-n_0} \sum_{r=0}^{\infty} \left(1 - e^{-\lambda t}\right)^r\right\}$$

$$= \frac{d}{dt}\left\{\frac{1}{(m-1)!}\left(e^{-\lambda t}\right)^m \left(1 - e^{-\lambda t}\right)^{n-n_0} \sum_{r=0}^{\infty} \left(1 - e^{-\lambda t}\right)^r\right\}$$

$$= \frac{d}{dt}\left\{\frac{1}{(m-1)!}\left(e^{-\lambda t}\right)^m \left(1 - e^{-\lambda t}\right)^{n-n_0} \left(1 - \left(1 - e^{-\lambda t}\right)^{-1}\right)\right\}$$

$$= \frac{d}{dt}\left\{\frac{1}{(m-1)!}\left(e^{-\lambda t}\right)^m \left(1 - e^{-\lambda t}\right)^{n-n_0} \left(e^{-\lambda t}\right)^{-1}\right\}$$

$$= \frac{d}{dt}\left\{\frac{1}{(m-1)!}\left(e^{-\lambda t}\right)^m \left(1 - e^{-\lambda t}\right)^{n-n_0} e^{\lambda t}\right\}$$

$$f_n(t) = \frac{d}{dt}\left\{\frac{1}{(m-1)!} e^{-\lambda t m + \lambda t}\left(1 - e^{-\lambda t}\right)^{n-n_0}\right\}$$

$$= \frac{d}{dt}\left\{\frac{1}{(m-1)!} e^{-\lambda t(m-1)}\left(1 - e^{-\lambda t}\right)^{n-n_0}\right\}$$

$$= \frac{1}{(m-1)!}\left\{-(m-1)\lambda\, e^{-\lambda t(m-1)}\left(1 - e^{-\lambda t}\right)^{n-n_0} + e^{-\lambda t(m-1)}\left(n - n_0\right)\left(1 - e^{-\lambda t}\right)^{n-n_0-1}.\lambda e^{-\lambda t}\right\}$$

$$= \frac{\lambda}{(m-1)!} e^{-\lambda t(m-1)}\left(1 - e^{-\lambda t}\right)^{n-n_0}\left\{-(m-1)+\left(n - n_0\right)e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{-1}\right\}$$

$$= \frac{\lambda}{(m-1)!} e^{-\lambda t(m-1)}\left(1 - e^{-\lambda t}\right)^{n-n_0-1}\left\{\left(n - n_0\right)e^{-\lambda t} - (m-1)\left(1 - e^{-\lambda t}\right)\right\}$$

$$= \frac{\lambda}{(m-1)!} e^{-\lambda t(m-1)}\left(1 - e^{-\lambda t}\right)^{n-n_0-1}\left\{\left(n - n_0\right)e^{-\lambda t} + (m-1)e^{-\lambda t} - m + 1\right\}$$

$$= \frac{\lambda}{(m-1)!} e^{-(m-1)\lambda t}\left(1 - e^{-\lambda t}\right)^{n-n_0-1}\left\{\left(n - n_0 + m - 1\right)e^{-\lambda t} - m + 1\right\}$$

But $m = n_0 + \frac{v}{\lambda}$

$$f_n(t) = \frac{\lambda}{\left(n_0 + \frac{v}{\lambda} - 1\right)!} e^{-\left(n_0 + \frac{v}{\lambda} - 1\right)\lambda t}\left(1 - e^{-\lambda t}\right)^{n-n_0-1}\left\{\left(n - n_0 + n_0 + \frac{v}{\lambda} - 1\right)e^{-\lambda t} - n_0 - \frac{v}{\lambda} + 1\right\}$$

$$= \frac{\lambda}{\left(n_0 + \frac{v}{\lambda} - 1\right)!} e^{-\left(n_0 + \frac{v}{\lambda} - 1\right)\lambda t}\left(1 - e^{-\lambda t}\right)^{n-n_0}\left\{\left(n + \frac{v}{\lambda} - 1\right)e^{-\lambda t} - \left(\frac{n_0\lambda + v - \lambda}{\lambda}\right)\right\}*$$

$$= \frac{\lambda}{\left(\frac{n_0\lambda + v - \lambda}{\lambda}\right)!\,\lambda} e^{-\left(n_0\lambda + v - \lambda\right)t}\left(1 - e^{-\lambda t}\right)^{n-n_0}\left\{\left(n\lambda + v - \lambda\right)e^{-\lambda t} - \left(n_0\lambda + v - \lambda\right)\right\}$$

$$= \frac{1}{\left(\frac{n_0\lambda + v - \lambda}{\lambda}\right)!} e^{-\left(n_0\lambda + v - \lambda\right)t}\left(1 - e^{-\lambda t}\right)^{n-n_0}\left\{\left(n\lambda + v - \lambda\right)e^{-\lambda t} - \left(n_0\lambda + v - \lambda\right)\right\}$$

Thus,

$$f_n(t) = \frac{1}{\left(\frac{n_0\lambda + v - \lambda}{\lambda}\right)!} e^{-\left(n_0\lambda + v - \lambda\right)t}\left(1 - e^{-\lambda t}\right)^{n-n_0}\left\{\left(n\lambda + v - \lambda\right)e^{-\lambda t} - \left(n_0\lambda + v - \lambda\right)\right\} \qquad \textbf{(6.9)}$$

When $n_0 = 1$,

$$f_n(t) = \frac{1}{\left(\frac{v}{\lambda}\right)!} e^{-vt} \left(1 - e^{-\lambda t}\right)^{n-1} \left\{ (n\lambda + v - \lambda) e^{-\lambda t} - v \right\}$$

$$= \frac{1}{\left(\frac{v}{\lambda}\right)!} \left(1 - e^{-\lambda t}\right)^{n-1} \left\{ (n\lambda + v - \lambda) e^{-(v+\lambda)t} - v e^{-vt} \right\} \qquad \textbf{(6.10)}$$

When $v = 0$,

$$f_n(t) = \frac{1}{(n_0 - 1)!} e^{-(n_0 - 1)\lambda t} \left(1 - e^{-\lambda t}\right)^{n - n_0} \left\{ \lambda(n-1) e^{-\lambda t} - \lambda(n_0 - 1) \right\}$$

$$= \frac{\lambda}{(n_0 - 1)!} e^{-(n_0 - 1)\lambda t} \left(1 - e^{-\lambda t}\right)^{n - n_0} \left\{ (n-1) e^{-\lambda t} - (n_0 - 1) \right\}$$

$$= \frac{\lambda}{(n_0 - 1)!} \left(1 - e^{-\lambda t}\right)^{n - n_0} \left\{ (n-1) e^{-n_0 \lambda t} - (n_0 - 1) e^{-(n_0 - 1)\lambda t} \right\} \qquad \textbf{(6.11)}$$

5.4.2 Mean and Variance

6.5 Polya Process

6.5.1 Determining $f_n(t)$ for the Polya Process

Initial Condition: X(0) = 0

$F_n(t) = 1 - \sum_{j=0}^{n-1} p_j(t)$ where $p_j(t) = \binom{j + \frac{1}{a} - 1}{j} p^{\frac{1}{a}} q^j$, $\quad j = 0, 1, 2, 3, \ldots$. $p_j(t)$ can also be

written as $p_j(t) = \binom{j + \frac{1}{a} - 1}{\frac{1}{a} - 1} \left(\frac{1}{1 + \lambda a t}\right)^{\frac{1}{a}} \left(\frac{\lambda a t}{1 + \lambda a t}\right)^j$; $\quad j = 0, 1, 2, 3, \ldots$

$$F_n(t) = 1 - \sum_{j=0}^{n-1} \binom{j + \frac{1}{a} - 1}{\frac{1}{a} - 1} \left(\frac{1}{1 + \lambda a t}\right)^{\frac{1}{a}} \left(\frac{\lambda a t}{1 + \lambda a t}\right)^j$$

$$= 1 - \sum_{j=0}^{n-1} \binom{j + \frac{1}{a} - 1}{\frac{1}{a} - 1} (\lambda a t)^j (1 + \lambda a t)^{-\left(j + \frac{1}{a}\right)}; \; t > 0 \text{ and } n = 1, 2, 3, \ldots$$

Differentiating both sides with respect to t, we have

$$f_n(t) = \frac{d}{dt} F_n(t) = -\sum_{j=0}^{n-1} \binom{j + \frac{1}{a} - 1}{\frac{1}{a} - 1} \frac{d}{dt} (\lambda a t)^j (1 + \lambda a t)^{-\left(j + \frac{1}{a}\right)}$$

$$= -\sum_{j=0}^{n-1} \binom{j + \frac{1}{a} - 1}{\frac{1}{a} - 1} \left\{ ja\lambda (\lambda a t)^{j-1} (1 + \lambda a t)^{-\left(j + \frac{1}{a}\right)} - \left(j + \tfrac{1}{a}\right) a \lambda (\lambda a t)^j (1 + \lambda a t)^{-\left(j + 1 + \frac{1}{a}\right)} \right\}$$

$$f_n(t) = \lambda \left\{ \sum_{j=0}^{n-1} \binom{j + \frac{1}{a} - 1}{\frac{1}{a} - 1} \left(j + \tfrac{1}{a}\right) a (\lambda a t)^j (1 + \lambda a t)^{-\left(j + 1 + \frac{1}{a}\right)} - \sum_{j=1}^{n-1} \binom{j + \frac{1}{a} - 1}{\frac{1}{a} - 1} ja (\lambda a t)^{j-1} (1 + \lambda a t)^{-\left(j + \frac{1}{a}\right)} \right\}$$

But

$$\binom{j + \frac{1}{a} - 1}{\frac{1}{a} - 1} \left(j + \tfrac{1}{a}\right) a = \frac{\left(j + \frac{1}{a} - 1\right)!}{\left(\frac{1}{a} - 1\right)!\, j!} \left(j + \tfrac{1}{a}\right) a = \frac{\left(j + \frac{1}{a} - 1\right)! \left(j + \frac{1}{a}\right)}{\frac{1}{a}\left(\frac{1}{a} - 1\right)!\, j!} = \frac{\left(j + \frac{1}{a}\right)!}{\frac{1}{a}!\, j!} = \binom{j + \frac{1}{a}}{\frac{1}{a}}$$

and
$$\binom{j + \frac{1}{a} - 1}{\frac{1}{a} - 1} a j = \frac{\left(j + \frac{1}{a} - 1\right)!\, a j}{\left(\frac{1}{a} - 1\right)!\, j!} = \frac{\left(j + \frac{1}{a} - 1\right)!}{\frac{1}{a}\left(\frac{1}{a} - 1\right)!\, (j - 1)!} = \frac{\left(j + \frac{1}{a} - 1\right)!}{\frac{1}{a}!\, (j - 1)!} = \binom{j + \frac{1}{a} - 1}{\frac{1}{a}}$$

$$f_n(t) = \lambda \left\{ \sum_{j=0}^{n-1} \binom{j + \frac{1}{a}}{\frac{1}{a}} \frac{(\lambda a t)^j}{(1 + \lambda a t)^{j + 1 + \frac{1}{a}}} - \sum_{j=1}^{n-1} \binom{j + \frac{1}{a} - 1}{\frac{1}{a}} \frac{(\lambda a t)^{j-1}}{(1 + \lambda a t)^{j + \frac{1}{a}}} \right\}$$

If we take $u = j - 1$, the above equation becomes

$$f_n(t) = \lambda \left\{ \sum_{j=0}^{n-1} \binom{j + \frac{1}{a}}{\frac{1}{a}} \frac{(\lambda a t)^j}{(1 + \lambda a t)^{j + 1 + \frac{1}{a}}} - \sum_{u=0}^{n-2} \binom{u - 1}{\frac{1}{a}} \frac{(\lambda a t)^u}{(1 + \lambda a t)^{u + 1 + \frac{1}{a}}} \right\}$$

$$f_n(t) = \lambda \binom{n-1+\frac{1}{a}}{\frac{1}{a}} \frac{(\lambda a t)^{n-1}}{(1+\lambda a t)^{n+\frac{1}{a}}}$$

$$= \frac{\lambda(n-1+\frac{1}{a})!}{\frac{1}{a}!(n-1)!} \frac{(\lambda a t)^{n-1}}{(1+\lambda a t)^{n+\frac{1}{a}}}$$

$$= \frac{\lambda \Gamma(n+\frac{1}{a})}{\Gamma(\frac{1}{a}+1)\Gamma n} \times \frac{(\lambda a t)^{n-1}}{(1+\lambda a t)^{n+\frac{1}{a}}}$$

$$= a \times \frac{\lambda}{\Gamma \frac{1}{a}} \times \frac{\Gamma(n+\frac{1}{a})}{\Gamma n} \times \frac{(\lambda a t)^{n-1}}{(1+\lambda a t)^{n+\frac{1}{a}}}, \ t>0, \ n>0$$

$$= \frac{a\lambda \Gamma(n+\frac{1}{a})}{\Gamma\frac{1}{a}\Gamma n} \times \frac{(\lambda a)^{n-1} t^{n-1}}{(1+\lambda a t)^{n+\frac{1}{a}}}, \ t>0, \ n>0$$

$$= \frac{\Gamma(n+\frac{1}{a})}{\Gamma\frac{1}{a}\Gamma n} \times \frac{(\lambda a)^{n} t^{n-1}}{(\lambda a)^{n+\frac{1}{a}}\left(t+\frac{1}{\lambda a}\right)^{n+\frac{1}{a}}}$$

$$f_n(t) = \frac{\Gamma(n+\frac{1}{a})}{\Gamma\frac{1}{a}\Gamma n} \times \frac{\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}} t^{n-1}}{\left(t+\frac{1}{\lambda a}\right)^{n+\frac{1}{a}}}, \qquad t>0, \ n>0, \ \frac{1}{a}>0, \ \frac{1}{\lambda a}>0 \qquad \textbf{(6.12)}$$

Definition of generalized Pareto distribution

Let $X \backslash \theta \sim \text{Gamma}(\alpha, \theta)$ and $\theta \sim G(\lambda, \beta)$.

Thus a gamma mixture of a gamma distribution is given by

$$f(x) = \int_0^\infty \frac{\theta^\alpha}{\Gamma\alpha} e^{-\theta x} x^{\alpha-1} g(\theta)\, d\theta$$

$$= \int_0^\infty \frac{\theta^\alpha}{\Gamma\alpha} e^{-\theta x} x^{\alpha-1} \frac{\beta^\lambda}{\Gamma\lambda} e^{-\beta\theta}\theta^{\lambda-1}\, d\theta$$

$$= \frac{x^{\alpha-1}\beta^\lambda}{\Gamma\alpha\,\Gamma\lambda} \int_0^\infty e^{-(x+\beta)\theta}\theta^{\alpha+\lambda-1}\, d\theta$$

Put $y = (x+\beta)\theta \quad \Rightarrow \quad \theta = \frac{y}{x+\beta}$ and $d\theta = \frac{dy}{x+\beta}$. Therefore,

$$f(x) = \int_0^\infty \left(\frac{x^{\alpha-1}\beta^{\lambda}}{\Gamma\alpha\ \Gamma\lambda} \right) e^{-y} \left(\frac{y}{x+\beta} \right)^{\alpha+\lambda-1} \frac{dy}{x+\beta}$$

$$= \frac{x^{\alpha-1}\beta^{\lambda}}{\Gamma\alpha\ \Gamma\lambda} \times \frac{1}{(x+\beta)^{\alpha+\lambda}} \int_0^\infty e^{-y} y^{\alpha+\lambda-1}\, dy$$

$$= \frac{x^{\alpha-1}\beta^{\lambda}}{\Gamma\alpha\ \Gamma\lambda} \times \frac{1}{(x+\beta)^{\alpha+\lambda}} \times \Gamma(\alpha+\lambda)$$

Therefore,

$$f(x) = \frac{\Gamma(\alpha+\lambda)}{\Gamma\alpha\ \Gamma\lambda} \times \beta^{\lambda} \times \frac{x^{\alpha-1}}{(x+\beta)^{\alpha+\lambda}}; \qquad x > 0$$

Back to equation **(6.12)**, replace x by t, α by n and $\lambda = \frac{1}{a}$. Therefore

$$f(x) = \frac{\Gamma\left(n+\frac{1}{a}\right)}{\Gamma n\ \Gamma\frac{1}{a}} \times \times \frac{\beta^{\frac{1}{a}}\ t^{n-1}}{(t+\beta)^{n+\frac{1}{a}}}; \quad \text{Further, let } \beta = \frac{1}{\lambda a}. \text{ Therefore}$$

$$f(x) = \frac{\Gamma\left(n+\frac{1}{a}\right)}{\Gamma n\ \Gamma\frac{1}{a}} \times \frac{\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}} t^{n-1}}{\left(t+\frac{1}{\lambda a}\right)^{n+\frac{1}{a}}}; \qquad t > 0,\ \frac{1}{a} > 0,\ n > 0 \text{ and } \frac{1}{\lambda a} > 0. \qquad \mathbf{(6.13)}$$

This is generalized Pareto (a Gamma mixture of Gamma) distribution.

Note that since $\int_0^\infty f(t)\,dt = 1$, then

$$f(x) = \int_0^\infty \frac{\Gamma\left(n+\frac{1}{a}\right)}{\Gamma n\ \Gamma\frac{1}{a}} \times \left(\frac{1}{\lambda a}\right)^{\frac{1}{a}} \times \frac{t^{n-1}}{\left(t+\frac{1}{\lambda a}\right)^{n+\frac{1}{a}}} = 1$$

$$\therefore \int_0^\infty \frac{t^{n-1}}{\left(t+\frac{1}{\lambda a}\right)^{n+\frac{1}{a}}} = \frac{\Gamma n}{\Gamma\left(n+\frac{1}{a}\right)} \times \frac{\Gamma\frac{1}{a}}{\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}}}$$

6.5.2 Mean and Variance of $f_n(t)$ for the Polya Process

Initial Condition: $X(0) = 0$

Mean

$$E(T_n) = \int_0^{\infty} t\ f_n(t)\ dt$$

$$= \int_0^{\infty} t \times \frac{\Gamma\left(n + \frac{1}{a}\right)}{\Gamma(n)\ \Gamma\left(\frac{1}{a}\right)} \times \frac{\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}}\ t^{n-1}}{\left(t + \frac{1}{\lambda a}\right)^{n + \frac{1}{a}}}\ dt$$

$$= \frac{\Gamma\left(n + \frac{1}{a}\right)\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}}}{\Gamma(n)\ \Gamma\left(\frac{1}{a}\right)} \int_0^{\infty} \frac{t^n}{\left(t + \frac{1}{\lambda a}\right)^{n + \frac{1}{a}}}\ dt$$

$$= \frac{\Gamma\left(n + \frac{1}{a}\right)\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}}}{\Gamma(n)\ \Gamma\left(\frac{1}{a}\right)} \int_0^{\infty} \frac{t^{(n+1)-1}}{\left(t + \frac{1}{\lambda a}\right)^{(n+1) + \left(\frac{1}{a} - 1\right)}}\ dt$$

$$= \frac{\Gamma\left(n + \frac{1}{a}\right)\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}}}{\Gamma(n)\ \Gamma\left(\frac{1}{a}\right)} \times \frac{\Gamma(n+1)\ \Gamma\left(\frac{1}{a} - 1\right)}{\Gamma\left(n + \frac{1}{a}\right)} \times \frac{1}{(\lambda a)^{\frac{1}{a} - 1}}$$

$$= \frac{\frac{1}{\lambda a}}{1} \times \frac{n}{\frac{1}{a} - 1} = \frac{n}{\lambda a\left(\frac{1}{a} - 1\right)} = \frac{n}{\lambda - \lambda a}$$

$$E(T_n) = \frac{n}{\lambda(1 - a)} \qquad \Rightarrow \qquad 0 < a < 1 \tag{6.14}$$

Variance

$$E(t^2) = \int_0^{\infty} t^2 \times \frac{\Gamma\left(n + \frac{1}{a}\right)}{\Gamma(n)\ \Gamma\left(\frac{1}{a}\right)} \times \frac{\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}}\ t^{n-1}}{\left(t + \frac{1}{\lambda a}\right)^{n + \frac{1}{a}}}\ dt$$

$$= \frac{\Gamma\left(n + \frac{1}{a}\right)\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}}}{\Gamma(n)\ \Gamma\left(\frac{1}{a}\right)} \int_0^{\infty} \frac{t^{n+1}}{\left(t + \frac{1}{\lambda a}\right)^{n + \frac{1}{a}}}\ dt$$

$$= \frac{\Gamma\left(n + \frac{1}{a}\right)\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}}}{\Gamma(n)\ \Gamma\left(\frac{1}{a}\right)} \int_0^{\infty} \frac{t^{(n+2)-1}}{\left(t + \frac{1}{\lambda a}\right)^{(n+2) + \left(\frac{1}{a} - 2\right)}}\ dt$$

$$E\left(t^2\right) = \frac{\Gamma\left(n + \frac{1}{a}\right) \times \left(\frac{1}{\lambda a}\right)^{\frac{1}{a}}}{\Gamma(n) \times \Gamma\left(\frac{1}{a}\right)} \times \frac{\Gamma(n+2) \times \Gamma\left(\frac{1}{a} - 2\right)}{\Gamma\left(n + \frac{1}{a}\right)} \times \frac{1}{(\lambda a)^{\frac{1}{a} - 2}}$$

$$= \frac{\left(\frac{1}{\lambda a}\right)^2}{1} \times \frac{(n+1)\times n}{\left(\frac{1}{a} - 1\right)\left(\frac{1}{a} - 2\right)} = \frac{(n+1)\times n}{\left[\lambda a\left(\frac{1}{a} - 1\right)\right]\left[\lambda a\left(\frac{1}{a} - 2\right)\right]}$$

$$= \frac{n(n+1)}{(\lambda - \lambda a)(\lambda - 2\lambda a)} = \frac{n(n+1)}{\lambda^2 (1-a)(1-2a)} \qquad \text{for } \; 0 < a < 1 \; \text{ and } \; 0 < 2a < 1.$$

or equivalently for $0 < a < 1$ and $0 < a < \frac{1}{2}$.

therefore $0 < a < \frac{1}{2}$.

$$\text{var}\left(T_n\right) = E\left(T_n^2\right) - \left[E\left(T_n\right)\right]^2$$

$$= \frac{n(n+1)}{\lambda^2(1-a)(1-2a)} - \frac{n^2}{\lambda^2(1-a)^2}$$

$$= \frac{n(n+1)(1-a) - n^2(1-2a)}{\lambda^2(1-a)^2(1-2a)}$$

$$= \frac{\left(n^2 + n\right)(1-a) - n^2 + 2an^2}{\lambda^2(1-a)^2(1-2a)}$$

$$= \frac{n^2 - n^2 a + n - na - n^2 + 2an^2}{\lambda^2(1-a)^2(1-2a)}$$

$$\text{var}\left(T_n\right) = \frac{an^2 + n - na}{\lambda^2(1-a)^2(1-2a)} = \frac{n(an + 1 - a)}{\lambda^2(1-a)^2(1-2a)}$$

$$\text{var}\left(T_n\right) = \frac{an\left(n + \frac{1}{a} - 1\right)}{\lambda^2(1-a)^2(1-2a)}, \qquad n > 0 \qquad \text{and} \qquad 0 < a < \frac{1}{2} \qquad \textbf{(6.15)}$$

CHAPTER SEVEN

SUMMARY AND CONCLUSION

7.1 Summary

In this thesis, we derived the Kolmogorov forward and backward differential equations for continuous time non homogenous pure birth process and then solved the forward Kolmogorov differential equation using three alternative approaches. We were able to generate distributions arising from increments from pure birth processes by the applying the three alternative approaches.

7.1.1 General Kolmogorov Forward and Backward Equations

The general forward Kolmogorov differential equation is;

$$\frac{\partial}{\partial t} p_{k,k+n}(s,t) + \lambda_{k+n}(t) p_{k,k+n}(s,t) = \lambda_{k+n-1}(t)\, p_{k,k+n-1}(s,t) \qquad \textbf{(2.11)}$$

With initial conditions: $p_{kk}(s,s) = 1$ and $p_{k,k+n}(s,s) = 0$ for $n > 0$.

The general Backward Kolmogorov differential equation is;

$$\frac{\partial}{\partial s} p_{ij}(s,t) = -\lambda_i(s) p_{ij}(s,t) + \lambda_i(s) p_{i+1,j}(s,t) \qquad \textbf{(2.14)}$$

7.1.2 Poisson Process

From the Poisson process, we found the distribution of the increments to be a Poisson distribution with parameter $\lambda(t-s)$

$$p_{k,k+n}(s,t) = e^{-\lambda(t-s)} \frac{\left[\lambda(t-s)\right]^n}{n!};\ n = 0, 1, 2, \ldots \qquad \textbf{(3.6, 4.5)}$$

This is independent the initial state and depends on the length of the time interval, thus for a Poisson processes the increments are independent and stationary.

The first passage distribution is

$$f_n(t) = \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} = \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} \text{ for } t > 0;\ n = 1, 2, 3, \ldots.. \qquad \textbf{(6.2)}$$

Which is gamma (n, λ)

The mean of the first passage distribution was found to be $E(T_n) = \frac{n}{\lambda}$ **(6.3)**

The Variance of the first passage distribution was found to be $\mathrm{Var}\ T = \frac{n}{\lambda^2}$ **(6.4)**

7.1.3 Simple Birth Process

From the Simple birth process, we found the distribution of the increments to be Negative binomial distribution with $p = e^{-\lambda(t-s)}$ and $q = 1 - e^{-\lambda(t-s)}$

$$p_{k,k+n}(s,t) = \binom{k+n-1}{n} e^{-\lambda k(t-s)} \left[1 - e^{-\lambda(t-s)}\right]^n . \quad \textbf{(3.15, 4.9, 6.8)}$$

It depends on the length of the time interval $t - s$ and on k and is thus stationary and not independent.

The first passage distribution when $n_0 = 1$ is

$$f_n(t) = \lambda(n-1)e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-2}; \quad t > 0,\ n = 2, 3, ...$$

which is an exponentiated exponential distribution with parameters λ and $n - 1$

The mean of the first passage distribution was found to be $E(T_n) = \frac{1}{l}\sum_{j=1}^{n-1}\frac{1}{j}$ **(7.7)**

7.1.4 Simple Birth with Immigration

From the Simple birth process with immigration, we found the distribution of the increments to be Negative binomial distribution with $p = e^{-\lambda(t-s)}$ and $q = 1 - e^{-\lambda(t-s)}$

$$p_{k,k+n}(s,t) = \binom{k+\frac{v}{\lambda}+n-1}{n} e^{-\left(k+\frac{v}{\lambda}+n\right)(t-s)\lambda}\left(1-e^{-\lambda(t-s)}\right)^n . \quad \textbf{(3.22, 4.13, 5.9)}$$

It depends on the length of the time interval $t - s$ and on k and is thus stationary and not independent.

The first passage distribution is

$$f_n(t) = \frac{1}{\left(\frac{n_0\lambda + v - \lambda}{\lambda}\right)!} e^{-(n_0\lambda + v - \lambda)t}\left(1 - e^{-\lambda t}\right)^{n-n_0}\left\{(n\lambda + v - \lambda)e^{-\lambda t} - (n_0\lambda + v - \lambda)\right\}$$

7.1.5 Polya Process

From the Polya process, we found the distribution of the increments to be Negative binomial distribution with $p = \frac{1 + \lambda as}{1 + \lambda at}$ and $q = 1 - \frac{1 + \lambda as}{1 + \lambda at} = \frac{\lambda a(t-s)}{1 + \lambda at}$.

$$p_{k,k+j}(s,t) = \binom{j + k + \frac{1}{a} - 1}{j} \left(\frac{1 + \lambda as}{1 + \lambda at}\right)^{k+\frac{1}{a}} \left(\frac{\lambda a(t - s)}{1 + \lambda at}\right)^{j} \quad \textbf{(3.28, 4.17, 5.10)}$$

It depends on the length of the time interval $t - s$ and on k and is thus stationary and not independent. The first passage distribution is $f_n(t) = \frac{\Gamma(n+\frac{1}{a})}{\Gamma\frac{1}{a}\,\Gamma n} \times \frac{\left(\frac{1}{\lambda a}\right)^{\frac{1}{a}} t^{n-1}}{\left(t + \frac{1}{\lambda a}\right)^{n+\frac{1}{a}}}$, $t > 0$, $n > 0$, $\frac{1}{a} > 0$, $\frac{1}{\lambda a} > 0$ which is a generalized Pareto distribution.

$$E(T_n) = \frac{n}{\lambda(1 - a)} \quad \Rightarrow \quad 0 < a < 1 \quad \textbf{(6.14)}$$

$$\text{var}(T_n) = \frac{a\,n(n+\frac{1}{a} - 1)}{\lambda^2 (1 - a)^2 (1 - 2a)}, \quad n > 0 \quad \text{and} \quad 0 < a < \frac{1}{2} \quad \textbf{(6.15)}$$

7.2 Conclusion

In each case of the birth processes, the various approaches lead to same distribution. The distributions that emerged from increments in pure birth process were power series distributions and were all discrete.

The first passage distributions that emerged from pure birth process were continuous distributions

7.3 Recommendation for Further Research

(1) Determine the distribution emerging from pure birth processes when the birth rate is a distribution function.

(2) Determine the distribution emerging from pure birth processes when the birth rate change over certain time intervals

(3) Determine the distribution emerging from pure birth processes when the birth rate is a survival function

(4) Determine the distribution emerging from pure birth processes when more than one births are allowed over between time t and $t + \Delta t$

7.4 References

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